On Stochastic Differential Games with Impulse Controls and Applications

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

Statement of co-authored work:

I confirm that Chapter 1, 2 and 3 are jointly co-authored with Professor Luciano Campi.

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Abstract

The thesis explores general stochastic differential games involving impulse controls and ultimately investigates competition in dealer markets.

The work begins with the first chapter on general non-zero stochastic differential games between an impulse controller and a stopper, providing the first model of such class of games using impulse controls. Nash equilibria are characterised through a verification theorem, which identifies a new system of quasi-variational inequalities whose solution gives equilibrium payoffs with the correspondent strategies. Then, in order to show how the verification theorem is meant to be applied, an example is shown and two different types of Nash equilibrium are fully characterised. To conclude, some numerical results describing the qualitative properties of both types of equilibrium are provided.

The dissertation continues with the second chapter on general zero-sum stochastic differential games with impulse controls. Here, two agents play feedback impulse control strategies instead of strategies defined in an Elliot-Kalton fashion, as commonly done in the literature, and are not allowed to apply impulses simultaneously, resulting in the upper value and lower value functions of the game being naturally associated with the cases in which either player has priority. The main objective is to apply the stochastic Perron's method in order to have the game value function as the viscosity solution to the double obstacle partial differential equation arising from the problem after a viscosity comparison result.

The third and final chapter is about the study of competition in dealer markets. The setting consists in two dealers trading at discrete times via market orders with price impact, resulting in one of the first nonzero-sum game with impulse controls applied to optimal trading. Similarly to the first chapter, a verification theorem identifying the system of quasi-variational inequalities providing the equilibrium payoff functions and strategies is given. Furthermore, a framework to look for equilibria where both players apply impulses simultaneously is introduced. This is very important as it is not possible to find equilibria when only one dealer trades at a time, whereas there exists at least a Nash equilibrium when both dealers trade simultaneously.

Contents

Introduction

This thesis consists of three self-contained chapters presenting new results on zero and nonzero-sum stochastic differential games with impulse controls.

Stochastic differential games constitute an interesting branch of mathematics as they allow to model the interaction between two or more agents when this happens over a state process evolving in continuous time. Two firms competing over their market shares, or traders maximising their P&L through strategies based on the price process fluctuations are some of the possible examples of such interactions. This is one of the reasons why they have been extensively studied since Isaacs' [50] pioneering work, although not much attention has been paid to the case when players utilise impulse controls rather than classic controls [1, 6, 9, 10, 31, 38, 43, 43, 63, 80]. Contrary to classic controls, which allows agents to continuously modify the state process dynamics, i.e. drift and volatility, impulse controls enable agents to induce a controlled jump, the impulse, on the state process at strategically selected discrete times. As such they provide more realistic models, especially when agents face fixed and proportional costs of action, see the introductions to Chapter 1, 2 and 3 for more details.

The thesis provides two general results on stochastic differential games involving impulse controls and one application to optimal trading in financial markets.

Main contributions of Chapter 1:

(i) Formulation of the first impulse controller vs stopper game.

(ii) Statement of a suitable system of quasi-variational inequalities and the corresponding verification theorem to find Nash equilibria.

(iii) Example with characterisation of two different types of Nash equilibria.

(iv) Qualitative analysis of both types of equilibria.

Chapter 1 presents a nonzero-sum game between an impulse controller and a stopper. This work is the first in the class of controller vs stopper games, introduced by Maitra and Sudderth [62], where the controller utilises impulse controls and it aims to inspire future research on its applications in finance, energy markets and real options. For instance, one could think of the stopper as a social planner or regulator who decides when to optimally shut down a business, the controller, which is maximising its profits manipulating either the quantity of goods produced or their prices. The main mathematical contribution lies on a system of quasi-variational inequalities and a verification theorem, inspired by [1], to be used to find Nash equilibria of such controller vs stopper games. Furthermore, the chapter contains an example showing how the system of quasi-variational inequalities and the verification theorem are meant to be applied, two different types of equilibria are identified and fully characterised. To conclude, some numerical analysis is carried on to investigate the qualitative properties of both types of equilibria.

Main contributions of Chapter 2:

(i) The stochastic Perron's method is adapted to stochastic differential games with impulse controls.

(ii) Symmetric formulation of the game where players use feedback impulse controls.

(iii) Comparison theorem for double obstacle partial differential equations to guarantee uniqueness of the viscosity solution.

In Chapter 2 a zero-sum game between two players playing impulse controls is studied by mean of viscosity solution theory. This choice is due to the fact that the value function is known to be non-smooth in most cases and viscosity solutions, introduced by Crandall and Lions in 1980, represent a generalisation of the concept of classical solutions to partial differential equations that allows to find solutions which don't need to be differentiable everywhere (smooth), see [39]. We look for such viscosity solutions via an adaptation of the stochastic Perron's method approach by Bayraktar and Sırbu [12, 15, 16, 74] as it is arguably more tractable. Indeed, the dynamic programming principle is obtained as a by-product. Moreover, the stochastic Perron's method is suitable to our definition of feedback strategies [48, 74], according to which, players take their decisions based on the evolution of the state process in an adapted fashion, resulting in a realistic and natural way to model their interaction, especially if compared with the asymmetric Elliot and Kalton formulation [42] commonly adopted in the literature. Finally, to guarantee uniqueness of the viscosity solution, a verification by comparison result for double-obstacle partial differential equations is provided.

Main contributions of Chapter 3:

(i) Application of stochastic differential games with impulse controls to optimal trading.

(ii) Formulation of a suitable system of quasi-variational inequalities and the corresponding verification theorem to find Nash equilibria when two players act simultaneously.

(iii) Characterisation of some equilibria.

Finally, Chapter 3 describes the competition between two dealers executing market orders to maximise their revenues over a finite time horizon. Market orders are used by traders to buy or sell a certain number of shares at a specific time at the best available price. Financial markets are not perfectly liquid, this means that it is usually not possible to buy or sell large amounts of shares at the same price as the market is made by a list of offers to buy or sell fixed numbers of shares at certain prices, called the limit order book. So, for example, if the limit order book is composed by offers to sell 5 shares at £5, 4 shares at £6 and 3 at £7, then, a market order to buy 10 shares is completed as follows: the first 5 are purchased at £5, the best available price, the second 4 at £6, the next best available price, and the final 1 at £7. Therefore, it is not usually convenient to complete large orders all at once as the bigger they are the higher the execution price is, due to the order riding the limit order book to be fulfilled, as in the example, generating what is usually called market impact. Given this set of characteristics, dealers' trades taking place at strategically chosen discrete times and costs proportional to their sizes, it seems natural to opt for an impulse control approach leading to the study of a nonzero-sum game with impulse controls. The research is carried in a fashion similar to Chapter 1 and [1], after the game description, two suitable systems of quasi-variational inequalities are provided together with the corresponding verification theorems to be used to search for Nash equilibria. Notably, the chapter contains the first system of quasi-variational inequalities which allows to find equilibria when both dealers trade simultaneously. This is crucial for two reasons:

- firstly, because it breaks one of the current limits in the literature on stochastic differential games with impulse controls, since agents are not allowed to intervene simultaneously in the existing models [1, 9, 10, 31, 38, 43, 43, 63, 80];
- secondly, because it is not possible to find equilibria where only one dealer trade at a time with this quasi-variational inequality approach.

The chapter ends with the characterisation of a few equilibria, followed by some promising work in progress.

Chapter 1

Nonzero-sum stochastic differential games between an impulse controller and a stopper

The content of this chapter is based on [30].

1.1 Introduction

Controller-stopper games are two-player stochastic dynamic games, whose payoffs depend on the evolution over time of some state variable, one player can control its dynamics, while the other player can stop the game. The study of these games started with Maitra and Sudderth's work [62] on a zero-sum discrete time setting. Later on, many authors investigated such games in continuous time, especially in the zero-sum case, while very little has been done in the nonzero-sum. Indeed, apart from Karatzas and Sudderth [54] and Karatzas and Li [53], all the other articles focus on the zero-sum case and in all of them the controller uses regular controls, i.e. absolutely continuous for the Lebesgue measure. Here, we mention Karatzas and Sudderth [55], who derived the explicit solution for a game with a one-dimensional diffusion with absorption at the endpoints of a bounded interval as a state process; Karatzas and Zamfirescu [57, 58] developed a martingale approach to a general class of controller-stopper games, while Bayraktar and Huang [12] showed that the value functions of such games is the unique viscosity solution to an appropriate Hamilton-Jacobi-Bellman equation. Moreover, Hernandez et al. [47] have analysed the case when the controller plays singular controls and derived a set of variational inequalities characterising the games value functions. On the whole, this class of games is motivated by a variety of applications in finance, insurance and economics. In view of this, we quote Bayraktar et al. [13] on convex risk measures, Nutz and Zhang [65] on sub-hedging of American options under volatility uncertainty, Bayraktar and Young [17] on minimisation of lifetime ruin probability and Karatzas and Wang [56] on pricing and hedging of American contingent claims among others.

Here, we consider the case of a controller facing fixed and proportional costs every time she moves the state variable, so that intervening continuously over time is clearly not feasible for her. In this context, the controller will make use of impulse controls, which are sequences of interventions times and corresponding intervention sizes, describing when and by how much will the controlled process be shifted. This kind of controls look like the natural choice in many concrete applications, from finance to energy markets and to real options. For this reason, they have been experiencing a comeback due to a demand for more realistic financial models (e.g. fixed transaction costs and liquidity risk), see for instance [8, 18, 27, 29, 35, 37, 61].

Impulse controls have been studied in stochastic differential games as well and, as in the controller-stopper case, most of the research has been done in the zero-sum framework. For this reason, it is worth mentioning the work by Aïd et al. [1], who developed a general model for non-zero sum impulse games implementing a verification theorem which provides an appropriate system of quasi-variational inequalities for the equilibrium payoffs and related strategies of the two players. Thereafter, Ferrari and Koch [43] produced a model of pollution control where the two players, the regulator and the energy producer, are assumed to face proportional and fixed costs and, as such, play an impulse nonzero-sum game which admits an equilibrium under some suitable conditions. Lastly, Basei et al. [9] studied the mean field game version of the nonzero-sum impulse game in [1] and proved the existence of ϵ-Nash equilibrium for the corresponding N-player game. Regarding the zero-sum case, here we quote Cosso [38], who examined a finite time horizon two-player game where both players act via impulse control strategies and showed that such games have a value which is the unique viscosity solution of the double-obstacle quasi-variational inequality. Furthermore, Azimzadeh [6] considered an asymmetric setting with one participant playing a regular control while the opponent is playing an impulse control with pre-commitment, meaning that at the beginning of the game the maximum number of impulses is declared, and proved that such a game has a value in the viscosity sense.

The content of this chapter is at the crossroad of the two streams of research we have discussed above: stopper-controller games and impulse games. Indeed, we study an impulse controller-stopper nonzero-sum game, focusing on the mathematical properties of Nash equilibria, while application to economics and finance are postponed to future research. Turning to the game's description, we consider a nonzero-sum stochastic differential game between two players, P1 and P2, where P1 can use impulse controls to affect a continuous-time stochastic process X while P2 can stop the game at any time. When P1 does not intervene, we assume X to diffuse according to a time homogeneous multidimensional diffusion process. Both players want to maximise their expected payoffs which are defined for every initial state $x \in \mathbb{R}^d$ and every couple (u, η) featuring, P1's intervention cost (gain for P2), running and terminal payoffs.

We adopt a PDE-based approach to characterise the Nash equilibria of this game, identifying a suitable system of quasi-variational inequalities (QVIs, for short) whose solution will give equilibrium payoffs. One of the main contributions of this chapter consists in the Verification Theorem 1.2.1 establishing that if two functions V_1 and V_2 are regular enough and they are solution to the system of QVIs, then they coincide with some equilibrium payoff functions of the game and a characterisation of the related equilibrium strategies is possible.

Furthermore, building on the verification theorem, we present an example of solvable impulse controller and stopper game. More in detail, we consider a game with a one-dimensional state variable X, modelled as a real-valued (scaled) Brownian motion. Both players have linear running payoffs. When P1 intervenes, she faces a penalty while P2 faces a gain, both characterised by a fixed and a variable part, proportional to the size of the impulse. Moreover, when P2 stops the game, she may suffer a loss proportional to the state variable, while P1 might gain something proportional to X as well. Some preliminary heuristics on the QVIs above leads us to consider two pairs of candidates for the functions V_i. Then, a careful application of the verification theorem shows that such candidates actually coincide with some equilibrium payoff functions. In particular, we are able to identify two kinds of Nash equilibria, both of threshold type, that can be shortly described as follows:

(i) in the first type of equilibrium, P1 intervenes when the state X is smaller than some threshold $\bar{x}_1$ and moves the process to some endogenously determined target x_1^*, while P2 terminates the game when the state X is bigger than some $\bar{x}_2$; in this kind of equilibrium the optimal target of P1, x_1^*, is strictly smaller than $\bar{x}_2$, so the two players intervene separately.

(ii) In the second type, P1 intervenes when the state X is smaller than some (possibly different) threshold $\bar{x}_1$ and move the state variable to the intervention region of P2, who is then forced by P1 to end the game. In this case, players' interventions are simultaneous.

We provide quasi-explicit expressions for the value functions and for the thresholds $\bar{x}_i$, x_1^* for both equilibria. Finally, we perform some numerical experiments providing several cases when one of the two equilibria emerges. The question if there are cases when the two types of equilibria can coexist is still open.

The chapter is organised as follows. Section 1.2 gives the general formulation of impulse controller and stopper game, in particular the notion of admissible strategies, and more importantly we state and prove a verification theorem giving sufficient condition in terms of the system of QVIs for a given couple of payoffs to be a Nash equilibrium. In Section 1.3, we consider the one-dimensional example with linear payoffs and provide quasi-explicitly characterisations for the two types of Nash equilibria sketched above. Finally, some numerical experiments illustrate the qualitative behaviour of such equilibria.

1.2 Description of the Game

In this section, we have gathered all main theoretical results on a general class of nonzero-sum impulse controller and stopper games. We start with a detailed description of the game, together with all technical assumptions and the definition of admissible strategies.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a complete and right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. On this space, we consider the uncontrolled state variable $X \equiv X^x$ defined as solution of the following time-homogeneous SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is an $\mathbb{F}$-Brownian motion and the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are assumed to be globally Lipschitz continuous, i.e. there exists a constant $C > 0$ such that for all x_1, $x_2 \in \mathbb{R}^d$ we have:

$$|b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2|,$$

so that existence of a unique strong solution is granted and X is well-defined.

We consider two players, that we call P1 and P2. Equation (3.1) describes the evolution of the state process in case of no intervention from both players. Let Z be a given subset of $\mathbb{R}^d$. During the game, P1 can affect X's dynamics applying some impulse $\delta \in Z$ in an additive fashion, moving the state variable from its left limit at τ, $X_{\tau-}$, to its new value $X_\tau = X_{\tau-} + \delta$, where τ denotes the intervention time. The controlled state variable is denoted by $X^{x,u}$:

$$X_t^{x,u} = x + \int_0^t b(X_s^{x,u})ds + \int_0^t \sigma(X_s^{x,u})dW_s + \sum_{n:\tau_n\leq t} \delta_n, \quad t \geq 0.$$

On the other hand, P2 can stop the game by choosing any stopping time η with values in $[0, \infty]$. We, now, give a proper definition of such strategies.

Definition 1.2.1 P1's strategy is any sequence $u = (\tau_n, \delta_n)_{n\geq 0}$, where $(\tau_n)_{n\geq 0}$ is a sequence of stopping times such that $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n \uparrow \infty$ and $\delta_n \in L^0(\mathcal{F}_{\tau_n})$ with values in Z. P2's strategy is any stopping time $\eta \in \mathcal{T}$, where $\mathcal{T}$ is the set of all $[0, \infty]$-valued $\mathbb{F}$-stopping times.

Remarks 1.2.1 We observe that simultaneous interventions are possible in this game. This is in contrast with games where both players intervenes with impulses, where simultaneous interventions are usually not allowed since they would be very difficult to handle with from a modelling perspective (cf. [1]). On the other hand here, due to the different nature of the strategies for the two players, one can safely allow for simultaneous actions. This has an interesting consequence on our analysis, as we will see in the linear game of the next section that at least two types of Nash equilibria are possible and in one of them P1 induces P2 to stop instantaneously.

The players want to maximise their respective objectives, featuring each of them three discounted terms: a running payoff, P1's intervention cost/gain and a terminal payoff. The players' discount factors can be different of each other. More precisely, for each $i = 1, 2$, $r_i > 0$ denotes the discount rate of player i, $f, g : \mathbb{R}^d \to \mathbb{R}$ are their running payoffs, $h, k : \mathbb{R}^d \to \mathbb{R}$ their terminal payoffs and $\phi, \psi : \mathbb{R}^d \times Z \to \mathbb{R}$ are the intervention cost and gain, respectively. Throughout the whole chapter, we work under the assumption that all these functions are continuous. Hence, we can define the payoffs as follows.

Definition 1.2.2 Let $x \in \mathbb{R}^d$, let (u, η) be a pair of strategies. Provided that the right-hand sides exist and are finite we set:

$$J_1(x; u, \eta) := \mathbb{E}_x \left[\int_0^\eta e^{-r_1 t} f(X_t^{x,u}) dt - \sum_{n:\tau_n \le \eta} e^{-r_1 \tau_n} \phi(X_{\tau_n -}^{x,u}, \delta_n) + e^{-r_1 \eta} h(X_\eta^{x,u}) \mathbb{1}_{(\eta<\infty)} \right]$$

$$J_2(x; u, \eta) := \mathbb{E}_x \left[\int_0^\eta e^{-r_2 t} g(X_t^{x,u}) dt + \sum_{n:\tau_n \le \eta} e^{-r_2 \tau_n} \psi(X_{\tau_n -}^{x,u}, \delta_n) + e^{-r_2 \eta} k(X_\eta^{x,u}) \mathbb{1}_{(\eta<\infty)} \right],$$

where the subscript in the expectation denotes the conditioning with respect to the starting point.

In order for J_1 and J_2 to be well defined, we now introduce the set of admissible strategies.

Definition 1.2.3 Let $x \in \mathbb{R}^d$ be some initial state and let (u, η) be some strategy profile. We say that the pair (u, η) is x-admissible if:

(i) the following random variables are all in $L^1(\Omega)$:

$$\int_0^\infty e^{-r_1 t} |f(X_t^{x,u})| dt, \qquad \int_0^\infty e^{-r_2 t} |g(X_t^{x,u})| dt,$$
$$e^{-r_1 \eta} |h(X_\eta^{x,u})|, \qquad e^{-r_2 \eta} |k(X_\eta^{x,u})|,$$
$$\sum_{k:\tau_k \le \infty} e^{-r_1 \tau_k} |\phi(X_{\tau_k -}^{x,u}, \delta_k)|, \qquad \sum_{k:\tau_k \le \infty} e^{-r_2 \tau_k} |\psi(X_{\tau_k -}^{x,u}, \delta_k)|;$$

(ii) for each $p \in \mathbb{N}$, the random variable $\|X^{x,u}\|_\infty := \sup_{t \ge 0} e^{-(r_1 \wedge r_2)t} |X_t^{x,u}|$ is in $L^p(\Omega)$.

We denote by $\mathcal{A}_x$ the set of all x-admissible pairs.

Remarks 1.2.2 Notice that, as it is formulated above, admissibility is a joint condition on the strategies of both players. Under condition (ii) above and if all functions f, g, h, k, ϕ and ψ have at most polynomial growth in their respective variables, the set of all jointly admissible strategies can be expressed as $\mathcal{A}_x^1 \times \mathcal{A}_x^2 = \mathcal{A}_x$, where $\mathcal{A}_x^i$ denotes Pi's set of (individually) admissible strategies for $i = 1, 2$, and is defined as follows: $\mathcal{A}_x^1$ is the set of all P1's strategies $u = (\tau_n, \delta_n)_{n\geq 0}$ such that $\sum_{n\geq 0} |\delta_n| \in L^p(\Omega)$ for all $p \geq 1$, while $\mathcal{A}_x^2$ is the set of all $[0, \infty]$-values stopping times.

Indeed, for P1's strategies for instance, using classical a-priori L^p-estimates of the (uncontrolled) state variable, there exists a constant $c > 0$ such that

$$\mathbb{E}\left[e^{-r_1\eta}|h(X_\eta)|\right] \leq c\mathbb{E}\left[e^{-r_1\eta}(1 + |X_\eta|^p)\right] \leq c(1 + \mathbb{E}[\|X\|_\infty^p]) < \infty.$$

Moreover, similar estimates can be performed for the other expectations in Definition 1.2.3(i).

We conclude this section with the classical definition of Nash equilibrium and the corresponding equilibrium payoffs.

Definition 1.2.4 (Nash Equilibrium) Given $x \in \mathbb{R}^d$, we say that $(u^*, \eta^*) \in \mathcal{A}_x$ is a Nash equilibrium if

$$\begin{aligned} J_1(x; u^*, \eta^*) &\geq J_1(x; u, \eta^*), \quad \text{for all } u \text{ s.t. } (u, \eta^*) \in \mathcal{A}_x, \\ J_2(x; u^*, \eta^*) &\geq J_2(x; u^*, \eta), \quad \text{for all } \eta \text{ s.t. } (u^*, \eta) \in \mathcal{A}_x. \end{aligned}$$

Finally, the equilibrium payoffs of any Nash equilibrium $(u^*, \eta^*) \in \mathcal{A}_x$ are defined as

$$V_i(x) := J_i(x; u^*, \eta^*), \quad i = 1, 2.$$

1.2.1 The System of Quasi-Variational Inequalities

Now, we introduce the differential problem that is satisfied by the equilibrium payoff functions of our game. Let $V_1, V_2 : \mathbb{R}^d \to \mathbb{R}$ be two measurable functions such that

$$\{\delta(x)\} := \text{argmax}_{\delta\in Z}\{V_1(x + \delta) - \phi(x, \delta)\}, \quad x \in \mathbb{R}^d, \tag{1.2}$$

for some measurable function $\delta : \mathbb{R}^d \to Z$. Moreover, we define the following two intervention operators:

$$\mathcal{M}V_1(x) := V_1(x + \delta(x)) - \phi(x, \delta(x)), \tag{1.3}$$

$$\mathcal{H}V_2(x) := V_2(x + \delta(x)) + \psi(x, \delta(x)), \tag{1.4}$$

for each $x \in \mathbb{R}^d$.

The expressions in (1.2), (1.3) and (1.4) have the following natural interpretation:

(1.2) let x be the current state of the process, if P1 intervenes immediately with impulse $\delta(x)$, P1's payoff after intervention changes to $V_1(x + \delta(x)) - \phi(x, \delta(x))$, given by the payoff in the new state minus the intervention cost. Therefore, $\delta(x)$ in (1.2) is the optimal impulse that P1 would apply in case of intervention.

(1.3) $\mathcal{M}V_1(x)$ represents P1's payoff just after her intervention.

(1.4) similarly, $\mathcal{H}V_2(x)$ represents P2's payoff following P1's intervention.

Moreover, for any functions V regular enough (specific assumptions will be given later) we can consider the infinitesimal generator of the uncontrolled state variable X:

$$\mathcal{A}V := b \cdot \nabla V + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top D^2 V),$$

where b, σ are as in (3.1), $\sigma^\top$ denotes the transposed of σ, ∇V and D^2V are the gradient and the Hessian matrix of V, respectively. We are interested in the following quasi-variational inequalities (QVIs, for short) for V_1, V_2:

$$\mathcal{M}V_1 - V_1 \leq 0 \qquad \text{everywhere} \tag{1.5}$$
$$V_2 - k \geq 0 \qquad \text{everywhere} \tag{1.6}$$
$$\mathcal{H}V_2 - V_2 = 0 \qquad \text{in } \{\mathcal{M}V_1 - V_1 = 0\} \tag{1.7}$$
$$V_1 = h \qquad \text{in } \{V_2 = k\} \tag{1.8}$$
$$\max\{\mathcal{A}V_1 - r_1V_1 + f, \mathcal{M}V_1 - V_1\} = 0 \qquad \text{in } \{V_2 > k\} \tag{1.9}$$
$$\max\{\mathcal{A}V_2 - r_2V_2 + g, k - V_2\} = 0 \qquad \text{in } \{\mathcal{M}V_1 - V_1 < 0\} \tag{1.10}$$

Each part of the QVIs system above can be interpreted in the following way:

(1.5) it means that it is not always optimal for P1 to intervene and it is a standard condition in impulse control theory [24, 29];

(1.6) if the current state is x and P2 chooses to stop the game, i.e. $\eta = 0$, she gains $k(x)$ and since this is a suboptimal strategy, we have $V_2(x) \geq k(x)$ for all $x \in \mathbb{R}^d$;

(1.7) by definition of Nash equilibrium we expect that P2 does not lose anything when P1 intervenes as in [1], otherwise P2 would like to deviate, by contradicting the notion of equilibrium;

(1.9) before P2 stops the game, P1 plays as in a classic impulse control problem (e.g. [29]);

(1.10) similarly as above, when P1 does not intervene P2 solves his own optimal stopping problem (e.g. [34]).

After all this preparation, we are ready to move to our main result, which is a verification theorem linking Nash equilibria and solutions to the QVIs system above.

1.2.2 The Verification Theorem

In this subsection, we state and prove our main verification theorem. This result will be key in order to compute Nash equilibria in specific examples.

THEOREM 1.2.1 Let $V_1, V_2 : \mathbb{R}^d \to \mathbb{R}$ be two given functions. Assume that (1.2) holds and set

$$\mathcal{C}_1 := \{\mathcal{M}V_1 - V_1 < 0\}, \quad \mathcal{C}_2 := \{V_2 - k > 0\},$$

with $\mathcal{M}V_1$ as in (1.3). Moreover, assume that:

- V_1 and V_2 are solutions of the system of QVIs;
- $V_i \in C^2(\mathcal{C}_j \setminus \partial\mathcal{C}_i) \cap C^1(\mathcal{C}_j) \cap C(\mathbb{R}^d)$, for $i \neq j$, and both functions have at most polynomial growth;
- $\partial\mathcal{C}_i$ is a Lipschitz surface, i.e. it is locally the graph of a Lipschitz function, and V_i's second order derivatives are locally bounded near $\partial\mathcal{C}_i$ for $i = 1, 2$.

Finally, let $x \in \mathbb{R}^d$ and assume that $(u^*, \eta^*) \in \mathcal{A}_x$, where $u^* = (\tau_n, \delta_n)_{n\geq 1}$ is given by

$$\tau_n := \inf\{t > \tau_{n-1} : X_t \in \mathcal{C}_1^c\}, \quad \{\delta_n\} := \text{argmax}_{\delta\in Z}\{V_1(X_{\tau_n -} + \delta) - \phi(X_{\tau_n -}, \delta)\}, \quad n \geq 0,$$

and

$$\eta^* := \inf\{t \geq 0 : V_2(X_t) = k(X_t)\},$$

with the convention $\tau_0 = 0$. Then, (u^*, η^*) is a Nash Equilibrium and $V_i = J_i(x; u^*, \eta^*)$ for $i = 1, 2$.

REMARKS 1.2.3 First, we stress that, unlike usual control problems, the candidates V_1, V_2 are not required to be twice differentiable everywhere, but only in $\{V_2 > k\}$ and $\{\mathcal{M}V_1 - V_1 < 0\}$ respectively. Moreover, we observe that for the equilibrium strategies in the theorem above the right-continuity of $(X_t^{x;u})_{t\geq 0}$ implies the following:

$$(\mathcal{M}V_1 - V_1)(X_s^{x,u^*}) < 0, \tag{1.11}$$

$$\delta_k = \delta(X_{\tau_k -}^{x,u^*}), \qquad \text{(where } \delta(\cdot) \text{ is as in (1.2))} \tag{1.12}$$

$$(\mathcal{M}V_1 - V_1)(X_{\tau_k -}^{x,u^*}) = 0, \qquad (\mathcal{H}V_2 - V_2)(X_{\tau_k -}^{x,u^*}) = 0 \tag{1.13}$$

$$(V_2 - k)(X_{\eta^*}) = 0, \qquad \text{(on } \{\eta^* < +\infty\}\text{)} \tag{1.14}$$

$$(V_2 - k)(X_s) > 0, \qquad \text{(when P2 plays } \eta^*\text{)} \tag{1.15}$$

for every strategies u and η such that both (u^*, η), (u, η^*) belong to $\mathcal{A}_x$, for every $s \in [0, \eta)$ and every $\tau_k < \infty$.

PROOF 1.2.1 Let $V_i(x) = J_i(x; u^*, \eta^*)$ for $i = 1, 2$. By definition of Nash Equilibrium we have to prove that $V_1(x) \geq J_1(x; u, \eta^*)$ and $V_2(x) \geq J_2(x; u^*, \eta)$ for every (u, η) such that both (u^*, η), (u, η^*) belong to $\mathcal{A}_x$. The proof is performed in three steps.

Step 1: We show that $V_1(x) \geq J_1(x; u, \eta^*)$. Let u be a strategy such that $(u, \eta^*) \in \mathcal{A}_x$. Thanks to the regularity assumptions and by approximation arguments of Theorem 2.1 in [67] (for more details see the proof of Theorem 3.3 in [1]), we can assume without loss of generality that $V_1 \in C^2(\mathcal{C}_2) \cap C(\mathbb{R}^d)$. For each $r > 0$ and $n \in \mathbb{N}$, we set

$$\tau_{r,n} := \tau_r \wedge n \wedge \eta^*$$

with $\tau_r := \inf\{s > 0 : X_s \notin B(x, r)\}$, where $B(x, r)$ is an open ball with radius r and centre in x. As usual, we adopt the convention $\inf \emptyset = +\infty$. Applying Itô's formula to $e^{-r_1 s} V_1(X_s)$ between time zero and $\tau_{r,n}$ and taking conditional expectations on both sides give

$$\begin{aligned} V_1(x) =& \mathbb{E}_x \Bigg[e^{-r_1 \tau_{r,n}} V_1(X_{\tau_{r,n}}) - \int_0^{\tau_{r,n}} e^{-r_1 s} (\mathcal{A} V_1 - r_1 V_1)(X_s) ds \\ & - \sum_{k: \tau_k \leq \tau_{r,n}} e^{-r_1 \tau_k} (V_1(X_{\tau_k}) - V_1(X_{\tau_k -})) \Bigg] . \end{aligned}$$

From (1.10) it follows that

$$(\mathcal{A} V_1 - r_1 V_1)(X_s) \leq -f(X_s)$$

for all $s \in [0, \eta^*)$. Moreover, using (1.5) we also have:

$$V_1(X_{\tau_k -}) \geq \mathcal{M} V_1(X_{\tau_k -}) \geq V_1(X_{\tau_k -} + \delta) - \phi(X_{\tau_k -}, \delta) = V_1(X_{\tau_k}) - \phi(X_{\tau_k -}, \delta).$$

Therefore,

$$V_1(x) \geq \mathbb{E}_x \left[e^{-r_1 \tau_{r,n}} V_1(X_{\tau_{r,n}}) + \int_0^{\tau_{r,n}} e^{-r_1 s} f(X_s) ds - \sum_{k: \tau_k \leq \tau_{r,n}} e^{-r_1 \tau_k} \phi(X_{\tau_k -}, \delta_k) \right] .$$

Observe that by admissibility we have

$$e^{-r_1 \tau_{r,n}} V_1(X_{\tau_{r,n}}) \leq e^{-r_1 \tau_{r,n}} C \left(1 + |X_{\tau_{r,n}}|^p\right) \leq C \left(1 + \|X\|_\infty^p\right) \in L^1(\Omega),$$

for some constants $C > 0$ and $p \in \mathbb{N}$. Thus, we can use dominated convergence theorem and pass to the limit, first as $r \to \infty$ and then for $n \to \infty$. Finally, because of (1.8), we obtain

$$\begin{aligned} V_1(x) \geq & \mathbb{E}_x \left[\int_0^{\eta^*} e^{-r_1 s} f(X_s) ds - \sum_{k: \tau_k \leq \eta^*} e^{-r_1 \tau_k} \phi(X_{\tau_k -}, \delta_k) + e^{-r_1 \eta^*} h(X_{\eta^*}) \mathbb{1}_{\{\eta^* < \infty\}} \right] \\ =& J_1(x; u, \eta^*). \end{aligned}$$

Step 2: We show that $V_2(x) \geq J_2(x; u^*, \eta)$. Let η be a $[0, \infty]$-valued stopping time such that $(u^*, \eta) \in \mathcal{A}_x$. Thanks to regularity assumptions and by the same approximation argument as before, we can assume again without loss of generality that $V_2 \in C^2(\mathcal{C}_1) \cap C(\mathbb{R}^d)$. Arguing exactly as in Step 1 we obtain

$$V_2(x) = \mathbb{E}_x \Bigg[e^{-r_2\tau_{r,n}} V_2(X_{\tau_{r,n}}) - \int_0^{\tau_{r,n}} e^{-r_2 s}(\mathcal{A}V_2 - r_2V_2)(X_s)ds \\ - \sum_{k:\tau_k \le \tau_{r,n}} e^{-r_2\tau_k} \left(V_2(X_{\tau_k}) - V_2(X_{\tau_k-})\right) \Bigg],$$

for the localising sequence $\tau_{r,n} := \tau_r \wedge n \wedge \eta$ ($r > 0$, $n \in \mathbb{N}$), where $\tau_r := \inf\{s > 0 : X_s \notin B(x,r)\}$. From (1.9) it follows that

$$(\mathcal{A}V_2 - r_2V_2)(X_s) \le -g(X_s)$$

for all $s \in [0,\eta)$. Moreover, due to (1.7) and (1.13) we obtain

$$V_2(X_{\tau_k-}) = \mathcal{H}V_2(X_{\tau_k-}) = V_2((X_{\tau_k-} + \delta_k) + \psi((X_{\tau_k-}, \delta_k) = V_2(X_{\tau_k}) + \psi(X_{\tau_k-}, \delta_k).$$

Then,

$$V_2(x) \ge \mathbb{E}_x \left[e^{-r_2\tau_{r,n}} V_2(X_{\tau_{r,n}}) + \int_0^{\tau_{r,n}} e^{-r_2 s} g(X_s)ds + \sum_{k:\tau_k \le \tau_{r,n}} e^{-r_2\tau_k}\psi(X_{\tau_k-}, \delta_k) \right]$$

and as before we can use dominated convergence theorem and pass to the limit so that using (1.8) we obtain

$$V_2(x) \ge \mathbb{E}_x \left[\int_0^{\eta} e^{-r_1 s} g(X_s)ds + \sum_{k:\tau_k \le \eta} e^{-r_1\tau_k}\psi(X_{\tau_k-}, \delta_k) + e^{-r_2\eta}k(X_\eta)\mathbb{1}_{\{\eta<\infty\}} \right] \\ = J_2(x; u^*, \eta).$$

Step 3: Let $V_1(x) = J_1(x; u^*, \eta^*)$. We argue as in Step 1, with equalities instead of inequalities by the property of u^*. Similarly for P2 with $V_2(x) = J_2(x; u^*, \eta^*)$. □

1.3 An Impulse Controller-Stopper Game with Linear Payoffs

In the next Sections 1.3.1-1.3.4, we provide an application of the verification theorem, Theorem 1.2.1, to an impulse game with a one-dimensional state variable evolving essentially as a Brownian motion, which can be shifted by P1's impulses and stopped by P2, and where both players want to maximise linear payoffs. We find two types of Nash equilibria for this game, depending on whether P1 finds it convenient or not to force P2 to stop the game. For both types, we provide quasi-explicit expressions for the equilibrium payoff functions and related strategies. Our findings will be illustrated by some numerical examples.

