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Дослідження розв'язків дисперсійних рівнянь старшого порядку з φ-субгауссовими початковими умовами

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Investigation of solutions to higher-order dispersive equations with φ-sub-Gaussian initial conditions

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У цій роботі досліджуються властивості траєкторій випадкових процесів, що задають розв'язки дисперсійних рівнянь старших порядків з φ-субгауссовими гармонізованими випадковими початковими умовами. Основний результат роботи – це оцінки для швидкості росту вказаних процесів на необмежених множинах. Клас φ-субгауссових випадкових процесів з функцією $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha \in (1, 2]$, є природним узагальненням гауссових процесів. Для таких початкових умов оцінки розподілів супремумів розв'язків можуть бути обчислені в досить простому вигляді. Оцінки для швидкості росту розв'язків диференціальних рівнянь з частковими похідними старших порядків у випадку загального вигляду N-функції Орліча φ отримано в [9], де виведення грунтувалося на результатах роботи [12]. Тут ми використовуємо дещо інший підхід, який грунтується на використанні іншого ентропійного інтегралу і дає нам можливість у випадку $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha \in (1, 2]$, отримати вирази для оцінок у явному вигляді.

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Ключові слова: φ-субгауссові процеси, дисперсійні рівняння старших порядків, випадкові початкові умови, швидкість росту

In this paper, there are studied sample paths properties of stochastic processes representing solutions of higher-order dispersive equations with random initial conditions given by φ-sub-Gaussian harmonizable processes. The main results are the bounds for the rate of growth of such stochastic processes considered over unbounded domains. The class of φ-sub-Gaussian processes with $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $1 < \alpha \le 2$, is a natural generalization of Gaussian processes. For such initial conditions the bounds for the distribution of supremum of solutions can be calculated in rather simple form. The bounds for the rate of growth of solution to higher-order partial differential equations with random initial conditions in the case of general φ were obtained in [9], the derivation was based on the results stated in [12]. Here we use another approach, which allows us, for the particular case $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha \in (1, 2]$, to present the expressions for the bounds in the closed form.

Key Words: φ-sub-Gaussian processes, higher-order dispersive equations, random initial condition, rate of growth

1 Introduction

Let $y(t)$, $t \in \mathbb{R}$, be a real strictly φ-sub-Gaussian process with $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $1 < \alpha \le 2$, determining constant C_y and its covariance $\mathsf{E}y(t)y(s) = \Gamma_y(s,t)$ has finite variation. Consider the related harmonizable process

$$\eta(x) = \int_{\mathbb{R}} \kappa(xu)\, dy(u), \; x \in \mathbb{R},$$

where $\kappa(v) = \cos v$ or $\kappa(v) = \sin v$. (The corresponding definitions are given in Appendix.)

Let

$$U(t,x)=\int_{-\infty}^{\infty} I(t,x,\lambda)\,dy(\lambda), \qquad (1.1)$$

where

$$I(t,x,\lambda)=\kappa\left(\lambda x+t\sum_{k=1}^{N}a_k\lambda^{2k+1}(-1)^k\right), \quad (1.2)$$

$t\geq 0$, $x\in\mathbb{R}$, and $\{a_k\}_{k=1}^N$ are some constants.

Suppose that the following condition holds

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\lambda\mu|^{(2N+1)(\rho+1)}\; d|\Gamma_y(\lambda,\mu)|<\infty, \quad (1.3)$$

for some $0<\rho\leq 1$.

In [9, 10] it was shown that under the above settings, the random field $U(t,x)$ given by (1.1) is the classical solution to the initial value problem

$$\sum_{k=1}^{N}a_k\frac{\partial^{2k+1}U(t,x)}{\partial x^{2k+1}}=\frac{\partial U(t,x)}{\partial t},\ t>0,\ x\in\mathbb{R}, \qquad (1.4)$$

$$U(0,x)=\eta(x),\ x\in\mathbb{R}. \qquad (1.5)$$

We refer to [1, 9, 10] for rigorous definition of the classical solution and for a discussion on the above equation and more general ones in random and non-random contexts. Note that the equation (1.4) belongs to the class of linear dispersive equations whose study is important from many points of view, both in theoretical and practical aspects (see, e.g., [14]).

