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A 2-approximation for 2D bin packing

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Abstract

In this paper, we study the two-dimensional geometrical bin packing problem (2DBP): given a list of rectangles, provide a packing of all these into the smallest possible number of 1×1 bins without rotating the rectangles.

We present a 2-approximate algorithm, which improves over the previous best known ratio of 3, matches the best results for the rotational case and also matches the known lower bound of approximability. Our approach makes strong use of a recently-discovered PTAS for a related knapsack problem and a new algorithm that can pack instances into $\mathrm{OPT} + 2$ bins for any constant OPT.

Keywords: bin packing, approximation, rectangle packing.

1 Introduction

In recent years, there has been increasing interest in extensions of packing problems such as strip packing [1, 2, 3, 4, 5], knapsack [6, 7, 8] and bin packing [9, 10, 11, 12, 13], to multiple criteria (vector packing) or multiple dimensions (geometric packing).

Two-dimensional bin packing, both with and without rotations, is one of the very classical problems in combinatorial optimization and its study has begun several decades ago. This is not only due to its theoretical appeal, but also to a large number of applications, ranging from print and web layout [14] (putting all ads and articles onto the minimum number of pages) to office planning (putting a fixed number of office cubicles into a small number of floors), to transportation problems (packing goods into the minimum number of standard-sized containers) and VLSI design [15].

It is easy to see that two-dimensional bin packing without rotation (2DBP) is strongly NP-hard as a generalization of its one-dimensional counterpart, hence the main focus is on algorithms with provable approximation quality.

Consider an algorithm A for 2DBP, and denote for each instance I with $A(I)$ the number of bins A produces and with $\mathrm{OPT}(I)$ the smallest number of bins into which I can be packed. A is an *α-approximation* for 2DBP if we have $\sup_I \{A(I)/\mathrm{OPT}(I)\} \leq \alpha$ over all instances I, and an *asymptotical α-approximation* if we have $\limsup_{\mathrm{OPT}(I)\to\infty} A(I)/\mathrm{OPT}(I) \leq \alpha$. A polynomial-time approximation scheme (PTAS) is a family $\{A_\epsilon : \epsilon > 0\}$ of $(1+\epsilon)$-approximation algorithms.

Remark 1. 2DBP is $(2-\epsilon)$-inapproximable for all $\epsilon > 0$, since the decision problem "can the instance be packed into a single bin?" contains the strongly NP-hard problem 3Partition as a special case (where all items have the same height $3/n$).

The best previously known result for the non-rotational case was a 3-approximation by Zhang [11]; for the rotational case, Harren and van Stee have recently given a 2-approximation in [13], the same ratio can be achieved using techniques by Jansen and Solis-Oba [5].

As to asymptotical approximation ratios, Bansal and Sviridenko showed in [10] that 2DBP does not admit an asymptotical PTAS. Caprara gave an algorithm with ratio of $1.69\ldots$ in [9], breaking the important barrier of 2. More recently, Bansal, Caprara and Sviridenko improved the rate to $1.52\ldots$ in [12] for both the rotational and non-rotational case.

A closely related problem is two-dimensional knapsack: here, every rectangle also has a profit and the objective is to pack a subset of high profit into a constant number (usually one) of target bins. The best currently known results here are a $2+\epsilon$-approximation by Jansen and Zhang [16] for the general case, and a PTAS by Jansen and Solis-Oba [17] if all items are squares. For our purposes, the special case that the profit equals the item's area is important. We have recently shown in [8, 18] that this problem admits a PTAS, and this algorithm is one of the corner stones of the algorithm presented here.

Our contribution We study the non-rotational geometric two-dimensional bin packing problem, i.e. we are given a list of rectangles (items) $r_1 = (w_i, h_i), \ldots, r_n = (w_n, h_n)$ with all w_i, h_i taken from the interval $]0,1]$, and the objective is to find a non-rotational non-overlapping packing of all items into the minimum number of containers of size 1×1. The main result of this paper is the following theorem:

Theorem 2. *There is a polynomial-time* 2*-approximation for two-dimensional geometric bin packing.*

This result is achieved using an asymptotic approximation algorithm for large optimal values; smaller (i.e. constant) values are solved by a recent breakthrough in the approximability of two-dimensional *knapsack* problems in [8, 18]: we have proven that there exists a PTAS for maximizing the area covered by rectangles within a 1×1 bin. This can be combined with other packing algorithms if the optimum is constant, but at least 2, to generate a packing into OPT $+ 2$ bins. If the optimal packing uses only one bin, we conduct a case study, again starting from a packing that covers $(1 - \epsilon)$ of the bin and generate a packing into OPT $+ 1 = 2$ bins.