1.3.1 Setting

We are in a more specific setting than before. This time, the state variable is one-dimensional, while the players have the following linear payoffs for $x \in \mathbb{R}$:

$$f(x) := x - s, \quad \phi(x) := c + \lambda|\delta|, \quad h(x) := ax,$$
$$g(x) := q - x, \quad \psi(x) := d + \gamma|\delta|, \quad k(x) := -bx,$$

with s, c, λ, a, q, d, γ, b positive constants fulfilling

$$a < \lambda \qquad \text{and} \qquad b < \gamma. \tag{1.16}$$

Hence, given an initial state x and an impulse strategy $u = (\tau_n, \delta_n)_{n \geq 1}$, we define the controlled process $X_t^{x;u}$ as

$$X_t = X_t^{x;u} = x + \sigma W_t + \sum_{n: \tau_n \leq t} \delta_n, \qquad t \geq 0,$$

where W is a standard one dimensional Brownian motion and $\sigma > 0$ is a fixed parameter. Moreover, we assume that the two players have the same discount factor $r_1 = r_2 = r$ such that

$$1 - \lambda r > 0 \qquad \text{and} \qquad 1 - br > 0. \tag{1.17}$$

The players' payoff functions are given by

$$J_1(x; u, \eta) = \mathbb{E}_x \left[\int_0^\eta e^{-rt}(X_t - s)dt - \sum_{n: \tau_n \leq \eta} e^{-r\tau_n}(c + \lambda|\delta_n|) + a e^{-r\eta} X_\eta \mathbb{1}_{\{\eta < \infty\}} \right],$$
$$J_2(x; u, \eta) = \mathbb{E}_x \left[\int_0^\eta e^{-rt}(q - X_t)dt + \sum_{n: \tau_n \leq \eta} e^{-r\tau_n}(d + \gamma|\delta_n|) - b e^{-r\eta} X_\eta \mathbb{1}_{\{\eta < \infty\}} \right].$$

Therefore, in this game P1 can shift the state variable X by intervening with impulses in order to keep it high enough, while paying some costs at each intervention time, until the end of the game, which is decided by P2. In addition to that, P2, who wants to keep X low, might gain something each time P1 intervenes. At the end of the game, P1 (resp. P2) receives (resp. loses) some amount proportional to X. Hence, depending on whether her terminal payoff is high enough, P1 might want to end the game soon, by forcing P2 to do that.

Our goal is to find some Nash equilibrium by solving the QVIs problem in (1.5)-(1.10). More specifically, a heuristic analysis of the QVIs system will help us finding a couple of quasi-explicit candidates W_1, W_2 for the equilibrium payoff functions of the game V_1, V_2. We recall the optimal impulse size and the intervention operators in this setting

$$\{\delta(x)\} = \text{argmax}_{\delta \in Z} \{W_1(x + \delta) - c - \lambda|\delta|\},$$
$$\mathcal{M}W_1(x) = W_1(x + \delta(x)) - c - \lambda|\delta(x)|,$$
$$\mathcal{H}W_2(x) = W_2(x + \delta(x)) + d + \gamma|\delta(x)|,$$

together with the infinitesimal generator of the uncontrolled state variable

$$\mathcal{A}V(x) = \frac{1}{2}\sigma^2 V''(x), \quad x \in \mathbb{R}.$$

Before giving the QVIs system in this case, let us introduce the continuation regions for both players

$$\begin{aligned}
\mathcal{C}_1 &= \{x \in \mathbb{R} : W_1(x + \delta(x)) - c - \lambda|\delta(x)| < W_1(x)\}, \\
\mathcal{C}_2 &= \{x \in \mathbb{R} : W_2(x) + bx > 0\},
\end{aligned}$$

so that the respective intervention regions are given by $\mathcal{C}_i^c$ for $i = 1, 2$. Now, the QVIs system becomes

$$\begin{aligned}
&W_1(x + \delta(x)) - c - \lambda|\delta(x)| - W_1(x) \le 0, && x \in \mathbb{R}, \\
&W_2(x) - bx \ge 0, && x \in \mathbb{R}, \\
&W_2(x + \delta(x)) + d + \gamma|\delta(x)| - W_2(x) = 0, && x \in \mathcal{C}_1^c, \\
&W_1(x) - ax = 0, && x \in \mathcal{C}_2^c, \\
&\max\left\{\frac{\sigma^2}{2} W_2''(x) - rW_2(x) + q - x, -xb - W_2(x)\right\} = 0, && x \in \mathcal{C}_1, \\
&\max\left\{\frac{\sigma^2}{2} W_1''(x) - rW_1(x) + x - s, (\mathcal{M}W_1 - W_1)(x)\right\} = 0, && x \in \mathcal{C}_2.
\end{aligned}$$

A first look at the system suggests the following representation for W_1 and W_2:

$$W_1(x) = \begin{cases} ax & x \in \mathcal{C}_2^c \\ \varphi_1(x) & x \in \mathcal{C}_1 \cap \mathcal{C}_2 \\ \mathcal{M}W_1(x) & x \in \mathcal{C}_1^c \cap \mathcal{C}_2 \end{cases} \tag{1.18}$$

$$W_2(x) = \begin{cases} -bx & x \in \mathcal{C}_2^c \\ \varphi_2(x) & x \in \mathcal{C}_1^c \cap \mathcal{C}_2 \\ \mathcal{H}W_2(x) & x \in \mathcal{C}_1^c \cap \mathcal{C}_2, \end{cases} \tag{1.19}$$

where φ_1 and φ_2 are solution to the ODEs

$$\frac{1}{2}\sigma^2 \varphi_1''(x) - r\varphi_1(x) + x - s = 0, \quad \frac{1}{2}\sigma^2 \varphi_2''(x) - r\varphi_2(x) + q - x = 0. \tag{1.20}$$

Hence, for each $x \in \mathbb{R}$, we have:

$$\varphi_1(x) = C_{11}e^{\theta x} + C_{12}e^{-\theta x} + \frac{x - s}{r}, \quad \varphi_2(x) = C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q - x}{r}, \tag{1.21}$$

where C_{11}, C_{12}, C_{21}, C_{22} are real parameters and $\theta := \sqrt{2r/\sigma^2}$.

1.3.2 An Equilibrium with no Simultaneous Interventions

In this subsection, we push our heuristics further by focusing on a first type of Nash equilibrium, where simultaneous interventions are not allowed. By this we mean that we are looking for an equilibrium of threshold type, where P1 intervenes each time X falls below a certain level, say $\bar{x}_1$, in which case P1 applies an impulse moving the state variable towards an optimal level x_1^* belonging to the continuation region of both players. On the other hand, P2 waits until X is too high for her, i.e. until X crosses some upper level, say $\bar{x}_2$, at which point P2 decides to stop the game. The heuristics will lead us to propose candidates for the equilibrium payoffs and related strategies, which will be then checked to be the correct ones subject to some additional conditions. Such additional conditions will be checked in some numerical examples.

Heuristics. Loosely speaking, since P1 is happy when X is high while P2 prefers it to be low, we make the following ansatz about the continuation regions:

$$\begin{aligned} \mathcal{C}_1^c = \ & (-\infty, \bar{x}_1] && \text{(P1 intervenes)}, \\ \mathcal{C}_1 \cap \mathcal{C}_2 = \ & (\bar{x}_1, \bar{x}_2) && \text{(no one intervenes)}, \\ \mathcal{C}_2^c = \ & [\bar{x}_2, \infty) && \text{(P2 intervenes)}. \end{aligned}$$

Hence, we can rewrite (1.18)-(1.19) as

$$W_1(x) = \begin{cases} ax, & x \in [\bar{x}_2, +\infty) \\ \varphi_1(x), & x \in (\bar{x}_1, \bar{x}_2) \\ \mathcal{M}W_1(x), & x \in (-\infty, \bar{x}_1] \end{cases} \tag{1.22}$$

$$W_2(x) = \begin{cases} -bx, & x \in [\bar{x}_2, +\infty) \\ \varphi_2(x), & x \in (\bar{x}_1, \bar{x}_2) \\ \mathcal{H}W_2(x), & x \in (-\infty, \bar{x}_1]. \end{cases} \tag{1.23}$$

Let us find more explicit expressions for the operators $\mathcal{M}W_1$ and $\mathcal{H}W_2$. In this example, it is natural to restrict the analysis to $\delta \geq 0$ since P1 prefers high values of $X^{x,u}$. Hence, whenever she intervenes she will always move the process X to the right, so that

$$\mathcal{M}W_1(x) = \sup_{\delta \geq 0} \{W_1(x+\delta) - c - \lambda|\delta|\} = \sup_{y \geq x} \{W_1(y) - c - \lambda(y-x)\}.$$

Here, we focus on the case where the maximum point belongs to $(\bar{x}_1, \bar{x}_2)$, in other words P1 does not force P2 to stop. In particular, we have $W_1(x_1^*) = \varphi(x_1^*)$ and

$$\varphi(x_1^*) = \max_{y \in (\bar{x}_1, \bar{x}_2)} \{\varphi(y) - \lambda y\}, \quad \text{i.e. } \varphi_1'(x_1^*) = \lambda, \ \varphi_1''(x_1^*) \leq 0, \ \bar{x}_1 < x_1^* < \bar{x}_2.$$

Therefore, we obtain

$$\mathcal{M}W_1(x) = \varphi_1(x_1^*) - c - \lambda(x_1^* - x), \quad \mathcal{H}W_2(x) = \varphi_2(x_1^*) + d + \gamma(x_1^* - x).$$

The parameters appearing in the expressions for W_1 and W_2 must be chosen so as to satisfy the regularity assumptions in the verification theorem, i.e.

$$\begin{aligned} & W_1 \in C^2((-\infty, \bar{x}_1] \cup (\bar{x}_1, \bar{x}_2)) \cap C^1((-\infty, \bar{x}_2]) \cap C(\mathbb{R}), \\ & W_2 \in C^2((\bar{x}_1, \bar{x}_2) \cup (\bar{x}_2, +\infty)) \cap C^1([\bar{x}_1, +\infty)) \cap C(\mathbb{R}). \end{aligned}$$

We can summarise the description of our candidates for equilibrium payoffs in the following

Ansatz 1.3.1 Let W_1 and W_2 be as in (1.22)-(1.23) where the parameters involved

$$(C_{11},\, C_{12},\, C_{21},\, C_{22},\, \bar{x}_1,\, \bar{x}_2,\, x_1^*)$$

satisfy the order condition

$$\bar{x}_1 < x_1^* < \bar{x}_2, \tag{1.24}$$

and the following equations

$$\begin{aligned}
&\varphi_1'(x_1^*) = \lambda \quad \text{and} \quad \varphi''(x_1^*) \le 0 && \text{(optimality of } x_1^*\text{)},\\
&\varphi_1'(\bar{x}_1) = \lambda && (C^1\text{-pasting in } \bar{x}_1),\\
&\varphi_2'(\bar{x}_2) = -b && (C^1\text{-pasting in } \bar{x}_2),\\
&\varphi_1(\bar{x}_1) = \varphi(x_1^*) - c - \lambda(x_1^* - \bar{x}_1) && (C^0\text{-pasting in } \bar{x}_1),\\
&\varphi_1(\bar{x}_2) = a\bar{x}_2 && (C^0\text{-pasting in } \bar{x}_2),\\
&\varphi_2(\bar{x}_1) = \varphi_2(x_1^*) + d + \gamma(x_1^* - \bar{x}_1) && (C^0\text{-pasting in } \bar{x}_1),\\
&\varphi_2(\bar{x}_2) = -b\bar{x}_2 && (C^0\text{-pasting in } \bar{x}_2).
\end{aligned} \tag{1.25}$$

Re-parametrisation. We will conveniently re-parametrise the equations above in order to reduce their complexity. Using the expressions in (1.21) we can rewrite (1.25) as follows

$$\theta C_{11} e^{\theta x_1^*} - \theta C_{12} e^{-\theta x_1^*} + \frac{1}{r} = \lambda \tag{1.26a}$$

$$\theta C_{11} e^{\theta \bar{x}_1} - \theta C_{12} e^{-\theta \bar{x}_1} + \frac{1}{r} = \lambda \tag{1.26b}$$

$$\theta C_{21} e^{\theta \bar{x}_2} - \theta C_{22} e^{-\theta \bar{x}_2} - \frac{1}{r} = -b \tag{1.26c}$$

$$C_{11} e^{\theta \bar{x}_1} + C_{12} e^{-\theta \bar{x}_1} + \frac{\bar{x}_1 - s}{r} = C_{11} e^{\theta x_1^*} + C_{12} e^{-\theta x_1^*} + \frac{x_1^* - s}{r} - c - \lambda(x_1^* - \bar{x}_1) \tag{1.26d}$$

$$C_{11} e^{\theta \bar{x}_2} + C_{12} e^{-\theta \bar{x}_2} + \frac{\bar{x}_2 - s}{r} = a\bar{x}_2 \tag{1.26e}$$

$$C_{21} e^{\theta \bar{x}_1} + C_{22} e^{-\theta \bar{x}_1} + \frac{q - \bar{x}_1}{r} = C_{21} e^{\theta x_1^*} + C_{22} e^{-\theta x_1^*} + \frac{q - x_1^*}{r} + d + \gamma(x_1^* - \bar{x}_1) \tag{1.26f}$$

$$C_{21} e^{\theta \bar{x}_2} + C_{22} e^{-\theta \bar{x}_2} + \frac{q - \bar{x}_2}{r} = -b\bar{x}_2 \tag{1.26g}$$

So, subtracting (1.26b) to (1.26a) we obtain

$$C_{11} = -\frac{1 - \lambda r}{r\theta} \frac{1}{e^{\theta x_1^*} + e^{\theta \bar{x}_1}}, \quad C_{12} = \frac{1 - \lambda r}{r\theta} \frac{e^{\theta(x_1^* + \bar{x}_1)}}{e^{\theta x_1^*} + e^{\theta \bar{x}_1}}. \tag{1.27}$$

Then, adding (1.26c) to (1.26g) we find

$$C_{21} = \frac{e^{-\theta \bar{x}_2}}{2r} \left[(1 - br) \left(\bar{x}_2 + \frac{1}{\theta} \right) - q \right], \quad C_{22} = \frac{e^{\theta \bar{x}_2}}{2r} \left[(1 - br) \left(\bar{x}_2 - \frac{1}{\theta} \right) - q \right].$$

Hence, by substitution, we are reduced to solving the following sub-system

$$-2\frac{1-\lambda r}{r\theta}\frac{e^{\theta\bar{x}_1}-e^{\theta x_1^*}}{e^{\theta\bar{x}_1}+e^{\theta x_1^*}}+\frac{1-\lambda r}{r}(\bar{x}_1-x_1^*)+c=0 \tag{1.28a}$$

$$-\frac{1-\lambda r}{r\theta}\frac{e^{2\theta\bar{x}_2}}{e^{\theta x_1^*}+e^{\theta\bar{x}_1}}+((1-ar)\bar{x}_2-s)\frac{e^{\theta\bar{x}_2}}{r}+\frac{1-\lambda r}{r\theta}\frac{e^{\theta(x_1^*+\bar{x}_1)}}{e^{\theta x_1^*}+e^{\theta\bar{x}_1}}=0 \tag{1.28b}$$

$$\begin{aligned}&\frac{e^{-\theta\bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q\right](e^{\theta\bar{x}_1}-e^{\theta x_1^*})+\frac{e^{\theta\bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]\\ &\quad\times(e^{-\theta\bar{x}_1}-e^{-\theta x_1^*})+\frac{1-\gamma r}{r}(x_1^*-\bar{x}_1)-d=0\end{aligned} \tag{1.28c}$$

Now, the change of variable $z=e^{\theta(x_1^*-\bar{x}_1)}$ turns equation (1.28a) into the following

$$\ln z-2\left(\frac{z-1}{z+1}\right)-\frac{cr\theta}{1-\lambda r}=0, \tag{1.29}$$

which has a unique solution $\tilde{z}>1$. Indeed, let $F(z):=\ln z-2(\frac{z-1}{z+1})-\frac{cr\theta}{1-\lambda r}$ and observe that it satisfies $F'(z)>0$ for all $z>1$. Moreover $z=e^{\theta(x_1^*-\bar{x}_1)}>1$ due to the order condition (1.24), $F(1)<0$ and $\lim_{z\to+\infty}F(z)=+\infty$. Therefore, there is only one value $\tilde{z}$ such that $F(\tilde{z})=0$, which can be easily computed numerically.

Now, in order to solve (1.28b) and (1.28c) we perform a second change of variable, $w=e^{\theta(\bar{x}_2-\bar{x}_1)}$, leading to the following equations

$$-\frac{1-\lambda r}{r\theta}\frac{w^2e^{\theta\bar{x}_1}}{\tilde{z}+1}+((1-ar)\bar{x}_2-s)\frac{e^{\theta\bar{x}_1}w}{r}+\frac{1-\lambda r}{r\theta}\frac{e^{\theta x_1^*}}{\tilde{z}+1}=0, \tag{1.30a}$$

$$\begin{aligned}&\frac{1-\tilde{z}}{2rw}\left[(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q\right]+\frac{w(\tilde{z}-1)}{2r\tilde{z}}\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]\\ &+\frac{1-\gamma r}{\theta r}\ln\tilde{z}-d=0.\end{aligned} \tag{1.30b}$$

Notice that (1.30a) is linear in $\bar{x}_2$, hence it can be easily solved in terms of $\tilde{z}$ and w, to get

$$\bar{x}_2=\left(\frac{1-\lambda r}{\theta w}\frac{w^2-\tilde{z}}{\tilde{z}+1}+s\right)\frac{1}{1-ar}. \tag{1.31}$$

Regarding (1.30b), it can be rewritten as

$$\begin{aligned}&w^4\frac{(1-br)(1-\lambda r)}{\theta(1-ar)(\tilde{z}+1)}+w^3\left[(1-br)\left(\frac{s}{1-ar}-\frac{1}{\theta}\right)-q\right]\\ &\quad+2\tilde{z}w^2\left(\frac{1}{\tilde{z}-1}\left(\frac{(1-\gamma r)}{\theta}\ln\tilde{z}-rd\right)-\frac{(1-br)(1-\lambda r)}{\theta(1-ar)(\tilde{z}+1)}\right)\\ &\quad\quad+\tilde{z}w\left(q-(1-br)\left(\frac{s}{1-ar}+\frac{1}{\theta}\right)\right)+\frac{(1-br)(1-\lambda r)\tilde{z}^2}{\theta(1-ar)(\tilde{z}+1)}=0.\end{aligned} \tag{1.32}$$

The equation for w above is a quartic equation for which explicit formulae for its roots are available. However, since they are quite cumbersome and not easy to use, we will solve it numerically, leaving the analysis for later. Once the two new parameters $\tilde{z}$ and $\tilde{w}$ are found, by solving numerically the respective equations above, the thresholds $\bar{x}_1,\bar{x}_2$ and the optimal level for P1, x_1^*, can be deduced automatically. It remains to check under which additional conditions such thresholds correspond to a Nash equilibrium of our original linear game. This will be done in the next paragraph.

Characterisation of the equilibrium and verification. The next proposition summarises our findings and establish the link between the solutions $\tilde{z}$ and $\tilde{w}$ to the equations above with the Nash equilibrium of threshold type we are looking for, provided some additional inequalities are fulfilled.

PROPOSITION 1.3.1 Assume that there exists a solution $(\tilde{z},\, \tilde{w})$ to (1.29)-(1.32) such that $1 < \tilde{z} < \tilde{w}$ and additionally

$$0 \le \frac{(1-br)(1-\lambda r)(\tilde{w}^2-\tilde{z})}{\theta\tilde{w}(1-ar)(\tilde{z}+1)} + \frac{1-br}{1-ar}s - q < \frac{1-br}{\theta}, \tag{1.33}$$

$$\left(\frac{1-br}{1-ar}\left(\frac{(1-\lambda r)(\tilde{w}^2-\tilde{z})}{\theta\tilde{w}(\tilde{z}+1)} + s\right) - q\right)(\tilde{w}-1)^2 + \frac{1-br}{\theta}(1+2\tilde{w}\ln\tilde{w}-\tilde{w}^2) > 0. \tag{1.34}$$

Then, a Nash equilibrium for the game in Section 1.3 exists and it is given by the pair (u^*, η^*), where $u^* = (\tau_n,\, \delta_n)_{n\ge 1}$ is defined by

$$\tau_n := \inf\left\{t > \tau_{n-1}; X_t \in (-\infty, \bar{x}_1]\right\}, \quad \delta_n := (x_1^* - x)\mathbf{1}_{(-\infty,\bar{x}_1]}(x),$$

and

$$\eta^* := \inf\{t \ge 0 : X_t \in [\bar{x}_2,\, +\infty)\},$$

where the thresholds $\bar{x}_1$, x_1^* and $\bar{x}_2$ satisfy

$$x_1^* = \bar{x}_2 + \frac{\ln z - \ln w}{\theta}, \quad \bar{x}_1 = \bar{x}_2 - \frac{\ln w}{\theta}, \quad \bar{x}_2 = \left(\frac{1-\lambda r}{\theta\tilde{w}}\frac{\tilde{w}^2-\tilde{z}}{\tilde{z}+1} + s\right)\frac{1}{1-ar}.$$

Moreover, the functions W_1, W_2 in Ansatz 1.3.1 coincide with the equilibrium payoff functions V_1, V_2, i.e.

$$V_1 \equiv W_1, \quad \text{and } V_2 \equiv W_2.$$

PROOF 1.3.1 The proof consists in checking all the conditions needed to apply the Verification Theorem (1.2.1). First, notice that by construction the functions W_1 and W_2 satisfy all required regularity properties, i.e. W_1 and W_2 have polynomial growth and

$$W_1 \in C^2\left((-\infty,\bar{x}_2)\setminus\{\bar{x}_1\}\right) \cap C^1\left((-\infty,\bar{x}_2)\right) \cap C(\mathbb{R}),$$
$$W_2 \in C^2\left((\bar{x}_1,+\infty)\setminus\{\bar{x}_2\}\right) \cap C^1\left((\bar{x}_1,+\infty)\right) \cap C(\mathbb{R}).$$

Moreover Lemmas 1.A.1 and 1.A.2 in the Appendix grant the optimality of the impulse $\delta(x)$, i.e.

$$\{\delta(x)\} = \operatorname{argmax}_{\delta\in Z}\left\{W_1(x+\delta) - c - \lambda|\delta|\right\}$$

together with the properties

$$\mathcal{M}W_1 - W_1 \le 0, \quad W_2(x) + bx \ge 0, \quad x \in \mathbb{R}.$$

Next, we show that for all $x \in \{\mathcal{M}W_1 - W_1 = 0\} = (-\infty,\, \bar{x}_1]$, we have $W_2(x) = \mathcal{H}W_2(x)$. Indeed, by definition of $\mathcal{H}W_2$ we have:

$$\begin{aligned}\mathcal{H}W_2(x) &= W_2(x+\delta(x)) + d + \gamma|\delta(x)| = W_2(x_1^*) + d + \gamma(x_1^* - x)\\ &= \varphi_2(x_1^*) + d + \gamma(x_1^* - x) = W_2(x) \qquad \forall\, x \in (-\infty,\, \bar{x}_1].\end{aligned}$$

Now, let $x \in \{\mathcal{M}W_1 - W_1 < 0\}$. We have to prove that

$$\max\{\mathcal{A}W_2(x) - rW_2(x) + q - x,\, -bx - W_2(x)\} = 0.$$

Since $\{\mathcal{M}W_1 - W_1 < 0\} = (\bar{x}_1, +\infty)$, we can consider two separate cases. In $(\bar{x}_1, \bar{x}_2)$ we have $-bx - W_2(x) < 0$ and

$$\mathcal{A}W_2(x) - rW_2(x) + q - x = \mathcal{A}\varphi_2(x) - r\varphi_2(x) + q - x = 0,$$

since φ_2 is solution to the ODE (1.20). On the other hand, in $[\bar{x}_2, +\infty)$ we know that $-bx = W_2(x)$, then we have to check that $\mathcal{A}W_2(x) - rW_2(x) + q - x \leq 0$ for all $x \in [\bar{x}_2, +\infty)$. First, notice that $W_2(x) = -bx$ and $\mathcal{A}W_2(x) = 0$. Hence, we are reduced to checking the inequality

$$\mathcal{A}W_2(x) - rW_2(x) + q - x = brx + q - x = q - (1 - br)x \leq 0. \tag{1.35}$$

Since by assumption $1 - br > 0$, the function $x \mapsto q - (1 - br)x$ is decreasing, so we just need to check whether the inequality holds in $\bar{x}_2$, i.e. $(1 - br)\bar{x}_2 - q \geq 0$ which is satisfied by (1.33).

To conclude our verification that the candidate equilibrium payoffs satisfy the QVIs system, we are left with checking that $-bx - W_2(x) = 0$ implies $W_1(x) = ax$, and that, on the other side, $-bx - W_2(x) < 0$ implies

$$\max\{\mathcal{A}W_1(x) - rW_1(x) + x - s, \mathcal{M}W_1(x) - W_1(x)\} = 0.$$

Now, the first implication holds by definition, while the second one boils down to proving

$$\max\{\mathcal{A}W_1(x) - rW_2(x) + x - s,\, \mathcal{M}W_1(x) - W_1(x)\} = 0, \quad x \in (-\infty, \bar{x}_2).$$

For $x \in (\bar{x}_1, \bar{x}_2)$ we have $\mathcal{M}W_1(x) - W_1(x) < 0$ and, as before,

$$\mathcal{A}W_1(x) - rW_1(x) + x - s = \mathcal{A}\varphi_1(x) - r\varphi_1(x) + x - s = 0$$

as φ_1 is solution to the ODE (1.20). For $x \in (-\infty, \bar{x}_1]$, we know that $\mathcal{M}W_1(x) - W_1(x) = 0$ and therefore we have to check that

$$\mathcal{A}W_1(x) - rW_1(x) + x - s \leq 0, \quad x \in (-\infty, \bar{x}_1].$$

To do that, recall first that $W_1(x) = \varphi_1(x_1^*) - c - \lambda(x_1^* - x)$ and $\mathcal{A}W_1(x) = 0$, which gives

$$\mathcal{A}W_1(x) - rW_1(x) + x - s = -r\varphi_1(\bar{x}_1) - r\lambda(x - \bar{x}_1) + x - s$$

since $\varphi_1(\bar{x}_1) = \varphi_1(x_1^*) - c - \lambda(x_1^* - \bar{x}_1)$. Notice that, since by assumption $1 - \lambda r > 0$, the function $x \mapsto -r\varphi_1(\bar{x}_1) - r\lambda(x - \bar{x}_1) + x - s$ is increasing in x. As a result, we only need to prove that the desired inequality holds for $x = \bar{x}_1$, i.e.

$$-r\varphi_1(\bar{x}_1) + \bar{x}_1 - s \leq 0,$$

which is verified since $\mathcal{A}\varphi_1(\bar{x}_1) - r\varphi_1(\bar{x}_1) + \bar{x}_1 - s = 0$ and $\mathcal{A}\varphi_1(\bar{x}_1) = r\varphi_1(\bar{x}_1) - \bar{x}_1 + s \geq 0$, due to $\varphi_1''(\bar{x}_1) \geq 0$.

To finish the proof, we check that equilibrium strategies are x-admissible for every $x \in \mathbb{R}$. By construction, the controlled process never exits from $(\bar{x}_1, \bar{x}_2) \cup \{x\}$, so that $\sup_{t\geq 0} e^{-rt}|X_t| \in L^p(\Omega)$ holds. It is easy to check that all the other conditions are satisfied provided we show the following:

$$\mathbb{E}_x\left[\sum_{k\geq 1} e^{-r\tau_k}(c + \lambda|\delta_k|)\right] < +\infty. \tag{1.36}$$

To start, let us assume that the initial state x is x_1^*. The idea is to write τ_k as a sum of independent and identically distributed copies of some exit time (as in the proof of Proposition 4.7 in [1]). Denote by μ the exit time of the process $x_1^* + \sigma W$ from $(\bar{x}_1, \bar{x}_2)$, where W is a one-dimensional Brownian motion. Then, each time τ_k can be decomposed as $\tau_k = \sum_{l\geq 1}^k \zeta_l$, where ζ_l are i.i.d. random variables with the same law as μ. We can now show (1.36). As $\delta_k = \delta_1 = x_1^* - \bar{x}_1$ for all $k \geq 1$, we have

$$\begin{aligned}
\mathbb{E}_{x_1^*}\left[\sum_{k\geq 1} e^{-r\tau_k}(c + \lambda|\delta_k|)\right] &\leq (c + \lambda\delta_1)\mathbb{E}_{x_1^*}\left[\sum_{k\geq 1} e^{-r\tau_1}\right] \\
&= (c + \lambda\delta_1)\mathbb{E}_{x_1^*}\left[\sum_{k\geq 1} e^{-r\sum_{l=1}^k \zeta_l}\right] \\
&= (c + \lambda\delta_1)\mathbb{E}_{x_1^*}\left[\sum_{k\geq 1}\prod_{l\geq 1}^k e^{-r\zeta_l}\right]
\end{aligned}$$

and, by the Fubini-Tonelli theorem and the independence of $(\zeta_l)_{l\geq 1}$, we get

$$\sum_{k\geq 1}\prod_{l\geq 1}^k \mathbb{E}_{x_1^*}\left[e^{-r\zeta_l}\right] \leq \sum_{k\geq 1}\left(\mathbb{E}_{x_1^*}\left[e^{-r\mu}\right]\right)^k,$$

which is a convergent geometric series, since $\mu > 0$. Then, for any $x \in (\bar{x}_1, \bar{x}_2)$ same arguments hold whereas, when $x \in [\bar{x}_2, +\infty)$, P2 stops as soon as the game starts and, as a consequence, P1 cannot apply any impulse, hence, the condition is satisfied. Finally, if $x \in (-\infty, \bar{x}_1]$ we have

$$\mathbb{E}_x\left[\sum_{k\geq 1} e^{-r\tau_k}(c + \lambda|\delta_k|)\right] = c + \lambda|x_1^* - x| + \mathbb{E}_{x_1^*}\left[\sum_{k\geq 1} e^{-r\tau_k}(c + \lambda|\delta_k|)\right] < +\infty.$$

since $\sup_{t\geq 0}|X_t| \in L^p(\Omega)$. □

1.3.3 An Equilibrium where the Controller Activates the Stopper

We turn now to another kind of Nash equilibrium, where P1 behaves similarly as in the previous type with the main difference that this time when the state variable X falls below a given threshold, she will intervene and send X directly to the stopping region of P2, hence forcing her to stop the game instantaneously. In particular, this would be an equilibrium in which the two players act at the same time. The approach we use to characterise such an equilibrium follows the same steps as in the previous subsection.

Heuristics. We start with some heuristics leading us to formulate a conjecture on the equations the thresholds characterising this equilibrium should reasonably satisfy. Arguing as before, we expect the candidates for equilibrium payoffs to be of the following type (1.18)-(1.19) as

$$W_1(x) = \begin{cases} ax & \text{in } [\bar{x}_2, +\infty) \\ \varphi_1(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ \mathcal{M}W_1(x) & \text{in } (-\infty, \bar{x}_1] \end{cases} \tag{1.37}$$

$$W_2(x) = \begin{cases} -bx & \text{in } [\bar{x}_2, +\infty) \\ \varphi_2(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ \mathcal{H}W_2(x) & \text{in } (-\infty, \bar{x}_1] \end{cases} \tag{1.38}$$

for suitable thresholds $\bar{x}_i$, $i = 1, 2$.

Now, according to the type of equilibrium we want to identify, we investigate the case in which the maximum point of the function $y \mapsto W_1(y) - \lambda y$ belongs to $[\bar{x}_2, +\infty)$, meaning that when P1 intervenes she is applying an optimal impulse moving the state variable to the stopping region of her competitor. Thus, in this case we have

$$\mathcal{M}W_1(x) = \sup_{y \geq \bar{x}_2} (ay - \lambda y).$$

Therefore, we have the following scenarios:

- if $a > \lambda \Rightarrow x_1^* \to \infty$;
- if $a = \lambda \Rightarrow x_1^*$ could be any $x \geq \bar{x}_2$;
- if $a < \lambda \Rightarrow x_1^* = \bar{x}_2$.

Clearly, the only interesting case is $a < \lambda$, so that $x_1^* = \bar{x}_2$. As a consequence, this type of equilibrium will be characterised only by two thresholds. Similarly as in the previous subsection, we characterise the parameters $(C_{11}, C_{12}, C_{21}, C_{22})$ and the thresholds $(\bar{x}_1, \bar{x}_2)$ by exploiting the smooth pasting conditions coming from the regularity assumptions postulated in Theorem 1.2.1. By doing so, we obtain

$$
\begin{aligned}
&\varphi_1'(\bar{x}_1) = \lambda && (C^1\text{-pasting in } \bar{x}_1),\\
&\varphi_1(\bar{x}_2) = a\bar{x}_2 && (C^0\text{-pasting in } \bar{x}_2),\\
&\varphi_1(\bar{x}_1) = a\bar{x}_2 - c - \lambda(\bar{x}_2 - \bar{x}_1) && (C^0\text{-pasting in } \bar{x}_1),\\
&\varphi_2'(\bar{x}_2) = -b && (C^1\text{-pasting in } \bar{x}_2),\\
&\varphi_2(\bar{x}_2) = -b\bar{x}_2 && (C^0\text{-pasting in } \bar{x}_2),\\
&\varphi_2(\bar{x}_1) = -b\bar{x}_2 + d + \gamma(\bar{x}_2 - \bar{x}_1) && (C^0\text{-pasting in } \bar{x}_1).
\end{aligned}
\tag{1.39}
$$

together with the order condition $\bar{x}_1 < \bar{x}_2$.

Re-parametrisation. We first rewrite (1.39) as

$$\theta C_{11}e^{\theta\bar{x}_1} - \theta C_{12}e^{-\theta\bar{x}_1} + \frac{1}{r} = \lambda \tag{1.40a}$$

$$\theta C_{21}e^{\theta\bar{x}_2} - \theta C_{22}e^{-\theta\bar{x}_2} - \frac{1}{r} = -b \tag{1.40b}$$

$$C_{11}e^{\theta\bar{x}_2} + C_{12}e^{-\theta\bar{x}_2} + \frac{\bar{x}_2 - s}{r} = a\bar{x}_2 \tag{1.40c}$$

$$C_{11}e^{\theta\bar{x}_1} + C_{12}e^{-\theta\bar{x}_1} + \frac{\bar{x}_1 - s}{r} = (a-\lambda)\bar{x}_2 + \lambda\bar{x}_1 - c \tag{1.40d}$$

$$C_{21}e^{\theta\bar{x}_2} + C_{22}e^{-\theta\bar{x}_2} + \frac{q - \bar{x}_2}{r} = -b\bar{x}_2 \tag{1.40e}$$

$$C_{21}e^{\theta\bar{x}_1} + C_{22}e^{-\theta\bar{x}_1} + \frac{q - \bar{x}_1}{r} = (\gamma - b)\bar{x}_2 + d - \gamma\bar{x}_1 \tag{1.40f}$$

Then, dividing (1.40a) by θ and adding it to (1.40d), we can solve the equation for C_{11} and consequently find C_{12} as in the previous case, (1.27). A similar manipulation of equations (1.40b) and (1.40e) yields C_{21} and C_{22}. At this point, plugging C_{11} and C_{12} in (1.40c) we obtain

$$
\begin{aligned}
&\frac{e^{\theta(\bar{x}_2-\bar{x}_1)}}{2}\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 + \frac{1}{\theta}\right)\frac{1-\lambda r}{r} - c + \frac{s}{r}\right] + \frac{e^{-\theta(\bar{x}_2-\bar{x}_1)}}{2}\\
&\qquad \times\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 - \frac{1}{\theta}\right)\frac{1-\lambda r}{r} - c + \frac{s}{r}\right] + \frac{1-ar}{r}\bar{x}_2 - \frac{s}{r} = 0
\end{aligned}
$$

which, noting that $\bar{x}_1 = \bar{x}_2 - \frac{\ln w}{\theta}$ and applying the change of variable $w = e^{\theta(\bar{x}_2-\bar{x}_1)}$, can be rewritten as

$$
\begin{aligned}
&w\left[\frac{(1-\lambda r)(\ln w - 1)}{r\theta} - \frac{1-ar}{r}\bar{x}_2 - c + \frac{s}{r}\right] + \frac{1}{w}\left[\frac{(1-\lambda r)(\ln w + 1)}{r\theta}\right.\\
&\qquad \left. - \frac{1-ar}{r}\bar{x}_2 - c + \frac{s}{r}\right] + 2\frac{(1-ar)\bar{x}_2 - s}{r} = 0.
\end{aligned}
$$

This is a linear equation in $\bar{x}_2$, yielding

$$\bar{x}_2 = \frac{(1-\lambda r)((\ln w - 1)w^2 + \ln w + 1) - cr\theta(w^2+1)}{\theta(1-ar)(w-1)^2} + \frac{s}{1-ar}. \tag{1.41}$$

Proceeding analogously with (1.40f), we obtain the following alternative expression for $\bar{x}_2$

$$\bar{x}_2 = \frac{q}{1-br} + \frac{w+1}{\theta(w-1)} + \frac{2(\theta rd - (1-\gamma r)\ln w)w}{\theta(1-br)(w-1)^2}. \tag{1.42}$$

Then, by equating (1.41) to (1.42), we obtain an equation in w:

$$\begin{aligned} G(w) :=& \frac{(1-\lambda r)((\ln w - 1)w^2 + \ln w + 1) - cr\theta(w^2+1)}{\theta(1-ar)(w-1)^2} + \frac{s}{1-ar} \\ & - \frac{q}{1-br} - \frac{w+1}{\theta(w-1)} - \frac{2(\theta rd - (1-\gamma r)\ln w)w}{\theta(1-br)(w-1)^2} = 0 \end{aligned} \tag{1.43}$$

which has at least a solution, say $\widehat{w} > 1$, due to $\lim_{w\to+\infty} G(w) = +\infty$ and $\lim_{w\to 1} G(w) = -\infty$. The first limit follows from the highest order term, $w^2 \ln w$, being multiplied by $\frac{1-\lambda r}{1-ar} > 0$ (cf. (1.17)). On the other hand, the second limit follows from (1.16):

$$\lim_{w\to 1} G(w) = \lim_{w\to 1} \frac{1}{(w-1)^2}\left[-\frac{2cr}{1-ar} - \frac{2rd}{1-br}\right] = -\infty.$$

Characterisation of the equilibrium and verification. The next proposition summarises our characterisation of this Nash equilibrium in terms of only one parameter, $\widehat{w}$, provided some further conditions, that will be checked numerically in the next subsection.

