INSTITUT FÜR INFORMATIK

UND PRAKTISCHE MATHEMATIK

The Twist Representation of Shape

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Bericht Nr. 0407
July 2004

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Abstract

We give a contribution to the representation problem of free-form curves and surfaces. Our proposal is an operational or kinematic approach based on the Lie group $SE(3)$. While in Euclidean space the modelling of shape as orbit of a point under the action of $SE(3)$ is limited, we are embedding our problem into the conformal geometric algebra $\mathbb{R}_{4,1}$ of the Euclidean space $\mathbb{R}^3$. This embedding results in a number of advantages which makes the proposed method a universal and flexible one with respect to applications. Especially advantagous is the equivalence of the proposed shape model to that of the Fourier representations.

Chapter 1

Introduction

Shape is a geometric conception of the appearance of an object, a data set or a function which "can be defined as the total of all information that is invariant under translations, rotations, and isotropic rescalings. Thus two objects can be said to have the same shape if they are similar in the sense of Euclidean geometry." This quotation from [43] is very general because of the consideration of scale invariance. By leaving out that property, we can define the shape of an object as that geometric conception that is invariant under the special Euclidean group, e.g. $SE(3)$ if we consider 3D shape in Euclidean space $\mathbb{R}^3$. Furthermore, we allow our objects to change their shape in a well-defined manner under the action of some external forces which may include also a re-normalization of size.

The literature on shape modelling and applications is vast. May it be visualization and animation in computer graphics or shape respectively motion recognition in computer vision. The central problem for the use- fulness in either field is the chosen representation of shape [34]. Let the representation be given statistically [24] or deterministically like algebraic or transcendental curves and surfaces (see e.g. [31] or [29]) by implicit or parametric functions [26], by a set of geometric primitive entities like points under constraints or finally by a set of features as invariants (see e.g. [52] and [25]). There is no general way of shape representation which satisfies all the requirements of an application. This makes shape modelling a continually fascinating area of research.

Here we present a new approach to the modelling of free-form shape of curves and surfaces which has some special features that make it especially attractive for computer vision and computer graphics. We get that new representation by the fusion of two different algebraic conceptions:

1) Free-form curves and surfaces are modelled as the orbit of a point under the action of the Lie group $SE(3)$, caused by a set of coupled infinitesimal generators of the group, called twists [35]. Hence, we choose a kinematic definition of shape.

2) These object models are embedded in the conformal geometric algebra (CGA) of the Euclidean space $\mathbb{R}^3$ [30] [21], that is $\mathbb{R}_{4,1}$. Only in conformal geometry the above mentioned modelling of shape unfolds their rich set of useful features which makes it so attractive for applications.

The conception of fusing a local with a global algebraic framework has been proposed already in [44]. But only the pioneering work of Hestenes, Li and Rockwood made it feasible to consider $se(3)$, the space of tangents to an object, embedded in $\mathbb{R}_{4,1}$, as the source of our shape model instead of using $se(3)$ in $\mathbb{R}^3$.

In the following we will present some motivations and aims of the conformal twist representation, organized as the topics of four sections. Doing this, we will use some terms and formulas which will be introduced in Chapter 2. As will become obvious, these topics will lead to remarkable generalizations in comparison to the state of the art.

1.1 Formal Equivalence of Shape and Motion

In behaviour based design of cognitive systems the internal representation of a behaviour is given by the perception-action cycle (PAC) [45]. Because in a PAC both perception and action are inseparable connected as both sides of a medal, the fusion of the geometry of objects with kinematics or the control of actions in the world would result in a more general starting point of system design. Sofar we are using in robot vision only the projections of a PAC if we are organizing action for vision or vision for action. That we can generate patterns by actions is not only known from the use of robot manipulators which are drawing a welding or color track. The tight relations of geometry and kinematics are known to the mathematicians for centuries, see e.g. [12] and [11]. As an example we want to mention the gearwheel mechanisms [23].

But in contrast to most applications in mechanical engineering we are not restricted in our approach by physically feasible motions nor will we get problems in generating spatial curves or surfaces.

By embedding our design method into CGA, both primitive geometric entities as points or objects on the one side and actions on the other side will have algebraic representations in one single framework. Furthermore, objects are defined by actions, and also actions can take on the role of operands.

1.2 Local (Infinitesimal) Generation of Global Patterns

This feature borrows its motivation again from the PAC. It is well-known that both attentive visual perception and motion are sequential processes. In both cases infinitesimal actions generate global patterns of low intrinsic dimension [6]. This phenomen corresponds the interpretation of the special Euclidean group in CGA, $SE(3)$, as a Lie group, where an element $g \in SE(3)$ performs a transformation of an entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$,

$$\underline{\boldsymbol{u}}' = \underline{\boldsymbol{u}}(\theta) = g\{\underline{\boldsymbol{u}}(0)\} \tag{1.1}$$

with respect to the parameter θ of g. Any $g \in SE(3)$ corresponds a Lie group operator $\boldsymbol{M}_s \in \mathbb{R}^+_{4,1}$ which is called a motor and which is applied by the bilinear spinor product,

$$\underline{\boldsymbol{u}}' = \boldsymbol{M}_s \underline{\boldsymbol{u}} \widetilde{\boldsymbol{M}}_s, \tag{1.2}$$

where $\widetilde{\boldsymbol{M}}_s$ is the reverse of $\boldsymbol{M}_s$. This indicates that $\boldsymbol{M}_s$ is an orthogonal operator. It represents the rigid body motion of the entity $\underline{\boldsymbol{u}}$ as a screw motion with six DOFs. But in our approach we are interpreting $\boldsymbol{M}_s$ as a general rotation, that is a rotation around an axis which is not passing the origin. Hence, we are using only five DOFs and we take the sign $\boldsymbol{M}$ for that type of motor. If g is an element of the Lie group $SE(3)$, than its infinitesimal generator, ξ, is defined in the corresponding Lie algebra, that is $\xi \in se(3)$. That Lie algebra element of the rigid body motion is geometrically interpreted as the rotation axis $\underline{\boldsymbol{l}}$ in conformal space. This axis has a

representation as bivector, it is of grade two, $\underline{\boldsymbol{l}} \in \langle \mathbb{R}_{4,1} \rangle_2$.
Then the motor $\boldsymbol{M}$ results from the exponential map of the generator $\underline{\boldsymbol{l}}$ of the group element, which is called a twist:

$$\boldsymbol{M} = \exp\left(-\frac{\theta}{2}\underline{\boldsymbol{l}}\right). \tag{1.3}$$

While θ is the rotation angle as the parameter of the motor, its generator is defined by the five degrees of freedom of a line in space.

In our approach, the motor $\boldsymbol{M}$ is the effective operator which causes arbitrarily complex object shape. This operator may result from the multiplicative coupling of a set of primitive motors $\{\boldsymbol{M}_i | i = n, ..., 1\}$,

$$\boldsymbol{M} = \boldsymbol{M}_n \boldsymbol{M}_{n-1} ... \boldsymbol{M}_2 \boldsymbol{M}_1. \tag{1.4}$$

Each of these motors $\boldsymbol{M}_i$ is representing a circular motion of a point.

1.3 Free-form Objects as Algebraic Entities

Central tasks of computer vision are the computation of geometric distance fits [47] between objects and the recognition of a rigid body motion. The last problem is also central to robotics [33] with respect to performing that motion. In [1] the authors are commenting the typical problems facing in context of pose estimation: "Most solutions are iterative and depend on nonlinear optimization of some geometric constraint, either on the world coordinates or on the projections to the image plane. For real-time applications, we are interested in linear or closed-form solutions free of initialization." This problem is of algebraic nature. It is impossible in Euclidean space to transform free-form objects in a covariant and linear way. The only linear geometric transformation which can be performed in Euclidean space is the rotation of a point. But free-form objects are far from being a point, and rotations are far from being sufficient in the above mentioned context.

Let us cite [4]: "A free-form surface has a well defined surface that is continuous almost everywhere except at vertices, edges and cusps." So what we are doing in practice, is decomposing a given object into a set of points and introducing certain constraints on this set, and finally reconstructing the features of the transformed object from those of the point set. But what we want to do instead, from both a cognitive and a numeric point of view, is to handle linearly complete objects as unique entities in the sense of the above transformations. That is what we mean by the term being algebraic entities.

In reality, the decomposition of objects into points is not sufficient if the rigid body motion is of interest. Instead, this makes necessary to formulate the problem in a homogeneous Euclidean space [32]. But it may be desirable to have at hand besides points also other higher order geometric entities as lines, planes, spheres etc. Just lines are no algebraic entities in Euclidean vector space. Instead, they can be handled only as subspaces. But in screw geometry besides point transformations also line transformations become linear ones, expressed in dual-number techniques, see [40] for an overview. In [3] it has been shown that the motor algebra $\mathbb{R}_{3,0,1}$, which is isomorphic to the dual- quaternion algebra, is a useful framework for representing the rigid body motion of both points and lines by linear operators. Its drawback is, just as in case of the dual-quaternion algebra, to be a degenerate algebra, that is giving no useful metric at hand.

The endeavour to overcome these representation problems by applying special algebraic embeddings can also be found in [48]. These authors used complex polynomial-coefficients for pose estimation of 2D implicit polynomial curves.

The conformal geometric algebra $\mathbb{R}_{4,1}$ overcomes that representation problem. This is caused by two essential facts. First, the representation of the special Euclidean group $SE(3)$ in $\mathbb{R}_{4,1}$ as a subgroup of the conformal group $C(3)$, is isomorphic to the special orthogonal group $SO^+(4,1)$. Hence, rigid body motion can be performed as rotation in CGA, see Chapter 1.2, and therefore has a covariant representation. Second, the basic geometric entity of the conformal geometric algebra is the sphere. All geometric entities derived by incidence operations from the sphere can be transformed in CGA by an element $g \in SE(3)$, that is a motor $\boldsymbol{M} \in \mathbb{R}^+_{4,1}$, in the same linear way, just as a point in the homogeneous Euclidean space $\mathbb{R}^4$. These entities can be points, lines, planes, circles and any higher order algebraic or transcendental curves or surfaces. This is caused by the outermorphism [20], which is not specific to CGA but can be observed in any geometric algebra and which means the preservation of the outer product under linear transformations.

