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SOME INEQUALITIES INVOLVING THE DISTANCE SIGNLESS LAPLACIAN EIGENVALUES OF GRAPHS

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Abstract. Given a simple graph G, the distance signlesss Laplacian $D^Q(G) = Tr(G) + D(G)$ is the sum of vertex transmissions matrix $Tr(G)$ and distance matrix $D(G)$. In this paper, thanks to the symmetry of $D^Q(G)$, we obtain novel sharp bounds on the distance signless Laplacian eigenvalues of G, and in particular the distance signless Laplacian spectral radius. The bounds are expressed through graph diameter, vertex covering number, edge covering number, clique number, independence number, domination number as well as extremal transmission degrees. The graphs achieving the corresponding bounds are delineated. In addition, we investigate the distance signless Laplacian spectrum induced by Indu-Bala product, Cartesian product as well as extended double cover graph.

1. Introduction

In this paper we consider a simple connected graph $G(V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and $E(G)$ is the edge set. We assume its *order* is $n = |V(G)|$ and *size* is $m = |E(G)|$. For a vertex $v \in V(G)$, $N(v)$ denotes its *neighborhood*. The *degree* of v is represented by $d_G(v)$ or d_v for brevity. The *distance* between two vertices u and v is denoted by d_{uv}. The *distance matrix* is defined as $D(G) = (d_{uv})_{u,v\in V(G)}$ [6]. The *complement* of G is represented by $\overline{G}$. The *transmission* of a vertex v is $Tr_G(v) = \sum\limits_{u\in V(G)} d_{uv}$. G is called *k-transmission regular* if for any vertex $Tr_G(v) \equiv k$. Let $\sigma(G)$ be the *Wiener index* or *transmission*. $Tr_G(v_i)$ (or Tr_i for short) is also known as *transmission degree*. $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is called *transmission degree sequence* and $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ is

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a diagonal matrix. The *second transmission degree* of v_i is expressed as $T_i = \sum_{j=1}^{n} d_{ij}Tr_j$. Standard terminology is utilized throughout the paper, see textbooks e.g. [15] or [16]. Some classical graphs are used as follows: K_n for the complete graph, $K_{s,t}$ for the complete bipartite graph, P_n for the path, C_n for the cycle, and S_n for the star.

In the work [7, 8, 9], the authors examined the (signless) Laplacian for graph distance matrix. *Distance Laplacian matrix* of G is defined as $D^L(G) = Tr(G) - D(G)$ and *distance signless Laplacian matrix* is $D^Q(G) = Tr(G) + D(G)$. When G is connected, $D^Q(G)$ is nonnegative and irreducible. We rank its eigenvalues as $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$, where $\rho_1(G)$ is the *distance signless Laplacian spectral radius*. Taking into consideration of the Perron-Frobenius theorem, $\rho_1(G)$ is simple and there is a unique positive unit eigenvector X associated with $\rho_1(G)$. This vector is known as the *distance signless Laplacian Perron vector*.

As the research of distance signless Laplacian spectra is of great significance for both algebraic graph theory and practical applications, determining eigenvalue bounds have received intensive attention in the literature. Xing et al. [27] recently determined minimum distance signless Laplacian spectral radius among trees with fixed order. Further in [28], Xing et al. identified minimum and second minimum distance signless Laplacian spectral radii among bicyclic graphs with fixed order. In [20], bounds for these spectral radius are presented through vertex transmissions. We refer the readers to [1, 2, 3, 4, 8, 9, 10, 11, 13, 22, 25, 27, 28] for more results related to such eigenvalues and spectral radii.

The rest of the paper is organized as follows. In Section 2, we obtain some bounds for the eigenvalues of the distance signless Laplacian matrix, in particular for the spectral radius through diameter, covering number, clique number, independence number, domination number, extremal transmission degrees. The graphs achieving the corresponding bounds are determined. In Section 3, we study the eigenvalues of distance signless Laplacian derived by graph operations including Indu-Bala product, Cartesian product and extended double cover graph.

2. Some bounds on the distance signless Laplacian eigenvalues

In this section, we are concerned with some bounds for the distance signless Laplacian eigenvalues, in particular those for spectral radius. We begin with the following lemmas.

Lemma 2.1. [27] *Suppose that G is connected.*

$$\rho_1(G) \geq \frac{4\sigma(G)}{n},$$

where the equality holds if and only if G is transmission regular.

Lemma 2.2. [5] *Suppose that G is connected. It has two distinct distance signless Laplacian eigenvalues if and only if it is a complete graph.*

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Lemma 2.3. [26] *If $A \in \mathbb{R}^{n\times n}$ is nonnegative with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \ldots, r_n$, then*

$$\min_{1\le i\le n} r_i \le \lambda(A) \le \max_{1\le i\le n} r_i.$$

When A is irreducible, the above equalities are true if and only if the row sums are all equivalent.

The following lemmas will be also helpful for proving of our main results in the sequel.

Lemma 2.4. [14] *Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ contain positive numbers. Then*

$$\frac{\Big(\sum_{i=1}^n a_i^2\Big)\Big(\sum_{i=1}^n b_i^2\Big)}{\Big(\sum_{i=1}^n a_ib_i\Big)^2} \le \frac{\Big(ab+AB\Big)^2}{4abAB},$$

where $0 < a \le a_i \le A$ and $0 < b \le b_i \le B$, $i = 1, \ldots, n$.

Lemma 2.5. [18] *Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ contain positive numbers. Then*

$$\frac{aA\sum_{i=1}^n b_i^2 + bB\sum_{i=1}^n a_i^2}{ab+AB} \le \sum_{i=1}^n a_ib_i,$$

where $0 < a \le a_i \le A$ and $0 < b \le b_i \le B$, $i = 1, \ldots, n$. The equality holds if and only if, for each i, either $(a_i, b_i) = (a, B)$ or $(a_i, b_i) = (A, b)$, where the alternative depends upon the particular value of i.

Our first theorem gives an inequality using transmission $\sigma(G)$, maximum transmission degree $Tr_{\max}$ and minimum transmission degree $Tr_{\min}$.

Theorem 2.6. *Suppose G is connected and let $Tr_{\max}$ and $Tr_{\min}$ be the two extremal transmission degrees.*

(i)

$$\sum_{i=1}^n Tr_i^2 \le \frac{\sigma^2(G)}{n}\left(\sqrt{\frac{Tr_{\max}}{Tr_{\min}}} + \sqrt{\frac{Tr_{\min}}{Tr_{\max}}}\right)^2, \tag{2.1}$$

with equality if G is transmission regular.

(ii)

$$\sum_{i=1}^n Tr_i^2 \le 2(Tr_{\max} + Tr_{\min})\sigma(G) - nTr_{max}Tr_{\min}, \tag{2.2}$$

where the equality holds if and only if G has only two type of transmission degrees $Tr_{\min}$ and $Tr_{\max}$.

Proof. (i) Let $(a_1, \ldots, a_n) = (Tr_1, \ldots, Tr_n)$ and $(b_1, \ldots, b_n) = (1, \ldots, 1)$. Applying Lemma 2.4 with $a = Tr_{\min}, A = Tr_{\max}$, and $b = B = 1$, we obtain the required result. If G is a transmission regular graph, then it is not difficult to verify that the equality in (2.1) holds.

(ii) Let $(a_1, \ldots, a_n) = (Tr_1, \ldots, Tr_n)$ and $(b_1, \ldots, b_n) = (1, \ldots, 1)$. Applying Lemma 2.5 with $a = Tr_{\min}, A = Tr_{\max}$ and $b = B = 1$, we obtain the result.