Proposition 1.3.2 Assume that there exists $\widehat{w}$ solution to (1.43) such that

$$(1-\lambda r)(\widehat{w} - \widehat{w}\ln\widehat{w} - 1) + cr\theta\widehat{w} > 0, \tag{1.44}$$

$$0 \le (1-br)(\widehat{w}^2 - 1) + 2(\theta rd - (1-\gamma r)\ln\widehat{w})\widehat{w} < (1-br)(\widehat{w}-1)^2. \tag{1.45}$$

Then, a Nash equilibrium for the game in Section 1.3 exists and it is given by the strategies (u^*, η^*), with $u^* = (\tau_n, \delta_n)_{n\ge 1}$ defined by

$$\tau_n := \inf\{t > \tau_{n-1}; X_t \in (-\infty, \bar{x}_1]\}, \quad \delta_n := (\bar{x}_2 - x)\,\mathbf{1}_{(-\infty,\bar{x}_1]}(x)$$

and

$$\eta^* := \inf\{t \ge 0 : X_t \in [\bar{x}_2, +\infty)\},$$

where the thresholds satisfy

$$\bar{x}_1 = \bar{x}_2 - \frac{\ln\widehat{w}}{\theta}, \quad \bar{x}_2 = \frac{q}{1-br} + \frac{\widehat{w}+1}{\theta(\widehat{w}-1)} + \frac{2(\theta rd - (1-\gamma r)\ln\widehat{w})\widehat{w}}{\theta(1-br)(\widehat{w}-1)^2}.$$

Moreover, the functions W_1, W_2 in Ansatz 1.3.1 coincide with the equilibrium payoff functions V_1, V_2, i.e.

$$V_1 \equiv W_1 \quad \text{and} \quad V_2 \equiv W_2.$$

Proof 1.3.2 We proceed as for the previous equilibrium, by checking all the conditions necessary to apply the verification theorem. First of all, the functions W_1, W_2 satisfy by construction all required regularity properties, i.e.

$$W_1 \in C^2\left((-\infty,\bar{x}_2) \setminus \{\bar{x}_1\}\right) \cap C^1\left((-\infty,\bar{x}_2)\right) \cap C(\mathbb{R}),$$

$$W_2 \in C^2\left((\bar{x}_1,+\infty) \setminus \{\bar{x}_2\}\right) \cap C^1\left((\bar{x}_1,+\infty)\right) \cap C(\mathbb{R})$$

and both have at most polynomial growth.

Next, Lemmas 1.A.3 and 1.A.4 give

$$\{\delta(x)\} = \operatorname{argmax}_{\delta \in Z} \{W_1(x+\delta) - c - \lambda|\delta|\}$$

together with

$$\mathcal{M}W_1(x) - W_1(x) \le 0, \quad W_2(x) + bx \ge 0,$$

for all $x \in \mathbb{R}$. Let $x \in \{\mathcal{M}W_1 - W_1 = 0\} = (-\infty, \bar{x}_1]$. By definition of $\mathcal{H}W_2$ we have:

$$\begin{aligned}\mathcal{H}W_2(x) &= W_2(x+\delta(x)) + d + \gamma|\delta(x)| = W_2(\bar{x}_2) + d + \gamma(\bar{x}_2 - x)\\ &= -b\bar{x}_2 + d + \gamma(\bar{x}_2 - x) = W_2(x).\end{aligned}$$

Now, in order to prove that

$$\max\{\mathcal{A}W_2(x) - rW_2(x) + q - x,\, -bx - W_2(x)\} = 0, \quad x \in (\bar{x}_1, +\infty),$$

we consider two separate cases as for the previous equilibrium. First, for $x \in (\bar{x}_1, \bar{x}_2)$, we have $-bx - W_2(x) < 0$ and

$$\mathcal{A}W_2(x) - rW_2(x) + q - x = \mathcal{A}\varphi_2(x) - r\varphi_2(x) + q - x = 0$$

since φ_2 is solution to the ODE (1.20), so the maximum between the two terms is zero. Second, we know that $-bx = W_2(x)$ for $x \in [\bar{x}_2, +\infty)$, then we have to check that $\mathcal{A}W_2(x) - rW_2(x) + q - x \le 0$ for any $x \in [\bar{x}_2, +\infty)$. Since $\mathcal{A}W_2(x) = 0$, we are reduced to verify the inequality

$$\mathcal{A}W_2(x) - rW_2(x) + q - x = brx + q - x = q - (1-br)x \le 0. \tag{1.46}$$

Given that $x \mapsto q - (1-br)x$ is decreasing due to $1 - br > 0$, it suffices to show the inequality above at the point $\bar{x}_2$, i.e. $(1-br)\bar{x}_2 - q \ge 0$, which is implied by (1.45).
To complete the verification that W_1, W_2 are solutions to the QVIs system, we show that in $-bx - W_2(x) = 0$ implies $W_1(x) = ax$ and that $-bx - W_2(x) < 0$ yields

$$\max\{\mathcal{A}W_1(x) - rW_1(x) + x - s, \mathcal{M}W_1(x) - W_1(x)\} = 0.$$

The first implication holds by definition. For the second one, we have to prove

$$\max\{\mathcal{A}W_1(x) - rW_2(x) + x - s,\, \mathcal{M}W_1(x) - W_1(x)\} = 0, \quad x \in (-\infty, \bar{x}_2).$$

For $x \in (\bar{x}_1, \bar{x}_2)$ we have $\mathcal{M}W_1(x) - W_1(x) < 0$ and as before

$$\mathcal{A}W_1(x) - rW_1(x) + x - s = \mathcal{A}\varphi_1(x) - r\varphi_1(x) + x - s = 0$$

as φ_1 is solution to the ODE (1.20). For any $x \in (-\infty, \bar{x}_1]$ we know that $\mathcal{M}W_1(x) - W_1(x) = 0$, hence, we have to check that

$$\mathcal{A}W_1(x) - rW_1(x) + x - s = (1-\lambda r)x + cr - s - (a-\lambda)r\bar{x}_2 \le 0, \quad x \in (-\infty, \bar{x}_1].$$

To do so, we notice that the function $x \mapsto (1-\lambda r)x + cr - s - (a-\lambda)r\bar{x}_2$ is increasing in x by assumption $1-\lambda r > 0$. Therefore, we only need to prove that the desired inequality for $x = \bar{x}_1$, i.e.

$$(1-ar)\bar{x}_2 - \frac{1-\lambda r}{\theta}\ln w + cr - s \leq 0,$$

which is given by Lemma 1.A.3. Finally, the optimal strategies are x-admissible for every $x \in \mathbb{R}$. Indeed, by construction, the controlled process never exits from $(\bar{x}_1, \bar{x}_2) \cup \{x\}$, and, as a consequence, $\sup_{t\geq 0} e^{-rt}|X_t| \in L^p(\Omega)$ holds for all $p \geq 1$. It is easy to check that all the other conditions are satisfied as in the first type of equilibrium. □

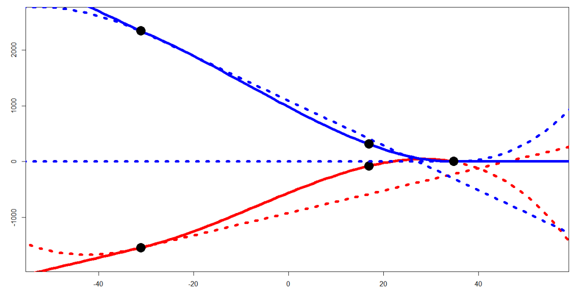
1.3.4 Numerical Experiments

In this section, we will give some numerical illustrations of the equilibrium payoff functions and a selection of comparative statics regarding the two types of Nash equilibria identified in the previous subsections (the numerical results in this section were obtained using R, rootSolve package). It is useful to remember that in order for the solutions to the QVIs system to be Nash equilibria of one of the two types, they have to satisfy either (1.33)-(1.34) or (1.44)-(1.45). Before we start, let us recall the meaning of the parameters involved:

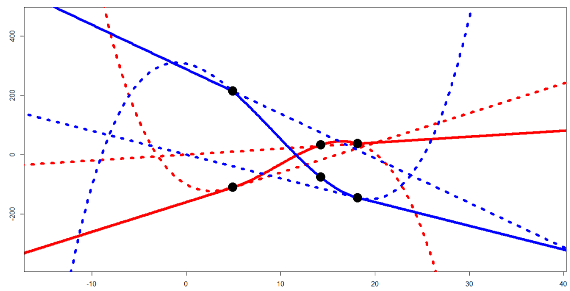
- s and q might be interpreted as exogenous costs and gains, respectively. Note that P1's running payoff $f(x) = x - s$, hence, in order to make profit P1 needs x to be greater than s, which can fairly be considered as P1's expense, an analogous reasoning applies for P2, but in the opposite direction since $g(x) = q - x$;
- a and b can be considered as terminal payoff sensitivity to the underlying process, X_t, as we have $h(x) = ax$ and $k(x) = -bx$ respectively;
- at each intervention time P1 faces a fixed cost, c, while P2 receives a fixed gain, d;
- moreover, λ is P1's proportional cost parameter, while γ is P2's proportional gain parameter;
- finally, r is the discount rate, the same for both players, and σ is the volatility of the state variable.

Equilibrium 1: no simultaneous interventions. In order to fulfill (1.33)-(1.34), we can observe that both inequalities are satisfied for high enough values of $\tilde{w}$. It is possible to show via graphical analysis that $\tilde{w}$, solution to (1.32), is decreasing in a, b, s and increasing in c, d, q, λ and γ. Therefore, we have chosen small values of a, b and s to obtain the first equilibrium, Scenario A, whereas for Scenario B we have looked for higher values and increased q and d in order to find an equilibrium. The table below provides the exact parameter settings, with $\bar{x}_1$, x_1^* and $\bar{x}_2$ are as in Proposition 1.3.1:

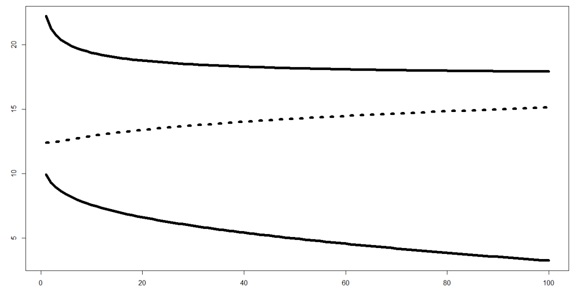
	r	σ	c	d	λ	γ	a	b	s	q	$\bar{x}_1$	x_1^*	$\bar{x}_2$
Scenario A	0.01	5	500	100	20	40	0	0	1	5	-31.11	16.95	34.84
Scenario B	0.01	1.5	50	150	10	15	2	8	10	10	4.95	14.26	18.18.



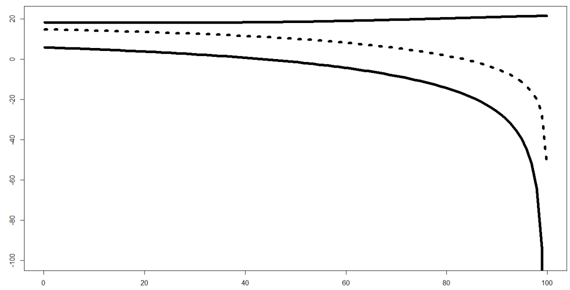
(i) $x \mapsto V_1(x)$ in red, $x \mapsto V_2(x)$ in blue for Scenario A



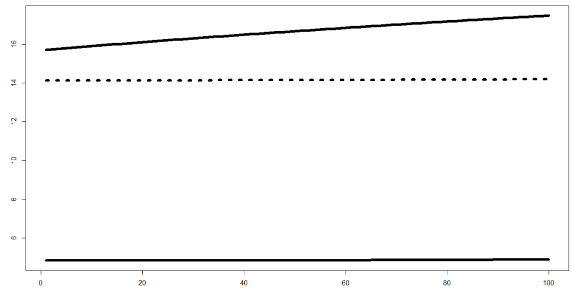
(ii) $x \mapsto V_1(x)$ in red, $x \mapsto V_2(x)$ in blue for Scenario B



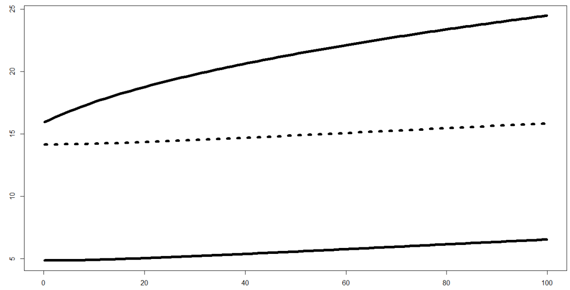
(iii) $c \mapsto \bar{x}_1, x_1^*, \bar{x}_2$ for Scenario B



(iv) $\lambda \mapsto \bar{x}_1, x_1^*, \bar{x}_2$ for Scenario B



(v) $d \mapsto \bar{x}_1, x_1^*, \bar{x}_2$ for Scenario B



(vi) $\gamma \mapsto \bar{x}_1, x_1^*, \bar{x}_2$ for Scenario B

Figure 1: Type I Equilibria

Figures 1(i)-1(ii) show how the equilibrium payoff functions behave in the selected scenarios, with the dashed lines showing the smooth-pasting of the three components of the payoff in (1.22) and (1.23). From Figure 1(i) to Figure 1(ii) we can see how a reduction in the volatility seems to shrink the continuation region, hence, the players become more cautious, reducing their intervention regions when there is more uncertainty. Another interesting fact to note is how the relative distance between $\bar{x}_1$ and $\bar{x}_2$ becomes smaller. This can be due to the increase in P2's terminal payoff sensitivity, b, and the increase in

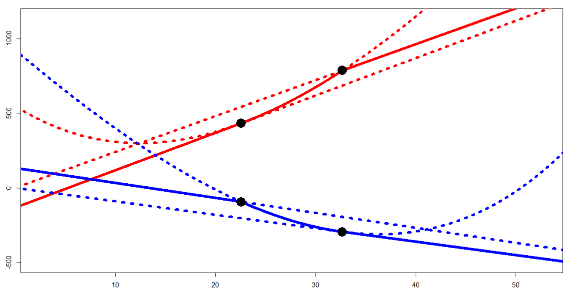
P1's exogenous cost, s. In one direction, P2 is losing more money when she decides to terminate the game, therefore she will not stop when the state process value is too high, hence she reduces her threshold $\bar{x}_2$. In the other, since P1 is facing higher exogenous costs, she pushes the target, x_1^*, as far as she can, making sure the state process is not going too low, rising the barrier $\bar{x}_1$.

Figures 1(iii)-1(iv)-1(v)-1(vi) represent some comparative statics of the thresholds $\bar{x}_1$, x_1^* and $\bar{x}_2$ for Scenario B. Similar graphs hold for Scenario A as well, therefore they are omitted. First, in Figure 1(iii) we can observe how an increase in P1's fixed cost expands the gap between $\bar{x}_1$ and x_1^*. The more P1 has to pay at any intervention time, the less often she will intervene, lowering the threshold, $\bar{x}_1$, and increasing the target, x_1^*. This allows P2, who does not like high values of x, to slightly lower her threshold, $\bar{x}_2$, so as to pay less when she will stop the game. In Figure 1(iv) the behaviour with respect to the proportional cost is quite different. P1 will reduce the interventions for higher λ, with the distance between $\bar{x}_1$ and x_1^* left nearly unchanged, while P2 keeps the barrier at a constant level $\bar{x}_2$. In particular, P1 tends to never intervening when the proportional cost reaches its maximum, set by the condition $1 - \lambda r > 0$. This behaviour shows how P1 is quite indifferent to changes in the proportional cost when this is not too big while she is really sensitive once it gets high. Finally, in Figures 1(v)-1(vi) we can see that, when P2's gains more each time P1 intervenes increases, P2 is happy playing for longer, heightening the threshold $\bar{x}_2$, since she is receiving more money.

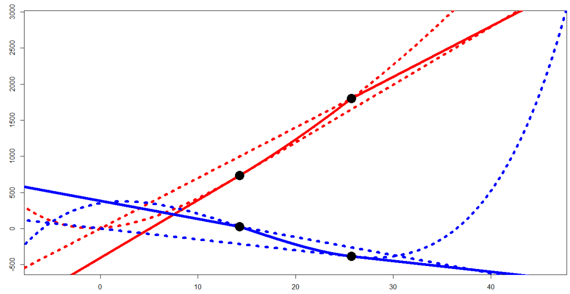
Equilibrium 2: P1 induces P2 to stop. To satisfy (1.44)-(1.45), we want $\widehat{w}$ to be neither too high nor too low, in particular, high λ should help in (1.44) as high $\widehat{w}$ in (1.45). As before, via graphical analysis it is possible to show that $\widehat{w}$, solution to (1.43), is decreasing in a, b, s and increasing in c, d, q, λ and γ. Therefore, the first instance of Nash equilibrium, Scenario B, has been selected to have high λ and $\widehat{w}$, choosing high values of c, d, q and γ and low values of b and s, whereas for Scenario A we have looked for lower values of λ and adapted the others. The table below shows the selected parameter settings, with $\bar{x}_1$ and $\bar{x}_2$ are as in Proposition 1.3.2:

	r	σ	c	d	λ	γ	a	b	s	q	$\bar{x}_1$	$\bar{x}_2$
Scenario A	0.01	5	100	100	25	10	24	9	45	0	22.56	32.68
Scenario B	0.01	1.5	150	125	80	25	70	15	10	15	14.27	25.72.

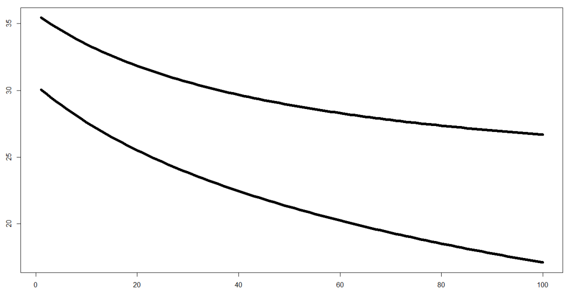
As before, Figure 2(i)-2(ii) represent the equilibrium payoff functions in the selected examples. First, we can observe that the continuation region in Scenario A is shifted to the right with respect to the one in Scenario B and we can observe that its width has not changed much from one case to the other. Furthermore, we can notice that Scenario B is more profitable for P2 and less profitable for P1. These two facts might be explained by the following changes from Scenario B to Scenario A: P1's exogenous cost, s, increases, so P1 cannot tolerate low levels of x, increasing her threshold $\bar{x}_1$. Moreover, although P2's



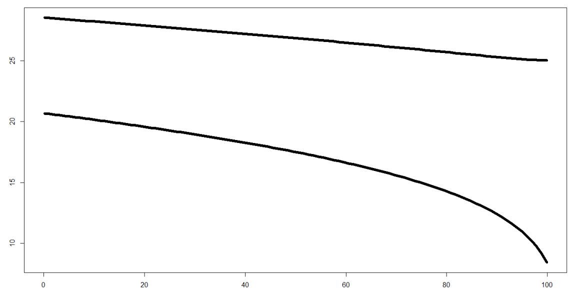
(i) $x \mapsto V_1(x)$ in red, $x \mapsto V_2(x)$ in blue for Scenario A



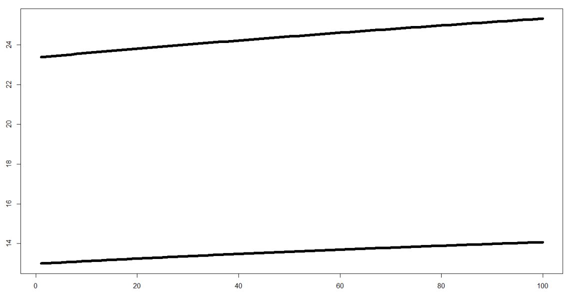
(ii) $x \mapsto V_1(x)$ in red, $x \mapsto V_2(x)$ in blue for Scenario B



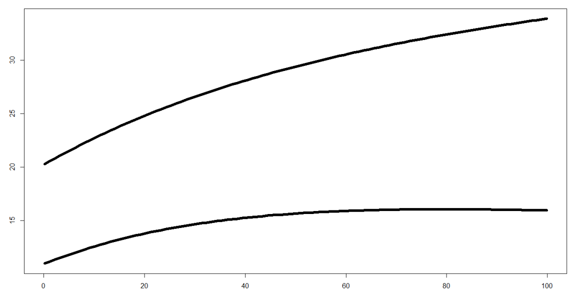
(iii) $c \mapsto \bar{x}_1, \bar{x}_2$ for Scenario B



(iv) $\lambda \mapsto \bar{x}_1, \bar{x}_2$ for Scenario B



(v) $d \mapsto \bar{x}_1, \bar{x}_2$ for Scenario B



(vi) $\gamma \mapsto \bar{x}_1, \bar{x}_2$ for Scenario B

Figure 2: Type II Equilibria

gains, q, d and γ, decrease we do not see her threshold scale down as it would be expected as the game is now less profitable. This is probably due to b's reduction, which leads P2 to stop for higher values of $\bar{x}_2$ since she is going to lose less when she decides to stop.

Now, let us spend some words on the comparative statics in Figures 2(iii)-2(iv)-2(v)-2(vi). When P1's costs, c and λ, increases, Figure 2(iii)-2(iv), P1 would intervene for lower values of x and the distance $\bar{x}_2 - \bar{x}_1$ will increase, even though $\bar{x}_2$ gets lower as well. This can be explained as follows, with the costs increasing, P1 is less willing to intervene, reducing $\bar{x}_1$, even though this shift allows P2 to lower her threshold, $\bar{x}_2$, since she likes low values of x. When the fixed gain, d, rises, Figure 2(v), P2 can afford the game to run for longer, increasing $\bar{x}_2$, as she will gain more when P1 will make her stop.

Moreover, this makes P1 heighten $\bar{x}_1$ in order to limit the proportional costs increment. Lastly, we have a similar behaviour to the one described above for the proportional gain, γ, Figure 2(vi). The main difference is the speed with which the distance between the thresholds increases, higher for proportional gain increments. This happens because, in case of proportional gain increments, P2 is more incentivised to push $\bar{x}_2$ far away since the bigger the impulse the more the revenue, whereas an analogous behaviour in case of fixed gain increments would lead to a loss in the terminal payoff outrunning the additional profit due to the fact that the gain, d, does not depend on the intensity of the impulse P1 is playing while the losses are increasing, since they depend on P2's threshold, $-b\bar{x}_2$.

Comparison between the two equilibria We conclude with a short discussion on the reasons why P1 would play aggressively, forcing P2 to stop. To do so we compare first the two scenarios A and B in both equilibria. So, going from Type I to Type II we see a reduction in the proportional gain, γ, an increase in P1 terminal payoff sensitivity, b, and a reduction in P2's exogenous gain, q, making P2 lower her threshold, $\bar{x}_2$, to reduce the losses at the end of the game. Then, P1's exogenous cost, s, increases making P1 rise both the threshold and the target, $\bar{x}_1$ and x_1^* respectively. Furthermore, P1 terminal payoff sensitivity, a, increases and, intuitively incentivise P1 to let P2 end the game sooner so to receive the terminal payoff. More specifically, since $\tilde{w}$ is decreasing in a, its increase makes $\ln \tilde{w} = \theta(\bar{x}_2 - \bar{x}_1)$ decrease, hence, since the distance between the two thresholds is now smaller, P1's target, x_1^*, is closer to P2's barrier up to the point they coincide, $x_1^* \equiv \bar{x}_2$.

Regarding Scenario B, again from Type I to Type II, we observe increments in the terminal payoff sensitivity of the two players, a and b, in particular P1's sensitivity rises much more than in the first scenario, hence, P1 is more incentivised to let P2 end the game. Another important change regards the proportional cost, λ, which is very high in case P1 induces P2 to stop. As we have seen before in the comparative statics in Figure 1(iv), P1 intervenes less and less when the proportional cost becomes higher and higher, so it is more convenient to intervene only once, inducing P2 to stop.

We finally observe that while we have managed to find numerical values for which only one of the two types of Nash equilibria emerges at a time, the problem of whether the two equilibria can coexist remains open.

1.4 Conclusions

In this chapter, we have introduced a general class of impulse controller vs stopper games whose state variable evolves according to a multi-dimensional Brownian motion driven diffusion. Moreover, we have provided a verification theorem giving sufficient conditions under which the solution of the suitable system of quasi-variational inequalities we implemented coincides with the two players' equilibrium payoff functions of the game. To show how the verification theorem and the system of quasi-variational inequalities are meant to be used, we have solved the game in a specific setting with linear payoffs and a one-dimensional scaled Brownian motion as a state variable, discovering the existence of two different types of equilibria which we have fully characterised. In particular, the one where player 1 forces player 2 to end the game could be considered as a limit case of the other equilibrium and further research in this direction might be interesting given that we did not prove if the two equilibria are alternative and we were not able to find any setting under which they could coexist.

1.A Appendix of Chapter 1

In this appendix, we have gathered some technical results used in the verification parts of Section 1.3 for both types of Nash equilibrium. We start with two lemmas on the continuation regions in the equilibrium where simultaneous actions are not allowed.

LEMMA 1.A.1 Let W_1 be as in (1.22). Then we have

$$\delta(x) = (x_1^* - x)\mathbf{1}_{(-\infty,\, x_1^*]}(x), \quad x \in \mathbb{R}.$$

Moreover

$$\{\mathcal{M}W_1 - W_1 < 0\} = (\bar{x}_1, +\infty) \quad \text{and} \quad \{\mathcal{M}W_1 - W_1 = 0\} = (-\infty, \bar{x}_1]. \tag{1.47}$$

PROOF 1.A.1 By a simple change of variable we obtain

$$\mathcal{M}W_1 = \max_{\delta \geq 0}\{W_1(x+\delta) - c - \lambda\delta\} = \max_{y \geq x}\{W_1(y) - c - \lambda(y-x)\}.$$

Let $\Gamma(y) := W_1(y) - \lambda y$. By definition of W_1 we have $\Gamma'(\bar{x}_1) = \Gamma'(x_1^*) = 0$. Moreover, the following properties are satisfied:

(i) $\Gamma'(x) = 0$ in $(-\infty, \bar{x}_1]$;

(ii) $\Gamma'(x) = a - \lambda < 0$ in $[\bar{x}_2, \infty)$;

(iii) $\Gamma'(x) > 0$ (resp. < 0) in $(\bar{x}_1, x_1^*)$ (resp. in $(x_1^*, \bar{x}_2)$).

Properties (i) and (ii) are easily checked. Regarding (iii), recall that

$$\Gamma'(x) = \varphi_1'(x) - \lambda = \theta C_{11}e^{\theta x} - \theta C_{12}e^{-\theta x} + \frac{1}{r} - \lambda, \quad x \in (\bar{x}_1, \bar{x}_2).$$

To study its sign, notice that $\Gamma''(x) = \theta^2 C_{11}e^{\theta x} + \theta^2 C_{12}e^{-\theta x} > 0$ for all $x \in (\bar{x}_1, \tilde{x})$,
where $\tilde{x}$ is such that $e^{\theta\tilde{x}} = \sqrt{-C_{12}/C_{11}} = e^{\frac{\theta}{2}(x_1^* + \bar{x}_1)}$. Moreover, since $\tilde{x} < x_1^*$ we have $\Gamma''(x_1^*) < 0$. Hence, it follows that $\Gamma'(x) > 0$ in $(\bar{x}_1, x_1^*)$, while $\Gamma'(x) < 0$ in $(x_1^*, \bar{x}_2)$.

As a consequence, Γ has a unique global maximum in x_1^*, so that

$$\max_{y \geq x}\Gamma(y) = \begin{cases} \Gamma(x_1^*) & \text{in } (-\infty, x_1^*] \\ \Gamma(x) & \text{in} (x_1^*, +\infty) \end{cases}$$

which gives

$$\text{argmax}_{\delta \geq 0}\{W_1(x+\delta) - c - \lambda\delta\} = \begin{cases} \{x_1^* - x\} & \text{in } (-\infty, x_1^*] \\ \{0\} & \text{in } (x_1^*, +\infty) \end{cases}$$

This implies the first part of our statement, i.e. $\delta(x) = (x_1^* - x)\mathbf{1}_{(-\infty, x_1^*]}(x)$. Now, to show (1.47), notice first that

$$\mathcal{M}W_1(x) = \begin{cases} W_1(x) - \zeta(x) & \text{in } (\bar{x}_1, \infty) \\ W_1(x) & \text{in } (-\infty, \bar{x}_1], \end{cases}$$

where we set

$$\zeta(x) := \begin{cases} \varphi_1(x) - \varphi_1(x_1^*) + c + \lambda(x_1^* - x) & \text{in } (\bar{x}_1,\, x_1^*] \\ c & \text{in } (x_1^*,\, +\infty) \end{cases}$$

Now, we prove that $\zeta > 0$. By C^0-pasting in $\bar{x}_1$ we have $\varphi_1(\bar{x}_1) = \varphi(x_1^*) - c - \lambda(x_1^* - \bar{x}_1)$, therefore

$$\zeta(x) = \varphi_1(x) - \varphi_1(\bar{x}_1) - \lambda(\bar{x}_1 - x) = \Gamma(x) - \Gamma(\bar{x}_1), \quad x \in (\bar{x}_1, x_1^*],$$

which is strictly positive since Γ is increasing in $(\bar{x}_1,\, x_1^*]$. Hence, ζ is strictly positive and we have

$$\{\mathcal{M}W_1 - W_1 < 0\} = (\bar{x}_1,\, +\infty), \quad \{\mathcal{M}W_1 - W_1 = 0\} = (-\infty,\, \bar{x}_1]. \qquad \square$$

Lemma 1.A.2 Let W_2 be as in (1.23). Assume there exists a solution $(\tilde{z},\, \tilde{w})$ to (1.29)-(1.32) such that $1 < \tilde{z} < \tilde{w}$ and

$$0 \le \frac{(1-br)(1-\lambda r)(\tilde{w}^2 - \tilde{z})}{\theta \tilde{w}(1-ar)(\tilde{z}+1)} + \frac{1-br}{1-ar} s - q < \frac{1-br}{\theta},$$
$$\left(\frac{1-br}{1-ar}\left(\frac{(1-\lambda r)(\tilde{w}^2-\tilde{z})}{\theta\tilde{w}(\tilde{z}+1)} + s\right) - q\right)(\tilde{w}-1)^2 + \frac{1-br}{\theta}(1 + 2\tilde{w}\ln\tilde{w} - \tilde{w}^2) > 0.$$

Then, we have

$$\{x \in \mathbb{R} : -bx - W_2(x) < 0\} = (-\infty,\, \bar{x}_2),$$
$$\{x \in \mathbb{R} : -bx - W_2(x) = 0\} = [\bar{x}_2,\, +\infty).$$

Proof 1.A.2 First, we recall that

$$W_2(x) = \begin{cases} -bx & \text{in } [\bar{x}_2,\, +\infty) \\ \varphi_2(x) & \text{in } (\bar{x}_1,\, \bar{x}_2) \\ \mathcal{H}W_2(x) & \text{in } (-\infty,\, \bar{x}_1] \end{cases}$$

where

$$\varphi_2(x) = C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q-x}{r}.$$

We want to prove that $\varphi_2(x) > -bx$ in $(\bar{x}_1,\, \bar{x}_2)$ and $\mathcal{H}W_2(x) > -bx$ in $(-\infty,\, \bar{x}_1]$. For the first inequality we are interested in the conditions such that, for all $x \in (\bar{x}_1,\, \bar{x}_2)$, we have

$$C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q-(1-br)x}{r} > 0, \tag{1.48}$$

or, equivalently,

$$e^{\theta(x-\bar{x}_2)}\left[(1-br)\left(\frac{1}{\theta} + \bar{x}_2\right) - q\right] + e^{\theta(\bar{x}_2 - x)}\left[(1-br)\left(\bar{x}_2 - \frac{1}{\theta}\right) - q\right] + 2\left(q - (1-br)x\right) > 0.$$

Now, applying the change of variable $e^{\theta(\bar{x}_2 - x)} = z > 1$ to the inequality above yields

$$(1-br)\left(\bar{x}_2 + \frac{1}{\theta}\right) - q + z^2\left[(1-br)\left(\bar{x}_2 - \frac{1}{\theta}\right) - q\right] + 2z\left(q - (1-br)\bar{x}_2 + \frac{1-br}{\theta}\ln z\right) > 0.$$

Since $\ln z > 0$ and $1 - br > 0$ by assumption, the left-side above is bigger than

$$(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q+z^2\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]+2z\left(q-(1-br)\bar{x}_2\right),$$

which is quadratic in z and it can be factorised as

$$(z-1)\left(z-\frac{(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q}{(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q}\right).$$

We show that our assumptions grant that the expression above is positive, which in turn will imply (1.48). Hence, the second factor is positive if the following holds:

$$(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q<0, \quad (1-br)\bar{x}_2-q\geq 0.$$

Then, using (1.31), the two inequalities above can be rewritten as

$$0\leq\frac{(1-br)(1-\lambda r)(\tilde{w}^2-\tilde{z})}{\theta\tilde{w}(1-ar)(\tilde{z}+1)}+\frac{1-br}{1-ar}s-q<\frac{1-br}{\theta},$$

which is true by assumption.

For showing the second inequality, i.e. $\mathcal{H}W_2(x) > -bx$ in $(-\infty, \bar{x}_1]$, we observe first that

$$\varphi_2(x_1^*)+d+\gamma(x_1^*-x)>-bx, \quad x\in(-\infty,\bar{x}_1]. \tag{1.49}$$

From the C^0-pasting condition in $\bar{x}_1$ we have that $\varphi_2(\bar{x}_1) = \varphi_2(x_1^*) + d + \gamma(x_1^* - \bar{x}_1)$, therefore we can rewrite (1.49) as

$$\varphi_2(\bar{x}_1)+\gamma(\bar{x}_1-x)>-bx.$$

Since $b < \gamma$ we only need to check that $F(\bar{x}_1) > 0$:

$$\begin{aligned}F(\bar{x}_1)=\varphi_2(\bar{x}_1)+b\bar{x}_1&=C_{21}e^{\theta\bar{x}_1}+C_{22}e^{-\theta\bar{x}_1}+\frac{q-(1-br)\bar{x}_1}{r},\\&=e^{-\theta(\bar{x}_2-\bar{x}_1)}\left[(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q\right]+e^{\theta(\bar{x}_2-\bar{x}_1)}\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]\\&\quad+2(q-(1-br)\bar{x}_1).\end{aligned}$$

Now, using again the change of variable $w = e^{\theta(\bar{x}_2-\bar{x}_1)}$, we have $\bar{x}_1 = \bar{x}_2 - \frac{\ln w}{\theta}$ and so $F(\bar{x}_1)e^{\theta(\bar{x}_2-\bar{x}_1)}$ can be re-expressed as

$$((1-br)\bar{x}_2-q)(\tilde{w}-1)^2+\frac{1-br}{\theta}(1+2\tilde{w}\ln\tilde{w}-\tilde{w}^2),$$

which, using (1.31), can be rewritten as

$$\left(\frac{1-br}{1-ar}\left(\frac{(1-\lambda r)(\tilde{w}^2-\tilde{z})}{\theta\tilde{w}(\tilde{z}+1)}+s\right)-q\right)(\tilde{w}-1)^2+\frac{1-br}{\theta}(1+2\tilde{w}\ln\tilde{w}-\tilde{w}^2),$$

which is positive by assumption. □

We conclude the appendix with two more lemmas on similar results for the other kind of equilibrium, where P1 forces P2 to stop the game.