Conditions of existence of the classical solution to the problem (1.4)–(1.5) with random φ-sub-Gaussian initial condition for the case of general φ are stated in [1, 9], properties of solutions are investigated in [9, 10], in particular, different estimates for the distribution of supremum of solution, and in [9] bounds are presented for the rate of growth of solution.

The solutions of heat equations with sub-Gaussian random initial conditions were investigated in [7, 8]. In the papers [4, 5], there were studied sample paths properties of stochastic processes representing solutions (in $L_2(\Omega)$ sense) of the heat equation with random initial conditions given by φ-sub-Gaussian stationary processes. The main results were the bounds for the distributions of the suprema for such stochastic processes considered over bounded and unbounded domains.

The estimates for the distribution of supremum of solution to the problem (1.4)–(1.5) with random φ-sub-Gaussian initial conditions for the particular case of $\varphi(x)=\frac{|x|^\alpha}{\alpha}$, $1<\alpha\leq 2$, are given in [10].

Proposition 1 ([10], Cor.3.1). *Let $y(u)$, $u\in\mathbb{R}$, be a real strictly φ-sub-Gaussian random process with $\varphi(x)=\frac{|x|^\alpha}{\alpha}$, $\alpha\in(1,2]$, determining constant C_y and $\mathsf{E}y(t)y(s)=\Gamma_y(s,t)$.*

Let $U(t,x)=\int\limits_{-\infty}^{\infty} I(t,x,\lambda)\,dy(\lambda)$, where $I(t,x,\lambda)$ is given by (1.2)*, $a\leq t\leq b$, $c\leq x\leq d$, and $\varepsilon_0=\sup_{\substack{a\leq t\leq b\\ c\leq x\leq d}}\tau_\varphi(U(t,x))$.*

Further, let the constant $\rho\in(0,1]$ be such that (1.3) *holds.*

Then

(i) $U(t,x)$ exists, is continuous with probability one and for its sample paths the Hölder continuity holds in the form

$$\sup_{\substack{t,t_1\in[a,b]:|t-t_1|\leq h\\ x,x_1\in[c,d]:|x-x_1|\leq h}}\tau_\varphi(U(t,x)-U(t_1,x_1))\leq 2C_ZC_yh^\rho,$$

where C_Z is defined in (1.8)*;*

(ii) for all $0<\theta<1$ such that $\theta\varepsilon_0<2C_ZC_y(\varkappa/2)^\rho$ with $\varkappa=\max(b-a,d-c)$, γ such that $\frac{1}{\alpha}+\frac{1}{\gamma}=1$, and $u>0$ the following inequality holds true

$$P\Big\{\sup_{\substack{a\leq t\leq b\\ c\leq x\leq d}}|U(t,x)|>u\Big\}\leq 2\exp\Big\{-\frac{u^\gamma(1-\theta)^\gamma}{\gamma\varepsilon_0^\gamma}\Big\} \times(2eC_ZC_y)^{2/\rho}\varkappa^2(\theta\varepsilon_0)^{-2/\rho}. \qquad (1.6)$$

In particular, if the process $y(u)$, $u\in\mathbb{R}\}$, is Gaussian, then

$$P\Big\{\sup_{\substack{a\leq t\leq b\\ c\leq x\leq d}}|U(t,x)|>u\Big\}\leq 2\exp\Big\{-\frac{u^2(1-\theta)^2}{2\varepsilon_0^2}\Big\} \times(2eC_Z)^{2/\rho}\varkappa^2(\theta\varepsilon)^{-2/\rho} \qquad (1.7)$$

for all $0<\theta<1$ such that $\theta\Gamma<2C_Z(\varkappa/2)^\rho$ and $u>0$.

