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Probabilistic ATL With Incomplete Information

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Abstract

Alternating-time Temporal Logic (ATL) [AHK02] is widely used to reason about strategic abilities of players. Aiming at strategies that can realistically be implemented in software, many variants of ATL [Jam04a] study a setting with incomplete information, where strategies may only take available information into account. Another generalization of ATL is Probabilistic ATL (see e.g., [CL07]), where strategies are required to achieve their goal with a certain probability.

We introduce a semantics of ATL that takes into account both of these aspects. We prove that our semantics allows simulation relations similar in spirit to usual bisimulations, and has a decidable model checking problem (in the case of memoryless strategies, with memory-dependent strategies the problem is undecidable).

1 Introduction

Alternating-time Temporal Logic (ATL) [AHK02] is widely recognized as a suitable logic to reason about strategic abilities. Using an operator $\langle\langle A\rangle\rangle\,\varphi$, it allows to formalize that a coalition (i.e., a set of players) A has a strategy to achieve the goal specified by the formula φ. In practice, requiring only the existence of a strategy is not sufficient: A player also needs to have enough information to implement the strategy. In a realistic environment, each player will only have partial information about the current state of the system. This leads to a restriction of the available strategies to so-called *uniform* ones [Jam04a], where strategies may only take into account information that is available to the player. A further important generalization is probabilistic ATL, where strategies are required to achieve a goal with a certain minimal probability [CL07].

We introduce a new semantics for ATL that takes both incomplete information and probabilism into account. In order to lead to reliable strategies in software implementations, our treatment of incomplete information states rather strict requirements for the admissible strategies: We demand that there is a *deterministic* way for the players to identify their strategies for each goal they want to achieve, given only the (potentially incomplete) knowledge about the current state of the system available to them. Our semantics also allows to require the players to *know* that their strategy is successful. However, we allow coalitions who will work together to agree on a joint set of strategies *before* the start of the game. To see why this is useful, consider a situation in which players are successful if both of them choose the same number (out of several available), and unsuccessful otherwise. Then—without communication—the coalition of these players does not have a success strategy; however if they anticipate this situation and agree on a number before the game is started, obviously they can be successful.

Prior agreement models coalitions where each player trusts each other and can rely on information about the behavior of others. In particular, the setting applies when the players are software programs that are developed together.

During the game, we assume that players may only communicate with each other using explicit moves. This allows to apply our semantics to situations where communication is an integral part of the problems the players want to solve, as the study of cryptographic protocols (see [KR03] and [KKT07] for studies of strategic properties of cryptographic protocols in a game-theoretic setting). Similarly, we treat storing of information as an explicit move and therefore focus on *memoryless* strategies: To determine their move, players only have access to the information they can currently observe, and not to the entire history of the game. In order to be able to model storing of information, we allow infinite game structures. The three main contributions of this paper are the following:

(i) We propose a new semantics for ATL that takes into account incomplete information and probabilism at the same time. We allow players to reach prior agreement about the strategies they will use during the game. We show that when requiring a natural "maximality" condition of the previously agreed strategies, then in the classical deterministic, complete-information setting, our our semantics is equivalent to standard ATL.

(ii) For our setting, we define a notion of simulation similar to the bisimulation obtained for standard ATL in [AHKV98]. These simulations allow to specify strategies on a finite "core" of an infinite structure, and then apply them in the original infinite one (if such a finite core exists). This result paves the way for software implementations of strategies for infinite systems. Our simulation notions may be of independent interest, as they can also be applied to standard (complete-information) semantics of probabilistic ATL. We are not aware of prior result on simulations or bisimulations for ATL in the probabilistic or incomplete-information setting; our definition covers both simultaneously.

(iii) We study the complexity of the model checking problem for our semantics. We prove that the problem is decidable for finite structures, which strengthens the above point of using simulations and small structures to represent infinite systems. The model checking problem is in 3EXPTIME and is 2EXPTIME-hard, but the complexity drops to PSPACE-complete in the case of deterministic game structures. The problem is undecidable for history-dependent strategies.

Related Work There is a rich literature on ATL with incomplete information, going back to the initial ATL introduction in [AHK97], where a variant of incomplete information is already introduced. The notion of uniform strategies that we use was first used in combination with ATL in [Jam04a] (there called incomplete information strategies), and later extended in [JvdH04]. They also discuss methods for a coalition to identify a correct strategy, and allow to separate the roles of the coalitions executing and identifying the strategy. In [Sch04], the model checking complexity of ATL with incomplete information and both history-dependent and memoryless strategies is studied. Further [HT06] discusses an extension of ATL where it is required that players know that they have a strategy, and can identify the corresponding strategy.

The goal of the above-mentioned works is different from ours: We do not consider planning and identifying strategies during the run of a game, but study what can be achieved by coalitions with help of an additional planning phase where coalitions may reach *prior agreement* (modeled in the form of a so-called strategy choice) on strategies for joint goals. This leads to a truth definition that cannot be specified in a purely local way (i.e., as a function of the game structure, the state, and the formula alone). To the best of our knowledge, prior agreement has not been addressed in combination with ATL before (note that in the usual semantics with complete information, prior agreement does not lead to a strategic advantage, see Proposition 2.1).

Probabilistic ATL has been studied in [BJ09], where the success of a coalition's strategy is measured depending on a probability measure describing the (likely) actions of the remainder of the players. In the current paper, we use the pessimistic worst-case assumption about the actions performed by the opponents of a coalition (see the conclusion for a discussion on alternatives). In [CL07], a model checking algorithm for history-dependent strategies for probabilistic ATL is introduced.

To the best of our knowledge our work is the first that studies ATL with incomplete information in a probabilistic setting.

The structure of the paper is as follows: In Section 2, we introduce our semantics for ATL and discuss various aspects of it. We also state the above-mentioned requirement of "maximal" prior agreements, which form a consistency condition for the chosen strategies. We prove that strategies satisfying this condition exist in a large class of game structures. Section 3 introduces our notion of simulation, and proves that it indeed allows to transfer previously agreed strategies. Section 4 contains our results on decidability and complexity of the model checking problem for our semantics. We conclude in Section 5 with a brief discussion of open questions.

2 Semantics for Probabilistic ATL* With Incomplete Information

We now introduce concurrent game structures, strategies, strategy choices (which model prior agreement) ATL*-formulas, and state our semantics definition (Section 2.1). In Section 2.2 we point out some simplifications for deterministic (i.e., not randomized) structures. Section 2.3 then discusses the motivation behind our semantics by way of an example, Section 2.4 shows how to model history-dependent strategies in our (essentially memoryless) setting in a straight-forward way that allows for a later decidability analysis. In Section 2.5 we discuss a subtle point of our semantics concerning "unsuitable" sets of previously-agreed strategies, and show that a natural condition on these strategies avoids the issue while at the same time embedding the standard semantics of ATL* into our framework.

2.1 Concurrent game structures, strategies, and ATL* with incomplete information

The following definition of a concurrent game structure is based on the one from [AHK02]. It extends the latter to infinite structures (see also [KKT07]), a probabilistic setting (see also [CL07]) and a mechanism to deal with incomplete information (see also [JvdH04]):

Definition A *concurrent game structure (CGS)* is a tuple $\mathcal{C} = (\Sigma, Q, \mathbb{P}, \pi, \Delta, \delta, \mathsf{eq})$, where

- Σ is a non-empty, finite set of *players*,
- Q is a non-empty set of *states*,
- $\mathbb{P}$ is a finite set of *propositional variables*,
- $\pi \colon \mathbb{P} \rightarrow \mathcal{P}(Q)$ is a *propositional assignment* (assigning each propositional variable the set of states in which it is true),
- Δ is a *move function* assigning to each state $q \in Q$ and player $a \in \Sigma$ a nonempty set $\Delta(q, a)$ of *moves* available at state q to player a. For $A \subseteq \Sigma$ and $q \in Q$, an (A, q)*-move* is a function c which maps each $a \in A$ to a move $c(a) \in \Delta(q, a)$. A (Σ, q)-move is a *total move.*

- δ is a *probabilistic transition function* which for each state q and total (Σ, q)-move c, returns a probability distribution $\delta(q, c)$ on Q (the state obtained when in q, all players perform their move as specified by c),
- eq is an *information function* $\mathsf{eq}\colon \{1, \ldots, n\} \times \Sigma \to \mathcal{P}(Q \times Q)$, where n is a natural number, and for each $i \in \{1, \ldots, n\}$ and $a \in \Sigma$, $\mathsf{eq}(i, a)$ is an equivalence relation on Q. We also call each $i \in \{1, \ldots, n\}$ a *degree of information.*

A subset $A \subseteq \Sigma$ is also called a *coalition of* $\mathcal{C}$. We often omit "of $\mathcal{C}$" when the game structure is clear from the context. The coalition $\Sigma \setminus A$ is denoted with $\overline{A}$. We often write $\Pr(\delta(q, c) = q')$ for $(\delta(q, c))(q')$. We only allow discrete probability distributions returned by δ, i.e., we require that for each $q \in Q$ and total (Σ, q)-move c, there is a countable set $Q' \subseteq Q$ such that $\sum_{q \in Q'} \Pr(\delta(q, c) = q) = 1$. The information function eq allows to reason about incomplete information: Often a player a will not have complete information about the current state. Hence for each player there is an associated "indistinguishability" relation which is an equivalence relation specifying the states between which a cannot distinguish. To be able to evaluate strategies with different degrees of information for the same player, we specify several relations $\mathsf{eq}(1, a), \ldots, \mathsf{eq}(n, a)$ for each player a. We often write $q_1 \sim_{\mathsf{eq}_i(A)} q_2$ for $(q_1, q_2) \in \cap_{a \in A} \mathsf{eq}(i, a)$ (i.e., no member of the coalition A can distinguish between q_1 and q_2), and write $q_1 \sim_{\mathsf{eq}_i(a)} q_2$ for $q_1 \sim_{\mathsf{eq}_i(\{a\})} q_2$. $\mathcal{C}$ is *deterministic* if its transition function δ is (i.e., the probability distributions returned by δ assigns 1 to a single state and 0 to all others). We say that a player a in $\mathcal{C}$ has *complete information* if $\mathsf{eq}(i, a)$ is the equality relation on the state set for all $i \in \mathbb{N}$. A CGS has complete information if every player has.

To shorten forthcoming examples, we will often assume that for each state q there is a propositional variable with the same name, which is true only in the state q.

We now define the set of formulas used to describe properties and strategic goals in a CGS. The syntax of our formulas is identical to the one of ATL* ([AHK02]), except for the addition of degrees of information and probabilities (the latter are handled as in [CL07]).

Definition Let $\mathcal{C}$ be a CGS with n degrees of information. Then the set of ATL*-*formulas for* $\mathcal{C}$ is defined as follows:

- A propositional variable of $\mathcal{C}$ is a state formula for $\mathcal{C}$,
- conjunctions and negations of state formulas for $\mathcal{C}$ are state formulas for $\mathcal{C}$,
- if A is a coalition, $1 \leq i \leq n$, $0 \leq \alpha \leq 1$, and $\blacktriangleleft$ is one of $\leq, <, \geq, >$, and ψ is a path formula for $\mathcal{C}$, then $\langle\langle A \rangle\rangle_i^{\blacktriangleleft \alpha}\, \psi$ is a state formula for $\mathcal{C}$,

– if A is a coalition, $1 \leq i \leq n$, and ψ is a state formula for $\mathcal{C}$, then $\mathcal{K}_i^A\psi$ is a state formula for $\mathcal{C}$,
– every state formula for $\mathcal{C}$ is a path formula for $\mathcal{C}$,
– conjunctions and negations of path formulas for $\mathcal{C}$ are path formulas for $\mathcal{C}$,
– If φ_1 and φ_2 are path formulas for $\mathcal{C}$, then $\mathsf{X}\varphi_1$ and $\varphi_1\mathsf{U}\varphi_2$ are path formulas for $\mathcal{C}$.

Again, we often omit the "for $\mathcal{C}$" when the structure is clear from the context. The operator $\langle\langle.\rangle\rangle$ is called *strategy operator*. An ATL*-formula is a state formula unless specified otherwise. We define the usual abbreviations, i.e., $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\Diamond\varphi = \mathsf{true}\,\mathsf{U}\varphi$, and $\Box\varphi = \neg\Diamond\neg\varphi$. We also simply write $\langle\langle a\rangle\rangle$ instead of $\langle\langle\{a\}\rangle\rangle$, etc. We say that a $\langle\langle.\rangle\rangle_i$-formula is one whose outmost operator is $\langle\langle A\rangle\rangle_i^{\blacktriangleleft\alpha}$ for some coalition A, and some $\blacktriangleleft\,\alpha$, etc. The set $pls(\varphi)$ is the set of players mentioned in the formula φ, i.e., it contains the player $a \in \Sigma$ if and only if $a \in A$ for some coalition A such that $\langle\langle A\rangle\rangle$ or $\mathcal{K}^A$ appears in φ. In a CGS with only one degree of information, we often omit the i subscript of the strategy operator, similarly in a deterministic CGS we usually omit the probability bound $\blacktriangleleft\,\alpha$ (and understand it to be read as ≥ 1 in deterministic structures).

Note that for technical reasons, we do not allow equality in the formulas: For a coalition to ensure that the probability of achieving a goal φ is *exactly* α requires to either lower or raise the probability of φ in an execution of the game, depending on what an adversial coalition does in parallel paths.

A *path* in a CGS $\mathcal{C}$ is a (possible infinite) sequence λ of states in $\mathcal{C}$. With $\lambda[i]$ we denote the ith state in λ, with $\lambda[i,k]$ the sequence $\lambda[i],\dots,\lambda[k]$, and with $\lambda[i,\infty]$ the (possibly infinite) sequence $\lambda[i],\lambda[i+1],\dots$. A *strategy* is a set of instructions for a player how to proceed in each state. A *uniform* strategy (see also, e.g., [Sch04]) requires these instructions to be identical in states which the player cannot distinguish—this ensures that a player has enough information to determine the move instructed by the strategy in the current state, given the information available to him.

Definition Let $\mathcal{C}$ be a CGS with state set Q and move function Δ, and n degrees of information. For a player a, an *a-strategy* in $\mathcal{C}$ is a function s_a assigning a move to each state such that $s_a(q) \in \Delta(q,a)$ for each $q \in Q$. For $1 \leq i \leq n$, s_a is *i-uniform*, if $q_1 \sim_{\mathsf{eq}_i(a)} q_2$ implies $s_a(q_1) = s_a(q_2)$. For a coalition A, an *A-strategy* is a family $(s_a)_{a\in A}$, where each s_a is an a-strategy.

As mentioned in the introduction, we only consider *memoryless* strategies: The action of a player may only depend on the current state. Usually, ATL* allows *history-dependent* strategies. Since we allow CGSs to be infinite, our semantics canonically allows the treatment of history-dependent strategies (see Sections 2.4 and 3.3), however we will see (Theorems 4.1 and 4.2) that model checking is decidable only for memoryless strategies.

Note that for i-uniform strategies to exists, it is required that for each state q, and player a, there is a move $m \in \cap_{q' \sim_{\mathsf{eq}_i(a)} q} \Delta(q', a)$. We will even assume that for each player a, there is a move $m_a \in \cap_{q \in Q} \Delta(q, a)$, i.e., some move that can be performed in *every* state. This assumption can be made without loss of generality, and can be ensured by simple renaming of moves. Similarly, a "natural" information function should ensure that in two "indistinguishable" states, a player has the same number of possible moves (this could always be satisfied by adding "dummy moves," we do not require this for our results).

A *response* to a coalition A is a function r such that for each $t \in \mathbb{N}$ and each $q \in Q$, $r(t, q)$ is a $(\overline{A}, q)$-move. Hence a response is an arbitrary reaction to the (probabilistic) outcomes of a possible strategy chosen by A. Given a strategy $s_A = (s_a)_{a \in A}$ and a response r to A, the resulting "game" is a Markov process, where the transition probabilities are determined by the probabilistic transition function of the CGS (the moves of the players in A are fixed by s_A, those by $\overline{A}$ are fixed by r: When in the i-th step, the game is in the state q, then a player $a \in A$ perform the move $s_a(q)$, and a player $b \in \overline{A}$ performs the move $r(i, q)$). Note that players in $\Sigma \setminus A$ are not bound by any strategy; they are not restricted to any uniformity conditions and also may act differently when the same state is reached twice during a run of the game. Since the resulting "game" is free of any strategic choices, we can state the following definition:

Definition Let $\mathcal{C}$ be a CGS, let s_A be an A-strategy, let r be a response to A. For a set M of paths over $\mathcal{C}$, and a state $q \in Q$,

$$\Pr\left(q \rightarrow M \mid s_A + r\right)$$

is the probability that in the Markov process resulting from $\mathcal{C}$, s_A, and r with initial state q, the resulting path is an element of M.

Strategies allow players to choose a move in a state. Uniform strategies ensure that a player has sufficient information to determine the correct move. Similarly, players also have to decide on a strategy for a given goal φ; we formalize this using *strategy choices*, that are required to fulfill an analogous uniformity requirement.

Definition Let $\mathcal{C}$ be a CGS with state set Q, and let A be a coalition. A *strategy choice for A in $\mathcal{C}$* is a function S such that for each $a \in A$, $q \in Q$, each $\langle\langle . \rangle\rangle_i$-formula φ for $\mathcal{C}$, $\mathsf{S}(a, q, \varphi)$ is an i-uniform a-strategy in $\mathcal{C}$, and if $q_1 \sim_{\mathsf{eq}_i(a)} q_2$, then $\mathsf{S}(a, q_1, \langle\langle A' \rangle\rangle_i \psi) = \mathsf{S}(a, q_2, \langle\langle A' \rangle\rangle_i \psi)$ for all $A' \subseteq A$.