As it turns out, the cornerstone of the proof will be the following Lemma, which is proven in Section 4.

Lemma 3. *There is a polynomial-time algorithm that finds a packing into two bins, provided that a packing into one bin exists.*

The case of larger optimum values is handled in Sections 2 and 3.

2 Large optimal value

As noted above, Bansal, Caprara and Sviridenko [12] have given a polynomial-time algorithm for 2DBP which has an asymptotic approximation ratio of $1.525 < 2$. From this, we immediately obtain a (non-asymptotic) approximation ratio of 2 for instances with a large optimum:

Corollary 4. There is a constant K so that for every instance I with $\mathrm{OPT}(I) \geq K$, the algorithm of Bansal et al. yields a packing into at most $2\mathrm{OPT}(I)$ bins.

An algorithm to solve the overall problem can now first run the algorithm of Bansal et al., and it can then run for each guessed OPT $= k < K$, of which there is only a constant number, a polynomial-time algorithm that tries to find a packing into $k + 2$ bins. For the case of OPT $= k = 1$, we generate a packing into two bins. Finally, the algorithm returns the best solution amongst these. The details of the algorithms for constant OPT are given in the next two sections.

We will in many cases fall back on Steinberg's algorithm, for which the following holds:

Theorem 5 (Steinberg [3])**.** *We can pack a set of items $\{r_i = (w_i, h_i), i = 1, \dots, n\}$ into a target area of size $u \times v$ if the following conditions hold:*

1. $\max\{w_i : i = 1, \dots, n\} \leq u$,
2. $\max\{h_i : i = 1, \dots, n\} \leq v$,
3. $2\sum_{i=1}^{n} w_i h_i \leq uv - (2\max\{w_i : i = 1, \dots, n\} - u)^+(2\max\{h_i : i = 1, \dots, n\} - v)^+$,

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$.

3 Solving for $1 < \mathrm{OPT} < K$

In this case, we can guess (by enumeration) the value of OPT. Supposing that we know the correct value, we can find a packing into $\mathrm{OPT} + 2$ bins using the following two key statements:

Theorem 6. *There is an algorithm that, given a set of rectangles $I = \{r_i = (w_i, h_i) : i = 1, \ldots, n\}$, a constant ϵ and a constant k such that there exists a packing of I into k bins, produces in polynomial time a packing of a subset $I' \subseteq I$ into k bins such that the total unpacked area $\sum\{w_i h_i : r_i \in I \setminus I'\}$ is bounded by $\epsilon \cdot \sum\{w_i h_i : r_i \in I\}$.*

Due to space constraints, the proof, which is mainly an extension of techniques of [18, 8], is only given in the full paper. We will only need the result for one single fixed $\epsilon \leq 1/2K$.

The other ingredient is a routine for packing all of the remaining items into at most 2 bins.

Lemma 7. *There is an algorithm that packs a set of rectangles with total area at most $1/2$ into 2 bins.*

Proof. We simply assign all items of width larger than $1/2$ to one bin, all items of (width at most $1/2$ and) height larger than $1/2$ to the other bin, and distribute the remaining items, which are at most $1/2$ in both directions, arbitrarily. Then, both bins can be packed using a simple extension [6] of Steinberg's algorithm [3] that permits a single large (i.e. both height and width larger than $1/2$) item to be present in the bin. □

Combining the two, we obtain:

Theorem 8. *For every constant k, we can check in polynomial time whether there exists a packing into $k + 2$ bins.*

4 Solving for $\mathrm{OPT} = 1$

In this section, we will show how an instance that admits a packing into one bin can be packed into two bins in polynomial time. Without loss of generality, we may assume that the total area of all items is exactly 1, by introducing dummy squares whose sidelength is $\gcd\{w_i, h_i : i = 1, \ldots, n\}$. In the following, all statements still hold when interchanging width and height, unless specifically noted otherwise.

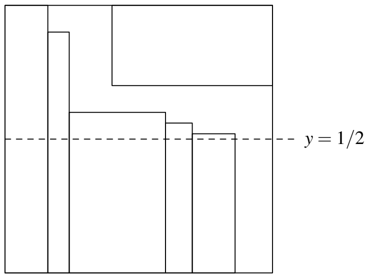


Figure 1: Tall items and one extra item

We will now study several cases separately and solve each in polynomial time. The algorithm will check which case applies and use the corresponding solution. We will mean "we can pack" to imply a step admits polynomial-time algorithms and "can/cannot be packed" to imply general feasibility.