Because there exists a dual representation of a sphere (and all derived entities) in CGA, which considers points as the basic geometric entity of the Euclidean space in the conformal space, all the known conceptions from Euclidean space can be transformed to the conformal one. This concerns also the Lie group interpretation of the rotation and the scheme of Chapter 1.2 to design arbitrary complex free-form curves from a set of coupled twists. Such objects can be linearly transformed in CGA.

1.4 Stratification of Spaces by CGA

Since the seminal paper [13] the purposive use of stratified geometries became an important design principle of vision systems. This means that an observer in dependence of its possibilities and needs can have access to different geometries as projective, affine or metric ones. Sofar this could hardly be realized, because each geometric conception, either metric, affine or projective requires its own algebraic representation, and the switch between these can result in some problems.

In CGA we have a quite other situation. The CGA $\mathbb{R}_{4,1}$ is a linear space of dimension 32. This mighty space represents not only conformal geometry but also affine geometry. Note that the special Euclidean group is a special affine group. Because the CGA $\mathbb{R}_{4,1}$ is representing the conformal geometry of the Euclidean space $\mathbb{R}^3$, it encloses also Euclidean geometry, which is represented by the geometric algebra $\mathbb{R}_{3,0}$. Furthermore, because $\mathbb{R}_{4,1}$ is the geometric algebra of the space $\mathbb{R}^{4,1} = \mathbb{R}^{3,0} \oplus \mathbb{R}^{1,1}$, $\oplus$ being the direct sum, $\mathbb{R}^{4,1}$ can be seen as a homogeneous extension of the Euclidean space $\mathbb{R}^{3,0}$ by a plane with Minkowski signature. It is well known that the projective space can be modelled as $\mathbb{R}^{3,1} = \mathbb{R}^{3,0} \oplus \mathbb{R}^{0,1}$. Hence, a projective geometric algebra $\mathbb{R}_{3,1}$ is also enclosed in $\mathbb{R}_{4,1}$. Thus, we have the stratification of the geometric algebras $\mathbb{R}_{3,0} \subset \mathbb{R}_{3,1} \subset \mathbb{R}_{4,1}$. This enables to consider metric (Euclidean), projective and kinematic (affine) problems in one single algebraic framework. If we think of the task of pose estimation as the estimation of a special Euclidean transformation in a fully projective scenario, this can be done without introducing any restrictions, see [41]. Nevertheless, the switch of representations of geometric entities between these spaces can simply be done by a set of operators. By embedding our twist representation of free-form objects into CGA, we can take advantage of that practically important feature.

In Chapter 2 we will give an introduction to the representation of the rigid body motion in CGA. This chapter will also give a brief sketch of the constructive principles of any geometric algebra and the essential algebraic features of CGA. The reader is advised to the papers [30] and [21] for more details on CGA. In addition, the paper [42] and the report [41] will enable the reader to get more insight into the twist representation and its application in pose estimation. As introduction to geometric algebra in general are recommended [38] and [8], [9].

In Chapter 3 we will present the conception of coupled twists to generate algebraic and transcendental curves and surfaces. The reverse viewpoint is taken on in Chapter 4. Here we assume that we have a given free-form curve or surface, either continuous or discrete, and we want to get the twist parameterization within the model of coupled twists. It will turn out that this problem is to a large extend equivalent to the Fourier series development or to the inverse Fourier transform representation of the object, respectively.

Chapter 2

Rigid Body Motion in Conformal Geometric Algebra

After giving a bird's eye view on the construction of a geometric algebra and on the features of the conformal geometric algebra, we will present the possibilities of representing the rigid body motion in CGA.

2.1 Some Constructive Principles of a Geometric Algebra

A geometric algebra (GA) $\mathbb{R}_{p,q,r}$ is a linear space of dimension 2^n, $n = p + q + r$, which results from a vector space $\mathbb{R}^{p,q,r}$. We call (p, q, r) the signature of the vector space of dimension n. This indicates that there are $p/q/r$ unit vectors $\boldsymbol{e}_i$ which square to $+1/-1/0$, respectively. While $n = p$ in case of the Euclidean space $\mathbb{R}^3$, $\mathbb{R}^{p,q,r}$ indicates a vector space with a different metric than the Euclidean one. In case of $r \neq 0$ there is a degenerate metric. We will omit the signature indexes from right if the interpretation is unique, as in the case of $\mathbb{R}^3$.

The basic product of a GA is the geometric product, indicated by juxtaposition of the operands. This product is associative and anticommutative. There can be used a lot of other product forms in CA too, as the outer product $(\wedge)$, the inner product $(\cdot)$, the commutator $(\underline{\times})$ and the anticommutator $\overline{\times}$. The space $\mathbb{R}_{p,q,r}$ is spanned by a set of 2^n linear subspaces of different grade called blades.

Giving the blades a geometric interpretation makes the difference of a GA to a Clifford algebra. A blade of grade k, a k-blade $\boldsymbol{B}_{\langle k \rangle}$, results from the outer product of k independent vectors $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_k\} \in \mathbb{R}^{p,q,r} \equiv \langle \mathbb{R}_{p,q,r} \rangle_1$,

$$\boldsymbol{B}_{\langle k \rangle} = \boldsymbol{a}_1 \wedge ... \wedge \boldsymbol{a}_k = \langle \boldsymbol{a}_1 ... \boldsymbol{a}_k \rangle_k, \tag{2.1}$$

where $\langle \cdot \rangle$ is the grade operator. There are $l_k = \binom{n}{k}$ different blades of grade k, $\boldsymbol{B}_{\langle k \rangle j} = 1, ..., l_k$. If $\boldsymbol{e}_0 \in \mathbb{R}_{p,q,r}$, $\boldsymbol{e}_0 \equiv 1$, is the unit scalar element and $\boldsymbol{e}_{1...n} \in \mathbb{R}_{p,q,r}$, $\boldsymbol{e}_{1...n} \equiv \boldsymbol{e}_1 ... \boldsymbol{e}_n \equiv \boldsymbol{I}$, is the unit pseudoscalar element of the GA, then $\boldsymbol{B}_{\langle 0 \rangle}$ is the scalar blade and $\boldsymbol{B}_{\langle n \rangle} \equiv \boldsymbol{I}$ is the pseudoscalar blade. Hence, $\sum_{k=0}^{n} l_k = 2^n$ is the dimension of the GA. Blades are directed numbers, thus $\boldsymbol{I}_{\langle k \rangle} = \boldsymbol{e}_{i_1} \wedge ... \wedge \boldsymbol{e}_{i_k}$ gives the direction of a blade.

The geometric algebra $\mathbb{R}_3$ of the Euclidean space $\mathbb{R}^3$ is of dimension 8. Its blade structure is given by 1 scalar, 3 vectors, 3 bivectors and 1 pseudoscalar: $\boldsymbol{B}_{\langle 0 \rangle} \equiv \boldsymbol{e}_0, \boldsymbol{B}_{\langle 1 \rangle 1} \equiv \boldsymbol{e}_1, \boldsymbol{B}_{\langle 1 \rangle 2} \equiv \boldsymbol{e}_2, \boldsymbol{B}_{\langle 1 \rangle 3} \equiv \boldsymbol{e}_3, \boldsymbol{B}_{\langle 2 \rangle 1} \equiv \boldsymbol{e}_{23} \equiv \boldsymbol{e}_2 \boldsymbol{e}_3, \boldsymbol{B}_{\langle 2 \rangle 2} \equiv \boldsymbol{e}_{31} \equiv \boldsymbol{e}_3 \boldsymbol{e}_1, \boldsymbol{B}_{\langle 2 \rangle 3} \equiv \boldsymbol{e}_{12} \equiv \boldsymbol{e}_1 \boldsymbol{e}_2$ and $\boldsymbol{B}_{\langle 3 \rangle} \equiv \boldsymbol{I} \equiv$

$\boldsymbol{e}_{123} \equiv \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ with $\boldsymbol{e}_0^2 = \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = 1$, $\boldsymbol{e}_{23}^2 = \boldsymbol{e}_{31}^2 = \boldsymbol{e}_{12}^2 = -1$ and $\boldsymbol{e}_{123}^2 = -1$.
Its even subalgebra, $\mathbb{R}_3^+$, is isomorphic to the quaternion algebra, $\mathbb{R}_3^+ \simeq \mathbb{R}_{0,2} \simeq \mathbb{H}$. It is composed by only even-grade elements and, thus of dimension 4.