A strategy choice S for a coalition A models "prior agreement:" Before the game starts, the coalition may agree on a set of suitable strategies to achieve strategic goals specified by ATL*-formulas. These sets of strategies are collected in a strategy choice S. The intended intuition is the following: When in the state q, the coalition A decides (or is instructed to) achieve the goal specified by φ, each player chooses the strategy $\mathsf{S}(a, q, \varphi)$. Hence the resulting A'-strategy is the family $(\mathsf{S}(a, q, \varphi))_{a \in A'}$, which (somewhat abusing notation) we denote with $\mathsf{S}(A', q, \varphi)$. The uniformity conditions for strategy choices and strategies ensure that players have "enough knowledge to *identify* and *execute*" the correct strategy ([JÅ06]).

The information-degree i specified in φ determines the amount of information that the coalition may use to achieve its goal (see the semantics definition below). This allows (by nesting operators) to express statements like "Coalition A does not even have a high-knowledge strategy to reach a state where coalition B has a low-knowledge strategy to achieve φ," even when A and B are not disjoint. We now give the definition of semantics of ATL*-formulas. We will discuss the definition briefly below, and give examples highlighting the main points in the forthcoming Section 2.3

Definition Let $\mathcal{C} = (\Sigma, Q, \mathbb{P}, \pi, \Delta, \delta, \mathsf{eq})$ be a CGS, and let S be a strategy choice for a coalition A in $\mathcal{C}$, let φ_1, φ_2 be state formulas for $\mathcal{C}$, let ψ_1, ψ_2 be path formulas for $\mathcal{C}$, such that $pls(\varphi_1), pls(\varphi_2), pls(\psi_1), pls(\psi_2) \subseteq A$, let $q \in Q$, and let λ be a path over Q. We define

- $\mathcal{C}, \mathsf{S}, q \models p$ iff $q \in \pi(p)$ for $p \in \mathbb{P}$,
- $\mathcal{C}, \mathsf{S}, q \models \neg\varphi_1$ iff $C, \mathsf{S}, q \not\models \varphi_1$,
- $\mathcal{C}, \mathsf{S}, q \models \varphi_1 \wedge \varphi_2$ iff $C, \mathsf{S}, q \models \varphi_1$ and $C, \mathsf{S}, q \models \varphi_2$,
- $\lambda, \mathsf{S} \models \varphi_1$ iff $\mathcal{C}, \mathsf{S}, \lambda[0] \models \varphi_1$,
- $\lambda, \mathsf{S} \models \neg\psi_1$ iff $\lambda, S \not\models \psi_1$,
- $\lambda, \mathsf{S} \models \psi_1 \wedge \psi_2$ iff $\lambda, \mathsf{S} \models \psi_1$ and $\lambda, \mathsf{S} \models \psi_2$,
- $\lambda, \mathsf{S} \models \mathsf{X}\psi_1$ iff $\lambda[1, \infty], S \models \psi_1$,
- $\lambda, \mathsf{S} \models \psi_1 \mathsf{U} \psi_2$ iff there is some $i \geq 0$ such that $\lambda[i, \infty], \mathsf{S} \models \psi_2$ and $\lambda[j, \infty], \mathsf{S} \models \psi_1$ for all $j < i$,
- If $\varphi_1 = \langle\langle A' \rangle\rangle_i^{\blacktriangleleft\alpha} \psi_1$, then $\mathcal{C}, \mathsf{S}, q \models \varphi_1$ iff for every response r to A', we have $\Pr\left(q \to \{\lambda \mid \lambda, \mathsf{S} \models \psi_1\} \mid \mathsf{S}(A', q, \varphi_1) + r\right) \blacktriangleleft \alpha$,
- $\mathcal{C}, \mathsf{S}, q \models \mathcal{K}_i^A \varphi_1$ if $\mathcal{C}, \mathsf{S}, q' \models \varphi_1$ for all $q' \in Q$ with $q' \sim_{\mathsf{eq}_i(A)} q$.

The fact that players may only have limited information about the current state is reflected in the above definition in three ways: First, by only considering uniform strategies, we ensure that players have enough information to *follow* the relevant strategy. Using strategy choices guarantees that players can *identify* their strategy. Finally, the *knowledge operator* $\mathcal{K}_\cdot$ allows to express further requirements about the available strategies: Consider the formula $\mathcal{K}_i^{A'}\,\langle\langle A'\rangle\rangle_i^{\blacktriangleleft\alpha}\,\psi$, for which we introduce the shorthand $\langle\langle \mathcal{K}A'\rangle\rangle_i^{\blacktriangleleft\alpha}$. By the above semantics, this formula is true in a state q with respect to a strategy choice S if and only if the strategies chosen for the formula ψ by S are successful in *every* state q' such that $q' \sim_{\mathsf{eq}_i(A')} q$. We believe that in "pessimistic" settings, this composed operator is in fact more useful than using $\langle\langle A\rangle\rangle_i^{\blacktriangleleft\alpha}$ without the knowledge-prefix: The intuition behind using this operator is that in the state q, the coalition A' cannot rule out that the actual state is q', hence if their chosen strategy is unsuccessful in q', the coalition A' cannot be sure about its success in q either. A similar requirement was made in [Sch04]. Essentially, $\langle\langle \mathcal{K}A'\rangle\rangle_i^{\blacktriangleleft\alpha}$ expresses that the coalition A' has a strategy to ensure that ψ is true with probability $\blacktriangleleft\,\alpha$, and with information degree i, the coalition can agree on the correct strategy, each player can execute the strategy, and the coalition has sufficient (distributed) knowledge to "know" that the strategy is successful.

2.2 Deterministic Game Structures

If $\mathcal{C}$ is a deterministic structure, several of the above definitions can be simplified. This increases readability for the deterministic case, and simplifies proofs of later results that are specific for deterministic structures (see the stronger simulation results that we obtain in this case). In the deterministic setting, the notion of a "response" of $\overline{A}$ to a strategy of a coalition A can be omitted, as for each total move there is exactly one possible follow-up state. Hence in this case the possible follow-up states of an (A,q)-move c is the set $\mathit{next}(q,c) = \{\delta(q,c') \mid c' \text{ is a } (\Sigma,q)\text{-move with } c'(a)=c(a) \text{ for all } a\in A\}$. The possible paths resulting from the application of an A-strategy $s_A = (s_a)_{a\in A}$ in a state q is the set $\mathit{out}(q,s_A)$ containing all paths λ such that $\lambda[0]=q$, and for each i, if c_i is the $(A,\lambda[i])$-move defined by $c_i(a) = s_a(\lambda[i])$ for all $a\in A$, we have that $\lambda[i+1]\in \mathit{next}(\lambda[i],c_i)$. The set of formulas is defined in the same way as in the probabilistic setting, except that we omit probabilities from the formulas. The modified semantics of the strategy operator is as follows: For a formula $\varphi = \langle\langle A'\rangle\rangle_i\,\psi$, we have that $\mathcal{C},\mathsf{S},q \models \langle\langle A'\rangle\rangle_i\,\varphi$ if and only if $\lambda,\mathsf{S}\models\psi$ for all $\lambda\in \mathit{out}(q,\mathsf{S}(A',q,\varphi))$.

2.3 Discussion of Semantics

The intention of our semantics for the $\langle\langle\mathcal{K}.\rangle\rangle$-operator is the following: For a CGS $\mathcal{C}$, a strategy choice S, and a state q of $\mathcal{C}$, the statement $\mathcal{C}, \mathsf{S}, q \models \underbrace{\langle\langle\mathcal{K}A\rangle\rangle_i^{\blacktriangleleft\alpha}\varphi}_{=:\psi}$ is true if and only if in the state q, the members of A can *identify* and *follow* a strategy that achieves φ with probability $p \blacktriangleleft \alpha$, and *know* that the strategy is correct.

This intuitive requirement is captured by our formal definition in the following way: Since in states $q \sim_{\mathsf{eq}_i(a)} q'$, we know that $\mathsf{S}(a, q, \psi) = \mathsf{S}(a, q', \psi)$, a player a is able to *identify* the strategy intended for the goal ψ by the strategy choice S. Since the strategy returned for ψ by S is i-uniform, a has enough information to *follow* the strategy. Finally, since we require that for every state q' such that $q' \sim_{\mathsf{eq}_i(A)} q$, the strategy $\mathsf{S}(A, q', \psi)$ (recall that this is the A-strategy chosen by S to reach the goal ψ in the state q') will achieve the goal, the coalition A also *knows* that the strategy will be successful—note that this does *not* imply that every single player knows this fact (this however can also be expressed in our semantics, by requiring $\bigwedge_{a\in A}\mathcal{K}_i^a \langle\langle A\rangle\rangle_i^{\blacktriangleleft\alpha}$ instead of $\langle\langle\mathcal{K}A\rangle\rangle_i^{\blacktriangleleft\alpha}$).

As an example, consider the CGS $\mathcal{C}$ shown in Figure 1, which revisits the classical "blind and lame agent" example: There is a blind player a who can turn a switch for a light bulb, but does not know whether the light is on or off. There is a second player b who can see, but cannot influence the switch. Formally, the moves of player a are 0 (do nothing), which does not change state, and 1 (turn the switch), which changes state from "On" to "Off" and vice versa. Player b has complete information, while for player a, both states are indistinguishable (there only is a single degree of information).

We now want to evaluate the available strategies for the formula XOn, i.e., the strategies of the players to turn on the light. Obviously, player b alone does not have a strategy to achieve XOn, as b cannot perform any relevant action. Since both states are indistinguishable for player a, any uniform strategy has to perform the same move $\beta \in \{0, 1\}$ in both the On and the Off state. We first consider the strategy s_a^1, which always performs the move 1 (i.e., toggles the switch). Let S_1 be the strategy choice that for the formula XOn returns this strategy.

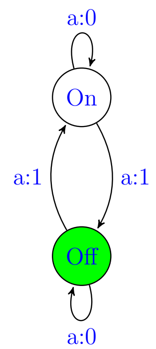

Figure 1: Example "Blind and Lame Agent"

Obviously, the strategy will be successful in the state Off, and will fail in the state On. Since On $\sim_{\mathsf{eq}(a)}$ Off, this implies that $\mathcal{C}, \mathsf{S}_1, \text{Off} \not\models \langle\langle\mathcal{K}a\rangle\rangle_1 \mathsf{X}\text{On}$, even though the selected strategy would be successful in the state - a does not have sufficient information to know that he will be successfully in turning on the light.

Now consider the coalition $A = \{a, b\}$. Since the moves of b are irrelevant, we use the same strategy choice as above (formally, add a dummy move for b). Obviously, this strategy choice is still unsuccessful in the state On, since the selected move does not result in the light being turned on. On the other hand, since the only state q' with $q' \sim_{\mathsf{eq}_1(A)}$ Off is the state Off itself - hence (since the move is successful in Off), it follows that $\mathcal{C}, \mathsf{S}_1, \mathrm{Off} \models \langle\langle \mathcal{K}A \rangle\rangle \, \mathsf{X}\mathrm{On}$. This expresses that *together*, the coalition A has enough information to know that their strategy is successful: When asked by an external environment whether they have a strategy to turn on the light, in the state Off, they would reply as follows (when S_1 is the previously agreed set of strategies):

Player a: "We have a strategy if and only if player b says so."
Player b: "Yes."

When asked the same question in the state On, player a would have to give the same answer (since he does not know whether the current state is On or Off), and player b would reply with "No." Hence the coalition possesses sufficient knowledge to determine whether the strategy is successful in both states.

Also note that when considering the strategy choice S_0, that always chooses the move 0 for a instead, the formula $\langle\langle a \rangle\rangle \, \mathsf{X}\mathrm{On}$ is satisfied in On, but not in Off. In particular this establishes the claim made in the introduction: Our semantics cannot be defined "locally," without fixing the set of previously agreed strategies in a strategy choice (also note that of these two strategy choices, none is strictly "better" than the other, both are equally valid).

Finally note that strategy choices are free to (somewhat counter-intuitively) define different strategies for the formula φ and the formula $\varphi \wedge \varphi$. Similarly one can define a strategy choice S such that for a $\mathcal{C}$ and a state q, and formulas $\varphi \wedge \psi$, we have that $\mathcal{C}, \mathsf{S}, q \models \langle\langle A \rangle\rangle_i^{\blacktriangleleft\alpha} (\varphi \wedge \psi)$, but $\mathcal{C}, \mathsf{S}, q \not\models \langle\langle A \rangle\rangle_i^{\blacktriangleleft\alpha} (\psi \wedge \varphi)$. Although unintuitive, this behavior can be useful: Consider the "blind and lame agent" example (see Figure 1). As argued there, there is a strategy choice S_1 allowing the coalition to turn on the light in the state Off. There is a *different* strategy choice achieving the same goal in the state On. By defining two formulas $\varphi_1 = \mathsf{X}\mathrm{On}$ and $\varphi_2 = \mathsf{X}(\mathrm{On} \wedge \mathrm{On})$, we can construct a strategy choice S such that $\mathcal{C}, \mathsf{S}, \mathrm{On} \models \langle\langle a, b \rangle\rangle \, \varphi_1$ and $\mathcal{C}, \mathsf{S}, \mathrm{Off} \models \langle\langle a, b \rangle\rangle \, \varphi_2$. This can be used to model the situation where an external "environment" decides which strategic choices the players should attempt to reach, and which uses formulas as commands. By sending the formula φ_1 or φ_2 in the states On or Off, the environment can pass additional information about the state to the players, allowing them to reach the goal "turn on the light" on both states —although not in a "uniform" way: the command to turn on the light is different in each state. This reflects that the command needs to contain additional information in order for the players to follow it. Although such a construction might be useful in some situations, we do not study the resulting possibilities in the present paper.

2.4 History Dependence

As mentioned before, ATL* usually allows so-called *history-dependent* strategies, where the action of a player in a state may depend on the entire previous history of the game. Although formally, our framework only considers memoryless strategies, history-dependence can be expressed in a straightforward way: We now define a "history-dependent version" $\mathcal{C}^{hst}$ of a given CGS $\mathcal{C}$, by simply encoding history into the states themselves. Note that if $\mathcal{C}$ is finite, then it also serves as a finite description of $\mathcal{C}^{hst}$, which allows us to algorithmically reason about the latter (see Section 4). The following definition uses the obvious treatment of incomplete information, where a player a can distinguish histories of they have different length of if he can distinguish between individual points in time.

Definition Let $\mathcal{C} = (\Sigma, Q, \mathbb{P}, \Pi, \Delta, \delta, \mathsf{eq})$ be a $\mathcal{C}$ with n degrees of information. Then $\mathcal{C}^{hst}$, the *history-dependent version of* $\mathcal{C}$, is defined as $\mathcal{C}^{hst} = (\Sigma, Q^+, \mathbb{P}, \Pi', \Delta', \delta', \mathsf{eq}')$, where

- Q^+ is the set of non-empty, finite sequences over Q,
- for $p \in \mathbb{P}$, $\Pi'(p) = \{q_1 \dots q_n q \mid q \in \Pi(p), q_1, \dots, q_n \in Q\}$,
- $\Delta'(q_1 \dots q_n, a) = \Delta(q_n, a)$,
- For a state $q_1 \dots q_n \in Q^+$ and a total move c, $\delta'(q_1 \dots q_n, c)$ is defined as $q_1 \dots q_n \delta(q_n, c)$,
- $\mathsf{eq}'(i, a)$ is defined as follows: For sequences $q^1 = q_1 \dots q_n$ and $q^2 = q'_1 \dots q'_m$ we have $q^1 \sim_{\mathsf{eq}'_i(a)} q^2$ if and only if $n = m$ and $q_j \sim_{\mathsf{eq}_i(a)} q'_j$ for all $j \leq n$.

We comment on the relationship between $\mathcal{C}$ and $\mathcal{C}^{hst}$ when we discuss simulation properties in Section 3.3.

2.5 Maximal Strategy Choices

We show that our semantics for ATL* admits strategy choices satisfying formulas which intuitively should be "unsatisfiable." This observation leads to a definition of a very natural subclass of "maximal" strategy choices avoiding this problem. Additionally, for maximal strategy choices our semantics of ATL* agrees with the standard semantics in the deterministic, complete-information case.

Consider the CGS $\mathcal{C}$ shown in Figure 2. $\mathcal{C}$ is a deterministic, complete-information CGS with a single player a. In q_0, there is a single move leading to q_1, in q_1, there are 2 moves leading to q_2 or q_3 respectively. Define φ as $\langle\langle a\rangle\rangle \mathsf{X}\neg \langle\langle a\rangle\rangle \mathsf{X} q_3$. Intuitively (and in standard ATL* semantics), φ expresses that there is a move by a such that in the resulting state, a cannot reach q_3. Intuitively (and in standard ATL*), φ is not satisfied in q_0, since the only available move for a leads to the state q_1, from which the state q_3 can be reached by the move 1. However, for the strategy choice S always returning the strategy that chooses the move 0 in every state, it follows that $\mathcal{C}, \mathsf{S}, q_0 \models \varphi$.

