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Aspects of Geometric Algebra in Euclidean, Projective and Conformal Space

Christian B.U. Perwass, Dietmar Hildenbrand

Bericht Nr. 0310
Version 1.0, September 2003

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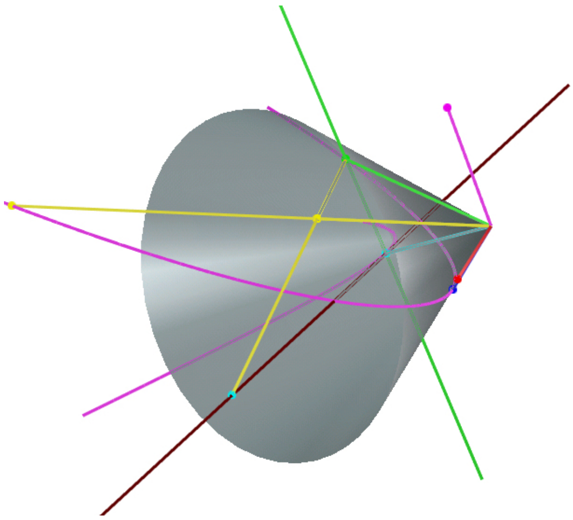
Abstract

This text is meant to be a script of a tutorial on Clifford (or Geometric) algebra. It is therefore not complete in the description of the algebra and neither completely rigorous. The reader is also not likely to be able to perform arbitrary calculations with Clifford algebra after reading this script. The goal of this text is to give the reader a feeling for what Clifford algebra is about and how it may be used. It is attempted to convey the basic ideas behind the use of Clifford algebra in the description of geometry in Euclidean, projective and conformal space.



Contents

Preface

This text is meant to be a script of a tutorial on Clifford (or Geometric) algebra. It is therefore not complete in the description of the algebra and neither completely rigorous. The reader is also not likely to be able to perform arbitrary calculations with Clifford algebra after reading this script. The goal of this text is to give the reader a feeling for what Clifford algebra is about and how it may be used. It is attempted to convey the basic ideas behind the use of Clifford algebra in the description of geometry in Euclidean, projective and conformal space.

There are also many other introductions to Clifford and Geometric algebra and its applications in Euclidean, projective and conformal space. Some of these are [19, 18, 20, 16, 25, 32, 15, 31, 21, 10, 26, 28, 9]. A collection of papers discussing in particular the conformal space in detail and applications of Geometric algebra in Computer Vision may be found in the book *Geometric Computing with Clifford Algebra* [38].

This text is separated into three main parts: "Introductions to Clifford Algebra", "Geometries" and "An Interactive Introduction to Geometric Algebra". The plural "Introductions" in the title of the first chapter is fully intentional, since two introductions will be given. The first concentrates on the geometric interpretation of Clifford algebra elements and the second on algebraic properties. The second chapter discusses the application of Geometric algebra to projective and conformal spaces. Here we will see how Geometric algebra can be used to represent points, lines, planes, circles and spheres. It will be shown that intersections between any of these objects can be expressed by a single operation and operations like reflections, rotations or inversions are equally expressed in a uniform way for *all* geometric entities. The third chapter recapitulates some important aspects of Geometric algebra in worked examples using the Geometric algebra visualization tool **CLUCalc**. This chapter should be particularly helpful, since it shows you how to explore important aspects of Geometric algebra interactively.

CLUCalc is of course not the only software available that deals with Clifford or Geometric algebra. Many software packages have been developed, because the numerical evaluation of Clifford algebra equations becomes more and more important as Clifford algebra becomes more prominent in applied fields like computer vision, computer graphics and robotics [21, 35, 22, 30, 10, 38, 8]. There are packages for the symbolic computer algebra systems Maple [1, 2] and Mathematica [5], a package for the numerical mathematics program MatLab called GABLE [9], the C++ software library GluCat [23], the C++ software library generator Gaigen [14], the Java library Clados [7] and a stand alone program called CLICAL [24], to name just a few.

In 1996, one of the authors (C. Perwass), started developing a C++ library to implement Clifford algebra operations. It has since grown to a whole suite of C++ libraries and stand alone programs for the manipulation and visualisation of Clifford algebras. This suite is called the *CLU-Project* [27]. 'CLU' stands for **C**lifford algebra **L**ibrary and **U**tilities. The goal of the CLU-Project is to offer an easy to use and yet powerful interface to work with and understand Clifford or Geometric algebra. All C++ libraries of the CLU-Project are Open Source and thus available to everybody.

CLUCalc is a user friendly frontend to these libraries. It is used in the "Interactive Introduction ..." and is available for download from [27]. In CLUCalc you can type your equations in a simple script language, called **CLUScript** and visualize the results immediately with OpenGL graphics. The program comes with a manual and a number of example scripts. It should serve as a good accompaniment to this script, helping you to understand the concepts behind Geometric algebra visually. The **CLUScript**s used in chapter three can also be downloaded through one of the following links:

```
www.dgm.informatik.tu-darmstadt.de/research/gat.zip
www.gris.informatik.tu-darmstadt.de/staff/hildenbrand.html
```

By the way, CLUCalc was also used to create all of the 2d and 3d graphics in this script. You can use it for the same purpose, illustrating your publications or web-pages, from the upcoming version 3.0 onwards. If you want to know more details, simply send an email to `clu@perwass.de`.

Christian Perwass
Kiel, September 2003

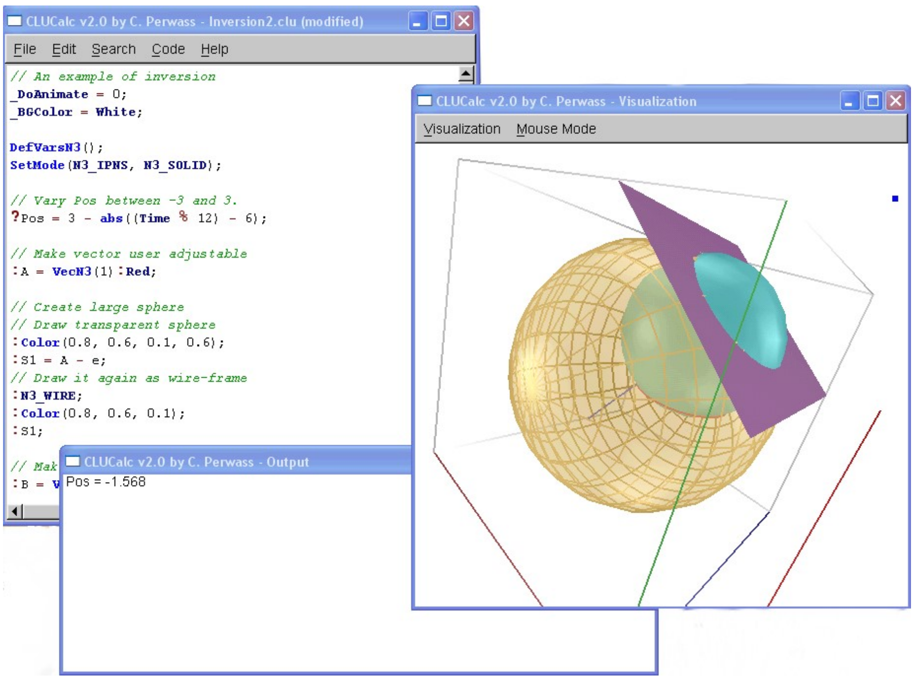


Figure 1: A screenshot of CLUCalc v2.0.

Chapter 1

Introductions to Clifford Algebra

by Dr. Christian Perwass

This chapter is separated into two main parts: "Introductions to Clifford Algebra" and "Geometries". The plural "Introductions" is fully intentional, since two introductions will be given. The first concentrates on the geometric interpretation of Clifford algebra elements and the second on algebraic properties. These two introductions also reflect the two terms mainly used for this algebra within the research community: "Geometric Algebra" and "Clifford Algebra". Roughly speaking, if somebody talks about Clifford algebra, he is more interested in the algebraic aspects. If someone talks about Geometric algebra, his interest lies more in the geometric interpretation of algebraic entities. Here we will start with the geometric interpretation of algebraic entities, since it is hoped that the reader's geometric intuition will further the understanding.

1.1 Geometric Algebra

In this introduction we will neglect many algebraic aspects and introduce Geometric algebra as an extension of the standard vector algebra. So let's start with a 3d Euclidean vector space denoted by $\mathbb{E}^3$. We will use the coordinate representation $\mathbb{R}^3$ for $\mathbb{E}^3$.

> **Aside.** Some notational aspects. $\mathbb{R}^{p,q}$ denotes a vector space of dimension $n = p+q$. An orthonormal basis of this vector space will be denoted by $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ and has p elements with a positive and q elements with a negative signature. The signature is, of course, only of importance once a scalar product is defined on the vector space. Note that by negative signature we mean that an orthonormal basis vector squares to minus one.

We assume that the standard scalar product is defined on $\mathbb{E}^3$. It will be denoted by $*$. Furthermore, the usual vector cross product exists on $\mathbb{E}^3$ and will be written as $\times$. Recall that the scalar product gives the length of the component two vectors have in common. The vector cross product, on the other hand, results in a vector perpendicular to both of the initial vectors. For example, let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$, then

$$\mathbf{a} * \mathbf{b} \in \mathbb{R} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} \in \mathbb{E}^3.$$

Furthermore,

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \;\Rightarrow\; \mathbf{c} \perp \mathbf{a} \text{ and } \mathbf{c} \perp \mathbf{b}.$$

A plane in $\mathbb{E}^3$ is typically represented by its normal and an offset vector. Given two vectors that are to span a plane, the vector cross product can be used to find the plane's normal. However, this only works in 3d. In higher dimensions the (standard) vector cross product of two vectors is not defined[1]. Nevertheless, we may be interested in describing the two dimensional subspace spanned by two vectors also in a n-dimensional vector space.

1.1.1 The Outer Product

Without explaining exactly what it is, we can define a Clifford algebra on $\mathbb{R}^n$, $\mathcal{C}\ell(\mathbb{R}^n)$ or simply $\mathcal{C}\ell_n$ if it is clear that we are forming the Clifford algebra over the reals. The latter will in fact be the case for the whole of this text.

The outer product is an operation defined within this algebra and is denoted by $\wedge$. Here are the properties of the outer product of vectors. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^n$.

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= -\mathbf{b} \wedge \mathbf{a} \\ (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} &= \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \\ \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} \wedge \mathbf{b}) + (\mathbf{a} \wedge \mathbf{c}). \end{aligned} \tag{1.1}$$

Another important property is

$$\mathbf{a} \wedge \mathbf{b} = 0 \iff \mathbf{a} \text{ and } \mathbf{b} \text{ are linearly dependent.} \tag{1.2}$$

Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subset \mathbb{R}^n$ be $k \leq n$ mutually linearly independent vectors. Then

$$(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k) \wedge \mathbf{b} = 0, \tag{1.3}$$

if and only if $\mathbf{b}$ is linearly dependent on $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$. The outer product of k vectors is called a k*-blade* and is denoted by

$$A_{\langle k \rangle} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k =: \bigwedge_{i=1}^{k} \mathbf{a}_i.$$

The *grade* of a blade is simply the number of vectors that "wedged" together give the blade. Hence, the outer product of k linearly independent vectors gives a blade of grade k, a k-blade.

[1]Note that in a n-dimensional vector space, one can define a vector cross product between $n-1$ vectors.

1.1.2 The Outer Product Null Space

In Geometric algebra, blades, as defined above, are given a geometric interpretation. This is based on their interpretation as linear subspaces. For example, given a vector $\mathbf{a} \in \mathbb{R}^n$, we can define a function $\mathcal{O}_{\mathbf{a}}$ as

$$\mathcal{O}_{\mathbf{a}} : \ \mathbf{x} \in \mathbb{R}^n \ \mapsto \ \mathbf{x} \wedge \mathbf{a} \in \mathcal{C}\ell(\mathbb{R}^n).$$

The kernel of this function is the set of vectors in $\mathbb{R}^n$ that $\mathcal{O}_{\mathbf{a}}$ maps to zero. This kernel will be called the *outer product null space* (OPNS) of $\mathbf{a}$ and denoted by $\mathbb{NO}(\mathbf{a})$. That is,

$$\text{kern } \mathcal{O}_{\mathbf{a}} = \mathbb{NO}(\mathbf{a}) := \big\{ \mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \wedge \mathbf{a} = 0 \in \mathcal{C}\ell(\mathbb{R}^n) \big\}. \tag{1.4}$$

We already know that $\mathbf{x} \wedge \mathbf{a}$ is zero if and only if $\mathbf{x}$ is linearly dependent on $\mathbf{a}$. Therefore, $\mathbb{NO}(\mathbf{a})$ can also be given in terms of $\mathbf{a}$ as

$$\mathbb{NO}(\mathbf{a}) = \big\{ \alpha\, \mathbf{a} \ : \ \alpha \in \mathbb{R} \big\},$$

which means that the OPNS of $\mathbf{a}$ is a *line through the origin with the direction of* $\mathbf{a}$. In Geometric algebra it is therefore said that a vector in $\mathbb{E}^n$ represents a line.

Given a 2-blade $\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^n)$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, a function $\mathcal{O}_{\mathbf{a}\wedge\mathbf{b}}$ can be defined as

$$\mathcal{O}_{\mathbf{a}\wedge\mathbf{b}} : \ \mathbf{x} \in \mathbb{R}^n \ \mapsto \ \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b} \ \in \mathcal{C}\ell(\mathbb{R}^n).$$

The kernel of this function is

$$\text{kern } \mathcal{O}_{\mathbf{a}\wedge\mathbf{b}} = \mathbb{NO}(\mathbf{a} \wedge \mathbf{b}) := \big\{ \mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b} = 0 \in \mathcal{C}\ell(\mathbb{R}^n) \big\}. \tag{1.5}$$

As before, it follows that the OPNS of $\mathbf{a} \wedge \mathbf{b}$ can be parameterized as follows

$$\mathbb{NO}(\mathbf{a} \wedge \mathbf{b}) = \big\{ \alpha\, \mathbf{a} + \beta\, \mathbf{b} \ : \ (\alpha,\, \beta) \in \mathbb{R}^2 \big\}.$$

Hence, $\mathbf{a} \wedge \mathbf{b}$ is said to represent the two-dimensional subspace of $\mathbb{R}^n$ spanned by $\mathbf{a}$ and $\mathbf{b}$, ie a plane through the origin. In general the OPNS of some k-blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$ is a k-dimensional linear subspace of $\mathbb{R}^n$.

$$\mathbb{NO}(A_{\langle k \rangle}) := \big\{ \mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \wedge A_{\langle k \rangle} = 0 \big\}.$$

Consider again the three-dimensional Euclidean space $\mathbb{E}^3$ with $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$ three mutually linearly independent vectors. Hence, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ form a basis of $\mathbb{E}^3$. Then

$$\begin{aligned} \mathbb{NO}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \ &:= \big\{ \mathbf{x} \in \mathbb{E}^3 \ : \ \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0 \in \mathcal{C}\ell(\mathbb{R}^3) \big\} \\ &= \big\{ \alpha\, \mathbf{a} + \beta\, \mathbf{b} + \gamma\, \mathbf{c} \in \mathbb{E}^3 \ : \ (\alpha,\, \beta,\, \gamma) \in \mathbb{R}^3 \big\}. \end{aligned}$$

Therefore, the OPNS of $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is the whole space $\mathbb{E}^3$. Since the OPNS of the outer product of any basis of $\mathbb{E}^3$ is the whole space $\mathbb{E}^3$, the blades created from different bases have to be similar. In fact, they only differ by a scalar factor. A blade of grade n in some $\mathcal{C}\ell(\mathbb{R}^n)$ is called a *pseudoscalar*. "Pseudoscalar" because all pseudoscalars only differ by a scalar factor, just like the scalar element $1 \in \mathcal{C}\ell(\mathbb{R}^n)$.

Aside. Note that the fact that $\mathbb{NO}\big(A_{\langle n \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)\big) = \mathbb{R}^n$, implies that no blades of grade higher than n can exist in $\mathcal{C}\ell(\mathbb{R}^n)$.

1.1.3 Magnitude of Blades

On the Euclidean space $\mathbb{E}^n$ the norm typically used is the L_2 norm. This is defined in terms of the scalar product. Let $\mathbf{a} \in \mathbb{E}^n$, then

$$\|\mathbf{a}\|_2 := \sqrt{\mathbf{a} * \mathbf{a}}. \tag{1.6}$$

This norm can also be extended to blades in $\mathcal{C}\ell(\mathbb{E}^n)$. We will not give a proper derivation here, but try to motivate the definition. In the following will also use $\|.\|$ instead $\|.\|_2$ for brevity. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and denote by $\mathbf{b}^\perp$ and $\mathbf{b}^\|$ the parts of $\mathbf{b} = \mathbf{b}^\perp + \mathbf{b}^\|$ that are perpendicular and parallel to $\mathbf{a}$, respectively. Then

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= \mathbf{a} \wedge (\mathbf{b}^\perp + \mathbf{b}^\|) \\ &= \mathbf{a} \wedge \mathbf{b}^\perp + \underbrace{\mathbf{a} \wedge \mathbf{b}^\|}_{=0} \\ &= \mathbf{a} \wedge \mathbf{b}^\perp. \end{aligned} \tag{1.7}$$

Similarly, for any k-blade $A_{\langle k \rangle} = \bigwedge_{i=1}^k \mathbf{a}_i$, we can find a set of k mutually orthogonal vectors $\{\mathbf{a}'_1, \ldots, \mathbf{a}'_k\}$, such that

$$A_{\langle k \rangle} = A'_{\langle k \rangle} := \bigwedge_{i=1}^{k} \mathbf{a}'_i.$$

Now, it may be shown that[2]

$$\|A_{\langle k \rangle}\| = \|A'_{\langle k \rangle}\| = \sqrt{\prod_{i=1}^{k} (\mathbf{a}'_i)^2} = \prod_{i=1}^{k} \|\mathbf{a}'_i\|, \tag{1.8}$$

with $k > 0$. Since the $\{\mathbf{a}'_i\}$ are mutually orthogonal, the norm or magnitude of $A_{\langle k \rangle}$ is the "volume" spanned by them. For $k = 1$ this reduces to the norm of a vector.

An illustrative example is the norm of a 2-blade (also called *bivector*). The bivector $\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^n)$ may also be written as $\mathbf{a} \wedge \mathbf{b}^\perp$, where $\mathbf{b}^\perp$ is the component of $\mathbf{b}$ that is perpendicular to $\mathbf{a}$. Then $\|\mathbf{b}^\perp\| = \sin\theta \, \|\mathbf{b}\|$, with $\theta = \angle(\mathbf{a}, \mathbf{b})$. Therefore,

$$\|\mathbf{a} \wedge \mathbf{b}\| = \|\mathbf{a} \wedge \mathbf{b}^\perp\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \sin\theta,$$

which is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Now consider a $n \times k$ matrix A, whose columns are the $\{\mathbf{a}_i\}_{i=1}^k \subset \mathbb{R}^n$. This will be written as $\mathsf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_k]$. We could now define the norm of such a matrix to be the "volume" of the parallelepiped spanned by its column vectors. This would then be in accordance with the norm of a blade of these vectors. In fact, for a matrix $\mathsf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$, where the $\{\mathbf{b}_i\}_{i=1}^n \subset \mathbb{R}^n$ are a basis of $\mathbb{R}^n$, the determinant of B, $\det(\mathsf{B})$ does give the volume of the parallelepiped spanned by the $\{\mathbf{b}_i\}_{i=1}^n$. Therefore, in this case,

$$\| \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_n \| = \det([\mathbf{b}_1, \ldots, \mathbf{b}_n]).$$

[2] In order to show this, the definition of the inner product is needed, which will be discussed later.

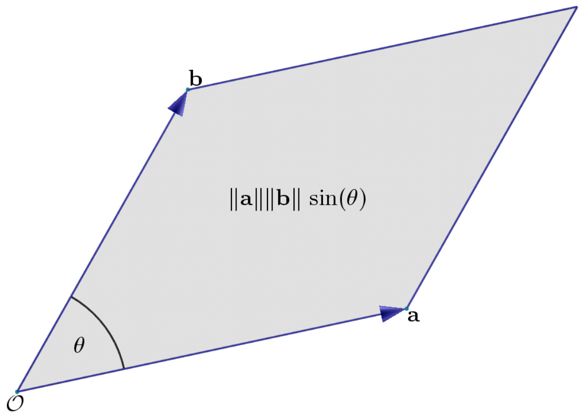


Figure 1.1: Area of bivector.