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Suppose that the equality in (2.2) holds. Applying Lemma 2.5, we observe that G has only two types of transmission degrees $Tr_{\min}$ and $Tr_{\max}$. Conversely, it is easy to verify that the equality in (2.2) holds if G has only two type of transmission degrees $Tr_{\min}$ and $Tr_{\max}$. $\square$

The following theorem gives bounds for the k-th largest distance signless Laplacian eigenvalue.

Theorem 2.7. *Suppose G is connected having diameter d and minimum degree δ. Let*

$$\begin{aligned}\varphi(G) \;\; &= \;\; \min\Big\{n^2(n-1)\Big(\frac{n^2(n-1)}{4}+d^2\Big)-4\sigma^2(G),\\ &\qquad n^2\Big(\big(nd-\frac{d(d-1)}{2}-1-\delta(d-1)\big)^2+(n-1)d^2\Big)-4\sigma^2(G)\Big\}.\end{aligned}$$

Then, for any $k=1,\ldots,n$, we have

$$\frac{1}{n}\left\{2\sigma(G)-\sqrt{\frac{k-1}{n-k+1}\varphi(G)}\right\}\leq \rho_k(G)\leq \frac{1}{n}\left\{2\sigma(G)+\sqrt{\frac{n-k}{k}\varphi(G)}\right\}. \tag{2.3}$$

Proof. First we prove the upper bound. Clearly,

$$\Big(\big(D^Q(G)\big)^2\Big)=\sum_{i=1}^{k}\rho_i^2+\sum_{i=k+1}^{n}\rho_i^2\geq\frac{\Big(\sum_{i=1}^{k}\rho_i\Big)^2}{k}+\frac{\Big(\sum_{i=k+1}^{n}\rho_i\Big)^2}{n-k}.$$

Let $M_k=\sum_{i=1}^{k}\rho_i$. Then

$$\Big(\big(D^Q(G)\big)^2\Big)\geq\frac{M_k^2}{k}+\frac{(2\sigma(G)-M_k)^2}{n-k},$$

which implies

$$\rho_k(G)\leq\frac{M_k}{k}\leq\frac{1}{n}\left\{2\sigma(G)+\sqrt{\frac{n-k}{k}\Big(n\cdot\Big(\big(D^Q(G)\big)^2\Big)-4\sigma^2(G)\Big)}\right\}.$$

Since for each $1\leq i\leq n$, we have $Tr_i\leq\frac{n(n-1)}{2}$, hence we observe that

$$\begin{aligned}n\cdot\Big(\big(D^Q(G)\big)^2\Big)-4\sigma^2(G) \;\; &= \;\; n\sum_{i=1}^{n}Tr_i^2+2n\sum_{1\leq i<j\leq n}(d_{ij})^2-4\sigma^2(G)\\ &\leq \;\; n\frac{n^3(n-1)^2}{4}+2n\frac{n(n-1)}{2}d^2-4\sigma^2(G)\\ &= \;\; n^2(n-1)\left(\frac{n^2(n-1)}{4}+d^2\right)-4\sigma^2(G).\end{aligned}$$

Also, since $Tr_i \leq nd - \frac{d(d-1)}{2} - 1 - d_i(d-1)$, we have

$$\begin{aligned}
& n \cdot \left(\left(D^Q(G)\right)^2 \right) - 4\sigma^2(G) \\
= \; & n \sum_{i=1}^{n} Tr_i^2 + 2n \sum_{1 \leq i < j \leq n} (d_{ij})^2 - 4\sigma^2(G) \\
\leq \; & n^2 \left(nd - \frac{d(d-1)}{2} - 1 - \delta(d-1) \right)^2 + 2n \frac{n(n-1)}{2} d^2 - 4\sigma^2(G) \\
= \; & n^2 \left(\left(nd - \frac{d(d-1)}{2} - 1 - \delta(d-1) \right)^2 + (n-1)d^2 \right) - 4\sigma^2(G).
\end{aligned}$$

Hence we get the right-hand side inequality of (2.3).

Now, we prove the lower bound. Let $N_k = \sum_{i=k}^{n} \rho_i$. We know

$$\begin{aligned}
\left(\left(D^Q(G)\right)^2 \right) = \sum_{i=1}^{k-1} \rho_i^2 + \sum_{i=k}^{n} \rho_i^2 \; & \geq \; \frac{\left(\sum_{i=1}^{k-1} \rho_i \right)^2}{k-1} + \frac{\left(\sum_{i=k}^{n} \rho_i \right)^2}{n-k+1} \\
& = \; \frac{(2\sigma(G) - N_k)^2}{k-1} + \frac{N_k^2}{n-k+1}.
\end{aligned}$$

Hence

$$\rho_k(G) \geq \frac{N_k}{n-k+1} \geq \frac{1}{n} \left\{ 2\sigma(G) - \sqrt{\frac{k-1}{n-k+1} \left(n \cdot \left(\left(D^Q(G)\right)^2 \right) - 4\sigma^2(G) \right)} \right\},$$

and we get the left-hand side inequality of (2.3). □

In particular, taking $k = 1$ and $k = n$ in Theorem 2.7, we have the following observations.

Corollary 2.8. *Suppose G is connected with the extremal distance signless Laplacian eigenvalues $\rho_n(G)$ and $\rho_1(G)$. We have*

$$\rho_1(G) \leq \frac{2\sigma(G)}{n} + \frac{\sqrt{(n-1) \left(n(2\sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2) - 4\sigma^2(G) \right)}}{n},$$

and

$$\rho_n(G) \geq \frac{2\sigma(G)}{n} - \frac{\sqrt{(n-1) \left(n(2\sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2) - 4\sigma^2(G) \right)}}{n}.$$

Now, we state the following observation.

Corollary 2.9. *Suppose G is connected having diameter d.*

$$\rho_1(G) \leq \frac{2\sigma(G) + \sqrt{(n-1)\left(n(n-1)W - 4\sigma^2(G)\right)}}{n},$$

where $W = nd^2 + (n+2)\sigma(G) - \frac{n^2(n-1)}{2}$.

Proof. Since for each $i = 1, 2, \dots, n$, we have $n-1 \leq Tr_i \leq \frac{n(n-1)}{2}$, hence by Theorem 2.6 (ii), we get

$$2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 \leq 2\frac{n(n-1)}{2} d^2,$$

$$\sum_{i=1}^{n} Tr_i^2 \leq (n^2+n-2)\sigma(G) - \frac{n^2(n-1)^2}{2}.$$

Then by Corollary 2.8, the result follows. □

The following shows an upper bound for $\rho_1(G)$ through transmission degrees, the second transmission degrees as well as a parameter α.

Theorem 2.10. *Recall that $\{Tr_1, Tr_2, \dots, Tr_n\}$ and $\{T_1, T_2, \dots, T_n\}$ are the transmission degree sequence and the second transmission degree sequence.*

$$\rho_1(G) \leq \max_{1 \leq i \leq n} \left\{ \frac{-\alpha + \sqrt{\alpha^2 + 8Tr_i(Tr_i + \frac{T_i}{Tr_i} + \alpha)}}{2} \right\}, \tag{2.4}$$

where $\alpha \geq 0$ is an unknown parameter. Equality occurs if and only if G is a transmission regular graph.