In the above proposition the following

constant is used

$$C_Z^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\Big(\frac{|\lambda|}{2}\Big)^{\rho} + \Big|\frac{1}{2}\sum_{k=1}^{N} a_k \lambda^{2k+1}(-1)^k\Big|^{\rho}\Big)$$
$$\times \Big(\Big(\frac{|\mu|}{2}\Big)^{\rho} + \Big|\frac{1}{2}\sum_{k=1}^{N} a_k \mu^{2k+1}(-1)^k\Big|^{\rho}\Big)\, d|\Gamma_y(\lambda,\mu)|$$
$$\le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2^{2\rho}}\Big(|\lambda|^{\rho} + \Big(\sum_{k=1}^{N} |a_k||\lambda|^{2k+1}\Big)^{\rho}\Big)$$
$$\times \Big(|\mu|^{\rho} + \Big(\sum_{k=1}^{N} |a_k||\mu|^{2k+1}\Big)^{\rho}\Big)\, d|\Gamma_y(\lambda,\mu)|, \quad (1.8)$$

the convergence of the last integral is guaranteed by the condition (1.3).

Class of φ-sub-Gaussian processes with $\varphi(x) = \frac{|x|^{\alpha}}{\alpha}$, $1 < \alpha \le 2$, is a natural generalization of Gaussian processes. For such initial conditions in (1.5) the bounds for the distribution of supremum of solution can be calculated in the simple closed form as given in Proposition 1 above.

In the present paper we continue the study of the behavior of solution in this particular case and obtain the bounds for the rate of growth of solution.

Note that bounds for the rate of growth of solution (1.1) in the case of general φ were obtained in [9], the derivation was based on the results stated in [12]. Here we use another approach based on the use of Theorem 1 stated below in Section 2. This allows us, for the particular case under consideration, to present in Section 3 the expressions for bounds in the closed form. Some necessary definitions and statements are given in Appendix.

2 Estimates for the rate of growth of φ-sub-Gaussian processes

In this section we present some general results for φ-sub-Gaussian processes defined over bounded and unbounded domains.

Let $(\mathbf{T},\rho)$ be a metric (pseudometric) space and $X(t), t \in \mathbf{T}$, be a φ-sub-Gaussian process. Introduce the following conditions.

Condition 1. Let the space $(\mathbf{T},\rho)$ be separable, the process X be separable on this space, $\varepsilon_0 = \sup_{t\in\mathbf{T}} \tau_\varphi(X(t)) < \infty$, and there exists a strictly increasing continuous function $\sigma(h), h \ge 0$, such that $\sigma(0) = 0$ and

$$\sup_{\rho(t,s)<h} \tau_\varphi(X(t) - X(s)) \le \sigma(h).$$

Condition 2. The function $r(x), x \ge 1$, is non-negative nondecreasing and such that $r(e^y), y \ge 0$ is a convex function.

Let $N(u) = N_{\mathbf{T}}(u)$, $u > 0$, be the metric massiveness of the space $(\mathbf{T},\rho)$, that is, $N(u)$ is the number of elements in the minimal u-covering of $(\mathbf{T},\rho)$.

Denote

$$I_r(\delta) = \int_0^{\delta} r(N(\sigma^{(-1)}(u)))du, \ \delta > 0. \quad (2.9)$$

For a function $\sigma(t)$, $t \ge 0$, we denote by $\sigma^{(-1)}(u), u \ge 0$, the inverse function.