The unit pseudoscalar of some $\mathcal{C}\ell(\mathbb{R}^n)$, is a blade of grade n with magnitude 1 and is usually denoted by I. Therefore, for example,

$$\mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_n = \|\, \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_n \,\|\ I = \det([\mathbf{b}_1,\ \ldots,\ \mathbf{b}_n])\ I.$$

1.1.4 The Inner Product

Another important operation in Geometric algebra is the *inner product*. The inner product will be denoted by $\cdot$. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, their inner product is just the same as their scalar product, ie

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} * \mathbf{b}.$$

This may be called the "metric" property of the inner product, since the result of the scalar product of two vectors depends on the metric of the vector space they lie in. However, the inner product also has some purely algebraic properties for elements in $\mathcal{C}\ell(\mathbb{R}^n)$, which are independent of the metric of the vector space $\mathbb{R}^n$. In the following a number these properties are stated without proof.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, then the bivector $\mathbf{b} \wedge \mathbf{c} \in \mathcal{C}\ell(\mathbb{R}^n)$. The inner product of $\mathbf{a}$ with this bivector gives,

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b}. \tag{1.9}$$

Since $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{a} \cdot \mathbf{c})$ are scalars, we see that the inner product of a vector with a bivector results in a vector. More generally it may be shown that for $k \geq 1$

$$\begin{aligned}
\mathbf{x} \cdot A_{\langle k \rangle} \quad = & \quad\; (\mathbf{x} \cdot \mathbf{a}_1)\,(\mathbf{a}_2 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \ldots \wedge \mathbf{a}_k) \\
& - \; (\mathbf{x} \cdot \mathbf{a}_2)\,(\mathbf{a}_1 \wedge \mathbf{a}_3 \wedge \mathbf{a}_4 \wedge \ldots \wedge \mathbf{a}_k) \\
& + \; (\mathbf{x} \cdot \mathbf{a}_3)\,(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_4 \wedge \ldots \wedge \mathbf{a}_k) \\
& - \; \text{etc.} \\
= & \sum_{i=1}^{k} (-1)^{(i+1)}\,(\mathbf{x} \cdot \mathbf{a}_i)\,\left[A_{\langle k \rangle} \setminus \mathbf{a}_i\right],
\end{aligned} \tag{1.10}$$

where $[A_{\langle k \rangle} \setminus \mathbf{a}_i]$ denotes the blade $A_{\langle k \rangle}$ without the vector $\mathbf{a}_i$. Here the inner product of a vector with a k-blade results in a $(k-1)$-blade. An example of another important rule is this

$$(\mathbf{a} \wedge \mathbf{b}) \cdot A_{\langle k \rangle} = \mathbf{a} \cdot \big(\mathbf{b} \cdot A_{\langle k \rangle}\big), \tag{1.11}$$

with $k \geq 2$. More generally, the inner product of blades $A_{\langle k \rangle}, B_{\langle l \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, with $0 < k \leq l \leq n$, can be expanded as

$$A_{\langle k \rangle} \cdot B_{\langle l \rangle} = \mathbf{a}_1 \cdot \Big(\mathbf{a}_2 \cdot \big(\ldots \cdot (\mathbf{a}_k \cdot B_{\langle l \rangle})\big)\Big). \tag{1.12}$$

Hence, the result of this inner product is a $(l-k)$-blade.

In comparison to the outer product we see that the inner and the outer product are antagonists: while the outer product increases the grade of a blade, the inner product reduces it.

1.1.5 The Inverse of a Blade

Similar to the formula for vectors, the inverse of a blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, $k \leq n$, is in general given by

$$A^{-1}_{\langle k \rangle} := \frac{\tilde{A}_{\langle k \rangle}}{\|A_{\langle k \rangle}\|^2},$$

as long as[3] $\|A_{\langle k \rangle}\| \neq 0$. Using this formula it may indeed be shown that

$$A_{\langle k \rangle} \cdot A^{-1}_{\langle k \rangle} = A^{-1}_{\langle k \rangle} \cdot A_{\langle k \rangle} = 1.$$

The symbol $\tilde{A}_{\langle k \rangle}$ denotes the *reverse* of a blade. The reverse is an operator that simply reverses the order of vectors in a blade. For example, if $A_{\langle k \rangle} = \bigwedge_{i=1}^{k} \mathbf{a}_i$ then

$$\tilde{A}_{\langle k \rangle} = \bigwedge_{i=k}^{1} \mathbf{a}_i = \mathbf{a}_k \wedge \mathbf{a}_{k-1} \wedge \ldots \wedge \mathbf{a}_1. \tag{1.13}$$

Since the outer product is associative and anti-commutative, the reordering of vectors in a blade can only change the blade's sign. For the reverse we find in particular

$$\tilde{A}_{\langle k \rangle} = (-1)^{k(k-1)/2}\,A_{\langle k \rangle}. \tag{1.14}$$

[3] The magnitude of a blade can in fact become zero in Minkowski spaces.

So, why do we need the reverse in the definition of the inverse of a blade? The answer is, that the reverse takes care of a sign that is introduced due to the grade of a blade. As an example consider the orthonormal basis $\{\mathsf{e}_i\}$ of $\mathbb{R}^n$. From equations (1.12) and (1.10) it follows that

$$\begin{aligned}(\mathsf{e}_1 \wedge \mathsf{e}_2) \cdot (\mathsf{e}_1 \wedge \mathsf{e}_2) &= \mathsf{e}_1 \cdot \big((\mathsf{e}_2 \cdot \mathsf{e}_1)\mathsf{e}_2 - (\mathsf{e}_2 \cdot \mathsf{e}_2)\mathsf{e}_1\big) \\ &= \mathsf{e}_1 \cdot \big(-\mathsf{e}_1\big) \\ &= -1.\end{aligned}$$

On the other hand, obviously $\mathsf{e}_1 \cdot \mathsf{e}_1 = 1$. That is, depending on the grade of a blade (a vector being a blade of grade 1), an additional sign is introduced or not. This is fixed by the reverse. Given any blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, then

$$A_{\langle k \rangle} \cdot \tilde{A}_{\langle k \rangle} = \|A_{\langle k \rangle}\|^2,$$

whereas

$$A_{\langle k \rangle} \cdot A_{\langle k \rangle} = (-1)^{k(k-1)/2} \, \|A_{\langle k \rangle}\|^2. \tag{1.15}$$

1.1.6 Geometric Interpretation of Inner Product

We can already get an idea of what is happening by looking at the Clifford algebra of $\mathbb{R}^2$, $\mathcal{C}\ell(\mathbb{R}^2)$ with orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2\}$. The outer product $\mathsf{e}_1 \wedge \mathsf{e}_2$ spans the whole space, ie a plane. Now let's look at the inner product of e_1 with this bivector.

$$\mathsf{e}_1 \cdot (\mathsf{e}_1 \wedge \mathsf{e}_2) = (\mathsf{e}_1 \cdot \mathsf{e}_1)\,\mathsf{e}_2 - (\mathsf{e}_1 \cdot \mathsf{e}_2)\,\mathsf{e}_1 = \mathsf{e}_2. \tag{1.16}$$

This may be interpreted as "subtracting" the subspace represented by e_1 from the subspace represented by $\mathsf{e}_1 \wedge \mathsf{e}_2$. What is left after the subtraction is, of course, perpendicular to e_1.

More generally, let $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and let

$$\mathbf{y} = \mathbf{x} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{x} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\,\mathbf{a}.$$

Now we find that

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot \big[(\mathbf{x} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\,\mathbf{a}\big] \\ &= (\mathbf{x} \cdot \mathbf{a})\,(\mathbf{x} \cdot \mathbf{b}) - (\mathbf{x} \cdot \mathbf{b})\,(\mathbf{x} \cdot \mathbf{a}) \\ &= 0.\end{aligned}$$

That is, $\mathbf{x}$ is perpendicular to $\mathbf{y}$, which again implies that the inner product $\mathbf{x} \cdot (\mathbf{a} \wedge \mathbf{b})$ "subtracted" the subspace represented by $\mathbf{x}$ from the subspace represented by $\mathbf{a} \wedge \mathbf{b}$. This can also be illustrated quite nicely in $\mathbb{E}^3$.

Let P denote the bivector $\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^3)$. In $\mathbb{E}^3$ this bivector represents a plane through the origin, as shown in figure 1.2. A vector $\mathbf{x} \in \mathbb{R}^3$ will in general have a component parallel to P, $\mathbf{x}^{\|}$, and a component perpendicular to P, $\mathbf{x}^{\perp}$, such that $\mathbf{x} = \mathbf{x}^{\|} + \mathbf{x}^{\perp}$. Therefore,

$$\mathbf{y} := \mathbf{x} \cdot P = (\mathbf{x}^{\|} + \mathbf{x}^{\perp}) \cdot P = \mathbf{x}^{\|} \cdot P.$$

The inner product $\mathbf{x}^{\|} \cdot P$ now "subtracts" the subspace represented by $\mathbf{x}^{\|}$ from the subspace represented by P, which results in a vector that lies in P and is perpendicular to $\mathbf{x}$, as shown in figure 1.2.

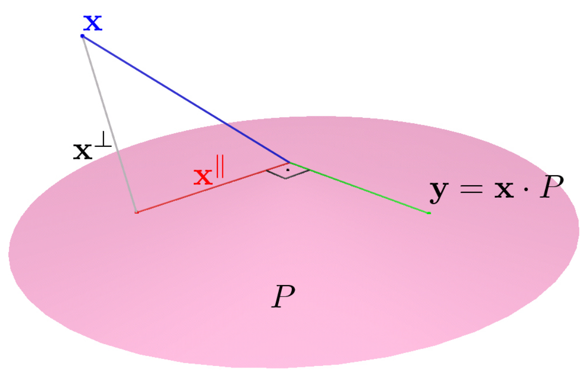


Figure 1.2: Inner product of vector and bivector.

1.1.7 The Inner Product Null Space

Just as for the outer product, we can also define the null space of blades with respect to the inner product. The *inner product null space* (IPNS) of a blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, denoted by $\mathbb{NI}(A_{\langle k \rangle})$, is the kernel of the function $\mathcal{I}_{A_{\langle k \rangle}}$ defined as

$$\mathcal{I}_{A_{\langle k \rangle}} : \mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{x} \cdot A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n), \tag{1.17}$$

and thus

$$\mathbb{NI}(A_{\langle k \rangle}) := \{ \, \mathbf{x} \in \mathbb{R}^n \; : \; \mathcal{I}_{A_{\langle k \rangle}}(\mathbf{x}) = 0 \in \mathcal{C}\ell(\mathbb{R}^n) \, \}. \tag{1.18}$$

For example, consider a vector $\mathbf{a} \in \mathbb{R}^3$, then $\mathbb{NI}(\mathbf{a})$ is given by

$$\mathbb{NI}(\mathbf{a}) := \{ \, \mathbf{x} \in \mathbb{R}^3 \; : \; \mathbf{x} \cdot \mathbf{a} = 0 \, \}.$$

That is, all vectors that are perpendicular to $\mathbf{a}$ belong to its IPNS. In $\mathbb{R}^3$ the IPNS of $\mathbf{a}$ is therefore a plane of which $\mathbf{a}$ is the *normal*. Earlier we already saw that the OPNS of a bivector represents a plane. This implies that there has to be some kind of relationship between the IPNS of a vector in $\mathbb{R}^3$ and the OPNS of a bivector in $\mathcal{C}\ell(\mathbb{R}^3)$.

1.1.8 The Dual

Let $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ denote again an orthonormal basis of $\mathbb{R}^3$. The IPNS of e_1 is the set of all vectors that are perpendicular to e_1. Hence,

$$\mathbb{NI}(\mathsf{e}_1) = \{ \, \alpha \, \mathsf{e}_2 + \beta \, \mathsf{e}_3 \; : \; (\alpha, \beta) \in \mathbb{R}^2 \, \},$$

the plane spanned by e_2 and e_3. However, we know that this is also the OPNS of $\mathsf{e}_2 \wedge \mathsf{e}_3$,

$$\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3) = \{ \, \alpha \, \mathsf{e}_2 + \beta \, \mathsf{e}_3 \; : \; (\alpha, \beta) \in \mathbb{R}^2 \, \}.$$

We may therefore ask whether there is a relation between the concepts of the IPNS and the OPNS. Such a relation does indeed exist and it is called *duality*. In the following we will see how this comes about.

Before we start with the actual calculations, we will introduce two set operations for sets of vectors that will become quite useful. The first is the direct sum of two sets of vectors denoted by $\oplus$. Given two sets $\mathbb{A} := \{\mathbf{a}_i\}_{i=1}^{k} \subset \mathbb{R}^n$ and $\mathbb{B} := \{\mathbf{b}_i\}_{i=1}^{l} \subset \mathbb{R}^n$ their direct sum is

$$\mathbb{A} \oplus \mathbb{B} := \left\{ \, \mathbf{a}_i + \mathbf{b}_j \, \in \mathbb{R}^n \; : \; 0 < i \leq k, \, 0 < j \leq l \, \right\}. \tag{1.19}$$

In particular this means for two infinite sets, ie one dimensional subspaces

$$\mathbb{A} := \left\{ \, \alpha \, \mathbf{a} \in \mathbb{R}^n \; : \; \alpha \in \mathbb{R} \, \right\}, \quad \text{and} \quad \mathbb{B} := \left\{ \, \beta \, \mathbf{b} \in \mathbb{R}^n \; : \; \beta \in \mathbb{R} \, \right\},$$

that their direct sum is the set of all linear combinations of the elements of $\mathbb{A}$ and $\mathbb{B}$. That is,

$$\mathbb{A} \oplus \mathbb{B} = \left\{ \, \alpha \, \mathbf{a} + \beta \, \mathbf{b} \, \in \mathbb{R}^n \; : \; (\alpha, \beta) \in \mathbb{R}^2 \, \right\}.$$

In this spirit it makes sense also to define a "direct subtraction" between two such sets as

$$\mathbb{A} \ominus \mathbb{B} := \left\{ \, \mathbf{x} \in \mathbb{A} \; : \; \mathbf{x} * \mathbf{y} = 0 \; \forall \mathbf{y} \in \mathbb{B} \, \right\}, \tag{1.20}$$

where we assume that a scalar product is defined on the elements of $\mathbb{A}$ and $\mathbb{B}$. Hence, the direct subtraction removes the linear dependence on elements of $\mathbb{B}$ from the elements of $\mathbb{A}$. Note that this is more than just to remove the elements of $\mathbb{B}$ from $\mathbb{A}$.

Now let us return to the question of duality. First of all note that the OPNS of e_1 is simply

$$\mathbb{NO}(\mathsf{e}_1) = \left\{ \, \alpha \, \mathsf{e}_1 \; : \; \alpha \in \mathbb{R} \, \right\},$$

a line through the origin with direction e_1. The direct sum of $\mathbb{NO}(\mathsf{e}_1)$ and $\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3)$ is the whole space $\mathbb{R}^3$,

$$\mathbb{NO}(\mathsf{e}_1) \oplus \mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3) = \left\{ \, \alpha \, \mathsf{e}_1 + \beta \, \mathsf{e}_2 + \gamma \, \mathsf{e}_3 \; : \; (\alpha, \beta, \gamma) \in \mathbb{R}^3 \, \right\} \equiv \mathbb{R}^3.$$

and, in particular, "removing" the linear dependence on $\mathbb{NO}(\mathsf{e}_1)$ from $\mathbb{R}^3$ gives $\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3)$,

$$\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3) = \mathbb{R}^3 \ominus \mathbb{NO}(\mathsf{e}_1).$$

With respect to $\mathbb{R}^3$, $\mathbb{NO}(\mathsf{e}_1)$ may therefore be called the *complement* set to $\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3)$. Furthermore,

$$\mathbb{NI}(\mathsf{e}_1) = \mathbb{R}^3 \ominus \mathbb{NO}(\mathsf{e}_1).$$

The question now is: can we find an operation in $\mathcal{C}\ell(\mathbb{R}^n)$ which transforms any blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$ into a complementary blade $B_{\langle n-k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, such that

$$\mathbb{NO}(A_{\langle k \rangle}) = \mathbb{R}^n \ominus \mathbb{NO}(B_{\langle n-k \rangle}).$$

Such an operation does indeed exist and is called the *dual*. The dual of a multivector $A \in \mathcal{C}\ell(\mathbb{R}^n)$ is written A^* and is defined as

$$A^* := A \cdot I^{-1}, \tag{1.21}$$

where I^{-1} is the inverse unit pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^n)$. It is a nice feature of Geometric algebra that the dual can be given as a standard product with a particular element of the algebra. However, this has also the drawback that the dual of the dual of a multivector may introduce an additional sign. That is,

$$\big(A^*\big)^* = \big(A \cdot I^{-1}\big) \cdot I^{-1} = A\big(I^{-1} \cdot I^{-1}\big).$$

Why the last step in this equation works will be shown later on in equation (1.31), page 16. If we believe this equation for the moment, then it shows that an additional sign is introduced whenever $I^{-1} \cdot I^{-1} = -1$. Since I^{-1} is a n-blade in $\mathcal{C}\ell(\mathbb{R}^n)$ we know from equations (1.14) and (1.15) that

$$I^{-1} \cdot I^{-1} = (-1)^{k(k-1)/2} \, \|I^{-1}\|^2 = (-1)^{k(k-1)/2}.$$

With respect to the orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ of $\mathbb{R}^3$, the dual operation has the following effect. Consider again the bivector $\mathsf{e}_2 \wedge \mathsf{e}_3$ which represents the plane spanned by e_1 and e_2 in its OPNS. The unit pseudoscalar of $\mathbb{R}^3$ and its inverse may be given as

$$I = \mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_3 \quad \text{and} \quad I^{-1} = \tilde{I} = \mathsf{e}_3 \wedge \mathsf{e}_2 \wedge \mathsf{e}_1 = -I.$$

Now, the dual of $\mathsf{e}_2 \wedge \mathsf{e}_3$ is

$$\begin{aligned}
(\mathsf{e}_2 \wedge \mathsf{e}_3)^* &= (\mathsf{e}_2 \wedge \mathsf{e}_3) \cdot I^{-1} \\
&= (\mathsf{e}_2 \wedge \mathsf{e}_3) \cdot (\mathsf{e}_3 \wedge \mathsf{e}_2 \wedge \mathsf{e}_1) \\
&= \mathsf{e}_2 \cdot \big(\mathsf{e}_3 \cdot (\mathsf{e}_3 \wedge \mathsf{e}_2 \wedge \mathsf{e}_1)\big),
\end{aligned}$$

where we used equation (1.12). We first evaluate the term within the outer brackets using equation (1.10).

$$\begin{aligned}
\mathsf{e}_3 \cdot (\mathsf{e}_3 \wedge \mathsf{e}_2 \wedge \mathsf{e}_1) &= (\mathsf{e}_3 \cdot \mathsf{e}_3)\,(\mathsf{e}_2 \wedge \mathsf{e}_1) - (\mathsf{e}_3 \cdot \mathsf{e}_2)\,(\mathsf{e}_3 \wedge \mathsf{e}_1) + (\mathsf{e}_3 \cdot \mathsf{e}_1)\,(\mathsf{e}_3 \wedge \mathsf{e}_2) \\
&= \mathsf{e}_2 \wedge \mathsf{e}_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathsf{e}_2 \wedge \mathsf{e}_3)^* &= \mathsf{e}_2 \cdot (\mathsf{e}_2 \wedge \mathsf{e}_1) \\
&= (\mathsf{e}_2 \cdot \mathsf{e}_2)\mathsf{e}_1 - (\mathsf{e}_2 \cdot \mathsf{e}_1)\mathsf{e}_2 \\
&= \mathsf{e}_1.
\end{aligned}$$

This is a nice example to see that the dual of a blade gives a blade complementing the whole space. In this case

$$(\mathsf{e}_2 \wedge \mathsf{e}_3) \;\wedge\; (\mathsf{e}_2 \wedge \mathsf{e}_3)^* = I,$$

the unit pseudoscalar. With respect to the OPNS we have

$$\mathbb{NO}\big(\mathsf{e}_2 \wedge \mathsf{e}_3\big) \oplus \mathbb{NO}\big((\mathsf{e}_2 \wedge \mathsf{e}_3)^*\big) = \mathbb{R}^3.$$

It is now also clear that the relation between the OPNS and IPNS is the duality. For example, we have seen before that

$$\mathbb{NO}\big(\mathsf{e}_2 \wedge \mathsf{e}_3\big) = \mathbb{R}^3 \ominus \mathbb{NO}\big(\mathsf{e}_1\big) = \mathbb{NI}\big(\mathsf{e}_1\big).$$

Since $\mathsf{e}_1 = (\mathsf{e}_2 \wedge \mathsf{e}_3)^*$ we have

$$\mathbb{NO}\big(\mathsf{e}_2 \wedge \mathsf{e}_3\big) = \mathbb{NI}\big((\mathsf{e}_2 \wedge \mathsf{e}_3)^*\big).$$

In general we have for some blade $A_{\langle k \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$

$$\mathbb{NO}\big(A_{\langle k \rangle}\big) = \mathbb{NI}\big(A^*_{\langle k \rangle}\big). \tag{1.22}$$

1.1.9 Geometric Interpretation of the IPNS

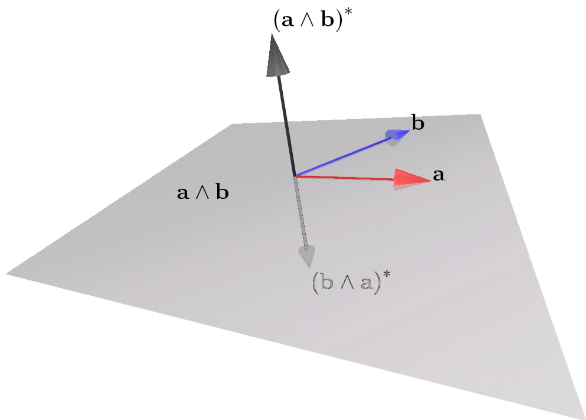


Figure 1.3: Dual of plane represented by bivector $\mathbf{a} \wedge \mathbf{b}$.