LEMMA 1.A.3 Let W_1 be as in (1.22). Assume there exists a solution $\widehat{w}$ to (1.43) such that

$$(1-\lambda r)(w - w\ln w - 1) + cr\theta w > 0.$$

Then we have

$$\delta(x) = (\bar{x}_2 - x)\mathbf{1}_{(-\infty,\,\bar{x}_2]}(x), \quad x \in \mathbb{R}.$$

Moreover, we have

$$\{\mathcal{M}W_1 - W_1 < 0\} = (\bar{x}_1,\, +\infty), \quad \{\mathcal{M}W_1 - W_1 = 0\} = (-\infty,\, \bar{x}_1].$$

PROOF 1.A.3 First, observe that

$$\mathcal{M}W_1 = \max_{\delta \geq 0}\{W_1(x+\delta) - c - \lambda\delta\} = \max_{y \geq x}\{W_1(y) - c - \lambda(y-x)\}.$$

Let us denote $\Gamma(y) := W_1(y) - \lambda y$. By definition of W_1 we have $\Gamma'(\bar{x}_1) = 0$. Moreover, the following properties hold true:

(i) $\Gamma'(x) = 0$ in $(-\infty,\, \bar{x}_1]$;

(ii) $\Gamma'(x) = a - \lambda < 0$ in $[\bar{x}_2,\, +\infty)$;

(iii) $\Gamma'(x) > 0$ in $(\bar{x}_1, \bar{x}_2)$.

As properties (i) and (ii) can be easily checked, we turn to showing (iii). Observe that, for all $x \in (\bar{x}_1, \bar{x}_2)$, one has $\Gamma'(x) = \varphi_1'(x) - \lambda = \theta C_{11}e^{\theta x} - \theta C_{12}e^{-\theta x} + \frac{1}{r} - \lambda$, hence

$$\begin{aligned}\Gamma'(x) =& \frac{\theta}{2}e^{\theta(x-\bar{x}_1)}\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 + \frac{1}{\theta}\right)\frac{1-\lambda r}{r} - c\frac{s}{r}\right] \\ &- \frac{\theta}{2}e^{-\theta(x-\bar{x}_1)}\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 - \frac{1}{\theta}\right)\frac{1-\lambda r}{r} - c + \frac{s}{r}\right] + \frac{1-\lambda r}{r}.\end{aligned}$$

Using the fact that $\bar{x}_1 = \bar{x}_2 - \frac{\ln\widehat{w}}{\theta}$ and setting $z = e^{\theta(x-\bar{x}_1)}$ we can rewrite it as

$$\left(-(1-ar)\bar{x}_2 + \frac{1-\lambda r}{\theta}\ln\widehat{w} - cr + s\right)(z^2-1) - \frac{1-\lambda r}{\theta}(z-1)^2 > 0,$$

which can be factorised as

$$(z-1)\left(z + \frac{\frac{1-\lambda r}{\theta}(\ln\widehat{w}+1) - (1-ar)\bar{x}_2 - cr + s}{\frac{1-\lambda r}{\theta}(\ln\widehat{w}-1) - (1-ar)\bar{x}_2 - cr + s}\right) > 0,$$

which is true whenever $\frac{1-\lambda r}{\theta}(\ln\widehat{w} - 1) - (1-ar)\bar{x}_2 - cr + s > 0$. Therefore, recalling (1.41), after some algebraic manipulation, we obtain the equivalent condition

$$(1-\lambda r)(\widehat{w} - \widehat{w}\ln\widehat{w} - 1) + cr\theta\widehat{w} > 0.$$

Hence property (iii) is fulfilled. As a consequence, Γ has a unique global maximum point in $\bar{x}_2$, and the rest of the proof follows the same lines as for Lemma 1.A.1. Hence, the details are omitted. □

LEMMA 1.A.4 Let W_2 be as in (1.23). For every $x \in \mathbb{R}$, assume there exists a solution $\widehat{w}$ to (1.43) such that:

$$0 \leq (1-br)(\widehat{w}^2-1) + 2(\theta rd - (1-\gamma r)\ln \widehat{w})\widehat{w} < (1-br)(\widehat{w}-1)^2.$$

Then, we have

$$\{x \in \mathbb{R} : W_2(x) > -bx\} = (-\infty,\, \bar{x}_2), \quad \{x \in \mathbb{R} : W_2(x) = -bx\} = [\bar{x}_2,\, +\infty).$$

PROOF 1.A.4 First, recall that

$$W_2(x) = \begin{cases} -bx & \text{in } [\bar{x}_2,\, +\infty) \\ \varphi_2(x) & \text{in } (\bar{x}_1,\, \bar{x}_2) \\ \mathcal{H}W_2(x) & \text{in } (-\infty,\, \bar{x}_1] \end{cases}$$

where

$$\varphi_2(x) = C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q-x}{r}.$$

Hence, we want to prove that $\varphi_2(x) > -bx$ in $(\bar{x}_1,\, \bar{x}_2)$ and $\mathcal{H}W_2(x) > -bx$ in $(-\infty,\, \bar{x}_1]$. For the first inequality we are interested in the conditions granting

$$C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q-(1-br)x}{r} > 0, \quad x \in (\bar{x}_1,\, \bar{x}_2),$$

or equivalently

$$e^{-\theta(\bar{x}_2-x)}\left[(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q\right] + e^{\theta(\bar{x}_2-x)}\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right] + 2(q-(1-br)x) > 0.$$

Letting $z = e^{\theta(\bar{x}_2-x)}$, the above inequality holds whenever

$$(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q+\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]z^2+2\left(q-(1-br)\left(\bar{x}_2-\frac{\ln z}{\theta}\right)\right)z > 0,$$

Since $\ln z > 0$ and $1-br > 0$ by assumption, the left-side above is bigger than

$$(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q+z^2\left[(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q\right]+2z\left(q-(1-br)\bar{x}_2\right),$$

which can be factorised as in the proof of Lemma 1.A.2 in

$$(z-1)\left(z-\frac{(1-br)\left(\bar{x}_2+\frac{1}{\theta}\right)-q}{(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q}\right).$$

We show that our assumptions grant that the expression above is positive. We proceed as in the proof of Lemma 1.A.2: the second factor above is positive if the following holds

$$(1-br)\left(\bar{x}_2-\frac{1}{\theta}\right)-q < 0, \quad (1-br)\bar{x}_2 - q \geq 0,$$

which, using (1.42), can be rewritten as

$$0 \leq (1-br)(\widehat{w}^2-1) + 2(\theta rd - (1-\gamma r)\ln \widehat{w})\widehat{w} < (1-br)(\widehat{w}-1)^2,$$

which is true by assumption.

For the second inequality we have

$$-b\bar{x}_2 + d + \gamma(\bar{x}_2 - x) > -bx, \quad x \in (-\infty, \bar{x}_1].$$

Since $\gamma > b$, the inequality holds whenever $(\gamma - b)(\bar{x}_2 - \bar{x}_1) + d > 0$, which is always true since $\bar{x}_2 > \bar{x}_1$ by the ordering condition. □

Chapter 2

Zero-sum Stochastic Differential Games with Impulse Controls: a Stochastic Perron's Method Approach

2.1 Introduction

Differential games have been widely studied since Isaacs' work [50] in 1965. In particular, we are interested in the branch of continuous time games with impulse controls, where players can act on the system only at discrete times, introduced by Bensoussan and Lions [21] in 1974. In the deterministic case, Yong [78] studied a zero-sum game involving impulse, continuous and switching controls proving the existence of the value of the game by mean of viscosity solutions and Farouq et al [41] later allowed for more general jumps, motivated by an application in mathematical finance. The first result combining the theory of viscosity solutions with stochastic differential games in which both players adopt impulse controls is by Cosso [38] in 2013, where he showed via the Dynamic Programming Principle that finite time zero-sum games admit a value. Thereafter, Mazid [63] and Zhang L. [80] generalised his work, weakening his assumption the first, and using a BSDE approach the second. Basu and Stettner [10] studied the zero-sum game when the state dynamics is a weak Feller-Markov process introducing the concept of shifted strategies, which allows them to restrict the game to a sequence of Dynkin games, to provide existence and uniqueness of a saddle point. Further research has been done in the zero-sum case when only one agent plays impulse controls while the other is playing classic continuous controls, see Azimzadeh [6] and Zhang F. [79]. It is important to point out that all these works, as most of the literature on stochastic differential games, rely on the Elliot and Kalton [42] formulation, according to which the upper and lower value functions are defined as the payoff when one player plays an open-loop control while the other is playing a best response strategy. This formulation is clearly asymmetric as it produces two value functions whose comparison is debatable. We believe a symmetric formulation where the upper and the lower value functions compare by definition is more natural and we refer to Sîrbu [74] for a thorough discussion. To achieve this goal, we model the strategical interaction between players using feedback strategies following the works of [48, 74]. In this way, players see how the other is acting, as they observe the path of the state up to the current time, and are able to respond accordingly, without the need for asymmetric formulations. Moreover, the viscosity solution approach with strategies à la Elliot and Kalton, which was first studied by Fleming and Souganidis [44] in their pioneering work in 1989, is in general quite complicated and the Dynamic Programming Principle (DPP) cannot be proven working directly with the value functions. Instead here, once we have proven that the stochastic Perron's Method can be adapted to differential games with impulse controls, we obtain the DPP as a by-product. The stochastic Perron's method was first introduced by Bayraktar and Sirbu [15] to construct viscosity solutions of linear parabolic equations associated with stochastic differential equations in a highly tractable way, and later applied to Dynkin games [16] and stochastic differential games [74].

Impulse strategies are well suited for all kind of situations in which fixed and proportional costs apply any time players control the state process. Indeed, the treatment with singular or classic controls would be faulty since the first would only capture proportional costs, whereas the second would miss discrete interventions. Furthermore, both would not be feasible since they would result in infinite costs due to players moving infinitely many times the state process, as they are controlling it continuously, paying each time some strictly positive fixed cost. In such cases, the controller would rather choose a sequence of intervention times at which he will induce a jump in the state dynamics. Such sequence of intervention times and jump sizes is called impulse control, as those jump sizes are commonly called impulses. This kind of controls have many applications, from finance to energy markets to real options [8, 35, 43, 60] and in particular, they have experienced a comeback in the latest years due to the research for more realistic financial models, from option pricing [22, 40, 75], to optimal portfolio selection [61, 68, 70], to options for long term insurance contracts [36], to control of exchange rates [24, 29] and finally, to order execution [27, 37]. Given the high interest, Aïd et al. [1] provided a general model for nonzero-sum stochastic differential games with impulse controls which was later generalised to the mean-field case by Basei et al [9].

This chapter is about finite time horizon zero-sum stochastic differential games with impulse controls and how to find their Nash equilibrium conveniently. More precisely, we study games in which the two players can act only at discrete times inducing jumps in a continuous time stochastic process which will be denoted by X, whose controlled dynamics is given by:

$$
\begin{aligned}
X_t^{s,x;u,v} =& x + \int_s^t b(t, X_r^{s,x;u,v})dr + \int_s^t \sigma(t, X_r^{s,x;u,v})dW_r + \\
& \sum_{n: s \le \tau_n(X_r^{s,x;u,v}) \le t} \delta_n(X_r^{s,x;u,v}) + \sum_{n: s \le \eta_n(X_r^{s,x;u,v}) \le t} \gamma_n(X_r^{s,x;u,v}), \qquad t \ge s.
\end{aligned}
$$

This game is zero-sum, which means that the net change in wealth is zero and one player's gain is equal to the other's loss; in the following we will refer to Player 1 (P1) as the maximiser and to Player 2 (P2) as the minimiser. Any time one of the two players applies an impulse to the system they face a cost, and, being the game zero-sum, it means they are paying a penalty to the other whenever they like to intervene. Then, the two players are going to respectively maximise and minimise the objective function which features a running and a terminal payoff together with the aforementioned interventions costs:

$$
\begin{aligned}
J(s, x; u, v) = \mathbb{E}\Bigg[& \int_s^T f(t, X_t^{s,x;u,v})dt - \sum_{n: \tau_n \le T} \phi(X_{\tau_n -}^{s,x;u,v}, \delta_n) \\
& + \sum_{n: \eta_n \le T} \psi(X_{\eta_n -}^{s,x;u,v}, \gamma_n) + g(X_T^{s,x;u,v}) \Bigg].
\end{aligned}
$$

In particular, we define the upper value of the game as the function associated with the players' optimisation problem when P1 has priority of intervention over P2, meaning that, if both want to apply an impulse at the same time, only P1's will work. Symmetrically, we define the lower value of the game when P2 has priority over P1. The upper and lower value of the game are related to two different double-obstacle quasi-variational inequalities, resulting from the Hamilton-Jacobi-Bellman-Isaacs (HJBI) Partial Differential Equation (PDE), which we will solve via viscosity solutions using the stochastic Perron's method.

The chapter is organised as follows. In Section 2 we will formally describe the game, in particular the definition of strategies according to which players intervene at stopping rules instead of stopping times, and apply the related impulses based solely on the information available up to then. Section 3 will adapt the stochastic Perron's method to such game and provides, in particular, the appropriate definitions of stochastic sub and super-solutions to construct the viscosity solution of the HJBI. Finally, in Section 4 we will prove a comparison result, so that the value function constructed via Perron's method exists and is unique.

2.2 Game Setting

Here we specify the framework of our zero-sum stochastic differential game with impulse controls, where two players can affect via impulses some given state variable evolving according to a possibly time inhomogeneous SDE as follows.

Uncontrolled state dynamics. Fix a finite time horizon $T > 0$ and some initial time $s \in [0, T]$, we are given a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting some k-dimensional Brownian motion W. Let $\mathbb{F}^s = (\mathcal{F}_t^s)_{t\in[s,T]}$ be the augmented filtration generated by W's increments starting at s and assume that the state process X takes values in $\mathbb{R}^d$ and satisfies

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_s = x, \tag{2.1}$$

for some initial value $x \in \mathbb{R}^d$, where the coefficients $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are assumed to be jointly continuous and locally Lipschitz continuous in x, uniformly in time, i.e. for all $K > 0$, there exists a constant $L_K > 0$ such that

$$|b(t, x_1) - b(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq L_K|x_1 - x_2|, \tag{2.2}$$

whenever $|x_1|, |x_2| \leq K$ and for all $t \in [0, T]$. Moreover, we assume they have at most linear growth in x, uniformly in time, i.e. there exists a constant $C > 0$ such that, for all $x \in \mathbb{R}^d$ and $t \in [0, T]$, we have

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|),$$

so that existence of a unique strong solution is granted and X is well-defined. Note that under the assumptions above X is well-defined even if the game starts at some stopping time greater than s.

Players' strategies and controlled dynamics. The goal of both players is to maximise their payoffs via the application of optimal impulses at some strategically chosen times. Before defining players' strategies we need to introduce, for a fixed starting time s, the Skorohod space $\mathcal{D}([s, T]) := \mathcal{D}([s, T], \mathbb{R}^d)$ and endow this path space with the filtration

$$\mathbb{B}^s := (\mathscr{B}_t^s)_{t\in[s,T]} \text{ defined by } \mathscr{B}_t^s := \mathscr{B}(\mathcal{D}([s, t])), \quad t \in [s, T],$$

where $\mathscr{B}_t^s := \mathscr{B}(\mathcal{D}([s, t]))$ is the Borel σ-algebra generated by the open sets in $\mathcal{D}([s, T])$. Elements of $\mathcal{D}([s, T])$ will be denoted by $y(\cdot)$ or y where there is no ambiguity. Stopping times on $\mathcal{D}([s, t])$ with respect to $\mathbb{B}^s$ are called stopping rules, as in Karatzas and Sudderth [55], i.e. a stopping rule is any mapping $\tau : \mathcal{D}([s, T]) \to [s, T] \cup \{+\infty\}$ such that

$$\{y \in \mathcal{D}([s, T]) : \tau(y) \leq t\} \in \mathscr{B}_t^s, \quad t \in [s, T].$$

Moreover, we denote by $\mathcal{T}_\rho$ the space of stopping rules greater than ρ, where ρ is itself a stopping rule. The last ingredient needed for the definition of players' strategies are the sets of values for the impulses.

Assumption 2.2.1 Let Δ and Γ be two given compact subsets of the state space $\mathbb{R}^d$.

We define the players' strategies as follows:

Definition 2.2.1 (Players' strategies) Let $s \in [0,T]$ and let $\rho \in \mathcal{T}_s$ be a given stopping rule. Then, a strategy for Player 1 (henceforth, P1) starting at ρ is any sequence $u = (\tau_n, \delta_n)_{n\geq 1}$, where

- $(\tau_n)_{n\geq 1} \subset \mathcal{T}_\rho$ is a strictly increasing sequence of stopping rules, i.e. $\rho \leq \tau_1 < \tau_2 < \cdots$, such that $\lim_{n\to+\infty} \tau_n = +\infty$ a.s.;
- $(\delta_n)_{n\geq 1}$ is a sequence of maps $\delta_n : \mathcal{D}([s,T]) \to \Delta$ such that $\delta_n \in L^0(\mathscr{B}_{\tau_n})$ for all $n \geq 1$.

Analogously, a strategy for Player 2 (P2) starting at ρ, is any sequence $v = (\eta_n, \gamma_n)_{n\geq 1}$, where

- $(\eta_n)_{n\geq 1} \subset \mathcal{T}_\rho$ is a strictly increasing sequence of stopping rules, i.e. $\rho \leq \eta_1 < \eta_2 < \cdots$, such that $\lim_{n\to+\infty} \eta_n = +\infty$ a.s.;
- $(\gamma_n)_{n\geq 1}$ is a sequence of maps $\gamma_n : \mathcal{D}([s,T]) \to \Gamma$ with $\gamma_n \in L^0(\mathscr{B}_{\eta_n})$ for all $n \geq 1$.

The set of all P1 (resp. P2) strategies starting at $\rho \in \mathcal{T}_s$, is denoted U^s_ρ (resp. V^s_ρ).

Remarks 2.2.1 The definition of strategies starting at some possibly later time will be very convenient in defining suitable notions of stochastic sub/super-solutions in order to extend the stochastic Perron's method to zero-sum impulse games. Moreover, note that when $\rho = s$ we have the usual notion of strategies starting at the initial time.

We do not allow for simultaneous impulses, hence, when P1 and P2 are playing some strategy u and v respectively, the controlled state variable evolves with either of the following dynamics, depending on which player has priority over the other:

$$
\begin{aligned}
X_t^{s,x;u,v,-} &= x + \int_s^t b(r, X_r^{s,x;u,v,-})dr + \int_s^t \sigma(r, X_r^{s,x;u,v,-})dW_r + \sum_{n: s\leq \eta_n(X^{s,x;u,v,-})\leq t} \gamma_n(X^{s,x;u,v,-}) \\
&\quad + \sum_{n: s\leq \tau_n(X^{s,x;u,v,-})\leq t} \delta_n(X^{s,x;u,v,-}) \prod_{l\geq 1} \mathbb{1}_{\{\tau_n(X^{s,x;u,v,-})\neq \eta_l(X^{s,x;u,v,-})\}}, \\
X_t^{s,x;u,v,+} &= x + \int_s^t b(r, X_r^{s,x;u,v,+})dr + \int_s^t \sigma(r, X_r^{s,x;u,v,+})dW_r + \sum_{n: s\leq \tau_n(X^{s,x;u,v,+})\leq t} \delta_n(X^{s,x;u,v,+}) \\
&\quad + \sum_{n: s\leq \eta_n(X^{s,x;u,v,+})\leq t} \gamma_n(X^{s,x;u,v,+}) \prod_{l\geq 1} \mathbb{1}_{\{\eta_n(X^{s,x;u,v,+})\neq \tau_l(X^{s,x;u,v,+})\}},
\end{aligned}
$$

with $t \in [s,T]$. The process $X^{s,x;u,v,-}$ represents the state dynamics in case P2 has priority of intervention over P1, whereas $X^{s,x;u,v,+}$ is otherwise. For ease of notation, we will refer to $\delta_n(X^{s,x;u,v,\pm})$ and $\gamma_n(X^{s,x;u,v,\pm})$ as δ_n and γ_n only.

The following lemma is needed for $(X_t^{s,x;u,v,\pm})_{t\in[s,T]}$ to be well-defined. First, notice that the controlled process evolves as the uncontrolled one in between impulses and that these occur according to players' stopping rules. Furthermore, we know that the uncontrolled process is well defined for any stopping time greater than s, hence, we need a result that links $\mathscr{B}$-stopping rules to $\mathcal{F}$-stopping times to apply recursively the existence and uniqueness result for the uncontrolled dynamics and obtain that $(X_t^{s,x;u,v,\pm})_{t\in[s,T]}$ is also well-defined.

LEMMA 2.2.1 Let $s \in [0,T]$ and let τ be a stopping rule in $\mathcal{T}_s$. Let $(X_t)_{t\in[s,T]}$ be a process with *càdlàg* paths, which is progressively measurable with respect to $\mathbb{F}^s$. Then, the random time $\tau_X : \Omega \to [s,T] \cup \{+\infty\}$ defined by $\tau_X(\omega) := \tau(X(\omega))$ is a stopping time with respect to $\mathbb{F}^s$. In addition $X_{\tau_X} \mathbb{1}_{\tau_X<\infty}$ is $\mathcal{F}^s_{\tau_X}$-measurable.

PROOF 2.2.1 By assumption, X is a process with càdlàg paths, i.e. $X(\omega) \in \mathcal{D}([s,T])$ for all ω. Since τ is a stopping rule, we have

$$\{y \in \mathcal{D}([s,T]) : \tau(y) \le t\} \in \mathscr{B}^s_t, \quad t \in [s,T].$$

Then, taking the inverse image through X of the set above we obtain

$$X^{-1}\left(\{y \in \mathcal{D}([s,T]) : \tau(y) \le t\}\right) = \{(r,\omega) \in [0,T] \times \Omega : r = \tau(X(\omega)) \le t\} \in \mathcal{F}^s_t$$

since, by definition of progressively measurable process, we have that for all $t \in [s,T]$ the mapping $(r,\omega) \mapsto X_r(\omega)$ is measurable on $[s,t] \times \Omega$ equipped with the product σ-field $\mathcal{B}([s,T]) \otimes \mathcal{F}^s_t$. □

Players' payoffs. As we mentioned above, both players play impulses in order to maximise their payoffs, but since the game is zero-sum we have that P1 (the maximiser) is going to receive a certain payoff from P2 (the minimiser) depending on the strategies they will be playing during the game. Such payoffs can be defined as follows:

$$\begin{aligned} J^-(s,x;u,v) = \mathbb{E}\Bigg[\int_s^T f(t, X_t^{s,x;u,v,-})dt - \sum_{n: s\le \tau_n \le T} \phi(X_{\tau_n -}^{s,x;u,v,-}, \delta_n) \prod_{l\ge 1} \mathbb{1}_{\{\tau_n \ne \eta_l\}} \\ + \sum_{n: s\le \eta_n \le T} \psi(X_{\eta_n -}^{s,x;u,v,-}, \gamma_n) + g(X_T^{s,x;u,v,-})\Bigg], \\ J^+(s,x;u,v) = \mathbb{E}\Bigg[\int_s^T f(t, X_t^{s,x;u,v,+})dt - \sum_{n: s\le \tau_n \le T} \phi(X_{\tau_n -}^{s,x;u,v,+}, \delta_n) \\ + \sum_{n: s\le \eta_n \le T} \psi(X_{\eta_n -}^{s,x;u,v,+}, \gamma_n) \prod_{l\ge 1} \mathbb{1}_{\{\eta_n \ne \tau_l\}} + g(X_T^{s,x;u,v,+})\Bigg], \end{aligned}$$

for any initial condition $(s,x) \in [0,T] \times \mathbb{R}^d$ and any pair of strategies $(u,v) \in U^s_s \times V^s_s$. Again, $J^-(s,x;u,v)$ represents the payoff in case P2 has priority over P1, whereas $J^+(s,x;u,v)$ is otherwise. The following assumptions on gains and costs grant in particular that the payoff functionals $J^\pm$ are bounded:

ASSUMPTION 2.2.2 (i) The running gain $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and the final gain $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and bounded.

(ii) The costs $\phi : \mathbb{R}^d \times \Delta \to \mathbb{R}_+$ and $\psi : \mathbb{R}^d \times \Gamma \to \mathbb{R}_+$ are continuous and bounded away from zero, i.e. $\inf_{(t,\delta)\in[0,T]\times\Delta} \phi(t,\delta) > 0$ and $\inf_{(t,\gamma)\in[0,T]\times\Gamma} \psi(t,\gamma) > 0$.

Remarks 2.2.2 The assumptions above are admittedly not the most general. However, they are consistent with the related literature on impulse games with viscosity solutions and the stochastic Perron's method. Indeed, regarding the first, our assumptions are similar to Cosso's [38], our only restriction concerns the spaces of impulses, Δ and Γ, which are assumed compact rather then just closed subsets of $\mathbb{R}^d$. Regarding the literature on stochastic Perron method, in most of it no running cost/gain is considered, while the terminal payoff is taken continuous and bounded [15, 16, 74].

Our goal is to prove that the game has a value under both instances of priority and to provide some sufficient conditions to show when the type of priority has no effect, namely the game has the same value regardless the priority rule. Now we can define the upper and lower value of the game as

$$V^-(s,x) = \sup_{u\in U_s^s} \inf_{v\in V_s^s} J^-(s,x;u,v), \quad V^+(s,x) = \inf_{v\in V_s^s} \sup_{u\in U_s^s} J^+(s,x;u,v).$$

We would heuristically expect $V^- \leq V^+$ since P2 is the minimiser and we will say that the priority of intervention is not relevant if the two values are equal, i.e. $V^- = V^+ =: V$.

HJBI equations and the concatenation property. At this point it is convenient to define the players' respective intervention operators, which will appear in the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, as

$$\mathcal{M}V(t,x) = \sup_{\delta\in\Delta} \left[V(t,x+\delta) - \phi(x,\delta)\right], \quad \mathcal{H}V(t,x) = \inf_{\gamma\in\Gamma} \left[V(t,x+\gamma) + \psi(x,\gamma)\right],$$

for any bounded measurable function $V : [0,T] \times \mathbb{R}^d \to \mathbb{R}$. These two operators describe the value of the game right after P1's and P2's optimal interventions respectively. In the next proposition we show that semi-continuity is preserved by the action of both operators.

Proposition 2.2.1 If $V : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a upper (lower) semi-continuous function then $\mathcal{M}V, \mathcal{H}V : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are upper (lower) semi-continuous as well.

Proof 2.2.2 *Lower semi-continuity of $\mathcal{H}V$.* Let us take a sequence $(t_n, x_n)_{n\geq 1} \in [s,T] \times \mathbb{R}^d$ and $(t,x) \in [s,T] \times \mathbb{R}^d$ so that (t_n,x_n) converges to (t,x). Since Γ is compact, for each $n \geq 1$ there exists a $\gamma_n \in \Gamma$ such that $\mathcal{H}V(t_n,x_n) = V(t_n,x_n+\gamma_n) + \psi(x_n,\gamma_n)$. Moreover, the sequence of $(\gamma_n)_{n\geq 1}$ is bounded, so by taking a subsequence if necessary we can suppose that there exists a $\gamma \in \Gamma$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. Therefore, since ψ is continuous

$$\liminf_{n\to\infty} \mathcal{H}V(t_n,x_n) = \liminf_{n\to\infty} (V(t_n,x_n+\gamma_n) + \psi(x_n,\gamma_n)) \geq V(t,x+\gamma) + \psi(x,\gamma) \geq \mathcal{H}V(t,x)$$

where the first inequality is due to V being lower semi-continuous.

Upper semi-continuity of $\mathcal{H}V$. As before we take a sequence $(t_n, x_n)_{n\geq 1}$ converging to some (t,x) in $[s,T]\times\mathbb{R}^d$ and fix a $\gamma=\gamma(t,x)\in\Gamma$ such that $\mathcal{H}V(t,x)=V(t,x+\gamma)+\psi(x,\gamma)$ and $\mathcal{H}V(t_n,x_n)\leq V(t_n,x_n+\gamma)+\psi(x_n,\gamma)$ for all $n\geq 1$. Hence,

$$\limsup_{n\to\infty}\mathcal{H}V(t_n,x_n)\leq\limsup_{n\to\infty}\left(V(t_n,x_n+\gamma)+\psi(x_n,\gamma)\right)\leq V(t,x+\gamma)+\psi(x,\gamma),$$

which proves that $\mathcal{H}V$ is upper semi-continuous.

The semi-continuity properties of $\mathcal{M}V$ can be proved in the same way, hence the details are omitted. □

The upper value function, V^+, will naturally be associated with the following double obstacle problem, which we will refer to as Upper Isaacs (UI):

$$\begin{cases}\min\left\{\max\left\{-\mathcal{A}V-V_t-f,V-\mathcal{H}V\right\},V-\mathcal{M}V\right\}=0, & \text{on } [0,T)\times\mathbb{R}^d,\\ V(T,x)=g(x), & x\in\mathbb{R}^d,\end{cases}\tag{2.3}$$

with $\mathcal{A}V=b\cdot\nabla V+\frac{1}{2}\mathrm{tr}(\sigma\sigma^\top D^2V)$ whereas the lower value function, V^-, will be associated with the following, which we will refer to as Lower Isaacs (LI):

$$\begin{cases}\max\left\{\min\left\{-\mathcal{A}V-V_t-f,V-\mathcal{M}V\right\},V-\mathcal{H}V\right\}=0, & \text{on } [0,T)\times\mathbb{R}^d,\\ V(T,x)=g(x), & x\in\mathbb{R}^d.\end{cases}\tag{2.4}$$

We say that the Isaacs condition holds whenever we have

$$\begin{aligned}&\max\left\{\min\left\{-\mathcal{A}V-V_t-f,V-\mathcal{M}V\right\},V-\mathcal{H}V\right\}\\ &\qquad=\min\left\{\max\left\{-\mathcal{A}V-V_t-f,V-\mathcal{H}V\right\},V-\mathcal{M}V\right\}\end{aligned}\tag{2.5}$$

on $[0,T)\times\mathbb{R}^d$.

Remarks 2.2.3 The Isaacs condition implies that the two values are the same, which means that it does not really matter who has priority over the other as, in equilibrium, it does not allow the players to achieve a better payoff.

Earlier we have defined the strategies as starting from a stopping rule $\rho\in\mathcal{T}_s$, although the game is starting at some time $s\leq\rho$. Therefore, in order to be able to use those strategies we need a result that allows us to concatenate them to strategies starting from an earlier time. Even though the result is stated for P1's strategies, an analogous one clearly holds for P2's as well.

Proposition 2.2.2 (concatenation property) Let $s\in[0,T]$, $\rho\in\mathcal{T}_s$ and $\tilde{u}=(\tilde{\tau}_n,\tilde{\delta}_n)\in U_\rho^s$. Then, for each $u=(\tau_n,\delta_n)\in U_s^s$, the mapping $u\otimes_\rho\tilde{u}:\mathcal{D}([s,T])\to([s,T]\cup\{+\infty\})^{\mathbb{N}}\times\Delta^{\mathbb{N}}$ defined by*

$$u\otimes_\rho\tilde{u}:=((\tau_1,\ldots,\tau_{n^*-1},\tilde{\tau}_1,\tilde{\tau}_2,\ldots),(\delta_1,\ldots,\delta_{n^*-1},\tilde{\delta}_1,\tilde{\delta}_2,\ldots)),\tag{2.6}$$

where $n^*:=\inf\{n\geq 1:\tau_n\geq\rho\}$, is a strategy starting at s, i.e. $u\otimes_\rho\tilde{u}\in U_s^s$.

*To ease the notation, we omit the dependence on the path $y(\cdot)$.

PROOF 2.2.3 By construction, the strategy $(\widehat{\tau}_n, \widehat{\delta}_n)_{n\geq 1}$ defined in (2.6) is composed of a strictly increasing sequence of intervention times satisfying $\lim_{n\to+\infty} \widehat{\tau}_n = +\infty$. Moreover, by definition of u and $\tilde{u}$, it follows that $\delta_n \in L^0(\mathscr{B}_{\tau_n})$ whenever $\tau_n < \tau_{n^*}$, and $\tilde{\delta}_n \in L^0(\mathscr{B}_{\tilde{\tau}_n})$ for $\tilde{\tau}_n \geq \tilde{\tau}_1$. Therefore, we deduce that $u \otimes_\rho \tilde{u} \in U_s^s$. □

2.3 Stochastic Perron's method

We want to find the value of the game, under both priority rules, as viscosity solution of the corresponding HJBI systems, (2.3)-(2.4), via the stochastic Perron's method. As such, we are going to define a suitable class of stochastic sub/super-solutions in a way they satisfy their respective half dynamic programming principle (DPP). Once they are defined properly, the stochastic Perron's method consists in showing that the infimum of such stochastic super-solutions is a viscosity sub-solution of (2.3)-(2.4), whereas the supremum of such sub-solutions is a viscosity super-solution of (2.3) or (2.4). Finally, to show that the game has a value, namely that the infimum of stochastic super-solutions is equal to the supremum of stochastic sub-solutions, we will have to perform only a verification by comparison. This last step will be done in Section 2.4.

Definition of stochastic super/sub-solutions and their properties. In this part we state the definitions of stochastic super/sub-solutions for UI and LI equations, together with some preliminary elementary properties. We start with the notion of stochastic super-solution of UI equation.

DEFINITION 2.3.1 A function $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called a stochastic super-solution of the UI equation if:

1. it is bounded and continuous, it satisfies $w(T,\cdot) \geq g(\cdot)$ and

$$(\mathcal{H}w - w)(t,x) \geq 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^d;$$

2. for each $s \in [0,T]$ and for each stopping rule $\rho \in \mathcal{T}_s$, there exists a P2 strategy $\tilde{v} = (\tilde{\eta}_n, \tilde{\gamma}_n) \in V_\rho^s$ such that for any $u = (\tau_n, \delta_n) \in U_s^s$, $v = (\eta_n, \gamma_n) \in V_s^s$, $x \in \mathbb{R}^d$ and each stopping rule ζ with $\rho \leq \zeta \leq T$, we have $\mathbb{P}$-a.s.

$$\begin{aligned} w(\rho, X_\rho) \geq \mathbb{E}\Bigg[\int_\rho^\zeta f(t,X_t)dt - \sum_{n:\rho\leq\tau_n<\zeta} \phi(X_{\tau_n-},\delta_n) \\ + \sum_{n:\rho\leq\tilde{\eta}_n<\zeta} \psi(X_{\tilde{\eta}_n-},\tilde{\gamma}_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\eta}_n\neq\tau_l\}} + w(\zeta, X_\zeta) \mid \mathcal{F}_\rho^s\Bigg], \end{aligned} \tag{2.7}$$

where we have used the simplifying notation $X := X^{s,x;u,v\otimes_\rho\tilde{v},+}$, $\rho := \rho(X)$, $\zeta := \zeta(X)$, and similarly for δ_n and $\tilde{\gamma}_n$.

An immediate consequence of being a stochastic super-solution of the UI equation is that, choosing $\rho = s$, there exists $\tilde{v} \in V_s^s$ such that, using the simplifying notation $X := X^{s,x;u,\tilde{v},+}$, we have $\mathbb{P}$-a.s.

$$\begin{aligned} w(s,x) \geq & \mathbb{E}\left[\int_s^{\zeta} f(t,X_t)dt - \sum_{n:s\leq\tau_n<\zeta} \phi(X_{\tau_n-},\delta_n) \right. \\ & \left. + \sum_{n:s\leq\tilde{\eta}_n<\zeta} \psi(X_{\tilde{\eta}_n-},\tilde{\gamma}_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\eta}_n\neq\tau_l\}} + w\left(\zeta,X_\zeta\right) \mid \mathcal{F}_s^s \right], \end{aligned}$$

for all $u \in U_s^s$ and $\zeta \in \mathcal{T}_s$. After taking the expectation, we can see that w satisfies the half DPP, with the notation $X := X^{s,x;u,v,+}$:

$$\begin{aligned} w(s,x) \geq \inf_{v\in V_s^s} \sup_{u\in U_s^s} \mathbb{E} & \left[\int_s^{\zeta} f(t,X_t)dt - \sum_{n:s\leq\tau_n<\zeta} \phi(X_{\tau_n-},\delta_n) \right. \\ & \left. + \sum_{n:s\leq\eta_n<\zeta} \psi(X_{\eta_n-},\gamma_n) \prod_{l\geq 1} \mathbb{1}_{\{\eta_n\neq\tau_l\}} + w\left(\zeta,X_\zeta\right) \right], \end{aligned} \tag{2.8}$$

for all $\zeta \in \mathcal{T}_s$. Moreover, since $w(T,\cdot) \geq g(\cdot)$, we have $w(s,x) \geq V^+(s,x)$ for all $(s,x) \in [0,T] \times \mathbb{R}^d$. Indeed, if we take $\zeta = T$ we have

$$\begin{aligned} w(s,x) \geq \inf_{v\in V_s^s} \sup_{u\in U_s^s} \mathbb{E} & \left[\int_s^{T} f(t,X_t)dt - \sum_{n:s\leq\tau_n<T} \phi(X_{\tau_n-},\delta_n) \right. \\ & \left. + \sum_{n:s\leq\eta_n<T} \psi(X_{\eta_n-},\gamma_n) \prod_{l\geq 1} \mathbb{1}_{\{\eta_n\neq\tau_l\}} + w\left(T,X_T\right) \right] \geq V^+(s,x). \end{aligned}$$

The stochastic sub-solutions of the LI equation are defined symmetrically as follows.