Any linear combination

$$\boldsymbol{A}_k = \sum_{j=1}^{l^*} \alpha_j \boldsymbol{B}_{\langle k \rangle j} \qquad , \quad l^* \leq l_k \ , \ \alpha_j \in \mathbb{R} \tag{2.2}$$

is called a k-vector, $\boldsymbol{A}_k \in \langle \mathbb{R}_{p,q,r} \rangle_k$.
This rich structure of a GA can be further increased by the linear combination of k-vectors,

$$\boldsymbol{A} = \sum_{k=k_*}^{k^*} \beta_k \boldsymbol{A}_k \qquad , \quad 0 \leq k_* < k^* \leq n \ , \beta_k \in \mathbb{R} \tag{2.3}$$

Here $\boldsymbol{A}$ is called a (general) multivector. It is composed of components of different grade. The multivector may result from the geometric product of a r-vector $\boldsymbol{A}_r$ with a s-vector $\boldsymbol{B}_s$,

$$\boldsymbol{A} = \boldsymbol{A}_r\boldsymbol{B}_s = \langle \boldsymbol{A}_r\boldsymbol{B}_r \rangle_{|r-s|} + \langle \boldsymbol{A}_r\boldsymbol{B}_s \rangle_{|r-s|+2} + \ldots + \langle \boldsymbol{A}_r\boldsymbol{B}_s \rangle_{r+s} \tag{2.4}$$

with the pure inner product

$$\boldsymbol{A}_r \cdot \boldsymbol{B}_s = \langle \boldsymbol{A}_r\boldsymbol{B}_s \rangle_{|r-s|} \tag{2.5}$$

and the pure outer product

$$\boldsymbol{A}_r \wedge \boldsymbol{B}_s = \langle \boldsymbol{A}_r\boldsymbol{B}_s \rangle_{r+s}. \tag{2.6}$$

All other components of $\boldsymbol{A}$ result from a mixture of inner and outer products. The product of two multivectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, can always be decomposed in the sum of an even and an odd component, which represent the anticommutator and the commutator products, respectively,

$$\boldsymbol{AB} = \frac{1}{2}(\boldsymbol{AB} + \boldsymbol{BA}) + \frac{1}{2}(\boldsymbol{AB} - \boldsymbol{BA}) =: \boldsymbol{A} \overline{\times} \boldsymbol{B} + \boldsymbol{A} \underline{\times} \boldsymbol{B}. \tag{2.7}$$

In case of the product of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}, \boldsymbol{b} \in \langle \mathbb{R}_{p,q,r} \rangle_1$, we get

$$\boldsymbol{ab} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) + \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{2.8}$$

$$= \langle \boldsymbol{ab} \rangle_0 + \langle \boldsymbol{ab} \rangle_2 = \alpha + \boldsymbol{A}_2 \tag{2.9}$$

with $\alpha \in \langle \mathbb{R}_{p,q,r} \rangle_0$ and $\boldsymbol{A}_2 \in \langle \mathbb{R}_{p,q,r} \rangle_2$.
An important conception of a GA is that of duality. This means that it is possible to change the blade base of a multivector $\boldsymbol{A} \in \mathbb{R}_{p,q,r}$. Its dual is written as $\boldsymbol{A}^*$ and is defined as

$$\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{I}^{-1}, \tag{2.10}$$

where $\boldsymbol{I}$ is the unit pseudoscalar of $\mathbb{R}_{p,q,r}$. In the case that $\boldsymbol{A}_k \in \langle \mathbb{R}_{p,q,r} \rangle_k$ the dual is given by $\boldsymbol{A}^*_{n-k} \in \langle \mathbb{R}_{p,q,r} \rangle_{n-k}$. The duality expresses the relations between the inner product null space, IPNS, and the outer product null space, OPNS, of a multivector, see [38]. The OPNS defines a collinear subspace of dimension k to a k-blade $\boldsymbol{B}_{\langle k \rangle} \subset \mathbb{R}_{p,q,r}$ which is given by all $\boldsymbol{x} \in \mathbb{R}^{p,q,r}$ so that

$$\boldsymbol{x} \wedge \boldsymbol{B}_{\langle k \rangle} = 0. \tag{2.11}$$

The IPNS defines a subspace of $\mathbb{R}_{p,q,r}$ which is orthogonal to a k-blade $\boldsymbol{B}_{\langle k \rangle} \subset \mathbb{R}_{p,q,r}$ and, hence

$$\boldsymbol{x} \cdot \boldsymbol{B}_{\langle k \rangle} = 0. \tag{2.12}$$

2.2 CGA of the Euclidean Space

The conformal geometry of Euclidean and non-Euclidean spaces is known for a long time [53] without giving strong impact on the modelling in engineering with the exception of electrical engineering. There are different representations of the conformal geometry. Most disseminated is a complex formulation [36]. Based on an idea in [22], in [30] and in two other papers of the same authors in [46], the conformal geometries of the Euclidean, spherical and hyperbolic spaces have been worked out in the framework of GA.

The basic approach is that a conformal geometric algebra (CGA) $\mathbb{R}_{p+1,q+1}$ is built from a pseudo-Euclidean space $\mathbb{R}^{p+1,q+1}$. If we start with an Euclidean space $\mathbb{R}^n$, the construction $\mathbb{R}^{n+1,1} = \mathbb{R}^n \oplus \mathbb{R}^{1,1}$, $\oplus$ being the direct sum, uses a plane with Minkowski signature for augmenting the basis of $\mathbb{R}^n$ by the additional basis vectors $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ with $\boldsymbol{e}_+^2 = 1$ and $\boldsymbol{e}_-^2 = -1$. Because that model can be interpreted as a homogeneous stereographic projection of all points $\boldsymbol{x} \in \mathbb{R}^n$ to points $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$, this space is called the homogeneous model of $\mathbb{R}^n$. Furthermore, by replacing the basis $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ with the basis $\{\boldsymbol{e}, \boldsymbol{e}_0\}$, the homogeneous stereographic representation will become a representation of null vectors. This is caused by the properties $\boldsymbol{e}^2 = \boldsymbol{e}_0^2 = 0$ and $\boldsymbol{e} \cdot \boldsymbol{e}_0 = -1$. The relation between the null basis $\{\boldsymbol{e}, \boldsymbol{e}_0\}$ and the basis $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ ist given by

$$\boldsymbol{e} := (\boldsymbol{e}_- + \boldsymbol{e}_+) \text{ and } \boldsymbol{e}_0 := \frac{1}{2}(\boldsymbol{e}_- - \boldsymbol{e}_+). \tag{2.13}$$

Any point $\boldsymbol{x} \in \mathbb{R}^n$ transforms to a point $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ according to

$$\underline{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^2\boldsymbol{e} + \boldsymbol{e}_0 \tag{2.14}$$

with $\underline{\boldsymbol{x}}^2 = 0$.

In fact, any point $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ is lying on a n-dimensional subspace $N_e^n \subset \mathbb{R}^{n+1,1}$, called horosphere [30]. The horosphere is a non-Euclidean model of the Euclidean space $\mathbb{R}^n$ with the following metrical relations between both [10]. Let be $d(\boldsymbol{x}, \boldsymbol{y})$ the (Euclidean) distance between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and let be $d(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})$, the distance of their mappings to the horosphere. Then,

$$d(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = -\frac{1}{2}d^2(\boldsymbol{x}, \boldsymbol{y}), \tag{2.15}$$

respectively

$$\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} = -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{y})^2. \tag{2.16}$$

It must be mentioned that the basis vectors $\boldsymbol{e}$ and $\boldsymbol{e}_0$ have a geometric interpretation. In fact, $\boldsymbol{e}$ corresponds the north pole and $\boldsymbol{e}_0$ corresponds the south pole of the hypersphere of the stereographic projection, embedded in $\mathbb{R}^{n+1,1}$. Thus, $\boldsymbol{e}$ is representing the points at infinity and $\boldsymbol{e}_0$ is representing the origin of $\mathbb{R}^n$ in the space $\mathbb{R}^{n+1,1}$.

By setting apart these two points from all others of the $\mathbb{R}^n$ makes $\mathbb{R}^{n+1,1}$ a homogeneous space in the sense that each $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ is a homogeneous null vector without having reference to the origin. This enables coordinate-free computing to a large extend. Hence, $\underline{\boldsymbol{x}} \in N_e^n$ constitutes an equivalence class $\{\lambda\underline{\boldsymbol{x}}, \lambda \in \mathbb{R}\}$ on the horosphere. The reduction of that equivalence class to a unique entity with metrical equivalence to the point $\boldsymbol{x} \in \mathbb{R}^n$ needs a normalization

$$\underline{\boldsymbol{x}} \longrightarrow -\frac{\underline{\boldsymbol{x}}}{\underline{\boldsymbol{x}} \cdot \boldsymbol{e}}. \tag{2.17}$$

The CGA $\mathbb{R}_{4,1}$, derived from the Euclidean space $\mathbb{R}^3$, offers 32 blades as basis of that linear space. This rich structure enables to represent low order geometric entities in a hierarchy of grades. These entities can be derived as solutions of either the IPNS or the OPNS in dependence of what we assume as basis geometric entity of the conformal space, see [38]. Sofar we only considered the mapping of an Euclidean point $\boldsymbol{x} \in \mathbb{R}^3$ to a point $\underline{\boldsymbol{x}} \in N_e^3 \subset \mathbb{R}^{4,1}$. But the null vectors on the horosphere are only a special subset of all the vectors of $\mathbb{R}^{4,1}$. These vectors are representing spheres as the basic entities of the conformal space. A sphere $\underline{\boldsymbol{s}} \in \mathbb{R}^{4,1}$ is defined by its center position, $\boldsymbol{c} \in \mathbb{R}^3$, and its radius $\rho \in \mathbb{R}$ according to

$$\underline{\boldsymbol{s}} = \boldsymbol{c} + \frac{1}{2}(\boldsymbol{c} - \rho)^2 \boldsymbol{e} + \boldsymbol{e}_0. \tag{2.18}$$