4.1 Mostly tall and wide items

Let us start by making a simple observation on the arrangement of tall items in the optimal solution:

Remark 9. Consider the set of 'tall' items of height more than $1/2$. We can always pack these items into a bin along with one arbitrary extra item r_i.

Proof. Note that no two tall items can intersect the same vertical line $x = x_0$. In particular, the total width of these items is at most 1. We sort the items by decreasing height, starting at the bottom-left corner, cf. Fig. 1, and place the extra item in the top right corner. Assume that r_i intersects the tall items. In particular, this means that items with height larger than $1 - h_i$ have total width of more than $1 - w_i$, which means that no feasible packing of these items along with r_i exists. □

Noting that Steinberg's algorithm will pack any set of items of total area at most $1/2$ into a bin if all items are bounded by $1/2$ in the same direction, we obtain:

Corollary 10. Consider the set of 'tall' items of height more than $1/2$. If their total area is at least $1/2 - \beta$ for some $-1/2 \leq \beta \leq 1/2$, then either every other item has area less than β or we can pack all items into two bins.

As a slight generalization, we can show:

Lemma 11. *For* $0 < \gamma \leq 1/2$ *arbitrary, set* $W := \sum\{w_j : j \in \{1, \dots, n\}, h_j > 1 - \gamma\}$. *Then,*

$$\sum\{w_j h_j \colon j \in \{1, \dots, n\}, w_j > 1 - W, h_j \leq 1 - \gamma\} \leq 2\gamma\,. \tag{1}$$

Proof. Consider a horizontal scanline $y = y_0$ for $y_0 \in [\gamma, 1 - \gamma]$ in any packing. Such a scanline intersects all items of height at least γ, hence, it will not admit an item of width larger than $1 - W$. Hence, all such items must be in the outermost regions of height γ at the top and bottom of the bin. □

Lemma 12. *If the total width of items taller than* $1/2$ *is larger than* $1 - \delta$ *for* $\delta = 3/4 - \sqrt{1/2} \approx 0.042$, *we can pack all items into two bins.*

Proof. We pack the tall items into the first bin, sorted by non-increasing height. They must fit next to each other, since no two of them can be atop one another, and the total area covered by these items is at least $1/2 - \delta/2$. Note that by Cor. 10, all other items have individual area bounded by $\delta/2$, or we are done. In particular, every other item is bounded by $\sqrt{\delta/2}$ in at least one direction.

We define $\delta' := -1/4 + \sqrt{1/16 + \delta/2}$. It is easy to verify the following statements:

1. $\sqrt{\delta/2} \leq (1/2 - \delta')/2$,
2. $(1/2 - \delta')^2 \geq 2\delta$,
3. we can pack a "virtual item" of size $(1/2 - \delta') \times (1/2 - \delta')$ into the upper right corner without intersecting the tall items,
4. we can pack a "virtual item" of size $1/2 \times (2\sqrt{\delta})$ into the upper right corner without intersecting the tall items.

Using the first three and Steinberg's algorithm, we are done if there are items of width at most $\sqrt{\delta/2}$ and height at most $1/2 - \delta'$ with total area at least $\delta/2$, and we are also done if there are items of height at most $\sqrt{\delta/2}$ and width at most $1/2 - \delta'$ with total area at least $\delta/2$.

Hence assume neither is the case. The total area of items not yet considered is at least $1 - (1/2 - \delta/2) - \delta/2 - \delta/2 = 1/2 - \delta/2$. Let us turn to items whose height is in $]1/2 - \delta', 1/2]$, i.e. they are "almost tall". Keeping in mind they cannot be packed atop an item of height larger

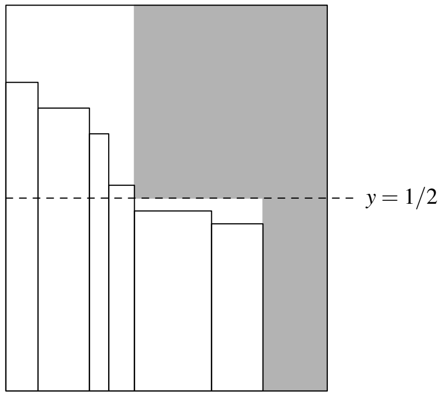


Figure 2: Space for items that are tall or almost tall

than $1/2 + \delta'$ in any packing, the total width of areas that can accomodate them in our packing, shaded in Fig. 2, is large enough for us to pack all but one of them there greedily into two shelves. Since the single item has area at most $\delta/2$, we are done if the total area of almost tall items is at least δ.