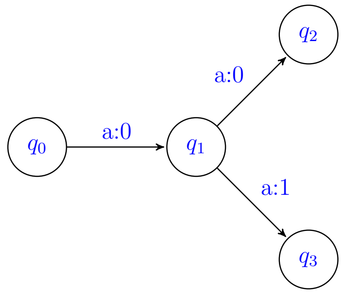

Figure 2: Example

Obviously, S is not very interesting, as it fails to achieve the goal $\mathsf{X} q_3$ (and hence succeeds in achieving φ) *deliberately*, by choosing an unsuccessful strategy although a successful one is available. In particular, the satisfaction of φ does not allow us to conclude that there is a move for a in q_0 such that in the next state, a cannot reach q_3 anymore *even when trying* (note that the example of a coalition A trying to achieve a situation where it is unable to reach a certain goal is not at all contrived: A might want to *commit* to, let's say, a secret value. Then a selection of strategies that could—but does not—violate the commitment is clearly unsatisfying: It is required that the commitment *cannot* be violated anymore, without the assumption of the good-will of A.)

In many situations, the above behavior is not natural: Often we want strategy choices to not deliberately choose "bad" strategies in "innermost" formulas in order to make "outermost" operators true; it should prioritize "innermost" formulas. To formalize this intuition, let $\mathsf{sd}(\varphi)$ (the *strategic depth* of a formula φ) be the maximal nesting degree of strategic operators in φ. For a strategy choice S for a coalition A in a CGS $\mathcal{C}$ and a formula φ with $pls(\varphi) \subseteq A$, with $\mathsf{sat}_\varphi(\mathsf{S}, j)$ we denote the set of pairs (q, ψ) such that ψ is a $\langle\langle . \rangle\rangle$-subformula of φ, $\mathsf{sd}(\psi) = j$, and $\mathcal{C}, S, q \models \psi$. Using this notation, we can define an order on strategy choices, where $\mathsf{S}_1 \leq \mathsf{S}_2$ should mean that S_2 does a a better job of prioritizing formulas with small depth than S_1 does.

Definition Let S_1 and S_2 be strategy choices for a coalition A in a CGS $\mathcal{C}$, let φ be an ATL*-formula for $\mathcal{C}$ with $pls(\varphi) \subseteq A$. Then $\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$ if 1. $\mathsf{sat}_\varphi(\mathsf{S}_1, i) = \mathsf{sat}_\varphi(\mathsf{S}_2, i)$ for all $i \leq \mathsf{sd}(\varphi)$, or 2. for the minimal i such that $\mathsf{sat}_\varphi(\mathsf{S}_1, i) \neq \mathsf{sat}_\varphi(\mathsf{S}_2, i)$, we have $\mathsf{sat}_\varphi(\mathsf{S}_1, i) \subsetneq \mathsf{sat}_\varphi(\mathsf{S}_2, i)$

As an example, in the above-described $\mathcal{C}$ from Figure 2, consider the strategy choice S' which always returns the strategy that in q_0 chooses the move 0 and in q_1, chooses the move 1. It can easily be verified that $\mathsf{S} \leq_\varphi \mathsf{S}'$. The order $\leq_\varphi$ can easily seen to be a preorder.

As usual, a strategy choice S is $\leq_\varphi$-maximal if $\mathsf{S} \leq_\varphi \mathsf{S}'$ implies $\mathsf{S}' \leq_\varphi \mathsf{S}$. In many situations, a restriction to maximal strategy choices is natural. In particular, for the deterministic, complete-information setting, our semantics restricted to maximal strategy choices is equivalent to the usual semantics of ATL* [AHK02]. In the following with $\mathcal{C}, q \models_{\text{ATL-ml}} \varphi$, we mean that the formula φ (with all indices i of an $\langle\langle A\rangle\rangle_i$-operator removed) is satisfied at the state q of the CGS $\mathcal{C}$ in the standard ATL* semantics restricted to memoryless strategies.[1] The following is very easy to show:

Proposition 2.1 *Let $\mathcal{C}$ be a deterministic CGS with complete information, let φ be a formula for $\mathcal{C}$, and let S be a $\leq_\varphi$-maximal strategy choice for pls(φ) in $\mathcal{C}$. Then for all subformulas ψ of φ, the following are equivalent:*

1. *$\mathcal{C}, \mathsf{S}, q \models \psi$,*
2. *$\mathcal{C}, q \models_{\text{ATL-ml}} \psi$.*

Proof. First now that in the complete-information setting, the knowledge operator can be disregarded: In this setting, a formula $\mathcal{K}_i^A\psi$ is obviously equivalent to ψ. Hence we only consider formulas in which this operator does not appear. We show the claim by induction on ψ. We also prove that if λ is a path in $\mathcal{C}$, then $\lambda, \mathsf{S} \models \psi$ if and only if $\lambda \models_{\text{ATL-ml}} \psi$ (where $\models_{\text{ATL-ml}}$ for paths is defined as the formula ψ, stripped of all its information-degree indices, being satisfied by the path λ in standard ATL* semantics). The case where ψ is a propositional variable is trivial, as is the induction step for propositional operators. When the result is true for sub-formulas ψ_1, ψ_2 of φ, the claim for paths and the formulas $\psi_1 \mathsf{U} \psi_2$ as well as $\mathsf{X}\psi_1$ follows trivially (since inductively, ψ_1, ψ_2, and φ are satisfied at the exact same indices of any path). Hence the only interesting case in the induction is when $\psi = \langle\langle A\rangle\rangle_1 \chi$ for some formula χ (note that we only have a single degree of information).

First assume that $\mathcal{C}, \mathsf{S}, q \models \psi$, and let $s_A = \mathsf{S}(A, q, \psi)$ be the joint strategy for A chosen by S. By definition, for all paths $\lambda \in out(q, s_A)$, we have that $\lambda, \mathsf{S} \models \chi$. Hence due to induction, for all $\lambda \in out(q, s_A)$, we have $\lambda \models_{\text{ATL-ml}} \chi$. Hence, $\mathcal{C}, q \models_{\text{ATL-ml}} \psi$, as the strategy s_A is also a valid strategy in standard ATL* semantics.

For the converse, assume that $\mathcal{C}, q \models_{\text{ATL-ml}} \psi$. By definition of standard ATL*-semantics, there is an A-strategy $s_A = \{s_a \mid a \in A\}$ such that $\lambda \models_{\text{ATL-ml}} \chi$ for

[1] We do not define standard ATL* semantics here, for the remainder of the paper the characterization given by Proposition 2.1 suffices.

all $\lambda \in out(q, s_A)$. By induction, it follows that for all $\lambda \in out(q, S_A)$, $\lambda, \mathsf{S} \models \chi$. Assume that $\mathcal{C}, \mathsf{S}, q \not\models \psi$. Then for the modified strategy choice S' which behaves as S except that in q, on input φ, for $a \in A$ returns the strategy s_a, it holds that $\mathsf{S} \leq_\varphi \mathsf{S}'$ and $\mathsf{S}' \not\leq_\varphi \mathsf{S}$ (note that since the only appearing equivalence relation is equality, the uniformity requirements on S' are trivially satisfied, hence S' is a strategy choice). This is a contradiction to the $\leq_\varphi$-maximality of S. □

Note that the same proof also gives the analogous result when considering probabilistic CGSs and the semantics as defined in [CL07], restricted to pure memoryless strategies.

Although maximal strategy choices are often the most natural ones to consider, they do not always exist. As an example consider the CGS $\mathcal{C}$ shown in Figure 3, where we have two players a and b, and all states q_i are indistinguishable for the player a. Player b has complete information. In each state q_i, a has a move j for every $j \in \mathbb{N}$. The resulting state when performing move j in q_i is r if $j \leq i$, and v if $j > i$ (the moves of b are irrelevant and therefore not shown). Consider the formula $\langle\langle \mathcal{K}a, b \rangle\rangle\, \mathsf{X}v$. Since all states q_i are indistinguishable for a, a uniform a-strategy s_a has to pick the same move j in all states q_i. Obviously, a strategy choice choosing such a strategy cannot be maximal, as the strategy choice choosing the strategy always returning $j+1$ ensures the success of $\mathsf{X}v$ in one additional state (note that the player b is required here; in his absence the formula would not be satisfied in any state since a alone does not know whether the strategy is successful, only the observer b has that knowledge).

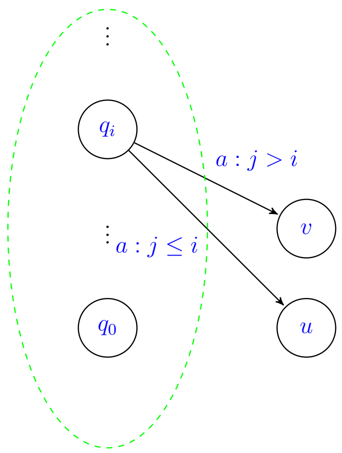

Figure 3: No maximal strategy choice

However, at least for countable structures the above situation of infinitely many indistinguishable states is the only one where maximal strategy choices may fail to exist. We say that a CGS has *finite index*, if for every equivalence relation $\mathsf{eq}(i, a)$, every equivalence class has finitely many elements. This criterion is clearly satisfied by finite CGSs, further if $\mathcal{C}$ has finite index, then $\mathcal{C}^{hst}$ has finite index as well. Trivially a CGS with complete information has finite index. In such CGSs, every strategy choice can be "enhanced" to obtain a maximal one:

Theorem 2.2 *Let $\mathcal{C}$ be a CGS with a countable state set and finite index, let φ be a formula for $\mathcal{C}$ with $pls(\varphi) \subseteq A$, and let S be a strategy choice for A in $\mathcal{C}$. Then there is a $\leq_\varphi$-maximal strategy choice S_{max} for A in $\mathcal{C}$ such that $\mathsf{S} \leq_\varphi \mathsf{S}_{max}$.*

Proof. Let Q denote the set of states of $\mathcal{C}$. Without loss of generality, assume that Q is infinite (otherwise, the theorem is trivial: With a finite state set, we can without loss of generality assume that there is only a finite set of moves, and thus there are only finitely many strategies and strategy choices, hence maximal strategy choices trivially exist). Let m denote the number of degrees of information in $\mathcal{C}$. For a strategy choice S and a subformula χ of φ, with $\mathsf{sat}(\mathsf{S},\chi)$, we denote the set $\{q \in Q \mid \mathcal{C},\mathsf{S},q \models \chi\}$. Let $q_1, q_2, \ldots$ be an enumeration of Q. Since $\mathcal{C}$ has finite index, for all $k \in \mathbb{N}$, there is a number d_k such that for all k', if there is some $a \in \Sigma$ and some $i \in \{1,\ldots,m\}$ with $q_{k'} \sim_{\mathsf{eq}_i(a)} q_k$, then $k' \leq d_k$. Let $\chi_1,\ldots,\chi_n$ be an enumeration of all $\langle\langle . \rangle\rangle$-subformulas of φ such that if $i \leq j$, then $\mathsf{sd}\,(\chi_i) \leq \mathsf{sd}\,(\chi_j)$. We construct S_{max} inductively, where we use (finite) induction on the formulas $\chi_1,\ldots,\chi_n$, and (infinite) induction over the sequence of states. For the partial constructions involved, we define a "partial" version of $\leq_\varphi$, which only considers subformulas up to a specific index, and only a prefix of $q_1, q_2, \ldots$. For $i \leq n$, $k \in \mathbb{N} \cup \{\infty\}$ and strategy choices S_1, S_2 for A, we write $\mathsf{S}_1 \leq_\varphi^{i,k} \mathsf{S}_2$ if for all j, l with $j \leq i-1$ or ($j = i$ and $l \leq k$), if $\mathcal{C},\mathsf{S}_1,q_l \models \chi_j$ and $\mathcal{C},\mathsf{S}_2,q_l \not\models \chi_j$, then there is some j' with $\mathsf{sd}\,(\chi_{j'}) < \mathsf{sd}\,(\chi_j)$ and $\mathsf{sat}(\mathsf{S}_1,\chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_2,\chi_{j'})$. This expresses that "up to formula χ_i and state q_l, the strategy choices S_1 and S_2 are comparable in a way compatible with $\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$." We define $\leq_\varphi^{i,k}$-maximality and $\equiv_\varphi^{i,k}$ in the obvious way. We state a number of straight-forward facts about this order:

Fact 1 *If $i < i'$ or ($i = i'$ and $k \leq k'$), then $\mathsf{S}_1 \leq_\varphi^{i',k'} \mathsf{S}_2$ implies $\mathsf{S}_1 \leq_\varphi^{i,k} \mathsf{S}_2$.*

Proof. Directly by definition. □

Fact 2 *$\mathsf{S}_1 \leq_\varphi^{i,\infty} \mathsf{S}_2$ if and only if $\mathsf{S}_1 \leq_\varphi^{i,k} \mathsf{S}_2$ for all $k \in \mathbb{N}$.*

Proof. This also follows directly from the definition. □

Fact 3 *For $i < n$, $\mathsf{S}_1 \leq_\varphi^{i,\infty} \mathsf{S}_2$ holds if and only if $\mathsf{S}_1 \leq_\varphi^{i+1,0} \mathsf{S}_2$.*

Proof. First assume $\mathsf{S}_1 \leq_\varphi^{i,\infty} \mathsf{S}_2$. To show $\mathsf{S}_1 \leq_\varphi^{i+1,0} \mathsf{S}_2$, let j, l be such that $j \leq i$ or ($j = i+1$ and $l = 0$) and $\mathcal{C},\mathsf{S}_1,q_l \models \chi_j$, $\mathcal{C},\mathsf{S}_2,q_l \not\models \chi_j$. Since q_0 does not exist, it follows that $j \leq i$. Since q_l is a valid state, it follows that $l \in \mathbb{N}$. Due to Fact 2, we know $\mathsf{S}_1 \leq_\varphi^{i,l} \mathsf{S}_2$. Since $j \leq i$, $l \leq l$, there is some j' with $\mathsf{sd}\,(\chi_{j'}) < \mathsf{sd}\,(\chi j)$ and $\mathsf{sat}(\mathsf{S}_1,\chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_2,\chi_{j'})$ as required. For the converse, assume that $\mathsf{S}_1 \leq_\varphi^{i+1,0} \mathsf{S}_2$. Due to Fact 1, it follows that for all $k \in \mathbb{N}$, $\mathsf{S}_1 \leq_\varphi^{i,k} \mathsf{S}_2$. By Fact 2, this implies $\mathsf{S}_1 \leq_\varphi^{i,\infty} \mathsf{S}_2$ as required. □

The following shows that $\leq_\varphi^{i,k}$ is a "partial version" of $\leq_\varphi$:

Fact 4 *$\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$ if and only if $\mathsf{S}_1 \leq_\varphi^{n,\infty} \mathsf{S}_2$.*

Proof. Let $\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$, we show $\mathsf{S}_1 \leq_\varphi^{n,\infty} \mathsf{S}_2$. Hence let $j \leq n$, let $l \in \mathbb{N}$, let $\mathcal{C}, \mathsf{S}_1, q_l \models \chi_j$ and $\mathcal{C}, \mathsf{S}_2, q_l \not\models \chi_j$. Let d be minimal with $\mathsf{sat}_\varphi(\mathsf{S}_1, d) \neq \mathsf{sat}_\varphi(\mathsf{S}_2, d)$. Obviously, $d \leq \mathsf{sd}(\chi_j)$. Since $\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$, we know $\mathsf{sat}_\varphi(\mathsf{S}_1, d) \subsetneq \mathsf{sat}_\varphi(\mathsf{S}_2, d)$. Since $\mathsf{sat}(\mathsf{S}_1, \chi_j) \nsubseteq \mathsf{sat}(\mathsf{S}_2, \chi_j)$, we know $d < \mathsf{sd}(\chi_j)$. Since $\mathsf{sat}_\varphi(\mathsf{S}_1, d) \subsetneq \mathsf{sat}_\varphi(\mathsf{S}_2, d)$, there is some j' with $\mathsf{sd}(\chi_{j'}) = d < \mathsf{sd}(\chi_j)$, and $\mathsf{sat}(\mathsf{S}_1, \chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_2, \chi_{j'})$ as required.

Now assume that $\mathsf{S}_1 \leq_\varphi^{n,\infty} \mathsf{S}_2$. To prove $\mathsf{S}_1 \leq_\varphi \mathsf{S}_2$, assume there is a minimal i with $\mathsf{sat}_\varphi(\mathsf{S}_1, i) \neq \mathsf{sat}_\varphi(\mathsf{S}_2, i)$. We need to show $\mathsf{sat}_\varphi(\mathsf{S}_1, i) \subseteq \mathsf{sat}_\varphi(\mathsf{S}_2, i)$. Indirectly assume there is some j with $\mathsf{sd}(\chi_j) = i$ and a state q_l such that $\mathcal{C}, \mathsf{S}_1, q_l \models \chi_j$, and $\mathcal{C}, \mathsf{S}_2, q_l \not\models \chi_j$. Since $\mathsf{S}_1 \leq_\varphi^{n,\infty} \mathsf{S}_2$, there is some j' with $i' := \mathsf{sd}(\chi_{j'}) < \mathsf{sd}(\chi_j) = i$ and $\mathsf{sat}(\mathsf{S}_1, \chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_2, \chi_{j'})$. This a contradiction, since $i' < i$, and i is minimal with $\mathsf{sat}_\varphi(\mathsf{S}_1, i) \neq \mathsf{sat}_\varphi(\mathsf{S}_2, i)$. ☐

We now construct the maximal strategy choice S_{max} inductively: For $0 \leq i \leq n$, we construct S_i such that

1. $\mathsf{S} \leq_\varphi^{i,\infty} \mathsf{S}_i$, and
2. S_i is maximal with respect to $\leq_\varphi^{i,\infty}$.