Proof. Let $X = (x_1, \dots, x_n)$ be the distance signless Laplacian Perron vector of G and $x_i = \max\{x_j | j = 1, 2, \dots, n\}$. Since

$$\rho_1(G)^2 X = \left(D^Q(G)\right)^2 X = (Tr + D)^2 X = Tr^2 X + TrDX + DTrX + D^2 X,$$

we have

$$\rho_1^2(G) x_i = Tr_i^2 x_i + Tr_i \sum_{j=1}^{n} d_{ij} x_j + \sum_{j=1}^{n} d_{ij} Tr_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ij} d_{jk} x_k.$$

Now, we consider a simple quadratic function of $\rho_1(G)$:

$$\left(\rho_1^2(G) + \alpha\rho_1(G)\right) X = (Tr^2 X + TrDX + DTrX + D^2 X) + \alpha(TrX + DX).$$

Considering the i-th equation, we have

$$\begin{aligned} \left(\rho_1^2(G) + \alpha\rho_1(G)\right) x_i &= Tr_i^2 x_i + Tr_i \sum_{j=1}^{n} d_{ij} x_j + \sum_{j=1}^{n} d_{ij} Tr_j x_j \\ &\quad + \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ij} d_{jk} x_k + \alpha(Tr_i x_i + \sum_{j=1}^{n} d_{ij} x_j). \end{aligned}$$

It is easy to check that the following inequalities are valid:

$$Tr_i \sum_{j=1}^{n} d_{ij} x_j \leq Tr_i^2 x_i, \ \sum_{j=1}^{n} d_{ij} Tr_j x_j \leq T_i x_i,$$

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$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \leq T_i x_i,\ \sum_{j=1}^{n} d_{ij}x_j \leq Tr_i x_i.$$

Hence, we have

$$\begin{aligned}
&\big(\rho_1^2(G)+\alpha\rho_1(G)\big)x_i \leq 2Tr_i^2x_i + 2T_ix_i + 2\alpha Tr_ix_i\\
\Rightarrow\ &\rho_1^2(G)+\alpha\rho_1(G) - (2Tr_i^2 + 2T_i + 2\alpha Tr_i) \leq 0\\
\Rightarrow\ &\rho_1(G) \leq \frac{-\alpha+\sqrt{\alpha^2+8Tr_i(Tr_i+\frac{T_i}{Tr_i}+\alpha)}}{2}.
\end{aligned}$$

From this the result follows.

Assume that equality occurs in (2.4), then each of the above inequalities in the above argument occur as equalities. Since each of the inequalities

$$Tr_i\sum_{j=1}^{n} d_{ij}x_j \leq Tr_i^2x_i,\ \sum_{j=1}^{n} d_{ij}Tr_jx_j \leq T_ix_i,$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \leq T_ix_i,\ \sum_{j=1}^{n} d_{ij}x_j \leq Tr_ix_i$$

occur as equalities if and only if G is a transmission regular graph, hence the equality occurs in (2.4) if and only if G is a transmission regular graph. □

The following upper bound regarding distance signless Laplacian spectral radius $\rho_1(G)$ was obtained in [22]:

$$\rho_1(G) \leq \max_{1\leq i\leq n}\left\{\sqrt{2Tr_i^2+2T_i}\right\}, \tag{2.5}$$

where the equality holds if and only if $Tr_i^2 + T_i$ is same for all i.

Remark 2.11. *For a connected graph G with the property that $T_i \leq Tr_i^2$, for all i. Then we have*

$$\frac{-\alpha+\sqrt{\alpha^2+8Tr_i(Tr_i+\frac{T_i}{Tr_i}+\alpha)}}{2} \leq \sqrt{2Tr_i^2+2T_i}.$$

Hence, the upper bound given by Theorem 2.10 improves the upper bound given by (2.5).

If in particular we take the parameter α in Theorem 2.10 as vertex covering number, edge covering number, clique number, independence number, domination number, minimum transmission degree, maximum transmission degree, then Theorem 2.10 leads to an upper bound for $\rho_1(G)$ in terms of vertex covering number, edge covering number, clique number, independence number, domination number, minimum transmission degree, maximum transmission degree, respectively.

Let $x_i = \min\{x_j|\ j = 1,\dots,n\}$ be the minimum among the elements of the distance signless Laplacian Perron vector $X = (x_1,\dots,x_n)$ of the graph G. Proceeding similar to Theorem 2.10, we derive the following lower bound for $\rho_1(G)$ via (second) transmission degrees as well as parameter α.

Theorem 2.12. *We have*

$$\rho_1(G) \geq \min_{1\leq i\leq n}\left\{\frac{-\alpha+\sqrt{\alpha^2+8Tr_i(Tr_i+\frac{T_i}{Tr_i}+\alpha)}}{2}\right\}, \tag{2.6}$$

where $\alpha \geq 0$ is an unknown parameter. The equality occurs if and only if G is a transmission regular graph.

The following lower bound for the distance signless Laplacian spectral radius $\rho_1(G)$ was obtained in [22]:

$$\rho_1(G) \geq \min_{1\leq i\leq n}\left\{\sqrt{2Tr_i^2+2T_i}\right\}, \tag{2.7}$$

where the equality holds if and only if $Tr_i^2 + T_i$ is same for all i.

Similar to Remark 2.11, it can be seen that the lower bound given by Theorem 2.12 improves the lower bound given by (2.7) for all graphs G with $T_i \geq Tr_i^2$, for all i.

Theorem 2.13. [25]

Suppose G has minimum degree δ_1 and second minimum degree δ_2.

$$\rho_1(G) \leq 2dn - d(d-1) - 2 - (d-1)(\delta_1+\delta_2),$$

where the equality holds if and only if G is a regular graph with diameter $d \leq 2$.

Remark 2.14. *It is worth noting that, if we take $\alpha = Tr_i$, then for any connected graph G of order n having minimum degree δ and diameter d, since*

$$Tr_i \leq nd - \frac{d(d-1)}{2} - 1 - d_i(d-1) \leq nd - \frac{d(d-1)}{2} - 1 - \delta(d-1).$$

Hence we have

$$\begin{aligned}\frac{-\alpha+\sqrt{\alpha^2+8Tr_i(Tr_i+\frac{T_i}{Tr_i}+\alpha)}}{2} &\leq 2Tr_i \\ &\leq 2Tr_{\max} \leq 2dn - d(d-1) - 2 - 2\delta(d-1).\end{aligned}$$

Therefore, the upper bound given by Theorem 2.10 improves the upper bound given by Theorem 2.13, provided that $T_i \leq Tr_i^2$, for all i.

Theorem 2.15. [25] *Suppose G has maximum degree Δ_1 and second maximum degree Δ_2.*

$$\rho_1(G) \geq 4n - 4 - \Delta_1 - \Delta_2,$$

where the equality holds if and only if G is a regular graph with diameter $d \leq 2$.

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Remark 2.16. *Similar to Remark 2.14, if we take $\alpha = Tr_i$, then for any connected graph G of order n having maximum degree Δ, since $Tr_i \geq d_i + 2(n-1-d_i) = 2n-2-d_i \geq 2n-2-\Delta$, hence we have*

$$\frac{-\alpha+\sqrt{\alpha^2+8Tr_i(Tr_i+\frac{T_i}{Tr_i}+\alpha)}}{2} \geq 2Tr_i \geq 2Tr_{\min} \geq 4n-4-2\Delta.$$

Therefore, the lower bound given by Theorem 2.12 improves the lower bound given by Theorem 2.15, provided that $T_i \geq Tr_i^2$, for all i.

Next we obtain a lower bound for $\rho_1(G)$ through transmission $\sigma(G)$ and maximum transmission degree $Tr_{\max}$.

Theorem 2.17. *Suppose G is connected having diameter d. If the transmission degree sequence of G is $\{Tr_1, Tr_2, \ldots, Tr_n\}$, then*

$$\begin{aligned} \rho_1(G) \geq \frac{1}{n}\Bigg[&8(\sigma(G)+n(n-1)-m)+\frac{4}{Tr_{\max}}\sum_{i=1}^{n}(2n-d_i-2)Tr_i^2 \\ &-\frac{8}{Tr_{\max}^2}(M_1+2M_2)\Bigg], \end{aligned} \tag{2.8}$$

where $M_1(G) = \sum_{i<j,d_{ij}=1} Tr_iTr_j$ and $M_2(G) = \sum_{i<j,d_{ij}\geq 2} Tr_iTr_j$. Moreover, the equality holds in (2.8) if and only if G is a complete graph K_n or G is isomorphic to a transmission regular graph of diameter 2.