Definition 2.3.2 A function $w : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is called a stochastic sub-solution of the LI equation if:

1. it is bounded and continuous, it satisfies $w(T,\cdot) \leq g(\cdot)$ and

$$(\mathcal{M}w - w)(t,x) \leq 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^d;$$

2. for each $s \in [0,T]$ and for each stopping rule $\rho \in \mathcal{T}_s$, there exists a strategy $\tilde{u} = (\tilde{\tau}_n, \tilde{\delta}_n) \in U_\rho^s$ such that for any $u = (\tau_n,\delta_n) \in U_s^s$, $v = (\eta_n,\gamma_n) \in V_s^s$, $x \in \mathbb{R}^d$ and each stopping rule ζ with $\rho \leq \zeta \leq T$, we have $\mathbb{P}$-a.s.

$$\begin{aligned} w(\rho,X_\rho) \leq \mathbb{E} & \left[\int_\rho^{\zeta} f(t,X_t)dt - \sum_{n:\rho\leq\tilde{\tau}_n<\zeta} \phi(X_{\tilde{\tau}_n-},\tilde{\delta}_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\tau}_n\neq\eta_l\}} \right. \\ & \left. + \sum_{n:\rho\leq\eta_n<\zeta} \psi(X_{\eta_n-},\gamma_n) + w(\zeta,X_\zeta) \mid \mathcal{F}_\rho^s \right], \end{aligned} \tag{2.9}$$

   where we have used the same simplifying notation as before with $X := X^{s,x;u\otimes_\rho\tilde{u},v,-}$.

By similar arguments as with the stochastic super-solution of the UI equation, we can observe that a stochastic sub-solution of the LI equation satisfies $w \leq V^-$ and the corresponding half DPP. As stated in Sîrbu [74], the two definitions are symmetric and they would be enough to proceed with the stochastic Perron's method in case the Isaacs condition (2.5) holds. For the general case we will have to use stochastic super/sub-solution of the LI/UI equations as well, which are defined as follows.

DEFINITION 2.3.3 A function $w : \mathbb{R}^d \to \mathbb{R}$ is called a stochastic sub-solution of the UI equation if:

1. it is bounded and continuous, it satisfies $w(T, \cdot) \leq g(\cdot)$ and

$$(\mathcal{M}w - w)(t,x) \leq 0, \qquad (t,x) \in [0,T] \times \mathbb{R}^d;$$

2. for each $s \in [0,T]$, for each stopping rule $\rho \in \mathcal{T}_s$ and for each $v = (\eta_n, \gamma_n) \in V_s^s$ there exists a strategy $\tilde{u} = (\tilde{\tau}_n, \tilde{\delta}_n) \in U_\rho^s$ such that for any $u = (\tau_n, \delta_n) \in U_s^s$, $x \in \mathbb{R}^d$ and each stopping rule ζ with $\rho \leq \zeta \leq T$, we have $\mathbb{P}$-a.s.

$$w(\rho, X_\rho) \leq \mathbb{E}\left[\int_\rho^\zeta f(t, X_t)dt - \sum_{n:\rho \leq \tilde{\tau}_n < \zeta} \phi(X_{\tilde{\tau}_n -}, \tilde{\delta}_n) \right.$$
$$\left. + \sum_{n:\rho \leq \eta_n < \zeta} \psi(X_{\eta_n -}, \gamma_n) \prod_{l \geq 1} \mathbb{1}_{\{\eta_n \neq \tilde{\tau}_l\}} + w(\zeta, X_\zeta) \mid \mathcal{F}_\rho^s \right],$$

with $X := X^{s,x;u \otimes_\rho \tilde{u}, v, +}$.

Proceeding in the usual way, choosing $\rho = s$ in the definition above and taking the expectation we get the half DPP

$$w(s,x) \leq \inf_{v \in V_s^s} \sup_{u \in U_s^s} \mathbb{E}\left[\int_s^\zeta f(t, X_t)dt - \sum_{n:s \leq \tau_n < \zeta} \phi(X_{\tau_n -}, \delta_n) \right.$$
$$\left. + \sum_{n:s \leq \eta_n < \zeta} \psi(X_{\eta_n -}, \gamma_n) \prod_{l \geq 1} \mathbb{1}_{\{\eta_n \neq \tau_l\}} + w(\zeta, X_\zeta) \right]. \tag{2.10}$$

Before proceeding any further, we need to introduce some more notation: we denote by

- $\mathcal{U}^{+/-}$ the set of stochastic super/sub-solutions of the UI,
- $\mathcal{L}^{+/-}$ the set of stochastic super/sub-solutions of the LI.

REMARKS 2.3.1 The sets $\mathcal{U}^{+/-}$ and $\mathcal{L}^{+/-}$ are non empty. For instance, let's focus on $\mathcal{U}^+$. Then, it contains all the functions of the form

$$w(\rho, x) = K + (T - \rho)C,$$

where $K \geq \sup g$ and $C \geq \sup f$. Here we check that all the conditions are satisfied:

- $w(T,x) = K \geq \sup g \geq g(x)$ for all $x \in \mathbb{R}^d$;
- $(w - \mathcal{H}w)(t,x) = -\psi(x,\delta) \leq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$;
- for each $s \in [0,T]$ and for each $\rho \in \mathcal{T}_s$ there exists P2's strategy $\tilde{v}$, consisting of no impulses from ρ onwards, such that for any $u \in U_s^s$ and any $v \in V_s^s$, $x \in \mathbb{R}^d$ and $\zeta : \rho \leq \zeta \leq T$ we have

$$\begin{aligned} w(\rho, X_\rho) &\geq (\zeta - \rho) \sup f + w(\zeta, X_\zeta) \\ &\geq \mathbb{E}\left[\int_\rho^\zeta f(t, X_t) dt - \sum_{n:\rho \leq \tau_n < \zeta} \phi(X_{\tau_n -}, \delta_n) + w(\zeta, X_\zeta) \mid \mathcal{F}_\rho^s \right]. \end{aligned}$$

  Note that the term $\sum_{n:\rho \leq \eta_n < \zeta} \psi(X_{\eta_n -}, \gamma_n)$ is absent due to $\tilde{v}$, since there are no impulses sent by P2.

The stochastic Perron's method suggests to take the supremum of sub-solutions and the infimum of super-solutions:

$$u^- := \sup_{w \in \mathcal{U}^-} w \leq V^+ \leq \inf_{w \in \mathcal{U}^+} w := u^+ \tag{2.11}$$

and

$$l^- := \sup_{w \in \mathcal{L}^-} w \leq V^- \leq \inf_{w \in \mathcal{L}^+} w := l^+, \tag{2.12}$$

so that $l^- \leq V^- \leq V^+ \leq u^+$. We want to prove that u^+ (resp. u^-) is a viscosity sub-solution (resp. super-solution) of the UI so that, after a comparison result, we obtain $u^- \geq u^+$. As a consequence we find the value of the game in which P1 has priority, $u^- = V^+ = u^+$. The same reasoning can be applied to $l^{-/+}$ in order to obtain $l^- = V^- = l^+$.

Remarks 2.3.2 The function u^+ is upper semi-continuous, being the pointwise infimum of continuous functions (each function $w \in \mathcal{U}^+$ is indeed continuous, hence upper semi-continuous), whereas u^- is lower semi-continuous since it is defined as the pointwise supremum of continuous functions. Analogous statements hold for l^+ and l^-.

In order to extend the stochastic Perron's method to our setting we need the following auxiliary properties adapted from Sîrbu [74, Lemmas 3.7-8].

Lemma 2.3.1 (i) If $w_1, w_2 \in \mathcal{U}^+$, then $w_1 \wedge w_2 \in \mathcal{U}^+$. If $w_1, w_2 \in \mathcal{U}^-$, then $w_1 \vee w_2 \in \mathcal{U}^-$.

(ii) There exists a non-increasing sequence $w_n \in \mathcal{U}^+$ such that $w_n \downarrow u^+$ and a non-decreasing sequence $w_n' \in \mathcal{U}^-$ such that $w_n' \uparrow u^-$.

Proof 2.3.1 (of Lemma 2.3.1) (i) First, we prove property 1 in Definition 2.3.1, i.e. $\mathcal{H}w - w \geq 0$ with $w := w_1 \wedge w_2$. Being Γ compact, for all $(t,x) \in [0,T] \times \mathbb{R}^d$ there exists some $\widehat{\gamma}(t,x)$ such that

$$\begin{aligned}(\mathcal{H}w - w)(t,x) &= w(t, x + \widehat{\gamma}(t,x)) + \psi(x, \widehat{\gamma}(t,x)) - w(t,x) \\ &\geq (\mathcal{H}w_1 - w)(t,x)\mathbb{1}_{\{w_1(t,x+\widehat{\gamma}(t,x)) \leq w_2(t,x+\widehat{\gamma}(t,x))\}} \\ &\quad + (\mathcal{H}w_2 - w)(t,x)\mathbb{1}_{\{w_1(t,x+\widehat{\gamma}(t,x)) > w_2(t,x+\widehat{\gamma}(t,x))\}},\end{aligned}$$

which is non-negative. Now we turn to property 2: let $\rho \in \mathcal{T}_s$ and consider two strategies $\tilde{v}^1 = (\tilde{\eta}^1_n, \tilde{\gamma}^1_n)_{n\geq 1}, \tilde{v}^2 = (\tilde{\eta}^2_n, \tilde{\gamma}^2_n)_{n\geq 1}$ belonging to V^s_ρ for P2 starting at time ρ and corresponding to the definitions of the UI stochastic super-solutions w_1, w_2. After defining a new strategy $\tilde{v}$ starting at ρ by combining the previous two as

$$\tilde{v}(y) = \tilde{v}^1(y)\mathbb{1}_{\{w_1(\rho(y), y(\rho(y))) \leq w_2(\rho(y), y(\rho(y)))\}} + \tilde{v}^2(y)\mathbb{1}_{\{w_1(\rho(y), y(\rho(y))) > w_2(\rho(y), y(\rho(y)))\}},$$

one can check that $\tilde{v} \in V^s_\rho$ satisfies the inequality (2.7) for $w = w_1 \wedge w_2$, so that w is a stochastic super-solution of the UI. Similar arguments can be used to prove the second property in (i).

(ii) Proposition 2.A.1 grants that there exists a sequence $(\tilde{w}_n)_{n\geq 1} \in \mathcal{U}^+$ such that $u^+ = \inf_{n\geq 1} \tilde{w}_n$. Now, we can just define $w_n = \tilde{w}_1 \wedge \tilde{w}_2 \wedge \ldots \wedge \tilde{w}_n \downarrow u^+$. A very similar proof leads to the analogous property of u^-. □

Lemma 2.3.2 Fix a compact $K \subset [s,T] \times \mathbb{R}^d$ and a non-increasing sequence $(w_n)_{n\geq 1}$ of stochastic super-solutions in $\mathcal{U}^+$ converging pointwise to u^+. Then, such sequence converges uniformly to u^+ in K, i.e. for every $\epsilon > 0$ there exists n_0 such that for all $n \geq n_0$

$$\sup_{(t,x)\in K} (w_n - u^+)(t,x) < \epsilon.$$

Proof 2.3.2 Let $\epsilon > 0$. For each $n \geq 1$, let us define the function $g_n(t,x) := (w_n - u^+)(t,x)$ and the set $A_n := \{(t,x) \in K \mid g_n(t,x) < \epsilon\}$. Due to the fact that g_n is non-increasing we have that $A_n \subseteq A_{n+1}$. Moreover, since g_n is lower semi-continuous, A_n is open. Hence, due to $g_n(t,x) \to 0$, $n \to \infty$, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $(A_n)_{n\geq 1}$ is an open cover of K. Then, by compactness of K there exists some $n_0 \geq 1$ such that $K \subset \cup_{n=1}^{n_0} A_n = A_{n_0}$, which means that $\sup_{(t,x)\in K}(w_n - u^+)(t,x) < \epsilon$. □

Corollary 2.3.1 Fix a sequence of $(w_n)_{n\geq 0}$ as in the lemma above. Then, for each $(t_0, x_0) \in [s,T] \times \mathbb{R}^d$ and $r > 0$ fixed there exists an $\epsilon > 0$ such that the sequence $(\mathcal{H}w_n)_{n\geq 0}$ converges uniformly to $\mathcal{H}u^+$ in $\overline{B_r(t_0,x_0)}$, i.e. for every $\epsilon > 0$ there exists $n_0 \geq 1$ such that for all $n \geq n_0$

$$\sup_{(t,x)\in \overline{B_r(t_0,x_0)}} (\mathcal{H}w_n - \mathcal{H}u^+)(t,x) < \epsilon.$$

Proof 2.3.3 To begin, let us note that we have

$$\begin{aligned}\sup_{(t,x)\in\overline{B_r(t_0,x_0)}}(\mathcal{H}w_n(t,x)-\mathcal{H}u^+(t,x)) &\le \sup_{(t,x)\in\overline{B_r(t_0,x_0)}} w_n(t,x+\gamma^*(t,x))-u^+(t,x+\gamma^*(t,x))\\ &\le \sup_{(t,x)\in\overline{B(t_0,r)}\times(\overline{B_r(x_0)}+\Gamma)}(w_n-u^+)(t,x)\end{aligned}$$

where $\gamma^*(t,x)\in\arg\min_{\gamma\in\Gamma}(u^+(t,x+\gamma)+\psi(x,\gamma))$ and $\overline{B_r(x_0)}+\Gamma := \{x+\gamma : \gamma\in\Gamma, x\in\overline{B_r(x_0)}\}$. Therefore, since $\overline{B_r(t_0)}\times(\overline{B_r(x_0)}+\Gamma)$ is compact, we only need to apply Lemma 2.3.2 to show

$$\sup_{(t,x)\in\overline{B_r(t_0,x_0)}}(\mathcal{H}w_n-\mathcal{H}u^+)(t,x)<\epsilon.$$

□

Viscosity solutions. Here we introduce the definition of viscosity solutions we are going to use throughout the rest of the chapter.

Definition 2.3.4 An upper semi-continuous function $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a viscosity sub-solution of (2.3) if, for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ and $\varphi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ such that $\varphi-u$ has a local minimum at (t_0,x_0) and $u(t_0,x_0)=\varphi(t_0,x_0)$, we have

$$\max\{\min\{-\mathcal{A}\varphi-\varphi_t-f,u-\mathcal{M}u\},u-\mathcal{H}u\}\le 0 \quad \text{in } (t_0,x_0)\in[0,T)\times\mathbb{R}^d, \tag{2.13}$$
$$\max\{\min\{u-g,u-\mathcal{M}u\},u-\mathcal{H}u\}\le 0 \quad \text{on } \{T\}\times\mathbb{R}^d. \tag{2.14}$$

A lower semi-continuous function $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a viscosity super-solution of (2.3) if, for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ and $\varphi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ such that $\varphi-u$ has a local maximum at (t_0,x_0) and $u(t_0,x_0)=\varphi(t_0,x_0)$, we have

$$\max\{\min\{-\mathcal{A}\varphi-\varphi_t-f,u-\mathcal{M}u\},u-\mathcal{H}u\}\ge 0 \quad \text{in } (t_0,x_0)\in[0,T)\times\mathbb{R}^d, \tag{2.15}$$
$$\max\{\min\{u-g,u-\mathcal{M}u\},u-\mathcal{H}u\}\ge 0 \quad \text{on } \{T\}\times\mathbb{R}^d. \tag{2.16}$$

A function u is a viscosity solution if it is both a sub and a super-solution. The definitions of viscosity super and sub-solutions of (2.4) are similar.

Main results of stochastic Perron's method. The following theorem is one of the main results of the stochastic Perron's method applied to impulse games. It provides a characterisation of the supremum (resp. infimum) of stochastic sub-solutions (resp. super-solutions) of the UI (resp. LI) in terms of viscosity solutions of the corresponding HJBI equations. We stress that one of the strengths of such an approach is that we obtain the DPP as by-product of such a characterisation.

Theorem 2.3.1 Under Assumption 2.2.2 the following hold:

1. The function l^+ is a bounded upper semi-continuous viscosity sub-solution of the LI equation and the function l^- is a bounded lower semi-continuous viscosity super-solution of the LI equation and they both satisfy the corresponding halves of the DPP for the lower equation.

2. The function u^+ is a bounded upper semi-continuous viscosity sub-solution of the UI equation and satisfies the half DPP (2.8). The function u^- is a bounded lower semi-continuous viscosity super-solution of the UI equation and satisfies the half DPP (2.10);

PROOF 2.3.4 *1. Viscosity sub-solution of the LI.*
1.1. Interior sub-solution property for l^+. We assume by contradiction that $l^+ = \inf_{w\in\mathcal{L}^+} w$ is not a viscosity sub-solution of LI in the parabolic interior, i.e. for some $r > 0$, $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ and some test function $\varphi \in \mathcal{C}^{1,2}([0, T) \times \mathbb{R}^d)$ such that

$$\min_{(t,x)\in B_r(t_0,x_0)} (\varphi - l^+)(t, x) = (\varphi - l^+)(t_0, x_0) = 0 \tag{2.17}$$

we have

$$\max\left\{\min\left\{-\mathcal{A}\varphi - \varphi_t - f, l^+ - \mathcal{M}l^+\right\}, l^+ - \mathcal{H}l^+\right\} > 0 \quad \text{at } (t_0, x_0).$$

This is equivalent to one of the following two cases:

(i) both $(-\mathcal{A}\varphi - \varphi_t - f)(t_0, x_0) > 0$ and $(l^+ - \mathcal{M}l^+)(t_0, x_0) > 0$;

(ii) $(l^+ - \mathcal{H}l^+)(t_0, x_0) > 0$.

Let us first analyse case (ii). We know that, by Definition 2.3.1, $w - \mathcal{H}w \leq 0$ for all $w \in \mathcal{L}^+$. By Lemma 2.3.2 we can select a $w \in \mathcal{L}^+$ such that $\mathcal{H}l^+(t_0, x_0) + \epsilon > \mathcal{H}w(t_0, x_0)$ for an arbitrarily small $\epsilon > 0$ so that we have

$$(w - \mathcal{H}l^+)(t_0, x_0) - \epsilon < (w - \mathcal{H}w)(t_0, x_0) \leq 0.$$

In particular, since by definition $l^+ = \inf_{w\in\mathcal{L}^+} w$, it follows that $w \geq l^+$ leading us to

$$(l^+ - \mathcal{H}l^+)(t_0, x_0) - \epsilon \leq (w - \mathcal{H}l^+)(t_0, x_0) - \epsilon < (w - \mathcal{H}w)(t_0, x_0) \leq 0,$$

which shows that $l^+ - \mathcal{H}l^+ > 0$ is impossible.

Let us turn to case (i). Assume that we have both inequalities

$$(-\mathcal{A}\varphi - \varphi_t - f)(t_0, x_0) > 0, \quad (l^+ - \mathcal{M}l^+)(t_0, x_0) > 0. \tag{2.18}$$

Fix $\xi > 0$, by continuity of the coefficients of the SDE (2.1), we can find a small enough open ball $B_\epsilon(t_0, x_0)$ for some $\epsilon > 0$, such that

$$\begin{aligned} &-\mathcal{A}\varphi - \varphi_t - f > 0 \quad \text{on } \overline{B_\epsilon(t_0, x_0)}, \\ &\varphi - \xi \geq l^+ \quad \text{on } \mathbb{T}_{\epsilon/2}(t_0, x_0), \end{aligned}$$

where the second inequality comes from (2.17) and $\mathbb{T}_{\epsilon/2}(t_0,x_0) := \overline{B_\epsilon(t_0,x_0)} \setminus B_{\epsilon/2}(t_0,x_0)$. Moreover, since φ is continuous and $\mathcal{M}l^+$ is upper semi-continuous (see Proposition 2.2.1), for ϵ' small enough, the second inequality in (2.18) implies

$$\varphi - \epsilon' \geq \mathcal{M}l^+ \quad \text{ on } \overline{B_\epsilon(t_0,x_0)}.$$

Let $\xi > \xi' > \xi/2$. By the property (ii) in Lemma 2.3.1, there exists a sequence $w_n \in \mathcal{L}^+$ with $w_n \downarrow l^+$ as $n \to \infty$. Using Lemma 2.A.1 and Lemma 2.3.2, we can find n sufficiently large such that $\varphi \geq w_n + \xi'$ on $\mathbb{T}_{\epsilon/2}(t_0,x_0)$ and $w_n - l^+ < \xi'$ on $\overline{B_\epsilon(t_0,x_0)}$. Note that the two inequalities are compatible on the torus since $\xi - \xi' < \xi'$, indeed

$$\begin{cases} l^+ - w_n + \xi' > 0 \\ \varphi - w_n - \xi' \geq 0 \end{cases} \Rightarrow \begin{cases} \varphi - w_n - (\xi - \xi') > 0 \\ \varphi - w_n - \xi' \geq 0 \end{cases}$$

due to $\varphi - l^+ \geq \xi$. Since such index n will remain fixed throughout the rest of the proof, we can conveniently set $w := w_n$. Then, we choose $0 < \mu < \xi' \wedge \epsilon'$ so that the function $\varphi^\mu := \varphi - \mu$ satisfies the properties

$$-\varphi^\mu_t - \mathcal{A}\varphi^\mu + f > 0 \quad \text{ on } \overline{B_\epsilon(t_0,x_0)} \tag{2.19}$$

$$\varphi^\mu > w \quad \text{ on } \mathbb{T}_{\epsilon/2}(t_0,x_0) \tag{2.20}$$

$$\varphi^\mu > \mathcal{M}l^+ \quad \text{ on } \overline{B_\epsilon(t_0,x_0)} \tag{2.21}$$

and

$$\varphi^\mu(t_0,x_0) = l^+(t_0,x_0) - \mu.$$

Now define

$$w^\mu := \begin{cases} \varphi^\mu \wedge w & \text{ on } \overline{B_\epsilon(t_0,x_0)} \\ w & \text{ outside} \end{cases}$$

given that $w^\mu(t_0,x_0) < l^+(t_0,x_0)$ we obtain a contradiction if we can show $w^\mu \in \mathcal{L}^+$.

Now, fix $s \in [0,T]$, $u = (\tau_n,\delta_n)_n$ and let $\rho \in \mathcal{T}_s$. We need to construct an impulse strategy $\tilde{v} \in V^s_\rho$ satisfying the properties as in the definition of the LI stochastic super-solution for w^μ (cf. Definition 2.3.1). We know already that w is a stochastic super-solution of the LI equation and, as such, there exists an impulse strategy $\tilde{v}_1 \in V^s_\rho$ satisfying (2.7) from ρ onwards. Then, we describe $\tilde{v}$ as follows:

1. If $\varphi^\mu < w$ at ρ, play the no impulse strategy, that we denote $\widehat{v} \equiv 0$.

2. If $\varphi^\mu \geq w$ at ρ, play $\tilde{v}_1$.

3. Play 1-2 until $\rho_1(y) \wedge \tau^u_1(y)$ where $\rho_1(y) := \inf\{t \in [\rho(y),T] : (t,y(t)) \in \partial B_{\epsilon/2}(t_0,x_0)\}$ (with the convention: $\inf \emptyset = +\infty$) and $\tau^u_1(y)$ is the first stopping rule according to the strategy $u = (\tau_n,\delta_n)_{n\geq 1} \in U^s_\rho$ previously fixed, such that $(\tau, y(\tau)) \in B_{\epsilon/2}(t_0,x_0)^c$. Here we know that $w^\mu = w$ by construction (2.20), either by continuity, if $\rho_1(y) \wedge \tau^u_1(y) = \rho_1(y)$, or by definition of τ^u_1 otherwise.

4. After $\rho_1(y) \wedge \tau_1^u(y)$, play the strategy $\tilde{v}_3 \in V^s_{\rho_1 \wedge \tau_1^u}$ such that the stochastic super-solution w satisfies (2.7) from $\rho_1(y) \wedge \tau_1^u(y)$.

The strategy doing 1-2 above, that we call $\tilde{v}_2 \in V^s_\rho$, can be written formally as

$$\tilde{v}_2(y) = \widehat{v}(y)\mathbb{1}_{\{\varphi^\mu(\rho(y),y(\rho(y)))<w(\rho(y),y(\rho(y)))\}} + \tilde{v}_1(y)\mathbb{1}_{\{\varphi^\mu(\rho(y),y(\rho(y)))\geq w(\rho(y),y(\rho(y)))\}}.$$

Now, to complete the definition of $\tilde{v} \in V^s_\rho$, it remains to concatenate $\tilde{v}_2 \in V^s_\rho$ with $\tilde{v}_3 \in V^s_{\rho_1(y)\wedge\tau_1^u(y)}$ as follows

$$\tilde{v} := \tilde{v}_2 \otimes_{\rho_1\wedge\tau_1^u} \tilde{v}_3 \in V^s_\rho.$$

At this point we are ready to use $\tilde{v}$ to show that w^μ satisfies (2.7).

Hence, let us fix $v \in V^s_s$, $x \in \mathbb{R}^d$ and $\zeta \in \mathcal{T}_\rho$. Denote by $X := X^{s,x;u,v\otimes_\rho\tilde{v},-}$, where $\tilde{v}$ was just defined above, while $\rho := \rho(X)$ and $\zeta := \zeta(X)$. Let also set $\rho_1 := \rho_1(X)$ and $\tau_1^u := \tau_1^u(X)$ and define the event $A := \{\varphi^\mu(\rho, X_\rho) < w(\rho, X_\rho)\} \in \mathcal{F}^s_\rho$. First, we observe that

$$X_t^{s,x;u,v\otimes_\rho\tilde{v}_2,-} = X_t^{s,x;u,v\otimes_\rho\widehat{v},-}\mathbb{1}_A + X_t^{s,x;u,v\otimes_\rho\tilde{v}_1,-}\mathbb{1}_{A^c} \quad \text{on } \{\rho \leq t \leq \rho_1 \wedge \tau_1^u\}.$$

Then, note that on the event A we have $w^\mu(\rho, X_\rho) = \varphi^\mu(\rho, X_\rho)$ whereas $w^\mu(\rho, X_\rho) = w(\rho, X_\rho)$ on A^c, which means we only need to show that (2.7) is satisfied on A since we know it is satisfied on A^c by definition of stochastic super-solution (recall that $w \in \mathcal{L}^+$). Hence, we apply Itô's formula on A from ρ to $\tau_1^u \wedge \rho_1$ and take conditional expectation to get

$$\begin{aligned}
&w^\mu(\rho, X_\rho) = \varphi^\mu(\rho, X_\rho)\\
&=\mathbb{E}\left[\left(w^\mu(\rho_1\wedge\tau_1^u, X_{\rho_1\wedge\tau_1^u}) - \int_\rho^{\rho_1\wedge\tau_1^u} (\varphi_t + \mathcal{A}\varphi)(t, X_t)dt - \sum_{\rho\leq s<\rho_1\wedge\tau_1^u} \Delta\varphi^\mu(s, X_s)\right)\mathbb{1}_{\{\rho_1\wedge\tau_1^u\leq\zeta\}} | \mathcal{F}^s_\rho\right]\\
&+\mathbb{E}\left[\left(\varphi^\mu(\zeta, X_\zeta) - \int_\rho^\zeta (\varphi_t + \mathcal{A}\varphi)(t, X_t)dt - \sum_{\rho\leq s<\zeta} \Delta\varphi^\mu(s, X_s)\right)\mathbb{1}_{\{\zeta<\tau_1^u\wedge\rho_1\}} \mid \mathcal{F}^s_\rho\right]\\
&= \text{(I)} + \text{(II)},
\end{aligned}$$

where we set $\Delta\varphi^\mu(s, X_s) := (\varphi^\mu(s, X_s) - \varphi^\mu(s-, X_{s-})) = (\varphi(s, X_s) - \varphi(s-, X_{s-})) =: \Delta\varphi(s, X_s)$. We consider the two summands on the RHS above separately:

$$\begin{aligned}
\text{(I)} \geq\ &\mathbb{E}\left[\left(w^\mu(\tau_1^u\wedge\rho_1, X_{\tau_1^u\wedge\rho_1}) + \int_\rho^{\tau_1^u\wedge\rho_1} f(t, X_t)dt - \sum_{n:\rho\leq\tau_n<\tau_1^u\wedge\rho_1} \Delta\varphi(\tau_n, X_{\tau_n})\right)\mathbb{1}_{\{\tau_1^u\wedge\rho_1\leq\zeta\}} \mid \mathcal{F}^s_\rho\right]\\
\geq\ &\mathbb{E}\left[\left(w^\mu(\tau_1^u\wedge\rho_1, X_{\tau_1^u\wedge\rho_1}) + \int_\rho^{\tau_1^u\wedge\rho_1} f(t, X_t)dt - \sum_{n:\rho\leq\tau_n<\tau_1^u\wedge\rho_1} \phi(X_{\tau_n-}, \delta_n)\prod_{l\geq1}\mathbb{1}_{\{\eta_l\neq\tau_n\}} + \right.\right.\\
&\left.\left.\sum_{n:\rho\leq\eta_n<\tau_1^u\wedge\rho_1} \psi(X_{\eta_n-}, \gamma_n)\right)\mathbb{1}_{\{\tau_1^u\wedge\rho_1\leq\zeta\}} \mid \mathcal{F}^s_\rho\right]
\end{aligned}$$

where the first inequality follows from (2.19) and $\widehat{v} \equiv 0$ (jumps occur only at times τ_n, $n \geq 1$), while the second one is due to the following two arguments. First, since there are no interventions coming from P2, the related costs vanish, i.e. $\sum \psi = 0$. Second, recall that w satisfies $w - l^+ < \xi'$ by construction so that, due to (2.21) and the definition of the stopping rule $\rho_1 \wedge \tau_1^u$† we have

$$\begin{aligned}\varphi^\mu(\tau_n-, X_{\tau_n-}) &> \mathcal{M}l^+(\tau_n-, X_{\tau_n-}) \geq l^+(\tau_n, X_{\tau_n-} + \delta_n) - \phi(X_{\tau_n-}, \delta_n)\\ &> w(\tau_n, X_{\tau_n-} + \delta_n) - \phi(X_{\tau_n-}, \delta_n) - \xi'\\ &> \varphi^\mu(\tau_n, X_{\tau_n-} + \delta_n) - \phi(X_{\tau_n-}, \delta_n) - \xi'\end{aligned}$$

and since $\xi > \xi'$ is arbitrary it follows

$$\varphi^\mu(\tau_n, X_{\tau_n-} + \delta_n) - \varphi^\mu(\tau_n, X_{\tau_n-}) \leq \phi(X_{\tau_n-}, \delta_n).$$

Now, for the other summand (II), observe that $w = \varphi^\mu$ over $[\rho, \zeta)$ along (s, X_s) so that

$$\begin{aligned}\text{(II)} \geq \mathbb{E}\Bigg[\Bigg(&\varphi^\mu(\zeta, X_\zeta) + \int_\rho^\zeta f(t, X_t)dt - \sum_{n:\rho\leq\tau_n<\zeta} \phi(X_{\tau_n-}, \delta_n) \prod_{l\geq 1} \mathbb{1}_{\{\eta_l\neq\tau_n\}}\\ &+ \sum_{n:\rho\leq\eta_n<\zeta} \psi(X_{\eta_n-}, \gamma_n)\Bigg) \mathbb{1}_{\{\zeta<\rho_1\wedge\tau_1^u\}} \mid \mathcal{F}_\rho^s\Bigg].\end{aligned}$$

As mentioned in step 4 of the construction of the impulse strategy $\tilde{v} \in V_\rho^s$, by definition of stochastic super-solution, $\tilde{v}_3 \in V^s_{\rho_1\wedge\tau_1^u}$ provides (2.7) concatenated with any previous strategy v and against any P1 strategy u so that, from $\rho_1 \wedge \tau_1^u$, we have

$$\begin{aligned}w^\mu(\rho_1 \wedge \tau_1^u, X_{\rho_1\wedge\tau_1^u}) &= w(\rho_1 \wedge \tau_1^u, X_{\rho_1\wedge\tau_1^u})\\ &\geq \mathbb{E}\Bigg[\int_{\rho_1\wedge\tau_1^u}^\zeta f(t, X_t)dt - \sum_{n:\rho_1\wedge\tau_1^u\leq\tau_n<\zeta} \phi(X_{\tau_n-}, \delta_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\eta}_l\neq\tau_n\}}\\ &\qquad + \sum_{n:\rho_1\wedge\tau_1^u\leq\tilde{\eta}_n<\zeta} \psi(X_{\tilde{\eta}_n-}, \tilde{\gamma}_n) + w(\zeta, X_\zeta) \mid \mathcal{F}^s_{\rho_1\wedge\tau_1^u}\Bigg]\\ &\geq \mathbb{E}\Bigg[\int_{\rho_1\wedge\tau_1^u}^\zeta f(t, X_t)dt - \sum_{n:\rho_1\wedge\tau_1^u\leq\tau_n<\zeta} \phi(X_{\tau_n-}, \delta_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\eta}_l\neq\tau_n\}}\\ &\qquad + \sum_{n:\rho_1\wedge\tau_1^u\leq\tilde{\eta}_n<\zeta} \psi(X_{\tilde{\eta}_n-}, \tilde{\gamma}_n) + w^\mu(\zeta, X_\zeta) \mid \mathcal{F}^s_{\rho_1\wedge\tau_1^u}\Bigg].\end{aligned}$$

Then, by the property of iterated conditional expectations we obtain $\mathbb{P}$-a.s.

$$\begin{aligned}w^\mu(\rho, X_\rho) \geq \mathbb{E}\Bigg[&\int_\rho^\zeta f(t, X_t)dt - \sum_{n:\rho\leq\tau_n<\zeta} \phi(X_{\tau_n-}, \delta_n) \prod_{l\geq 1} \mathbb{1}_{\{\tilde{\eta}_l\neq\tau_n\}}\\ &+ \sum_{n:\rho\leq\tilde{\eta}_n<\zeta} \psi(X_{\tilde{\eta}_n-}, \tilde{\gamma}_n) + w^\mu(\zeta, X_\zeta) \mid \mathcal{F}^s_\rho\Bigg].\end{aligned}$$

†Note that the stopping rule $\rho_1 \wedge \tau_1^u$ guarantees that $\varphi^\mu(t, X_t) < w(t, X_t)$ for all $t \in [\rho, \rho_1 \wedge \tau_1^u)$ on A, which would not necessarily be true if we picked ρ_1 alone instead, as in the works by Sîrbu [74] and Bayraktar et al. [11]. This is due to the fact that in our case we do not necessarily get to the boundary $\partial B_{\epsilon/2}(t_0, x_0)$ in a diffusive manner due to the presence of the jumps induced by players' impulses.