And because $\underline{\boldsymbol{s}}^2 = \rho^2 > 0$, it must be a non-null vector. A point $\underline{\boldsymbol{x}} \in N_e^3$ can be considered as a degenerate sphere of radius zero and a plane $\underline{\boldsymbol{p}} \in \mathbb{R}^{4,1}$ can be interpreted as a sphere of infinite radius. Hence, spheres $\underline{\boldsymbol{s}}$, points $\underline{\boldsymbol{x}}$ and planes $\underline{\boldsymbol{p}}$ are entities of grade 1. By taking the outer product of spheres $\underline{\boldsymbol{s}}_i$, other entities can be constructed.
So we get a circle $\underline{\boldsymbol{z}}$, a point pair $\underline{\boldsymbol{q}}$ and a point $\underline{\boldsymbol{y}}$ as entities of grade 2, 3 and 4, respectively, which exist outside the null cone in $\mathbb{R}^{4,1}$,

$$\underline{\boldsymbol{z}} = \underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2 \tag{2.19}$$

$$\underline{\boldsymbol{q}} = \underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2 \wedge \underline{\boldsymbol{s}}_3 \tag{2.20}$$

$$\underline{\boldsymbol{y}} = \underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2 \wedge \underline{\boldsymbol{s}}_3 \wedge \underline{\boldsymbol{s}}_4 \tag{2.21}$$

as solutions of the IPNS. If we consider the OPNS on the other hand, we are starting with points $\underline{\boldsymbol{x}}_i \in N_e^3$ and can proceed similarly to define a point pair $\underline{\boldsymbol{Q}}$, a circle $\underline{\boldsymbol{Z}}$ and a sphere $\underline{\boldsymbol{S}}$ as entities of grade 2, 3 and 4 derived from points $\underline{\boldsymbol{x}}_i$ on the null cone of $\mathbb{R}_{4,1}$ according to

$$\underline{\boldsymbol{Q}} = \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \tag{2.22}$$

$$\underline{\boldsymbol{Z}} = \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \wedge \underline{\boldsymbol{x}}_3 \tag{2.23}$$

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \wedge \underline{\boldsymbol{x}}_3 \wedge \underline{\boldsymbol{x}}_4. \tag{2.24}$$

These sets of entities are obviously related by the duality $\underline{\boldsymbol{u}}^* = \underline{\boldsymbol{U}}$.
In OPNS, for lines $\underline{\boldsymbol{L}}$ and planes $\underline{\boldsymbol{P}}$, we have the definitions

$$\underline{\boldsymbol{L}} = \boldsymbol{e} \wedge \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \tag{2.25}$$

$$\underline{\boldsymbol{P}} = \boldsymbol{e} \wedge \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \wedge \underline{\boldsymbol{x}}_3 \tag{2.26}$$

and in IPNS we get the lines $\underline{\boldsymbol{l}}$ and the planes $\underline{\boldsymbol{p}}$ as entities of grade 2 and 1 as the dual of $\underline{\boldsymbol{L}}$ and $\underline{\boldsymbol{P}}$, respectively. Finally,

$$\underline{\boldsymbol{X}} = \boldsymbol{e} \wedge \underline{\boldsymbol{x}}$$

is called the affine representation of a point [30]. This representation of a point is used if the interplay of the projective with the conformal representation is of interest in applications as in [41]. The same is with the line $\underline{\boldsymbol{L}}$ and the plane $\underline{\boldsymbol{P}}$.

Let us come back to the stratification of spaces mentioned in Chapter 1.4. Let be $\boldsymbol{x} \in \mathbb{R}^n$ a point of the Euclidean space, $\boldsymbol{X} \in \mathbb{R}^{n,1}$ a point of the projective space and $\underline{\boldsymbol{X}} \in \mathbb{R}^{n+1,1}$ a point of the conformal space. Then the operations which transform the representation between the spaces are for $\mathbb{R}_3 \longrightarrow \mathbb{R}_{3,1} \longrightarrow \mathbb{R}_{4,1}$

$$\underline{\boldsymbol{X}} = \boldsymbol{e} \wedge \boldsymbol{X} = \boldsymbol{e} \wedge (\boldsymbol{x} + \boldsymbol{e}_-), \tag{2.27}$$

respectively for $\mathbb{R}_{4,1} \longrightarrow \mathbb{R}_{3,1} \longrightarrow \mathbb{R}_3$

$$\boldsymbol{x} = -\frac{\boldsymbol{X}}{\boldsymbol{X} \cdot \boldsymbol{e}_-} = \frac{((\boldsymbol{e}_+ \cdot \underline{\boldsymbol{X}}) \wedge \boldsymbol{e}_-) \cdot \boldsymbol{e}_-}{(\boldsymbol{e}_+ \cdot \underline{\boldsymbol{X}}) \cdot \boldsymbol{e}_-}. \tag{2.28}$$

2.3 The Special Euclidean Group in CGA

A geometry is defined by its basic entities, the geometric transformation group which is acting in a linear and covariant manner on all the entities which are constructed from the basic entity by incidence operations, and the resulting invariances with respect to that group. The search for such a geometry was motivated in Chapter 1.3. Next we want to specify the required features of the group.

The invariants of the conformal group $C(3)$ in $\mathbb{R}^3$ are angles. But to make a geometry a proper one, we have to require that any action $\mathcal{A}$ of that group on an entity, say $\boldsymbol{u}$, is grade preserving, respectively structure preserving. This makes necessary that the operator $\boldsymbol{A}$ applies as versor product [39]

$$\mathcal{A}\{\boldsymbol{u}\} = \boldsymbol{A}\boldsymbol{u}\boldsymbol{A}^{-1}. \tag{2.29}$$

This means that the entity $\boldsymbol{u}$ should transform covariantly [28], [7]. If $\boldsymbol{u}$ is composed by e.g. two representants $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of the basis entities of the geometry, then $\boldsymbol{u}$ should transform according to

$$\mathcal{A}\{\boldsymbol{u}\} = \mathcal{A}\{\boldsymbol{u}_1 \circ \boldsymbol{u}_2\} = (\boldsymbol{A}\boldsymbol{u}_1\boldsymbol{A}^{-1}) \circ (\boldsymbol{A}\boldsymbol{u}_2\boldsymbol{A}^{-1}) = \boldsymbol{A}\boldsymbol{u}\boldsymbol{A}^{-1}. \tag{2.30}$$

The conformal group $C(3)$ is mighty [36], but other than (2.29) and (2.30) it is nonlinear and transforms not covariantly in $\mathbb{R}^3$. Besides, in $\mathbb{R}^3$ there exist no entities other than points which could be transformed.

As we have shown in Chapter 2.2, in $\mathbb{R}_{4,1}$ the situation is quite different because all the geometric entities derived there can be seen also as algebraic entities in the sense of Chapter 1.3. Not only the elements of the null cone transform covariantly but also those of the dual space of $\mathbb{R}_{4,1}$. Furthermore, the representation of the conformal group $C(3)$ in $\mathbb{R}_{4,1}$ has the required properties of (2.29) and (2.30), see [30] and [28]. All vectors with positive signature in $\mathbb{R}_{4,1}$, that is a sphere, a plane as well as the components inversion and reflection of $C(3)$ compose a multiplicative group. That is called the versor representation of $C(3)$. This group is isomorphic to the Lorentz group of $\mathbb{R}_{4,1}$. The subgroup, which is composed by products of an even number of these vectors, is the spin group $Spin^+(4,1)$, the spin representation of $O^+(4,1)$. To that group belong the subgroups rotation, translation, dilatation, and transversion of $C(3)$. They are applied as a spinor $\boldsymbol{S}$, $\boldsymbol{S} \in \mathbb{R}^+_{4,1}$ and $\boldsymbol{S}\widetilde{\boldsymbol{S}} = |\boldsymbol{S}|^2$. A rotor $\boldsymbol{R}$, $\boldsymbol{R} \in \langle\mathbb{R}_{4,1}\rangle_2$ and $\boldsymbol{R}\boldsymbol{R}^2 = 1$ is a special spinor. Rotation and translation are represented in $\mathbb{R}_{4,1}$ as rotors.

The special Euclidean group $SE(3)$ is defined by $SE(3) = SO(3) \oplus \mathbb{R}^3$. Therefore, the rigid body motion $g = (R, \boldsymbol{t})$, $g \in SE(3)$ of a point $\boldsymbol{x} \in \mathbb{R}^3$ writes in Euclidean space

$$\boldsymbol{x}' = g\{\boldsymbol{x}\} = R\boldsymbol{x} + \boldsymbol{t}. \tag{2.31}$$

Here R is a rotation matrix and $\boldsymbol{t}$ is a translation vector. Because $SE(3) \subset C(3)$, in our choice of a special rigid body motion the representation of $SE(3)$ in CGA is isomorphic to the special orthogonal group, $SO^+(4,1)$. Hence, such $g \in SE(3)$, which does not represent the full screw, is represented as rotation in $\mathbb{R}_{4,1}$. This rotation is a general one, that is the rotation axis in $\mathbb{R}^3$ is

shifted out of the origin by the translation vector $\boldsymbol{t}$.
That $g \in SE(3)$ is represented by a special rotor $\boldsymbol{M}$ [28], called a motor, $\boldsymbol{M} \in \langle \mathbb{R}_{4,1} \rangle_2$. The motor may be written

$$\boldsymbol{M} = \exp\left(-\frac{\theta}{2}\underline{\boldsymbol{l}}\right), \tag{2.32}$$

where $\theta \in \mathbb{R}$ is the rotation angle and $\underline{\boldsymbol{l}} \in \langle \mathbb{R}_{4,1} \rangle_2$ is indicating the line of the general rotation. To specify $\underline{\boldsymbol{l}}$ by the rotation and translation in $\mathbb{R}^3$, the motor has to be decomposed into its rotation and translation components. The normal rotation in CGA is given by the rotor

$$\boldsymbol{R} = \exp\left(-\frac{\theta}{2}\boldsymbol{l}\right) \tag{2.33}$$

with $\boldsymbol{l} \in \langle \mathbb{R}_3 \rangle_2$ indicating the rotation plane which passes the origin. The translation in CGA is given by a special rotor, called a translator,

$$\boldsymbol{T} = \exp\left(\frac{\boldsymbol{e}\boldsymbol{t}}{2}\right) \tag{2.34}$$

with $\boldsymbol{t} \in \langle \mathbb{R}_3 \rangle_1$ as the translation vector. Because rotors constitute a multiplicative group, a naive formulation of the coupling of $\boldsymbol{R}$ and $\boldsymbol{T}$ would be

$$\boldsymbol{M} = \boldsymbol{T}\boldsymbol{R}. \tag{2.35}$$

But if we interprete the rotor $\boldsymbol{R}$ as that entity of $\mathbb{R}_{4,1}$ which should be transformed by translation in a covariant manner, a better choice is

$$\boldsymbol{M} = \boldsymbol{T}\boldsymbol{R}\widetilde{\boldsymbol{T}}. \tag{2.36}$$