Proof. Let $D^Q(G) = [q_{ij}]$ and $X = (x_1, \ldots, x_n)^T$ be any unit vector. As the spectral radius of $D^Q(G)$ and $Tr(G)^{-1}D^Q(G)Tr(G)$ are same, we have

$$X^T\left\{Tr(G)^{-1}D^Q(G)Tr(G)\right\}X \leq \rho_1(G)X^TX. \tag{2.9}$$

Since $\sum_{i=1}^{n} x_i^2 = 1$, hence from (2.9) we get

$$\rho_1(G) \geq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{Tr_j}{Tr_i}q_{ij}(x_i+x_j)^2. \tag{2.10}$$

As X is a unit vector, we assume that $X = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^T$. From (2.10), we get

$$\begin{aligned} \rho_1(G) &\geq \frac{4}{n}\sum_{i<j}d_{ij}\left(\frac{Tr_i}{Tr_j}+\frac{Tr_j}{Tr_i}\right)+\frac{4}{n}\sum_{i=1}^{n}Tr_i \\ &\geq \frac{4}{n}\sum_{i<j,d_{ij}=1}\left(\frac{Tr_i}{Tr_j}+\frac{Tr_j}{Tr_i}\right) \\ &\quad +\frac{8}{n}\sum_{i<j,d_{ij}\geq 2}\left(\frac{Tr_i}{Tr_j}+\frac{Tr_j}{Tr_i}\right)+\frac{8}{n}\sigma(G). \end{aligned} \tag{2.11}$$

Now

$$\begin{aligned}
\sum_{i<j, d_{ij}=1}\left(\frac{Tr_i}{Tr_j}+\frac{Tr_j}{Tr_i}\right) &= 2m+\sum_{i<j, d_{ij}=1}\frac{(Tr_i-Tr_j)^2}{Tr_iTr_j}\\
&\geq 2m+\frac{1}{Tr^2_{\max}}\Bigg(\sum_{i<j, d_{ij}=1}(Tr_i^2+Tr_j^2)\\
&\qquad -2\sum_{i<j, d_{ij}=1}Tr_iTr_j\Bigg) \qquad (2.12)\\
&= 2m+\frac{1}{Tr^2_{\max}}\Bigg(\sum_{i=1}^{n}d_iTr_i^2\\
&\qquad -2\sum_{i<j, d_{ij}=1}Tr_iTr_j\Bigg), \qquad (2.13)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i<j, d_{ij}\geq 2}\left(\frac{Tr_i}{Tr_j}+\frac{Tr_j}{Tr_i}\right) = n(n-1)-2m+\sum_{i<j, d_{ij}\geq 2}\frac{(Tr_i-Tr_j)^2}{Tr_iTr_j}\\
\geq\ & n(n-1)-2m\\
& +\frac{1}{Tr^2_{\max}}\left(\sum_{i<j, d_{ij}\geq 2}(Tr_i^2+Tr_j^2)-2\sum_{i<j, d_{ij}\geq 2}Tr_iTr_j\right) \qquad (2.14)\\
=\ & n(n-1)-2m\\
& +\frac{1}{Tr^2_{\max}}\left(\sum_{i=1}^{n}(n-1-d_i)Tr_i^2-2\sum_{i<j, d_{ij}\geq 2}Tr_iTr_j\right). \qquad (2.15)
\end{aligned}$$

Using (2.13) and (2.15) in (2.11), we get the required result (2.8).

Suppose that equality holds in (2.8). Then $X=(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}})^T$ is an eigenvector corresponding to eigenvalue $\rho_1(G)$ of $Tr(G)^{-1}D^Q(G)Tr(G)$. From equality in (2.11), it is seen $d\leq 2$. From equality in (2.12), it is seen $Tr_1=\cdots=Tr_n$. Similarly, from equality in (2.14), we have $Tr_1=\cdots=Tr_n$. Consequently, $G\cong K_n$ or G is isomorphic to a transmission regular graph of diameter 2.

On the other hand, one can easily see that the equality holds in (2.8) if $G=K_n$ or G is a transmission regular graph of diameter 2. $\square$

Suppose the matrix of a graph G takes the form

$$M=\begin{pmatrix} X & \beta & \cdots & \beta & \beta\\ \beta^t & B & \cdots & C & C\\ \vdots & \vdots & \ldots & \vdots & \vdots\\ \beta^t & C & \cdots & B & C\\ \beta^t & C & \cdots & C & B \end{pmatrix}, \qquad (2.16)$$

where $X \in R^{t\times t}$, $\beta \in R^{t\times s}$ and $B, C \in R^{s\times s}$, satisfying $n = t + cs$, where c is the number of copies of B. Then the following technique given in [21] can be applied in order to extract the spectrum via the union of building blocks. Let $\sigma^{[k]}(Y)$ represent the multi-set formed by k copies of the spectrum of Y (i.e. $\sigma(Y)$).

Lemma 2.18. [21] *Let M be a matrix as in (2.16), with $c \geq 1$ copies of the block B. Therefore,*
(i) $\sigma^{[c-1]}(B - C) \subseteq \sigma(M)$;
(ii) $\sigma(M) \setminus \sigma^{[c-1]}(B - C) = \sigma(M')$ *is the set of the remaining $t + s$ eigenvalues of M, where* $M' = \begin{pmatrix} X & \sqrt{c}.\beta \\ \sqrt{c}.\beta^t & B + (c-1)C \end{pmatrix}$.

Let $T_{a,b}$, with $a + b = n - 2$ and $a \geq b \geq 1$ be the tree obtained by joining an edge between the root vertices of stars $K_{1,a}$ and $K_{1,b}$(the vertex of degree greater than one in a star is called root vertex). It is clear that a tree with diameter $d = 3$ is always of the form $T_{a,b}$. The following gives the distance signless Laplacian spectrum of $T_{a,b}$.

Lemma 2.19. *The distance signless Laplacian spectrum of $T_{a,b}$ is*

$$\{y_1 - 2^{[b-1]}, y_2 - 2^{[a-1]}, x_1, x_2, x_3, x_4\},\ y_1 = 2a + 3b + 1, y_2 = 2b + 3a + 1,$$

where $x_1 \geq x_2 \geq x_3 \geq x_4$ are the eigenvalues of the matrix

$$M_2 = \begin{pmatrix} 2a+b+1 & 1 & 2\sqrt{a} & \sqrt{b} \\ 1 & 2b+a+1 & \sqrt{a} & 2\sqrt{b} \\ 2\sqrt{a} & \sqrt{a} & y_1 + 2(a-1) & 3\sqrt{ab} \\ \sqrt{b} & 2\sqrt{b} & 3\sqrt{ab} & y_2 + 2(b-1) \end{pmatrix}.$$