Finally, it is left to be shown that

$$(w^\mu - \mathcal{H}w^\mu)(t,x) \le 0 \qquad \text{for all } (t,x) \in [s,T] \times \mathbb{R}^d.$$

First, we note that

$$\begin{aligned}
&(w^\mu - \mathcal{H}w^\mu)(t,x) \le (w - \mathcal{H}w^\mu)(t,x) \\
&= \begin{cases} \max\{(w - \mathcal{H}w)(t,x), (w - \mathcal{H}\varphi^\mu)(t,x)\} & \text{if } (t, x + \gamma^*(t,x)) \in \overline{B_\epsilon(t_0,x_0)}, \\ (w - \mathcal{H}w)(t,x) & \text{otherwise,} \end{cases}
\end{aligned}$$

where $\gamma^* = \gamma^*(t,x) \in \arg\min_{\gamma\in\Gamma} [w^\mu(t, x+\gamma) + \psi(x,\gamma)]$. Hence, we only need to show $w - \mathcal{H}\varphi^\mu \le 0$ for $(t, x+\gamma^*) \in \overline{B_\epsilon(t_0,x_0)}$ since $w - \mathcal{H}w \le 0$ follows by definition of $w \in \mathcal{L}^+$. Then, recall that, $\varphi \ge l^+$ and $w < l^+ + \xi'$ hold on $\overline{B_\epsilon(t_0,x_0)}$. Therefore, by definition of $\varphi^\mu := \varphi - \mu$ and w we have

$$\begin{aligned}
(w - \mathcal{H}\varphi^\mu)(t,x) &= w(t,x) - \varphi^\mu(t, x+\gamma^*) - \psi(x,\gamma^*) \\
&= w(t,x) - \varphi(t, x+\gamma^*) - \psi(x,\gamma^*) + \mu \\
&\le w(t,x) - l^+(t, x+\gamma^*) - \psi(x,\gamma^*) + \mu \\
&< w(t,x) - w(t, x+\gamma^*) - \psi(x,\gamma^*) + \mu + \xi' \\
&< (w - \mathcal{H}w)(t,x) + 2\xi' \le 2\xi'
\end{aligned}$$

as $\mathcal{H}w(t,x) \le w(t, x+\gamma^*) + \psi(x,\gamma^*)$ and $\mu < \xi'$, to show that $(w - \mathcal{H}\varphi^\mu)(t,x) \le 0$ by letting ξ' tend to zero.

1.2. The terminal condition property for l^+. We assume by contradiction that there exists $x_0 \in \mathbb{R}^d$ such that

$$\max\{\min\{l^+ - g, l^+ - \mathcal{M}l^+\}, l^+ - \mathcal{H}l^+\} > 0 \qquad \text{at } (T, x_0),$$

which is satisfied whenever we have one of the two cases:

(i) $(l^+ - g)(T, x_0) > 0$ and $(l^+ - \mathcal{M}l^+)(T, x_0) > 0$;

(ii) $(l^+ - \mathcal{H}l^+)(T, x_0) > 0$.

As shown in part 1.1, case (ii) is not possible. Therefore, let us consider case (i). Due to continuity of g and upper semi-continuity of $\mathcal{M}l^+$ there exists an $\epsilon > 0$ such that

$$l^+(T, x_0) \ge \max\{g, \mathcal{M}l^+\}(t,x) + \epsilon \qquad \text{for all } (t,x) \in \overline{B_\epsilon(T,x_0)},$$

where, with a slight abuse of notation, we denote as $B_\epsilon(T,x_0)$ the restriction $B_\epsilon(T,x_0) \cap ([s,T] \times \mathbb{R}^d)$. Since l^+ is bounded by construction, it is bounded on the torus $\mathbb{T}_{\epsilon/2}(T,x_0)$ as well, so that, for $\nu > 0$ small enough and $\xi > 0$, we have

$$l^+(T, x_0) + \frac{\epsilon^2}{4\nu} > \xi + \sup_{(t,x)\in\mathbb{T}_{\epsilon/2}(T,x_0)} l^+(t,x).$$

Then, for $k > 0$ define the function

$$\varphi^{\nu,\epsilon,k}(t,x) := l^+(T,x_0) + \frac{(x-x_0)^2}{\nu} + k(T-t)$$

and apply Lemma 2.A.1 and Lemma 2.3.2 to a sequence of stochastic super-solutions $(w_n)_{n\geq 1} \in \mathcal{L}^+$, as done previously in part 1.1, to find n sufficiently large so that $\varphi^{\nu,\epsilon,k} \geq w_n + \xi'$ on $\mathbb{T}_{\epsilon/2}(T,x_0)$ and $w_n - l^+ < \xi'$ on $\overline{B_\epsilon(T,x_0)}$ for some $\xi/2 < \xi' < \xi$. Therefore, for $w := w_n$ we have

$$l^+(T,x_0) + \frac{\epsilon^2}{4\nu} > \sup_{(t,x)\in\mathbb{T}_{\epsilon/2}(T,x_0)} w(t,x).$$

Moreover, for $k > 0$ large enough $\varphi^{\nu,\epsilon,k}$ satisfies

$$-\varphi_t^{\nu,\epsilon,k} - \mathcal{A}\varphi^{\nu,\epsilon,k} - f > 0 \qquad \text{on } \overline{B_\epsilon(T,x_0)}$$

and

$$\varphi^{\nu,\epsilon,k}(t,x) \geq l^+(T,x_0) \geq \mathcal{M}l^+(t,x) \qquad \text{on } \overline{B_\epsilon(T,x_0)}.$$

Hence, for $0 < \mu < \xi \wedge \epsilon$ we define the function $w^{\mu,\nu,\epsilon,k}$ as

$$w^{\mu,\nu,\epsilon,k} := \begin{cases} w \wedge (\varphi^{\nu,\epsilon,k} - \mu) & \text{on } \overline{B_\epsilon(T,x_0)} \\ w & \text{otherwise} \end{cases}$$

to replicate the arguments in part 1.1 to show $w^{\mu,\nu,\epsilon,k} \in \mathcal{L}^+$ as

$$w^{\mu,\nu,\epsilon,k}(T,x) \geq \varphi^{\nu,\epsilon,k}(T,x) - \mu \geq l^+(T,x_0) - \mu \geq g(x) + \epsilon - \mu \geq g(x)$$

for $(T,x) \in \overline{B_\epsilon(T,x_0)}$ and reach a contradiction since

$$w^{\mu,\nu,\epsilon,k}(T,x_0) = l^+(T,x_0) - \mu < l^+(T,x_0) = \inf_{w\in\mathcal{L}^+} w.$$

2. l^- is a viscosity super-solution of LI.
2.1. The interior super-solution property for l^-. Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$ in the parabolic interior such that, for some $r > 0$, a smooth function φ strictly touches l^- from below at (t_0,x_0). i.e.

$$\max_{(t,x)\in B_r(t_0,x_0)} (\varphi - l^-)(t,x) = (\varphi - l^-)(t_0,x_0) = 0.$$

Assume by contradiction that

$$\max\left\{\min\left\{-\varphi_t - \mathcal{A}\varphi - f, l^- - \mathcal{M}l^-\right\}, l^- - \mathcal{H}l^-\right\} < 0 \quad \text{at } (t_0,x_0)$$

which happens in one of the following two cases:

(i) both $-\varphi_t - \mathcal{A}\varphi - f < 0$ and $l^- - \mathcal{H}l^- < 0$;

(ii) both $l^- - \mathcal{M}l^- < 0$ and $l^- - \mathcal{H}l^- < 0$.

Let us first analyse case (ii). Similarly to part 1.1, we know that $w - \mathcal{M}w \geq 0$ for all $w \in \mathcal{L}^-$ by definition. By Lemma 2.3.2, we can select a $w \in \mathcal{L}^-$ such that $\mathcal{M}l^-(t_0, x_0) - \epsilon < \mathcal{M}w(t_0, x_0)$ for an arbitrarily small $\epsilon > 0$ so that we have

$$(w - \mathcal{M}l^-)(t_0, x_0) + \epsilon > (w - \mathcal{M}w)(t_0, x_0) \geq 0.$$

In particular, since by definition of l^- we have $l^- = \sup_{w \in \mathcal{L}^-} w$, it follows that $w \leq l^-$ leading us to

$$(l^- - \mathcal{M}l^-)(t_0, x_0) + \epsilon \geq (w - \mathcal{M}l^-)(t_0, x_0) + \epsilon > (w - \mathcal{M}w)(t_0, x_0) \geq 0$$

to show that $l^- - \mathcal{M}l^- < 0$ is impossible.

Let us turn to case (i). Assume that we have both $-\varphi_t - \mathcal{A}\varphi - f < 0$ and $l^- - \mathcal{H}l^- < 0$ at (t_0, x_0). As in part 1.1, fix $\xi > 0$, by continuity we can find a small ball $B_\epsilon(t_0, x_0)$ for some $\epsilon > 0$ such that

$$\begin{aligned} &- \mathcal{A}\varphi - \varphi_t - f < 0 \quad \text{on } \overline{B_\epsilon(t_0, x_0)}, \\ &\varphi + \xi \leq l^- \qquad \text{on } \mathbb{T}_{\epsilon/2}(t_0, x_0). \end{aligned}$$

Moreover, since φ is continuous and $\mathcal{H}l^-$ is lower semi-continuous, for ϵ' small enough we have

$$\varphi + \epsilon' \leq \mathcal{H}l^- \quad \text{on } \overline{B_\epsilon(t_0, x_0)}.$$

Let $\xi > \xi' > \xi/2$. By the property (ii) in Lemma 2.3.1, there exists a sequence $w_n \in \mathcal{L}^-$ with $w_n \uparrow l^-$ as $n \to \infty$. Using Lemma 2.A.1 and Lemma 2.3.2, we can find n sufficiently large such that $\varphi \leq w_n - \xi'$ on $\mathbb{T}_{\epsilon/2}(t_0, x_0)$ and $w_n - l^- > -\xi'$ on $\overline{B_\epsilon(t_0, x_0)}$ as in part 1.1. Set $w := w_n$ and choose $0 < \mu < \xi' \wedge \epsilon'$ so that the function $\varphi^\mu := \varphi + \mu$ satisfies the properties

$$-\varphi_t^\mu - \mathcal{A}\varphi^\mu - f < 0 \text{ on } B_\epsilon(t_0, x_0) \tag{2.22}$$

$$\varphi^\mu < w \text{ on } \mathbb{T}_{\epsilon/2}(t_0, x_0) \tag{2.23}$$

$$\varphi^\mu < \mathcal{H}l^- \text{ on } B_\epsilon(t_0, x_0) \tag{2.24}$$

and

$$\varphi^\mu(t_0, x_0) = l^-(t_0, x_0) + \mu.$$

Now we define

$$w^\mu := \begin{cases} \varphi^\mu \vee w & \text{on } B_\epsilon(t_0, x_0) \\ w & \text{outside.} \end{cases}$$

Since $w^\mu(t_0, x_0) > l^-(t_0, x_0)$ we obtain a contradiction if we can show $w^\mu \in \mathcal{L}^-$. Then, fix s and let $\rho \in \mathcal{T}_s$. We need to construct an impulse strategy $\tilde{u} \in U_s^s$ satisfying the properties as in the definition of stochastic sub-solution of the LI for w^μ. We know already that w is a stochastic sub-solution of the LI equation and, as such, there exists an impulse strategy $\tilde{u}_1 \in U_\rho^s$ satisfying (2.9) from ρ onwards. Then we formulate $\tilde{u}$ as follows

1. if $\varphi^\mu > w$ at ρ, play the no impulse strategy $\bar{u} \equiv 0$;

2. if $\varphi^\mu \le w$ at ρ, follow the strategy $\tilde{u}_1$;

3. Follow 1-2 until $(\rho_1 \wedge \eta_1^v)(y)$ where $\rho_1(y) := \inf\{t \in [\rho(y), T] : (t, y(t)) \in \partial B_{\epsilon/2}(t_0, x_0)\}$ (with the convention $\inf \emptyset = +\infty$) and $\eta_1^v(y)$ is the first stopping rule according to $v = (\eta_n, \gamma_n)_n \in V_s^s$ such that $(\eta, y(\eta)) \in B_{\epsilon/2}(t_0, x_0)^c$. Here we know that $w^\mu = w$ by construction (2.23);

4. After $(\rho_1 \wedge \eta_1^v)(y)$, follow $\tilde{u}_3 \in U^s_{\rho_1 \wedge \eta_1^v}$ such that the stochastic sub-solution w satisfies (2.9) from $(\rho_1 \wedge \eta_1^v)(y)$.

Let's write formally the strategy doing 1-2 above as $\tilde{u}_2 \in U_\rho^s$ by

$$\tilde{u}_2(y) = \widehat{u}(y) \mathbb{1}_{\{\varphi^\mu(\rho(y), y(\rho(y))) > w(\rho(y), y(\rho(y)))\}} + \tilde{u}_1(y) \mathbb{1}_{\{\varphi^\mu(\rho(y), y(\rho(y))) \le w(\rho(y), y(\rho(y)))\}}.$$

Now, to complete the definition of $\tilde{u} \in U_\rho^s$ it is left to concatenate $\tilde{u}_2 \in U_\rho^s$ with $\tilde{u}_3 \in U^s_{\rho_1(y) \wedge \eta_1^v(y)}$ as follows

$$\tilde{u} := \tilde{u}_2 \otimes_{\rho_1} \tilde{u}_3 \in U_\rho^s.$$

At this point we are ready to use $\tilde{u}$ to show that w^μ satisfies (2.9).

Hence, let us fix $u = (\tau_n, \delta_n)_{n \ge 1} \in U_s^s$, $v = (\eta_n, \gamma_n)_{n \ge 1} \in V_s^s$, $x \in \mathbb{R}^d$ and $\zeta \in \mathcal{T}_\rho$. Denote by $X := X^{s,x;u \otimes_\rho \tilde{u}, v, -}$, where $\tilde{u}$ was just defined above, while $\rho := \rho(X)$ and $\zeta := \zeta(X)$. Let also set $\rho_1 := \rho_1(X)$ and $\eta_1^v := \eta_1^v(X)$(note $\rho_1 \wedge \eta_1^u \ge \rho$ by definition) and define the event $A := \{\varphi^\mu(\rho, X_\rho) > w(\rho, X_\rho)\} \in \mathcal{F}_\rho^s$. First, we observe that

$$X_t^{s,x;u \otimes_\rho \tilde{u}_2, v, -} = X_t^{s,x;u \otimes_\rho \widehat{u}, v, -} \mathbb{1}_A + X_t^{s,x;u \otimes_\rho \tilde{u}_1, v, -} \mathbb{1}_{A^c} \quad \text{on } \{\rho \le t \le \rho_1 \wedge \eta_1^v\}.$$

Then, note that on the event A we have $w^\mu(\rho, X_\rho) = \varphi^\mu(\rho, X_\rho)$ whereas $w^\mu(\rho, X_\rho) = w(\rho, X_\rho)$ on A^c, which means we only need to show that (2.9) is satisfied on A since we know it is satisfied on A^c by definition of stochastic sub-solution, $w \in \mathcal{L}^-$. Hence, we apply Itô's formula on A from ρ to $\eta_1^v \wedge \rho_1$ and take conditional expectation to get

$$\begin{aligned}
&w^\mu(\rho, X_\rho) = \varphi^\mu(\rho, X_\rho) \\
&= \mathbb{E}\left[\left(w^\mu(\rho_1 \wedge \eta_1^v, X_{\rho_1 \wedge \eta_1^v}) - \int_\rho^{\rho_1 \wedge \eta_1^v} (\varphi_t + \mathcal{A}\varphi)(t, X_t) dt - \sum_{\rho \le s < \rho_1 \wedge \eta_1^v} \Delta \varphi^\mu(s, X_s)\right) \mathbb{1}_{\{\rho_1 \wedge \eta_1^v \le \zeta\}} \,|\, \mathcal{F}_\rho^s\right] \\
&\quad + \mathbb{E}\left[\left(\varphi^\mu(\zeta, X_\zeta) - \int_\rho^\zeta (\varphi_t + \mathcal{A}\varphi)(t, X_t) dt - \sum_{\rho \le s < \zeta} \Delta \varphi^\mu(s, X_s)\right) \mathbb{1}_{\{\zeta < \eta_1^v \wedge \rho_1\}} \,|\, \mathcal{F}_\rho^s\right] \\
&= \text{(I)} + \text{(II)}.
\end{aligned}$$

We consider the two summands on the RHS above separately:

$$
\begin{aligned}
(\mathrm{I}) \leq & \mathbb{E}\left[\left(w^\mu(\eta_1^v\wedge\rho_1, X_{\eta_1^v\wedge\rho_1})+\int_\rho^{\eta_1^v\wedge\rho_1} f(t,X_t)dt-\sum_{n:\rho\leq\eta_n<\eta_1^v\wedge\rho_1}\Delta\varphi^\mu(\eta_n,X_{\eta_n})\right)\mathbb{1}_{\{\eta_1^v\wedge\rho_1\leq\zeta\}}\,|\,\mathcal{F}_\rho^s\right]\\
\leq & \mathbb{E}\left[\left(w^\mu(\eta_1^v\wedge\rho_1, X_{\eta_1^v\wedge\rho_1})+\int_\rho^{\eta_1^v\wedge\rho_1} f(t,X_t)dt-\sum_{n:\rho\leq\tau_n<\eta_1^v\wedge\rho_1}\phi(X_{\tau_n-},\delta_n)\prod_{l\geq1}\mathbb{1}_{\{\eta_l\neq\tau_n\}}\right.\right.\\
&\left.\left.+\sum_{n:\rho\leq\eta_n<\eta_1^v\wedge\rho_1}\psi(X_{\eta_n-},\gamma_n)\right)\mathbb{1}_{\{\eta_1^v\wedge\rho_1\leq\zeta\}}\,|\,\mathcal{F}_\rho^s\right]
\end{aligned}
$$

where the first inequality follows from (2.22) and $\widehat{u}\equiv0$ (jumps occur only at η_ns), while the second one is due to the following two arguments. First, since there are no interventions coming from P1, the related costs vanish, i.e. $\sum\phi=0$. Second, due to (2.24), similarly to part 1.1 we have

$$
\begin{aligned}
\varphi^\mu(\eta_n-,X_{\eta_n-}) &< \mathcal{H}l^-(\eta_n-,X_{\eta_n-})\leq l^-(\eta_n,X_{\eta_n-}+\gamma_n)+\psi(X_{\eta_n-},\gamma_n)\\
&< w(\eta_n,X_{\eta_n-}+\gamma_n)+\psi(X_{\eta_n-},\gamma_n)+\xi'\\
&< \varphi^\mu(\eta_n,X_{\eta_n-}+\gamma_n)+\psi(X_{\eta_n-},\gamma_n)+\xi'
\end{aligned}
$$

and since $\xi>\xi'$ is arbitrary it follows

$$
\varphi^\mu(\eta_n,X_{\eta_n-}+\gamma_n)-\varphi^\mu(\eta_n-,X_{\eta_n-})\geq-\psi(X_{\eta_n-},\gamma_n).
$$

Now, for the other summand (II), observe that $w=\varphi^\mu$ over $[\rho,\zeta)$ along (s,X_s) so that

$$
\begin{aligned}
(\mathrm{II}) \leq & \mathbb{E}\left[\left(\varphi^\mu(\zeta,X_\zeta)+\int_\rho^\zeta f(t,X_t)dt-\sum_{n:\rho\leq\tau_n<\zeta}\phi(X_{\tau_n-},\delta_n)\prod_{l\geq1}\mathbb{1}_{\{\eta_l\neq\tau_n\}}\right.\right.\\
&\left.\left.+\sum_{n:\rho\leq\eta_n<\zeta}\psi(X_{\eta_n-},\gamma_n)\right)\mathbb{1}_{\{\zeta<\rho_1\wedge\eta_1^u\}}\,|\,\mathcal{F}_\rho^s\right].
\end{aligned}
$$

As mentioned in step 4 of the construction of the strategy $\tilde{u}\in U_\rho^s$, by definition of stochastic super-solution, $\tilde{u}_3\in U^s_{\rho_1\wedge\eta_1^v}$ provides (2.9) concatenated with any previous strategy u and against any P2 strategy v so that, from $\rho_1\wedge\eta_1^v$, we have

$$
\begin{aligned}
w^\mu(\rho_1\wedge\eta_1^v,X_{\rho_1\wedge\eta_1^v}) &= w(\rho_1\wedge\eta_1^v,X_{\rho_1\wedge\eta_1^v})\\
&\leq \mathbb{E}\left[\int_{\rho_1\wedge\eta_1^v}^\zeta f(t,X_t)dt-\sum_{n:\rho_1\wedge\eta_1^v\leq\tilde{\tau}_n<\zeta}\phi(X_{\tilde{\tau}_n-},\tilde{\delta}_n)\prod_{l\geq1}\mathbb{1}_{\{\eta_l\neq\tilde{\tau}_n\}}\right.\\
&\qquad\left.+\sum_{n:\rho_1\wedge\eta_1^v\leq\eta_n<\zeta}\psi(X_{\eta_n-},\gamma_n)+w(\zeta,X_\zeta)\,|\,\mathcal{F}^s_{\rho_1\wedge\eta_1^v}\right]\\
&\leq \mathbb{E}\left[\int_{\rho_1\wedge\eta_1^v}^\zeta f(t,X_t)dt-\sum_{n:\rho_1\wedge\eta_1^v\leq\tilde{\tau}_n<\zeta}\phi(X_{\tilde{\tau}_n-},\tilde{\delta}_n)\prod_{l\geq1}\mathbb{1}_{\{\eta_l\neq\tilde{\tau}_n\}}\right.\\
&\qquad\left.+\sum_{n:\rho_1\wedge\eta_1^v\leq\eta_n<\zeta}\psi(X_{\eta_n-},\gamma_n)+w^\mu(\zeta,X_\zeta)\,|\,\mathcal{F}^s_{\rho_1\wedge\eta_1^v}\right].
\end{aligned}
$$

Then, by the property of iterated conditional expectations we obtain $\mathbb{P}$-a.s.

$$w^\mu(\rho, X_\rho) \leq \mathbb{E}\left[\int_\rho^\zeta f(t, X_t)dt - \sum_{n:\rho\leq\tilde{\tau}_n<\zeta} \phi(X_{\tilde{\tau}_n-}, \tilde{\delta}_n) \prod_{l\geq 1} \mathbb{1}_{\{\eta_l\neq\tilde{\tau}_n\}} \right.$$
$$\left. + \sum_{n:\rho\leq\eta_n<\zeta} \psi(X_{\eta_n-}, \gamma_n) + w^\mu(\zeta, X_\zeta) \mid \mathcal{F}^s_\rho \right].$$

Finally, it is left to be shown that

$$(w^\mu - \mathcal{M}w^\mu)(t,x) \geq 0 \quad \forall\, (t,x) \in [s,T] \times \mathbb{R}^d,$$

which can be proven as in part 1.1.

2.2. The terminal condition property for l^-. First we argue by contradiction, similar to the analogous step in 1.2, and then we construct a strategy $\tilde{u} = (\tilde{\tau}_n, \tilde{\delta}_n)$ as in 2.1 above depending on the fixed ρ. Then, alike part 2.1, we apply Itô's formula and conditioning to finish the proof. □

2.4 Verification by comparison

The last step needed to prove that the game has a value is the verification by comparison, so that the infimum of stochastic super-solutions of, respectively, the LI and UI (i.e. l^+/u^+) is equal to the supremum of stochastic sub-solutions of the LI and UI (i.e. l^-/u^-). The verification consists in proving that $u^- \geq u^+$ which, by definition, implies $u^- = V^+ = u^+$ (analogously we get $l^- = V^- = l^+$). The auxiliary Lemma below is an adaptation from the optimal control case, see Ishii [51] Lemma 3.3 and Seydel [73] Lemma 5.8.

LEMMA 2.4.1 Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be an upper semi-continuous viscosity sub-solution of (2.13)-(2.14). Let $s \in [0,T]$ and assume that there exist $w \in C^{1,2}([s,T] \times \mathbb{R}^d)$ and a positive function $k : [s,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$(\mathcal{A}w + w_t + f)(t,x) \geq k(t,x) \qquad (t,x) \in [s,T) \times \mathbb{R}^d$$
$$\max\left\{(w - \mathcal{H}w)(t,x), w(t,x) - \inf_{\delta\in\Delta} [w(t, x+\delta) - \phi(x,\delta)]\right\} \leq -k(t,x) \quad (t,x) \in [s,T] \times \mathbb{R}^d$$
$$w(T,x) - g(x) \leq -k(T,x) \qquad x \in \mathbb{R}^d.$$

Then, for each $m \in \mathbb{N}$, $u_m(t,x) = \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m}$ is a viscosity sub-solution of

$$\begin{aligned}
&\max\left\{\min\left\{(-\mathcal{A}\varphi - \varphi_t - f)(t,x), (u_m - \mathcal{M}u_m)(t,x)\right\},\right.\\
&\qquad\qquad \left.(u_m - \mathcal{H}u_m)(t,x)\right\} + \frac{k(t,x)}{m} = 0 \quad (t,x) \in [s,T) \times \mathbb{R}^d,\\
&\max\left\{\min\left\{u_m(T,x) - g(x), (u_m - \mathcal{M}u_m)(T,x)\right\},\right.\\
&\qquad\qquad \left.(u_m - \mathcal{H}u_m)(T,x)\right\} + \frac{k(T,x)}{m} = 0 \qquad x \in \mathbb{R}^d.
\end{aligned} \tag{2.25}$$

PROOF 2.4.1 For any $\varphi \in C^{1,2}([s,T]\times\mathbb{R}^d)$ we suppose $u_m - \varphi$ attains a local maximum at $(t_0, x_0) \in [s,T]\times\mathbb{R}^d$. Then, we have

$$
\begin{aligned}
(u_m - \varphi)(t,x) &= \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m} - \varphi(t,x) \\
&= \left(1 - \frac{1}{m}\right)\left[u(t,x) + \frac{w(t,x)}{m-1} - \frac{m}{m-1}\varphi(t,x)\right]
\end{aligned}
$$

and we obtain that $u(t,x) - \left(\frac{m}{m-1}\varphi(t,x) - \frac{w(t,x)}{m-1}\right)$ attains a local maximum at (t_0, x_0). Hence, using the fact that u is a viscosity sub-solution of (2.13) we get

$$
\max\left\{\min\left\{\frac{m}{m-1}(-\mathcal{A}\varphi - \varphi_t) - f + \frac{\mathcal{A}w + w_t}{m-1}, u - \mathcal{M}u\right\}, u - \mathcal{H}u\right\} \le 0 \qquad \text{in } (t_0, x_0).
$$

Thus, we know that

$$
\begin{aligned}
&\min\left\{\frac{m}{m-1}(-\mathcal{A}\varphi - \varphi_t) - f + \frac{\mathcal{A}w + w_t}{m-1}, u - \mathcal{M}u\right\} \le 0 \\
&u - \mathcal{H}u \le 0
\end{aligned}
$$

hold. So, when

$$
\frac{m}{m-1}(-\mathcal{A}\varphi - \varphi_t) - f + \frac{\mathcal{A}w + w_t}{m-1} \le 0,
$$

via multiplying by $(1 - 1/m)$ we get

$$
-\mathcal{A}\varphi - \varphi_t - f + \frac{k}{m} \le -\mathcal{A}\varphi - \varphi_t - f + \frac{f + \mathcal{A}w + w_t}{m} \le 0,
$$

from which we obtain

$$
-\mathcal{A}\varphi - \varphi_t - f \le -\frac{k}{m}.
$$

When $u - \mathcal{M}u \le 0$ we have the following:

$$
\begin{aligned}
&(u_m - \mathcal{M}u_m)(t,x) \\
&= \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m} - \sup_{\delta}\left[\left(1 - \frac{1}{m}\right) u(t,x+\delta) + \frac{w(t,x+\delta)}{m} - \phi(x,\delta)\right] \\
&= \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m} - \sup_{\delta}\left[\left(1 - \frac{1}{m}\right)(u(t,x+\delta) - \phi(x,\delta)) \right.\\
&\quad \left. + \frac{w(t,x+\delta) - \phi(x,\delta)}{m}\right] \\
&\le \left(1 - \frac{1}{m}\right)(u - \mathcal{M}u)(t_0, x_0) + \frac{w(t,x) - w(t,x+\delta^*) + \phi(x,\delta^*)}{m} \\
&\le \frac{w(t,x) - \inf_{\delta}[w(t,x+\delta) - \phi(x,\delta)]}{m} \le -\frac{k}{m} < 0,
\end{aligned}
$$

where δ^* is such that $\sup_{\delta}[u(t,x+\delta) - \phi(x,\delta)] = u(t,x+\delta^*) - \phi(x,\delta^*)$.

Finally, in case $u - \mathcal{H}u \le 0$ we have

$$
\begin{aligned}
&(u_m - \mathcal{H}u_m)(t,x) \\
&= \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m} - \inf_\gamma \left[\left(1 - \frac{1}{m}\right) u(t,x+\gamma) + \frac{w(t,x+\gamma)}{m} + \psi(x,\gamma)\right] \\
&= \left(1 - \frac{1}{m}\right) u(t,x) + \frac{w(t,x)}{m} - \inf_\gamma \left[\left(1 - \frac{1}{m}\right) (u(t,x+\gamma) + \psi(x,\gamma)) \right. \\
&\quad \left. + \frac{\psi(x,\gamma) + w(t,x+\gamma)}{m}\right] \\
&\le \left(1 - \frac{1}{m}\right) (u - \mathcal{H}u)(t_0,x_0) + \frac{(w - \mathcal{H}w)(t,x)}{m} \le -\frac{k}{m} < 0.
\end{aligned}
$$

Regarding the terminal condition, we know

$$\min\{u - g, u - \mathcal{M}u\} \le 0, \qquad u - \mathcal{H}u \le 0.$$

Hence, we only need to check $u_m - g + k/m \le 0$ when $u - g \le 0$:

$$
\begin{aligned}
\left(1 - \frac{1}{m}\right) u(T,x) + \frac{w(T,x)}{m} - g(x) &= \left(1 - \frac{1}{m}\right) (u(T,x) - g(x)) + \frac{w(T,x) - g(x)}{m} \\
&\le -\frac{k}{m} < 0.
\end{aligned}
$$

□

Before we proceed with the comparison theorem we provide the definition of viscosity solution to the HJBI (2.3)-(2.4) by means of jets, as it is needed in the proof. By $\mathcal{S}(d)$ we denote the set of symmetric matrices of dimension d.

DEFINITION 2.4.1 (FROM SECTION 8 [39]) Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a lower semi-continuous function, then we denote by $J^{2,-}v(t,x)$ the parabolic sub-jet of v at $(t,x) \in [0,T) \times \mathbb{R}^d$ as the set of triples $(p,q,X) \in \mathbb{R}^{d+1} \times \mathcal{S}(d)$ such that

$$v(s,y) \ge v(t,x) + p(s-t) + \langle q, (y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2),$$

as $s \to t$ ($s \to t+$, when $t = 0$) and $y \to x$. We also introduce the parabolic limiting sub-jet of v at $(t,x) \in [0,T) \times \mathbb{R}^d$:

$$
\begin{aligned}
\bar{J}^{2,-}v(t,x) = \big\{(p,q,X) \in \mathbb{R}^{d+1} \times \mathcal{S}(n) : \exists\, (t_n, x_n, p_n, q_n, X_n) \in [0,T) \times \mathbb{R}^{2d+1} \times \mathcal{S}(n) \\
\text{such that } (p_n, q_n, X_n) \in J^{2,-}v(t_n,x_n) \text{ and} \\
(t_n, x_n, v(t_n,x_n), p_n, q_n, X_n) \to (t, x, v(t,x), p, q, X)\big\}.
\end{aligned}
$$

When v is an upper semi-continuous function on $[0,T] \times \mathbb{R}^d$, we similarly define the parabolic super-jet $J^{2,+}v(t,x)$ and the parabolic limiting super-jet $\bar{J}^{2,+}v(t,x)$ of v at $(t,x) \in [0,T] \times \mathbb{R}^d$ by

$$J^{2,+}v(t,x) = -J^{2,}(-v)(t,x), \qquad \bar{J}^{2,+}v(t,x) = -\bar{J}^{2,}(-v)(t,x).$$

Then, we have the following result from [38]

LEMMA 2.4.2 Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a lower (resp. upper) semi-continuous function. Then v is a viscosity super-solution (resp. sub-solution) to the Lower Isaacs (2.4) if and only if

- for every $(t,x) \in [0,T) \times \mathbb{R}^d$ and $(p,q,X) \in \bar{J}^{2,-}v(t,x)$ (resp. $\bar{J}^{2,+}v(t,x)$) we have

$$\max\left\{\min\left\{p - \langle b(t,x), q\rangle - \frac{1}{2}tr[(\sigma^\top\sigma)(t,x)X] - f(t,x),\right.\right.$$
$$\left.\left.(v - \mathcal{M}v)(t,x)\right\}, (v - \mathcal{H}v)(t,x)\right\} \geq 0 \quad (\leq 0).$$

- for every $x \in \mathbb{R}^d$ we have

$$\max\left\{\min\left\{v(T,x) - g(x), (v - \mathcal{M}v)(T,x)\right\}, (v - \mathcal{H}v)(T,x)\right\} \geq 0 \quad (\leq 0).$$

An analogous statement hold for the UI equation (2.3).

THEOREM 2.4.1 Let u and v be a bounded viscosity sub-solution and a bounded viscosity super-solution to the LI equation (2.4) respectively and w be as in Lemma 2.4.1. Suppose all the assumptions in the lemma hold, then, $u \leq v$ on $[0,T] \times \mathbb{R}^d$.

REMARKS 2.4.1 An analogous result can be proven for the UI case.

PROOF 2.4.2 Let k be as in Lemma 2.4.1 and u_m be a viscosity sub-solution of the perturbed PDE (2.25). Here we will prove the inequality $u_m \leq v$ for any $m \geq 1$ as we obtain the desired result by letting $m \to \infty$ since $u = \lim_{m\to\infty} u_m$. To the contrary, we suppose $\max_{[s,T]\times\mathbb{R}^d} u_m - v > 0$ for some $m \geq 1$ and shall get a contradiction. Then, there exists $(\widehat{t}, \widehat{x}) \in [s,T] \times \mathbb{R}^d$ such that $(u_m - v)(\widehat{t}, \widehat{x}) = \theta > 0$.