Proposition 2. *Let $X = \{X(t), t \in \mathbf{T}\}$ be a φ-sub-Gaussian process, conditions 1–2 hold, $\rho(t,s) = \tau_\varphi(X(t) - X(s))$, $t,s \in T$. Suppose $I_r(\varepsilon_0) < \infty$.*

Then for all $0 < \theta < 1$ and $u > 0, \lambda > 0$

$$\mathsf{E}\exp\Big\{\lambda \sup_{t\in\mathbf{T}} |X(t)|\Big\} \le 2Q(\lambda,\theta),$$

and

$$P\Big\{\sup_{t\in\mathbf{T}} |X(t)| \ge u\Big\} \le 2A(\theta,u),$$

where

$$Q(\lambda,\theta) = \exp\Big\{\varphi\Big(\frac{\lambda\varepsilon_0}{1-\theta}\Big)\Big\} r^{(-1)}\Big(\frac{I_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\Big),$$

$$A(\theta,u) = \exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\} r^{(-1)}\Big(\frac{I_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\Big).$$

Proposition 2 is a variant of the result stated in [2, Theorem 4.4, p. 107] (see also [10, Theorem 2.3]).

Condition 3. Let $f(t)$, $t \ge 0$, be a continuous strictly increasing function such that $f(t) > 0$ and $f(t) \to \infty$ as $t \to \infty$. Introduce the sequence $b_0 = 0, b_{k+1} > b_k, b_k \to \infty, k \to \infty$, $b_{k+1} - b_k \ge 2A$. Denote $V_k = [b_k, b_{k+1}] \times [-A, A], k = 0, 1, \ldots,$ $f_k = f(b_k)$, $\varepsilon_k = \sup_{(t,x)\in V_k} \tau_\varphi(\xi(t,x))$, and suppose that $0 < \varepsilon_k < \infty$.

Denote $\gamma_k = \sigma_k(b_{k+1} - b_k)$, where σ_k are introduced in the next theorem, $\tilde{\theta} = \inf_k \frac{\gamma_k}{\varepsilon_k}$.

Theorem 1. *Let $\xi(t,x)$, $(t,x) \in V$, $V = [0,+\infty) \times [-A,A]$, be a φ-sub-Gaussian separable random field and Conditions 2 and 3 hold. Suppose futher that:*

(*i*) *there exist the increasing continuous functions $\sigma_k(h), h > 0$, such that $\sigma_k(h) \to 0$ as $h \to 0$,*

$$\sup_{\substack{(t_i,x_i)\in V_k, i=1,2\\ |t_1-t_2|\le h,\\ |x_1-x_2|\le h}} \tau_\varphi(\xi(t_1,x_1)-\xi(t_2,x_2)) \le \sigma_k(h)$$

and for $k = 0, 1, \dots$

$$I_{r,k}(\gamma_k) = \int_0^{\gamma_k} r(N_{V_k}(\sigma_k^{(-1)}(u)))du < \infty;$$

(*ii*) $C = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{f_k} < \infty$;

(*iii*) *for any $\theta \in (0,1)$*

$$S(\theta, r) = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{f_k} \log\Big(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\Big) < \infty.$$

Then

(*i*) *for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $\lambda > 0$*

$$\mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|\xi(t,x)|}{f(t)}\Big\} \le 2\exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\}\exp\Big\{\frac{S(\theta,r)}{C}\Big\}; \tag{2.10}$$

(*ii*) *for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $u > 0$*

$$P\Big\{\sup_{(t,x)\in V} \frac{|\xi(t,x)|}{f(t)} > u\Big\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{C}\Big)\Big\}\exp\Big\{\frac{S(\theta,r)}{C}\Big\}. \tag{2.11}$$

Proof. For the proof we use the same arguments as in [5], extended for the case of the domain of the form $[0, +\infty) \times [-A, A]$. Let $r_k > 0, k = 0, 1, \dots$ and $\sum_{k=0}^{\infty} \frac{1}{r_k} = 1$. Then for any $\lambda > 0$

$$I(\lambda) = \mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|\xi(t,x)|}{f(t)}\Big\} \le \mathsf{E}\exp\Big\{\lambda \sum_{k=0}^{\infty} \sup_{(t,x)\in V_k} \frac{|\xi(t,x)|}{f_k}\Big\} \le \prod_{k=0}^{\infty}\Big(\mathsf{E}\exp\Big\{\lambda r_k \sup_{(t,x)\in V_k} \frac{|\xi(t,x)|}{f_k}\Big\}\Big)^{\frac{1}{r_k}}.$$