We have already seen that the IPNS of some vector $\mathbf{n} \in \mathbb{R}^3$ is a plane through the origin, whereby $\mathbf{n}$ is the plane's normal. With respect to the dual operation, it was shown in the previous section that the normal of a plane spanned by $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, is given by $(\mathbf{a} \wedge \mathbf{b})^*$. Suppose that $\mathbf{n} = (\mathbf{a} \wedge \mathbf{b})^*$. The side of the plane $\mathbf{a} \wedge \mathbf{b}$ from which the normal $\mathbf{n}$ sticks out from is usually regarded as the "front"-side of the plane. Thus, a bivector represents a *sided* plane. For example, the normal $\mathbf{m}$ of $\mathbf{b} \wedge \mathbf{a}$ is given by

$$\mathbf{m} = (\mathbf{b} \wedge \mathbf{a})^* = -(\mathbf{a} \wedge \mathbf{b})^* = -\mathbf{n}.$$

Hence, the plane represented by $\mathbf{b} \wedge \mathbf{a}$ consists of the same subspace in $\mathbb{R}^3$ as the plane represented by $\mathbf{a} \wedge \mathbf{b}$, but their front-sides point in opposite directions. This situation is shown in figure 1.3. This also shows the relation between the vector cross product and the outer product:

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a} \wedge \mathbf{b})^*.$$

Aside. Note that the idea of a plane normal vector does only work in $\mathbb{R}^3$. In any dimension higher than three the set of vectors perpendicular to one vector spans a higher dimensional space than a plane. Nevertheless, a bivector always describes a plane, independent of the dimension it is embedded in.

Now that we are happy that a vector in $\mathbb{R}^3$ represents a plane with respect to its IPNS, we can ask what the IPNS of blades of higher grade is. Consider the non-zero bivector $\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^3)$. In order to give its IPNS we have to find which vectors $\mathbf{x} \in \mathbb{R}^3$ satisfy $\mathbf{x} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$. With the help of equation (1.10) we find

$$\mathbf{x} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{x} \cdot \mathbf{a})\, \mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\, \mathbf{a}.$$

Since we assumed that $\mathbf{a} \wedge \mathbf{b} \neq 0$, $\mathbf{a}$ and $\mathbf{b}$ have to be linearly independent. Therefore, the above expression can only become zero if and only if

$$\mathbf{x} \cdot \mathbf{a} = 0 \quad \text{and} \quad \mathbf{x} \cdot \mathbf{b} = 0.$$

Geometrically this means that $\mathbf{x}$ has to lie on the plane represented by $\mathbf{a}$ *and* on the plane represented by $\mathbf{b}$, in their IPNS. Hence, $\mathbf{x}$ lies on the intersection of the two planes represented by $\mathbf{a}$ and $\mathbf{b}$. This shows that the outer product of two vectors represents the intersection of their separately represented geometric entities. In terms of sets this reads

$$\mathbb{NI}(\mathbf{a} \wedge \mathbf{b}) = \mathbb{NI}(\mathbf{a}) \ \cap \ \mathbb{NI}(\mathbf{b}). \tag{1.23}$$

Such an intersection line also has an orientation, which in this case is given by $(\mathbf{b} \wedge \mathbf{a})^*$.

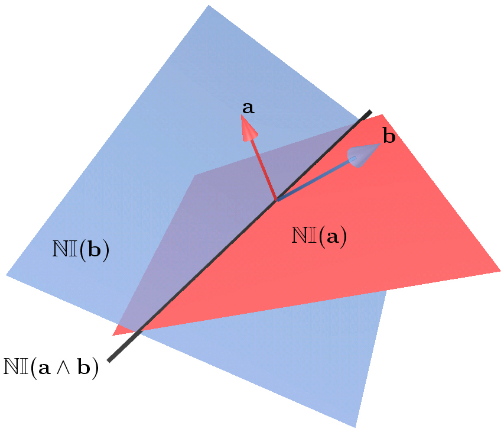


Figure 1.4: Intersection of two planes in terms of IPNS.

> **Aside.** Note that in $\mathbb{R}^3$ we cannot represent two parallel but not identical planes through the IPNS of two vectors, since all such planes go through the origin.

The last type of blade we can discuss in $\mathbb{R}^3$ with respect to its IPNS is a 3-blade, or *trivector*. As we have seen already a trivector $A_{\langle 3 \rangle} \in \mathcal{C}\ell(\mathbb{R}^3)$ is a pseudoscalar and thus

$$A_{\langle 3 \rangle} = \|A_{\langle 3 \rangle}\|\, I,$$

where I is the unit-pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^3)$. Let $A_{\langle 3 \rangle}$ be given by

$$A_{\langle 3 \rangle} := \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}.$$

If $A_{\langle 3 \rangle} \neq 0$ then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. In order to find the IPNS of $A_{\langle 3 \rangle}$, we need to find which vectors $\mathbf{x}$ satisfy $\mathbf{x} \cdot A_{\langle 3 \rangle} = 0$. Using again equation (1.10) it follows

$$
\begin{aligned}
\mathbf{x} \cdot A_{\langle 3 \rangle} = & \quad (\mathbf{x} \cdot \mathbf{a})\,(\mathbf{b} \wedge \mathbf{c}) \\
- & \quad (\mathbf{x} \cdot \mathbf{b})\,(\mathbf{a} \wedge \mathbf{c}) \\
+ & \quad (\mathbf{x} \cdot \mathbf{c})\,(\mathbf{a} \wedge \mathbf{b}).
\end{aligned}
$$

The bivectors $(\mathbf{b} \wedge \mathbf{c})$, $(\mathbf{a} \wedge \mathbf{c})$ and $(\mathbf{a} \wedge \mathbf{b})$ are linearly independent and thus $\mathbf{x} \cdot A_{\langle 3 \rangle} = 0$ if and only if

$$\mathbf{x} \cdot \mathbf{a} = 0 \quad \text{and} \quad \mathbf{x} \cdot \mathbf{b} = 0 \quad \text{and} \quad \mathbf{x} \cdot \mathbf{c} = 0.$$

Geometrically this means that $\mathbf{x} \cdot A_{\langle 3 \rangle} = 0$ if and only if $\mathbf{x}$ lies on the intersection of the three planes represented by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Since all planes represented through the IPNS of vectors pass through the origin, the only point all three planes can meet in is the origin. Hence, the only solution for $\mathbf{x}$ to $\mathbf{x} \cdot A_{\langle 3 \rangle} = 0$ is the trivial solution $\mathbf{x} = \mathbf{0} \in \mathbb{R}^3$. Figure 1.5 illustrates this.

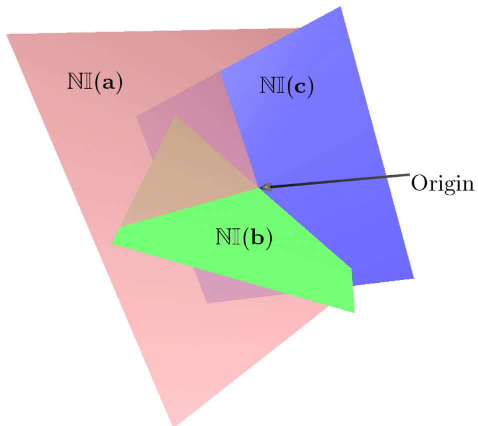


Figure 1.5: Intersection of three planes in terms of IPNS.

1.1.10 The Meet Operation

We have seen that we can intersect subspaces quite easily, if they are represented through the IPNS of blades. For example, two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ represent two planes in their IPNS. The intersection of these two planes is simply represented by $\mathbf{a} \wedge \mathbf{b}$. (Recall figure 1.4) The question we would like to answer in this section is: *is there an operation that evaluates the intersection of subspaces represented through the OPNS of blades?*

The short answer is: *yes*. The longer answer will follow now. First we need to remember how the OPNS and IPNS are connected. Given a bivector $\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^3)$ representing a plane in its OPNS, we can find the respective representation of the plane in term of the IPNS by taking the dual of the bivector. Suppose that $\mathbf{c} \in \mathbb{R}^3$ is given by $\mathbf{c} = (\mathbf{a} \wedge \mathbf{b})^*$, then

$$\mathbb{NI}(\mathbf{c}) = \mathbb{NI}\big((\mathbf{a} \wedge \mathbf{b})^*\big) = \mathbb{NO}\big(\mathbf{a} \wedge \mathbf{b}\big).$$

Using a so far unproven property of the inner product (equation (1.31)), we can also write

$$\begin{aligned}\mathbf{c}\cdot I &= (\mathbf{a}\wedge\mathbf{b})^*\cdot I\\ &= \big((\mathbf{a}\wedge\mathbf{b})\cdot I^{-1}\big)\cdot I\\ &= (\mathbf{a}\wedge\mathbf{b})\,(I^{-1}\cdot I)\\ &= \mathbf{a}\wedge\mathbf{b},\end{aligned}$$

where I is again the unit pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^3)$. That means, in order to transform an IPNS representation into an OPNS representation, we have to multiply with the unit pseudoscalar, a kind of "inverse" dual. In terms of sets,

$$\mathbb{NO}\big(\mathbf{a}\wedge\mathbf{b}\big) = \mathbb{NO}\big(\mathbf{c}\cdot I\big) = \mathbb{NI}\big(\mathbf{c}\big).$$

Now we can see how to express the intersection of two subspaces in terms of the OPNS of two blades. Suppose $\mathbf{a}_1\wedge\mathbf{a}_2,\ \mathbf{b}_1\wedge\mathbf{b}_2 \in \mathcal{C}\ell(\mathbb{R}^3)$ represent two planes in terms of their OPNS. Let their respective normals be denoted by $\mathbf{n}_a = (\mathbf{a}_1\wedge\mathbf{a}_2)^*$ and $\mathbf{n}_b = (\mathbf{b}_1\wedge\mathbf{b}_2)^*$. Then in terms of the IPNS the intersection of the two planes is given by $\mathbf{n}_a\wedge\mathbf{n}_b$. As we have seen above, the corresponding expression of the intersection line in terms of the OPNS is simply $(\mathbf{n}_a\wedge\mathbf{n}_b)\cdot I$. Substituting now for $\mathbf{n}_a$ and $\mathbf{n}_b$ gives,

$$\big[(\mathbf{a}_1\wedge\mathbf{a}_2)^* \ \wedge\ (\mathbf{b}_1\wedge\mathbf{b}_2)^*\big]\cdot I.$$

This is actually not quite the general intersection operation we were looking for, but it is already pretty good and is thus given its own name: the *regressive* product. Here is the proper definition.

Let $A, B \in \mathcal{C}\ell(\mathbb{R}^n)$ be two arbitrary multivectors and let I denote the unit pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^n)$. The *regressive* product is denoted by $\triangledown$ and is defined as

$$A\triangledown B := \big[A^*\ \wedge\ B^*\big]\cdot I. \tag{1.24}$$

For the above example this means that given the bivectors $\mathbf{a}_1\wedge\mathbf{a}_2$ and $\mathbf{b}_1\wedge\mathbf{b}_2$, representing two planes in their OPNS, the intersection of these planes in the OPNS is given by

$$(\mathbf{a}_1\wedge\mathbf{a}_2)\ \triangledown\ (\mathbf{b}_1\wedge\mathbf{b}_2).$$

Unfortunately, their is a problem. Let $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ again denote an orthonormal basis of $\mathbb{R}^3$. Now suppose we wanted to find the intersection of a line represented by e_2 and a plane represented by $\mathsf{e}_2\wedge\mathsf{e}_3$, through their OPNS. We see immediately that since e_2 is also contained in the bivector $\mathsf{e}_2\wedge\mathsf{e}_3$, the line is completely contained within the plane and thus their intersection should be the line e_2 itself. However, the regressive product gives

$$\begin{aligned}\mathsf{e}_2\,\triangledown\,(\mathsf{e}_2\wedge\mathsf{e}_3) &= \big[\mathsf{e}_2^*\wedge(\mathsf{e}_2\wedge\mathsf{e}_3)^*\big]\cdot I\\ &= \big[(\mathsf{e}_1\wedge\mathsf{e}_3)\wedge\mathsf{e}_1\big]\cdot I\\ &= \big[-(\mathsf{e}_1\wedge\mathsf{e}_1)\wedge\mathsf{e}_3\big]\cdot I\\ &= 0,\end{aligned}$$

where I is the pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^3)$. The problem is that the line $\mathbb{NO}(\mathsf{e}_2)$ and the plane $\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3)$ live in a 2d-subspace of $\mathbb{R}^3$ spanned by e_2 and e_3. The dimension e_1 is of no importance for the evaluation of their intersection. Suppose now that we work in the subalgebra $\mathcal{C}\ell(\mathbb{R}^2) \subset \mathcal{C}\ell(\mathbb{R}^3)$, where $\{\mathsf{e}_2, \mathsf{e}_3\}$ give an orthonormal basis of $\mathbb{R}^2$. Then the respective unit pseudoscalar is $I = \mathsf{e}_2 \wedge \mathsf{e}_3$ and $I^{-1} = \mathsf{e}_3 \wedge \mathsf{e}_2$, and we obtain

$$\mathsf{e}_2^* = -\mathsf{e}_3 \quad \text{and} \quad (\mathsf{e}_2 \wedge \mathsf{e}_3)^* = 1.$$

Hence, the regressive product now gives

$$\begin{aligned} \mathsf{e}_2 \,\triangledown\, (\mathsf{e}_2 \wedge \mathsf{e}_3) &= \big[\mathsf{e}_2^* \wedge (\mathsf{e}_2 \wedge \mathsf{e}_3)^*\big] \cdot I \\ &= \big[-\mathsf{e}_3 \wedge 1\big] \cdot I \\ &= -\mathsf{e}_3 \cdot I \\ &= \mathsf{e}_2, \end{aligned}$$

which is what we want. This shows that the regressive product works, if we evaluate it in the correct subalgebra. This notion is captured in the general intersection operation: the *meet*.

The meet is basically the regressive product where the pseudoscalar is chosen appropriately. "Appropriately" means that instead of the pseudoscalar of the whole space, the pseudoscalar of the space spanned by the two blades of which the meet is to be evaluated, is used. This introduces the concept of the *join*.

Given two blades $A_{\langle k\rangle}, B_{\langle l\rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, then their join is a *unit* blade $J \in \mathcal{C}\ell(\mathbb{R}^n)$ such that

$$\mathbb{NO}(J) = \mathbb{NO}(A_{\langle k\rangle}) \;\oplus\; \mathbb{NO}(B_{\langle l\rangle}).$$

The join is sometimes also written as an operator, denoted by $\dot\wedge$. For example, the join of e_2 and $\mathsf{e}_2 \wedge \mathsf{e}_3$ is simply

$$\mathsf{e}_2 \;\dot\wedge\; (\mathsf{e}_2 \wedge \mathsf{e}_3) = \mathsf{e}_2 \wedge \mathsf{e}_3,$$

since $\|\mathsf{e}_2 \wedge \mathsf{e}_3\| = 1$ and

$$\mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3) = \mathbb{NO}(\mathsf{e}_2) \;\oplus\; \mathbb{NO}(\mathsf{e}_2 \wedge \mathsf{e}_3).$$

> **Aside.** Note that this definition of the join does not fix the sign of J. This is just as for the unit pseudoscalar I, where we only demanded that its magnitude is unity, but we did not say anything about its sign. We will not discuss this problem further apart from noting that it becomes irrelevant when working in projective spaces.

We can now define the meet in terms of the join. Let $A_{\langle k\rangle}, B_{\langle l\rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$ and let $J = A_{\langle k\rangle} \dot\wedge B_{\langle l\rangle}$ be their join. Then the *meet* of $A_{\langle k\rangle}$ and $B_{\langle l\rangle}$ is denoted by $\vee$ and defined as

$$A_{\langle k\rangle} \vee B_{\langle l\rangle} := \Big[\big(A_{\langle k\rangle} \cdot J^{-1}\big) \;\wedge\; \big(B_{\langle l\rangle} \cdot J^{-1}\big)\Big] \cdot J. \tag{1.25}$$

In terms of sets this is

$$\mathbb{NO}\big(A_{\langle k\rangle} \vee B_{\langle l\rangle}\big) = \mathbb{NO}\big(A_{\langle k\rangle}\big) \;\cap\; \mathbb{NO}\big(B_{\langle l\rangle}\big).$$

Note that the meet is only defined for blades and it becomes the regressive product, if the join is the pseudoscalar. Equation (1.25) can also be simplified to read

$$A_{\langle k\rangle} \vee B_{\langle l\rangle} = \big(A_{\langle k\rangle} \cdot J^{-1}\big) \cdot B_{\langle l\rangle}. \tag{1.26}$$

1.1.11 The Geometric Product

We have already seen a lot of features of Geometric algebra. However, so far, we managed to avoid the actual algebra product, the *geometric product*. This product will be discussed in more detail in the Clifford algebra introduction later on. At this point only some basic features are introduced.

The formula most often shown right in the beginning of a Geometric algebra introduction is

$$\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{1.27}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are two vectors, and juxtaposition of two vectors, as in $\mathbf{a}\,\mathbf{b}$, denotes the geometric product. **It is important** to note that this equation is **only** valid for vectors, **not** for blades or multivectors in general. It might at first seem strange to add a scalar $(\mathbf{a} \cdot \mathbf{b})$ and a bivector $(\mathbf{a} \wedge \mathbf{b})$, but they are just different elements of the Geometric algebra. This is just like for complex numbers, where a real and an imaginary part are added.

A somewhat more general form of equation (1.27) is

$$\mathbf{a}\,B_{\langle l \rangle} = \mathbf{a} \cdot B_{\langle l \rangle} + \mathbf{a} \wedge B_{\langle l \rangle}, \tag{1.28}$$

with $B_{\langle l \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$ and $l > 0$. For $l = 0$, ie $B_{\langle l \rangle}$ a scalar, we have

$$\mathbf{a}\,B_{\langle 0 \rangle} = \mathbf{a} \wedge B_{\langle 0 \rangle}.$$

In general we always have for a scalar $\alpha \in \mathbb{R}$ and a multivector $A \in \mathcal{C}\ell(\mathbb{R}^n)$ that their inner product is **identically** zero,

$$\alpha \cdot A \equiv 0.$$

This turns out to be a necessary definition to keep the system of operations in Geometric algebra self-consistent.

The geometric product is associative and distributive but in general not commutative. That is, for multivectors $A, B, C \in \mathcal{C}\ell(\mathbb{R}^n)$

$$\begin{aligned}
(A\,B)\,C &= A\,(B\,C),\\
A\,(B + C) &= (A\,B) + (A\,C),\\
(B + C)\,A &= (B\,A) + (C\,A),\\
A\,B &\neq B\,A, \quad \text{in general.}
\end{aligned} \tag{1.29}$$

Two further useful properties of the geometric product are the following. Given two blades $A_{\langle k \rangle}, B_{\langle l \rangle} \in \mathcal{C}\ell(\mathbb{R}^n)$, then

$$\mathbb{NO}(A_{\langle k \rangle}) \cap \mathbb{NO}(B_{\langle l \rangle}) = \emptyset \iff A_{\langle k \rangle}\,B_{\langle l \rangle} = A_{\langle k \rangle} \wedge B_{\langle l \rangle}, \tag{1.30}$$

and

$$\left.\begin{aligned}
&\mathbb{NO}(A_{\langle k \rangle}) \subseteq \mathbb{NO}(B_{\langle l \rangle})\\
\text{or}\quad &\mathbb{NO}(B_{\langle l \rangle}) \subseteq \mathbb{NO}(A_{\langle k \rangle})
\end{aligned}\right\} \iff A_{\langle k \rangle}\,B_{\langle l \rangle} = A_{\langle k \rangle} \cdot B_{\langle l \rangle}. \tag{1.31}$$

Equation (1.31) for example implies that for some vector $\mathbf{a} \in \mathbb{R}^n$,

$$\mathbf{a}^* \cdot I = (\mathbf{a} \cdot I^{-1}) \cdot I = (\mathbf{a}\,I^{-1})\,I = \mathbf{a}\,(I^{-1}\,I) = \mathbf{a},$$

where I is the pseudoscalar of $\mathcal{C}\ell(\mathbb{R}^n)$.

1.1.12 Reflection

So far we have seen how to construct linear subspaces using the outer product and to subtract linear subspaces from one another using the inner product. We also now know how to intersect linear subspaces using the meet and how to form their union with the join. We now would like to operate on subspaces while keeping their dimensionality unchanged. For example, rotating a line results in another line, not in a point or a plane. An operation on a blade that does not change its grade, is called *grade preserving*.