We choose $\mathsf{S}_0 = \mathsf{S}$ (note that $\mathsf{S}_1 \leq_\varphi^{0,\infty} \mathsf{S}_2$ is true for all S_1, S_2, hence our choice of S_0 satisfies the required conditions). Assume that S_i has been defined. For S_{i+1}, we again use an inductive construction, where induction is over the sequence of states: For all $k \in \mathbb{N}$, we define $\mathsf{S}_{i+1,k}$ such that 1. $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1,k}$, and 2. $\mathsf{S}_{i+1,k}$ is $\leq_\varphi^{i+1,k}$-maximal among those S' with $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}'$.

We define $\mathsf{S}_{i+1,0}$ as follows: For a player $a \in A$, a number $1 \leq j \leq n$, and a state q, let

$$\mathsf{S}_{i+1,0}(a, q, \chi_j) = \begin{cases} \mathsf{S}_i(a, q, \chi_j), & \text{if } j \leq i, \\ \mathsf{S}(a, q, \chi_j), & \text{otherwise.} \end{cases}$$

Obviously, $\mathsf{S}_{i+1,0}$ satisfies the uniformity requirements of strategy choices (as for different formulas, different existing strategy choices are used). We claim that $\mathsf{S}_{i+1,0}$ satisfies the required conditions:

1. We show $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1,0}$. For this, let $j \leq i+1$, let $l \in \mathbb{N}$ such that $\mathcal{C}, \mathsf{S}, q_l \models \chi_j$, and $\mathcal{C}, \mathsf{S}_{i+1,0}, q_l \not\models \chi_j$. Let j be minimal such that a corresponding l exists.

   Case 1: $j \leq i$. Then for the formula χ_j and its subformulas (with must appear before χ_j in the enumeration $\chi_1, \ldots, \chi_n$), $\mathsf{S}_{i+1,0}$ behaves as S_i does. Hence $\mathcal{C}, \mathsf{S}_i, q_l \not\models \chi_j$. Since inductively, we know $\mathsf{S} \leq_\varphi^{i,\infty} \mathsf{S}_i$, there is some j' with $\mathsf{sd}(\chi_{j'}) < \mathsf{sd}(\chi_j)$, and $\mathsf{sat}(\mathsf{S}, \chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_i, \chi_{j'}) = \mathsf{sat}(\mathsf{S}_{i+1,0}, \chi_{j'})$ as required.

Case 2: $j = i + 1$. Due to the minimality of j, we know that $\mathsf{sat}(\mathsf{S}, \chi_{j'}) \subseteq \mathsf{sat}(\mathsf{S}_{i+1,0}, \chi_{j'})$ for all $j' \leq i$. If there is some j' with $\mathsf{sd}(\chi_{j'}) < \mathsf{sd}(\chi_{i+1})$ and $\mathsf{sat}(\mathsf{S}, \chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_{i+1,0}, \chi_{j'})$, the claim follows. Hence assume that $\mathsf{sat}(\mathsf{S}, \chi_{j'}) = \mathsf{sat}(\mathsf{S}_{i+1,0}, \chi_{j'})$ for all j' with $\mathsf{sd}(\chi_{j'}) < \mathsf{sd}(\chi_{i+1})$. In particular, for all proper $\langle\langle . \rangle\rangle$-subformulas of χ_j, the strategy choices S and S_{i+1} lead to the exact same set of satisfying states. Since $\mathsf{S}_{i+1,0}$ uses the same strategy as S, the strategy of $\mathsf{S}_{i+1,0}$ is successful as well, which is a contradiction to $\mathcal{C}, \mathsf{S}, q_l \models \chi_j$ and $\mathcal{C}, \mathsf{S}_{i+1,0}, q_l \not\models \chi_j$.

2. We show that $\mathsf{S}_{i+1,0}$ is $\leq_\varphi^{i+1,0}$-maximal (even unconditionally). For this, assume that $\mathsf{S}_{i+1,0} \leq_\varphi^{i+1,0} \mathsf{S}'$. We need to show $\mathsf{S}' \leq_\varphi^{i+1,0} \mathsf{S}_{i+1,0}$. Due to Fact 3, it suffices to show $\mathsf{S}' \leq_\varphi^{i,\infty} \mathsf{S}_{i+1,0}$. By construction, for formulas χ_j with $j \leq i$, the strategy choices $\mathsf{S}_{i+1,0}$ and S_i behave identically. Hence $\mathsf{S}_i \equiv_\varphi^{i,\infty} \mathsf{S}_{i+1,0}$. Since by induction, S_i is $\leq_\varphi^{i,\infty}$-maximal, and (by the above and Fact 3), we know that $\mathsf{S}_{i+1,0} \equiv_\varphi^{i,\infty} \mathsf{S}_i \leq_\varphi^{i,\infty} \mathsf{S}'$, it follows that $\mathsf{S}' \leq_\varphi^{i,\infty} \mathsf{S}_i \equiv_\varphi^{i,\infty} \mathsf{S}_{i+1,0}$, as required.

Now assume that for some $k \geq 0$, $\mathsf{S}_{i+1,k}$ has been defined satisfying the above. We define $\mathsf{S}_{i+1,k+1}$ as follows: Let

$$Q_{i+1,k+1} = \left\{ q \in Q \mid \text{there is some } a \in \Sigma, t \text{ such that } q \sim_{\mathsf{eq}_t(a)} q_{k+1} \right\}.$$

Since $\mathcal{C}$ has finite index, it follows that $Q_{i+1,k+1}$ is finite. $Q_{i+1,k+1}$ is the set of states that may be relevant for the strategies chosen in the state q_{k+1} (as a player must choose the same strategy in indistinguishable states). Let $M_{i+1,k+1}$ be maximal such that there is a strategy choice $\mathsf{S}_{i+1,k+1}$ such that 1. $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1,k+1}$, 2. $\mathsf{S}_{i+1,k} \leq_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1}$, 3. $\mathcal{C}, \mathsf{S}_{i+1,k+1}, q \models \chi_{i+1}$ for all $q \in M_{i+1,k+1}$. If possible with the first two points, choose $\mathsf{S}_{i+1,k+1}$ such that $\mathcal{C}, \mathsf{S}_{i+1,k+1}, q_{k+1} \models \chi_{i+1}$. A strategy choice satisfying the first two obviously exists, as $\mathsf{S}_{i+1,k}$ satisfies both points by induction. Since $Q_{i+1,k+1}$ is finite, a maximal subset $M_{i+1,k+1}$ exists and thus $\mathsf{S}_{i+1,k+1}$ can be chosen satisfying the above. We require an additional condition of $\mathsf{S}_{i+1,k+1}$—this is important, as it ensures that the strategy chosen for χ_{i+1} reaches a "stable" point in the construction (note that $\mathsf{S}_{i+1,k+1}$ and $\mathsf{S}_{i+1,k}$ may result in different strategies even for states q_l with $l < k$, as the state q_{k+1} might be equivalent to q_l for some of the involved players, and the previous strategy may not be "optimal" in the state q_{k+1}).

Fact 5 $\mathsf{S}_{i+1,k+1}$ *can be chosen such that* $\mathsf{S}_{i+1,k+1}(a, \chi_j, q_l) = \mathsf{S}_{i+1,k}(a, \chi_j, q_l)$ *for all* j, l *with* $j \leq i$ *or* ($j = i$ *and* $l \leq k$ *and* $q_l \neq Q_{i+1,k+1}$).

Proof. Since $\mathsf{S}_{i+1,k} \leq_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1}$ and the inductive maximality of the former among those strategy choices that are $\leq_\varphi^{i+1,\infty}$-above S (which the latter also is), we know that $\mathsf{S}_{i+1,k} \equiv_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1}$. Hence for $j \leq i$, we have $\mathsf{sat}(\mathsf{S}_{i+1,k}, \chi_j) =$

$\mathsf{sat}(\mathsf{S}_{i+1,k+1}, \chi_j)$. For those formulas, we can thus let $\mathsf{S}_{i+1,k+1}$ choose the same strategies as $\mathsf{S}_{i+1,k}$ without changing any relevant property. Further, it is obviously irrelevant how $\mathsf{S}_{i+1,k+1}$ is defined for the formula χ_{i+1} at states that are no elements of $Q_{i+1,k+1}$ for the above maximality condition. Hence $\mathsf{S}_{i+1,k+1}$ can be chosen to behave in the same way as $\mathsf{S}_{i+1,k}$ for these arguments. $\square$

We now show that $\mathsf{S}_{i+1,k+1}$ satisfies the claimed conditions. The first condition ($\mathsf{S} \leq_\varphi^{i,\infty} \mathsf{S}_i$) is trivial by choice of $\mathsf{S}_{i+1,k+1}$. The second condition requires that $\mathsf{S}_{i+1,k+1}$ is $\leq_\varphi^{i+1,k+1}$-maximal among all S' with $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}'$. Hence let $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}'$, and let $\mathsf{S}_{i+1,k+1} \leq_\varphi^{i+1,k+1} \mathsf{S}'$. We show $\mathsf{S}' \leq_\varphi^{i+1,k+1} \mathsf{S}_{i+1,k+1}$. Since $\mathsf{S}_{i+1,k} \leq_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1} \leq_\varphi^{i+1,k} \mathsf{S}'$, and $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1,k+1}, \mathsf{S}'$, and $\mathsf{S}_{i+1,k}$ is (by induction) $\leq_\varphi^{i+1,k}$ maximal among all strategy choices that are $\leq_\varphi^{i+1,\infty}$-above S, it follows that $\mathsf{S}_{i+1,k} \equiv_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1} \equiv_\varphi^{i+1,k} \mathsf{S}'$. Indirectly assume that $\mathsf{S}' \not\leq_\varphi^{i+1,k+1} \mathsf{S}_{i+1,k+1}$. Due to the $\equiv_\varphi^{i+1,k}$-equivalence above, it then follows that $\mathcal{C}, \mathsf{S}', q_{k+1} \models \chi_{i+1}$, and $\mathcal{C}, \mathsf{S}_{i+1,k+1}, q_{k+1} \not\models \chi_{i+1}$. Since $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}'$ and $\mathsf{S}_{i+1,k} \equiv_\varphi^{i+1,k} \mathsf{S}_{i+1,k+1} \equiv_\varphi^{i+1,k} \mathsf{S}'$, in the choice of $\mathsf{S}_{i+1,k+1}$, we are in the "if possible" case, and thus have chosen $\mathsf{S}_{i+1,k+1}$ such that $\mathcal{C}, \mathsf{S}_{i+1,k+1}, q_{k+1} \models \chi_{i+1}$. This is a contradiction, hence $\mathsf{S}_{i+1,k+1}$ satisfies the required conditions.

We now define S_{i+1} as follows: Let $a \in A$, let χ_j be one of the subformulas of φ, and let $q_k \in Q$. If $j \leq i$, simply define $\mathsf{S}_{i+1}(a, \chi_j, q_k) = \mathsf{S}_i(a, \chi_j, q_k)$. Otherwise, let d_k be maximal such that $q_{d_k} \sim_{\mathsf{eq}_t(a)} q_k$ for some t, and let $\mathsf{S}_{i+1}(a, \chi_j, q_k) = \mathsf{S}_{i+1,d_k}(a, \chi_j, q_k)$. Since the choice of d_k depends only on the equivalence class of q_k, and S_{i+1,d_k} is a strategy choice, the thus-constructed S_{i+1} obviously is a strategy choice as well. We show that S_{i+1} satisfies the necessary conditions.

1. We show $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1}$. Let $j \leq i+1$, $l \in \mathbb{N}$ such that $\mathcal{C}, \mathsf{S}, q_l \models \chi_j$, and $\mathcal{C}, \mathsf{S}_{i+1}, q_l \not\models \chi_j$. It follows that $\mathcal{C}, \mathsf{S}_{i+1,d_l}, q_l \not\models \chi_j$. Since by choice of S_{i+1,d_l}, we know that $\mathsf{S} \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1,d_l}$, there is some j' with $\mathsf{sd}(\chi_{j'}) < \mathsf{sd}(\chi_j)$ and $\mathsf{sat}(\mathsf{S}, \chi_{j'}) \subsetneq \mathsf{sat}(\mathsf{S}_{i+1,d_l}, \chi_{j'}) = \mathsf{sat}(\mathsf{S}_{i+1}, \chi_{j'})$ as required.
2. We show that S_{i+1} is $\leq_\varphi^{i+1,\infty}$-maximal. Hence let $\mathsf{S}_{i+1} \leq_\varphi^{i+1,\infty} \mathsf{S}'$. We show $\mathsf{S}' \leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1}$. Obviously, $\mathsf{S}_i \leq_\varphi^{i,\infty} \mathsf{S}_{i+1} \leq_\varphi^{i,\infty} \mathsf{S}'$. Due to the former's maximality (by induction), we have $\mathsf{S}_i \equiv_\varphi^{i,\infty} \mathsf{S}_{i+1} \equiv_\varphi^{i,\infty} \mathsf{S}'$. Assume $\mathsf{S}' \not\leq_\varphi^{i+1,\infty} \mathsf{S}_{i+1}$. Since $\mathsf{S}_{i+1} \equiv_\varphi^{i,\infty} \mathsf{S}'$ and $\mathsf{S}_{i+1} \leq_\varphi^{i+1,\infty} \mathsf{S}'$, we know $\mathsf{sat}(\mathsf{S}_{i+1}, \chi_{i+1}) \subsetneq \mathsf{sat}(\mathsf{S}', \chi_{i+1})$. Hence there is some k such that $\mathcal{C}, \mathsf{S}', q_k \models \chi_{i+1}$, and $\mathcal{C}, \mathsf{S}_{i+1}, q_k \not\models \chi_{i+1}$. Let d_k be minimal such that $q_l \in Q_{i+1,k}$ implies $l \leq d_k$. By construction, for all players $a \in A$ and all $q \in Q_{i+1,d_k}$, we have $\mathsf{S}_{i+1}(a, \chi_{i+1}, q) = \mathsf{S}_{i+1,d_k}(a, \chi_{i+1}, q)$. Recall that M_{i+1,d_k} is defined as $\{q \in Q_{i+1,d_k} \mid \mathcal{C}, \mathsf{S}_{i+1,d_k}, q \models \chi_{i+1}\}$. Let $M = \{q \in Q_{i+1,d_k} \mid \mathcal{C}, \mathsf{S}', q \models \chi_{i+1}\}$. Since $\mathsf{S}_{i+1,d_k} \leq_\varphi^{i+1,d_k} \mathsf{S}_{i+1} \leq_\varphi^{i+1,d_k} \mathsf{S}'$, we know that $M_{i+1,d_k} \subseteq M$, and $q_l \in Q_{i+1,d_k}$ implies $l \leq d_k$. Further, we know that $q_k \in M \setminus M_{i+1,d_k}$. This is a contradiction to the maximality of M_{i+1,d_k} (note that since $\mathsf{S}_{i+1} \leq_\varphi^{i+1,\infty} \mathsf{S}'$, S' satisfies both conditions necessary for the choice of S_{i+1,d_k}). Hence S_{i+1} is $\leq_\varphi^{i+1,\infty}$-maximal as required.

We now choose $\mathsf{S}_{max} = \mathsf{S}_n$. It follows that $\mathsf{S} \leq_\varphi^{n,\infty} \mathsf{S}_{max}$, and S_{max} is $\leq_\varphi^{n,\infty}$-maximal. By Fact 4, this implies that $\mathsf{S} \leq_\varphi \mathsf{S}_{max}$, and S_{max} is $\leq_\varphi$-maximal, as required. □

3 Simulation relations

Simulations and Bisimulations are often used to relate structures to one another in a way preserving "interesting" features: A bisimulation between structures S_1 and S_2 with state sets Q_1 and Q_2 is usually a relation $Z \subseteq Q_1 \times Q_2$ such that when $(q_1, q_2) \in Z$, then q_1 and q_2 satisfy the same properties (e.g., the same logical formulas). In our case, a simulation allows to "translate" a strategy choice from one (potentially "easy") CGS to another (potentially "complicated") one. This allows players to construct their joint strategy choice on an "easy" structure and apply it in the "complicated" one (when the description of the simulation relation itself is of manageable size). This feature is particularly attractive since model checking for finite structures is decidable, see Section 4.

Bisimulations for ATL* were originally defined in [AHKV98] (see also [LMO08] for a definition which is closer to our adaption). The additional requirements that we make of our simulations are required to deal with incomplete information, probabilism, and explicit strategies:

1. We require certain *uniformity* conditions similar to the ones required for strategies and strategy choices,
2. we demand that moves between related states can be transferred in a deterministic (and uniform) way,
3. we handle probabilities in the natural way,
4. since we want to transfer strategy choices, our simulations only need to work for a particular coalition (and then allows to transfer strategy choices for that coalition).

We only state definitions for (unidirectional) simulations; a bisimulation analogously can be defined as a relation that is a simulation in both directions. A bisimulation then allows to translate strategy choices in both directions, hence essentially establishing "strategic equivalence" of the structures for the coalition in question.

In Section 3.1 we give the formal definition of our simulation relations and state the mentioned result allowing to use simulations as a way to transfer strategies. This result is then proven in Section 3.2. Section 3.3 then discusses properties of simulations.

3.1 Definitions and Basic Properties

We give two separate definitions for the deterministic and the probabilistic case, since the first one allows a somewhat relaxed set of conditions. The basic properties are the same in both cases: As mentioned above, we need to require that principals can "transfer" their moves from one CGS to the other with only the information available to them. We first give the definition for the more involved probabilistic case. In the following, when Z is a binary relation and λ_1, λ_2 are paths of the same length, we write $\lambda_1 \sim_Z \lambda_2$ to indicate that $(\lambda_1[i], \lambda_2[i]) \in Z$ for all relevant i. Also, for a state q, we write $Z(q)$ to denote the set $\{q' \mid (q, q') \in Z\}$.