By using Proposition 2 we obtain

$$I(\lambda) \le \prod_{k=0}^{\infty} 2^{1/r_k}\Big(\exp\Big\{\varphi\Big(\frac{\lambda r_k \varepsilon_k}{(1-\theta)f_k}\Big)\Big\}\Big)^{\frac{1}{r_k}} \times \Big(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\Big)^{\frac{1}{r_k}}$$

$$= \exp\Big\{\sum_{k=0}^{\infty}\varphi\Big(\frac{\lambda r_k \varepsilon_k}{(1-\theta)f_k}\Big)\frac{1}{r_k}\Big\} \times \prod_{k=0}^{\infty} 2^{1/r_k}\Big(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\Big)^{\frac{1}{r_k}}$$

$$= 2\exp\Big\{\sum_{k=0}^{\infty}\varphi\Big(\frac{\lambda r_k \varepsilon_k}{(1-\theta)f_k}\Big)\frac{1}{r_k}\Big\} \times \exp\Big\{\sum_{k=0}^{\infty}\frac{1}{r_k}\log\Big(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\Big)\Big\}.$$

Let $r_k = \frac{Cf_k}{\varepsilon_k}$. Then we obtain the estimate (2.10):

$$I(\lambda) \le 2\exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\} \times \exp\Big\{\frac{1}{C}\sum_{k=0}^{\infty}\frac{\varepsilon_k}{a_k}\log\Big(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\Big)\Big\}.$$

The estimate (2.11) follows then by Chebyshev's inequality.

Corollary 1. *Let conditions of Theorem 1 hold with $\sigma_k(h) = c_k h^\beta$, $c_k > 0$, $0 < \beta \le 1$, and $|b_{k+1} - b_k| \ge 2A$, but condition (iii) is replaced by the following one*

(*iv*) *There exists $0 < \gamma \le 1$ such that*

$$S_1 = \sum_{k=0}^{\infty} \frac{\varepsilon_k^{1-\frac{2\gamma}{\beta}}(b_{k+1}-b_k)^{2\gamma} c_k^{\frac{2\gamma}{\beta}}}{f_k} < \infty.$$

Then

(*i*) *for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $\lambda > 0$*

$$\mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|\xi(t,x)|}{f(t)}\Big\} \le 2^{\frac{4}{\beta}}\exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\}A_1(\theta);$$

(*ii*) *for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $u > 0$*

$$P\Big\{\sup_{(t,x)\in V} \frac{|\xi(t,x))|}{f(t)} > u\Big\} \le 2^{\frac{4}{\beta}}\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{C}\Big)\Big\}A_1(\theta),$$

where

$$A_1(\theta) = \exp\Big\{\frac{S_1}{\gamma C}\Big(\frac{2^{\frac{4}{\beta}-2}}{\theta^{\frac{2}{\beta}}}\Big)^{\gamma}\Big\}.$$

Proof. Under the conditions of Theorem 1 we can write the estimate

$$I_{r,k}(\varepsilon_k) \le \hat{I}_{r,k}(\varepsilon_k) = \int_0^{\varepsilon_k} r\Big(\Big(\frac{b_{k+1}-b_k}{2\sigma_k^{(-1)}(u)}+1\Big)\Big(\frac{A}{\sigma_k^{(-1)}(u)}+1\Big)\Big)du.$$

Therefore, we can apply conditions of Theorem 1 using the integral $\hat{I}_{r,k}$.