Without much further ado, we will look at such a grade preserving operation. Let $\mathbf{a}, \mathbf{n} \in \mathbb{R}^n$ denote two vectors, whereby $\|\mathbf{n}\| = 1$. Also write $\mathbf{a} = \mathbf{a}^{\|} + \mathbf{a}^{\perp}$, where $\mathbf{a}^{\|}$ is the component of $\mathbf{a}$ parallel and $\mathbf{a}^{\perp}$ the component perpendicular to $\mathbf{n}$. Note that the following calculation is valid for all dimensions $n \geq 2$ of the vector space.

$$\begin{aligned} \mathbf{n}\,\mathbf{a}\,\mathbf{n} &= (\mathbf{n}\,\mathbf{a})\,\mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{a} + \mathbf{n} \wedge \mathbf{a})\,\mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{a})\,\mathbf{n} + (\mathbf{n} \wedge \mathbf{a}) \cdot \mathbf{n} + \underbrace{(\mathbf{n} \wedge \mathbf{a}) \wedge \mathbf{n}}_{=0}. \end{aligned}$$

So far we only applied the associativity of the geometric product and equation (1.27). Using equation (1.10) we see that

$$(\mathbf{n} \wedge \mathbf{a}) \cdot \mathbf{n} = (\mathbf{a} \cdot \mathbf{n})\,\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\,\mathbf{a}.$$

Hence,

$$\begin{aligned} \mathbf{n}\,\mathbf{a}\,\mathbf{n} &= (\mathbf{n} \cdot \mathbf{a})\,\mathbf{n} + (\mathbf{a} \cdot \mathbf{n})\,\mathbf{n} - \underbrace{(\mathbf{n} \cdot \mathbf{n})}_{=1}\,\mathbf{a} \\ &= 2\,(\mathbf{n} \cdot \mathbf{a})\,\mathbf{n} - \mathbf{a}. \end{aligned}$$

Clearly we have $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{a}^{\|}$, and since $\mathbf{a}^{\|}$ is the component of $\mathbf{a}$ parallel to $\mathbf{n}$, we can also write $\mathbf{a}^{\|} = \|\mathbf{a}^{\|}\|\,\mathbf{n}$. Thus,

$$\begin{aligned} \mathbf{n}\,\mathbf{a}\,\mathbf{n} &= 2\,(\mathbf{n} \cdot \mathbf{a}^{\|})\,\mathbf{n} - \mathbf{a} \\ &= 2\,\|\mathbf{a}^{\|}\|\,\mathbf{n} - \mathbf{a} \\ &= 2\,\mathbf{a}^{\|} - \mathbf{a}\| - \mathbf{a}^{\perp} \\ &= \mathbf{a}^{\|} - \mathbf{a}^{\perp}. \end{aligned}$$

That is, the component of $\mathbf{a}$ perpendicular to $\mathbf{n}$ has been negated, while the parallel component $\mathbf{a}^{\|}$ remained unchanged. Geometrically this is a *reflection* of the vector $\mathbf{a}$ on the line through the origin with direction $\mathbf{n}$. This is illustrated in figure 1.6.

The really nice thing about this reflection operation is that it can be applied to any blade. For example, given a plane as bivector $A_{\langle 2 \rangle} \in \mathcal{C}\ell(\mathbb{R}^3)$, it can be reflected in a normalized vector $\mathbf{n} \in \mathbb{R}^3$ simply by evaluating $\mathbf{n}\,A_{\langle 2 \rangle}\,\mathbf{n}$. This is shown in figure 1.7.

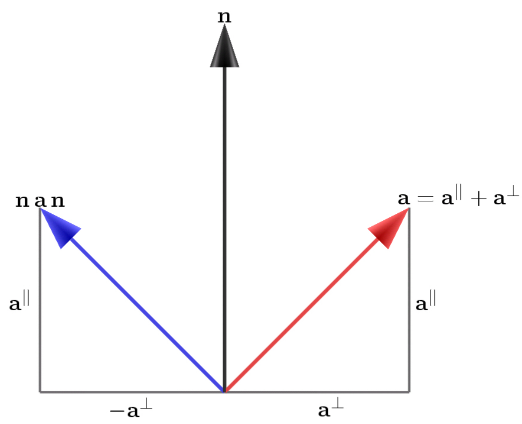


Figure 1.6: Reflection of vector $\mathbf{a}$ on vector $\mathbf{n}$.

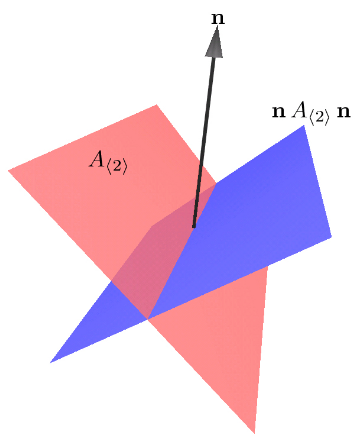


Figure 1.7: Reflection of bivector $A_{\langle 2\rangle}$ on vector $\mathbf{n}$.

Let $A_{\langle 2\rangle} = \mathbf{a}_1 \wedge \mathbf{a}_2$ with $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$, then it may in fact be shown that

$$\mathbf{n}\,A_{\langle 2\rangle}\,\mathbf{n} = \big(\mathbf{n}\,\mathbf{a}_1\,\mathbf{n}\big) \;\wedge\; \big(\mathbf{n}\,\mathbf{a}_2\,\mathbf{n}\big).$$

That is, the reflection of the outer product of two vectors, is the outer product of the separately reflected vectors. By the way, this property is also called *outer-morphism*, not to be confused with auto-morphism.

A blade may also be reflected on another blade. Figure 1.8 shows the reflection of a vector $\mathbf{a} \in \mathbb{R}^3$ on a bivector $N_{\langle 2\rangle} \in \mathcal{C}\ell(\mathbb{R}^3)$ by evaluating $N_{\langle 2\rangle}\mathbf{a}N_{\langle 2\rangle}$. This operation again results in

$$N_{\langle 2\rangle}\mathbf{a}N_{\langle 2\rangle} = \mathbf{a}^{\parallel} - \mathbf{a}^{\perp},$$

where $\mathbf{a}^{\parallel}$ and $\mathbf{a}^{\perp}$ are this time the parallel and perpendicular components of $\mathbf{a}$ with respect to $N_{\langle 2\rangle}$.

The reflection operation is in fact the only operation we will ever be using in Geometric algebra. Any other operation needed will be obtained by combining a number of different reflections. In Euclidean space this confines us in fact to reflections and rotations about axes that

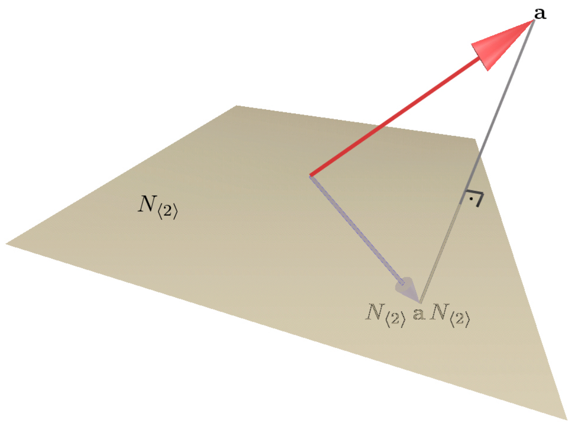


Figure 1.8: Reflection of vector $\mathbf{a}$ on bivector $N_{\langle 2 \rangle}$.

pass through the origin, as will be shown in the next section. To extend the set of available operations Euclidean space will have to be embedded in other spaces, which will be discussed later on.

1.1.13 Rotation

Reflections with respect to a normalized vector $\mathbf{n}$ are always reflections on a line with direction $\mathbf{n}$, passing through the origin. It may be shown that two consecutive reflections on different, normalized vectors $\mathbf{n}$ and $\mathbf{m}$ are equivalent to a rotation of twice the angle between $\mathbf{n}$ and $\mathbf{m}$.

Figure 1.9 shows such a setup in 3d-Euclidean space. The normalized vectors $\mathbf{n}, \mathbf{m} \in \mathbb{R}^3$ enclose an angle $\angle(\mathbf{n}, \mathbf{m}) = \theta$ and define a rotation plane through their outer product $\mathbf{n} \wedge \mathbf{m}$. Reflecting a vector $\mathbf{a} \in \mathbb{R}^3$ first on $\mathbf{n}$ and then on $\mathbf{m}$, rotates the component of $\mathbf{a}$ that lies in the rotation plane by 2θ. The component of $\mathbf{a}$ perpendicular to the rotation plane remains unchanged.

The rotation of vector $\mathbf{a}$ in the plane $\mathbf{n} \wedge \mathbf{m}$ by an angle 2θ may then be written as

$$\mathbf{b} = \mathbf{m}\,\mathbf{n}\,\mathbf{a}\,\mathbf{n}\,\mathbf{m}. \tag{1.32}$$

From the definition of the geometric product we find that

$$\mathbf{m}\,\mathbf{n} = \mathbf{m} \cdot \mathbf{n} + \mathbf{m} \wedge \mathbf{n}$$

and also

$$\mathbf{n}\,\mathbf{m} = \mathbf{n} \cdot \mathbf{m} + \mathbf{n} \wedge \mathbf{m} = \mathbf{n} \cdot \mathbf{m} + (\mathbf{m} \wedge \mathbf{n})\tilde{}.$$

Since the reverse of a scalar is still the same scalar it follows

$$\mathbf{m}\,\mathbf{n} = (\mathbf{n}\,\mathbf{m})\tilde{}.$$

Equation (1.32) may therefore also be written more succinctly as

$$\mathbf{b} = R\,\mathbf{a}\,\tilde{R}, \quad \text{with } R := \mathbf{m}\,\mathbf{n}. \tag{1.33}$$

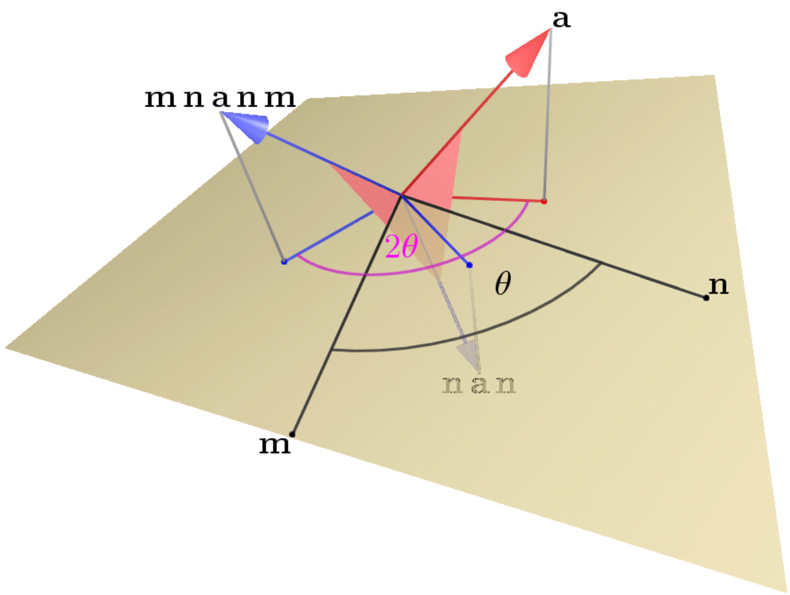


Figure 1.9: Rotation of vector $\mathbf{a}$ by consecutive reflections of $\mathbf{a}$ on $\mathbf{n}$ and $\mathbf{m}$.

Since applying R as above has the effect of a rotation, R is called a *rotor*. Note that a rotor has to satisfy the equation

$$R\,\tilde{R} = 1,$$

because it would otherwise also scale the entity it is applied to. We can actually recognize this as something familiar, by expanding R as

$$\begin{aligned} R &= \mathbf{m}\,\mathbf{n} \\ &= \mathbf{m}\cdot\mathbf{n} + \mathbf{m}\wedge\mathbf{n} \\ &= \cos\theta + \sin\theta\, U_{\langle 2\rangle}, \end{aligned} \tag{1.34}$$

where $\theta = \angle(\mathbf{m},\mathbf{n})$ and $U_{\langle 2\rangle}$ is the normalized version of $\mathbf{m}\wedge\mathbf{n}$, ie

$$U_{\langle 2\rangle} := \frac{\mathbf{m}\wedge\mathbf{n}}{\|\mathbf{m}\wedge\mathbf{n}\|}.$$

From equation (1.15) we know that

$$U_{\langle 2\rangle}\cdot U_{\langle 2\rangle} = (-1)^{2(2-1)/2}\,\|U_{\langle 2\rangle}\|^2 = -1.$$

Since $U_{\langle 2\rangle}$ squares to -1, the expression for R in equation (1.34) is similar to that of a complex number z in the polar representation

$$z = r\,(\cos\theta + \mathrm{i}\,\sin\theta),$$

where $\mathsf{i} = \sqrt{-1}$ represents the imaginary unit and $r \in \mathbb{R}$ is the radius. For complex numbers it is well known that the above expression can also be written as

$$z = r\ \exp(\mathsf{i}\,\theta).$$

The definition of the exponential function can be extended to Geometric algebra, and it can be shown that the Taylor series of $\exp(\theta\, U_{\langle 2\rangle})$ does indeed converge to

$$\exp\big(\theta\, U_{\langle 2\rangle}\big) = \cos\theta + \sin\theta\, U_{\langle 2\rangle} = R. \tag{1.35}$$

It turns out that $R = \exp(\theta U_{\langle 2\rangle})$ actually represents a clockwise rotation by an angle 2θ in the plane $U_{\langle 2\rangle}$. The term "clockwise" only makes really sense in 3d-space. Here it means clockwise relative to the rotation axis given by $U^*_{\langle 2\rangle}$. If we want to represent a mathematically positive, ie *anti*-clockwise, rotation about an angle θ, within the plane $U_{\langle 2\rangle}$, we need to write the corresponding rotor as

$$R = \exp\big(-\tfrac{\theta}{2}\, U_{\langle 2\rangle}\big). \tag{1.36}$$

Just as for reflections, a rotor represents a rotation in any dimension. A rotor can also rotate any blade. That is, with the same rotor we can rotate vectors, bivectors, etc. It turns out that for a rotor we also have an outer-morphism. This means that given a blade $A_{\langle k\rangle} = \bigwedge_{i=1}^{k} \mathbf{a}_i$, with $\{\mathbf{a}_i\} \subset \mathbb{R}^n$, and a rotor R, we can expand the expression $RA_{\langle k\rangle}\tilde{R}$ as

$$R\,A_{\langle k\rangle}\,\tilde{R} = \big(R\,\mathbf{a}_1\,\tilde{R}\big) \wedge \big(R\,\mathbf{a}_2\,\tilde{R}\big) \wedge \ldots \wedge \big(R\,\mathbf{a}_k\,\tilde{R}\big). \tag{1.37}$$

Hence, the rotation of the outer product of a number of vectors is the same as the outer product of a number of rotated vectors.

1.2 Clifford Algebra

In the previous section we mainly discussed the geometric interpretation of elements of Geometric algebra. However, we did not say very much about the algebra itself. We will do this now, and since we are in the following mainly interested in algebraic aspects, we will talk about Clifford algebra instead of Geometric algebra. Recall that these are just two names for the same thing. The only difference is that when we talk about Geometric algebra we would like to emphasize the geometric interpretation of the elements of that algebra. Note that in the following we will not be a hundred percent mathematically rigorous. For a ”proper”, pure mathematical introduction see for example [15, 31, 17, 32, 25]. An even more abstract but very interesting approach to Clifford algebra can be found in [36, 37].

William K. Clifford (1845-1879) introduced what we now call Geometric or Clifford Algebra, in a paper entitled ”On the classification of geometric algebras,” [6]. He realized (as Grassmann did) that Grassmann's exterior algebra and Hamilton's quaternions can be brought into the same algebra by a slight change of the exterior product. With this new product, which we will call the geometric product, the multiplication rules of the quaternions follow directly from combinations of basis vectors (more details later), while Grassmann's exterior algebra is not lost. Furthermore, complex numbers and the Pauli matrices, as used in Quantum mechanics, have also a natural representation in Clifford algebra.

1.2.1 The Geometric Product Revisited

Let $\mathbb{V}^n$ be some n-dimensional vector space over a field $\mathbb{F}$, where n is finite[4]. Furthermore, let a scalar product, denoted by $*$, be defined on $\mathbb{V}^n$. That is, for two elements $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$,

$$\mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a} \in \mathbb{F}.$$

Note that a Hilbert space also satisfies these properties.

The Clifford algebra over $\mathbb{V}^n$, denoted by $\mathcal{C}\ell(\mathbb{V}^n)$ or simply $\mathcal{C}\ell_n$, is an algebra that also contains the elements of $\mathbb{V}^n$ and the field $\mathbb{F}$. The algebra product is called the *Clifford* or *geometric* product and will for the moment be denoted by $\circ$. Later on the geometric product will be represented by the juxtaposition of two elements. At the moment an explicit symbol is hoped to further the reader's understanding.

In order to clarify what we mean by algebra, here are all the axioms of $\mathcal{C}\ell(\mathbb{V}^n)$. First of all, the elements of some $\mathcal{C}\ell(\mathbb{V}^n)$, which will be called *multivectors*, satisfy the axioms of a vector space over the field $\mathbb{F}$.

1. **Multivector addition.** For any two elements $A, B \in \mathcal{C}\ell(\mathbb{V}^n)$ there exists an element $C = A + B \in \mathcal{C}\ell(\mathbb{V}^n)$, their sum.

2. **Scalar multiplication.** For any element $A \in \mathcal{C}\ell(\mathbb{V}^n)$ and any scalar $\alpha \in \mathbb{F}$, there exists an element $\alpha A \in \mathcal{C}\ell(\mathbb{V}^n)$, the α-multiple of A.

[4]We will only discuss finite dimensional Clifford algebras. Infinite dimensional Clifford algebras pose some additional problems which we would like to avoid here.

Now the axioms of the vector space. In the following let $\mathcal{C}\ell_n$ denote $\mathcal{C}\ell(\mathbb{V}^n)$. Also let $A, B, C \in \mathcal{C}\ell_n$ and $\alpha, \beta \in \mathbb{F}$.

1. **Associativity** of multivector addition

$$(A+B)+C = A+(B+C).$$

2. **Commutativity**

$$A+B = B+A.$$

3. **Identity element of addition.** There exists an element $0 \in \mathcal{C}\ell_n$, the zero element, such that $A+0=A$.

4. **Associativity** of scalar multiplication

$$\alpha(\beta A) = (\alpha\beta)A.$$

5. **Commutativity** of scalar multiplication

$$\alpha A = A\alpha.$$

6. **Identity element of scalar multiplication.** The identity element $1 \in \mathbb{F}$ satisfies

$$1\,A = A.$$

7. **Distributivity** of multivector sums.

$$\alpha\,(A+B) = \alpha A + \alpha B.$$

8. **Distributivity** of scalar sums.

$$(\alpha+\beta)\,A = \alpha\,A + \beta\,A.$$

If we choose the field $\mathbb{F}$ to be the reals $\mathbb{R}$, then it follows from these axioms that for each $A \in \mathcal{C}\ell_n$ there exists an element $-A := (-1)A$ such that

$$A - A := A + (-A) = A + (-1)A = \big(1+(-1)\big)\,A = 0\,A = 0.$$

Now we come to the axioms of the algebra product, the geometric product. Again let $A, B, C \in \mathcal{C}\ell_n$ and $\alpha, \beta \in \mathbb{F}$.

1. **The algebra is closed** under the geometric product

$$(A \circ B) \in \mathcal{C}\ell_n.$$

2. **Associativity.**

$$(A \circ B) \circ C = A \circ (B \circ C).$$

3. **Distributivity.**

$$A \circ (B + C) = A \circ B + A \circ C \quad \text{and} \quad (B + C) \circ A = B \circ A + C \circ A.$$

4. **Scalar multiplication.**

$$\alpha \circ A = A \circ \alpha = \alpha A.$$

So far, all the axioms we gave simply define a fairly general algebra. What actually separates Clifford algebra from other algebras is its *defining equation*. We said before that $\mathbb{V}^n \subset \mathcal{C}\ell(\mathbb{V}^n)$, which is mathematically not quite rigorous but good enough to understand what is going on. The defining equation of Clifford algebra is that for all vectors $\mathbf{a} \in \mathbb{V}^n \subset \mathcal{C}\ell(\mathbb{V}^n)$ the following equation holds

$$\mathbf{a} \circ \mathbf{a} = \mathbf{a} * \mathbf{a} \ \in \mathbb{F}. \tag{1.38}$$

That is, the geometric product of a vector (*not* multivector in general) with itself maps to an element of the field $\mathbb{F}$. From now we will only consider Clifford algebras over the reals, ie we set $\mathbb{F} \equiv \mathbb{R}$.