If this is not the case, we know that all remaining items, i.e. those of height at most $\sqrt{\delta/2}$ and width larger than $1/2 - \delta'$, cover a total area of at least $1/2 - 3/2\delta$. If the subset of these of width at most $1/2$ is at least $\delta/2$, we can use property 4 to pack some of them in the top right corner. If all of this does not happen, we know that all items of width larger than $1/2$ cover an area of at least $1/2 - 2\delta$. We now finally claim that we can pack a selection of these items of area at least $\delta/2$ along with the tall items. Namely, greedily select wide items of minimal width until their area is at least $\delta/2$ (and at most δ), and pack them vertically, starting in the top right corner, in non-increasing order of width. The total height of this stack is at most 2δ. Assume this stack overlaps the tall items at some coordinate (x, y). In particular, this means that items taller than y have total width at least x. By Lemma 11, this means that there is only a total area of at most $2 - 2y \leq 4\delta$ wider than $1 - x$. At the same time, all unselected wide items are at least this wide and have total area at least $1/2 - 3\delta$, which contradicts the fact that $\delta < 1/14$. □

4.2 General properties of packings.

In this and all subsequent cases, we will first apply the algorithm in [8, 18], which will pack items with total area at least $1 - \epsilon$ into one bin. The remaining items have the additional property that each item is bounded

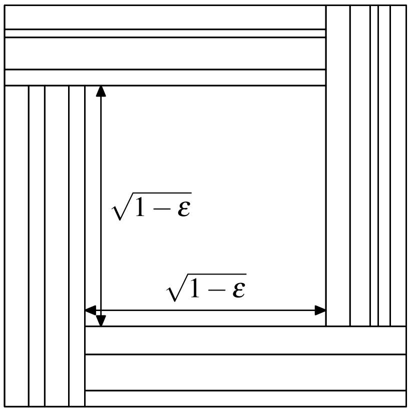


Figure 3: Partition instances in a 2DBP instance with area ϵ

in at least one direction by ϵ.

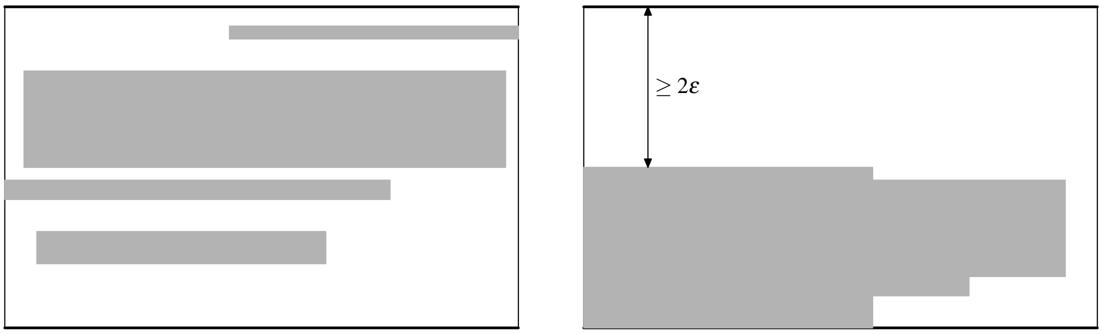
While we know these items can be packed into the one more bin we are allowed, it is in general NP-hard to do so, since the items could encode two instances of Partition, see Fig. 3.

However, we can draw conclusions about the arrangement of items in this $(1-\epsilon)$-packing:

Lemma 13. *Consider a cutting line $y = y_0$ for arbitrary y_0. If the total area of items intersected by this line is at least $1/2$, then we can pack all items in two bins.*

Proof. We move all items intersected by the cutting line into the second bin. W.l.o.g. all items are bottom-aligned and sorted by decreasing height as in 9. Note that we might not include all items of height more than $1/2$ yet. Let us assume the total width of these missing items is some $w > 0$. We insert these items into their proper sorted position. This will shift all smaller items to the right, possibly removing them from the bin. However, the area covered in the bin will not decrease. At the end, one item might intersect the right boundary of the bin. This item has height at most $1/2$, since all taller items can be packed side by side. Using this item as the 'extra' item, we invoke Lemma 9. The remaining items are not tall and can be re-packed in the first bin using Steinberg's algorithm. □

Our angle of attack is the following: we will identify a suitable strip of height at least 2ϵ in the packed bin and move all items that intersect this strip into the second bin. This creates empty space that can be used to pack all unpacked items of height no more than ϵ in the first bin using



(a) Before reordering (b) After reordering

Figure 4: Repacking wide items in a good strip

Steinberg's algorithm. The remaining unpacked items have height at least ϵ, width at most ϵ and total area at most ϵ, and we will add them in the second bin.