We choose $r(x) = x^\alpha - 1, x \ge 1, 0 < \alpha < \beta$ in Theorem 1, then using similar calculations as in [5] (see Corollary 7 therein), we obtain

$$r^{(-1)}\Big(\frac{\hat{I}_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big) \le 2^{\frac{4}{\beta}-1}\Big(\frac{(b_{k+1}-b_k)^2 c_k^{\frac{2}{\beta}} 2^{2(\frac{2}{\beta}-1)}}{(\theta\varepsilon_k)^{\frac{2}{\beta}}}+1\Big).$$

Applying the inequality $\log(1+x) \le \frac{x^\gamma}{\gamma}$, for $0<\gamma\le 1$ and $x \ge 0$, we can write the estimate:

$$S(\theta, r) \le C\log(2^{\frac{4}{\beta}-1}) + \frac{1}{\gamma}\sum_{k=0}^{\infty}\frac{\varepsilon_k^{1-\frac{2\gamma}{\beta}}}{f_k}(b_{k+1}-b_k)^{2\gamma}\Big(\frac{c_k^{\frac{2}{\beta}}2^{\frac{4}{\beta}-2}}{\theta^{\frac{2}{\beta}}}\Big)^\gamma,$$

from which we obtain the expression for $A_1(\theta)$.

Statement (ii) follows from (i) in view of Chebyshev's inequality.

3 Estimates for the rate of growth of the fields $U(t,x)$ over unbounded domains

We apply now the results of the previous section for the filed $U(t,x)$ given by (1.1) with $(t,x) \in V = [0,+\infty)\times[-A,A]$.

Theorem 2. *Let $y(u)$, $u \in \mathbb{R}$, be a real strictly φ-sub-Gaussian random process with $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha \in (1,2]$, determining constant C_y and* $\mathsf{E}y(t)y(s) = \Gamma_y(s,t)$.

Let $U(t,x) = \int\limits_{-\infty}^{\infty} I(t,x,\lambda)\,dy(\lambda)$, where $I(t,x,\lambda)$ is given by (1.2)*, $(t,x) \in V = [0,+\infty)\times[-A,A]$. Further, let the constant $\rho \in (0,1]$ be such that* (1.3) *holds.*

Suppose that:

(i) Condition 2 is satisfied and Condition 3 holds with $\varepsilon_k = \sup_{(t,x)\in V}\tau_\varphi(U(t,x)) < \infty$;

(ii) $C = \sum\limits_{k=0}^{\infty}\frac{\varepsilon_k}{f_k} < \infty$;

(iii) There exists $0<\gamma\le 1$ such that

$$\widehat{S}_1 = \sum_{k=0}^{\infty}\frac{\varepsilon_k^{1-\frac{2\gamma}{\rho}}(b_{k+1}-b_k)^{2\gamma}}{f_k} < \infty.$$

Then for any $\theta \in (0, \min(1,\tilde{\theta}))$ and any $u>0$

$$P\Big\{\sup_{(t,x)\in V}\frac{|U(t,x))|}{f(t)} > u\Big\} \le 2^{\frac{4}{\rho}}\exp\Big\{-\frac{u^\beta(1-\theta)^\beta}{\beta C^\beta}\Big\}\widehat{A}_1(\theta),$$

where

$$\widehat{A}_1(\theta) = \exp\Big\{\frac{\widehat{S}_1(2C_ZC_y)^{2\gamma/\rho}}{\gamma C}\Big(\frac{2^{4/\rho-2}}{\theta^{2/\rho}}\Big)^\gamma\Big\},$$

β is such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$, C_Z is given by (1.8).

Proof. We apply Corollary 1. Conditions of Theorem 1 hold with $\sigma_k(h) = 2C_ZC_yh^\rho$, $\rho \in (0,1]$, therefore $c_k = 2C_ZC_y$. All other conditions are modified correspondingly and lead to the result.

Remark. The bounds for the rate of grows of the random field $U(t,x)$, $(t,x) \in V = [0,+\infty)\times[-A,A]$, which represents the solution to the problem (1.4)–(1.5), were studied in [9] for the case of a general φ. The approach in the present paper is based (following [10]) on consideration of the different entropy integral given by (2.9), which leads to the presentation of bounds in the explicit form, for $\varphi(x) = \frac{|x|^\alpha}{\alpha}$, $\alpha \in (1,2]$. The case of general φ should be further investigated.