In order to work with Clifford algebra we would also like to know whether the scalar product of two different vectors $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ can also be expressed in terms of the geometric product. Well, using the defining equation (1.38) we find

$$\begin{aligned} & (\mathbf{a}+\mathbf{b}) \circ (\mathbf{a}+\mathbf{b}) &&= (\mathbf{a}+\mathbf{b}) * (\mathbf{a}+\mathbf{b}) \\ \iff \quad & \mathbf{a}\circ\mathbf{a} + \mathbf{a}\circ\mathbf{b} + \mathbf{b}\circ\mathbf{a} + \mathbf{b}\circ\mathbf{b} &&= (\mathbf{a}*\mathbf{a}) + 2\,\mathbf{a}*\mathbf{b} + \mathbf{b}*\mathbf{b} \\ \iff \quad & \tfrac{1}{2}(\mathbf{a}\circ\mathbf{b} + \mathbf{b}\circ\mathbf{a}) &&= \mathbf{a}*\mathbf{b}. \end{aligned} \tag{1.39}$$

The expression $\frac{1}{2}(\mathbf{a} \circ \mathbf{b} + \mathbf{b} \circ \mathbf{a})$ is also called the *anti-commutator product*. We will also write this as

$$\mathbf{a} \barwedge \mathbf{b} := \tfrac{1}{2}(\mathbf{a} \circ \mathbf{b} + \mathbf{b} \circ \mathbf{a}), \quad \textbf{anti-commutator product}. \tag{1.40}$$

Similarly we can also define the *commutator product* as

$$\mathbf{a} \veebar \mathbf{b} := \tfrac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a}), \quad \textbf{commutator product}. \tag{1.41}$$

In the literature the commutator product of two multivectors $A, B \in \mathcal{C}\ell_n$ would usually be written as $[A, B]$ and the anti-commutator product as $\{A, B\}$. In this text we will use the symbols introduced above to emphasize the operator quality of these products. By applying the properties of the geometric product we can see immediately that the geometric product of two multivectors can be written as the sum of the commutator and anti-commutator product.

$$A \circ B = A \barwedge B + A \veebar B. \tag{1.42}$$

Usually vectors in $\mathbb{V}^n$ are expressed as linear combinations of a set $\{\mathsf{e}_1, \mathsf{e}_2, \ldots\}$ of orthonormal basis vectors of $\mathbb{V}^n$. However, in this formal setting we havn't even defined what we mean by "orthogonal". So let's do this now. Two vectors are said to be *orthogonal* iff

$$\mathbf{a} \barwedge \mathbf{b} = \mathbf{a} * \mathbf{b} = 0.$$

A set of n orthonormal vectors $\{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_n\} \subset \mathbb{V}^n$ therefore has the properties,

$$\mathsf{e}_i \overline{\times} \mathsf{e}_i = 1 \quad \text{and} \quad \mathsf{e}_i \overline{\times} \mathsf{e}_j = 0\,,\ i \neq j.$$

From this it also follows that for $i \neq j$,

$$\begin{aligned} \mathsf{e}_i \circ \mathsf{e}_j &= \mathsf{e}_i \overline{\times} \mathsf{e}_j + \mathsf{e}_i \underline{\times} \mathsf{e}_j = \mathsf{e}_i \underline{\times} \mathsf{e}_j\,, \\ \mathsf{e}_j \circ \mathsf{e}_i &= \mathsf{e}_j \overline{\times} \mathsf{e}_i + \mathsf{e}_j \underline{\times} \mathsf{e}_i = \mathsf{e}_j \underline{\times} \mathsf{e}_i\,, \end{aligned} \tag{1.43}$$

and since $A \underline{\times} B = -B \underline{\times} A$ by definition, we have $\mathsf{e}_i \circ \mathsf{e}_j = -\mathsf{e}_j \circ \mathsf{e}_i$. This is also one of the properties of the outer product which we introduced in equation (1.1). Now that we know the properties of the $\{\mathsf{e}_i\}$, we can take a first look at the geometric product of two general vectors. Let $\mathbf{a}, \mathbf{b} \in \mathcal{C}\ell(\mathbb{R}^2)$ be given by $\mathbf{a} = \alpha^i \mathsf{e}_i$ and $\mathbf{b} = \beta^i \mathsf{e}_i$, where $i \in \{1, 2\}$. Note that we use the Einstein summation convention here, which states that a superscript index repeated as a subscript index, or vice versa, implies a summation over the range of the index. In this case $\alpha^i \mathsf{e}_i \equiv \sum_{i=1}^{2} \alpha^i \mathsf{e}_i$.

$$\begin{aligned} \mathbf{a} \circ \mathbf{b} &= (\alpha^1 \mathsf{e}_1 + \alpha^2 \mathsf{e}_2)\,(\beta^1 \mathsf{e}_1 + \beta^2 \mathsf{e}_2) \\ &= \Big(\alpha^1 \beta^1\, \mathsf{e}_1 \circ \mathsf{e}_1 + \alpha^2 \beta^2\, \mathsf{e}_2 \circ \mathsf{e}_2\Big) + \Big(\alpha^1 \beta^2\, \mathsf{e}_1 \circ \mathsf{e}_2 + \alpha^2 \beta^1\, \mathsf{e}_2 \circ \mathsf{e}_1\Big) \\ &= \Big(\alpha^1 \beta^1 + \alpha^2 \beta^2\Big) + \Big(\alpha^1 \beta^2 - \alpha^2 \beta^1\Big)\, \mathsf{e}_1 \circ \mathsf{e}_2\,, \end{aligned} \tag{1.44a}$$

$$\begin{aligned} \mathbf{b} \circ \mathbf{a} &= (\beta^1 \mathsf{e}_1 + \beta^2 \mathsf{e}_2)\,(\alpha^1 \mathsf{e}_1 + \alpha^2 \mathsf{e}_2) \\ &= \Big(\beta^1 \alpha^1\, \mathsf{e}_1 \circ \mathsf{e}_1 + \beta^2 \alpha^2\, \mathsf{e}_2 \circ \mathsf{e}_2\Big) + \Big(\beta^1 \alpha^2\, \mathsf{e}_1 \circ \mathsf{e}_2 + \beta^2 \alpha^1\, \mathsf{e}_2 \circ \mathsf{e}_1\Big) \\ &= \Big(\alpha^1 \beta^1 + \alpha^2 \beta^2\Big) - \Big(\alpha^1 \beta^2 - \alpha^2 \beta^1\Big)\, \mathsf{e}_1 \circ \mathsf{e}_2. \end{aligned} \tag{1.44b}$$

We therefore see that

$$\mathbf{a} \overline{\times} \mathbf{b} = \tfrac{1}{2}(\mathbf{a} \circ \mathbf{b} + \mathbf{b} \circ \mathbf{a}) = \alpha^1 \beta^1 + \alpha^2 \beta^2 = \mathbf{a} * \mathbf{b}\,, \tag{1.45a}$$

$$\mathbf{a} \underline{\times} \mathbf{b} = \tfrac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a}) = \Big(\alpha^1 \beta^2 - \alpha^2 \beta^1\Big)\, \mathsf{e}_1 \circ \mathsf{e}_2 = \mathbf{a} \wedge \mathbf{b}. \tag{1.45b}$$

1.2.2 The Basis of $\mathcal{C}\ell_n$

The question still remains what the geometric algebra of a vector space is. Given a vector space $\mathbb{R}^n$ with an orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_n\}$ there are 2^n ways to combine the $\{\mathsf{e}_i\}$ with the geometric product such that no two of these products are linearly dependent. Each of these products is called a *basis blade*. Together they form the (algebraic) basis of $\mathcal{C}\ell_n(\mathbb{R}^n)$ denoted by $\mathbb{B}_n$. This will become more clear in the following example.

From now on we will write the geometric product again by juxtaposition of two elements. For example, the geometric product of $A, B \in \mathcal{C}\ell_n$ will no longer be written as $A \circ B$ but as $A\,B$.

Consider the vector space $\mathbb{R}^3$ with orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$. A set of linearly independent combinations of these basis elements using the geometric product is for example given by,

$$\mathbb{B}_3 := \{ \underbrace{1}_{\text{scalar}}, \underbrace{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3}_{\text{vectors}}, \underbrace{\mathsf{e}_2\mathsf{e}_3, \mathsf{e}_3\mathsf{e}_1, \mathsf{e}_1\mathsf{e}_2}_{\text{bivectors}}, \underbrace{\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3}_{\text{trivector}} \} \tag{1.46}$$

Recall that the geometric product is associative. Hence, we can write $(\mathsf{e}_1\mathsf{e}_2)\mathsf{e}_3$ simply as $\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3$. Also recall that $\mathsf{e}_i\mathsf{e}_j = -\mathsf{e}_j\mathsf{e}_i$ for $i \neq j$. Therefore, using a different order for the $\{\mathsf{e}_i\}$ in the basis blades can at most change the sign of the basis blades.

Given a basis $\mathbb{B}_n := \{E_i\}$ of some $\mathcal{C}\ell_n$, we can write a multivector explicitly as

$$A = \alpha^i E_i \; ; \quad i \in \{1, 2, \ldots, 2^n\}, \tag{1.47}$$

where we used the Einstein summation convention, which, as was already mentioned above, says that a superscript index repeated as a subscript index, or vice versa, within a product implies a sum over the range of the index. That is,

$$\sum_{i=1}^{2^n} \alpha^i E_i \equiv \alpha^i E_i \; ; \quad i \in \{1, 2, \ldots, 2^n\}. \tag{1.48}$$

In our $\mathcal{C}\ell_3$ example the elements of $\mathbb{B}_3$ may be defined as

$$\begin{aligned} &E_1 := 1, \\ &E_2 := \mathsf{e}_1,\ E_3 := \mathsf{e}_2,\ E_4 := \mathsf{e}_3, \\ &E_5 := \mathsf{e}_2\mathsf{e}_3,\ E_6 := \mathsf{e}_3\mathsf{e}_1,\ E_7 := \mathsf{e}_1\mathsf{e}_2, \\ &E_8 := \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3. \end{aligned} \tag{1.49}$$

Therefore, a general multivector in $\mathcal{C}\ell_3$ looks like this.

$$\begin{aligned} A &= \alpha^i E_i \\ &= \alpha^1 + \\ &\quad \alpha^2\, \mathsf{e}_1 + \alpha^3\, \mathsf{e}_2 + \alpha^4\, \mathsf{e}_3 + \\ &\quad \alpha^5\, \mathsf{e}_2\mathsf{e}_3 + \alpha^6\, \mathsf{e}_3\mathsf{e}_1 + \alpha^7\, \mathsf{e}_1\mathsf{e}_2 + \\ &\quad \alpha^8\, \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3. \end{aligned} \tag{1.50}$$

The *grade* of a basis blade is defined as the number of different e-elements the basis blade contains. Hence, the grade of $\mathsf{e}_1\mathsf{e}_2$ is 2 and the grade of $\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3$ is 3. Consequently the grade of the scalar 1 is zero. The basis blade of highest grade in a particular geometric algebra is called the *pseudoscalar* of that algebra. It plays an important role in the context of the *dual* operation, as we have already seen. A linear combination of basis blades all of some grade k is called a *vector of grade* k or a k*-vector*. Thus the name *multivector* for an arbitrary element of a geometric algebra: it is a linear combination of vectors of different grades.

The basis blades $\{E_i\}$ of some Clifford algebra $\mathcal{C}\ell_n$ satisfy the following properties.

1. There exists an identity element denoted by E_1 such that

$$E_1 E_i = E_i E_1 = E_i$$

2. $E_i E_i = \lambda_i E_1$, where $\lambda_i \in \{-1, 1\}$.

3. $E_i E_j = g_{ij}{}^k E_k$, where $g_{ij}{}^k \in \{-1, 0, 1\}$ and for given i and j, $g_{ij}{}^k$ is non zero for exactly one value of k.

From these properties it follows that every basis blade of some $\mathcal{C}\ell_n$ is invertible, that is for all E_i there exists an E_i^{-1} such that $E_i E_i^{-1} = E_i^{-1} E_i = E_1$.

Let's take a look at some examples. The following calculations employ the associativity of the geometric product and the property $\mathsf{e}_i \mathsf{e}_j = -\mathsf{e}_j \mathsf{e}_i$ for $i \neq j$.

$$\begin{aligned} &E_2 E_2 = \mathsf{e}_1 \mathsf{e}_1 = 1; \\ &E_5 E_5 = (\mathsf{e}_2 \mathsf{e}_3)\,(\mathsf{e}_2 \mathsf{e}_3) = -\mathsf{e}_2 (\mathsf{e}_3 \mathsf{e}_3) \mathsf{e}_2 = -\mathsf{e}_2 \mathsf{e}_2 = -1 \end{aligned} \tag{1.51}$$

This shows that there exist basis blades that square to -1. This is an important property that has far reaching consequences. It allows us for example to create multivectors that behave like complex numbers, without using the imaginary unit $\mathrm{i} = \sqrt{-1}$.

For example, consider $\mathcal{C}\ell_2$ with pseudoscalar $I := \mathsf{e}_1 \mathsf{e}_2$. From our previous considerations it is clear that $I^2 = -1$ if $\mathsf{e}_1 \mathsf{e}_1 = \mathsf{e}_2 \mathsf{e}_2 = 1$. Define two multivectors $A, B \in \mathcal{C}\ell_2$ as $A := \alpha_1 + \beta_1 I$ and $B = \alpha_2 + \beta_2 I$. The geometric product of A and B becomes

$$\begin{aligned} AB &= (\alpha_1 + \beta_1 I)\,(\alpha_2 + \beta_2 I) \\ &= \alpha_1 \alpha_2 + \beta_1 \beta_2 I^2 + \alpha_1 \beta_2 I + \alpha_2 \beta_1 I \\ &= (\alpha_1 \alpha_2 - \beta_1 \beta_2) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) I \end{aligned}$$

Comparing this with the multiplication rules for complex numbers, it shows that the multivectors A, B in conjunction with the geometric product behave just like complex numbers.

Let us now return to the properties of basis blades. Here are some examples to clarify the third property.

$$\begin{aligned} &E_2 E_5 = \mathsf{e}_1 (\mathsf{e}_2 \mathsf{e}_3) = E_8 \\ &E_5 E_6 = (\mathsf{e}_2 \mathsf{e}_3)\,(\mathsf{e}_3 \mathsf{e}_1) = \mathsf{e}_2 (\mathsf{e}_3 \mathsf{e}_3) \mathsf{e}_1 = -\mathsf{e}_1 \mathsf{e}_2 = -E_7 \\ &E_6 E_5 = (\mathsf{e}_3 \mathsf{e}_1)\,(\mathsf{e}_2 \mathsf{e}_3) = \mathsf{e}_1 (\mathsf{e}_3 \mathsf{e}_3) \mathsf{e}_2 = \mathsf{e}_1 \mathsf{e}_2 = E_7 \end{aligned} \tag{1.52}$$

The last two equations show that basis blades do not necessarily commute. Hence, multivectors may not commute.

Even though every basis blade is invertible, multivectors may not be. Consider for example $A \in \mathcal{C}\ell_2$ defined as $A := \frac{1}{2}(1 + \mathsf{e}_1)$.

$$
\begin{aligned}
A^2 &= \tfrac{1}{4}(1+\mathsf{e}_1)(1+\mathsf{e}_1) \\
&= \tfrac{1}{4}(1+\mathsf{e}_1+\mathsf{e}_1+\mathsf{e}_1\mathsf{e}_1) \\
&= \tfrac{1}{4}(2+2\mathsf{e}_1) \\
&= \tfrac{1}{2}(1+\mathsf{e}_1) \\
&= A
\end{aligned}
$$

That is, A squares to itself. It can be shown that this implies that A has no inverse. Therefore, if we talk about multivectors in general we cannot assume that they always have an inverse. It can also be shown that if a multivector has no inverse there exists another multivector that multiplied with the first gives zero. For example, let $B \in \mathcal{C}\ell_2$ be defined as $B := \frac{1}{2}(1 - \mathsf{e}_1)$. Then

$$
\begin{aligned}
AB &= \tfrac{1}{4}(1+\mathsf{e}_1)(1-\mathsf{e}_1) \\
&= \tfrac{1}{4}(1-\mathsf{e}_1+\mathsf{e}_1-\mathsf{e}_1\mathsf{e}_1) \\
&= 0
\end{aligned}
$$

Also note the following "curiosity". For the *coset* of multivectors $\mathcal{C}\ell_2 A := \{XA : X \in \mathcal{C}\ell_2\}$, A is a right idempotent, since $(XA)A = X(AA) = XA$. In other words, right multiplying an element of $\mathcal{C}\ell_2 A$ with A leaves the initial multivector unchanged. Furthermore, for all $Y \in \mathcal{C}\ell_2 A$, $YB = (XA)B = X(AB) = 0$.

1.2.3 Inverting Multivectors

So far we have mainly dealt with vectors and blades. This is mainly because they offer a nice way to deal with subspaces. However, we have not done much with general multivectors. General multivectors are linear combinations of subspaces. Therefore, simple operations like increasing or decreasing a grade are not immediately useful. Nevertheless, we may still be interested in solving multivector equations of the type $AX = B$ for X given A and B, where $A, B, X \in \mathcal{C}\ell_{p,q}$. The solution is obviously $X = A^{-1}B$. However, can we always invert the multivector A? And what if X has to satisfy a number of equation simultaneously? In order to solve these problems in general, we need to look at multivectors from a more general point of view.

Earlier we denoted the basis of a Clifford algebra by a set of basis blades. That is, the basis $\mathbb{B}_n$ of $\mathcal{C}\ell_n$ is given by $\mathbb{B}_n = \{E_i\}$. Let us repeat the basic properties of the $\{E_i\}$.

1. There exists an identity element denoted by E_1 such that

$$E_1 E_i = E_i E_1 = E_i,$$

2. $E_i E_i = \lambda_i \, E_1$, where $\lambda_i \in \{-1, 1\}$,

3. $E_i E_j = g_{ij}{}^k \, E_k$, where $g_{ij}{}^k \in \{-1, 0, 1\}$ and for any given i and j, $g_{ij}{}^k$ is non zero for exactly one value of k. Recall that $g_{ij}{}^k \, E_k$ implies a summation over the range of k which is $\{1, 2, \ldots, 2^n\}$.

The last property ensures that the $\{E_i\}$ are invertible. It is also the key to inverting multivectors. The point is, that we can regard multivectors as 2^n dimensional vectors, and the geometric product is evaluated by contraction with the tensor $g_{ij}{}^k$.

For example, let $A, B \in \mathcal{C}\ell_n$ be given by $A = \alpha^i E_i$ and $B = \beta^i E_i$, where $\alpha^i, \beta^i \in \mathbb{R}$. Given the basis $\mathbb{B}_n$ we can therefore represent A and B by $(\alpha^1, \alpha^2, \ldots, \alpha^{2^n})$ and $(\beta^1, \beta^2, \ldots, \beta^{2^n})$, respectively. Then the resultant multivector $C \in \mathcal{C}\ell_n$, with $C = \eta^i E_i$, of the geometric product of A and B is given by

$$\eta^k = \alpha^i \beta^j \, g_{ij}{}^k. \tag{1.53}$$

Recall again that there is an implicit summation over indices i and j. Now suppose multivectors B and C are given, and we would like to evaluate A. We can do this by first contracting β^j with $g_{ij}{}^k$ and then inverting the resultant matrix. That is, first define

$$h_i{}^k := \beta^j \, g_{ij}{}^k, \tag{1.54}$$

and then solve for (α^i) via

$$\alpha^i = \sum_k \eta^k \left(h_i{}^k\right)^{-1} = \eta^k \, \bar{h}^i{}_k, \tag{1.55}$$

where $\bar{h}^i{}_k := (h_i{}^k)^{-1}$. In future we will write the inverse of any tensor $a_{ijk\ldots}{}^{pqr\ldots}$ as $\bar{a}^{ijk\ldots}{}_{pqr\ldots}$.

Clearly, the problem with equation (1.55) is that $h_i{}^k$ does not necessarily have an inverse. However, if we apply a singular value decomposition to $h_i{}^k$, we can see whether a multivector is invertible (if no singular value is zero), if yes invert it and otherwise find a pseudo-inverse. If a multivector is not invertible we also call it a *singular* multivector. With regard to the matrix $h_i{}^k$ from above, it may be shown that the rank of $h_i{}^k$ is always a power of two. Since $h_i{}^k$ is a $2^n \times 2^n$ matrix, also the dimension of the null space of $h_i{}^k$ is a power of two.

1.2.4 Solving for a Versor

In the last section we saw how to invert multivectors if they are invertible and how to solve multivector equations of the type $AX = B$. Another type of equation which we will encounter quite frequently, is that of a versor equation. That is, we are looking for the versor $V \in \mathcal{C}\ell_n$, which solves $VA\tilde{V} = B$, given $A, B \in \mathcal{C}\ell_n$. At first it might seem that this is not a linear equation anymore, since V appears twice on the left hand side. However, since a versor is always invertible and its inverse is its reverse, we can write

$$VA\tilde{V} = B \iff VA = BV \iff VA - BV = 0, \tag{1.56}$$

which is again a linear equation. Unfortunately, we cannot write this equation in the form $XV = Y$. Nevertheless, we can still solve this equation numerically. Before we show how to do this, let us first see whether the solution for V is unique.