Proof. Let $V(K_{1,b}) = \{v_1, u_1, \ldots, u_b\}$ and $V(K_{1,a}) = \{v_2, w_1, \ldots, w_a\}$. The vertex set of $T_{a,b}$ is $V(T_{a,b}) = \{v_1, v_2, u_1, \ldots, u_b, w_1, \ldots, w_a\}$. It is not hard to see that $Tr(v_1) = 2a + b + 1$, $Tr(v_2) = 2b + a + 1$, $Tr(u_i) = 2b + 3a + 1 = y_2$ and $Tr(w_j) = 2a + 3b + 1 = y_1$, for $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, a$. With this labeling, the distance signless Laplacian matrix of $T_{a,b}$ takes the form $D^Q(T_{a,b}) = \begin{pmatrix} X & \beta & \beta & \cdots & \beta \\ \beta^t & y_1 & 2 & \cdots & 2 \\ \beta^t & 2 & y_1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta^t & 2 & 2 & \cdots & y_1 \end{pmatrix}$, where $\beta = \begin{pmatrix} 2 \\ 1 \\ 3 \\ \vdots \\ 3 \end{pmatrix}$ and $X = \begin{pmatrix} 2a+b+1 & 1 & 1 & \cdots & 1 \\ 1 & 2b+a+1 & 2 & \cdots & 2 \\ 1 & 2 & y_2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2 & 2 & \cdots & y_2 \end{pmatrix}$. Using Lemma 2.18, with $B = [y_1]$, $C = [2]$ and $c = a$, it follows that $\sigma(D^Q(T_{a,b})) = \sigma^{[a-1]}(B - C) \cup \sigma(M_1) = \sigma^{[a-1]}([y_1 - 2]) \cup \sigma(M_1)$, where $M_1 = \begin{pmatrix} X & \sqrt{a}\beta \\ \sqrt{a}\beta & y_1 + 2(a-1) \end{pmatrix}$. Interchanging the third and last column of M_1 and then third and last row of the resulting matrix, we obtain a matrix similar to M_1. In the resulting matrix taking

$$X = \begin{pmatrix} 2a+b+1 & 1 & 2\sqrt{a} \\ 1 & 2b+a+1 & \sqrt{a} \\ 2\sqrt{a} & \sqrt{a} & y_1+2(a-1) \end{pmatrix}, \beta = \begin{pmatrix} 1 \\ 2 \\ 3\sqrt{a} \end{pmatrix},$$

$B = [y_2]$, $C = [2]$ and $c = b$ in Lemma 2.18, It follows that $\sigma(M_1) = \sigma^{[b-1]}(B-C) \cup \sigma(M_2) = \sigma^{[b-1]}([y_2-2]) \cup \sigma(M_2)$, where M_2 is the matrix given in the statement. That completes the proof. □

The next result concerns with the distance signless Laplacian spectral radius of trees.

Theorem 2.20. *Let T be a tree of order $n \geq 2$ having diameter d.*
(i) *If $d = 1$, then $\rho_1(T) = 1$.*
(ii) *If $d = 2$, then $\rho_1(T) = \frac{5n-8+\sqrt{9(n-2)^2+4(n-1)}}{2}$.*
(iii) *If $d = 3$, then $\rho_1(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 defined in Lemma 2.19.*
(iv) *If $d \geq 4$, then let $P = v_1v_2\cdots v_dv_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\rho_1(T) \geq \frac{6n + d(d-7) + (a_1+a_2)(d-4) + 2\sqrt{(a_2-a_1)^2(d-2)^2 + 4d^2}}{2}.$$

Proof. If T is a tree of diameter $d = 1$, then $T \cong K_2$ and so $\rho_1(T) = 1$. If T is a tree of diameter $d = 2$, then $T \cong K_{1,n-1}$ and so $\rho_1(T) = \frac{5n-8+\sqrt{9(n-2)^2+4(n-1)}}{2}$, (see [7]). If T is a tree of diameter $d = 3$, then $T \cong T_{a,b}$ and so using Lemma 2.19, it follows that $\rho_1(T) = x_1$, where x_1 is the largest eigenvalue of M_2 given in Lemma 2.19. So, suppose that diameter of tree T is at least 4, then $n \geq 5$. Let $v_1v_2\ldots v_{d+1}$ be a diametral path of T, and let a_1 and a_2 be the number of pendent neighbors of v_2 and v_d, respectively. We have

$$\begin{aligned} Tr(v_1) &\geq 2(a_1-1) + 1 + 2 + \cdots + (d-1) + da_2 + 3(n - a_1 - a_2 - d + 1) \\ &= 3n - a_1 + a_2(d-3) - 3d + 1 + \frac{d(d-1)}{2}. \end{aligned}$$

Similarly

$$Tr(v_{d+1}) \geq 3n - a_2 + a_1(d-3) - 3d + 1 + \frac{d(d-1)}{2}.$$

Let M be the principal sub-matrix of $D^Q(T)$ induced by the vertices v_1 and v_{d+1}. Then

$$M = \begin{pmatrix} Tr(v_1) & d \\ d & Tr(v_{d+1}) \end{pmatrix},$$

thus

$$\begin{aligned} \rho_1(M) &= \frac{Tr(v_1) + Tr(v_{d+1}) + \sqrt{(Tr(v_1) - Tr(v_{d+1}))^2 + 4d^2}}{2} \\ &\geq \frac{6n + d(d-7) + (a_1+a_2)(d-4) + 2) + \sqrt{(a_2-a_1)^2(d-2)^2 + 4d^2}}{2}. \end{aligned}$$

Now, by Interlacing Theorem [15], we have $\rho_1(T) \geq \rho_1(M)$. □

The following observation follows from Theorem 2.20.

Corollary 2.21. *Suppose a tree T has diameter $d \geq 4$.*

$$\rho_1(T) \geq \frac{1}{2}\Big(6n + d^2 - 5d + 2\Big).$$

Proof. Using $a_1, a_2 \geq 0$ in Theorem 2.20 (iv), the result follows. □

3. Distance signless Laplacian spectrum of some graph classes

It is well recognized that some of the graph families can be determined by their spectra. In this section, we are interested in investigating the distance signless Laplacian spectrum of graphs with diameter 2 and 3 which are derived from graph operations. In particular, we consider Cartesian product, Indu-Bala product and extended double cover graph. Some lemmas are in order.

Lemma 3.1. [19] *Suppose*

$$A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

is a symmetric block matrix. The spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

We first consider the Cartesian product of graphs. The *Cartesian product* $G \times H$ of two graphs is built up over vertex set $V(G) \times V(H)$ in which (u_1, u_2) and (v_1, v_2) are adjacent when $u_1 = v_1$ and $u_2 v_2 \in E(H)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G)$.

Theorem 3.2. *Suppose G is r-regular with diameter 1 or 2 and adjacency spectrum $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$. The distance signless Laplacian spectrum of $H = G \times K_2$ is of the form $\big\{2(5n - 2r - 4)$, $2(2n - r - 2)$, $5n - 2r - 4^{[n-1]}, 5n - 2\lambda_i - 2r - 8\big\}$, $i = 2, 3, \ldots, n$.*

Proof. Since G is a graph with diameter no more than 2, diameter of H is 2 or 3 and H is $(r+1)$-regular. The distance signless Laplacian matrix of H must be of the form

$$\begin{pmatrix} A + 2\bar{A} + (5n - 2r - 4)I & A + 2\bar{A} + J \\ A + 2\bar{A} + J & A + 2\bar{A} + (5n - 2r - 4)I \end{pmatrix},$$

in which A is the adjacency matrix of G, $\bar{A}$ is the adjacency matrix of $\bar{G}$, J is the all one matrix and I is the identity matrix. Recall that $\bar{A} = J - I - A$, and the result follows from Lemma 3.1. □

The extended double cover graph of a graph is introduced by Alon [12]. For results of eigenvalues of such graphs we refer to [17]. The *extended double cover graph* of G can be viewed as a bipartite structure with partitions $X = \{x_1, x_2, \ldots x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The two vertices x_i and y_j are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G. The extended double cover graph is denoted by G^*.