Appendix

Definition 1. [2, 13] A continuous even convex function φ is *an Orlicz N-function* if $\varphi(0)=0$, $\varphi(x)>0$, $x\ne 0$, and $\lim\limits_{x\to 0}\frac{\varphi(x)}{x}=0$, $\lim\limits_{x\to\infty}\frac{\varphi(x)}{x}=\infty$.

Condition Q. Let φ be an N-function which satisfies $\liminf\limits_{x\to 0}\frac{\varphi(x)}{x^2}=c>0$, where the case $c=\infty$ is possible.

Definition 2. [2, 3, 11] Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathbf{P}\}$ be a standard probability space. The random variable ζ is φ-sub-Gaussian, or belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, if $\mathsf{E}\zeta = 0$, $\mathsf{E}\exp\{\lambda\zeta\}$ exists for all $\lambda\in\mathbb{R}$ and there exists a constant $a>0$ such that the following inequality holds for all $\lambda\in\mathbb{R}$

$$\mathsf{E}\exp\{\lambda\zeta\} \le \exp\{\varphi(\lambda a)\}.$$

The random process $\zeta = \{\zeta(t), t\in T\}$ is called φ-sub-Gaussian if the random variables $\{\zeta(t), t\in T\}$ are φ-sub-Gaussian.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm (see [2, 11]):

$$\tau_\varphi(\zeta) = \inf\{a > 0 : \mathsf{E}\exp\{\lambda\zeta\} \le \exp\{\varphi(a\lambda)\}.$$

Definition 3. [2, 13] The function φ^* defined by

$$\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$$

is called *the Young-Fenchel transform (or convex conjugate)* of the function φ.

Definition 4. [3, 6] A family Δ of random variables $\zeta \in Sub_\varphi(\Omega)$ is called strictly φ-sub-Gaussian if there exists a constant C_Δ such that for all countable sets I of random variables $\zeta_i \in \Delta$, $i \in I$, the following inequality holds:

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\zeta_i\right) \le C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\zeta_i\right)^2\right)^{1/2}. \tag{3.12}$$

The constant C_Δ is called the *determining* constant of the family Δ.

The linear closure of a strictly φ-sub-Gaussian family Δ in $L_2(\Omega)$ is strictly φ-sub-Gaussian with the same determining constant ([6]).

Definition 5. [3, 6] Random process $\zeta(t), t \in T$ is called (strictly) φ-sub-Gaussian if the family of random variables $\{\zeta(t), t \in T\}$ is (strictly) φ-sub-Gaussian with a determining constant C_ζ.

Let K be a deterministic kernel and $X(t) = \int_T K(t,s)\,d\xi(s)$, $t \in T$, where ξ is a strictly φ-sub-Gaussian random process, and the integral is defined in the mean-square sense. Then X is strictly φ-sub-Gaussian with the same determining constant.

Definition 6. Real-valued second order random function $X(t)$, $t \in \mathbb{R}$, is called harmonizable, if there exists a real-valued centered second order function $y(u)$, $u \in \mathbb{R}$, such that $X(t) = \int_{-\infty}^{\infty} \sin tu\, dy(u)$ or $X(t) = \int_{-\infty}^{\infty} \cos tu\, dy(u)$ and $\Gamma_y(t,s) = \mathsf{E}y(t)y(s)$ has finite variation. The integral is defined in the mean-square sense.

Proposition 3. *Real-valued second order function $X(t), t \in \mathbb{R}$, $\mathsf{E}X(t) = 0$, is harmonizable if and only if there exists the covariance function $\Gamma_y(u,v)$ with finite variation such that*

$$\Gamma_x(t,s) = \mathsf{E}X(t)X(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} \kappa(tu)\kappa(sv)\, d\Gamma_y(u,v),$$

where $\kappa(v) = \cos v$ or $\kappa(v) = \sin v$.

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