Consider a horizontal scanline $y = y_0$ in the packed bin. We call the scanline *good* if it does not intersect the interior of a rectangle with width more than $1/2$. Similarly, a vertical scanline $x = x_0$ is good if it does not intersect a rectangle of height larger than $1/2$. The sets $H := \{y_0 \in [0,1] : y = y_0 \text{ is good}\}$ and $V := \{x_0 \in [0,1] : x = x_0 \text{ is good}\}$ of horizontal and vertical good scanlines are easily seen to be finite unions of closed disjoint intervals. We extend the notion of the length of an interval to disjoint intervals in the obvious way and note that unless Lemma 12 applies, both H and V have length at least δ.

In particular, the following Lemma can be applied for $\varepsilon := 2\epsilon$:

Lemma 14. *For an arbitrary fixed $w \in \mathbb{N}$, set $k := 1 + \lceil \delta^{-1}(w-1) \rceil$ and $\varepsilon := \min\{k^{-1}, (2k^2)^{-1}(k\delta - w + 1)\}$. Partition the bin into k horizontal strips of equal height. Then, w of these strips contain good intervals of height at least ε.*

Proof. Assume this is not the case. Then, the total sum of good intervals clearly is upperbounded by

$$(w-1)/k + k\varepsilon < (w-1)/k + (k\delta - w + 1)/k = \delta,$$

a contradiction. □

Let us now consider one such strip that contains 2ϵ in good intervals. We may assume the top and bottom edge of the strip to be good scanlines, otherwise, we move the top edge down and the bottom edge up until they are without losing any length of good interval in the strip. We can now move all items that intersect the strip and have width at most $1/2$ to the second bin and re-stack the wider items that are not moved in the first bin, thus obtaining a contiguous empty area of at least size $1 \times 2\epsilon$ (see Fig. 4).

Note that all items moved from the first to the second strip are of width at most $1/2$. Hence, if their total area is at most $1/2 - \epsilon$, they can be packed together with the remaining unpacked items (which are all bounded in width by ϵ and have total area at most ϵ) using Steinberg's algorithm.

4.3 One big item

In this section, we consider instances which contain one *big* item r_1 with $w_1, h_1 \geq 1/2$, located at (x_1, y_1). By the previous discussion, we may also assume that $w_1, h_1 < 1 - \delta$. To help rearranging items which are very limited in one dimension, the following observation will be useful:

Lemma 15. *Given a set $\{a_1, \ldots, a_n\}$ of numbers and a minimal target value T such that $S := \sum_{i=1}^{n} a_i \geq 2T + \max_{i=1\ldots,n} a_i$, we can identify a subset $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} a_i \geq T$ and $\sum_{i \notin I} a_i \geq T$.*

Proof. Consider a fractional optimal solution to the Partition problem on the a_i's, for example obtained by greedy packing. Note the solution contains only at most one fractional item a_j. Since $a_j \leq \max\{a_i : i = 1 \ldots n\}$, both parts have size at least T even if we assigne a_j to the other. □

By symmetry, we assume that $y_1 \leq 1 - h_1 - y_1$, i.e. r_1 is somewhat in the lower half of the bin. We consider the horizontal strip from y_1 to $y_1 + 2\epsilon$ and all the items that intersect for movement to the second bin, see Fig. 5, and note that all items that do not intersect $y = y_1 + 2\epsilon$ are of bounded height at most $y_1 + 2\epsilon \leq h_1 - 2\epsilon$. In particular, we can pack these items into the 'hole' left in the first bin by moving r_1 without obstructing the horizontal strip at its bottom.

By previous discussion, we can assume that the total width of items higher than $1/2$ that intersect $y = y_1 + 2\epsilon$ is at most $1 - \delta$, in particular, we assume that there is a (continuous by reordering) interval of length at least δ which contains only items of height at most $1/2$. If we can apply

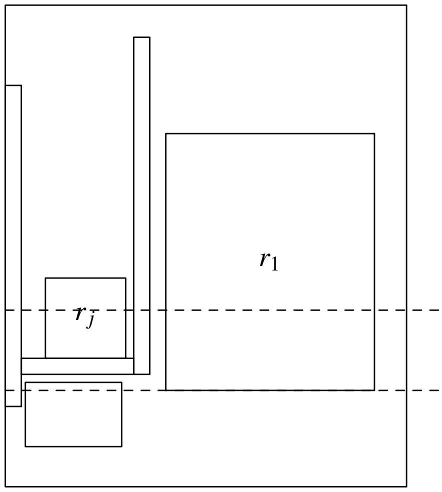


Figure 5: Packing with one big item

Lemma 15 with target value 2ϵ, we have cleared a vertical strip of width 2ϵ.