Theorem 3.3. *Suppose G is r-regular with diameter 2. Let $r, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the adjacency spectrum of G. The distance signless Laplacian spectrum of G^* is of the form $\{10n-4r-8, 4n-4, 5n-2\lambda_i-2r-8, 5n+2\lambda_i-2r-4\}$, $i=2,3,\ldots,n$.*

Proof. It can be seen that G^* is $r+1$ regular graph with diameter 3. A vertex $v \in V(G^*)$ has reciprocal transmission $5n-2r-4$. Thereby, $D^Q(G^*)$ has the form

$$\begin{pmatrix} (5n-2r-6)I+2J & A+3\bar{A}+I \\ A+3\bar{A}+I & (5n-2r-6)I+2J \end{pmatrix},$$

where A, $\bar{A}$, $\bar{G}$, J, I have the same definition as in Theorem 3.2. Now the result follows from Lemma 3.1 and $\bar{A} = J-I-A$. □

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are defined on disjoint sets of V_1 and V_2 with $|V_1| = n_1$ and $|V_2| = n_2$, then the *union* is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \nabla G_2$ is characterized by $G_1 \cup G_2$ and all those edges linking V_1 to V_2.

Corollary 3.4. *The distance signless Laplacian spectrum of the extended double cover graph $C_n \nabla C_n$, is of the form $\{16(n-1), 4(2n-1), 2(5n-8), 2(3n-2), 2(4n-\lambda_i-6)^{[2]}, 2(4n+\lambda_i-4)^{[2]}\}$, $i=2,3,\ldots,n$.*

Proof. The join of C_n with another copy is a regular graph of diameter 2 having adjacency eigenvalues $n+2$, $2-n$, λ_i, (2 times) for $i=2,3,\ldots,n$, where $\{2, \lambda_2, \ldots, \lambda_n\}$ is the adjacency spectrum of C_n. Then, by Theorem 3.4, the distance signless Laplacian spectrum of extended double cover graph $C_n \nabla C_n$ is $16(n-1)$, $4(2n-1)$, $2(5n-8)$, $2(3n-2)$, $2(4n-\lambda_i-6)$, (twice) for $i=2,3,\ldots,n$ and $2(4n+\lambda_i-4)$, (twice) for $i=2,3,\ldots,n$. Hence the result follows. □

The following results concerns distance signless Laplacian spectrum of join graphs.

Theorem 3.5. *Let G_i be r_i-regular graph with order n_i and adjacency eigenvalues $\lambda_{i,1} = r_i \geq -\lambda_{i,2} \geq \ldots \geq \lambda_{i,n_i}$ for $i = 0,1,2,$. The distance signless Laplacian spectrum of $G_0 \nabla (G_1 \cup G_2)$ has eigenvalues $m+n_0-\lambda_{0,j}-r_0-4$ for $j=2,\ldots,n_0$, and $2m-n_0-\lambda_{i,j}-r_i-4$, for $i=1,2$ and $j=2,3,\ldots,n_i$, where $m=\sum_{i=0}^{2} n_i$, and eigenvalues of the following matrix*

$$\begin{pmatrix} m+3n_0-2r_0-4 & n_1 & n_2 \\ n_0 & 2m+2n_1-n_0-2r_1-4 & 2n_2 \\ n_0 & 2n_1 & 2m+2n_2-n_0-2r_2-4 \end{pmatrix}. \tag{3.1}$$

Proof. The distance signless Laplacian matrix of $F = G_0 \nabla (G_1 \cup G_2)$ takes the form

$$\begin{pmatrix} S_0 & J & J \\ J & S_1 & 2J \\ J & 2J & S_2 \end{pmatrix},$$

where $S_0 = 2J - A(G_0) + (m+n_0-r_0-4)I$, and $S_i = 2J - A(G_i) + (2m-n_0-r_i-4)I$ for $i=1,2$.

G_0 is regular and it has the all-one vector $\mathbf{1}$ as an eigenvector associated with eigenvalue r_0. Other eigenvectors are orthogonal to $\mathbf{1}$. If λ is an arbitrary adjacency eigenvalue of G_0 associated with X satisfying $\mathbf{1}^T X = 0$, $[X^T \quad 0 \quad 0]^T$ is an eigenvector of $D^Q(F)$ associated with $m + n_0 - \lambda - r_0 - 4$.

Set μ, ξ as arbitrary adjacency eigenvalues of G_1 and G_2 with corresponding eigenvectors Y and Z. Likewise, the vectors $[0 \quad X^T \quad 0]^T$ and $[0 \quad 0 \quad X^T]^T$ are eigenvectors of $D^Q(F)$ associated with $2m - n_0 - \mu - r_1 - 4$ and $2m - n_0 - \xi - r_2 - 4$, respectively.

As such we obtain eigenvectors of the form $[X^T \quad 0 \quad 0]^T$, $[0 \quad X^T \quad 0]^T$ and $[0 \quad 0 \quad X^T]^T$. They are $m - 3$ eigenvectors. All of them are orthogonal to $[\mathbf{1}^T \quad 0 \quad 0]^T$, $[0 \quad \mathbf{1}^T \quad 0]^T$ and $[0 \quad 0 \quad \mathbf{1}^T]^T$. The rest three eigenvectors of $D^Q(F)$ are of the form $[\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1}]^T$ for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

If ν is an eigenvalue of $D^Q(F)$ associated with an eigenvector $(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T$, from D^Q $(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T$ $= \nu(\alpha\mathbf{1}, \beta\mathbf{1}, \gamma\mathbf{1})^T$, and $A(G_i)\mathbf{1} = r_i\mathbf{1}$ for $i = 0, 1, 2$, we know that

$$\begin{aligned}
&(m + 3n_0 - 2r_0 - 4)\alpha + n_1\beta + n_2\gamma = \nu\alpha,\\
&n_0\alpha + (2m + 2n_1 - n_0 - 2r_1 - 4)\beta + 2n_2\gamma = \nu\beta,\\
&n_0\alpha + 2n_1\beta + (2m + 2n_2 - n_0 - 2r_2 - 4)\gamma = \nu\gamma.
\end{aligned}$$

This system has a nontrivial solution if and only if ν is an eigenvalue of (3.1). Any nontrivial solution of it forms an eigenvector of $D^Q(F)$ associated with ν. Since all the rest eigenvectors of $D^Q(F)$ are formed as such, we know that each eigenvalue of (3.1) is also an eigenvalue of $D^Q(F)$. $\square$

For $G(n_0, n_1, n_2) = K_{n_0}\nabla(K_{n_1} \cup K_{n_2})$, its distance signless Laplacian spectrum can be derived on the basis of Theorem 3.5.

Corollary 3.6. *The distance signless Laplacian eigenvalues of $G(n_0, n_1, n_2)$ consists of eigenvalue $m-2$, with multiplicity $n_0 - 1$, the eigenvalue $2m - n_0 - n_1 - 2$, with multiplicity $n_1 - 1$, the eigenvalue $2m - n_0 - n_2 - 2$, with multiplicity $n_2 - 1$ and all eigenvalues of the following matrix*

$$\begin{pmatrix} m + n_0 - 2 & n_1 & n_2 \\ n_0 & 2m - n_0 - 2 & 2n_2 \\ n_0 & 2n_1 & 2m - n_0 - 2 \end{pmatrix},$$

where $m = \sum_{i=0}^{2} n_i$.