Case 1: $\widehat{t} \in [s,T)$. Let $r > 0$, and introduce $\tilde{u}_m(t,x) := e^{rt}u_m(t,x)$ and $\tilde{v}(t,x) := e^{rt}v(t,x)$. Then, they are viscosity sub and super-solution to

$$\max\{\min\{rW - W_t - \mathcal{A}W - \tilde{f}, W - \tilde{\mathcal{M}}W\}, W - \tilde{\mathcal{H}}W\} + \frac{\tilde{k}}{m} = 0$$
$$\max\{\min\{rW - W_t - \mathcal{A}W - \tilde{f}, W - \tilde{\mathcal{M}}W\}, W - \tilde{\mathcal{H}}W\} = 0$$

respectively, with $\tilde{f} := e^{rt}f$, $\tilde{k} := e^{rt}k$ and

$$\tilde{\mathcal{M}}W(t,x) = \sup_{\delta\in\Delta}[W(t,x+\delta) - e^{rt}\phi(x,\delta)] \quad \tilde{\mathcal{H}}W(t,x) = \inf_{\gamma\in\Gamma}[W(t,x+\gamma) + e^{rt}\psi(x,\gamma)].$$

We note that the function $\tilde{u}_m(t,x) - (t-\widehat{t})^2 - \tilde{v}(t,x)$ takes the maximum $\tilde{\theta} := e^{rt}\theta$ and $(\widehat{t}, \widehat{x}) \in [s,T) \times \mathbb{R}^d$ is a unique maximum point. For each $\eta > 0$ we define the function Φ on $[s,T] \times \overline{B(\widehat{x}, R)}^2$ for some $R > 0$ by

$$\Phi(t,x,y) = \tilde{u}_m(t,x) - (t-\widehat{t})^2 - \frac{1}{2\eta}(x-y)^2 - \tilde{v}(t,y) \tag{2.26}$$

and let $(t_\eta, x_\eta, y_\eta) \in [s,T] \times \overline{B(\widehat{x}, R)}^2$ be a maximum point. We observe that the inequality $\Phi(\widehat{t}, \widehat{x}, \widehat{x}) \leq \Phi(t_\eta, x_\eta, y_\eta)$ implies

$$\begin{aligned} \tilde{u}_m(\widehat{t}, \widehat{x}) - \tilde{v}(\widehat{t}, \widehat{x}) &\leq \tilde{u}_m(\widehat{t}, \widehat{x}) - \tilde{v}(\widehat{t}, \widehat{x}) + \frac{1}{2\eta}(x_\eta - y_\eta)^2 \\ &\leq \tilde{u}_m(t_\eta, x_\eta) - (t_\eta - \widehat{t})^2 - \tilde{v}(t_\eta, y_\eta). \end{aligned} \tag{2.27}$$

Since $\tilde{u}_m$ and $\tilde{v}$ are bounded we have that $|x_\eta - y_\eta| \to 0$ as $\eta \to 0$. By compactness of $[s,T] \times \overline{B(\widehat{x}, R)}^2$ we see that $(t_{\eta_n}, x_{\eta_n}, y_{\eta_n}) \to (\bar{t}, \bar{x}, \bar{y}) \in [s,T] \times \overline{B(\widehat{x}_0, R)}^2$, as $n \to +\infty$, for a suitable sequence $(\eta_n)_n$ converging to zero. Using (2.27) together with the semi-continuity of $\tilde{u}_m$ and $\tilde{v}$ we get

$$\tilde{\theta} \leq \tilde{u}_m(\bar{t}, \bar{x}) - (\bar{t} - \widehat{t})^2 - \tilde{v}(\bar{t}, \bar{y}).$$

Hence, we get $(\bar{t}, \bar{x}) = (\widehat{t}, \widehat{x})$ and $(t_\eta, x_\eta, y_\eta) \to (\widehat{t}, \widehat{x}, \widehat{x})$ since $(\widehat{t}, \widehat{x})$ is the unique maximum point of (2.26). Moreover, we obtain

$$\begin{aligned} \tilde{u}_m(\widehat{t}, \widehat{x}) - \tilde{v}(\widehat{t}, \widehat{x}) &\leq \liminf_{\eta \to 0} \tilde{u}_m(t_\eta, x_\eta) - \limsup_{\eta \to 0} \tilde{v}(t_\eta, y_\eta) \\ &\leq \limsup_{\eta \to 0} \tilde{u}_m(t_\eta, x_\eta) - \liminf_{\eta \to 0} \tilde{v}(t_\eta, y_\eta) \leq \tilde{u}_m(\widehat{t}, \widehat{x}) - \tilde{v}(\widehat{t}, \widehat{x}). \end{aligned}$$

Thus, we have

$$\liminf_{\eta \to 0} \tilde{u}_m(t_\eta, x_\eta) - \limsup_{\eta \to 0} \tilde{v}(t_\eta, y_\eta) = \limsup_{\eta \to 0} \tilde{u}_m(t_\eta, x_\eta) - \liminf_{\eta \to 0} \tilde{v}(t_\eta, y_\eta)$$

which implies

$$0 \leq \limsup \tilde{u}_m - \liminf \tilde{u}_m \leq \liminf \tilde{v} - \limsup \tilde{v} \leq 0.$$

Therefore,

$$\lim_{\eta \to 0} \tilde{u}_m(t_\eta, x_\eta) = \tilde{u}_m(\widehat{t}, \widehat{x}) \quad \text{and} \quad \lim_{\eta \to 0} \tilde{v}(t_\eta, y_\eta) = \tilde{v}(\widehat{t}, \widehat{x}).$$

As $(t_\eta, x_\eta, y_\eta) \to (\widehat{t}, \widehat{x}, \widehat{x}) \in [s,T) \times B(\widehat{x}, R)^2$ we have that $(t_\eta, x_\eta, y_\eta) \in [s,T) \times B(\widehat{x}, R)^2$ for η small. Then, by Ishii's lemma [39] there exist $X_\eta, Y_\eta \in \mathcal{S}^d$ such that

$$\begin{aligned} &(p_u, \frac{1}{\eta}(x_\eta - y_\eta), X_\eta) \in \bar{J}^{2,+}(\tilde{u}_m(t_\eta, x_\eta)) \\ &(p_v, \frac{1}{\eta}(x_\eta - y_\eta), Y_\eta) \in \bar{J}^{2,-}(\tilde{v}(t_\eta, x_\eta)) \end{aligned}$$

satisfying

$$\begin{aligned} &p_u - p_v = 2(t_\eta - \widehat{t}) \\ &\begin{pmatrix} X_\eta & 0 \\ 0 & -Y_\eta \end{pmatrix} \leq \frac{3}{\eta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{aligned}$$

where I denotes the identity matrix. Then, by sub and super-solution properties of $\tilde{u}_m$ and $\tilde{v}$ we get

$$\max\left\{\min\left\{r\tilde{u}_m(t_\eta,x_\eta)-p_u-b(t_\eta,x_\eta)\frac{1}{\eta}(x_\eta-y_\eta)-\frac{1}{2}tr[(\sigma^\top\sigma)(t_\eta,x_\eta)X_\eta]-\tilde{f}(t_\eta,x_\eta),\right.\right.$$
$$\left.\left.(\tilde{u}_m-\tilde{\mathcal{M}}\tilde{u}_m)(t_\eta,x_\eta)\right\},(\tilde{u}_m-\tilde{\mathcal{H}}\tilde{u}_m)(t_\eta,x_\eta)\right\}\le-\frac{\tilde{k}}{m}$$
$$\max\left\{\min\left\{r\tilde{v}(t_\eta,y_\eta)-p_v-b(t_\eta,y_\eta)\frac{1}{\eta}(x_\eta-y_\eta)-\frac{1}{2}tr[(\sigma^\top\sigma)(t_\eta,y_\eta)Y_\eta]-\tilde{f}(t_\eta,y_\eta),\right.\right.$$
$$\left.\left.(\tilde{v}-\tilde{\mathcal{M}}\tilde{v})(t_\eta,y_\eta)\right\},(\tilde{v}-\tilde{\mathcal{H}}\tilde{v})(t_\eta,y_\eta)\right\}\ge0$$

We first solve the case in which $\min\{r\tilde{v}(t_\eta,y_\eta)-p_v-b(t_\eta,y_\eta)\frac{1}{\eta}(x_\eta-y_\eta)-\frac{1}{2}tr[(\sigma^\top\sigma)(t_\eta,y_\eta)Y_\eta]-\tilde{f}(t_\eta,y_\eta),(\tilde{v}-\tilde{\mathcal{M}}\tilde{v})(t_\eta,y_\eta)\}\ge0$. This implies either

$$r\tilde{u}_m(t_\eta,x_\eta)-p_u-b(t_\eta,x_\eta)\frac{1}{\eta}(x_\eta-y_\eta)-\frac{1}{2}tr[(\sigma^\top\sigma)(t_\eta,x_\eta)X_\eta]-\tilde{f}(t_\eta,x_\eta)+\frac{\tilde{k}}{m}$$
$$\le r\tilde{v}(t_\eta,y_\eta)-p_v-b(t_\eta,y_\eta)\frac{1}{\eta}(x_\eta-y_\eta)-\frac{1}{2}tr[(\sigma^\top\sigma)(t_\eta,y_\eta)Y_\eta]-\tilde{f}(t_\eta,y_\eta),$$

or

$$(\tilde{u}_m-\tilde{\mathcal{M}}\tilde{u}_m)(t_\eta,x_\eta)+\frac{\tilde{k}}{m}\le(\tilde{v}-\tilde{\mathcal{M}}\tilde{v})(t_\eta,y_\eta).$$

We begin with the first inequality

$$r(\tilde{u}_m(t_\eta,x_\eta)-\tilde{v}(t_\eta,y_\eta))\le2(t_\eta-\widehat{t})-\frac{1}{\eta}(x_\eta-y_\eta)(b(t_\eta,y_\eta)-b(t_\eta,x_\eta))+$$
$$+\frac{3}{2\eta}tr\left[(\sigma(t_\eta,x_\eta)-\sigma(t_\eta,y_\eta))^\top(\sigma(t_\eta,x_\eta)-\sigma(t_\eta,y_\eta))\right]+\tilde{f}(t_\eta,x_\eta)-\tilde{f}(t_\eta,y_\eta)-\frac{\tilde{k}}{m}$$
$$\le2(t_\eta-\widehat{t})+\frac{K}{\eta}(x_\eta-y_\eta)^2+\tilde{f}(t_\eta,x_\eta)-\tilde{f}(t_\eta,y_\eta)-\frac{\tilde{k}}{m}$$

where K denotes a positive constant depending only on the Lipschitz constants of b and σ. Letting $\eta\to0$ we obtain a contradiction.
Then, the second inequality implies

$$\tilde{u}_m(t_\eta,x_\eta)-\tilde{v}(t_\eta,y_\eta)\le\tilde{\mathcal{M}}\tilde{u}_m(t_\eta,x_\eta)-\tilde{\mathcal{M}}\tilde{v}(t_\eta,y_\eta)-\frac{\tilde{k}}{m}\le\tilde{u}_m(t_\eta,x_\eta+\delta^*)-\tilde{v}(t_\eta,y_\eta+\delta^*)-\frac{\tilde{k}}{m}$$

where δ^* is such that $\sup_\delta[\tilde{u}_m(t_\eta,x_\eta+\delta)-\phi(x_\eta,\delta)]=\tilde{u}_m(t_\eta,x_\eta+\delta^*)-\phi(x_\eta,\delta^*)$. Therefore, we get a contradiction by letting $\eta\to0$ as $(\widehat{t},\widehat{x})$ is the unique global maximum of $\tilde{u}_m-v$. Finally, when $(\tilde{v}-\tilde{\mathcal{H}}\tilde{v})(t_\eta,y_\eta)\ge0$ we have

$$(\tilde{u}_m-\tilde{\mathcal{H}}\tilde{u}_m)(t_\eta,x_\eta)+\frac{\tilde{k}}{m}\le(\tilde{v}-\tilde{\mathcal{H}}\tilde{v})(t_\eta,y_\eta)$$

which implies

$$\tilde{u}_m(t_\eta,x_\eta)-\tilde{v}_m(t_\eta,y_\eta)\le\tilde{\mathcal{H}}\tilde{u}_m(t_\eta,x_\eta)-\tilde{\mathcal{H}}\tilde{v}(t_\eta,y_\eta)-\frac{\tilde{k}}{m}\le\tilde{u}_m(t_\eta,x_\eta+\gamma^*)-\tilde{v}(t_\eta,y_\eta+\gamma^*)-\frac{\tilde{k}}{m}$$

where γ^* is such that $\inf_\gamma[\tilde{v}(t_\eta, y_\eta+\gamma)+\psi(y_\eta,\gamma)] = \tilde{v}(t_\eta, y_\eta+\gamma^*)+\psi(y_\eta,\gamma^*)$. Similarly to the case above, we get a contradiction by letting $\eta \to 0$.

Case 2: $\widehat{t} = T$. Here we have

$$\max\{\min\{(u_m - g)(T,\widehat{x}), (u_m - \mathcal{M}u_m)(T,\widehat{x})\}, (u_m - \mathcal{H}u_m)(T,\widehat{x})\} + \frac{k}{m} \leq 0$$
$$\max\{\min\{(v - g)(T,\widehat{x}), (v - \mathcal{M}v)(T,\widehat{x})\}, (v - \mathcal{H}v)(T,\widehat{x})\} \geq 0.$$

To conclude the proof we will show the contradiction when $\min\{v-g, v-\mathcal{M}v\} = v-g \geq 0$ as the other instances can be derived as in Case 1. Here we have

$$(v-g)(T,\widehat{x}) \geq (u_m - g)(T,\widehat{x}) + \frac{k}{m}$$

which implies

$$(u_m - v)(T,\widehat{x}) + \frac{k}{m} \leq 0 < \theta = (u_m - v)(T,\widehat{x}) = \max_{x\in\mathbb{R}} \left[u_m(T,x) - v(T,x)\right].$$

Therefore, we have shown that $\max_{(t,x)\in[0,T]\times\mathbb{R}^d}(u_m - v)(t,x) \leq 0$ for all $m \geq 1$. At this point we only need to let $m \to \infty$ to complete the proof. □

We summarise all our founding in the following theorem:

THEOREM 2.4.2 Under Assumptions 2.2.1-2.2.2 and assumptions in Lemma 2.4.1 we have that $V^-(t,x) = l^+(t,x) = l^-(t,x)$ is the unique continuous viscosity solution of the Lower Isaacs equation (2.4). Moreover, the lower value function V^-, of the game where P2 has priority, satisfies the DPP:

$$V^-(s,x) = \sup_{u\in U_s^s}\inf_{v\in V_s^s} \mathbb{E}\left[\int_0^\rho f\left(t, X_\cdot^{s,x,u,v,-}\right)dt - \sum_{n:s\leq\tau_n<\rho}\phi\left(X_{\tau_n-}^{s,x,u,v,-},\delta_n\right)\prod_{l\geq 1}\mathbb{1}_{\{\tau_n\neq\eta_l\}}\right.$$
$$\left. + \sum_{n:s\leq\eta_n<\rho}\psi\left(X_{\eta_n-}^{s,x,u,v,-},\gamma_n\right) + V^-\left(\rho(X_\cdot^{s,x,u,v,-}), X_{\rho(X_\cdot^{s,x,u,v,-})}^{s,x,u,v,-}\right)\right] \quad \forall\, \rho \in \mathcal{T}_s.$$

Similarly, we have that $V^+(t,x) = u^+(t,x) = u^-(t,x)$ is the unique continuous viscosity solution of the Upper Isaacs equation (2.3) and it satisfies the DPP:

$$V^+(s,x) = \inf_{v\in V_s^s}\sup_{u\in U_s^s} \mathbb{E}\left[\int_0^\rho f\left(t, X_\cdot^{s,x,u,v,+}\right)dt - \sum_{n:s\leq\tau_n<\rho}\phi\left(X_{\tau_n-}^{s,x,u,v,+},\delta_n\right)\right.$$
$$\left. + \sum_{n:s\leq\eta_n<\rho}\psi\left(X_{\eta_n-}^{s,x,u,v,+},\gamma_n\right)\prod_{l\geq 1}\mathbb{1}_{\{\tau_l\neq\eta_n\}} + V^+\left(\rho(X_\cdot^{s,x,u,v,+}), X_{\rho(X_\cdot^{s,x,u,v,+})}^{s,x,u,v,+}\right)\right] \quad \forall\, \rho \in \mathcal{T}_s.$$

If the Isaacs condition (2.5) holds, then the game has a value regardless who has priority of intervention and the value is

$$l^- = V^- = V = V^+ = u^+.$$

PROOF 2.4.3 The DPP is due to the way u^+, u^-, l^+ and l^- are constructed since the stochastic sub/super-solutions of the Lower/Upper Isaacs satisfy the corresponding half DPP by definition. Apply Theorem 2.4.1 to $v = l^-$ and $u = l^+$ so that $l^- \geq l^+$. Then, since $l^- \leq l^+$ by construction (2.12), it follows that $l^- = l^+$. Similarly we get $u^- = u^+$.□

2.A Appendix of Chapter 2

The following propositions collect, for reader's convenience, two auxiliary results from Bayraktar and Sîrbu [15, Proposition 4.1 and Lemma 4.1], which have been used in the proofs of Section 2.3.

PROPOSITION 2.A.1 Assume that (M, d) is a separable metric space (or less, a topological space with a countable base) and $\mathcal{G}$ is a class of functions $f : M \to \mathbb{R} \cup \{\pm\infty\}$. Assume also that each function in the class $\mathcal{G}$ is upper semi-continuous. Then, there exists a countable subclass $\mathcal{H} \subset \mathcal{G}$ such that

$$f_*(x) := \inf_{f \in \mathcal{G}} f(x) = \inf_{f \in \mathcal{H}} f(x), \qquad \text{for each } x \in M.$$

Moreover, let $g : M \to \mathbb{R} \cup \{\pm\infty\}$. Then, the following conditions are equivalent:

(i) $g(x) = \inf_{f \in \mathcal{G}} f(x)$, for each $x \in M$;

(ii) $\{x \in M \mid g(x) < q\} = \cup_{f \in \mathcal{G}} \{x \in M \mid f(x) < q\}$, for each $q \in \mathbb{Q}$.

The next lemma is the result of a modification of Bayraktar and Sîrbu [16, Lemma 2.4] and [15, Lemma 4.1].

LEMMA 2.A.1 Let $0 < \xi' < \xi$ and $\epsilon > 0$. Moreover, let $\varphi \in C^{1,2}([s, T] \times \mathbb{R}^d)$ and let (t_0, x_0) be a local minimum of $\varphi - u^+$ such that

$$\varphi - \xi \geq u^+ \text{ on } \mathbb{T}_{\epsilon/2}(t_0, x_0).$$

Then there exists a stochastic super-solution $w \in \mathcal{U}^+$ such that

$$\varphi - \xi' \geq w \text{ on } \mathbb{T}_{\epsilon/2}(t_0, x_0).$$

PROOF 2.A.1 Using Lemma 2.3.1 and Proposition 2.A.1 we can choose a decreasing sequence $(w_n)_{n \geq 1} \subset \mathcal{U}^+$ of stochastic super-solutions such that $w_n \downarrow u^+$. We denote by

$$A_n := \{w_n \geq \varphi - \xi'\} \cap \left(\mathbb{T}_{\epsilon/2}(t_0, x_0)\right).$$

We have that $A_{n+1} \subseteq A_n$ and $\cap_{n=0}^{\infty} A_n = \emptyset$ since $(w_n)_n$ is monotonically decreasing converging pointwise to u^+ and $u^+(t_0, x_0) = \varphi(t_0, x_0)$. In addition, since both w_n and φ are continuous, each A_n is closed. By compactness, we get that there exists an $n_0 \geq 1$ such that $A_{n_0} = \emptyset$, which means that

$$\varphi - \xi' > w_{n_0}, \text{ on } \overline{B_\epsilon(t_0, x_0)} \setminus \{t_0, x_0\}.$$

We now choose $w := w_{n_0}$. □

Chapter 3

Competition in dealer markets: an impulse game approach

3.1 Introduction

The scope of this work is to study the strategical interaction between two dealers trading to maximise their profits over a finite time horizon. In particular, the competition happens via trade execution, meaning that, when they place an order, they can't just take into account their own market impact as the one generated by their competitor has to be considered as well. Optimal trading for singular agents has been widely studied since the pioneering works by Bertsimas and Lo [23] and Almgren and Chriss [5] for the discrete time case and Almgren [4] in continuous time. Then, many authors have built on top of their models with various settings: discrete time [2, 14, 66, 69], continuous time impulse controls [18, 20, 25, 59], continuous time trading rate control [19, 33] and continuous time trading rate control with impulses [46, 66], the list of references is not exhaustive. The extensions to situations with several competing traders have been researched since Brunner and Pedersen's paper [28]. In particular, Bank et al. [7] studied a liquidity model analysing the interactions between dealers, their clients and an end-users market, Schied and Zhang [72] considered the case when N players try to optimally execute their trades, Carlin et al. [32] worked on cooperative equilibria and when they break, Moallemi [64] looked at asymmetry of information, and the list goes on. Moreover, there are many papers on applications of mean field games, see for instance Cardaliaguet and Lehalle [31] on crowd trading with impulse controls and Jaimungal et al. [49] on optimal execution, among others.

An important modelling choice when dealing with optimal trading regards market impact, namely how the price process is affected by order sizes. We can divide the market impact models available in two generations: according to the first [4, 5, 23], market impact has two components, one permanent, affecting the price of all current and future trades equally, and one temporary, affecting only the price of the trade that triggered it. The second generation follows recent research in market microstructure, which has showed that market impact is rather transient, see Bouchaud [26] and Taranto et al. [76, 77], meaning that the effect of each trade on the price is temporary but long lasting, the price process has long-memory. Hence, we will consider a model with transient price impact in line with Obizhaeva and Wang [66], Gatheral et al. [46] and others.

In practice, dealers trade at discrete times, according to their strategies, facing some fixed and proportional costs, depending on the exchange they trade in, the order size and the liquidity in the market. Given the context, it seems natural to opt for impulse controls when choosing how to model players' trades as they are sequences of intervention times and impulses [18, 20, 25, 31, 59]. In particular, in our game dealers trade placing Market Orders, so that at each trading time they will send an order to buy or sell a certain number of shares, the impulse, causing a market impact and hence, a cost proportional to its size, as it is riding the limit order book to be fulfilled. Other ways proportional and fixed costs have been studied are stochastic control with viscosity solutions, dual approach based on shadow prices in a frictionless market and asymptotics for vanishing costs, see [52, 71] and references therein.

One challenge of stochastic differential games with impulse controls is to manage the case when players want to apply impulses at the same time. Indeed, in all existing works there are only results in case one player has priority over the other, see [1, 9, 10, 38, 63]. Here we allow players to intervene simultaneously, to do so we provide a new system of quasi-variational inequalities (QVIs) where at each trading time a static game is played. This allows for a more realistic model, breaking one of the existing limits to the analysis of stochastic differential games with impulse controls.

The chapter is organised as follows. In Section 1 we formally define the game and discuss the property of the model. In Section 2 we provide a QVIs system and related verification theorem to obtain equilibria where only one dealer trades at a time and we show that we cannot find any with this approach. In Section 3, after introducing a new system of QVIs and related verification theorem to allow for simultaneous interventions, we prove that there exists at least one equilibrium. In Section 4 we discuss interesting areas for future research.

3.2 Game Setting

The Game. We consider a game in which two players, two dealer firms, compete over a fixed time period $[0, T]$. The two dealers start the game with amount of shares x and y respectively and want to maximise their revenues by time T. Both players are able to observe the information flow, which we model by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, satisfying the usual conditions, on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We model trading using impulse controls, namely dealers place their order at some trading times τ^i, $i = 1, 2$, at which they trade the amount $\delta^i \in \mathbb{R}$, hence we will denote as $u_i := (\tau_n^i, \delta_n^i)_{n\geq 1}$ the dealers' trading strategies.

We assume that the unaffected price process S^0 evolves according to a Brownian motion à la Bachelier (as in most of the literature on optimal execution/liquidation, see [31, 33, 46]):

$$dS_t^0 = b(t)dt + \sigma dW_t, \quad S_0^0 = s, \tag{3.1}$$

for some initial value $s \in \mathbb{R}^+$, where b is deterministic and continuous and $\sigma \geq 0$. One drawback of such dynamics is that it would allow the price to be negative, regardless of dealers' trades. This is justified in the literature as it does not happen for a reasonable parameters choice and it would occur with negligible probability. Moreover, this model is usually considered in short time horizon execution problems so that it is less likely that the price will go negative, see Almgren and Chriss [5], Almgren [4] and Gatheral et al [46].

Hence, the Dealers' holdings are

$$X_t = x - \sum_{n:\tau_n^1 \leq t} \delta_n^1, \qquad Y_t = y - \sum_{n:\tau_n^2 \leq t} \delta_n^2,$$

with $\delta_n^i > 0$ a sell order and $\delta_n^i < 0$ a buy order.

Each trade creates some transient market impact and the affected price process in the trading-in-continuous-time literature is usually of the form

$$S_t = S_t^0 + \int_0^t G(t-s)dX_s,$$

or similarly the Gatheral model [45]

$$S_t = S_0 + \int_0^t f(v_s)G(t-s)ds + \int_0^t \sigma dW_s,$$

where v_t is the rate of trading, for a suitable decay kernel G. The most popular kernel choices are $G(t) = t^{-\gamma}$ for $0 < \gamma < 1$ and $G(t) = e^{-\rho t}$ with $\rho > 0$, see Obizhaeva and Wang, Gatheral et al. [46, 66]. Hence, we adopt a consistent adaptation to our trading at discrete times model, selecting the exponential kernel $G(t) = e^{-\rho t}$:

$$S_t = S_t^0 - \beta \left(\sum_{\tau_n^1 \leq t} \delta_n^1 e^{\rho(\tau_n^1 - t)} + \sum_{\tau_n^2 \leq t} \delta_n^2 e^{\rho(\tau_n^2 - t)} \right)$$

for some positive $\beta > 0$ and $\rho > 0$. Note that, the higher the β the worse the execution price is going to be, meaning that a small β represents more liquid markets. Then, since trading incurs in costs proportional to the order size due to market liquidity and the effect of hitting the limit order book, it might not be optimal to play aggressively, i.e. sell all the shares in one trade for example. Moreover, dealers have to choose the optimal trading frequency keeping account of inventory costs, which we will denote as $\gamma > 0$ for the running inventory and $\Gamma > 0$ for the terminal inventory. Hence, players will maximise their respective goal functionals:

$$
\begin{aligned}
J^1(0,x,s;u_1,u_2) :=& \mathbb{E}_{x,y,s}\left[\sum_{n:0\le\tau_n^1\le T}\delta_n^1 S_{\tau_n^1} + S_T X_T - \int_0^T \gamma X_t^2 dt - \Gamma X_T^2\right],\\
J^2(0,y,s;u_1,u_2) :=& \mathbb{E}_{x,y,s}\left[\sum_{n:0\le\tau_n^2\le T}\delta_n^2 S_{\tau_n^2} + S_T Y_T - \int_0^T \gamma Y_t^2 dt - \Gamma Y_T^2\right],
\end{aligned}
$$

where the subscripts in the expectations denote conditioning with respect to the starting point. Notice that the payoff functions depend on the other players' controls indirectly via S_t. The inventory costs are quadratic to make up for some degree of inventory aversion, as such the payoff functions are linear quadratic as in [4, 19, 31, 33, 49].

The Strategies. As we mentioned in the game description, the dealers trade at some suitably chosen trading times $(\tau_n^i)_{n\ge1}$ an amount of shares $(\delta_n^i)_{n\ge1}$, where each δ_n^i is a random variable with real values. This leads us to the following definition of strategies for both players.

Definition 3.2.1 Player i's strategy is an impulse control $u_i = (\tau_n^i, \delta_n^i)_{n\ge1}$ where $(\tau_n^i)_{n\ge1}$ is a sequence of stopping times such that $0 \le \tau_1^i < \tau_2^i < \ldots$ and $\lim_{n\to\infty}\tau_n^i = \infty$ a.s., with $\delta_n^i \in L^0(\mathcal{F}_n)$ for each $n \ge 1$.We will refer to $\mathcal{U}$ as the set of strategies.

Before proceeding with the analysis we need to introduce the class of admissible strategies.

Definition 3.2.2 Two strategies, $u_1 = (\tau_n^1, \delta_n^1)_n \in \mathcal{U}$ and $u_2 = (\tau_n^2, \delta_n^2)_n \in \mathcal{U}$, are admissible if they are such that the payoff functions are well defined:

$$
\begin{aligned}
&\mathbb{E}\left[\Big|\sum_{n:0\le\tau_n^i\le T}\delta_n^i S_{\tau_n^i}\Big|\right] < \infty \ (i=\{1,2\}); \quad \mathbb{E}[|X_T S_T|] < \infty; \quad \mathbb{E}[|Y_T S_T|] < \infty;\\
&\mathbb{E}\left[\int_0^T \gamma X_t^2 dt\right] < \infty; \qquad \mathbb{E}\left[\int_0^T \gamma Y_t^2 dt\right] < \infty,
\end{aligned}
$$

together with

$$\|X\|_\infty, \|Y\|_\infty, \|S\|_\infty \in L^2(\Omega),$$

where $\|X\|_\infty = \sup_{t\in[0,T]}|X_t|$. We denote $\mathbb{U}$ the set of admissible strategies.

Our goal is to find the Nash equilibria, namely the couple of strategies (u_1^*, u_2^*), such that

$$
\begin{aligned}
J^1(0,x,y,s;u_1^*;u_2^*) &\ge J^1(0,x,y,s;u_1,u_2^*) \quad \forall\, u_1 \in \mathbb{U},\\
J^2(0,x,y,s;u_1^*,u_2^*) &\ge J^2(0,x,y,s;u_1^*,u_2) \quad \forall\, u_2 \in \mathbb{U}.
\end{aligned}
$$

3.3 No Simultaneous Trading Equilibria

In this section we adapt the arguments from [1] to obtain a suitable system of QVIs, which, together with a verification theorem, will provide a framework to find Nash equilibria when only one dealer trades at a time. Finally, we will show how the restrictions to no simultaneous trades turns out to be fatal as there are no candidate equilibrium payoff functions.

The QVIs system. When either player intervenes, he is going to make a gain of $\delta_n^i S_{\tau_n} = \delta_n^i (S^0_{\tau_n^i} - \beta\delta_n^i)$, recall $\delta_n^i > 0$ is a sell order. Hence, Dealer 1's equilibrium payoff function satisfies at each intervention time τ

$$V^1(\tau, x, y, s) = \sup_{\delta^1 \in \mathbb{R}} \left[\delta^1 (s - \beta\delta^1) + V^1(\tau, x - \delta^1, y, s - \beta\delta^1) \right],$$

as when he trades he does so maximising his returns. As such, we define each player's intervention operator accordingly

$$\mathcal{M}^1 V^1(t, x, y, s) = \sup_{\delta^1 \in \mathbb{R}} \left[\delta^1 (s - \beta\delta^1) + V^1(t, x - \delta^1, y, s - \beta\delta^1) \right],$$
$$\mathcal{M}^2 V^2(t, x, y, s) = \sup_{\delta^2 \in \mathbb{R}} \left[\delta^2 (s - \beta\delta^2) + V^2(t, x, y - \delta^2, s - \beta\delta^2) \right].$$

Moreover, since dealers' trades cannot increase the value of the game the value functions have to satisfy

$$(V^i - \mathcal{M}^i V^i)(t, x, y, s) \geq 0 \quad (t, x, y, s) \in [0, T] \times \mathbb{R}^3,\ i = 1, 2.$$

We are interested in Nash equilibria, according to which, one Dealer's equilibrium payoff function should not get worse when the other trades, in order to avoid deviations from the equilibrium strategy. In mathematical terms, this translates to

$$V^1(t, x, y, s) = V^1(t, x, y - \delta^2, s - \beta\delta^2), \quad V^2(t, x, y, s) = V^2(t, x - \delta^1, y, s - \beta\delta^1). \tag{3.2}$$

Finally, for any V regular enough we can consider the infinitesimal generator of the uncontrolled state variable S:

$$\mathcal{A}V = bV_s + \frac{1}{2}\sigma^2 V_{ss},$$

where b, σ are as in (3.1). Given the information above, the system of QVIs to be satisfied by the dealers' equilibrium payoff functions has to be the following

$$
\begin{aligned}
&(V^i - \mathcal{M}^i V^i)(t,x,y,s) \geq 0 && \text{in } [0,T]\times\mathbb{R}^3 \\
&V^1(t,x,y,s) = V^1(t,x,y-\delta^2, s-\beta\delta^2), && \text{in } \{(V^2-\mathcal{M}^2V^2)(t,x,y,s)=0\}, \qquad (3.3a)\\
&V^2(t,x,y,s) = V^2(t,x-\delta^1,y, s-\beta\delta^1), && \text{in } \{(V^1-\mathcal{M}^1V^1)(t,x,y,s)=0\},\\
&\max\{\mathcal{A}V^i + V^i_t - \gamma x^2, \mathcal{M}^iV^i - V^i\} = 0 && \text{in } \{(V^j-\mathcal{M}^jV^j)(t,x,y,s)>0\}\cap[0,T)\times\mathbb{R}^3, \\
& && (3.3b)\\
&\max\{sx - \Gamma x^2 - V^1, \mathcal{M}^1V^1 - V^1\} = 0 && \text{in } \{(V^2-\mathcal{M}^2V^2)(t,x,y,s)>0\}\cap\{T\}\times\mathbb{R}^3, \\
& && (3.3c)\\
&\max\{sy - \Gamma y^2 - V^2, \mathcal{M}^2V^2 - V^2\} = 0 && \text{in } \{(V^1-\mathcal{M}^1V^1)(t,x,y,s)>0\}\cap\{T\}\times\mathbb{R}^3.
\end{aligned}
$$

The Verification Theorem

THEOREM 3.3.1 Let $V^{1,2} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ be two given functions. Assume that

$$
\begin{aligned}
&\{\delta^1(t,x,y,s)\} \in \operatorname{argmax}\{V^1(t,x-\delta^1,y,s-\beta\delta^1) - \delta^1(s-\beta\delta^1)\}, (t,x,y,s)\in[0,T]\times\mathbb{R}^3,\\
&\{\delta^2(t,x,y,s)\} \in \operatorname{argmax}\{V^2(t,x,y-\delta^2,s-\beta\delta^2) - \delta^2(s-\beta\delta^2)\}, (t,x,y,s)\in[0,T]\times\mathbb{R}^3,
\end{aligned}
$$

hold and set $\mathcal{C}^i := \{V^i - \mathcal{M}^iV^i > 0\}$. Moreover, assume that

- V^1, V^2 are solutions of the system of QVI;
- $V^i \in C^{1,0,0,2}(\mathcal{C}_j \setminus \partial\mathcal{C}_i) \cap C^{1,0,0,1}(\mathcal{C}_j) \cap C^0([0,T]\times\mathbb{R}^3)$ for $i,j=1,2, i\neq j$ and have at most quadratic growth;
- $\partial\mathcal{C}_i$ is a Lipschitz surface and V^i's second derivatives are locally bounded near $\partial\mathcal{C}_i$ for $i=1,2$.

Finally, let $(t,x,y,s)\in[0,T]\times\mathbb{R}^3$, $(u_1^*, u_2^*)\in\mathbb{U}$ where $u_i^* = (\tau_n^i, \delta_n^i)$ $i=1,2$, are given by

$$
\begin{aligned}
&\tau_n^1 = \inf\{t > \tau_{n-1}^1 : (t, X_t, Y_t, S_t) \in \mathcal{C}_1^c\}, \quad \delta_n^1 \in \operatorname{argmax}\{V^1(\tau_n^1, X_{\tau_n^1}, Y_{\tau_n^1}, S_{\tau_n^1}) - S_{\tau_n^1}\delta\},\\
&\tau_n^2 = \inf\{t > \tau_{n-1}^2 : (t, X_t, , Y_t, S_t) \in \mathcal{C}_2^c\}, \quad \delta_n^2 \in \operatorname{argmax}\{V^2(\tau_n^2, X_{\tau_n^2}, Y_{\tau_n^2}, S_{\tau_n^2}) - S_{\tau_n^2}\delta\}
\end{aligned}
$$

with the convention $\tau_0^i = 0, i=1,2$. Then (u_1^*, u_2^*) is a NE and $V^1(t,x,y,s) = J^1(t,x,y,s;u_1^*,u_2^*)$, $V^2(t,x,y,s) = J^2(t,x,y,s;u_1^*,u_2^*)$.