Otherwise, we know that there is an item r_j of width at least $\delta - 4\epsilon > 2\epsilon$ and height at most $1/2$ on this scanline. We can now construct a packing as follows: we pack all tall items, including r_1 and those that were not yet packed at all, along with r_j as extra item, by Lemma 9. Note that there might be horizontal overlap of r_k and the tall items, but it is bounded by 2ϵ, since it is only caused by tall unpacked items. We can pack the non-tall items of the scanline below r_k in width 2ϵ, and the remaining unpacked non-tall items, we pack into a container sized $4\epsilon \times 1/2$, which we can position in the lower-right corner, since $8\epsilon \leq w_j$ for $\epsilon \leq \delta/10$.

4.4 At least one medium-sized item

For this case, assume that there exists an item $r_i = (w_i, h_i)$ such that $w_i, h_i \geq 12\epsilon$ and $h_i \leq 1/2$ at some position (x_i, y_i). Without loss of generality, suppose that $y_i \leq 1/2 - 1/2h_i$. We will now consider three consecutive strips I, II and III that intersect the item, cf. Fig. 6a. Since $h_i \geq 12\epsilon$, we may assume that the bottom strip is still 2ϵ away from the bottom of the bin. It is also easy to verify that in any case, the top of the third strip has a distance of at least h_i to the top of the bin.

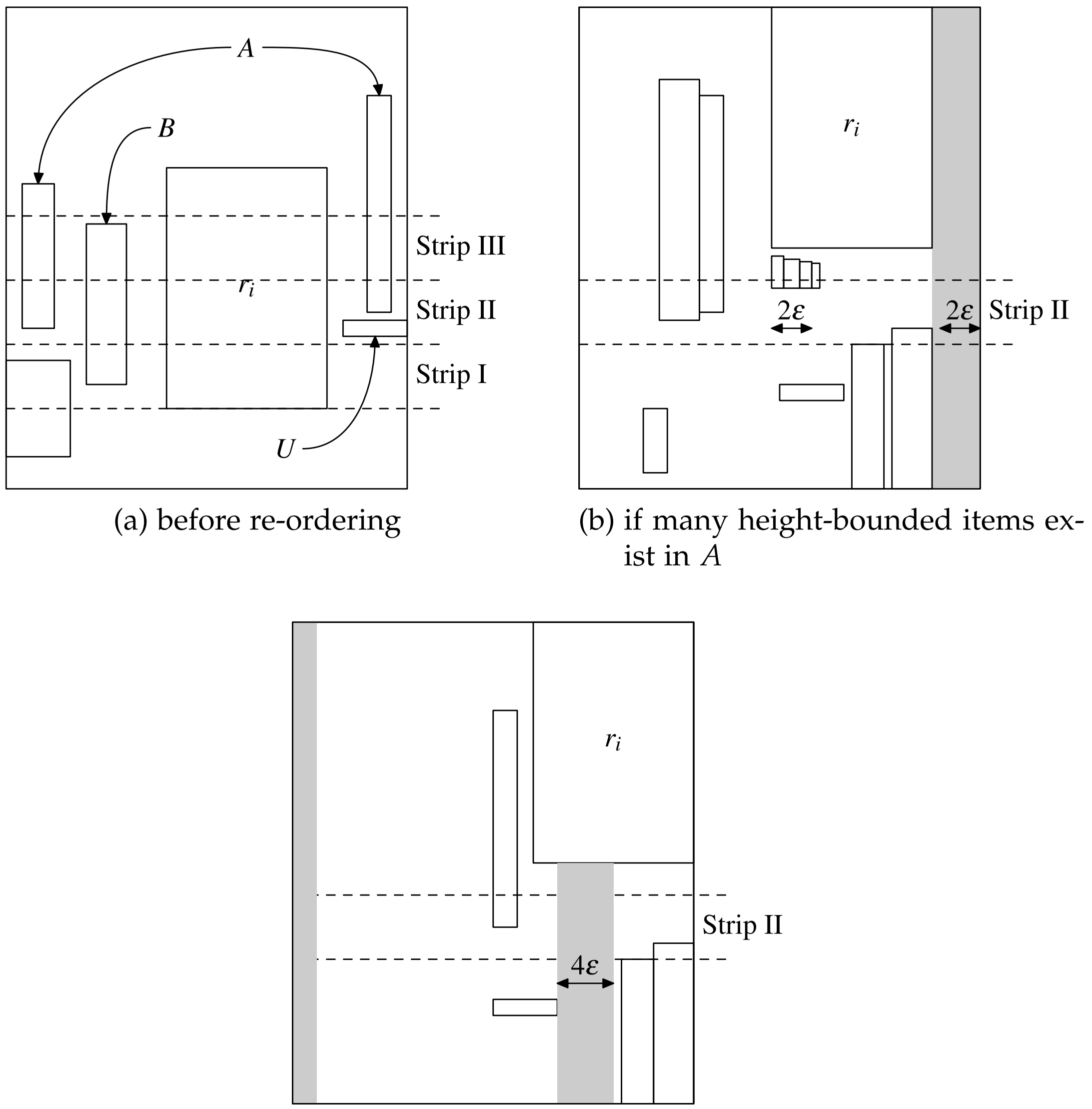
Remark 16. If one of Strips I, II, III contains items other than r_i of height at most $1 - h_i$ and total width 2ϵ that totally bisect the strip, we can pack the instance into two bins.