We call this special motor representation the twist representation. Its exponential form is given by

$$\boldsymbol{M} = \exp\left(\frac{1}{2}\boldsymbol{e}\boldsymbol{t}\right) \exp\left(-\frac{\theta}{2}\boldsymbol{l}\right) \exp\left(-\frac{1}{2}\boldsymbol{e}\boldsymbol{t}\right). \tag{2.37}$$

This equation expresses the shift of the rotation axis $\boldsymbol{l}^*$ in the plane $\boldsymbol{l}$ by the vector $\boldsymbol{t}$ to perform the normal rotation and finally shifting back the axis.
Because $SE(3)$ is a Lie group, the line $\underline{\boldsymbol{l}} \in \langle \mathbb{R}_{4,1} \rangle_2$ is the representation of the infinitesimal generator of $\boldsymbol{M}$, $\xi \in se(3)$. We call the generator representation a twist because it represents rigid body motion as general rotation. It is parameterized by the position and orientation of $\underline{\boldsymbol{l}}$ which are the six Plücker coordinates, represented by the rotation plane $\boldsymbol{l}$ and the inner product $(\boldsymbol{t} \cdot \boldsymbol{l})$, [41],

$$\underline{\boldsymbol{l}} = \boldsymbol{l} + \boldsymbol{e}(\boldsymbol{t} \cdot \boldsymbol{l}). \tag{2.38}$$

The most general formulation of the rigid body motion is the screw motion [40]. It is formulated in CGA as

$$\boldsymbol{M}_s = \boldsymbol{T}_s\boldsymbol{T}\boldsymbol{R}\widetilde{\boldsymbol{T}} \tag{2.39}$$

with the pitch translator

$$\boldsymbol{T}_s = \exp\left(\frac{d}{2}\boldsymbol{e}\boldsymbol{l}^*\right), \tag{2.40}$$

where $\boldsymbol{l}^* \in \langle\mathbb{R}_3\rangle_1$ is the screw axis as the dual of $\boldsymbol{l}$ and $\boldsymbol{t}_s = d\boldsymbol{l}^*$ is a translation vector parallel to that axis. If we formulate $\boldsymbol{M}_s$ as

$$\boldsymbol{M}_s = \exp\left(-\frac{\theta}{2}(\boldsymbol{l} + \boldsymbol{e}\boldsymbol{m})\right) \tag{2.41}$$

with the vector $\boldsymbol{m} \in \mathbb{R}^3$

$$\boldsymbol{m} = \boldsymbol{t} \cdot \boldsymbol{l} - \frac{d}{\theta}\boldsymbol{l}^*, \tag{2.42}$$

then all special cases of the rigid body motion, represented in the CGA $\mathbb{R}_{4,1}$ can be derived from (2.42):

$\boldsymbol{m} = 0$	:	pure rotation ($\boldsymbol{M}_s = \boldsymbol{R}$)
$\boldsymbol{m} = \boldsymbol{t}, \theta \longrightarrow 0$	:	pure translation ($\boldsymbol{M}_s = \boldsymbol{T}$)
$\boldsymbol{m} \perp \boldsymbol{l}^*$	:	general rotation ($\boldsymbol{M}_s = \boldsymbol{M}$)
$\boldsymbol{m} \not\perp \boldsymbol{l}^*$	:	general screw motion

A motor $\boldsymbol{M}$ transforms covariantly any entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$ according to

$$\underline{\boldsymbol{u}}' = \boldsymbol{M}\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}} \tag{2.43}$$

with $\underline{\boldsymbol{u}}' \in \mathbb{R}_{4,1}$. An equivalent equation is valid for the dual entity $\underline{\boldsymbol{U}} \in \mathbb{R}_{4,1}$. Because motors concatenate multiplicatively, a multiple-motor transformation of $\underline{\boldsymbol{u}}$ resolves recursively. Let be $\boldsymbol{M} = \boldsymbol{M}_2\boldsymbol{M}_1$, then

$$\underline{\boldsymbol{u}}'' = \boldsymbol{M}\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}} = \boldsymbol{M}_2\boldsymbol{M}_1\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}}_1\widetilde{\boldsymbol{M}}_2 = \boldsymbol{M}_2\underline{\boldsymbol{u}}'\widetilde{\boldsymbol{M}}_2. \tag{2.44}$$

It is a feature of any GA that also composed entities, which are built by the outer product of other ones, transform covariantly by a linear transformation. This is called outermorphism [20]. Following Chapter 1.3, this is an important feature of the chosen algebraic embedding that will be demonstrated in Chapter 3. Let be $\underline{\boldsymbol{z}} \in \langle\mathbb{R}_{4,1}\rangle_2$ a circle, which is composed by two spheres $\underline{\boldsymbol{s}}_1, \underline{\boldsymbol{s}}_2 \in \langle\mathbb{R}_{4,1}\rangle_1$ according to $\underline{\boldsymbol{z}} = \underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2$. Then the transformed circle computes as

$$\underline{\boldsymbol{z}}' = \boldsymbol{M}\underline{\boldsymbol{z}}\widetilde{\boldsymbol{M}} = \boldsymbol{M}(\underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2)\widetilde{\boldsymbol{M}} = \langle\boldsymbol{M}(\underline{\boldsymbol{s}}_1\underline{\boldsymbol{s}}_2)\widetilde{\boldsymbol{M}}\rangle_2 \tag{2.45}$$

$$= \langle\boldsymbol{M}\underline{\boldsymbol{s}}_1\widetilde{\boldsymbol{M}}\boldsymbol{M}\underline{\boldsymbol{s}}_2\widetilde{\boldsymbol{M}}\rangle_2 = \boldsymbol{M}\underline{\boldsymbol{s}}_1\widetilde{\boldsymbol{M}} \wedge \boldsymbol{M}\underline{\boldsymbol{s}}_2\widetilde{\boldsymbol{M}} \tag{2.46}$$

$$= \underline{\boldsymbol{s}}_1' \wedge \underline{\boldsymbol{s}}_2'. \tag{2.47}$$

Chapter 3

Shape Models from Coupled Twists

In this chapter we will approach step by step the kinematic design of algebraic and transcendental curves and surfaces by coupling a certain set of twists as generators of a multiple-parameter Lie group action.

3.1 Kinematic Chain as Model of Constrained Motion

In the preceding chapter we argued that each entity $\underline{\boldsymbol{u}}_i$ contributing to the rigid model of another entity $\underline{\boldsymbol{u}}$ is performing the same transformation, represented by the motor $\boldsymbol{M}$. Now we assume an ordered set of non-rigidly coupled rigid components of an object. This is for example a model of bar-shaped mechanisms [35] if the components are coupled by either revolute or prismatic joints. Such model is called a kinematic chain [33]. In a kinematic chain the task is to formulate the net movement of the end-effector at the n-th joint by movements of the j-th joints, $j = 1, ..., n-1$, if the 0-th joint is fixed coupled with a world coordinate system. These movements are discribed by the motors $\boldsymbol{M}_j$. Let be $\mathcal{T}_j$ the transformation of an attached joint j with respect to the base coordinate system, then for $j = 1, ..., n$ the point $\underline{\boldsymbol{x}}_{j,i_j}, i_j = 1, ..., m_j$, transforms according to

$$\mathcal{T}_j(\underline{\boldsymbol{x}}_{j,i_j}, \boldsymbol{M}_j) = \boldsymbol{M}_1 ... \boldsymbol{M}_j \underline{\boldsymbol{x}}_{j,i_j} \widetilde{\boldsymbol{M}}_j ... \widetilde{\boldsymbol{M}}_1 \tag{3.1}$$

and

$$\mathcal{T}_0(\underline{\boldsymbol{x}}_{0,i_0}) = \underline{\boldsymbol{x}}_{0,i_0}. \tag{3.2}$$

These motors $\boldsymbol{M}_j$ are describing the flexible geometry of the kinematic chain very efficiently. This results in an object model $\mathcal{O}$ defined by a kinematic chain with n segments and described by any geometric entity $\underline{\boldsymbol{u}}_{j,i_j} \in \mathbb{R}_{4,1}$ attached to the j-th segment,

$$\mathcal{O} = \left\{ \mathcal{T}_0(\underline{\boldsymbol{u}}_{0,i_0}), \mathcal{T}_1(\underline{\boldsymbol{u}}_{1,i_1}, \boldsymbol{M}_1), ..., \mathcal{T}_n(\underline{\boldsymbol{u}}_{n,i_n}, \boldsymbol{M}_n) | n, i_0, ..., i_n \in \mathbb{N} \right\}. \tag{3.3}$$

If $\underline{\boldsymbol{u}}_{j,i_j}$ is performing a motion caused by the motor $\boldsymbol{M}$, then

$$\underline{\boldsymbol{u}}'_{j,i_j} = \boldsymbol{M} \left(\mathcal{T}_j(\underline{\boldsymbol{u}}_{j,i_j}, \boldsymbol{M}_j) \right) \widetilde{\boldsymbol{M}} \tag{3.4}$$

$$= \boldsymbol{M}(\boldsymbol{M}_1 ... \boldsymbol{M}_j \underline{\boldsymbol{u}}_{j,i_j} \widetilde{\boldsymbol{M}}_j ... \widetilde{\boldsymbol{M}}_1) \widetilde{\boldsymbol{M}} \tag{3.5}$$

is the net movement of $\underline{\boldsymbol{u}}_{j,i_j}$.
Obviously, this equation describes a constrained motion of the considered entities.