Definition Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CGSs with state sets Q_1 and Q_2, the same set of players, the same set of propositional variables, and n degrees of information. Then a relation $Z \subseteq Q_1 \times Q_2$ is a *probabilistic uniform strong alternating simulation for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$* if for all $(q_1, q_2) \in Z$, all $i \in \{1, \ldots, n\}$, and all players $a \in A$, there is a function $\Delta^{1\to 2}_{(i,a,q_1,q_2)}$ such that for all $A' \subseteq A$ we have

- *propositional equivalence:* q_1 and q_2 satisfy the same propositional variables,
- for all (A', q_1)-moves c_1, the (A', q_2)-move c_2 with $c_2(a) = \Delta^{1\to 2}_{(i,a,q_1,q_2)}(c_1(a))$ has the
  1. *Forward Move Property:* for each $(\overline{A'}, q_1)$-move $c_1^{\overline{A'}}$, there is a $(\overline{A'}, q_2)$-move $c_2^{\overline{A'}}$ such that for all $q_1' \in Q_1$, we have
  $$\Pr\left(\delta(q_2, c_2 \cup c_2^{\overline{A'}}) \in Z(q_1')\right) = \Pr\left(\delta(q_1, c_1 \cup c_1^{\overline{A'}}) = q_1'\right).$$
  2. *Backward Move Property:* for each $(\overline{A'}, q_2)$-move $c_2^{\overline{A'}}$, there is a $(\overline{A'}, q_1)$-move $c_1^{\overline{A'}}$ such that for all $q_1' \in Q_1$, we have
  $$\Pr\left(\delta(q_2, c_2 \cup c_{q_2}^{\overline{A'}}) \in Z(q_1')\right) = \Pr\left(\delta(q_1, c_1 \cup c_1^{\overline{A'}}) = q_1'\right).$$
- *Move Uniformity:* If $(q_1, q_2), (q_1', q_2') \in Z$ with $q_1 \sim_{\mathsf{eq}_1^i(a)} q_1'$ and $q_2 \sim_{\mathsf{eq}_1^i(a)} q_2'$, then $\Delta^{1\to 2}_{(i,a,q_1,q_2)} = \Delta^{1\to 2}_{(i,a,q_1',q_2')}$,
- *Uniformity:* for all $a \in A$, and all $(q_1', q_2') \in Z$, if $q_2 \sim_{\mathsf{eq}_2^i(a)} q_2'$, then $q_1 \sim_{\mathsf{eq}_1^i(a)} q_1'$.
- *Knowledge Transfer:* if $q_1' \sim_{\mathsf{eq}_i^1(A')} q_1$, then there is some $q_2' \in Q_2$ such that $q_2' \sim_{\mathsf{eq}_i^2(A')} q_2$ and $(q_1', q_2') \in Z$.
- *Uniqueness:* For all $q_2 \in Q_2$, there is exactly one $q_1 \in Q_1$ with $(q_1, q_2) \in Z$ (i.e., $Z^{-1} \colon Q_2 \to Q_1$ is a function).

Note that even though if Z is a probabilistic uniform strong alternating simulation then Z^{-1} is a function, we still write Z as a relation in order to be able to treat the probabilistic and deterministic cases in a uniform way (see Theorem 3.1 and its proof in Section 3.2). We comment a bit on the requirements in the above definition:

- Propositional equivalence is obviously necessary if Z-related states should have the same properties.
- The requirement of the existence of $\Delta_{\dots}^{1\to 2}$ ensures that in states related via Z, players have a method to transfer their moves from $\mathcal{C}_1$ to $\mathcal{C}_2$ that does not depend on how other players act in the same state, and (due to the move uniformity requirement) only depends on the equivalence class of the current state; this ensures that a player has enough information to determine the move suggested by applying the simulation. The existence of this function, together with the forward and backward move properties forms the "core" of the simulation: These requirements ensure that every move in one of the structures can be "mirrored" in the other such that for a potential follow-up state $q_1' \in Q_1$, the probability of reaching q_1' in $\mathcal{C}_1$ is the same as the one for reaching a state Z-related to q_1' in $\mathcal{C}_2$.
- Uniformity is a basic "compatibility" requirement between the involved equivalence relations and the relation Z. It implies that the function Z^{-1} can be "computed" by the players in question given their local information: Given a state q_2 of $\mathcal{C}_2$, a player a can determine the equivalence class of $Z^{-1}(q_2)$ with respect to its own indistinguishability relation.
- Knowledge transfer ensures that if $(q_1, q_2) \in Z$, and a group of principals cannot distinguish between q_1 and q_1', then in the "simulated world," there also is a pair of states (related in the same way by Z) that the principals cannot distinguish. This implies that in $\mathcal{C}_2$, the players have the same amount of information as in $\mathcal{C}_1$. It is necessary to be able to transform truth of formulas using the knowledge operator, from the later proofs it is obvious that for a restricted language omitting this operator, requiring knowledge transfer is unnecessary.
- Z^{-1} is required to be a function in order to be able to make precise statements about the involved probabilities—see the proof of the later Lemma 3.3, where this property is used crucially to relate sums of probabilities in CGSs related by a probabilistic uniform strong alternating simulation.

In the deterministic case, we can relax the conditions: Since we do not have to take care of exactly-matching probabilities, we do not have to require Z^{-1} to be a function. In order to increase readability, we give the full definition of the deterministic case instead of merely pointing out the differences. We refer to the simplified definitions for the deterministic case introduced in Section 2.2.

Definition Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CGSs with state sets Q_1 and Q_2, the same set of players and the same set of propositional variables, and n degrees of information. Then a relation $\emptyset \neq Z \subseteq Q_1 \times Q_2$ is a *uniform strong alternating simulation for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$* if for all $(q_1, q_2) \in Z$, all $i \in \{1, \ldots, n\}$, and and all players $a \in A$, there is a function $\Delta^{1\to 2}_{(i,a,q_1,q_2)}$ such that for all $A' \subseteq A$,

- Z satisfies propositional equivalence, move uniformity, uniformity, and knowledge transfer.
- for all (A', q_1)-moves c_1, the (A', q_2)-move c_2 with $c_2(a) = \Delta^{1\to 2}_{(i,a,q_1,q_2)}(c_1(a))$ has the
  1. *Forward Move Property:* for all $q_1' \in \mathit{next}(q_1, c_1)$, there is some $q_2' \in \mathit{next}(q_2, c_2)$ with $(q_1', q_2') \in Z$.
  2. *Backward Move Property:* for all $q_2' \in \mathit{next}(q_2, c_2)$, there is some $q_1' \in \mathit{next}(q_1, c_1)$ with $(q_1', q_2') \in Z$.
- *Surjectivity:* For all $q_2 \in Q_2$, there is some $q_1 \in Q_1$ with $(q_1, q_2) \in Z$.

A probabilistic uniform strong alternating simulation trivially also is a uniform strong alternating simulation. Simulations allow to transfer strategies in the canonical way; a natural application is when $\mathcal{C}_1$ is a "small representation" of $\mathcal{C}_2$, for example $\mathcal{C}_1$ may be the "finite core" of $\mathcal{C}_2$. We show the following theorem (note that for uncountable deterministic structures, the proof relies on the Axiom of Choice):

Theorem 3.1 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be (deterministic) CGSs, let A be a coalition, and Z a probabilistic uniform strong alternating simulation (uniform strong alternating simulation) for A from $\mathcal{C}_1$ to $\mathcal{C}_2$. Then for all strategy choices S_1 for A in $\mathcal{C}_1$, there is a strategy choice S_2 for A in $\mathcal{C}_2$ such that for all formulas φ for $\mathcal{C}_1/\mathcal{C}_2$ with $\mathit{pls}(\varphi) \subseteq A$, and for all pairs $(q_1, q_2) \in Z$, it holds that $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \varphi$ iff $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \varphi$.*

The above theorem should *not* be read as stating that $\mathcal{C}_1$ and $\mathcal{C}_2$ are "strategically equivalent:" this is only the case when there are simulations in *both* directions. We will discuss some properties of our notion of simulation in Section 3.3 with an example. In the following section, we prove Theorem 3.1.

Finally note the existential statement proven by Theorem 3.1 does not immediately yield a practical way to construct S_2 from S_1. However the proof gives an explicit construction, where the complexity of the description of S_2 is essentially the sum of the complexities of the descriptions of S_1 and the simulation Z, together with its associated move transfer function $\Delta^{1\to 2}_{\ldots}$.

3.2 Proof of Theorem 3.1

For the proof of the theorem, it is useful to study the following variant of a simulation relation. In the following, for a set M of paths, we write $Z(M)$ for the set containing all paths λ_2 such that there is some $\lambda_1 \in M$ and $\lambda_1 \sim_Z \lambda_2$. The following definition will only be used in the proof of Theorem 3.1.

Definition Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CGSs with state sets Q_1 and Q_2, the same player set and the same set of propositional variables. Then $Z \subseteq Q_1 \times Q_2$ is a *probabilistic strategy simulation* for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$ if the following holds:

1. For all $(q_1, q_2) \in Z$, q_1 and q_2 satisfy the same propositional variables,
2. For every strategy choice S_1 for A in $\mathcal{C}_1$, there is a strategy choice S_2 for A in $\mathcal{C}_2$ such that for all $\langle\langle A'\rangle\rangle_i$-formulas φ with $pls(\varphi) \subseteq A$ for $\mathcal{C}_1/\mathcal{C}_2$, and all $(q_1, q_2) \in Z$, we have
   (a) For all responses r_1 to A', and $\trianglelefteq \in \{\leq, \geq\}$, there is some response r_2 to A' such that for all sets M of paths over $\mathcal{C}_1$, we have
   $$\Pr(q_1 \to M \mid \mathsf{S}_1(A', q_1, \varphi) + r_1) \trianglelefteq \Pr(q_2 \to Z(M) \mid \mathsf{S}_2(A', q_2, \varphi) + r_2).$$
   (b) For all responses r_2 to A', and $\trianglelefteq \in \{\leq, \geq\}$, there is some response r_1 to A' such that for all sets M of paths over $\mathcal{C}_1$, we have
   $$\Pr(q_2 \to Z(M) \mid \mathsf{S}_2(A', q_2, \varphi) + r_2) \trianglelefteq \Pr(q_1 \to M \mid \mathsf{S}_1(A', q_1, \varphi) + r_1).$$
3. Z satisfies uniformity, surjectivity, and knowledge transfer.

Note that it is enough to consider the reflexive operators $\leq$ and $\geq$ here, the result for their irreflexive variants $<$ and $>$ in Theorem 3.1 also follows. Again, the deterministic case allows a slightly simpler definition:

Definition Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be CGSs with state sets Q_1 and Q_2, the same player set and the same set of propositional variables. Then $Z \subseteq Q_1 \times Q_2$ is a *strategy simulation* for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$ if the following holds:

1. For all $(q_1, q_2) \in Z$, q_1 and q_2 satisfy the same propositional variables,
2. For every strategy choice S_1 for A in $\mathcal{C}_1$, there is a strategy choice S_2 for A in $\mathcal{C}_2$ such that for all $\langle\langle A'\rangle\rangle_i$-formulas φ with $pls(\varphi) \subseteq A$ for $\mathcal{C}_1/\mathcal{C}_2$, and all $(q_1, q_2) \in Z$, we have
   (a) For all $\lambda_1 \in out(q_1, \mathsf{S}_1(A', q_1, \varphi))$, there is some $\lambda_2 \in out(q_2, \mathsf{S}_2(A', q_2, \varphi))$ with $\lambda_1 \sim_Z \lambda_2$,

(b) For all $\lambda_2 \in out(q_2, \mathsf{S}_2(A', q_2, \varphi))$, there is some $\lambda_1 \in out(q_1, \mathsf{S}_1(A', q_1, \varphi))$ with $\lambda_1 \sim_Z \lambda_2$.

3. Z satisfies uniformity, surjectivity, and knowledge transfer.

It is easy to see that a strategy simulation or a probabilistic strategy simulation allows to transfer strategy choices:

Lemma 3.2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be (deterministic) CGSs, let A be a coalition, and Z a probabilistic strategy simulation (strategy simulation) for A from $\mathcal{C}_1$ to $\mathcal{C}_2$. Then for all strategy choices S_1 for A in $\mathcal{C}_1$, there is a strategy choice S_2 for A in $\mathcal{C}_2$ such that for all formulas φ for $\mathcal{C}_1/\mathcal{C}_2$ with $pls(\varphi) \subseteq A$, and for all pairs $(q_1, q_2) \in Z$, it holds that $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \varphi$ iff $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \varphi$.*

Proof. We first cover the deterministic case: Let S_2 be the strategy choice that exists for S_1 and has the properties due to the definition of a strategy simulation. We show that for every path λ_1 over $\mathcal{C}_1$ and λ_2 over $\mathcal{C}_2$ with $\lambda_1 \sim_Z \lambda_2$ and every subformula ψ of φ that $\lambda_1, \mathsf{S}_1 \models \psi$ if and only if $\lambda_2, \mathsf{S}_2 \models \psi$. This in particular proves the result. We proceed inductively. The case where ψ is a propositional variable is trivial, as is the induction step for propositional operators. In the case that $\varphi = \mathsf{X}\psi$ or $\varphi = \psi \mathsf{U} \chi$ the claim follows, since due to induction the formulas ψ and χ are satisfied at the exact same indices of λ_1 and λ_2. Now consider the case $\varphi = \langle\langle A' \rangle\rangle_i\, \psi$. First assume that $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \varphi$, we show that $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \varphi$. Hence let $\lambda_2 \in out(q_2, \mathsf{S}_2(A', q_2, \varphi))$. We show $\lambda_2, \mathsf{S}_2 \models \psi$. Due to the choice of S_2, there is some $\lambda_1 \in out(q_1, \mathsf{S}_1(A', q_1, \varphi))$ such that $\lambda_1 \sim_Z \lambda_2$. Since $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \varphi$, we know that $\lambda_1, \mathsf{S}_1 \models \psi$, and due to induction it follows that $\lambda_2, \mathsf{S}_2 \models \psi$ as required. The other direction follows symmetrically. Finally assume that $\varphi = \mathcal{K}_i^{A'} \psi$ for some $A' \subseteq A$ and i. First assume that $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \mathcal{K}_i^{A'} \psi$. To prove that $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \mathcal{K}_i^{A'} \psi$, let $q_2' \sim_{\mathsf{eq}_i A'} q_2$, we need to show that $\mathcal{C}_2, \mathsf{S}_2, q_2' \models \psi$. Due to the surjectivity of a strategy simulation, there is a state q_1' of $\mathcal{C}_1$ such that $(q_1', q_2') \in Z$. Due to the uniformity condition, it follows that $q_1' \sim_{\mathsf{eq}_i(A')} q_1$. Since $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \mathcal{K}_i^{A'} \psi$, it follows that $\mathcal{C}_1, \mathsf{S}_1, q_1' \models \psi$. Since $(q_1', q_2') \in Z$, induction implies that $\mathcal{C}_2, \mathsf{S}_2, q_2' \models \psi$ as required. Now assume that $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \mathcal{K}_i^{A'} \psi$. To prove that $\mathcal{C}_1, \mathsf{S}_1, q_1 \models \mathcal{K}_i^{A'} \psi$, let $q_1' \sim_{\mathsf{eq}_i(A')} q_1$, we need to show that $\mathcal{C}_1, \mathsf{S}_1, q_1' \models \psi$. Due to knowledge transfer, there is some state q_2' of $\mathcal{C}_2$ such that $q_2' \sim_{\mathsf{eq}_i(A')} q_2$, and $(q_1', q_2') \in Z$. Since $\mathcal{C}_2, \mathsf{S}_2, q_2 \models \mathcal{K}_i^{A'} \psi$, it follows that $\mathcal{C}_2, \mathsf{S}_2, q_2' \models \psi$. Due to induction, this implies that $\mathcal{C}_1, \mathsf{S}_1, q_1' \models \psi$ as required. This concludes the proof for the deterministic case.