Proof. Proof follows from Theorem 3.5, by taking $r_0 = n_0 - 1, r_1 = n_1 - 1, r_2 = n_2 - 1, \lambda_{i,j} = -1$, for all $i = 0, 1, 2$ and $j = 2, 3, \ldots, n_i$. $\square$

Let $K_n - e$ be the graph obtained from K_n by discarding an edge e. Taking $n_0 = n - 2, n_1 = n_2 = 1$ and $m = n$, in Corollary 3.6, we know that the distance signless Laplacian spectrum of $K_n - e$ comprises of $\{n - 2^{[n-3]}, x_1, x_2, x_3\}$, where x_1, x_2 and x_3 are the roots of $f(x) = x^3 - 4(n-1)x^2 + 5n(n-2) - 2n^3 + 6n^2 - 8 = 0$.

Next, we consider Indu-Bala product of graphs [23]. The *Indu-Bala product* of two graphs G_1 and G_2, $G_1 \blacktriangledown G_2$, can be defined as follows. Assume $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) =$

$\{v_1, v_2, \ldots, v_{n_2}\}$. Take a disjoint copy $G_1'\nabla G_2'$ of $G_1\nabla G_2$ with vertex sets $V(G_1') = \{u_1', u_2', \ldots, u_{n_1}'\}$ and $V(G_2') = \{v_1', v_2', \ldots, v_{n_2}'\}$. v_i is adjacent to v_i' for any $i = 1, 2, \ldots, n_2$. In Figure 1, we sketch the Indu-Bala product of P_4 and K_3.

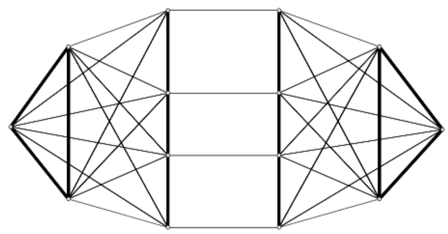

FIGURE 1. *The graph $K_3\blacktriangledown P_4$*

Theorem 3.7. *Suppose G_i is r_i-regular with n_i vertices and $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,n_i}$ is adjacency eigenvalues for $i = 1, 2$. The distance signless Laplacian spectrum of $G_1\blacktriangledown G_2$ is given by $2m - \lambda_{1,j} + n_1 - n_2 - r_1 - 4$ for $j = 2, 3, \ldots, n_1$ each with multiplicity 2, $2m - 2\lambda_{2,j} + n_2 - n_1 - 2r_2 - 8$ for $j = 2, 3, \ldots, n_2$; $2m + n_2 - n_1 - 2r_2 - 4$ with multiplicity $(n_2 - 1)$, where $m = 2(n_1 + n_2)$, also all eigenvalues of the matrix*

$$\begin{pmatrix} m_1 & n_2 & 2n_2 & 3n_1 \\ n_1 & m_2 & 3n_2 - r_2 - 2 & 2n_1 \\ 2n_1 & 3n_2 - r_2 - 2 & m_2 & n_1 \\ 3n_1 & 2n_2 & n_2 & m_1 \end{pmatrix}, \tag{3.2}$$

where $m_1 = 2m + 3n_1 - n_2 - 2r_1 - 4$ and $m_2 = 2m + 3n_2 - n_1 - 3r_2 - 6$.

Proof. The distance signless Laplacian matrix of the graph $H = G_1\blacktriangledown G_2$ has the form

$$\begin{pmatrix} S_1 & J & 2J & 3J \\ J & S_2 & 3J - 2I - A(G_2) & 2J \\ 2J & 3J - 2I - A(G_2) & S_3 & J \\ 3J & 2J & J & S_4 \end{pmatrix},$$

where $S_i = 2J - A(G_1) + (2m + n_1 - n_2 - r_1 - 4)I, i = 1, 4$ and $S_i = 2J - A(G_2) + (2m + n_2 - n_1 - 2r_2 - 6)I, i = 2, 3$.

As in Theorem 3.5, let λ be an adjacency eigenvalue of G_1 associated with X, satisfying $\mathbf{1}^T X = 0$. $[X^T \quad 0 \quad 0 \quad 0]^T$ is an eigenvector of $D^Q(H)$ associated with $2m - \lambda + n_1 - n_2 - r_1 - 4$. Likewise, the vector $[0 \quad 0 \quad 0 \quad X^T]^T$ is an eigenvector of $D^Q(H)$ associated with $2m - \lambda + n_1 - n_2 - r_1 - 4$. Let μ be an arbitrary adjacency eigenvalue of G_2 with eigenvector Y, satisfying $\mathbf{1}^T Y = 0$. The vectors $[0 \quad Y^T \quad Y^T \quad 0]^T$ and $[0 \quad -Y^T \quad Y^T \quad 0]^T$ are similarly eigenvectors of $D^Q(H)$ with eigenvalues $2m - 2\mu + n_2 - n_1 - 2r_2 - 8$ and $2m + n_2 - n_1 - 2r_2 - 4$ respectively. Therefore, we have

$[X^T \quad 0 \quad 0 \quad 0]^T$, $[0 \quad 0 \quad 0 \quad X^T]^T$, $[0 \quad Y^T \quad Y^T \quad 0]^T$ and $[0 \quad -Y^T \quad Y^T \quad 0]^T$ as $m-4$ eigenvectors. Other eigenvectors are orthogonal to $[\mathbf{1}^T \quad 0 \quad 0 \quad 0]^T$, $[0 \quad \mathbf{1}^T \quad 0 \quad 0]^T$, $[0 \quad 0 \quad \mathbf{1}^T \quad 0]^T$ and $[0 \quad 0 \quad 0 \quad \mathbf{1}^T]^T$. They must span the space spanned by the rest four eigenvectors of $D^Q(H)$. They are of the form $[\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1}]^T$ for some $(\alpha,\beta,\gamma,\delta) \neq (0,0,0,0)$. If ν is an eigenvalue of $D^Q(H)$ with an eigenvector $(\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1})^T$, using $D^Q(\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1})^T = \nu(\alpha\mathbf{1} \quad \beta\mathbf{1} \quad \gamma\mathbf{1} \quad \delta\mathbf{1})^T$, and $A(G_i)\mathbf{1} = r_i\mathbf{1}$ for $i=1,2$, we arrive at:

$$\begin{aligned}
&(2m+3n_1-n_2-2r_1-4)\alpha + n_2\beta + 2n_2\gamma + 3n_1\delta = \nu\alpha,\\
&n_1\alpha + (2m+3n_2-n_1-3r_2-6)\beta + (3n_2-r_2-2)\gamma + 2n_1\delta = \nu\beta,\\
&2n_1\alpha + (3n_2-r_2-2)\beta + (2m+3n_2-n_1-3r_2-6)\gamma + n_1\delta = \nu\gamma,\\
&3n_1\alpha + 2n_2\beta + n_2\gamma + (2m+3n_1-n_2-2r_1-4)\delta = \nu\delta.
\end{aligned}$$

The system has a nontrivial solution when ν is an eigenvalue of (3.2). Any nontrivial solution of the system must be an eigenvector of $D^Q(H)$ associated with ν. Because all four rest eigenvectors of $D^Q(H)$ are as such, any eigenvalue of (3.2) is also an eigenvalue of $D^Q(H)$. □

Recall that the *k-th power* G^k has the same set of vertices as G. Two vertices in G^k forms an edge if the distance between them in G is no more than k. We will obtain the distance signless Laplacian spectrum of the square of cycle and square of hypercube of dimension n. We show that the square of hypercube of dimension n has three distinct distance signless Laplacian eigenvalues.