3.2 The Operational Model of Shape

We will now introduce another type of constrained motion, which cannot be realized with physical systems but should be understood as a generalization of a kinematic chain. This is our proposed model of operational or kinematic shape [41]. Operational shape means that a shape results as the net effect, that is the orbit, of a point under the action of a set of coupled operators. So the operators at the end are the representations of the shape. Kinematic shape means that these operators are the motors as representations of $SE(3)$ in $\mathbb{R}_{4,1}$. The principle is simple. It goes back to the interpretation of any $g \in SE(3)$ as a Lie group action [35], see equation (1.1). But only in $\mathbb{R}_{4,1}$ we can take advantage from its representation as rotation around the axis $\underline{\boldsymbol{l}}$, see equations (2.32), (2.36) and (2.37).

In Chapter 2.2 we introduced the sphere and the circle from IPNS and OPNS, respectively. We call these definitions the canonical ones. On the other hand, a circle has an operational definition which is given by the following.
Let be $\underline{\boldsymbol{x}}_\phi$ a point which is a mapping of another point $\underline{\boldsymbol{x}}_0$ by $g \in SE(3)$ in $\mathbb{R}_{4,1}$. This may be written as

$$\underline{\boldsymbol{x}}_\phi = \boldsymbol{M}_\phi \underline{\boldsymbol{x}}_0 \widetilde{\boldsymbol{M}}_\phi \tag{3.6}$$

with $\boldsymbol{M}_\phi$ being the motor which rotates $\underline{\boldsymbol{x}}$ by an angle ϕ,

$$\boldsymbol{M}_\phi = \exp\left(-\frac{\phi}{2}\Psi\right). \tag{3.7}$$

Here again is Ψ the twist as generator of the rotation around the axis $\underline{\boldsymbol{l}}$, see equation (2.32). Note that $\Psi = \alpha \boldsymbol{l}, \alpha \in \mathbb{R}$. If ϕ covers densely the whole span $[0, ..., 2\pi]$, then the generated set of points $\{\underline{\boldsymbol{x}}_\phi\}$ is also dense. The infinite set $\{\underline{\boldsymbol{x}}_\phi\}$ is the orbit of a rotation caused by the infinite set $\{\boldsymbol{M}_\phi\}$, which has the shape of a circle in $\mathbb{R}^3$. The set $\{\boldsymbol{x}_\phi\}$ represents the well-known subset conception in a vector space of geometric objects in analytic geometry. In fact, that circle is on the horosphere N_e^3 because it is composed only by points. We will write $\underline{\boldsymbol{z}}_{\{1\}}$ instead of $\{\underline{\boldsymbol{x}}_\phi\}$ to indicate the different nature of that circle in comparison to either $\underline{\boldsymbol{z}}$ or $\underline{\boldsymbol{Z}}$ of Chapter 2.2. The index $\{1\}$ means that the circle is generated by one twist from a continuous argument ϕ. So the circle, embedded in $\mathbb{R}_{4,1}$, is defined by

$$\underline{\boldsymbol{z}}_{\{1\}} = \left\{\underline{\boldsymbol{x}}_\phi | \text{ for all } \phi \in [0, ..., 2\pi]\right\}. \tag{3.8}$$

Its radius is given by the distance of the chosen point $\underline{\boldsymbol{x}}_0$ to the axis $\underline{\boldsymbol{l}}$ and its orientation in space depends on the parameterization of $\underline{\boldsymbol{l}}$. That $\underline{\boldsymbol{z}}_{\{1\}}$ is defined by an infinite set of arguments is no real problem in case of computational geometry or applications where only discretized shape is of interest. Besides, the approximation can be done locally as good as computing time allows. More interesting is the fact that in the canonical definitions of Chapter 2.2 the geometric entities are all derived from either spheres or points. In case of the operational definition of shape, the circle is the basic geometric entity, respectively rotation is the basic operation.

A sphere results from the coupling of two motors, $\boldsymbol{M}_{\phi_1}$ and $\boldsymbol{M}_{\phi_2}$, whose twist axes meet at the center of the sphere and which are perpendicularly arranged. The following twists are possible generators, but any other orientation is even good,

$$\Psi_1 = \boldsymbol{e}_{12} + \boldsymbol{e}(\boldsymbol{c} \cdot \boldsymbol{e}_{12}) \tag{3.9}$$

$$\Psi_2 = \boldsymbol{e}_{31} + \boldsymbol{e}(\boldsymbol{c} \cdot \boldsymbol{e}_{31}) \tag{3.10}$$

with the sphere center $\boldsymbol{c}$, and $\boldsymbol{e}_{12}$, $\boldsymbol{e}_{31}$ are two orthogonal planes.

The resulting constrained motion of a point $\underline{\boldsymbol{x}}_{0,0}$ performs a rotation on a sphere given by $\phi_1 \in [0, ..., 2\pi]$ and $\phi_2 \in [0, ..., \pi]$,

$$\underline{\boldsymbol{x}}_{\phi_1,\phi_2} = \boldsymbol{M}_{\phi_2}\boldsymbol{M}_{\phi_1}\underline{\boldsymbol{x}}_{0,0}\boldsymbol{M}_{\phi_1}\boldsymbol{M}_{\phi_2}. \tag{3.11}$$

The complete orbit of a sphere is given by

$$\underline{\boldsymbol{s}}_{\{2\}} = \left\{\underline{\boldsymbol{x}}_{\phi_1,\phi_2} | \text{ for all } \phi_1 \in [0, ..., 2\pi], \phi_2 \in [0, ..., \pi]\right\}. \tag{3.12}$$

Let us come back to the point of generalization of the well-known kinematic chains. These models of linked bar mechanisms have to be physically feasible. Instead, our model of coupled twists is not limited by that constraint. Therefore, the sphere expresses a virtual coupling of twists. This includes both location and orientation in space, and the possibility of fixating several twists at the same location, for any dimension of the space $\mathbb{R}^n$. There are several extensions of the introduced kinematic model which are only possible in CGA.

First, while the group $SE(3)$ can only act on points, its representation in $\mathbb{R}_{4,1}$ may act in the same way on any entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$ derived from either points or spheres. This results in high complex free-form shapes caused from the motion of relative simple generating entities and low order sets of coupled twists.

Second, only by coupling a certain set of twists, high complex free-form shapes may be generated from a complex enough constrained motion of a point.

Let be $\underline{\boldsymbol{u}}_{\{n\}}$ the shape generated by n motors $\boldsymbol{M}_{\phi_1}, ..., \boldsymbol{M}_{\phi_n}$. We call it the n-twist model,

$$\underline{\boldsymbol{u}}_{\{n\}} = \left\{\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} | \text{ for all } \phi_1, ..., \phi_n \in [0, ..., 2\pi]\right\} \tag{3.13}$$

with

$$\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} = \boldsymbol{M}_{\phi_n}...\boldsymbol{M}_{\phi_1}\underline{\boldsymbol{x}}_{0,...,0}\widetilde{\boldsymbol{M}}_{\phi_1}...\widetilde{\boldsymbol{M}}_{\phi_n}. \tag{3.14}$$

Then the last equation may also be written

$$\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} = \boldsymbol{M}_{\phi_n}...\boldsymbol{M}_{\phi_2}\underline{\boldsymbol{x}}_{\{\phi_1,0,...,0\}}\widetilde{\boldsymbol{M}}_{\phi_2}...\widetilde{\boldsymbol{M}}_{\phi_n}. \tag{3.15}$$

By continuing that reformulation, we get

$$\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} = \boldsymbol{M}_{\phi_n}\underline{\boldsymbol{x}}_{\{\phi_1,...,\phi_{n-1},0\}}\widetilde{\boldsymbol{M}}_{\phi_n}. \tag{3.16}$$

This corresponds also to

$$\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} = \boldsymbol{M}\underline{\boldsymbol{x}}_{0,...,0}\widetilde{\boldsymbol{M}} \tag{3.17}$$

with $\boldsymbol{M} = \boldsymbol{M}_{\phi_n}...\boldsymbol{M}_{\phi_1}$. The set $\left\{\underline{\boldsymbol{x}}_{\phi_1,0,...0}\right\}$ represents a circle and the set $\left\{\underline{\boldsymbol{x}}_{\phi_1,...,\phi_{n-1},0}\right\}$ represents an entity which, coupled with a circle, will result in $\left\{\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n}\right\}$. While equation (3.14) is representing a multiple-parameter Lie group form of $SE(3)$, where the nested motors are carrying the complexity of $\underline{\boldsymbol{u}}_{\{n\}}$, the complexity is stepwise shifted to $\underline{\boldsymbol{u}}_{\{n-1\}}$ in equation (3.16). Furthermore, while both equations (3.16) and (3.17) are linear in the motors, equation (3.17) looks so simple because the parameters of the resulting motor $\boldsymbol{M}$ are now a function in the space spanned by the parameters of the generating twists $\Psi_1, ..., \Psi_n$ and the arguments of the motors $\phi_1, ..., \phi_n$. Instead of using the single-parameter form (3.17), we prefer equation (3.14).