The probabilistic case is very similar: Let S_2 be the strategy choice for A in $\mathcal{C}_2$ obtained from Z. We proceed in the same way as for the deterministic case, and all cases are identical to that one, except for the following: It remains to show that if $\varphi = \langle\langle A' \rangle\rangle_i^{\blacktriangleleft\alpha} \psi$ for some $A' \subseteq A$, then $\mathcal{C}_1, \mathsf{S}_1, \lambda_1[0] \models \varphi$ if and only if $\mathcal{C}_2, \mathsf{S}_2, \lambda_2[0] \models \varphi$. Let $\underline{\blacktriangleleft}$ denote the reflexive closure of $\blacktriangleleft$. First assume that $\mathcal{C}_1, \mathsf{S}_1, \lambda_1[0] \models \varphi$. To prove that $\mathcal{C}_2, \mathsf{S}_2, \lambda_2[0] \models \varphi$, let $s_2 = \mathsf{S}_2(A', \lambda_2[0], \varphi)$, and let r_2 be a response to A'. We need to show that

$$\Pr\left(q_2 \to \{\lambda_2 \mid \lambda_2, \mathsf{S}_2 \models \psi\} \mid s_2 + r_2\right) \blacktriangleleft \alpha.$$

For $j \in \{1, 2\}$, let $M_j := \{\lambda \text{ path over } \mathcal{C}_j \mid \lambda, \mathsf{S}_j \models \psi\}$. Since Z is a probabilistic strategy simulation, and inductively we know that $M_2 = Z(M_1)$, and $(\lambda_1[0], \lambda_2[0]) \in Z$, there is some response r_1 to A' such that (when $s_1 = \mathsf{S}_1(A', \lambda_1[0], \varphi)$)

$$\Pr\left(\lambda_2[0] \to M_2 \mid s_2 + r_2\right) \underline{\blacktriangleleft} \Pr\left(\lambda_1[0] \to M_1 \mid s_1 + r_1\right).$$

Obviously,

$$P_j := \Pr\left(\lambda_j[0] \to \{\lambda \mid \lambda, \mathsf{S}_j \models \psi\} \mid s_j + r_j\right) = \Pr\left(\lambda_j[0] \to M_j \mid s_j + r_j\right),$$

due to the above it follows that $P_2 \underline{\blacktriangleleft} P_1$. Since $\mathcal{C}_1, \mathsf{S}_1, \lambda_1[0] \models \varphi$, we know that $P_1 \blacktriangleleft \alpha$. Hence $P_2 \underline{\blacktriangleleft} P_1 \blacktriangleleft \alpha$, and thus $P_2 \blacktriangleleft \alpha$ as required. The converse direction is virtually identical. □

Theorem 3.1 now follows directly from the above and the following Lemma:

Lemma 3.3 *Every probabilistic uniform strong alternating simulation (uniform strong alternating simulation) Z for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a probabilistic strategy simulation (strategy simulation) for A from $\mathcal{C}_1$ to $\mathcal{C}_2$ for A.*

Proof. The construction of the required strategy choice S_2 is identical for the deterministic and the probabilistic case, hence we first state it and prove a straightforward property of it, and then give separate correctness proofs for each case. Hence let Q_1 and Q_2 be the state sets for $\mathcal{C}_1$ and $\mathcal{C}_2$, let Σ be the player set and $\mathbb{P}$ the set of propositional variables, and let n be the number of degrees of information in $\mathcal{C}_1$ and $\mathcal{C}_2$. Let δ_1 and δ_2 be the transition functions of $\mathcal{C}_1$ and $\mathcal{C}_2$. Since Z is a uniform strong alternating simulation (or a probabilistic uniform strong alternating simulation), the same propositional variables from $\mathbb{P}$ hold in states q_1 and q_2 if $(q_1, q_2) \in Z$. In order to show that Z is a strategy simulation (or a probabilistic strategy simulation), we need to establish a correspondence between strategy choices for $\mathcal{C}_1$ and $\mathcal{C}_2$. We start with establishing a map between the states of the CGSs, and use this (together with the maps $\Delta_{\dots}^{1\to 2}$ on moves implied by Z) to define a map on strategies, which then allows us to transfer strategy choices (note that we make a slight abuse of notation here in calling all of these functions Z, this indicates that they are canonically obtained from the relation Z).

Transferring States For a state $q_2 \in Q_2$, let $Z^{-1}(q_2)$ be some state $q_1 \in Q_1$ with $(q_1, q_2) \in Z$. Since Z is surjective, such a state exists, and with the axiom of choice, Z^{-1} is a function. Note that the Axiom of Choice is not required for countable structures, and in the probabilistic case Z^{-1} is a function by definition of a probabilistic uniform strong alternating simulation.

Transferring Strategies Let s_a be an a-strategy for a player $a \in A$ in $\mathcal{C}_1$, let $i \in \{1, \ldots, n\}$, We define the a-strategy $Z_i(s_a)$ in $\mathcal{C}_2$ as follows:

$$(Z_i(s_a))(q_2) = \Delta^{1\to 2}_{(i,a,Z^{-1}(q_2),q_2)}(s_a(Z^{-1}(q_2))) \text{ for all } q_2 \in Q_2.$$

We claim that if s_a is i-uniform, then so is $Z_i(s_a)$. Hence let $q_2' \sim_{\mathsf{eq}_i^2(a)} q_2$, we show $(Z_i(s_a))(q_2') = (Z_i(s_a))(q_2)$. Let $q_1' := Z^{-1}(q_2')$, and let $q_1 := Z^{-1}(q_2)$. Since $(q_1, q_2), (q_1', q_2') \in Z$, and $q_2' \sim_{\mathsf{eq}_i^2(a)} q_2$, the uniformity of Z implies that $q_1 \sim_{\mathsf{eq}_i^1(a)} q_1'$ and $\Delta^{1\to 2}_{(i,a,q_1,q_2)} = \Delta^{1\to 2}_{(i,a,q_1',q_2')}$. Since s_a is i-uniform, we also have $s_a(q_1') = s_a(q_1)$. It now follows that

$$\begin{aligned}
(Z_i(s_a))(q_2') &= \Delta^{1\to 2}_{(i,a,Z^{-1}(q_2'),q_2')}(s_a(Z^{-1}(q_2'))) &&= \Delta^{1\to 2}_{(i,a,q_1',q_2')}(s_a(q_1')) \\
&= \Delta^{1\to 2}_{(i,a,q_1,q_2)}(s_a(q_1)) &&= \Delta^{1\to 2}_{(i,a,Z^{-1}(q_2),q_2)}(s_a(Z^{-1}(q_2))) \\
&= (Z_i(s_a))(q_2).
\end{aligned}$$

Hence $Z(s_a)$ is i-uniform as required.

Transferring Strategy Choices Let S_1 be a strategy choice for A in $\mathcal{C}_1$. We define $Z(\mathsf{S}_1)$ as follows: For a $\langle\langle.\rangle\rangle_i$-formula φ, let

$$(Z(\mathsf{S}_1))(a, q_2, \varphi) = Z_i(\mathsf{S}_1(a, Z^{-1}(q_2), \varphi)).$$

To show that $Z(\mathsf{S}_1)$ is a strategy choice, we prove that the uniformity conditions are satisfied. By construction, for a $\langle\langle.\rangle\rangle_i$-formula φ, a strategy of the form $Z_i(s_a)$ for an i-uniform strategy s_a is returned (since S_1 is a strategy choice). Due to the above, this implies that every strategy returned by $Z(\mathsf{S}_1)$ for a $\langle\langle.\rangle\rangle_i$-formula φ is i-uniform. Now let $a \in A$, and let $q_2' \sim_{\mathsf{eq}_i^2(a)} q_2$. Since Z is uniform, it follows that $Z^{-1}(q_2) \sim_{\mathsf{eq}_i^1} Z^{-1}(q_2')$. Since S_1 is a strategy choice, this implies that $\mathsf{S}_1(a, Z^{-1}(q_2), \varphi) = \mathsf{S}_1(a, Z^{-1}(q_2'), \varphi)$, and thus $(Z(\mathsf{S}_1))(a, q_2, \varphi) = Z_i(\mathsf{S}_1(a, Z^{-1}(q_2), \varphi)) = Z_i(\mathsf{S}_1(a, Z^{-1}(q_2'), \varphi)) = (Z(\mathsf{S}_1))(a, q_2', \varphi)$ as required.

Proving Strategy Simulation We now prove that Z is a strategy simulation (probabilistic strategy simulation) for A from $\mathcal{C}_1$ to $\mathcal{C}_2$. Due to the above, we know that

$Z(\mathsf{S}_1)$ is a strategy choice. It remains to show that $\mathsf{S}_2 := Z(\mathsf{S}_1)$ satisfies the simulation requirements. We first show the following auxiliary claim, which we use in the proof for both the deterministic and the probabilistic case (note that in the latter case, the claim is trivial):

Claim Let $(q_1, q_2) \in Z$, let $(\lambda_1[k], \lambda_2[k]) \in Z$, let φ be a $\langle\langle A'\rangle\rangle_i$-formula, let $A' \subseteq A$, and let c_1 and c_2 be moves in $\lambda_1[k]$ and $\lambda_2[k]$ such that $c_1(a) = (\mathsf{S}_1(a, q_1, \varphi))(\lambda_1[k])$ and $c_2(a) = (\mathsf{S}_2(a, q_2, \varphi))(\lambda_2[k])$ for all $a \in A'$. Then for all $a \in A'$, we have that $c_2(a) = \Delta^{1\to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}(c_1(a))$.

Proof. Let $q_1' := Z^{-1}(q_2)$. Then (q_1, q_2), $(q_1', q_2) \in Z$. Due to the uniformity of Z, it follows that $q_1 \sim_{\mathsf{eq}_i^1(a)} q_1'$ for all $a \in A$. Also, let $q_1^k := Z^{-1}(\lambda_2[k])$. Then $(q_1^k, \lambda_2[k])$, $(\lambda_1[k], \lambda_2[k]) \in Z$. The uniformity of Z now implies $q_1^k \sim_{\mathsf{eq}_i^1(a)} \lambda_1[k]$ for all $a \in A$. We now have 1. $(q_1^k, \lambda_2[k]) \in Z$, 2. $(\lambda_1[k], \lambda_2[k]) \in Z$, 3. $q_1^k \sim_{\mathsf{eq}_i^1(a)} \lambda_1[k]$ for all $a \in A'$, 4. $\lambda_2[k] \sim_{\mathsf{eq}_i^2(a)} \lambda_2[k]$ for all $a \in A'$. Therefore, the move-uniformity of Z implies that $\Delta^{1\to 2}_{(i,a,q_1^k,\lambda_2[k])} = \Delta^{1\to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}$. Since S_1 is a strategy choice, and $q_1 \sim_{\mathsf{eq}_i^1(a)} q_1'$ for all $a \in A$, and φ is an $\langle\langle.\rangle\rangle_i$-formula, we know that $\mathsf{S}_1(a, q_1, \varphi) = \mathsf{S}_1(a, q_1', \varphi)$. We denote this strategy with s_a. Since S_1 is a strategy choice, it follows that s_1 is i-uniform. Hence $s_a(q_1^k) = s_a(\lambda_1[k])$. It now follows that

$$\begin{aligned}
c_2(a) &= (\mathsf{S}_2(a, q_2, \varphi))(\lambda_2[k]) \\
&= ((Z(\mathsf{S}_1))(a, q_2, \varphi))(\lambda_2[k]) \\
&= (Z_i(\mathsf{S}_1(a, Z^{-1}(q_2), \varphi)))(\lambda_2[k]) \\
&= (Z_i(\mathsf{S}_1(a, q_1', \varphi)))(\lambda_2[k]) \\
&= (Z_i(s_a))(\lambda_2[k]) \\
&= \Delta^{1\to 2}_{(i,a,Z^{-1}(\lambda_2[k]),\lambda_2[k])}(s_a(Z^{-1}(\lambda_2[k]))) \\
&= \Delta^{1\to 2}_{(i,a,q_1^k,\lambda_2[k])}(s_a(q_1^k)) \\
&= \Delta^{1\to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}(s_a(\lambda_1[k])) \\
&= \Delta^{1\to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}((\mathsf{S}_1(a, q_1, \varphi))(\lambda_1[k]) \\
&= \Delta^{1\to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}(c_1(a)),
\end{aligned}$$

as required. □

Using this claim, we can now prove that Z is indeed a strategy simulation if Z is a uniform strong alternating simulation, and that Z is a probabilistic strategy simulation if Z is a probabilistic uniform strong alternating simulation. From now on, the proofs of the probabilistic and deterministic cases differ, we start with the deterministic one. Let $\varphi = \langle\langle A'\rangle\rangle_i \, \psi$ for some formula ψ, and let $(q_1, q_2) \in Z$.

(a) Let $\lambda_1 \in \mathit{out}(q_1, \mathsf{S}_1(A', q_1, \varphi))$. We need to show that there is some $\lambda_2 \in \mathit{out}(q_2, \mathsf{S}_2(A', q_2, \varphi))$ with $\lambda_1 \sim_Z \lambda_2$.
We construct the path λ_2 inductively. Choose $\lambda_2[0] = q_2$, and assume that for some $k \geq 0$, we have constructed a path $\lambda_2[0 \dots k]$ such that for all

$j \leq k$, $(\lambda_1[j], \lambda_2[j]) \in Z$, and $\lambda_2[0 \ldots k]$ is an initial segment of a path from $out(q_2, \mathsf{S}_2(A', q_2, \varphi))$. We need to show that there is some $\lambda_2[k+1] \in Q_2$ such that

1. $(\lambda_1[k+1], \lambda_2[k+1]) \in Z$,
2. $\lambda_2[k+1] \in next(\lambda_2[k], c_2)$, where c_2 is the move played by coalition A' when following the strategy selected by S_2 for the formula φ when started in q_2, i.e., $c_2(a) = (\mathsf{S}_2(a, q_2, \varphi))(\lambda_2[k])$ for all $a \in A'$.

Similarly, let c_1 be the A'-move played by the coalition A' in the state $\lambda_1[k]$, i.e., for $a \in A'$, $c_1(a) = (\mathsf{S}_1(a, q_1, \varphi))(\lambda_1[k])$. Due to the above claim, we know that for each $a \in A'$, we have $c_2(a) = \Delta^{1 \to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}(c_1(a))$. Since $\lambda_1 \in out(q_1, \mathsf{S}_1(A', q_1, \varphi))$, we know that $\lambda_1[k+1] \in next(\lambda_1[k], c_1)$. Hence, due to the forward move property of Z, there is some $\lambda_2[k+1] \in next(\lambda_2[k], c_2)$ such that $(\lambda_1[k+1], \lambda_2[k+1]) \in Z$, as required.

(b) This direction is very similar. Let $\lambda_2 \in out(q_2, \mathsf{S}_2(A', q_2, \varphi))$. We need to show that there is some $\lambda_1 \in out(q_1, \mathsf{S}_1(A', q_1, \varphi))$ with $\lambda_1 \sim_Z \lambda_2$. Let $\lambda_1[0] := q_1$, and inductively assume that $\lambda_1[0 \ldots k]$ has been defined such that for all $j \leq k$, we have $(\lambda_1[j], \lambda_2[j]) \in Z$, and $\lambda_1[0 \ldots k]$ is an initial segment of a path in $out(q_1, \mathsf{S}(A', q_1, \varphi))$. Again, let c_1 and c_2 be the moves played by the coalition A' in the states $\lambda_1[k]$ and $\lambda_2[k]$, i.e., $c_j(a) = (\mathsf{S}_j(a, q_j, \varphi))(\lambda_j[k])$. Due to the claim above, it again follows that $c_2(a) = \Delta^{1 \to 2}_{(i,a,\lambda_1[k],\lambda_2[k])}(c_1(a))$ for all $a \in A'$. Hence due to the backward move property of Z, since $\lambda_2[k+1] \in out(\lambda_2[k], c_2)$ and $(\lambda_1[k], \lambda_2[k]) \in Z$, there is some $\lambda_1[k+1] \in next(\lambda_1[k], c_1)$ with $(\lambda_1[k+1], \lambda_2[k+1]) \in Z$, as required.

Hence S_2 satisfies the conditions, and thus Z is a strategy simulation as claimed. This completes the proof for the deterministic case.

The probabilistic case requires more care since we do not only need to show the existence of a corresponding "result" of a game, but need to show that the involved probabilities are related in the required manner. Since in a probabilistic uniform strong alternating simulation we require that Z^{-1} is a function, i.e., for every state q_2 in $\mathcal{C}_2$, there is a *unique* state q_1 from $\mathcal{C}_1$ such that $(q_1, q_2) \in Z$, we can relate the "successful" resulting plays in a one-to-one fashion that allows to transfer probabilities. To show that Z is a probabilistic strategy simulation, let φ be a $\langle\langle A' \rangle\rangle_i$-formula with $pls(\varphi) \subseteq A$, and let $(q_1, q_2) \in Z$. We introduce some notation required for both directions of the proof: For a set M of paths over $\mathcal{C}_1$, and states $q_1 \in Q_1$, $q_2 \in Q_2$, we use the following: Let $M_1 = M$, let $M_2 = Z(M_1)$, let $Z^{-1}(\lambda)$ for a path λ be the path resulting from applying Z^{-1} to each state in λ. Further, for $t \in \mathbb{N}$, and $j \in \{1, 2\}$, let $M^j_t = \{\lambda \mid |\lambda| = t+1,\ \lambda \text{ is a prefix of a path in } M_j\}$, and let $F^j_t = \left\{q \in Q_j \mid \text{there is some } \lambda \in M^j_t \text{ with } \lambda[t] = q\right\}$. In the following we need to construct the relevant responses r_1, r_2 according to the definition of a probabilistic strategy simulation. Note that it suffices to define $r_j(t, q)$ for states

$q \in F_t^j$, since a path λ over $\mathcal{C}_j$ with $\lambda[t] \notin F_t^j$ does not contribute to the relevant probabilities. In the following, let s_j be the strategy played by the coalition A' in $\mathcal{C}_j$, i.e., $s_j := \mathsf{S}_j(A', q_j, \varphi)$. Let $s_j = (s_a^j)_{a \in A'}$. With slight abuse of notation, for a state $q \in Q_j$ we write $s_j(q)$ for the (A', q)-move c defined by $c(a) = s_a^j(q)$ for all $a \in A'$. Also, with $\Pr_t(.)$ we denote the probability measure in the involved CGSs after t steps of the game (the relevant CGS will always be clear from the context).