Theorem 3.8. *Let* $\{\frac{n^2}{4}, 0, \lambda_3, \ldots, \lambda_n\}$ *or* $\{\frac{n^2}{4}, -1, \lambda_3, \ldots, \lambda_n\}$ *be the distance spectrum of* C_n *depending on whether* $\frac{n}{2}$ *is even or odd. Then, the distance signless Laplacian spectrum of* C_n^2 *is given by*

$$\left\{\frac{n^2+2n}{4}, \frac{n^2}{8}, \frac{n^2+2n+4\lambda_3}{8}, \ldots, \frac{n^2+2n+4\lambda_n}{8}\right\},$$

if $\frac{n}{2}$ *is even and by*

$$\left\{\frac{n^2+2n}{4}, \frac{n^2-4}{8}, \frac{n^2+2n+4\lambda_3}{8}, \ldots, \frac{n^2+2n+4\lambda_n}{8}\right\},$$

if $\frac{n}{2}$ *is odd.*

Proof. Suppose C_n has vertex set $\{u_1, u_2, \ldots, u_n\}$. We partition it as $V_1 \cup V_2$ where V_1 has vertices of even index and V_2 has those of odd index. Each pair of vertices in V_1 or V_2 has even distance between. A vertex of V_1 and a vertex of V_2 has odd distance between them. We index the rows and columns of distance signless Laplacian matrix taking those in V_1 followed by those in V_2. With an appropriate ordering, the distance signless Laplacian matrix becomes

$$D^Q(C_n) = \begin{pmatrix} K+S & U \\ U & K+S \end{pmatrix},$$

where each entry of the block S is even and any row in S is given by the sum of distances from a vertex in V_1 to other vertices in V_1. S has constant row sum $r(S)$. Elements in the block U are even and rows in U are given by the sum of distances from a vertex in V_1 to other vertices in V_2. U has constant row sum $r(U)$ with

$$r(S)=\left\{\begin{array}{ll}\frac{n^2-4}{8} & \text{if } \frac{n}{2} \text{ is odd}\\ \frac{n^2}{8} & \text{if } \frac{n}{2} \text{ is even}\end{array}\right\},\quad r(U)=\left\{\begin{array}{ll}\frac{n^2+4}{8} & \text{if } \frac{n}{2} \text{ is odd}\\ \frac{n^2}{8} & \text{if } \frac{n}{2} \text{ is even.}\end{array}\right.$$

Also $K=\left(\frac{n^2}{4}\right)I$. Therefore, the distance signless Laplacian matrix of C_n^2 has the form

$$D^Q(C_n^2)=\frac{1}{2}\begin{pmatrix}P+S & U+J_{\frac{n}{2}\times\frac{n}{2}}\\ U+J_{\frac{n}{2}\times\frac{n}{2}} & P+S\end{pmatrix},$$

where $P=\left(\frac{n^2+2n}{4}\right)I$. Now, using Lemma 3.1, the eigenvalues of $D^Q(C_n^2)$ are the union of the eigenvalues of $\frac{1}{2}(P+S+U+J)$ and $\frac{1}{2}(P+S-U-J)$. Hence, if $\dfrac{n}{2}$ is even, then

$$\begin{aligned}spec(D^Q(C_n^2)) &= \left\{\frac{n^2+2n}{8}+\frac{n^2}{8}+\frac{n}{4},\frac{n^2+2n}{8}-\frac{n}{4},\frac{n^2+2n}{8}+\frac{\lambda_3}{2},\right.\\ &\qquad \left.\ldots,\frac{n^2+2n}{8}+\frac{\lambda_n}{2}\right\},\end{aligned}$$

and if $\dfrac{n}{2}$ is odd, then

$$\begin{aligned}spec(D^Q(C_n^2)) &= \left\{\frac{n^2+2n}{8}+\frac{n^2}{8}+\frac{n}{4},\frac{n^2+2n}{8}-\left(\frac{1}{2}+\frac{n}{4}\right),\frac{n^2+2n}{8}+\frac{\lambda_3}{2},\right.\\ &\qquad \left.\ldots,\frac{n^2+2n}{8}+\frac{\lambda_n}{2}\right\}.\end{aligned}$$

Hence, the proof is complete. □

Finally, we consider the distance signless Laplacian spectrum of the square of hypercube of dimension n. The *n-dimensional hypercube* Q_n admits $V(Q_n)=\{(a_1,a_2,\ldots,a_n): a_i=0 \text{ or } 1\}$ and an edge in it means the two end points differ with precisely one coordinate. For $u,v\in V(Q_n)$, $d(u,v)=r$ if and only if u and v have coordinates different in precisely r locations.

The *Hamming graph* $H(n,d)$ has vertex set X^n with $d=|X|\geq 2$, and an edge in it means the two end points differ with just one coordinate. The n-dimensional hypercube Q_n can be thought of as $H(n,2)$.

Lemma 3.9. [24] *The distance spectrum of $H(n,d)$ is* $\left\{nd^{n-1}(d-1)^{[1]},0^{[d^n-n(d-1)-1]},-(d^{n-1})^{[n(d-1)]}\right\}$.

Theorem 3.10. *Suppose Q_n is the hypercube with dimension n. The distance signless Laplacian spectrum of Q_n^2 is*

$$\left\{\sum_{i=1}^{n} i\binom{n}{i}+(2^{n-2})^{[1]},\frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i}+(2^{n-2})^{[2^n-(n+2)]},\frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i}^{[n+1]}\right\}.$$

Proof. For a vertex $x = (0, 0, \ldots, 0)$, let V_1 have vertices with even distance from x and V_2 have vertices with odd distance from x. Vertices in V_1 and those in V_2 have even distance between them. A vertex in V_1 and a vertex in V_2 have odd distance between them. With a suitable order in V_1 and V_2, the distance signless Laplacian matrix becomes

$$D^Q(Q_n) = \begin{pmatrix} K+S & U \\ U & K+S \end{pmatrix},$$

where U and S are just as in Theorem 3.8 and $K = \left(\sum_{i=1}^{n} i\binom{n}{i}\right) I$, because the sum of distances from a vertex in V_1 to others in V_1 is

$$k_1 = \begin{cases} \sum_i i\binom{n}{i}, \quad i \in 2k,\ k = 1, 2, \ldots, \frac{n-1}{2}, & \text{if } n \text{ is odd} \\ \sum_i i\binom{n}{i}, \quad i \in 2k,\ k = 1, 2, \ldots, \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

The sum of the distances from a vertex in V_1 to those inV_2 is

$$k_2 = \begin{cases} \sum_i i\binom{n}{i}, \quad i \in 2k-1,\ k = 1, 2, \ldots, \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \sum_i i\binom{n}{i}, \quad i \in 2k-1,\ k = 1, 2, \ldots, \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

The matrix $U + S$ has constant row sum $k_1 + k_2 = \sum_{i=1}^{n} i\binom{n}{i}$ and the matrix $U - S$ has constant row sum $k_1 - k_2 = 0$. for all n. Therefore, the distance signless Laplacian matrix of Q_n^2 has the form

$$D^Q(Q_n^2) = \frac{1}{2}\begin{pmatrix} F+S & U + J_{2^{n-1}\times 2^{n-1}} \\ U + J_{2^{n-1}\times 2^{n-1}} & F+S \end{pmatrix},$$

where $F = \left(\sum_{i=1}^{n} i\binom{n}{i} + 2^{n-1}\right) I$. Now, using Lemma 3.1, the eigenvalues of $D^Q(Q_n^2)$ are the union of the eigenvalues of $\frac{1}{2}(F + S + U + J)$ and $\frac{1}{2}(F + S - U - J)$. Hence, the distance signless Laplacian spectrum is

$$\Big\{\frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i} + 2^{n-2} + \frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i} + 2^{n-2}, \frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i} + 2^{n-2} + 0,$$
$$\frac{1}{2}\sum_{i=1}^{n} i\binom{n}{i} + 2^{n-2} - 2^{n-2}\Big\}.$$

Thanks to Lemma 3.9, we derive the conclusion. □

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