Table 3.1: Simple geometric entities generated from up to three twists

Entity	Generation	Class
point	twist axis intersected with point	0twist curve
circle	twist axis not collinear with point	1twist curve
line	twist axis is at infinity	1twist curve
conic	2 parallel not collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = -2$
line segment	2 twists, building a degenerate conic	2twist curve $\lambda_1 = 1, \lambda_2 = -2$
cardioid	2 parallel not collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = 1$
nephroid	2 parallel not collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = 2$
rose	2 parallel not collinear twists, j loops	2twist curve $\lambda_1 = 1, \lambda_2 = -j$
spiral	1 finite and 1 infinite twist	2twist curve $\lambda_1 = 1, \lambda_2 = 1$
sphere	2 perpendicular twists	2twist surface $\lambda_1 = 1, \lambda_2 = 1$
plane	2 parallel twists at infinity	2twist surface
cylinder	2 twists, one at infinity	2twist surface
cone	2 twists, one at infinity	2twist surface
quadric	a conic rotated with third twist	3twist surface

3.3 Free-form Objects

There is a lot of more degrees of freedom to design free-form objects embedded in $\mathbb{R}_{4,1}$ by the motion of a point caused by coupled twists.
While a single rotation-like motor generates a circle, a single translation-like motor generates a line as root of non-curved objects. Of course, several of both variants can be mixed.
Other degrees of freedom of the design result from following extensions:

- Introducing an individual angular frequency λ_i to the motor $\boldsymbol{M}_{\phi_i}$ influences also the synchronization of the rotation angles ϕ_i.
- Rotation within limited angular segments: $\phi_i \in [\alpha_{i_1}, ..., \alpha_{i_2}]$ with $0 \leq \alpha_{i_1} < \alpha_{i_2} \leq 2\pi$.

Let us consider the simple example of a 2-twist model of shape,

$$\underline{\boldsymbol{u}}_{\{2\}} = \left\{ \underline{\boldsymbol{x}}_{\phi_1,\phi_2} | \text{ for all } \phi_1, \phi_2 \in [0, ..., 2\pi] \right\} \tag{3.18}$$

with

$$\underline{\boldsymbol{x}}_{\phi_1,\phi_2} = \boldsymbol{M}_{\lambda_2\phi_2}\boldsymbol{M}_{\lambda_1\phi_1}\underline{\boldsymbol{x}}_0\widetilde{\boldsymbol{M}}_{\lambda_1\phi_1}\widetilde{\boldsymbol{M}}_{\lambda_2\phi_2}, \tag{3.19}$$

$\lambda_1, \lambda_2 \in \mathbb{R}$ and $\phi_1 = \phi_2 = \phi \in [0, ..., 2\pi]$.

That model can generate not only a sphere, but an ellipse ($\lambda_1 = -2, \lambda_2 = 1$), different well-known algebraic curves (in space), see [41], as cardioid, nephroid or deltoid, transcendental curves like a spiral, or surfaces. For a list of examples see Table 3.1.

Interestingly, the order of nonlinearity of algebraic curves grows faster than the number of the generating motors.

3.4 Extensions of the Conceptions

By replacing the initial point $\underline{\boldsymbol{x}}_0$ by any other geometric entity, $\underline{\boldsymbol{u}}_0$, built from either points or spheres by applying the outer product, the conceptions remain the same. This makes the kinematic object model in conformal space a recursive one.

The infinite set of arguments ϕ_i of the motor $\boldsymbol{M}_{\phi_i}$ to generate the entity $\underline{\boldsymbol{u}}_{\{n\}}$ will in practice be reduced to a finite one, which results in a discrete entity $\underline{\boldsymbol{u}}_{[n]}$. The index $[n]$ indicates that n twists are used with a finite set of arguments $\{\phi_{i,j_i}|j_i \in \{0, ..., m_i\}\}$.

The previous formulations of free-form shape did assume a rigid model. As in the case of the kinematic chain, the model can be made flexible. This happens by encapsulating the entity $\underline{\boldsymbol{u}}_{[n]}$ into a set of motors $\left\{\boldsymbol{M}_j^d|j = J, ..., 1\right\}$, which results in a deformation of the object.

$$\underline{\boldsymbol{u}}_{[n]}^d = \boldsymbol{M}_J^d ... \boldsymbol{M}_1^d \underline{\boldsymbol{u}}_{[n]} \widetilde{\boldsymbol{M}}_1^d ... \widetilde{\boldsymbol{M}}_J^d \tag{3.20}$$

Finally, the entity $\underline{\boldsymbol{u}}_{[n]}^d$ may perform a motion under the action of a motor $\boldsymbol{M}$, which itself may be composed by a set of motors $\{\boldsymbol{M}_i|i = I, ..., 1\}$ according to equation (1.4),

$$\underline{\boldsymbol{u}}_{[n]}^{d'} = \boldsymbol{M} u_{[n]}^d \widetilde{\boldsymbol{M}}. \tag{3.21}$$

But a twist is not only an operator but may take on in CGA also the role of an operand,

$$\Psi' = \boldsymbol{M} \Psi \boldsymbol{M}. \tag{3.22}$$

This causes a dynamic shape model as an alternative to (3.20). If the angular argument of $\boldsymbol{M}$ is specified by, e.g. $\theta = 2\pi t$, $t_* \leq t \leq t^*$, then the twist axis $\underline{\boldsymbol{l}}$ may move on the arc of a circle. An interesting application is the so-called ball-and-socket joint required to accurately model shoulder and hip joints of articulated persons [18].

Sofar, the entity $\boldsymbol{u}_{\{n\}}$ was embedded in the Euclidean space. Lifting up the entity to the conformal space, $\underline{\boldsymbol{u}}_{\{n\}} \in \mathbb{R}_{4,1}$, is simply done by

$$\underline{\boldsymbol{u}}_{\{n\}} = \boldsymbol{e} \wedge \left(\boldsymbol{u}_{\{n\}} + \boldsymbol{e}_-\right) = \boldsymbol{e} \wedge \boldsymbol{U}_{\{n\}} \tag{3.23}$$

with $\boldsymbol{U}_{\{n\}}$ being the shape in the projective space $\mathbb{R}_{3,1}$.

Chapter 4

Twist Models and Fourier Representations

The message of the last section is the following. A finite set of coupled twist (or nested motors) performs a constrained motion of any set of geometric entities, whose orbit uniquely represents either a curve, a surface or a volume of arbitrary complexity. This needs a parameterized model of the generators of the shape. In some applications the reverse problem may be of interest. That is to find a parameterized twist model for a given shape. Also that task can be solved: Any curve, surface or volume of arbitrary complexity can be mapped to a finite set of coupled twists, but in a non-unique manner. That means, that there are different models which generate the same shape.

We will show here that there is a direct and intuitive relation between the twist model of shape and the Fourier representations. The Fourier series decomposition and the Fourier transforms in their different representations are well-known techniques of signal analysis and image processing [37]. The interesting fact that this equivalence of representations results in a fusion of conceptions from geometry, kinematics, and signal theory is of great importance in engineering. Furthermore, because the presented modelling of shape is embedded in a conformal space, there will be a single access also for embedding the Fourier representations in either conformal or projective geometry. This is quite different to the recent publications [49], [50]. It will hopefully enable an image processing which is conformally embedded and, in case of image sequences, the pose in space can be coupled to image analysis in a better way that in [51]. Our first attempt to formulate projective Fourier coefficients offered some serious problems which have to be overcome [41].

4.1 The Case of a Closed Planar Curve

Let us consider a closed curve $\boldsymbol{c} \in \mathbb{R}^2$ in a parametric representation with $t \in \mathbb{R}$. Then its Fourier series representation is given by

$$\boldsymbol{c}(t) = \sum_{\nu=-\infty}^{\infty} \gamma_\nu \exp\left(\frac{j2\pi\nu t}{T}\right) \tag{4.1}$$

with the Fourier coefficients γ_ν, $\nu \in \mathbb{Z}$ as frequency and j, $j^2 = -1$, as imaginary unit and the curve length T.

This model of a curve has been used for a long time in image processing for shape analysis by Fourier descriptors (these are the Fourier coefficients) [54], [17]. Furthermore, affine invariant Fourier descriptors can be used [2] to couple a space curve to its affine image.

We will translate this spectral representation into the model of an infinite number of coupled twists by following the method presented in [42]. Because equation (4.1) is valid in an Euclidean space, the twist model has to be reformulated accordingly. This will be shown for the case of a 2-twist curve $\underline{\boldsymbol{c}}_{\{2\}}$. Then equation (3.19) can be written in $\mathbb{R}_3$ for $\phi_1 = \phi_2 = \phi$ as

$$\begin{aligned} \boldsymbol{x}_\phi &= \boldsymbol{R}_{\lambda_2\phi}\left((\boldsymbol{R}_{\lambda_1\phi}(\boldsymbol{x}_0 - \boldsymbol{t}_1)\widetilde{\boldsymbol{R}}_{\lambda_1\phi} + \boldsymbol{t}_1) - \boldsymbol{t}_2\right)\widetilde{\boldsymbol{R}}_{\lambda_2\phi} + \boldsymbol{t}_2 && (4.2)\\ &= \boldsymbol{p}_0 + \boldsymbol{V}_{1,\phi}\boldsymbol{p}_1\widetilde{\boldsymbol{V}}_{1,\phi} + \boldsymbol{V}_{2,\phi}p_2\widetilde{\boldsymbol{V}}_{2,\phi}. && (4.3)\end{aligned}$$

Here the translation vectors have been absorbed by the vectors $\boldsymbol{p}_i$ and the $\boldsymbol{V}_i$ are built by certain products of the rotors $\boldsymbol{R}_{\lambda_i\phi}$. We call the $\boldsymbol{p}_i$ the phase vectors. Next, for the aim of interpreting that equation as a Fourier series expansion, we rewrite the Fourier basis functions as rotors of an angular frequency $i \in \mathbb{Z}$, in the plane $\boldsymbol{l} \in \mathbb{R}_2,\ \boldsymbol{l}^2 = -1$,

$$\boldsymbol{R}_{\lambda_i\phi} = \exp\left(-\frac{\lambda_i\phi}{2}\boldsymbol{l}\right) = \exp\left(-\frac{\pi i\phi}{T}\boldsymbol{l}\right). \tag{4.4}$$