First let r_1 be a response to A', let $\trianglelefteq$ be one of the reflexive operators $\leq$, $\geq$. We need to show that there is some response r_2 to A' such that for all sets M of paths over $\mathcal{C}_1$, we have

$$P_1 := \Pr(q_1 \to M_1 \mid s_1 + r_1) \trianglelefteq \Pr(q_2 \to M_2 \mid s_2 + r_2) =: P_2.$$

Note that this is the easier direction of the proof, since here we essentially prove that every response in the (potentially "easy") CGS $\mathcal{C}_1$ can be mirrored in the (potentially "more complicated") CGS $\mathcal{C}_2$. We construct r_2 such that for all $t \in \mathbb{N}$, we have

$$P_t^1 := \Pr_t\left(q_1 \to M_t^1 \mid s_1 + r_1\right) = \Pr_t\left(q_2 \to M_t^2 \mid s_2 + r_2\right) =: P_t^2.$$

If this equality is satisfied for all $t \in \mathbb{N}$, then $P_1 = P_2$ follows, which clearly implies $P_1 \trianglelefteq P_2$ as required. Note that if $q_1 \notin F_0^1$, then (since $Z^{-1}(q_2) = q_1$) it follows that $q_2 \notin F_0^2$, and both P_1 and P_2 are zero, in particular $P_2 = P_1$ as required. Hence assume that $q_1 \in F_0^1$, then also $q_2 \in F_0^2$. Then for all responses r_2, we have that $P_0^2 = P_0^1 = 1$. Now assume inductively that $r_2(t', q)$ has been defined for all $t' < t$ and all $q \in F_{t'}^2$ such that $P_{t'}^2 = P_{t'}^1$. We define $r_2(t, q)$ for all $q \in F_t^2$ such that $P_t^2 = P_t^1$ (recall that values $r_2(t', q)$ for $q \notin F_{t'}^2$ are irrelevant). Hence let $q \in F_t^2$, and let $c_1^{A'}$ be the move played by A' in $Z^{-1}(q)$, then by construction

$$c_1^{A'}(a) = s_a^1(Z^{-1}(q)) = (\mathsf{S}_1(a, q_1, \varphi))(Z^{-1}(q)).$$

Similarly, let $c_2^{A'}$ be the move played by A' in q, i.e.,

$$c_2^{A'}(a) = s_a^2(q) = (\mathsf{S}_2(a, q_2, \varphi))(q).$$

Since $(Z^{-1}(q), q) \in Z$, the claim above implies (note that for the probabilistic case, this can be shown easily without the claim)

$$c_2^{A'}(a) = \Delta^{1\to 2}_{(i,a,Z^{-1}(q),q)}(c_1^{A'}(a)) \text{ for all } a \in A'.$$

Since $r_1(t, Z^{-1}(q))$ is an $(\overline{A'}, Z^{-1}(q))$-move, the forward move property implies that there is an $(\overline{A'}, q)$-move $r_2(t, q)$ such that for all $q_1'' \in Q_1$,

$$\Pr(\delta(q, s_2(q) \cup r_2(t, q)) \in Z(q_1'')) = \Pr\left(\delta(Z^{-1}(q), s_1(Z^{-1}(q)) \cup r_1(t, Z^{-1}(q))) = q_1''\right).$$

For this choice of r_2, P^2_{t+1} is obtained as

$$\begin{aligned}
& P^2_{t+1} \\
= \quad & \sum_{\lambda_2\in M^2_{t+1}} \Pr_{t+1}\left(q_2 \to \{\lambda_2\} \mid s_2+r_2\right) \\
= \quad & \sum_{\lambda_1\in M^1_{t+1}} \sum_{\lambda_2\in M^2_t, Z^{-1}(\lambda_2)=\lambda_1} \Bigg(\Pr_t\left(q_2 \to \{\lambda_2[0,t]\} \mid s_2+r_2\right) \\
& \quad \cdot \sum_{q_2''\in Z(\lambda_1[t+1])} \Pr\left(\delta(\lambda_2[t], s_2(\lambda_2[t]) \cup r_2(t,\lambda_2[t]) = q_2'')\right)\Bigg) \\
= \quad & \sum_{\lambda_1\in M^1_{t+1}} \sum_{\lambda_2\in M^2_t, Z^{-1}(\lambda_2)=\lambda_1} \left(\Pr_t\left(q_2 \to \{\lambda_2[0,t]\} \mid s_2+r_2\right)\right. \\
& \quad \left.\cdot \Pr\left(\delta(\lambda_2[t], s_2(\lambda_2[t]) \cup r_2(t,\lambda_2[t])) \in Z(\lambda_1[t+1])\right)\right) \\
(*) = \quad & \sum_{\lambda_1\in M^1_{t+1}} \sum_{\lambda_2\in M^2_t, Z^{-1}(\lambda_2)=\lambda_1} \left(\Pr_t\left(q_2 \to \{\lambda_2[0,t]\} \mid s_2+r_2\right)\right. \\
& \quad \left.\cdot \underbrace{\Pr\left(\delta(\lambda_1[t], s_1(\lambda_1[t]) \cup r_1(t,\lambda_1[t])) = \lambda_1[t+1]\right)}_{=:A}\right) \\
= \quad & \sum_{\lambda_1\in M^1_{t+1}} A \cdot \left(\sum_{\lambda_2\in M^2_t, Z^{-1}(\lambda_2)=\lambda_1} \Pr_t\left(q_2 \to \{\lambda_2[0,t]\} \mid r_2+s_2\right)\right) \\
(**) = \quad & \sum_{\lambda_1\in M^1_{t+1}} A \cdot \Pr_t\left(q_1 \to \{\lambda_1[0,t]\} \mid r_1+s_1\right) \\
= \quad & \sum_{\lambda_1\in M^1_{t+1}} \Pr\left(\delta(\lambda_1[t], s_1(\lambda_1[t]) \cup r_1(t,\lambda_1[t])) = \lambda_1[t+1]\right) \\
& \quad \cdot \Pr_t\left(q_1 \to \{\lambda_1[0,t]\} \mid s_1+r_1\right) \\
= \quad & \sum_{\lambda_1\in M^1_{t+1}} \Pr_{t+1}\left(q_1 \to \{\lambda_1\} \mid s_1+r_1\right) \\
= \quad & \Pr_{t+1}\left(q_1 \to M^1_{t+1} \mid s_1+r_1\right) \\
= \quad & P^1_{t+1}
\end{aligned}$$

as required (to obtain the equality $(**)$ we use induction applied to the set $M = \{\lambda_1\}$, note that the construction of r_2 does not depend on the choice of M, hence we can apply induction for this choice of M; the equality $(*)$ follows from the choice of r_2). Note that since we allow only discrete probability distributions, for each finite length there is only a countable number of paths with a non-zero probability, hence the above sums are well-defined.

The second case is more difficult, since here we have to show that every response in the (potentially "complicated") CGS $\mathcal{C}_2$ can be mirrored in the (potentially "easier" one) $\mathcal{C}_1$. Also note that, contrary to the above case, we do not get equality of the involved probabilities here. In the following, assume that $\blacktriangleleft$ is $\leq$. The case $\geq$ is virtually identical. Let r_2 be a response to A'. We need to show that there is some response r_1 to A' such that for all sets M of paths over $\mathcal{C}_1$, we have

$$\Pr\left(q_2 \rightarrow Z(M_1) \mid \mathsf{S}_2(A', q_2, \varphi) + r_2\right) \leq \Pr\left(q_1 \rightarrow M \mid \mathsf{S}_1(A', q_1, \varphi) + r_1\right).$$

Note that we can exchange r_2 with any response r_2' such that

$$\Pr\left(q_2 \rightarrow \{\lambda_2 \mid \lambda_1 \sim_Z \lambda_2 \text{ for some } \lambda_1 \in M\} \mid \mathsf{S}_2(A', q_2, \varphi) + r_2'\right)$$
$$\geq$$
$$\Pr\left(q_2 \rightarrow \{\lambda_2 \mid \lambda_1 \sim_Z \lambda_2 \text{ for some } \lambda_1 \in M\} \mid \mathsf{S}_2(A', q_2, \varphi) + r_2\right),$$

if we then construct a response r_1 such that

$$\Pr\left(q_1 \rightarrow M \mid \mathsf{S}_1(A', q_1, \varphi) + r_1\right) = \Pr\left(q_2 \rightarrow Z(M) \mid \mathsf{S}_2(A', q_2, \varphi) + r_2'\right),$$

then the original inequality is true as well. In particular, we may exchange r_2 for a response r_2' that achieves the highest possible probability of the game following a path in $Z(M)$, if one exists. For the construction of this r_2' (which from now on we denote by r_2), note that the actions of the players in A' are completely determined by the strategy choice S_2, which by construction has the following property: In two states q_a and q_b with $Z^{-1}(q_a) = Z^{-1}(q_b) = q$, each member a of the coalition A' performs the move obtained by its move-transfer function applied to the move selected by S_1 in the state q. By construction of the move-transfer functions $\Delta_{\dots}^{1\rightarrow 2}$, the thus-obtained moves in q_a and q_b are equivalent with regard to the possible probability distributions of $Z^{-1}(q_n)$, where q_n is the follow-up state of q: For the question whether the followed path is an element of M_2, only the Z-preimages of the current and next states are relevant. Hence an optimal r_2 will use the same probability distribution for the results on the Z-preimage of the follow-up state. Since the response r_2 may be constructed with the full knowledge of the strategies of A', and is not bound by any uniformity- or memorylessness constraints, we thus can assume that in an "optimal" response r_2, and for states q_a and q_b as above, for all states $q_1'' \in Q_1$, and all $t \in \mathbb{N}$, we have

$$\Pr\left(\delta(q_a, s_2(q_a) \cup r_2(t, q_a)) \in Z(q_1'')\right) = \Pr\left(\delta(q_b, s_2(q_b) \cup r_2(t, q_b)) \in Z(q_1'')\right).$$

Even when no "optimal" response exists, we may still assume the equality above and obtain a response with a probability at least as high as that of the original r_2. Hence without loss of generality, we can assume that the original response r_2 satisfies this property, and now construct a response r_1 such that the obtained probabilities for r_1 and r_2 are the same (and hence the one for r_1 is bounded by the one for the original r_2 as explained above).

We again use the notation M^j, M^j_t, F^j_t, P^j_t as introduced above, and with the same arguments as above we can assume that $q_j \in F^j_0$, and hence for any choice of r_1, the initial probabilities P^1_0 and P^2_0 are identical. Again assume that $r_1(t', q)$ has been defined for all $t' < t$ and all $q \in F^1_{t'}$ such that $P^1_{t'} = P^2_{t'}$. We now define $r_1(t, q)$ for all $q \in F^1_t$ (note that again, choices where $q \notin F^1_t$ are irrelevant).

Let $s_1(q)$ be the (A', q)-move played by coalition A' in q, i.e., $(s_1(q))(a) = (\mathsf{S}_1(a, q_1, \varphi))(q)$. For a state $q_z \in Z(q)$, let $s_2(q_z)$ be the (A', q_z)-move played by A' in q_z, i.e., $(s_2(q_z))(a) = (\mathsf{S}(a, q_2, \varphi))(q_z)$. The claim above (again, the probabilistic case can be shown easier without using the claim) implies that $(s_2(q_z))(a) = \Delta^{1\to 2}_{(i,a,q,q_z)}((s_1(q))(a))$.

Hence due to the backward move property, there is an $(\overline{A'}, q)$-move $r_1(t, q)$ such that for one $q_z \in Z(q)$ and the $(\overline{A'}, q_z)$-move $r_2(t, q_z)$, we have that for all $q''_1 \in Q_1$,

$$\Pr\left(\delta(q_z, s_2(q_z) \cup r_2(t, q_z)) \in Z(q''_1)\right) = \Pr\left(\delta(q, s_1(q) \cup r_1(t, q)) = q''_1\right).$$

Due to the choice of r_2, we also know that for q_z, $q'_z \in Z(q)$, and $q''_1 \in Q_1$, the following equality holds:

$$\Pr\left(\delta(q_z, s_2(q_z) \cup r_2(t, q_z)) \in Z(q''_1)\right) = \Pr\left(\delta(q'_z, s_2(q'_z) \cup r_2(t, q'_z)) \in Z(q''_1)\right).$$

Hence the single move $r_1(t, q)$ in fact satisfies that simultaneously for *all* $q_z \in Z(q)$, we have that for all $q''_1 \in Q_1$,

$$\Pr\left(\delta(q_z, s_2(q_z) \cup r_2(t, q_z)) \in Z(q''_1)\right) = \Pr\left(\delta(q, s_1(q) \cup r_1(t, q)) = q''_1\right).$$

Analogously to the direction above, we show that this choice of $r_1(t, q)$ for all $q \in F^1_t$ ensures that $P^1_{t+1} = P^2_{t+1}$. The probability P^2_{t+1} is obtained exactly as in the case above, the only difference is that the equation $(*)$ is now true due to the choice of r_1 rather than that of r_2. As explained above, the equality of the involved probabilities for the (modified) response r_2 leads to the originally required inequality for the original r_2. This concludes the proof of Lemma 3.3 and thus of Theorem 3.1. □

3.3 Discussion of Simulation Properties

We show the following trivial result, which on first sight may be surprising:

Proposition 3.4 *For every CGS $\mathcal{C}$ and every coalition A, there is a probabilistic uniform strong alternating simulation (and thus a uniform strong alternating simulation) for A from $\mathcal{C}$ to $\mathcal{C}^{hst}$.*

Proof. Let $\mathcal{C} = (\Sigma, Q, \mathbb{P}, \Pi, \delta, \sigma, \mathsf{eq})$, then $\mathcal{C}^{hst} = (\Sigma, Q^+, \mathbb{P}, \Pi', \delta', \sigma', \mathsf{eq}')$ as defined earlier. We define the probabilistic uniform strong alternating simulation Z as follows: $(q, q_1 \dots q_n) \in Z$ if and only if $q = q_n$. All $\Delta^{1\to 2}$-functions are the identity. One routinely checks that the required conditions are satisfied. $\square$

A (false) way of reading the above is that $\mathcal{C}$ and $\mathcal{C}^{hst}$ are strategically equivalent. However, this is completely incorrect: A probabilistic uniform strong alternating simulation allows to *transfer* a strategy choice (see Theorem 3.1), but since the translation is only in one direction, no equivalence is obtained. Hence Proposition 3.4 merely states that if a group of players has agreed on a set of joint strategies to achieve their respective goals in the basic CGS $\mathcal{C}$, then they are free to apply the same strategies even if they are given the additional ability to remember the history of the game, thereby ignoring this capability. Stated in this way, Proposition 3.4 is entirely unsurprising (hence the trivial proof). In particular, it does *not* state that with the additional capabilities, the players could not achieve *more* in $\mathcal{C}^{hst}$ than in the original CGS $\mathcal{C}$.

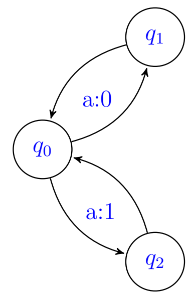

Figure 4: Example

As an example, consider the CGS $\mathcal{C}$ with three states q_0, q_1, q_2, where in q_0, the player can freely choose whether the successor state should be q_1 or q_2, and from the latter two states, every move leads back to q_0. Assume that the only player a in the game has complete information. Now consider the formula $\varphi = \langle\langle a \rangle\rangle_1 \Box(\Diamond q_1 \wedge \Diamond q_2)$. There is no strategy (and hence no strategy choice) satisfying φ in $\mathcal{C}$, as a, when following a (memoryless) strategy, would have to make the same move every time the game is in q_0, and hence only one of the states q_1, q_2 is visited infinitely often. In the history-dependent version $\mathcal{C}^{hst}$, the player may remember which choice he made the last time that the state q_0 was visited, and act accordingly. This not only shows that (as is well-known) history-dependent strategies are strictly stronger than memoryless ones, but also implies that a φ-maximal strategy choice in a $\mathcal{C}_1$, when transferred to some CGS $\mathcal{C}_2$ via simulation (Theorem 3.1), does not necessarily remain φ-maximal: In the above example, as φ is unsatisfiable and does not have any proper $\langle\langle . \rangle\rangle$-subformulas, any strategy choice is φ-maximal. When transferred to $\mathcal{C}^{hst}$, it is not maximal, as there it would be required to satisfy φ, since this clearly is possible. However, note that for any finite CGS $\mathcal{C}$, and any formula φ, its history-dependent version $\mathcal{C}^{hst}$ always allows $\leq_\varphi$-maximal strategy choices—this follows directly from Theorem 2.2, as clearly, any such $\mathcal{C}^{hst}$ has finite index.

The above example also clearly shows that in general, there is no probabilistic uniform strong alternating simulation or uniform strong alternating simulation from $\mathcal{C}^{hst}$ to $\mathcal{C}$.

Note also that a simulation between CGSs canonically transfers to their history-dependent versions—while trivial, this result is helpful as it allows to specify simulations on the (potentially finite) CGSs $\mathcal{C}_1$ and $\mathcal{C}_2$, and obtain the simulation between the (infinite) CGSs ${\mathcal{C}_1}^{hst}$ and ${\mathcal{C}_2}^{hst}$ in a generic manner:

Proposition 3.5 *If there is a probabilistic uniform strong alternating simulation (uniform strong alternating simulation) for a coalition A from $\mathcal{C}_1$ to $\mathcal{C}_2$, then there also is one from ${\mathcal{C}_1}^{hst}$ to ${\mathcal{C}_2}^{hst}$.*

Proof. Let Z be the simulation from $\mathcal{C}_1$ to $\mathcal{C}_2$, and let $\Delta^{1\to 2}_{\dots}$ be the corresponding move transfer functions. For states $q_1 = (q_1^1 \dots q_n^1)$ of ${\mathcal{C}_1}^{hst}$ and $q_2 = (q_1^2 \dots q_m^2)$ of ${\mathcal{C}_2}^{hst}$, let $(q_1, q_2) \in Z'$ if and only if $(q_j^1, q_j^2) \in Z$ for all $j \leq n$, and $n = m$, and in this case let ${\Delta^{1\to 2}_{(i,a,q_1,q_2)}}' = {\Delta_{(i,a,q_n^1,q_m^2)}}^{1\to 2}$ for all i and a. It can easily be verified that these choices satisfy the required conditions. □

The converse of Proposition 3.5 does not hold: For every CGS $\mathcal{C}$, the structures $\mathcal{C}^{hst}$ and ${\mathcal{C}^{hst}}^{hst}$ are essentially identical, in particular there is a probabilistic uniform strong alternating simulation (and thus a uniform strong alternating simulation) from ${\mathcal{C}^{hst}}^{hst}$ to $\mathcal{C}^{hst}$, but due to the above example there is not necessarily a simulation from $\mathcal{C}^{hst}$ to $\mathcal{C}$.