Let $\mathbb{U}_B \subset \mathcal{C}\ell_n$ be a set of invertible, linearly independent multivectors that commute with B, i.e.

$$\mathbb{U}_B := \{X \in \mathcal{C}\ell_n \ : \ XB = BX, \ \exists X^{-1} \in \mathcal{C}\ell_n \ \Rightarrow \ XX^{-1} = 1\}.$$

For $X \in \mathbb{U}_B$ we therefore have

$$X(VA - BV) = 0 \iff (XV)A - B(XV) = 0. \tag{1.57}$$

That is, if V is a solution to $VA - BV = 0$, then so is the coset $\mathbb{U}_B V = \{XV \ : \ X \in \mathbb{U}_B\}$. Again we have that if $X_1, X_2 \in \mathbb{U}_B$, then $(X_1 X_2) \in \mathbb{U}_B$. Hence, $\mathbb{U}_B$ is the basis of a subalgebra $\mathcal{C}\ell(\mathbb{U}_B) \subset \mathcal{C}\ell_n$. Therefore, the number of elements in $\mathbb{U}_B$ is a power of two. The following example should clarify this.

Let $\mathbf{a}, \mathbf{b} \in \mathcal{C}\ell_3$ be two unit vectors in Euclidean 3d-space $\mathbb{R}^3$. We are looking for the rotor R such that $R\mathbf{a}\tilde{R} = \mathbf{b}$. We will denote the solution rotor as R_{ab}. However, we find that a basis set of the subalgebra that commutes with $\mathbf{b}$ is given by

$$\mathbb{U}_{\mathbf{b}} = \{1, \mathbf{b}, \mathbf{b}^*, I\}, \tag{1.58}$$

where I is the pseudoscalar of $\mathcal{C}\ell_3$. Note that in $\mathcal{C}\ell_3$ the pseudoscalar commutes with all multivectors. Therefore, the solution set of R that solves $RA - BR = 0$, is the coset

$$\mathbb{U}_{\mathbf{b}} R_{ab} = \{R_{ab}, \mathbf{b}R_{ab}, \mathbf{b}^* R_{ab}, IR_{ab}\}. \tag{1.59}$$

The solution of $RA - BR = 0$ is thus not unique. If we introduce a second vector pair $\{\mathbf{a}', \mathbf{b}'\}$ that is also related by $R_{ab}\mathbf{a}'\tilde{R}_{ab} = \mathbf{b}'$, and demand that R is a solution of $R\mathbf{a}\tilde{R} = \mathbf{b}$ and $R\mathbf{a}'\tilde{R} = \mathbf{b}'$ simultaneously, then the solution set of R is $\{R_{ab}, IR_{ab}\}$. Even if we introduce a third vector pair related by R_{ab} as before, we cannot constrain the solution set for R further. Instead we have to demand that the solution for R lies in the even subalgebra of $\mathcal{C}\ell_3$. The even subalgebra of $\mathcal{C}\ell_3$ contains all linear combinations of blades of even grade. It is indeed a subalgebra, since the geometric product of two even grade elements results again in an even grade element.

We want R to lie in the even subalgebra of $\mathcal{C}\ell_3$, since this is how we usually express a rotor: a scalar plus a bivector component. This then reduces the solution set for R to $\{R_{ab}\}$, since IR_{ab} contains only odd grade blades: a vector and a trivector.

In fact, it is possible to evaluate a rotor from two vectors, so this analysis might seem somewhat superfluous. However, in applications we typically have vector estimates that contain noise and we want to find the best rotor from a set of noisy vector pairs. This can be achieved through a numerical method.

Let us consider again the general problem, We have two multivectors $A, B \in \mathcal{C}\ell_n$ which are related by a versor $V \in \mathcal{C}\ell_n$ via $VA\tilde{V} = B$. In order to solve this problem numerically, we again express A, B and V as 2^n-dimensional vectors, $A = \alpha^i E_i$, $B = \beta^i E_i$ and $V = \eta^i E_i$. Then the equation $VA - AV = 0$ becomes

$$\begin{aligned} VA - BV &= \eta^i \alpha^j \, g_{ij}{}^k - \beta^j \eta^i \, g_{ji}{}^k \\ &= \eta^i \, (\alpha^j \, g_{ij}^k - \beta^j \, g_{ji}^k) \\ &= \eta^i \, t_i{}^k, \end{aligned} \tag{1.60}$$

where ${t_i}^k := \alpha^j\, g^k_{ij} - \beta^j\, g^k_{ji}$. That is, in order to solve for V we have to solve $\eta^i {t_i}^k = 0$ for η^i. In other words, we are looking for the null-space of the matrix ${t_i}^k$. ${t_i}^k$ is a $2^n \times 2^n$ matrix. From the above analysis it also follows that the dimension of the solution space of V is a power of two. We can constrain the solution space by introducing more multivector pairs $X, Y \in \mathcal{C}\ell_n$ such that $VX - YV = 0$. However, at some point we will probably want to restrain the solution space to some subalgebra of $\mathcal{C}\ell_n$ or even to certain basis blades. This can be done quite easily by reducing the matrix ${t_i}^k$ in the index i appropriately.

For example, if we are looking for a rotor R, we know that it only contains a scalar and a bivector component. Accordingly we could reduce the respective matrix ${t_i}^k$ in index i to those indices that refer to the scalar and the bivector components.

This will not be discussed further here. However, a C++ implementation of this algorithm is part of the CLU library. It is also used to invert multivectors in CLUCalc.

1.3 Relation to other Geometric Algebras

Clearly, Clifford (or Geometric) algebra is not the only algebra describing geometry. In this section we will take a look at other algebras that relate to geometry and see how they are related to Clifford algebra.

1.3.1 Gibbs' Vector Algebra

Basically, the inner product between vectors in Clifford algebra is equivalent to the scalar product of vectors in Gibbs' vector algebra. Furthermore, since the dual of the outer product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ gives the vector perpendicular to the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$, it should be no surprise that the outer product is related to the cross product in the following way.

$$\boldsymbol{a} \times \boldsymbol{b} = (\boldsymbol{a} \wedge \boldsymbol{b})^*. \tag{1.61}$$

We can also translate identities of Gibbs vector algebra into Clifford algebra. For example, the triple scalar product of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$.

$$\begin{aligned} \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= \boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})^* \\ &= \boldsymbol{a} \cdot \big((\boldsymbol{b} \wedge \boldsymbol{c}) \cdot I^{-1}\big) \\ &= (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}) \cdot I^{-1} \\ &= (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})^* \\ &= \det([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]). \end{aligned} \tag{1.62}$$

Recall that the magnitude of $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$ is the volume of the parallelepiped spanned by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. This shows again that the outer product of three vectors spans a volume element. Another often used identity is the triple vector product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. This is usually expanded as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}).$$

Translating this expression into Clifford algebra gives,

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \Big(\boldsymbol{a} \wedge \big((\boldsymbol{b} \wedge \boldsymbol{c})I^{-1}\big)\Big)I^{-1} \\ &= \boldsymbol{a} \cdot \Big(\big((\boldsymbol{b} \wedge \boldsymbol{c})I^{-1}\big)I^{-1}\Big) \\ &= -\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) \\ &= \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}). \end{aligned} \tag{1.63}$$

The expansion in Clifford algebra is valid in any dimension, whereas the vector cross product is only defined in a 3d-vector space.

1.3.2 Complex Numbers

Complex numbers may also be regarded as a geometric algebra, if we interpret the real and imaginary part of a complex number as the two coordinates of a point in a 2d-space. A complex number $z \in \mathbb{C}$ can be expressed in two equivalent ways.

$$z = \alpha + \mathrm{i}\beta = \varrho\, \exp(\mathrm{i}\,\theta),$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit, and $\alpha, \beta, \varrho, \theta \in \mathbb{R}$. The relation between α, β and ϱ and θ is $\varrho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$. When we discussed rotors we argued that since a unit bivector in $\mathcal{C}\ell_n$ squares to minus one, it may replace the imaginary unit i. Accordingly, we extended the definition of the exponential function to multivectors, in order to write a rotor in exponential form. We can also use the exponential function to write any multivector $A \in \mathcal{C}\ell_n$ which is defined as $A = \alpha + U_{\langle 2\rangle}\beta$, where $U_{\langle 2\rangle} \in \mathcal{C}\ell_n$ is a unit bivector, as

$$A = \varrho\, \exp(U_{\langle 2\rangle}\, \theta).$$

Note that A is an element of a subalgebra $\mathcal{C}\ell_2 \subseteq \mathcal{C}\ell_n$, $n \geq 2$. More precisely, it is an element of the even subalgebra $\mathcal{C}\ell_2^+ \subset \mathcal{C}\ell_2$, which consists of the linear combinations of the even grade elements of $\mathcal{C}\ell_2$. The even subalgebra $\mathcal{C}\ell_2^+$ of $\mathcal{C}\ell_2$ has basis $\{1, U_{\langle 2\rangle}\}$, where $U_{\langle 2\rangle}$ is also the pseudoscalar of $\mathcal{C}\ell_2$. $\mathcal{C}\ell_2^+$ is indeed a subalgebra, since it is closed under the geometric product. Therefore, we have found an isomorphism between the complex numbers $\mathbb{C}$ and the geometric algebra $\mathcal{C}\ell_2^+$, where the product between complex numbers becomes the geometric product. Note that the complex conjugate becomes the reverse, since the reverse of A is

$$\tilde{A} = \varrho\, \exp(\tilde{U}_{\langle 2\rangle}\, \theta) = \varrho\, \exp(-U_{\langle 2\rangle}\, \theta),$$

which is equivalent to

$$z^* = \varrho\, \exp(-\mathrm{i}\,\theta).$$

We will not go any deeper into complex analysis at this point. In any case, since there is an isomorphism between $\mathbb{C}$ and $\mathcal{C}\ell_2^+$, everything from complex analysis carries over. However, simply replacing i by a bivector is in itself not particularly interesting, since it does not give us anything we did not have before. Nevertheless, it shows that we can regard the complex number geometric algebra as part of Clifford algebra.

1.3.3 Quaternions

The interesting aspect of the isomorphism between $\mathbb{C}$ and $\mathcal{C}\ell_2^+$ is, that $\mathcal{C}\ell_n$ has $\binom{n}{k}$ bivectors and thus the same number of different even subalgebras $\mathcal{C}\ell_2^+$. That is, in Clifford algebra we can combine different complex spaces. One effect of this is that there is an isomorphism between quaternions ($\mathbb{H}$) and $\mathcal{C}\ell_3^+$. Before we show this isomorphism, we should probably recapitulate quaternions.

The name 'quaternion' literally means a combination of four parts. The quaternions we are talking about here consist of a scalar component and three imaginary components. The imaginary components are typically denoted by $\mathsf{i}, \mathsf{j}, \mathsf{k}$ and they satisfy the following relations.

$$\begin{aligned}
&\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1, \\
&\mathsf{ij} = \mathsf{k},\ \mathsf{jk} = \mathsf{i},\ \mathsf{ki} = \mathsf{j}, \\
&\mathsf{ij} = -\mathsf{ji},\ \mathsf{jk} = -\mathsf{kj},\ \mathsf{ki} = -\mathsf{ik}, \\
&\mathsf{ijk} = -1.
\end{aligned} \tag{1.64}$$

A general quaternion is then given by

$$a = \alpha_0 + \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k},$$

with $\{\alpha_i\} \subset \mathbb{R}$. A *pure* quaternion is one with no scalar component, i.e. $\bar{a} = \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k}$ is a pure quaternion. The square of a pure quaternion gives

$$\bar{a}^2 = (\alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k})^2 = -\big((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2\big).$$

The complex conjugate of a quaternion a is denoted by a^*. It negates all imaginary components. Therefore,

$$\begin{aligned}
aa^* &= (\alpha_0 + \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k})\,(\alpha_0 - \alpha_1 \mathsf{i} - \alpha_2 \mathsf{j} - \alpha_3 \mathsf{k}) \\
&= (\alpha_0)^2 + (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2.
\end{aligned}$$

A unit pure quaternion $\hat{\bar{a}}$ satisfies $\hat{\bar{a}}\hat{\bar{a}}^* = 1$ and thus $\hat{\bar{a}}\hat{\bar{a}} = -1$. We can therefore write the quaternion a also as

$$\begin{aligned}
a &= (\alpha_0 + \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k}) \\
&= \varrho\,(\cos\theta + \hat{\bar{a}}\,\sin\theta),
\end{aligned}$$

where $\varrho = \sqrt{aa^*}$, $\theta = \tan^{-1}(\bar{a}\bar{a}^*/\alpha_0)$, $\bar{a} = \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k}$ and $\hat{\bar{a}} = \bar{a}/\sqrt{\bar{a}\bar{a}^*}$. Since $\hat{\bar{a}}$ squares to minus one, we have again an isomorphism between the complex numbers $\mathbb{C}$ and a subalgebra of $\mathbb{H}$. We can also extent the definition of the exponential function to quaternions to find

$$a = (\alpha_0 + \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k}) = \varrho\, \exp(\theta\,\hat{\bar{a}}),$$

where ϱ, θ and $\hat{\bar{a}}$ are given as before. It can be shown that the operation $\hat{r}\bar{a}\hat{r}^*$ between a unit quaternion $\hat{r} = \exp(\frac{1}{2}\theta\hat{\bar{r}})$ and a pure quaternion $\bar{a}$, represents a rotation of $\bar{a}$. That is, if we regard $\bar{a} = \alpha_1 \mathsf{i} + \alpha_2 \mathsf{j} + \alpha_3 \mathsf{k}$ as a vector $(\alpha_1, \alpha_2, \alpha_3)$, then $\hat{r}\bar{a}\hat{r}^*$ rotates this vector by an angle θ about the vector represented by $\hat{\bar{r}}$.

Let us take a look at two simple examples of this. We assume $(\mathsf{i}, \mathsf{j}, \mathsf{k})$ to form the basis of a right-handed coordinate system. The pure quaternion k can be written in exponential form as $\mathsf{k} = \exp(\frac{1}{2}\pi\,\mathsf{k})$. Therefore, it should rotate the pure quaternion i about 180 degrees, if applied as kik^*.

$$\mathsf{kik}^* = -\mathsf{ikk}^* = -\mathsf{i}.$$

Note that this example also shows that operators and elements we operate on can be of the same type. Let us consider now this somewhat more complex example.

Consider the rotation operator for a rotation about the k-axis, $r = \exp(\frac{1}{2}\theta\mathsf{k})$. We can expand r to read $r = \cos\frac{1}{2}\theta + \mathsf{k}\sin\frac{1}{2}\theta$. If we apply r to i it should rotate i in the ij-plane by an angle θ.

$$\begin{aligned}
r\,\mathsf{i}\,r^* &= (\cos\tfrac{1}{2}\theta + \mathsf{k}\sin\tfrac{1}{2}\theta)\,\mathsf{i}\,(\cos\tfrac{1}{2}\theta - \mathsf{k}\sin\tfrac{1}{2}\theta) \\
&= \cos^2\tfrac{1}{2}\theta\,\mathsf{i} - \cos\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta\,\mathsf{ik} + \cos\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta\,\mathsf{ki} - \sin^2\tfrac{1}{2}\theta\,\mathsf{kik} \\
&= (\cos^2\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}\theta)\mathsf{i} + 2\cos\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta\,\mathsf{j} \\
&= \cos\theta\,\mathsf{i} + \sin\theta\,\mathsf{j}.
\end{aligned}$$

This shows that $r = \exp(\frac{1}{2}\theta\,\mathsf{k})$ is indeed a rotation operator about a mathematically positive angle θ. If we compare this with rotors in Clifford algebra, we see that there is a difference in sign. Recall that a rotor for a rotation about a mathematically positive angle θ is given by $\exp(-\frac{1}{2}\theta U_{\langle 2\rangle})$. This difference in sign stems from the way in which we interpreted bivectors. This will become clear once we have given the isomorphism between quaternions and a Clifford algebra.

What we have discussed so far about quaternions already shows how similar they are to rotors, which we discussed earlier. This also gives us a hint on how to find an isomorphism. Basically, we need to find multivectors in a Clifford algebra which have the same properties as i, j and k, and form together with the unit scalar the basis of a Clifford subalgebra. To cut a long story short, we can identify the imaginary units i, j and k with the following bivectors in $\mathcal{C}\ell_3$.

$$\mathsf{i} \to e_2e_3, \quad \mathsf{j} \to e_1e_2, \quad \mathsf{k} \to e_3e_1, \tag{1.65}$$

where the $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$ are an orthonormal basis of $\mathbb{R}^3$. Therefore, the Clifford algebra $\mathcal{C}\ell_3^+$ with basis $\{1, e_2e_3, e_1e_2, e_3e_1\}$, is isomorph to the quaternions $\mathbb{H}$, if we make the above identifications. Note that this is only one possible isomorphism. Let us check one property of the quaternions.

$$\begin{aligned}
&\mathsf{ij} \to e_2e_3\,e_1e_2 = e_3e_1 \to \mathsf{k}, \\
&\mathsf{jk} \to e_1e_2\,e_3e_1 = e_2e_3 \to \mathsf{i}, \\
&\mathsf{ki} \to e_3e_1\,e_2e_3 = e_1e_2 \to \mathsf{j}.
\end{aligned} \tag{1.66}$$

We can now see where the sign difference in the rotation operators comes from. When we work with vectors we usually assume that we are working in a right-handed system and the coordinates are given in order of the x-, y- and z-axis, e.g. $(\alpha_1, \alpha_2, \alpha_3)$. When we use quaternions, we identify i, j and k with the three coordinate axes in this order. In Clifford algebra, on the other hand, we denote the three axes by e_1, e_2 and e_3. Now, recall that the rotation plane is given by a unit bivector, e.g. $U_{\langle 2\rangle} \in \mathcal{C}\ell_3^+$. We have also seen that the corresponding rotation axis in 3d is given by $U_{\langle 2\rangle}^*$. Note that

$$(e_2e_3)^* = e_1,\ (e_1e_2)^* = e_3,\ (e_3e_1)^* = e_2.$$

Therefore, the rotation axis $(\alpha_1\mathsf{i}+\alpha_2\mathsf{j}+\alpha_3\mathsf{k})$ corresponds to the rotation axis $(\alpha_1 e_1+\alpha_2 e_3+\alpha_3 e_2)$ in the Clifford algebra using the above identification for i, j and k. That is, the y- and z-axes are exchanged. Therefore, if we embed quaternions into Clifford algebra, we cannot apply them to vectors, only to other quaternions. If we translate the quaternions to rotors, we need to make the appropriate exchange of axes, which also introduces the minus sign into the rotor.

We have seen that quaternions are basically the space of rotors in $\mathcal{C}\ell_3$, which is the even subalgebra $\mathcal{C}\ell_3^+ \subset \mathcal{C}\ell_3$. The main advantages of rotors in Clifford algebra over quaternions are that rotors may be defined in any dimension and that a rotor can rotate blades of any grade. That is, we can not only rotate vectors but also lines, planes and any other geometric object that can be represented by a blade.

1.3.4 Grassmann Algebra

Today Grassmann algebra is usually taken as a synonym for *exterior algebra*. Although Grassmann also developed exterior algebra, he looked at the whole subject from a much more general point of view. In fact, he developed some fundamental results of what is today known as *universal algebra*. In his book "Die lineare Ausdehnungslehre dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch auf die Statistik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert", Grassmann basically developed linear algebra with the theory of basis and dimension for finite-dimensional linear spaces. He called vectors *extensive quantities* and a basis $\{e_1, e_2, \dots, e_n\}$ a *system of units*. The vector space spanned by a basis he called *region*. He then introduced a very general product on the extensive quantities (vectors). Given two vectors $\boldsymbol{a} = \alpha^i e_i$ and $\boldsymbol{b} = \beta^i e_i$, a general product of the two is written as

$$\boldsymbol{a}\boldsymbol{b} = \alpha^i \beta^j\,(e_i e_j).$$

Recall that there is an implicit sum here over i and j. He makes no additional assumptions at first about the elements $(e_i e_j)$, apart from noting that they are extensive quantities themselves. The set of products that can be formed with extensive quantities he called a *product structure*. For example, for a vector basis $\{e_1, e_2\}$ the set of products is

$$\{e_1, e_2, (e_1e_1), (e_1e_2), (e_2e_1), (e_2e_2), e_1(e_1e_1), e_1(e_1e_2), \dots\}.$$

This product structure may then be constrained by a *determining equation*. That is, if we denote the elements of the product structure by $\{E_i\}$, a determining equation is $\alpha^i E_i = 0$, $\alpha^i \in \mathbb{R}$. For example, we could use as determining equation $(e_1e_2) + (e_2e_1) = 0$. Then (e_1e_2) is linearly dependent on (e_2e_1). Or, more generally, $(e_ie_j) + (e_je_i) = 0$, for all i and j. This also implies that $e_ie_i = 0$. If we also assume associativity of the product, then the basis for the algebra generated by $\{e_1, e_2\}$ becomes

$$\{e_1, e_2, (e_1e_2)\}.$$

Grassmann found that the only determining equations that stay invariant under a change of basis are, for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}\,\boldsymbol{b} = 0$, $\boldsymbol{a}\,\boldsymbol{b} - \boldsymbol{b}\,\boldsymbol{a} = 0$ and $\boldsymbol{a}\,\boldsymbol{b} + \boldsymbol{b}\,\boldsymbol{a} = 0$. He then considered in some length the algebra generated by the determining equation $\boldsymbol{a}\,\boldsymbol{b} + \boldsymbol{b}\,\boldsymbol{a} = 0$. This algebra is today called exterior algebra and the product which satisfies this determining equation is called the exterior product. In the following we will denote the exterior product by $\wedge$, just like the outer product. In fact, "outer product" is just another name for exterior product.