All rotors of a planar curve lie in the same plane as the phase vectors $\boldsymbol{p}_i$. After some algebra, see [42], we get for the transformed point

$$\boldsymbol{x}_\phi = \sum_{i=0}^{2}\boldsymbol{p}_i \exp\left(\frac{2\pi i\phi}{T}\boldsymbol{l}\right) \tag{4.5}$$

and for the curve as subspace of $\mathbb{R}^3$ the infinite set of points

$$\boldsymbol{c}_{\{2\}} = \{\boldsymbol{x}_\phi | \text{ for all } \phi \in [0,...,2\pi] \text{ and for all } i \in \{0,1,2\}\}. \tag{4.6}$$

A general (planar) curve is given by

$$\boldsymbol{c}_{\{\infty\}} = \{\boldsymbol{x}_\phi | \text{ for all } \phi \in [0,...,2\pi] \text{ and for all } i \in \mathbb{Z}\}, \tag{4.7}$$

respectively as Fourier series expansion, written in the language of kinematics

$$\boldsymbol{c}_{\{\infty\}} = \left\{\lim_{n\longrightarrow\infty}\sum_{i=-n}^{n}\boldsymbol{p}_i \exp\left(\frac{2\pi i\phi}{T}\boldsymbol{l}\right)\right\} = \left\{\lim_{n\longrightarrow\infty}\sum_{i=-n}^{n}\boldsymbol{R}_{\lambda_i\phi}\boldsymbol{p}_i\widetilde{\boldsymbol{R}}_{\lambda_i\phi}\right\}. \tag{4.8}$$

A discretized curve is called a contour. In that case equation (4.8) has to consider a finite model of n twists and the Fourier series expansion becomes the inverse discrete Fourier transform. Hence, a planar contour is given by the finite sequence $\boldsymbol{c}_{[n]}$ with the contour points $c_k, -n \le k \le n$, in parametric representation

$$c_k = \sum_{i=-n}^{n}\boldsymbol{p}_i \exp\left(\frac{2\pi ik}{2n+1}\boldsymbol{l}\right), \tag{4.9}$$

and the phase vectors are computed as discrete Fourier transform of the contour

$$\boldsymbol{p}_i = \frac{1}{2n+1}\sum_{k=-n}^{n}c_k \exp\left(-\frac{2\pi ik}{2n+1}\boldsymbol{l}\right). \tag{4.10}$$

These equations imply that the angular argument ϕ_k is replaced by k. If the contour can be interpreted as a satisfactory sampled curve [37], the curve $\boldsymbol{c}_{\{\infty\}}$ can be reconstructed from $\boldsymbol{c}_{[n]}$.

4.2 Extensions of the Conceptions

The extension of the case of a planar curve, embedded in $\mathbb{R}^3$, to a 3D curve is easily done. This happens by taking its projections to either $\boldsymbol{e}_{12}$, $\boldsymbol{e}_{23}$, or $\boldsymbol{e}_{31}$ as periodic planar curves. Hence, we get the superposition of these three components. Let these components in case of a 3D-contour be $\boldsymbol{c}^j_{[n]}$, indicating that the rotation axes $\boldsymbol{l}^*_j$ are perpendicular to the rotation planes $\boldsymbol{l}_j$, then

$$\boldsymbol{c}_{[n]} = \sum_{j=1}^{3} \boldsymbol{c}^j_{[n]} \tag{4.11}$$

with the contour points of the projections c^j_k, $j = 1, 2, 3$ and $-n \leq k \leq n$,

$$c^j_k = \sum_{i=-n}^{n} \boldsymbol{p}^j_i \exp\left(\frac{2\pi i k}{2n+1}\boldsymbol{l}_j\right). \tag{4.12}$$

Another useful extension is with respect to surface representations, see [42]. If this surface is a 2D function orthogonal to a plane spanned by the bivectors $\boldsymbol{e}_{ij}$, then the twist model corresponds to the 2D inverse FT. In the case of an arbitrary orientation of the rotation planes $\boldsymbol{l}_j$ instead, or in the case of the surface of a 3D object, the procedure is comparable to that of equation (4.12). The surface is represented as a two-parametric surface $\boldsymbol{s}(t_1, t_2)$ as superposition of three projections $\boldsymbol{s}^j(t_1, t_2)$.
In the case of a discrete surface in a two-parametric representation we have the finite surface representation $\boldsymbol{s}_{[n_1,n_2]}$,

$$\boldsymbol{s}_{[n_1,n_2]} = \sum_{j=1}^{3} \boldsymbol{s}^j_{[n_1,n_2]} \tag{4.13}$$

with the surface points of the projections $s^j_{k_1,k_2}$, $j = 1, 2, 3$ and $-n_1 \leq k_1 \leq n_1$, $-n_2 \leq k_2 \leq n_2$,

$$s^j_{k_1,k_2} = \sum_{i_1=-n_1}^{n_1} \sum_{i_2=-n_2}^{n_2} \boldsymbol{p}^j_{i_1,i_2} \exp\left(\frac{2\pi i_1 k_1}{2n_1+1}\boldsymbol{l}_j\right) \exp\left(\frac{2\pi i_2 k_2}{2n_2+1}\boldsymbol{l}_j\right) \tag{4.14}$$

and the phase vectors, respectively,

$$\boldsymbol{p}^j_{i_1,i_2} = \frac{1}{2n_1+1}\frac{1}{2n_2+1}\boldsymbol{p}^{j'}_{i_1,i_2} \tag{4.15}$$

$$\boldsymbol{p}^{j'}_{i_1,i_2} = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} s^j_{k_1,k_2} \exp\left(-\frac{2\pi i_1 k_1}{2n_1+1}\boldsymbol{l}_j\right) \exp\left(-\frac{2\pi i_2 k_2}{2n_2+1}\boldsymbol{l}_j\right). \tag{4.16}$$

Finally, we will formulate an alternative model of a curve $\underline{\boldsymbol{c}} \in \mathbb{R}_{4,1}$ [41]. While equation (4.8) expresses the additive superposition of rotated phase vectors in Euclidean space, the following model expresses a multiplicative coupling of the twists directly in conformal space.

$$\underline{\boldsymbol{c}}_{\{\infty\}} = \left\{ \lim_{n\longrightarrow\infty} \left(\prod_{i=n}^{-n} \boldsymbol{T}_{\lambda_i\phi}\right) \underline{\boldsymbol{O}} \left(\prod_{i=-n}^{n} \widetilde{\boldsymbol{T}}_{\lambda_i\phi}\right) \right\}. \tag{4.17}$$

This equation results from the assumption that the point $\boldsymbol{x}_0 = (0,0,0) \in \mathbb{R}^3$, expressed as the affine point $\underline{\boldsymbol{O}} \in \mathbb{R}_{4,1}$, $\underline{\boldsymbol{O}} = \boldsymbol{e} \wedge \boldsymbol{e}_0$, is translated $2n+1$ times by the translators

$$\boldsymbol{T}_{\lambda_i \phi} = \frac{1 + \boldsymbol{e}\boldsymbol{t}_{\lambda_i \phi}}{2} \tag{4.18}$$

with the Euclidean vector

$$\boldsymbol{t}_{\lambda_i \phi} = \boldsymbol{R}_{\lambda_i \phi} \boldsymbol{p}_i \widetilde{\boldsymbol{R}}_{\lambda_i \phi}. \tag{4.19}$$

It turns out that equation (4.17) has some numeric advantages in application [41].

The discussed equivalence of the twist model and the Fourier representation has several advantages in practical use of the model. The most important may be to apply low-frequency approximations of the shape. For instance in pose estimation [41] it turned out that the estimations of the motion parameters of non-convex objects can be regularized efficiently in that way. Instead of estimating motors, the parameters of the twists are estimated because of numeric problems.

It turned out that [27] already proposed an elliptic approximation to a contour. The authors call their Fourier transform (FT) the elliptic FT. The generating model of shape is that of coupled ellipses. Later on, after Bracewell [5] rediscovered that this type of FT has been already proposed in [19] as real-valued FT, it is well-known as Hartley transform. Taking the Hartley transform instead of the complex-valued FT has the advantage of reducing the computational complexity by a factor of two.

In some applications it is not necessary to have at hand the global shape of an object as inverse discrete Fourier transform. Instead a local spectral representation of the shape would be sufficient. The Gabor transform [16] could be a candidate. But a better choice is the monogenic signal [14]. This is computed by scale adaptive filters [15] for getting the spectral representations of oriented lines in the plane. That approach is comparable to the model of coupled twists. But in contrast to our former assumption, the orientation of the orientation plane $\underline{\boldsymbol{l}}$ is not fixed but adapted to the orientation of the shape tangents in the plane. An extension to lines in $\mathbb{R}^3$ and to its coupling to the twist model is work in progress.

Chapter 5

Summary and Conclusions

We presented an operational, respectively kinematic model of shape in $\mathbb{R}^3$. This model is based on the Lie group $SE(3)$, embedded in the conformal geometric algebra $\mathbb{R}_{4,1}$ of the Euclidean space. While the modelling of shape in $\mathbb{R}^3$ caused by actions of $SE(3)$ is limited, a lot of advantages result from the chosen algebraic embedding. With respect to applications the possibility of conformal (and projective) shape models should be mentioned. We did not discuss applications. Instead, we refer the reader to the website http://www.ks.informatik.uni-kiel.de with respect to the problem of pose estimation. Because the chosen twist model is equivalent to the Fourier representation (in some aspects it overcomes that), the proposed shape representation unifies geometry, kinematics, and signal theory. It can be assumed that this will have great impact on either theory and practice in computer vision, computer graphics and modelling of mechanisms.

Acknowledgment

This work has been partially supported (G.S. an C.P.) by EC Grant IST-2001-3422 (VISATEC), by DFG Grant RO 2497/1-1 (B.R.), and by DFG Graduiertenkolleg No. 357 (B.R. and C.P.)

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