4 Complexity and Decidability

Strategy choices represent agreement of a coalition prior to a game: The coalition has to decide on a suitable strategy for every relevant goal, these strategies are then pooled in the strategy choice. Hence the "planning" of suitable strategies consists of determining a strategy choice achieving these goals for a given a CGS and a set of goals. In this section, we study the computational complexity of this problem. This situation is an example for the approach known as *planning as model checking*, see also [Jam04b]. Formally, we consider the following decision problems—depending on whether we allow all strategy choices or are only interested in maximal ones (see Section 2.5). Note that the algorithms we provide not only determine whether an appropriate strategy choice exists, but also compute one.

Problem:	$\exists$Choice
Input:	A CGS $\mathcal{C}$, a state q of $\mathcal{C}$, a state-formula φ
Question:	Is there a strategy choice S for $pls(\varphi)$ in $\mathcal{C}$ such that $\mathcal{C}, \mathsf{S}, q \models \varphi$?

Problem:	$\exists$maxChoice
Input:	A CGS $\mathcal{C}$, a state q of $\mathcal{C}$, a state-formula φ
Question:	Is there a $\leq_\varphi$-maximal strategy choice S for $pls(\varphi)$ in $\mathcal{C}$ such that $\mathcal{C}, \mathsf{S}, q \models \varphi$?

For studying the complexity of these problems, we assume that the transition function is specified as a complete table in the encoding of $\mathcal{C}$. For finite structures, the model checking problem is decidable, where the complexity in the deterministic case is considerably lower than in the probabilistic setting:

Theorem 4.1 $\exists$maxChoice *and* $\exists$Choice *are*

1. PSPACE-*complete for deterministic structures,*
2. *solvable in* 3EXPTIME *and* 2EXPTIME-*hard for probabilistic structures.*

Proof. We first cover the deterministic case and start by showing that both deterministic problems are in PSPACE. We present (almost identical) algorithms that solve them in polynomial space. On input $\mathcal{C}$, q, a state-formula φ, and a state q of $\mathcal{C}$, the algorithm proceeds as follows: Let Q be the state set of $\mathcal{C}$, let Σ be the set of players. First, we guess a candidate S for a strategy choice for the coalition A. Since a strategy is merely a function from the set of states into the set of moves (and thus can be represented with size quadratic in $\mathcal{C}$), and obviously only the strategies specified for the subformulas of φ are relevant, a strategy choice is of polynomial size (number of players in A multiplied by $|Q|$ multiplied by number of subformulas of φ multiplied by the size of a single strategy) and thus can be represented (and hence guessed) polynomially. We can nondeterministically guess S, since nondeterministic PSPACE is the same complexity class as PSPACE [Sav73]. Obviously, the uniformity conditions on S can be verified in polynomial time. It remains to check that $\mathcal{C}, \mathsf{S}, q_0 \models \varphi$, and that S is $\leq_\varphi$-maximal.

To verify that $\mathcal{C}, \mathsf{S}, q_0 \models \varphi$, we proceed inductively: For each subformula $\psi = \langle\langle A'\rangle\rangle_i \, \psi'$ of φ and each subformula χ of ψ, and each pair (q_1, q_2) of states in Q, we determine whether for all $\lambda \in out(q_2, \mathsf{S}(A', q_1, \psi))$, we have that $\mathcal{C}, \mathsf{S}, \lambda \models \chi$. In this case (slightly abusing notation) we write $\mathcal{C}, \mathsf{S}(q_1, \psi), q_2 \models \chi$. This is true if, when the players in A' keep following the strategies chosen by the strategy choice S in the state q_1 on input ψ, all possible outcomes from the state q_2 on satisfy χ.

We determine whether this holds for all combinations inductively over ψ and χ. First assume that $\chi = \langle\langle A''\rangle\rangle_j \, \varphi'$. By definition,

$\mathcal{C}, \mathsf{S}(q_1, \psi), q_2 \models \chi$
iff $\mathcal{C}, \mathsf{S}, \lambda \models \chi$ for all $\lambda \in out(q_2, \mathsf{S}(A', q_1, \psi))$
iff $\mathcal{C}, \mathsf{S}, q_2 \models \langle\langle A'' \rangle\rangle_j \, \varphi'$
iff $\mathcal{C}, \mathsf{S}, \lambda \models \varphi'$ for all $\lambda \in out(q_2, \mathsf{S}(A'', q_2, \chi))$
iff $\mathcal{C}, \mathsf{S}(q_2, \chi), q_2 \models \varphi'$.

By induction, we know whether the latter is true. Hence we can determine whether $\mathcal{C}, \mathsf{S}(q_1, \psi), q_2 \models \chi$. If the outmost operator of χ is the knowledge-operator $\mathcal{K}_i^{A'}$, then the truth of χ can easily be verified by checking all states that are $\sim_{\mathsf{eq}_i(A')}$-equivalent to q, for the latter we inductively have the required information. Now assume that the outmost operator of χ is neither a strategy- now a knowledge-operator. Inductively, we know for every $\langle\langle . \rangle\rangle$- or $\mathcal{K}$-subformula χ' of χ the set $S_{\chi'}$ of states q such that $\mathcal{C}, \mathsf{S}(q_1, \psi), q \models \chi'$. Hence we can replace every such formula χ' in χ with a new propositional variable that is exactly true in the states from $S_{\chi'}$. This modification of χ results in a formula in which no strategy- or knowledge-operator appears, hence the formula is an LTL-formula. From $\mathcal{C}$, we further obtain a CGS $\mathcal{C}'$, where in every state q, the moves of the coalition A' are hard-coded to use the strategy $\mathsf{S}(A', q_1, \psi)$: For every state q', the one-step reachable states are exactly those in $next(q', \mathsf{S}(A', q_1, \psi)(q'))$ (where again $\mathsf{S}(A', q_1, \psi)(q')$ denotes the (A', q')-move according to the strategy $\mathsf{S}(A', q_1, \psi)$). The question whether $\mathcal{C}, \mathsf{S}(q_1, \psi), q_2 \models \chi$ now is the problem whether in $\mathcal{C}'$, there is *no* path λ starting in q_2 such that $\lambda \models \neg\chi$, where satisfaction here is defined as in standard LTL. This problem can be solved in PSPACE due to [SC85, Theorem 4.1]. Since PSPACE with a PSPACE-oracle is again PSPACE, this implies that we can fill the table in polynomial space, and determine whether $\mathcal{C}, \mathsf{S}(q_1, \psi), q_2 \models \chi$ for all relevant combinations. As shown in the $\langle\langle A'' \rangle\rangle_j$-case above, this also allows us to decide whether a $\langle\langle . \rangle\rangle$-subformula of φ is satisfied at a state in $\mathcal{C}$, and thus using Boolean combinations we can determine whether $\mathcal{C}, \mathsf{S}, q_0 \models \varphi$.

It remains to verify that S is indeed $\leq_\varphi$-maximal. If this is not the case, then there exists a strategy choice S' such that $\mathsf{S} \leq_\varphi \mathsf{S}'$, and $\mathsf{S}' \not\leq_\varphi \mathsf{S}$. This S' can be nondeterministically guessed, and then we can (in the same way as for S above) determine for all subformulas ψ and states q whether $\mathcal{C}, \mathsf{S}', q \models \psi$, and thus compare S and S' with respect to $\leq_\varphi$ according to the definition. This completes the PSPACE-decision procedure for $\exists$maxChoice. Note that when leaving out the final step, we obtain a polynomial-space algorithm for $\exists$Choice.

We now show PSPACE-hardness of both deterministic problems, and again use [SC85, Theorem 4.1]. That theorem establishes PSPACE-completeness of the following problem: Given a CGS $\mathcal{C}$ with one player, a state q, and an LTL-formula φ, is there a path λ starting in q such that $\lambda \models \varphi$? (Here again satisfaction is the standard LTL notion of satisfaction which is equivalent to the one for ATL* for formulas that do not contain strategy operators). This problem is a special case of both $\exists\mathsf{Choice}$ and of $\exists\mathsf{maxChoice}$, where we ask whether the empty coalition has a strategy choice to ensure $\neg\varphi$—the input instance to $\exists\mathsf{Choice}$ and $\exists\mathsf{maxChoice}$ this contains of $\mathcal{C}$, the state q, and the formula $\langle\langle\emptyset\rangle\rangle\,\neg\varphi$ (where $\mathcal{C}$ has complete information). Note that since in this formula, no nested strategy operators appear, the question whether a strategy choice or a maximal strategy choice exist are equivalent. Since coPSPACE = PSPACE, this completes the proof of PSPACE-completeness for both $\exists\mathsf{Choice}$ and $\exists\mathsf{maxChoice}$ for the deterministic case.

For the probabilistic case, we make use of the model checking algorithm presented in [CL07]. In 3EXPTIME, we can replace the nondeterministic guessing of the strategy choice S with a complete search over all possible strategy choices which only adds a multiplicative factor of 2^{n^k} to the overall complexity. It suffices to show that each candidate can be verified in 3EXPTIME. Similarly as in the deterministic case, for each $\langle\langle A'\rangle\rangle_i$-subformula ψ, each subformula χ of ψ, and each pair (q_1, q_2) of states in Q, we determine

$$\mathrm{maxProb}\,(q_1 \to \chi \mid \mathsf{S}(\psi, q_2)) = \max\left\{\Pr\left(q_1 \to \{\lambda \mid \lambda, \mathsf{S} \models \chi\} \;\mid \mathsf{S}(A', q_2, \psi) + r\right)\right\},$$

where r ranges over all responses to A', as well as the analogously defined $\mathrm{minProb}\,(q_1 \to \chi \mid \mathsf{S}(\psi, q_2))$.

More precisely, we determine for each $\langle\langle.\rangle\rangle_.^{\blacktriangleleft\alpha}$ appearing in φ whether $\mathrm{maxProb}\,(q_1 \to \chi \mid \mathsf{S}(\psi, q_2)) \blacktriangleleft \alpha$ and whether $\mathrm{minProb}\,(q_1 \to \chi \mid \mathsf{S}(\psi, q_2)) \blacktriangleleft \alpha$. For propositional variables, this is trivial. If $\chi = \langle\langle A''\rangle\rangle_i^{\blacktriangleleft\alpha}\,\chi'$ for some $A'' \subseteq A$ and some path formula χ', it follows that

$$\mathrm{maxProb}\,(q_1 \to \chi \mid \mathsf{S}(\psi, q_2)) = \begin{cases} 1 & \text{if } \mathcal{C}, \mathsf{S}, q_1 \models \chi, \\ 0 & \text{otherwise.} \end{cases}$$

Also by definition, $\mathcal{C}, \mathsf{S}, q_1 \models \chi$ if and only if for all suitable responses r, we have that $\Pr\left(q_1 \to \{\lambda \mid \lambda, \mathsf{S} \models \chi'\} \;\mid \mathsf{S}(A', q_1, \chi) + r\right) \blacktriangleleft \alpha$. If $\blacktriangleleft$ is either $\leq$ or $<$, this is true if and only if $\mathrm{maxProb}\,(q_1 \to \chi' \mid \mathsf{S}(\chi, q_1)) \blacktriangleleft \alpha$ (in the case that $\blacktriangleleft$ is $\geq$ or $>$, we use $\mathrm{minProb}\,(q_1 \to \chi' \mid \mathsf{S}(\chi, q_1))$ analogously). By induction, we know whether this is true. The knowledge-operator $\mathcal{K}$ is handled in the same obvious way as in the deterministic case.

Hence assume that the outmost operator of χ is neither a strategy- now a knowledge operator. We transform χ to an LTL-formula in a similar way as in the deterministic case: Since due to induction we know for all states q of $\mathcal{C}$ and all strategy-subformulas χ' of χ the exact of states in which χ' is satisfied, we can introduce new propositional variables that indicate the truth of each of these formulas χ', and thus obtain a modifies game structure and formula where

- the moves of the coalition A' are fixed by $\mathsf{S}(A', q_2, \psi)$,

– strategy operators are replaced with propositional variables,
– there is a single player who controls the moves of $\overline{A'}$.

The thus-modified formula χ is an LTL-formula (Note that the player controlling the moves of $\overline{A'}$ cannot be replaced by a simple set of "next states" as in the deterministic case, since this does not lead to a well-defined probability measure). Following the construction in the full version of [CL07], truth of this formula can be determined in 3EXPTIME. Hence the entire verification can be performed in $\mathrm{P}^{\mathrm{3EXPTIME}} = \mathrm{3EXPTIME}$.

The verification whether the thus-obtained S is indeed φ-maximal can again be carried out in the same way as in the deterministic case (this essentially squares the time complexity of the algorithm, and still remains in 3EXPTIME).

For the hardness result, note that already the problem to determine whether an LTL-formula is satisfied by a probabilistic game structure with probability 1 is hard for 2EXPTIME (see [CY88]). This clearly reduces to both $\exists\mathsf{maxChoice}$ and $\exists\mathsf{Choice}$, by considering a CGS with a single player. For such a formula, obviously the unique existing strategy choice is φ-maximal. □

The above, together with the result from Schobbens [Sch04] that model checking for memoryless ATL* is PSPACE-complete, shows that the model checking complexity of our semantics comes at no additional cost compared to that of standard ATL* with memoryless strategies in the deterministic setting (recall that due to Proposition 2.1, our semantics are a generalization of memoryless ATL*). As expected due to results from Courcoubetis and Yannakakis [CY88], model checking for the probabilistic case is significantly more complex.

The situation is different for history-dependent strategies: While model checking for standard ATL* (which as mentioned allows history-dependent strategies) is 2EXPTIME-complete ([AHK02]), the problem becomes undecidable in the setting with incomplete information. Formally, we define the following decision problems:

Problem:	$\exists\mathsf{Choice}^{hst}$
Input:	A CGS $\mathcal{C}$, a state q of $\mathcal{C}$, a state-formula φ, a coalition A
Question:	Is there a strategy choice S for $\mathcal{C}^{hst}$ such that $\mathcal{C}^{hst}, S, q \models \varphi$?

Problem:	$\exists\mathsf{Choice}^{hst}$
Input:	A CGS $\mathcal{C}$, a state q of $\mathcal{C}$, a state-formula φ, a coalition A
Question:	Is there a $\leq_\varphi$-maximal strategy choice S for $\mathcal{C}^{hst}$ such that $\mathcal{C}^{hst}, S, q \models \varphi$?

The undecidability result now easily follows from [AHK02]:

Theorem 4.2 *$\exists\mathsf{Choice}^{hst}$ and $\exists\mathsf{Choice}^{hst}$ are undecidable for the deterministic and the probabilistic case.*

Proof. Obviously it suffices to prove the result for the deterministic case. In [AHK02, Theorem 7.1], it was shown that model checking of a basic version of ATL* with incomplete information is already undecidable. Due to Proposition 2.1, and since our semantics are a generalization of the restricted case considered there, it follows that $\exists\mathsf{Choice}^{hst}$ is undecidable as well. The proof of the mentioned theorem in [AHK02] shows undecidability even for a fixed formula which has the very simple form $\langle\langle A\rangle\rangle \Diamond p$. Since no nesting occurs, obviously every strategy choice satisfying this formula is maximal already. Hence the undecidability result also holds for $\exists\mathsf{Choice}^{hst}$. □

5 Conclusion and future research

We have considered the situation in which a coalition A agrees on a set of strategies prior to the game, which are collected in a *strategy choice*. In the evaluation of the success probability of such a set of strategies, we adopted the usual pessimistic convention that the remaining players follow their best-possible strategy, which is allowed to use information not available to the coalition A (including being allowed to be history-dependent). It would be interesting to relax this worst-case assumption and assume that the counter-coalition also has only bounded resources available. In our setting, one might wish to consider statements like "there is a strategy choice S for A such that for all strategy choices S' for $\overline{A}$, S achieves ..." It is an interesting question to investigate whether this leads to a sound semantics for reasoning about resource-bounded adversial coalitions. A further relaxation of the "worst-case" assumption is the following: When a sub-coalition $A' \subseteq A$ tries to achieve a goal, then the players in $A \setminus A'$ are treated as adversial. One could also consider a relaxed semantics where the coalition $A \setminus A'$ continue to play their previously-chosen strategies.

It also may be interesting to determine the complexity of the model checking problem for basic ATL, where compared with the variant ATL* that we consider, the nesting of strategic and temporal operators is restricted. It is likely that this restriction leads to an easier model checking problem in the same way as it does for standard ATL [AHK02].

Finally, considering mixed strategies is obviously an interesting issue. Note that a basic form of mixed strategies is possible in our semantics, when we introduce intermediate states in which the next move by a player is chosen at random, where the probability distribution may be different in each state, hence the players may be given control over the distributions. However, a general treatment of mixed strategies remains open.

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