PROOF 3.3.1 We do the proof for V^1 only since the steps for V^2 are the same.
Let $V^1(t,x,y,s) = J^1(t,x,y,s;u_1^*,u_2^*)$. Step 1: we have to prove that $V^1(t,x,y,s) \geq J^1(t,x,y,s;u_1,u_2^*)$ for all $u_1 : (u_1,u_2^*)\in\mathbb{U}$. Thanks to regularity assumptions plus approximation arguments as in [1], we can assume without loss of generality that $V^1 \in C^{1,0,0,2}(\mathcal{C}_2)\cap C^0([0,T)\times\mathbb{R}^3)$. Then, fix $r\geq 0$ and define the stopping time

$$\tau_{r,T} := T \wedge \tau_r,$$

where $\tau_r := \inf\{t > 0 : S_t \notin B(s,r)\}$ is the exit time from the ball with radius r and centre s, with the convention $\inf \emptyset = +\infty$. We apply Itô's formula to $V(t, X_t, Y_t, S_t)$ between time 0 and $\tau_{r,T}$ and take conditional expectation on both sides

$$V^1(0,x,y,s) = \mathbb{E}_{x,y,s}\Bigg[V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) - \int_0^{\tau_{r,T}} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t)dt \\ - \sum_{k:0\le\tau_k^i\le\tau_{r,T}} \Big(V^1(\tau_k^i-, X_{\tau_k^i-}, Y_{\tau_k^i-}, S_{\tau_k^i-}) - V^1(\tau_k^i, X_{\tau_k^i}, Y_{\tau_k^i}, S_{\tau_k^i})\Big)\Bigg].$$

First, let's note that

$$\int_0^{\tau_{r,T}} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t)dt = \int_0^{\tau_{r,T}} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t)\mathbb{1}_{\{V^2 > \mathcal{M}V^2\}}dt$$

as from (3.3a)

$$\int_0^{\tau_{r,T}} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t)\mathbb{1}_{\{V^2 = \mathcal{M}V^2\}}dt \\ = \sum_{\tau_n^2 \le \tau_{r,T}} \int_{\tau_n^2}^{\tau_n^2} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t)\mathbb{1}_{\{V^2 = \mathcal{M}V^2\}}dt = 0.$$

Then, from (3.3b) we get

$$(\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t) \le \gamma X_t^2$$

and

$$V^1(\tau_k^1, X_{\tau_k^1-}, Y_{\tau_k^1-}, S_{\tau_k^1-}) \ge \mathcal{M}^1 V^1(\tau_k^1, X_{\tau_k^1-}, Y_{\tau_k^1-}, S_{\tau_k^1-}) \\ \ge V(\tau_k^1, X_{\tau_k^1}, Y_{\tau_k^1}, S_{\tau_k^1}) + \delta_k^1 S_{\tau_k^1}.$$

Therefore, we can rewrite the previous inequality as

$$V^1(0,x,y,s) \ge \mathbb{E}_{x,y,s}\left[V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) - \int_0^{\tau_{r,T}} \gamma X_t^2 dt + \sum_{k:0\le\tau_k^1\le\tau_{r,T}} \delta_k^1 S_{\tau_k}\right].$$

By polynomial growth assumptions we have

$$V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) \le C(1 + |X_{\tau_{r,T}}|^2 + |Y_{\tau_{r,T}}|^2 + |S_{\tau_{r,T}}|^2) \\ \le C(1 + \|X\|_\infty^2 + \|Y\|_\infty^2 + \|S\|_\infty^2) \in L^1$$

for some $C > 0$. Then, we apply the dominated convergence theorem and pass to the limit as $r \to \infty$. Finally, because of the terminal condition (3.3c) we have

$$V^1(0,x,y,s) \ge \mathbb{E}_{x,y,s}\left[\sum_{k:0\le\tau_k^1\le T} \delta S_{\tau_k} - \int_0^T \gamma X_t^2 dt + S_T X_T - \Gamma X_T^2\right] = J^1(0,x,y,s;u_1,u_2^*).$$

Step 2: the equality follows by properties of u_1^*. In particular by

$$\tau_n^1 = \inf\{t > \tau_{n-1}^1 : (t, X_t, Y_t, S_t) \in \mathcal{C}_1^c\}, \quad \delta_n^1 \in \operatorname{argmax}\{V^1(\tau_n^1, X_{\tau_n^1}, Y_{\tau_n^1}, S_{\tau_n^1}) - S_{\tau_n^1}\delta\},$$

thanks to which, from (3.3b) we get

$$\begin{aligned}(\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t) &= \gamma X_t^2,\\ V^1(\tau_k^1-, X_{\tau_k^1-}, Y_{\tau_k^1-}, S_{\tau_k^1-}) &= \mathcal{M}^1 V^1(\tau_k^1-, X_{\tau_k^1-}, Y_{\tau_k^1-}, S_{\tau_k^1-}).\end{aligned}$$

Therefore, we are able to substitute the inequalities in step 1 with equalities to get the desired result. □

No equilibria with no-simultaneous trades. In the following we will show that in case simultaneous interventions are not allowed we are not able to find any Nash equilibria with this QVIs approach when the solution of the PDE is of quadratic form. The system of QVIs suggests that our candidates have the form

$$W^1(t,x,y,s) = \begin{cases} \varphi^1(t,x,s) & (t,x,y,s) \in \mathcal{C}_1 \cap \mathcal{C}_2 \\ \varphi^1(t, x - \delta_1^*, s - \beta\delta_1^*) + \delta_1^*(s - \beta\delta_1^*) & (t,x,y,s) \in \mathcal{C}_1^c \\ \varphi^1(t, x, s - \beta\delta_2^*) & (t,x,y,s) \in \mathcal{C}_2^c \end{cases}$$

$$W^2(t,x,y,s) = \begin{cases} \varphi^2(t,y,s) & (t,x,y,s) \in \mathcal{C}_1 \cap \mathcal{C}_2 \\ \varphi^2(t, y - \delta_2^*, s - \beta\delta_2^*) + \delta_2^*(s - \beta\delta_2^*) & (t,x,y,s) \in \mathcal{C}_2^c \\ \varphi^2(t, y, s - \beta\delta_1^*) & (t,x,y,s) \in \mathcal{C}_1^c \end{cases}$$

where φ^1, φ^2 are the solution to the PDEs

$$\mathcal{A}\varphi^1 + \varphi_t^1 - \gamma x^2 = 0, \qquad \varphi^1(T,x,s) = sx - \Gamma x^2, \tag{3.4}$$

$$\mathcal{A}\varphi^2 + \varphi_t^2 - \gamma y^2 = 0, \qquad \varphi^2(T,y,s) = sy - \Gamma y^2, \tag{3.5}$$

and δ_1^*, δ_2^* are the equilibrium impulses. First, we notice that functions of the following form are solutions to (3.4)-(3.5):

$$\varphi^1(t,x,s) = C_1 s + C_2(T-t) + C_3, \qquad \varphi^2(t,y,s) = K_1 s + K_2(T-t) + K_3;$$

so that, solving for the boundary conditions we get

$$\varphi^1(t,x,s) = sx + (bx - \gamma x^2)(T-t) - \Gamma x^2, \quad \varphi^2(t,y,s) = sy + (by - \gamma y^2)(T-t) - \Gamma y^2.$$

Here comes the critical point. Assume Dealer 1 trades at $\tau \in [0,T]$, then, in order to have a Nash equilibrium, Dealer 2 should not get worse, otherwise he would deviate, this means that

$$\varphi^2(\tau, Y_\tau, S_\tau) = \varphi^2(\tau, Y_{\tau-}, S_{\tau-}).$$

Since, only Dealer 1 is trading we have $Y_\tau = Y_{\tau-}$ and $S_\tau = S_{\tau-} - \beta\delta_1$. Hence, rewriting the equality above we get

$$(S_{\tau-} - \beta\delta_1)Y_\tau + (bY_\tau - \gamma Y_\tau^2)(T-\tau) - \Gamma Y_\tau^2 = S_{\tau-}Y_\tau + (bY_\tau - \gamma Y_\tau^2)(T-\tau) - \Gamma Y_\tau^2$$

which results in

$$-\beta\delta_1 Y_\tau = 0$$

showing that anytime one of the two dealers will trade the other would deviate. This means we are not able to find any equilibria under this approach, although some equilibria may still exist.

3.4 Equilibria with Simultaneous Trading

In this section we provide an alternative QVIs system and related verification theorem in order to find Nash equilibria where dealers can trade at the same time. Finally, we solve the new QVIs system and apply the verification theorem to find Nash equilibria.

The QVIs system. In order to have simultaneous interventions we need to introduce different operators from the ones used in the previous section. To begin with, instead of dealing with players' trading times we will consider intervention times at which at least one of the two dealers is buying or selling the stock. This approach will lead to a similar but different system of QVIs.

According to this framework at trading times the dealers are playing a static nonzero-sum game. Hence, we introduce the Nash operator $\mathcal{N}$ such that, for any $V := (V^1, V^2)$, $\mathcal{N}V(t, x, y, s)$ returns the set of all Nash equilibria payoffs of the static game in $(t, x, y, s) \in [0, T] \times \mathbb{R}^3$, which we denote by $\nu = (\nu_1, \nu_2)$, where dealers maximise

$$V^1(t, x - \delta_1, y - \delta_2, s - \beta(\delta_1 + \delta_2)) + \delta_1(s - \beta(\delta_1 + \delta_2))$$
$$V^2(t, x - \delta_1, y - \delta_2, s - \beta(\delta_1 + \delta_2)) + \delta_2(s - \beta(\delta_1 + \delta_2))$$

respectively. Given the information above, the system of QVIs to be satisfied by the dealers' equilibrium payoff functions has to be the following. Let $\nu = (\nu_1, \nu_2)$ be a measurable selector of $\mathcal{N}V$, i.e. $\nu : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^2$ measurable such that $\nu(t, x, y, s) \in \mathcal{N}V(t, x, y, s)$ for all $(t, x, y, s) \in [0, T] \times \mathbb{R}^3$

$$
\begin{aligned}
&V - \nu \geq 0 && \text{in } [0, T] \times \mathbb{R}^3 \\
&\max\{\mathcal{A}V^1 + V^1_t - \gamma x^2, \nu_1 - V^1\} = 0 && \text{in } [0, T] \times \mathbb{R}^3 \qquad (3.6a)\\
&\max\{\mathcal{A}V^2 + V^2_t - \gamma y^2, \nu_2 - V^2\} = 0 && \text{in } [0, T] \times \mathbb{R}^3 \\
&\max\{sx - \Gamma x^2 - V^1, \nu_1 - V^1\} = 0 && \text{in } \{T\} \times \mathbb{R}^3 \qquad (3.6b)\\
&\max\{sy - \Gamma y^2 - V^2, \nu_2 - V^2\} = 0 && \text{in } \{T\} \times \mathbb{R}^3.
\end{aligned}
$$

Note that $\nu_i - V^i$ coincide with $\mathcal{M}V^i - V^i$ in case the other player is passive, $\delta_j = 0, j \neq i$, whereas it coincides with (3.2) whether he is passive, $\delta_i = 0$ while the other dealer is trading, $\delta_j \neq 0, i \neq j$.

The Verification Theorem

THEOREM 3.4.1 Let $V^{1,2} : [0,T] \times \mathbb{R}^3$ be two given functions. Let $\mathcal{N}V$ be a non-empty compact-valued correspondence. Then, there exists a measurable selector $\nu \in \mathcal{N}V$. Set $\mathcal{C} := \{V - \nu > 0\}$ and $\mathcal{C}^c \equiv \{V - \nu = 0\}$ and assume that

- V^1, V^2 are strong solutions of the system of QVIs (3.6);
- $V^i \in C^{1,0,0,2}(\mathcal{C}) \cap C^0([0,T] \times \mathbb{R}^3)$ for $i,j = 1,2$ and have at most quadratic growth;

Finally, let $(t,x,y,s) \in [0,T] \times \mathbb{R}^3$, $(u_1^*, u_2^*) \in \mathbb{U}$ where $u_i^* = (\tau_n, \delta_n^i)$, $i = 1,2$, are given by

$$\tau_n = \inf\{t > \tau_{n-1} : (t, X_t, Y_t, S_t) \in \mathcal{C}^c\},$$

and $\{\delta_n\} = \{(\delta_n^1, \delta_n^2)\}$ are such that $V(\tau_n -, X_{\tau_n -}, Y_{\tau_n -}, S_{\tau_n -}) = \nu(\tau_n, X_{\tau_n}, Y_{\tau_n}, S_{\tau_n})$ with the convention $\tau_0^i = 0, i = 1,2$. Then (u_1^*, u_2^*) is a NE and $V^1(t,x,y,s) = J^1(t,x,y,s; u_1^*, u_2^*)$, $V^2(t,x,y,s) = J^2(t,x,y,s; u_1^*, u_2^*)$.

PROOF 3.4.1 First we have to prove existence of a measurable selector $\nu \in \mathcal{N}V$. Recall $\mathcal{N}V(t,x,y,s) : ([0,T] \times \mathbb{R}^3, \mathcal{B}([0,T] \times \mathbb{R}^3)) \twoheadrightarrow \mathbb{R}^2$, where $\mathcal{B}$ is the Borel σ-algebra and $\twoheadrightarrow$ is used to distinguish a correspondence, or set-valued function, from a function. Since $([0,T] \times \mathbb{R}^3, \mathcal{B}([0,T] \times \mathbb{R}^3))$ is a measurable space, $\mathbb{R}^2$ is a metrizable space and $\mathcal{N}V$ is non-empty with compact values, then $\mathcal{N}V$ is also measurable by Theorem 18.10 in [3]. Moreover, since $\mathbb{R}^2$ is a Polish space, by the Kuratowski-Ryll-Nardzewski Selection Theorem (Theorem 18.13 [3]) there exists a measurable selector $\nu \in \mathcal{N}V$.

We do the proof for V^1 only since the steps for V^2 are the same. Let $V^1(t,x,y,s) = J^1(t,x,y,s; u_1^*, u_2^*)$. We have to prove that $V^1(t,x,y,s) \geq J^1(t,x,y,s; u_1, u_2^*)$ for all $u_1 : (u_1, u_2^*) \in \mathbb{U}$. Then, fix $r \geq 0$ and define the stopping time

$$\tau_{r,T} := T \wedge \tau_r$$

with $\tau_r := \inf\{t > 0 : S_t \notin B(s,r)\}$ with the convention $\inf \emptyset = +\infty$. We apply Ito's formula to $V(t, X_t, Y_t, S_t)$ between time 0 and $\tau_{r,T}$ and take conditional expectation on both sides gives

$$\begin{aligned} V^1(0,x,y,s) = &\mathbb{E}_{x,y,s} \Bigg[V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) - \int_0^{\tau_{r,T}} (\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t) dt \\ &- \sum_{k: 0 \leq \tau_k \leq \tau_{r,T}} \left(V^1(\tau_k -, X_{\tau_k -}, Y_{\tau_k -}, S_{\tau_k -}) - V^1(\tau_k, X_{\tau_k}, Y_{\tau_k}, S_{\tau_k}) \right) \Bigg]. \end{aligned}$$

From (3.6a) we get

$$(\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t) \leq \gamma X_t^2$$

and

$$V^1(\tau_k, X_{\tau_k -}, Y_{\tau_k -}, S_{\tau_k -}) \geq \nu_1(\tau_k, X_{\tau_k -}, Y_{\tau_k -}, S_{\tau_k -}) = V^1(\tau_k, X_{\tau_k}, Y_{\tau_k}, S_{\tau_k}) + \delta_k^1 S_{\tau_k}.$$

Therefore, we can rewrite the previous inequality as

$$V^1(0,x,y,s) \geq \mathbb{E}_{x,y,s}\left[V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) - \int_0^{\tau_{r,T}} \gamma X_t^2 dt + \sum_{k:0\leq\tau_k\leq\tau_{r,T}} \delta_k^1 S_{\tau_k}\right].$$

By quadratic growth assumptions we have

$$\begin{aligned} V^1(\tau_{r,T}, X_{\tau_{r,T}}, Y_{\tau_{r,T}}, S_{\tau_{r,T}}) &\leq C(1+|X_{\tau_{r,T}}|^2 + |Y_{\tau_{r,T}}|^2 + |S_{\tau_{r,T}}|^2) \\ &\leq C(1+\|X\|_\infty^2 + \|Y\|_\infty^2 + \|S\|_\infty^2) \in L^1 \end{aligned}$$

for some $C > 0$. Then, we apply the dominated convergence theorem and pass to the limit as $r \to \infty$. To conclude this step, because of the terminal condition (3.6b) we have

$$V^1(0,x,y,s) \geq \mathbb{E}_{x,y,s}\left[\sum_{k:0\leq\tau_k\leq T} \delta S_{\tau_k} - \int_0^T \gamma X_t^2 dt + S_T X_T - \Gamma X_T^2\right] = J^1(0,x,y,s;u_1,u_2^*).$$

Now we obtain the desired equality by properties of u_1^*. According to it we have that (δ_n^1, δ_n^2) is a Nash equilibrium of the static game played at each trading time, i.e.

$$V^1(\tau_k, X_{\tau_k -}, Y_{\tau_k -}, S_{\tau_k -}) = V^1(\tau_k, X_{\tau_k}, Y_{\tau_k}, S_{\tau_k}) + \delta_k^1 S_{\tau_k}.$$

Hence, by (3.6a) we get

$$(\mathcal{A}V^1 + V_t^1)(t, X_t, Y_t, S_t) = \gamma X_t^2,$$

form which we obtain $V^1(t,x,y,s) = J^1(t,x,y,s;u_1^*,u_2^*)$ following the same steps as above.
□

The Equilibrium Candidates. In order to find Nash equilibria, we first need to find some suitable candidates via solving the QVIs system. Once they are identified, we apply Theorem 3.4.1 to verify that they are indeed Nash equilibria. We begin looking for equilibria for long-short dealers to later analyse the long-only case. Let's first write the system of QVIs we want to solve. Let $\nu = (\nu_1, \nu_2) \in \mathcal{N}W$

$$\begin{aligned} &W - \nu \geq 0 && \text{in } [0,T]\times\mathbb{R}^3 \\ &\max\{\mathcal{A}W^1 + W_t^1 - \gamma x^2, \nu_1 - W^1\} = 0 && \text{in } [0,T]\times\mathbb{R}^3 \\ &\max\{\mathcal{A}W^2 + W_t^2 - \gamma y^2, \nu_2 - W^2\} = 0 && \text{in } [0,T]\times\mathbb{R}^3 \\ &\max\{sx - \Gamma x^2 - W^1, \nu_1 - W^1\} = 0 && \text{in } \{T\}\times\mathbb{R}^3 \\ &\max\{sy - \Gamma y^2 - W^2, \nu_2 - W^2\} = 0 && \text{in } \{T\}\times\mathbb{R}^3. \end{aligned}$$

A careful look at the QVIs system suggests the following functional form for the solution

$$W^1(t,x,y,s) = \begin{cases} \varphi^1(t,x,s) & (t,x,y,s) \in \mathcal{C}_1 \\ \nu_1(t,x,y,s) & (t,x,y,s) \in \mathcal{C}_1^c \end{cases}$$

$$W^2(t,x,y,s) = \begin{cases} \varphi^2(t,y,s) & (t,x,y,s) \in \mathcal{C}_2 \\ \nu_2(t,x,y,s) & (t,x,y,s) \in \mathcal{C}_2^c \end{cases}$$

where φ^1, φ^2 are the solution to (3.4)-(3.5), $\nu = (\nu_1, \nu_2)$ is such that

$$\nu_1(t,x,y,s) = \varphi^1(t, x - \delta_1^*, s^*) + \delta_1^* s^*, \qquad \nu_2(t,x,y,s) = \varphi^2(t, y - \delta_2^*, s^*) + \delta_2^* s^*,$$

with δ_1^* and δ_2^* the equilibrium impulses, $s^* = s - \beta(\delta_1^* + \delta_2^*)$ and $\mathcal{C}_1$, $\mathcal{C}_2$ such that:

$$\begin{aligned}
\mathcal{C}_1 &:= \{(t,x,y,s) \in [0,T] \times \mathbb{R}^3 : \varphi^1(t,x,s) - \varphi^1(t, x - \delta_1^*, s^*) + \delta_1^* s^* > 0\}, \\
\mathcal{C}_2 &:= \{(t,x,y,s) \in [0,T] \times \mathbb{R}^3 : \varphi^2(t,y,s) - \varphi^2(t, y - \delta_2^*, s^*) + \delta_2^* s^* > 0\}.
\end{aligned}$$

As in the case with no simultaneous interventions in the previous section, we get

$$\varphi^1(t,x,s) = sx + (bx - \gamma x^2)(T-t) - \Gamma x^2, \quad \varphi^2(t,y,s) = sy + (by - \gamma y^2)(T-t) - \Gamma y^2$$

since the PDEs are identical.

Static Nash Equilibrium. Now we look for Nash equilibria in the static games played at trading times. We start finding Dealer 1's best response:

$$\max_{\delta_1 \in \mathbb{R}} \left[s - \beta(\delta_1 + \delta_2)\right] x + \left[b(x - \delta_1) - \gamma(x - \delta_1)^2\right](T-t) - \Gamma(x - \delta_1)^2.$$

The first order condition gives us:

$$-\beta x - b(T-t) + 2\gamma(x - \delta_1)(T-t) + 2\Gamma(x - \delta_1) = 0.$$

So, Dealer 1's optimal trade is

$$\delta_1 = x - \frac{\beta x + b(T-t)}{2[\gamma(T-t) + \Gamma]} \tag{3.7}$$

as the second derivative is negative. Symmetrically, Dealer 2's best response is

$$\delta_2 = y - \frac{\beta y + b(T-t)}{2[\gamma(T-t) + \Gamma]}. \tag{3.8}$$

Moreover, as they do not depend on each other they are equilibrium strategies of the static game. Notice that the position held by the dealers after each trading time τ is

$$X_\tau = X_{\tau-} - \delta_1^* = \frac{\beta X_{\tau-} + b(T-t)}{2[\gamma(T-t) + \Gamma]}, \qquad Y_\tau = Y_{\tau-} - \delta_2^* = \frac{\beta Y_{\tau-} + b(T-t)}{2[\gamma(T-t) + \Gamma]}, \tag{3.9}$$

showing that the dealers' holding are decreasing in the inventory costs γ, Γ, increasing in the market impact, β, as it makes them send smaller orders, whereas the impact of the trend, b, on their holdings is decreasing in time. Moreover, it is important to notice that, in case dealers start the game from long positions $X_0 = x, Y_0 = y \geq 0$, then, from (3.9), they will never go short, namely $X_t, Y_t \geq 0$ for all $t \in [0,T]$, if the trend is positive, $b \geq 0$. Finally, we derive both dealers' continuation regions $\mathcal{C}_1, \mathcal{C}_2$

$$\begin{aligned}
\mathcal{C}_1 &= \{(t,x,y,s) \in [0,T] \times \mathbb{R}^3 : \beta x \delta_2 - [\gamma(T-t) + \Gamma]\delta_1^2 > 0\}, \\
\mathcal{C}_2 &= \{(t,x,y,s) \in [0,T] \times \mathbb{R}^3 : \beta y \delta_1 - [\gamma(T-t) + \Gamma]\delta_2^2 > 0\}.
\end{aligned}$$

Application of the verification theorem (work in progress). Now we verify under which conditions dealers trading according to (3.7)-(3.8) is a Nash equilibrium. To begin with, we can note that our candidates W_1, W_2 satisfy the regularity properties by construction (quadratic growth and $W^i \in C^{1,0,0,2}(\mathcal{C}) \cap C^0([0,T] \times \mathbb{R}^d)$). Then, we need to check that δ_1^* and δ_2^* are equilibrium strategies of the static game played at trading times:

$$\begin{aligned}
&\varphi^1(t, x - \delta_1^*, s - \beta(\delta_1^* + \delta_2^*)) + \delta_1^*(s - \beta(\delta_1^* + \delta_2^*)) \\
&\qquad\qquad\qquad \geq \varphi^1(t, x - \delta_1, s - \beta(\delta_1 + \delta_2^*)) + \delta_1(s - \beta(\delta_1 + \delta_2^*)) \\
&\varphi^2(t, y - \delta_2^*, s - \beta(\delta_1^* + \delta_2^*)) + \delta_2^*(s - \beta(\delta_1^* + \delta_2^*)) \\
&\qquad\qquad\qquad \geq \varphi^2(t, y - \delta_2, s - \beta(\delta_1^* + \delta_2)) + \delta_2(s - \beta(\delta_1^* + \delta_2)),
\end{aligned}$$

for all $\delta_1, \delta_2 \in \mathbb{R}$, which is satisfied as δ_1^* is a maximum point of $\varphi^1(t, x - \delta_1, s - \beta(\delta_1 + \delta_2^*)) + \delta_1(s - \beta(\delta_1 + \delta_2^*))$ (analogously δ_2^*).

It remains to show that we have $W - \mathcal{N}W \geq 0$ everywhere so that $W = (W_1, W_2)$ is a solution to the system as the remaining conditions follow by construction. Fundamentally, we want to verify that $\{W_1 - v_1 > 0\} \equiv \{W_2 - v_2 > 0\}$ as the continuation region is $\{W - \mathcal{N}W > 0\}$ and the intervention region is $\{W - \mathcal{N}W = 0\}$. To do so, let's define $\xi_1 := W_1 - v_1$ and $\xi_2 := W_2 - v_2$. We want to have $\xi_1 = k\xi_2$, with $k > 0$. First, let's write ξ_1 and ξ_2 explicitly

$$\begin{aligned}
\xi_1 &= \beta x \delta_2 - [\gamma(T-t) + \Gamma]\delta_1^2 \\
\xi_2 &= \beta y \delta_1 - [\gamma(T-t) + \Gamma]\delta_2^2.
\end{aligned}$$

Hence, we want to find under which conditions we have

$$\beta x \delta_2 - [\gamma(T-t) + \Gamma]\delta_1^2 = k\{\beta y \delta_1 - [\gamma(T-t) + \Gamma]\delta_2^2\}.$$

Plugging in (3.7) and (3.8) we get

$$\begin{aligned}
&2\beta x\{2y[\gamma(T-t) + \Gamma] - \beta y - b(T-t)\} - \{2x[\gamma(T-t) + \Gamma] - \beta x - b(T-t)\}^2 = \\
&k\left\{2\beta y\{2x[\gamma(T-t) + \Gamma] - \beta x - b(T-t)\} - \{2y[\gamma(T-t) + \Gamma] - \beta y - b(T-t)\}^2\right\}
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
&\{4\beta xy[\gamma(T-t) + \Gamma] - 2\beta^2 xy\}(1-k) - 4b\beta(T-t)(x - ky) - 4[\gamma(T-t) + \Gamma]^2(x^2 - ky^2) \\
&\quad - \beta^2(x^2 - ky^2) - b^2(T-t)^2(1-k) + 4\beta[\gamma(T-t) + \Gamma](x^2 - ky^2) \\
&\qquad + 4b(T-t)[\gamma(T-t) + \Gamma](x - ky) = 0.
\end{aligned}$$

Notice that we need to fix $k = 1$ as one of the conditions to let the equality hold. Now we fix $k = 1$, divide by $(x - y)$ and rewrite the equation

$$\begin{aligned}
&-4b\beta(T-t) - 4[\gamma(T-t) + \Gamma]^2(x+y) - \beta^2(x+y) \\
&\qquad + 4\beta[\gamma(T-t) + \Gamma](x+y) + 4b(T-t)[\gamma(T-t) + \Gamma] = 0.
\end{aligned}$$

To finally find the conditions we need to collect the terms in x, y, t:

$$\begin{aligned} &- 4b(T-t)(\beta-\Gamma) - 4\gamma^2(x+y)(T-t)^2 + 4\gamma(x+y)(T-t)(\beta-2\Gamma) \\ &\quad - (x+y)(\beta-2\Gamma)^2 + 4b\gamma(T-t)^2 = 0 \end{aligned}$$

which is satisfied under the following

- $\beta = 2\Gamma$, $\gamma = 0$, and $b = 0$, which means no-one is trading as $\delta_1^* = \delta_2^* = 0$.
- $x = y$ as we have divided earlier by $x - y$,
- $x = -y$ and $b = 0$, which means the dealers are making the market to each other as this condition implies $\delta_1^* = -\delta_2^*$ so that $x = -y$ holds for all $t \in [0, T]$.

Now it is left to be shown that once dealers trade they enter the continuation region $\mathcal{C}$, avoiding potentially infinitely many trades in one instant.
We begin focusing on the case $x = -y$ with $b = 0$. Here $\mathcal{C}$ writes:

$$\mathcal{C} = \left\{(t, x, -x, s) \in [0, T] \times \mathbb{R}^3 : -\beta x\delta_1 - [\gamma(T-t)+\Gamma]\delta_1^2 > 0\right\}.$$

We are interested in the inequality which, after plugging in (3.7) and some manipulations, can be represented as

$$\{\beta^2 - 4[\gamma(T-t)+\Gamma]\}x^2 > 0.$$

So that, we can write the continuation region as

$$\mathcal{C} = \left\{(t, x, -x, s) \in [0, T] \times \mathbb{R}^3 : (T-t) < \frac{\beta - 2\Gamma}{2\gamma}, \quad x \neq 0\right\}$$

as $T - t \geq 0$ and $-\frac{\beta+2\Gamma}{2\gamma} < 0$. For $\mathcal{C}$ to be non-empty and such that there is trading we need $2(\gamma T + \Gamma) > \beta > 2\Gamma$. Under this regime both dealers trade at the very first time $t = 0$, liquidate their positions as $\delta_1^* = x$ according to (3.7) and then don't trade any more, as the equilibrium trade is now equal to zero.

Regarding the perfectly symmetric case, $x = y$, we have

$$\mathcal{C} = \left\{(t, x, x, s) \in [0, T] \times \mathbb{R}^3 : \beta x\delta_1 - [\gamma(T-t)+\Gamma]\delta_1^2 > 0\right\},$$

whose inequality, substituting (3.7), rewrites

$$\beta x\left(x - \frac{\beta x + b(T-t)}{2[\gamma(T-t)+\Gamma]}\right) - [\gamma(T-t)+\Gamma]\left(x - \frac{\beta x + b(T-t)}{2[\gamma(T-t)+\Gamma]}\right) > 0.$$

After some computations we get the more usable form

$$\{\{2[\gamma(T-t)+\Gamma] - \beta\}x - b(t-t)\}\{\{2[\gamma(T-t)+\Gamma] - 3\beta\}x - b(t-t)\} < 0. \quad (3.10)$$

Therefore, dealers potentially hit the trading region in two curves

$$\tilde{x}_1(t) = \frac{b(T-t)}{2[\gamma(T-t)+\Gamma] - \beta}, \quad (3.11)$$

$$\tilde{x}_2(t) = \frac{b(T-t)}{2[\gamma(T-t)+\Gamma] - 3\beta}. \quad (3.12)$$

Below we analyse how dealers trades when they hit the curves above.

1. *Hitting $\tilde{x}_1$.* First, plugging (3.4) into (3.9), we find that holdings after the trade haven't changed and $\delta_1 = 0$.

2. *Hitting $\tilde{x}_2$.* Similarly, we find that after trading Dealer 1 owns

$$x^*(t) = \frac{b(T-t)}{\gamma(T-t)+\Gamma}.$$

Now, to understand the trading behaviour we need to study the inequality (3.10).

- if $\Gamma > \frac{3}{2}\beta$ and $b > 0$ (if $b < 0$ change the order of $\tilde{x}_1, \tilde{x}_2$) the continuation region is

$$\mathcal{C} = \left\{(t,x,x,s) \in [0,T] \times \mathbb{R}^3 : \tilde{x}_1(t) < x < \tilde{x}_2(t)\right\}.$$

  In order to avoid infinitely many trades we need to make sure x^* is in $\mathcal{C}$, which means we need

$$2[\gamma(T-t)+\Gamma] - 3\beta < \gamma(T-t)+\Gamma < 2[\gamma(T-t)+\Gamma] - \beta, \tag{3.13}$$

  which holds whenever γ and T are such that $\gamma T + \Gamma < 3\beta$. For instance, $X_0 = x \in \mathcal{C}$, $\beta \in \mathbb{R}$, $\Gamma = 2\beta$, $\gamma = \beta/2$ and $T = 1$ is an equilibrium.

- if $\beta > 2[\gamma T + \Gamma]$ and $b > 0$ (if $b < 0$ change the order of $\tilde{x}_1, \tilde{x}_2$) the continuation region is

$$\mathcal{C} = \left\{(t,x,x,s) \in [0,T] \times \mathbb{R}^3 : \tilde{x}_2(t) < x < \tilde{x}_1(t)\right\}.$$

  Similarly, we need (3.13) to hold, we need $\Gamma > \beta$ which is not compatible with $\beta > 2[\gamma T + \Gamma]$, hence, no equilibria.

- if $2\Gamma > \beta$ and $2(\gamma T + \Gamma) < 3\beta$ and $b > 0$ (if $b < 0$ change the order of $\tilde{x}_1, \tilde{x}_2$) the continuation region assume a counter-intuitive shape:

$$\mathcal{C} = \left\{(t,x,x,s) \in [0,T] \times \mathbb{R}^3 : x < \tilde{x}_2(t), \quad x > \tilde{x}_1(t)\right\},$$

  as dealers don't trade when they have very long or very short positions, which they hold until maturity as $\tilde{x}_i(t) \to 0$ as $t \to T$ for all $i = 1, 2$, so that near maturity we have that the continuation region is very close to the whole space $[0,T] \times \mathbb{R}^3$. In this case, we can only force $x^* > \tilde{x}_1$ as $x^* \geq 0$ and $\tilde{x}_2 \leq 0$ when $b \geq 0$, analogously when $b < 0$. Then, $x^* > \tilde{x}_1$ holds when $\Gamma > \beta$. One instance of equilibrium in this setting is $X_0 = \tilde{x}_2$, $\Gamma = \beta + \epsilon$ and $\beta > 2\gamma T + 2\epsilon$ for any $\epsilon > 0$. Notice that, if we pick any $X_0 = x \in \mathcal{C}$ there won't be trades as the intervention region is shrinking with time.

3.5 Work in Progress.

Generalising the current setting. As we understand that a perfectly symmetric setting where both dealers have the same risk aversion/inventory costs we would like to consider a setting in which we allow for different coefficients, namely $\gamma_1 \geq \gamma_2$ and $\Gamma_1 \geq \Gamma_2$ instead of γ and Γ.

Mixed Strategies. One very interesting change would be to allow dealers to play at each trading time a mixed strategy instead of a pure one. To do so we would need to rearrange the definition of strategies, for instance we could define them as follows. Let's denote $\mathcal{P}(\mathbb{R})$ the space of probability measures on $\mathbb{R}$, equipped with the weak convergence topology.

Definition 3.5.1 Player i's strategy is a sequence $u_i = (\tau_n^i, \Delta_n^i)_{n\geq 0}$, where $(\tau_n^i)_{n\geq 0}$ is a sequence of stopping times such that $0 \leq \tau_1^i < \tau_2^i < \ldots$ and $\lim_{n\to\infty} \tau_n^i = \infty$ a.s., with $\Delta_n^i \in L^0(\mathcal{F}_n)$ taking values in $\mathcal{P}(\mathbb{R})$ for each $n \geq 1$. We denote $\mathcal{U}$ the players' set of strategies.

In this setting dealers would maximise

$$\int\int \big(V^1(t, x-\delta_1, y-\delta_2, s-\beta(\delta_1+\delta_2)) + \delta_1(s-\beta(\delta_1+\delta_2))\big)\,\Delta_1(d\delta_1)\Delta_2(d\delta_2)$$
$$\int\int \big(V^2(t, x-\delta_1, y-\delta_2, s-\beta(\delta_1+\delta_2)) + \delta_2(s-\beta(\delta_1+\delta_2))\big)\,\Delta_1(d\delta_1)\Delta_2(d\delta_2).$$

Once we suitably adapt the verification theorem we have access to a big pool of candidates. For instance, below we consider the case when dealers trade according to a Poisson, a Uniform or a Normal distribution.

Poisson mixing. In this case Dealer 1 is selecting the intensity parameter for a Poisson distribution, λ

$$\max_{\lambda} \left[s-\beta(\lambda+\delta_2)\right] x + \left[b(x-\lambda) - \gamma(x^2 - 2x\lambda + \lambda + \lambda^2)\right](T-t) - \Gamma(x^2 - 2x\lambda + \lambda + \lambda^2)^2.$$

Then the first order condition would give us

$$\beta x + [-b - \gamma(-2x+1+2\lambda)](T-t) - \Gamma(-2x+1+2\lambda) = 0,$$

from which we derive the equilibrium rate, as it does not depend on the other dealer's trades:

$$\lambda = x - \frac{1}{2} - \frac{\beta x - b(T-t)}{2[\gamma(T-t)+\Gamma]}.$$

Uniform mixing. Now we allow Dealer 1 to mix according to a Uniform$(\underline{x}, \overline{x})$. Hence, the dealer has to find the optimal lower and upper bound $\underline{x}, \overline{x}$:

$$\max_{\underline{x}, \overline{x}} \left[s - \beta \left(\frac{\underline{x} + \overline{x}}{2} + \delta_2 \right) \right] x + \left[b \left(x - \frac{\underline{x} + \overline{x}}{2} \right) - \gamma \left(x^2 - 2x(\underline{x} + \overline{x}) + \frac{(\underline{x} - \overline{x})^2}{12} \right. \right.$$
$$\left. \left. + \frac{(\underline{x} + \overline{x})^2}{4} \right) \right] (T - t) - \Gamma \left(x^2 - 2x(\underline{x} + \overline{x}) + \frac{(\underline{x} - \overline{x})^2}{12} + \frac{(\underline{x} + \overline{x})^2}{4} \right).$$

Then, the first order condition for $\underline{x}$ is

$$-\frac{\beta}{2} x + \left[-\frac{b}{2} - \gamma \left(-x + \frac{2}{3} \underline{x} + \frac{2}{3} \overline{x} \right) \right] (T - t) - \Gamma \left(-x + \frac{2}{3} \underline{x} + \frac{2}{3} \overline{x} \right) = 0,$$

from which we get

$$\frac{2}{3}(\underline{x} + \overline{x}) = x - \frac{\beta x - b(T - t)}{2[\gamma(T - t) + \Gamma]}.$$

Similarly, the FOC for $\overline{x}$ returns

$$\frac{2}{3}(\underline{x} + \overline{x}) = x - \frac{\beta x - b(T - t)}{2[\gamma(T - t) + \Gamma]},$$

meaning that there exists infinitely many equilibria. For instance, taking $\overline{x} = 2\underline{x}$ returns

$$\underline{x} = \frac{1}{2} \left\{ x - \frac{\beta x - b(T - t)}{2[\gamma(T - t) + \Gamma]} \right\}, \qquad \overline{x} = x - \frac{\beta x - b(T - t)}{2[\gamma(T - t) + \Gamma]}.$$

Normal mixing. When a dealer chose to mix according to a Normal(μ, σ^2) distribution, then he has to maximise over μ and σ^2:

$$\max_{\mu, \sigma^2} [s - \beta(\mu + \delta_2)] x + \left[b(x - \mu) - \gamma(x^2 - 2x\mu + \mu^2 + \sigma^2) \right] (T - t) - \Gamma(x^2 - 2x\mu + \mu^2 + \sigma^2).$$

The FOCs in μ and σ return:

$$\mu = x - \frac{\beta x - b(T - t)}{2[\gamma(T - t) + \Gamma]}, \qquad -2[\gamma(T - t) + \Gamma]\sigma^2 = 0,$$

resulting in the pure strategy (3.7).

All the examples above show that, once everything is suitably adapted to allow for mixing strategies, it will be possible to look for way more candidates. For instance, as any of the strategies above don't depend on the other dealer's one, it is possible to look for equilibria when one dealer is playing a pure strategy while the other is playing a mixed strategy or when both mix according to different probability distributions.

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