Proof. Move the strip in question to the second bin. Assume by reordering that all bisecting items are adjacent to r_i. It is then possible to shift r_i to one side by at least 2ϵ, which frees a vertical strip of width 2ϵ. □

In the following, we assume that Remark 16 cannot be applied. We move Strip II and all items that intersect it to the second bin. By reordering, we assume that $x_i + w_i = 1$, and we re-set $y_i := 1 - h_i$. All other items in the second bin can be partitioned into three groups: B, the set of items that completely bisect the strip, A, the set of items that intersect the upper boundary of the strip, but not the lower (i.e. they are 'above the strip'), and U, the set of remaining items, which are entirely in or partially under the strip. By reordering, we assume all elements of B are at the left side of the bin, ordered by non-increasing height.

Note that since Strip II is in the lower half of the bin, all items in U are bounded in height by $1/2$, and in particular by $1 - h_i$. Since Remark 16 did not apply to Strip I, those items in U that bisected Strip I have total width at most 2ϵ and can be packed below r_i. All others are bounded in height by 4ϵ. Since Strip I still had at least 2ϵ space beneath it, we can drop these items by 2ϵ. As a consequence, none of them intersects Strip II anymore, so we re-sort the items in A by non-increasing height. We now consider those items in A and B that bisected Strip III entirely and have height at most $1 - h_i$. Again by Remark 16, their total width is bounded by 2ϵ and we pack them below r_i as well.

At this point, all items in Strip II that are not re-packed below r_i have height greater than $1 - h_1$ or belong to A and have height at most 4ϵ. If the total width of the latter is at least 2ϵ, we can shift r_i and all items packed below it to the left by 2ϵ, generating a vertical strip of width 2ϵ at the right of the bin, cf. Fig. 6b. Otherwise, we can pack them below r_i. The total free width remaining below r_i is now at least 6ϵ, which we use to shift the entire packing of U to the right by 2ϵ and finally to pack all unpacked items of height at most $1 - h_i$, width at most ϵ and total area at most ϵ into a target area sized $4\epsilon \times (1 - h_i)$. Now, all items that are not packed below r_i, including the remaining unpacked items, have height greater than $1 - h_1 > 1/2$, and hence, by Lemma 9, they fit next to r_1 as sketched in Fig. 6c.



(a) before re-ordering

(b) if many height-bounded items exist in A

(c) if few height-bounded items exist in A

Figure 6: Packing with one medium item

4.5 All small or elongated items

If the previous discussion does not apply, we know that all items in the instance are bounded by 12ϵ in one direction. We show that there are few items of 'medium' sidelength, and items that are small in both directions can be packed efficiently using NFDH:

Lemma 17. *The total area of packed items of height at least* 12ϵ *and at most* $1/2$ *and width at most* 12ϵ *is bounded by* 54ϵ*, or else we can pack all items into two bins.*

Proof. Partition the bins into horizontal strips of height 2ϵ, and note that each item of height at least 12ϵ will bisect at least four of these strips and intersect up to two more. In particular, at least two thirds of its area is spent in completely bisecting strips. If there is a total width of 36ϵ of such bisecting items in one strip, we can use Lemma 15 to re-pack them, freeing a vertical strip of width 2ϵ, and we obtain a packing. Otherwise, the total area of the items is at most $36\epsilon \cdot 3/2 = 54\epsilon$. □