Today exterior algebra is introduced in much the same way, albeit more generally and rigorously. The general product Grassmann introduced is replaced by the *tensor product*.

Grassmann also introduced an inner product between extensive quantities of the same grade. He did this in a very interesting way, by first defining what is essentially the dual. For an extensive quantity E the dual is denoted by E^* and is defined such that $E^* \wedge E$ is an extensive quantity of highest grade, i.e. a pseudoscalar. Since the pseudoscalars span a one dimensional subspace he equated the extensive quantity $e_1 \wedge e_2 \wedge \ldots \wedge e_n$ with the scalar 1. With this definition $E^* \wedge E$ is indeed a scalar. The inner product of two extensive quantities E, F of same grade is then defined as

$$< E, F >:= E^* \wedge F.$$

1.3.5 Grassmann-Cayley Algebra

The main difference between Grassmann and Grassmann-Cayley algebra is that there is also a grade reducing inner product defined between blades of different grade. This product may also be called the shuffle or the regressive product. Sometimes this product is also called the meet and the exterior product is called the join. This should not be confused with the meet and join defined previously in this text. Another source of confusion is the meaning of the symbols $\wedge$ and $\vee$, which is exactly the opposite to what they mean in Clifford algebra. The symbol $\wedge$ usually stands for the meet (inner product) and the $\vee$ stands for the join (outer product). This is actually somewhat more logical than the use in Clifford algebra, since it compares with the use of the symbols for union ($\cup$) and intersection ($\cap$). Unfortunately, not all authors that use Grassmann-Cayley algebra follow this convention. Sometimes Grassmann algebra is also taken to mean Grassmann-Cayley algebra. At times even completely different symbols ($\triangledown$, $\triangle$) are used for meet and join.

Despite these notational differences Grassmann-Cayley algebra and Clifford algebra are equivalent in the sense that anything expressed in one of them can also be expressed in the other. Which one you prefer is probably a matter of taste.

The shuffle product is defined with respect to the bracket operator $[]$. The bracket operator is defined for elements of highest grade in an algebra (pseudoscalars), for which it evaluates their magnitude. In the following we will use the Clifford algebra notation. If $A_{\langle k \rangle}, B_{\langle l \rangle} \in \mathcal{C}\ell_n$ are given by $A_{\langle k \rangle} = \bigwedge_{i=1}^{k} \boldsymbol{a}_i$ and $B_{\langle l \rangle} = \bigwedge_{i=1}^{l} \boldsymbol{b}_i$, with $k + l \geq n$ and $k \geq l$, then the shuffle product of $A_{\langle k \rangle}$ and $B_{\langle l \rangle}$, which we will temporarily denote by $\odot$, is defined as

$$A_{\langle k \rangle} \odot B_{\langle l \rangle} := \sum_{\sigma} \mathsf{sgn}(\sigma) \left[\boldsymbol{a}_{\sigma(1)} \boldsymbol{a}_{\sigma(2)} \ldots \boldsymbol{a}_{\sigma(n-l)}\, \boldsymbol{b}_1 \wedge \ldots \boldsymbol{b}_l \right] \boldsymbol{a}_{\sigma(n-l+1)} \wedge \ldots \boldsymbol{a}_{\sigma(k)}. \tag{1.67}$$

The sum is taken over all permutations σ of $\{1, \ldots, k\}$, such that $\sigma(1) < \sigma(2) < \ldots \sigma(n-l)$ and $\sigma(n-l+1) < \sigma(n-l+2) < \ldots \sigma(n)$. These type of permutations are called *shuffles* of the $(n-l,\, k-(n-l))$ split of $A_{\langle k \rangle}$. If σ is an even permutation of $\{1, \ldots, k\}$ then $\mathsf{sgn}(\sigma) = +1$, otherwise $\mathsf{sgn}(\sigma) = -1$. For example, the shuffles of a $(2, 1)$ split of $\{1, 2, 3\}$ are

$$\big(\{1, 2\},\, \{3\}\big),\; \big(\{1, 3\},\, \{2\}\big),\; \big(\{2, 3\},\, \{1\}\big),$$

where

$$\mathsf{sgn}(\{1, 2, 3\}) = +1,\; \mathsf{sgn}(\{1, 3, 2\}) = -1,\; \mathsf{sgn}(\{2, 3, 1\}) = +1.$$

Therefore, for $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} \subset \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$ we find

$$(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3) \odot \boldsymbol{b} = [\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{b}]\,\boldsymbol{a}_3 - [\boldsymbol{a}_1\boldsymbol{a}_3\boldsymbol{b}]\,\boldsymbol{a}_2 + [\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{b}]\,\boldsymbol{a}_1.$$

If $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$, and $\boldsymbol{b} = \beta^i e_i$, then we find

$$\begin{aligned}(e_1 \wedge e_2 \wedge e_3) \odot \boldsymbol{b} &= [e_1e_2\boldsymbol{b}]\,e_3 - [e_1e_3\boldsymbol{b}]\,e_2 + [e_2e_3\boldsymbol{b}]\,e_1 \\ &= [e_1e_2\,\beta^3e_3]\,e_3 - [e_1e_3\,\beta^2e_2]\,e_2 + [e_2e_3\,\beta^1e_1]\,e_1 \\ &= \boldsymbol{b},\end{aligned}$$

since $[e_1e_2e_3] = 1$. This shows that the pseudoscalar is the unit element with respect to the shuffle product. We have seen this before when we introduced the regressive product in definition 1.24 (page 14). In fact, it can be shown that the regressive product as we defined it *is* the shuffle product. That is,

$$A_{\langle k\rangle} \odot B_{\langle l\rangle} \equiv A_{\langle k\rangle} \triangledown B_{\langle l\rangle} = \left(A^*_{\langle k\rangle} \wedge B^*_{\langle l\rangle}\right) I.$$

The shuffle product is usually used to evaluate the intersection of subspaces. As we have seen in the discussion of the meet and join, this is only the case if the join of the two subspaces is the whole space. The shuffle product also cannot fully replace the Clifford algebra inner product, since it is defined to be zero for two blades $A_{\langle k\rangle}, B_{\langle l\rangle} \in \mathcal{C}\ell_n$ if $k+l < n$. It is nonetheless possible to recover the inner product from the shuffle product through the definition of the Hodge dual. This is basically the same as the dual we defined here. The only difference is that the Hodge dual of the Hodge dual of a blade is again the blade in any space. The dual of the dual of blade in Clifford algebra is either the blade or the negated blade. The Clifford algebra inner product may then be expressed in terms of the shuffle product as

$$A_{\langle k\rangle} \cdot B_{\langle l\rangle} \iff A^*_{\langle k\rangle} \odot B_{\langle l\rangle}.$$

This follows right away from the definition of the regressive product. If we translate the Hodge dual of $A_{\langle k\rangle}$ as $A_{\langle k\rangle} I$ then

$$A^*_{\langle k\rangle} \odot B_{\langle l\rangle} \Rightarrow (A_{\langle k\rangle} I) \triangledown B_{\langle l\rangle} = (A_{\langle k\rangle} \wedge B^*_{\langle l\rangle}) I = A_{\langle k\rangle} \cdot B_{\langle l\rangle}.$$

Grassmann-Cayley algebra is probably most widely used in the area of computer vision [12, 13] and robotics [39, 40]. There is still a lively, ongoing discussion within the research community, whether Grassmann-Cayley or Clifford algebra is better suited for these fields. To a large extend this is probably a matter of personal preference, and we will leave this decision to the reader's intuition.

Chapter 2

Geometries

by Dr. Christian Perwass

In the previous chapter we first talked about Geometric algebra and how elements of that algebra are taken to represent geometric entities. We also saw how we can operate on such entities in order to reflect or rotate them. In the second part of the previous chapter we then looked at Geometric algebra from an algebraic point of view, ie we introduced the axioms of Clifford algebra. In this chapter we would like to return to the geometric interpretation of the algebra.

Although we will talk in the following about spaces which embed Euclidean space in some way, the basic meaning of blades as linear subspaces and the reflection operator remain the same within these spaces. However, their effect on the embedded Euclidean space, or rather their interpretation in terms of the embedded Euclidean space may change quite substantially.

2.1 Projective Space

We will denote the homogeneous embedding of Euclidean space $\mathbb{E}^n$ by $\mathbb{PE}^n$. $\mathbb{PE}^n$ is also called a projective space. The properties of $\mathbb{PE}^n$ basically derive from the way Euclidean space is embedded in it. The projective space $\mathbb{PE}^n$ will be represented by $\mathbb{R}^{n+1} \backslash \mathbf{0}$, ie a $(n+1)$-dimensional vector space without the origin. The canonical (orthonormal) basis of $\mathbb{R}^{n+1}$ will be denoted by $\{\mathsf{e}_1, \ldots, \mathsf{e}_n, \mathsf{e}_{n+1}\}$. The basis vector e_{n+1} is also called the *homogeneous* component or dimension.

2.1.1 The Setup

The transformation operator from Euclidean to the corresponding projective space will be denoted by $\mathcal{P}$ and its inverse by $\mathcal{P}^{-1}$. The operator $\mathcal{P}$ is defined as

$$\mathcal{P} \,:\, \mathbf{x} \in \mathbb{E}^n \;\mapsto\; \mathbf{x} + \mathsf{e}_{n+1} \in \mathbb{PE}^n. \tag{2.1}$$

That is, Euclidean space is embedded as a particular hyperplane $\mathcal{P}(\mathbb{E}^n)$ in projective space. A vector in $\mathbb{PE}^n$ will also be called a *homogeneous* vector. Note that the origin of Euclidean space becomes e_{n+1} in projective space. This means that the origin of Euclidean space, as represented in projective space is not a special point any more. For example, while the scalar product of a vector with the origin in Euclidean space is always identically zero, this is not necessarily the case in projective space.

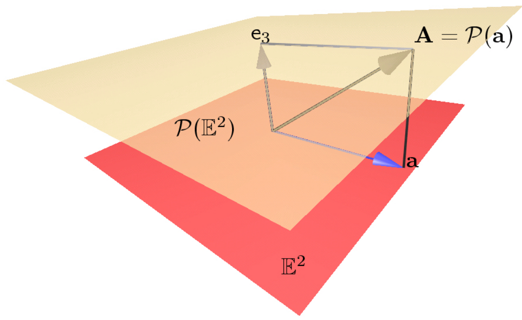


Figure 2.1: Embedding of Euclidean vector $\mathbf{a} \in \mathbb{E}^2$ in projective space $\mathbb{PE}^2$ as $\mathbf{A} = \mathcal{P}(\mathbf{a})$.

Figure 2.1 illustrates the embedding of Euclidean vectors in projective space for the case of $\mathbb{E}^2$. A vector $\mathbf{a} \in \mathbb{E}^2$ from Euclidean space is embedded in projective space $\mathbb{PE}^2$ by adding the homogeneous dimension e_3. The homogeneous representation of $\mathbf{a}$ in $\mathbb{PE}^2$ is then denoted by $\mathbf{A} = \mathcal{P}(\mathbf{a})$.

Although Euclidean vectors are mapped to a hyperplane in projective space, a general homogeneous vector may lie anywhere in $\mathbb{PE}^n \equiv \mathbb{R}^{n+1} \setminus \mathbf{0}$. Therefore, the question is how homogeneous vectors that do not lie on $\mathcal{P}(\mathbb{E}^n)$ are projected back to $\mathbb{E}^n$. This projection is in fact the key to the power of the homogeneous representation.

The transformation from $\mathbb{PE}^n$ to $\mathbb{E}^n$ is denoted by $\mathcal{P}^{-1}$ and is defined as

$$\mathcal{P}^{-1} : \mathbf{A} \in \mathbb{PE}^n \mapsto \frac{1}{\mathbf{A} \cdot \mathsf{e}_{n+1}} \sum_{i=1}^{n} \big(\mathbf{A} \cdot \mathsf{e}_i\big)\, \mathsf{e}_i \in \mathbb{E}^n. \tag{2.2}$$

Clearly, this transformation is only valid for homogeneous vectors that have a non-zero homogeneous component. Those homogeneous vectors that do have a zero homogeneous component would map to infinity and are accordingly called points at infinity or *direction* vectors.

Using the transformation $\mathcal{P}^{-1}$ the whole of $\mathbb{PE}^n$ apart from the plane $\mathsf{e}_{n+1} = 0$ is mapped to $\mathbb{E}^n$. What does this mean for a particular homogeneous vector? Well, the homogeneous vector is first scaled such that its homogeneous component is unity, and then its first n components are taken as the n components of the corresponding Euclidean vector. This is illustrated in figure 2.2.

The effect of $\mathcal{P}^{-1}$ is that the overall scale of a homogeneous vector in projective space is of

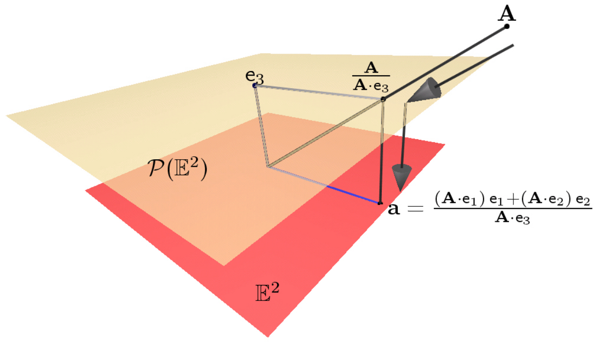


Figure 2.2: Projections of a homogeneous vector $\mathbf{A} \in \mathbb{PE}^2$ into the corresponding Euclidean space $\mathbb{E}^2$ as $\mathbf{a} = \mathcal{P}^{-1}(\mathbf{A})$.

no importance. For example, given a vector $\mathbf{a} \in \mathbb{E}^n$ and a scale $\alpha \in \mathbb{R} \setminus 0$, then

$$\mathcal{P}^{-1}\Big(\alpha\, \mathcal{P}(\mathbf{a})\Big) = \mathbf{a}.$$

Hence, the name "projective space": homogeneous vectors are projected onto the hyperplane $\mathcal{P}(\mathbb{E}^n)$ before they are "orthographically" projected into $\mathbb{E}^n$. The hyperplane $\mathcal{P}(\mathbb{E}^n)$ is also called the *affine* plane.

> **Aside.** Affine transformations are in fact just those that when applied to a point on $\mathcal{P}(\mathbb{E}^n)$ leave the point on that plane. Projective transformations on the other may move points through the whole space $\mathbb{PE}^n$.

2.1.2 Geometric Algebra on $\mathbb{PE}^n$

Recall that elements of Geometric algebra are given geometric meaning by looking at their OPNS or IPNS, the outer or inner product null space. When we write down a blade, its OPNS always represents a linear subspace. For example, a bivector in $\mathbb{PE}^2$ is a two dimensional subspace, since we represent $\mathbb{PE}^2$ by $\mathbb{R}^{2+1}$. However, we are not really interested in what this bivector represents in $\mathbb{PE}^2$. We would like to know what it represents in the corresponding $\mathbb{E}^2$. How do we do that? Well, we need to be more precise about which null space we are actually interested in.

Given a bivector $A_{\langle 2\rangle} \in \mathcal{C}\ell(\mathbb{PE}^2)$, we are only interested in those vectors in $\mathbb{PE}^2$ that lie in one of its null spaces, which we can also map back to Euclidean space. The other way around: we ask which vectors in $\mathbb{E}^2$ when transformed to $\mathbb{PE}^2$ lie in the null space of $A_{\langle 2\rangle}$. We therefore introduce the concept of the *Euclidean* outer and inner product null space, denoted by $\mathbb{NO}_E$ and $\mathbb{NI}_E$, respectively. For $\mathcal{C}\ell(\mathbb{PE}^n)$ they are defined as follows.

$$\mathbb{NO}_E(A_{\langle k\rangle} \in \mathcal{C}\ell(\mathbb{PE}^n)) := \{\, \mathbf{a} \in \mathbb{E}^n \;:\; \mathcal{P}(\mathbf{a}) \wedge A_{\langle k\rangle} = 0 \in \mathcal{C}\ell(\mathbb{PE}^n) \,\},$$
$$\text{and} \quad \mathbb{NI}_E(A_{\langle k\rangle} \in \mathcal{C}\ell(\mathbb{PE}^n)) := \{\, \mathbf{a} \in \mathbb{E}^n \;:\; \mathcal{P}(\mathbf{a}) \cdot A_{\langle k\rangle} = 0 \in \mathcal{C}\ell(\mathbb{PE}^n) \,\}. \tag{2.3}$$

2.1.3 The Euclidean OPNS

So how can we evaluate the Euclidean IPNS or OPNS of a blade in projective space? Consider, for example, a vector $\mathbf{a} \in \mathbb{E}^n$ with homogeneous representation $\mathbf{A} = \mathcal{P}(\mathbf{a}) \in \mathbb{PE}^n$. The OPNS of $\mathbf{A}$ is simply given by

$$\mathbb{NO}(\mathbf{A}) = \{\, \alpha\, \mathbf{A} \in \mathbb{PE}^n \;:\; \alpha \in \mathbb{R} \setminus 0 \,\},$$

a projective line in $\mathbb{PE}^n$. The factor α must not be zero since the origin of $\mathbb{R}^{n+1}$ is not an element of $\mathbb{PE}^n$. Since all elements of $\mathbb{NO}(\mathbf{A})$ can be mapped to $\mathbb{E}^n$ by $\mathcal{P}^{-1}$, we find that

$$\begin{aligned}
\mathbb{NO}_E(\mathbf{A}) &= \mathcal{P}^{-1}\Big(\mathbb{NO}(\mathbf{A})\Big) \\
&= \Big\{\, \tfrac{1}{(\alpha A)\cdot \mathsf{e}_{n+1}} \textstyle\sum_{i=1}^{n} \Big((\alpha A)\cdot \mathsf{e}_i\Big)\, \mathsf{e}_i \;:\; \alpha \in \mathbb{R} \setminus 0 \,\Big\} \\
&= \{\, \mathcal{P}^{-1}(\mathbf{A}) \;:\; \alpha \in \mathbb{R} \setminus 0 \,\} \\
&= \mathbf{a}.
\end{aligned}$$

This shows that even though the OPNS of $\mathbf{A}$ is a (projective) line in $\mathbb{PE}^n$, the Euclidean OPNS of $\mathbf{A}$ is only the vector $\mathbf{a} \in \mathbb{E}^n$. This is great, since it enables us to represent a zero-dimensional object, ie a point, in $\mathbb{E}^n$ by a line in $\mathbb{PE}^n$.

An example of this has already been shown for the case of $\mathbb{E}^2$ in figure 2.2. All points in $\mathbb{PE}^2$ along the line from, but excluding, the origin of $\mathbb{PE}^2$ to the homogeneous vector $\mathbf{A}$, represent the same point $\mathbf{a}$ in $\mathbb{E}^2$.

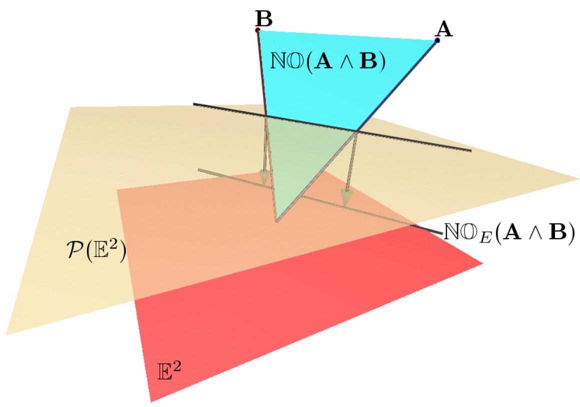


Figure 2.3: Representation of line in $\mathbb{E}^2$ through bivector in $\mathcal{C}\ell(\mathbb{PE}^2)$.

Figure 2.3 illustrates the OPNS and Euclidean OPNS of a bivector in $\mathbb{PE}^2$. The OPNS of the outer product of two homogeneous vectors $\mathbf{A}, \mathbf{B} \in \mathbb{PE}^2$ is a plane in $\mathbb{PE}^2$. The orthographic projection of the intersection of $\mathbb{NO}(\mathbf{A} \wedge \mathbf{B})$ with the plane $\mathcal{P}(\mathbb{E}^2)$, then gives the Euclidean OPNS of $\mathbf{A} \wedge \mathbf{B}$: a line in $\mathbb{E}^2$. Note that this line does not pass through the origin. This shows one of the advantages of working in $\mathcal{C}\ell(\mathbb{PE}^2)$ instead of $\mathcal{C}\ell(\mathbb{E}^2)$. In $\mathcal{C}\ell(\mathbb{E}^2)$ we could only represent lines through the origin, whereas in $\mathcal{C}\ell(\mathbb{PE}^2)$ we can represent arbitrary lines in the corresponding $\mathbb{E}^2$.