From this, we conclude that in the only case left to consider, the majority of items are either tall or wide or very small in both dimensions. We will use this knowledge to construct an entire packing from scratch. More precisely, denote with A_{wide} the total area of items of width at least $1/2$ and with H their total height, with A_{tall} the total area of items of height at least $1/2$ and with W their total width, and with A_{small} the total area of items which are bounded in both directions by 12ϵ. By the previous lemma, we obtain that

$$A_{tall} + A_{wide} + A_{small} \geq 1 - \epsilon - 108\epsilon\,. \quad (2)$$

For convenience, we assume $A_{wide} \leq A_{tall}$ and construct the packing illustrated in Fig. 7 for the first bin: all tall items are packed at the left side in non-increasing order. In the top right corner, we pack a subset of wide items of height almost $H/2$ (but for one fractional item): by arguments similar to the proof of Lemmas 12 and 11, they do not intersect the tall items. Their area is at least $A_{wide}/3 - 12\epsilon$, since wider items might all have width 1. In the bottom right corner, we reserve a target area of size $w \times h$, at least $\delta \times \delta$, which touches the tall and wide items. We fill this area with small items using NFDH. If we run out of small items doing this, the area remaining for the second bin is bounded by

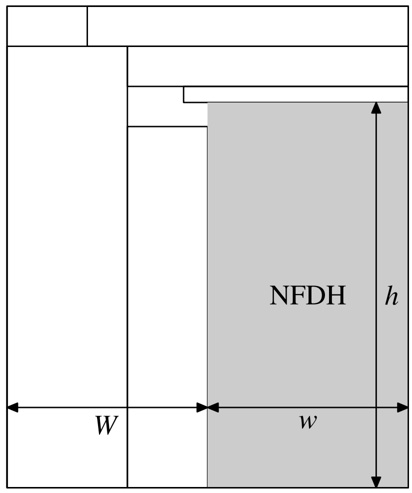


Figure 7: Packing elongated items

$108\epsilon + 2/3A_{wide} + 12\epsilon \leq 120\epsilon + 1/3$, which is less than $1/2$ for $\epsilon \leq 1/720$, and hence, we can pack the second bin with Steinberg's algorithm.

If we do not run out of small items, we have covered at least an area of $1/2$ in the first bin: the left part of the bin is filled at least half by tall items, the top part is filled at least half by wide items. The remaining part is filled with NFDH. Note that each shelf will be packed to at least $w - 12\epsilon$. We might lose 12ϵ at the top of the area and not account for one shelf by standard shifting arguments, but still, the covered area is at least

$$(w - 12\epsilon)(h - 24\epsilon) > wh - 36\epsilon \geq wh/2 \tag{3}$$

if we set $\epsilon < \delta^2/72 \approx 2.4 \cdot 10^{-5}$. The unpacked items can hence be packed into the second bin using Steinberg's algorithm.

Summing up, the overall algorithm works as outlined in Fig. 8.

5 Conclusion

We have presented an algorithm that generates 2-approximate solutions for two-dimensional geometric bin packing, which matches the rate known for the rotational problem. Since both the rotational and non-rotational problem are not approximable to any $2 - \epsilon$ unless $P = NP$, this concludes the study of absolute approximability of these problems. For practical applications, it would be interesting to find faster algorithms: our algorithm relies heavily on the knapsack PTAS in [8, 18] and techniques in [5] with a doubly-exponential dependency on ϵ, in particular when compared to

1. Run the algorithm of Bansal et al. [12].
2. For each $k = 2, \ldots, K$ run the algorithm of Sect. 3.
3. If the area of all items is at most 1:
   a) Apply Lemma 12, if possible.
   b) Else, generate a packing of $(1 - \epsilon)$ area in the first bin using [8, 18].
   c) If this packing contains a big item, apply the algorithm in Sect. 4.3.
   d) Else, if this packing contains an item of at least 12ϵ in both directions, apply the algorithm in Sect. 4.4.
   e) Else, apply the algorithm in Sect. 4.5.
4. Return the best solution found.

Figure 8: The overall algorithm

the running time of Zhang's 3-approximation in [11]. Still, our result is an important step in the study of two-dimensional packing problems.

Another important open problem is the gap in asymptotic behaviour between the non-existence of an APTAS and the best known algorithm with asymptotic quality of $1.525\ldots$.

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