Without going into any more detail, it may be shown that the Euclidean OPNS of the outer product of three homogeneous vectors in $\mathcal{C}\ell(\mathbb{PE}^3)$ represents a plane in $\mathbb{E}^3$. That is, given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$ and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{PE}^3$ with

$$\mathbf{A} = \mathcal{P}(\mathbf{a}) \quad \text{and} \quad \mathbf{B} = \mathcal{P}(\mathbf{b}) \quad \text{and} \quad \mathbf{C} = \mathcal{P}(\mathbf{c}),$$

it may be shown that $\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})$ is a plane in $\mathbb{E}^3$ which passes through the points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. To summarize, we have

$\mathbb{NO}_E(\mathbf{A})$	**Point** $\mathbf{a}$
$\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B})$	**Line** through $\mathbf{a}$ and $\mathbf{b}$
$\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})$	**Plane** through $\mathbf{a}$, $\mathbf{b}$ and $cvec$

2.1.4 The Euclidean IPNS

We can also consider the Euclidean IPNS of blades of $\mathcal{C}\ell(\mathbb{E}^3)$. We will do this in some detail for a homogeneous vector. Let $\mathbf{A} \in \mathbb{PE}^3$ be given by

$$\mathbf{A} = \hat{\mathbf{a}} - \alpha\, \mathsf{e}_o,$$

where $\hat{\mathbf{a}} \in \mathbb{E}^3$ and $\|\hat{\mathbf{a}}\| = 1$. Furthermore, $\alpha \in \mathbb{R}$ and e_o denotes the homogeneous dimension e_{3+1}, in order to emphasize its meaning as the vector in $\mathbb{PE}^3$ representing the origin of $\mathbb{E}^n$. Let us now try to evaluate the Euclidean IPNS of $\mathbf{A}$. That is, we are looking for all those vectors $\mathbf{x} \in \mathbb{E}^3$ that satisfy $\mathbf{A} \cdot \mathcal{P}(\mathbf{x}) = 0$.

$$\begin{aligned}
\mathbf{A} \cdot \mathcal{P}(\mathbf{x}) = 0 \quad &\iff \quad (\hat{\mathbf{a}} - \alpha\, \mathsf{e}_o) \cdot (\mathbf{x} + \mathsf{e}_o) = 0 \\
&\iff \quad \hat{\mathbf{a}} \cdot \mathbf{x} - \alpha = 0 \\
&\iff \quad \hat{\mathbf{a}} \cdot \mathbf{x}^{\|} - \alpha = 0 \\
&\iff \quad \mathbf{x}^{\|} = \alpha\, \hat{\mathbf{a}},
\end{aligned}$$

where $\mathbf{x}^{\|}$ is the component of $\mathbf{x}$ parallel to $\hat{\mathbf{a}}$. If we write the component of $\mathbf{x}$ perpendicular to $\hat{\mathbf{a}}$ as $\mathbf{x}^{\perp}$, then it follows that any vector $\mathbf{x} \in \mathbb{E}^3$ of the form

$$\mathbf{x} = \alpha\, \hat{\mathbf{a}} + \mathbf{x}^{\perp},$$

lies in the Euclidean IPNS of $\mathbf{A}$. Hence, $\mathbf{A}$ represents a plane with normal $\hat{\mathbf{a}}$ and distance α from the origin in $\mathbb{E}^3$. As for Euclidean space it may also be shown that for homogeneous vectors $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{PE}^3$, we have

Plane:	$\mathbb{NI}_E(\mathbf{A})$
Line:	$\mathbb{NI}_E(\mathbf{A} \wedge \mathbf{B}) = \mathbb{NI}_E(A) \;\cap\; \mathbb{NI}_E(B)$
Point:	$\mathbb{NI}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}) = \mathbb{NI}_E(A) \;\cap\; \mathbb{NI}_E(B) \;\cap\; \mathbb{NI}_E(C)$

2.1.5 The Pinhole Camera Model

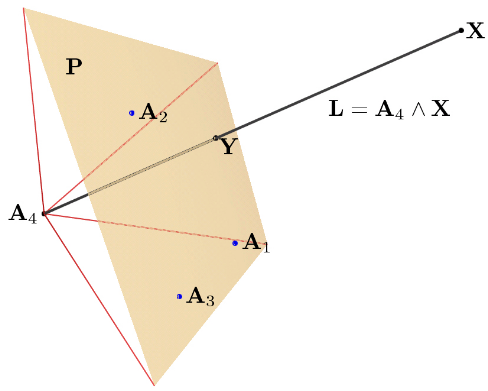


Figure 2.4: Model of a pinhole camera in $\mathbb{PE}^3$.

The Geometric algebra of projective space is very useful to represent projections in the pinhole camera model. Figure 2.4 show such a setup. Homogeneous vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{PE}^3$ form a basis of $\mathbb{PE}^3$. The homogeneous vector $\mathbf{A}_4$ represents the optical center of the pinhole camera, while $P = \mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3$ represents the image plane. In order to project a homogeneous vector $\mathbf{X}$ onto the image plane, we simply have to intersect the image plane P with the line L connecting $\mathbf{X}$ with the optical center $\mathbf{A}_4$, ie $L = \mathbf{A}_4 \wedge \mathbf{X}$. We can do this with the meet operation,

$$\mathbf{Y} = \mathbf{L} \vee \mathbf{P} = (\mathbf{A}_4 \wedge \mathbf{X}) \ \vee \ (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3).$$

Since the join of L and P is the whole space $\mathbb{PE}^3$, we can also use the regressive product instead of the meet, which simplifies the evaluation of the meet.

By using such simple geometric constructions, which can be readily translated into Geometric algebra equations, also the relations between two, three or more cameras can be analyzed. This then leads, for example, to the fundamental matrix and the trifocal tensor as was shown in [22, 29, 30, 28].

2.1.6 Reflections in Projective Space

By going from Euclidean to projective space, an additional dimension, the homogeneous dimension, is introduced. We may therefore wonder what effect this has when using the reflection operator as introduced earlier. First of all consider a vector $\mathbf{a} \in \mathbb{E}^2$ and its homogeneous representation

$$\mathbf{A} = \mathcal{P}(\mathbf{a}) = \mathbf{a} + \mathsf{e}_o \ \in \mathbb{PE}^2,$$

where e_o denotes again the homogeneous dimension $\mathsf{e}_3 \in \mathbb{PE}^2$. A reflection about e_o gives

$$\begin{aligned}
\mathsf{e}_o\,\mathbf{A}\,\mathsf{e}_o &= \mathsf{e}_o\,\mathbf{a}\,\mathsf{e}_o + \mathsf{e}_o\,\mathsf{e}_o\,\mathsf{e}_o \\
&= -\mathbf{a}\,\mathsf{e}_o\,\mathsf{e}_o + \mathsf{e}_o \\
&= -\mathbf{a} + \mathsf{e}_o,
\end{aligned}$$

where we used the fact that e_o is perpendicular to all vectors in $\mathbb{E}^2$. Therefore,

$$\mathsf{e}_o\,\mathbf{a} = \mathsf{e}_o \wedge \mathbf{a} = -\mathbf{a} \wedge \mathsf{e}_o = -\mathbf{a}\mathsf{e}_o.$$

We thus have

$$\mathcal{P}^{-1}\Big(\mathsf{e}_o\,\mathcal{P}\big(\mathbf{a}\big)\,\mathsf{e}_o\Big) = -\mathbf{a},$$

which shows that a reflection of $\mathbf{A}$ about e_o represents a reflection about the origin of $\mathbf{a}$.

Next consider a vector $\mathbf{n} \in \mathbb{E}^2$, with $\|\mathbf{n}\| = 1$. Although this is mathematically not quite rigorous, we can regard the vector $\mathbf{n}$ also as a direction vector of $\mathbb{PE}^2$, since it has no e_o component. If we take $\mathbf{A}$ as given above, we can ask what a reflection of a homogeneous vector $\mathbf{A}$ on a direction vector $\mathbf{n}$ in $\mathbb{PE}^2$ means.

$$\begin{aligned}
\mathbf{n}\,\mathbf{A}\,\mathbf{n} &= \mathbf{n}\,(\mathbf{a} + \mathsf{e}_o)\,\mathbf{n} \\
&= \mathbf{n}\,\mathbf{a}\,\mathbf{n} + \mathbf{n}\,\mathsf{e}_0\,\mathbf{n} \\
&= \mathbf{n}\,\mathbf{a}\,\mathbf{n} - \mathsf{e}_o\,\mathbf{n}^2 \\
&= \mathbf{n}\,\mathbf{a}\,\mathbf{n} - \mathsf{e}_o.
\end{aligned}$$

For convenience, let us at this point introduce an operator $\mathcal{A}$ that projects homogeneous vectors in $\mathbb{PE}^n$ onto the affine plane $\mathcal{P}(\mathbb{E}^n) \subset \mathbb{PE}^n$. The operator is therefore defined as

$$\mathcal{A}\,:\; \mathbf{A} \in \mathbb{PE}^n \;\mapsto\; \frac{\mathbf{A}}{\mathbf{A}\cdot\mathsf{e}_o} \in \mathbb{PE}^n, \tag{2.4}$$

where e_o is again the homogeneous dimension. We may also say that $\mathcal{A}$ transforms homogeneous vectors to affine vectors. This operator is also useful, since homogeneous vectors on $\mathcal{P}(\mathbb{E}^n)$ can be immediately identified with their corresponding Euclidean vectors in $\mathbb{E}^n$. For our reflection example from above we find,

$$\begin{aligned}
\mathcal{A}(\mathbf{n}\,\mathbf{A}\,\mathbf{n}) &= -\mathbf{n}\,\mathbf{a}\,\mathbf{n} + \mathsf{e}_o \\
&= -(\mathbf{a}^{\|} + \mathbf{a}^{\perp}) + \mathsf{e}_o \\
&= \mathbf{a}^{\perp} - \mathbf{a}^{\|} + \mathsf{e}_o,
\end{aligned}$$

where $\mathbf{a}^{\|}$ and $\mathbf{a}^{\perp}$ are the orthogonal and parallel components of $\mathbf{a}$ with respect to $\mathbf{n}$, respectively. This shows that the component of the homogeneous vector $\mathbf{A}$ that is *parallel* to the reflection direction $\mathbf{n}$, is reflected and not the part perpendicular to it. Figure 2.5 shows this setup.

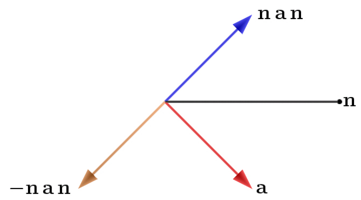


Figure 2.5: Effect in $\mathbb{E}^2$ of reflection of homogeneous vector on direction vector in $\mathbb{PE}^2$.

This is not really what we wanted to achieve. However, we can remedy the situation by reflecting $\mathbf{n}\,\mathbf{A}\,\mathbf{n}$ again through the origin. That is, in order to reflect a homogeneous vector on a line with direction $\mathbf{n}$, we have to use as operator $(\mathbf{n}\,\mathsf{e}_0)$ instead of $\mathbf{n}$.

$$\begin{aligned}(\mathbf{n}\,\mathsf{e}_0)\,\mathbf{A}\,(\mathsf{e}_0\,\mathbf{n}) &= \mathbf{n}\,(-\mathbf{a}+\mathsf{e}_0)\,\mathbf{n}\\ &= -\mathbf{n}\,\mathbf{a}\,\mathbf{n}+\mathbf{n}\,\mathsf{e}_o\,\mathbf{n}\\ &= -\mathbf{n}\,\mathbf{a}\,\mathbf{n}-\mathsf{e}_o,\end{aligned}$$

and thus

$$\mathcal{A}\Big((\mathbf{n}\,\mathsf{e}_0)\,\mathbf{A}\,(\mathsf{e}_0\,\mathbf{n})\Big) = \mathbf{n}\,\mathbf{a}\,\mathbf{n}+\mathsf{e}_o.$$

2.1.7 Rotations in projective space

In the last section we saw how a reflection in $\mathbb{E}^2$ has to be expressed in projective space $\mathbb{PE}^2$ when applied to homogeneous vectors. Since a rotation expressed by rotor is nothing else than two consecutive reflections, a rotor may also take on a different form in projective space.

Suppose we want to rotate the vector $\mathbf{a}\in\mathbb{E}^2$ by reflecting it first on $\mathbf{n}\in\mathbb{E}^2$ and then on $\mathbf{m}\in\mathbb{E}^2$. However, we want to do this in projective space where $\mathbf{A}=\mathcal{P}(\mathbf{a})\in\mathbb{PE}^2$. Since a reflection on $\mathbf{n}$ has to be expressed as $(\mathbf{n}\,\mathsf{e}_o)$ and a reflection on $\mathbf{m}$ as $(\mathbf{m}\,\mathsf{e}_o)$, the rotation of $\mathbf{A}$ has to look like this

$$(\mathbf{m}\,\mathsf{e}_o)\,(\mathbf{n}\,\mathsf{e}_o)\,\mathbf{A}\,(\mathsf{e}_o\,\mathbf{n})\,(\mathsf{e}_o\,\mathbf{m}) = R\,\mathbf{A}\,\tilde{R},\quad R:=(\mathbf{m}\,\mathsf{e}_o)\,(\mathbf{n}\,\mathsf{e}_o).$$

Such a double reflection is illustrated in figure 2.6. Here vector $\mathbf{a}\in\mathbb{E}^2$ is represented in $\mathbb{PE}^2$ by $\mathbf{A}$. A first reflection of $\mathbf{A}$ on $\mathbf{n}\,\mathsf{e}_o$ gives $\mathbf{B}$. A further reflection of $\mathbf{B}$ on $\mathbf{m}\,\mathsf{e}_o$ gives $\mathbf{C}$.

However, the expression for R can be simplified.

$$\begin{aligned}R &= (\mathbf{m}\,\mathsf{e}_o)\,(\mathbf{n}\,\mathsf{e}_o)\\ &= -\mathbf{m}\,\mathbf{n}\,\mathsf{e}_o\,\mathsf{e}_o\\ &= -\mathbf{m}\,\mathbf{n}.\end{aligned}$$

That is, compared to the expression of the rotor in $\mathbb{E}^2$, a minus sign is introduced. This, however, cancels out when the rotor is applied.

$$R\,\mathbf{A}\,\tilde{R} = (-\mathbf{m}\,\mathbf{n})\;\mathbf{A}\;(-\mathbf{n}\,\mathbf{m}) = (\mathbf{m}\,\mathbf{n})\;\mathbf{A}\;(\mathbf{n}\,\mathbf{m}).$$

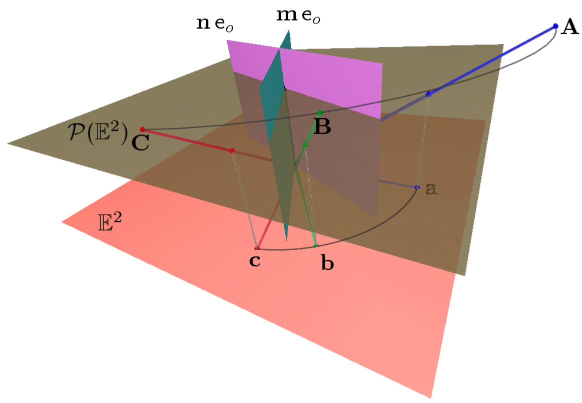


Figure 2.6: Double reflection of homogenous vector $\mathbf{A}$ on reflection planes $\mathbf{n}\,\mathsf{e}_o$ and $\mathbf{m}\,\mathsf{e}_o$ in $\mathbb{PE}^2$.

We may also argue that since an overall scalar factor is of no importance for homogeneous vectors with respect to their projection into Euclidean space, the minus sign of the rotor in projective space may be neglected. Hence, **we can use the same representation of a rotor in Euclidean and projective space**.

2.1.8 A Strange Reflection in Projective Space

We have so far looked at reflections of homogeneous vector on the homogeneous dimension e_o and on direction vectors, ie homogeneous vectors with a zero e_o component. However, what does a reflection of a homogeneous vector on another homogeneous vector look like?

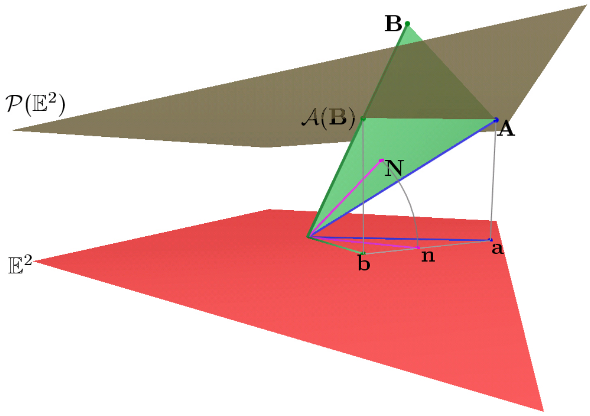


Figure 2.7: Effect in $\mathbb{E}^2$ and $\mathbb{PE}^2$ of the reflection of a homogeneous vector $\mathbf{A}$ on another homogeneous vector $\mathbf{N}$.

The answer to this question is illustrated in figure 2.7. Vector $\mathbf{a} \in \mathbb{E}^2$ is embedded in projective space as $\mathbf{A} = \mathcal{P}(\mathbf{a}) \in \mathbb{PE}^2$. Instead of reflecting $\mathbf{a}$ on $\mathbf{n}$ in $\mathbb{E}^2$, $\mathbf{A}$ is reflected on $\mathbf{N}$ in $\mathbb{PE}^2$.

In this example $\mathbf{n} = \mathsf{e}_1$ and $\mathbf{N} = \frac{1}{\sqrt{2}}\,(\mathbf{n} + \mathsf{e}_o)$, ie $\mathbf{N}$ is of unit "length", if we regard $\mathbb{PE}^2 \equiv \mathbb{R}^3$ as a three dimensional Euclidean space for a moment. A reflection of $\mathbf{A}$ on $\mathbf{N}$ will thus negate the component of $\mathbf{A}$ perpendicular to $\mathbf{N}$, which results in $\mathbf{B}$. This vector however lies off the affine plane $\mathcal{P}(\mathbb{E}^2)$. A projection of $\mathbf{B}$ into Euclidean space $\mathbb{E}^2$ then results in $\mathbf{b}$, which is not the reflection of $\mathbf{a}$ on $\mathbf{n}$.

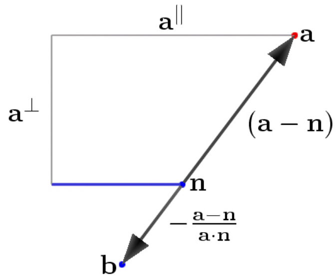


Figure 2.8: Effect in $\mathbb{E}^2$ of reflection of homogeneous vector on another homogeneous vector in $\mathbb{PE}^2$.

Analytically we find the following equation for this type of reflection.

$$\begin{aligned}\mathcal{P}^{-1}\Big(\mathbf{N}\,\mathcal{P}(\mathbf{a})\,\mathbf{N}\Big) &= \mathcal{P}^{-1}\Big(\tfrac{1}{\sqrt{2}}\,(\mathbf{n} + \mathsf{e}_o)\;(\mathbf{a} + \mathsf{e}_o)\;\tfrac{1}{\sqrt{2}}\,(\mathbf{n} + \mathsf{e}_o)\Big)\\ &= \ldots\ \text{exercise :-)}\\ &= (\mathbf{a}^{\|})^{-1} - \tan\theta\,\hat{\mathbf{a}}^{\perp},\end{aligned}$$

where $\mathbf{a}^{\|}$ and $\mathbf{a}^{\perp}$ are again the parallel and perpendicular components of $\mathbf{a}$ with respect to $\mathbf{n}$, $\hat{\mathbf{a}}^{\perp} = \mathbf{a}^{\perp}/\|\mathbf{a}^{\perp}\|$ and $\theta = \angle(\mathbf{a},\,\mathbf{n})$. A geometrically more informative expansion of the above equation is the following

$$\mathcal{P}^{-1}\Big(\mathbf{N}\,\mathcal{P}(\mathbf{a})\,\mathbf{N}\Big) = \mathbf{n} - \frac{\mathbf{a} - \mathbf{n}}{\mathbf{a}\cdot\mathbf{n}}.$$

This latter formula is illustrated in figure 2.8.

Figure 2.9 shows the effect of this type of reflection on the points of the unit circle centered on the origin. The non-central point moving from left to right is the projection into $\mathbb{E}^2$ of the homogeneous vector about which the points on the unit circle were reflected. The reflection of the unit circle becomes an ellipse, a hyperbola or the unit circle if the homogeneous reflection vector becomes the origin.

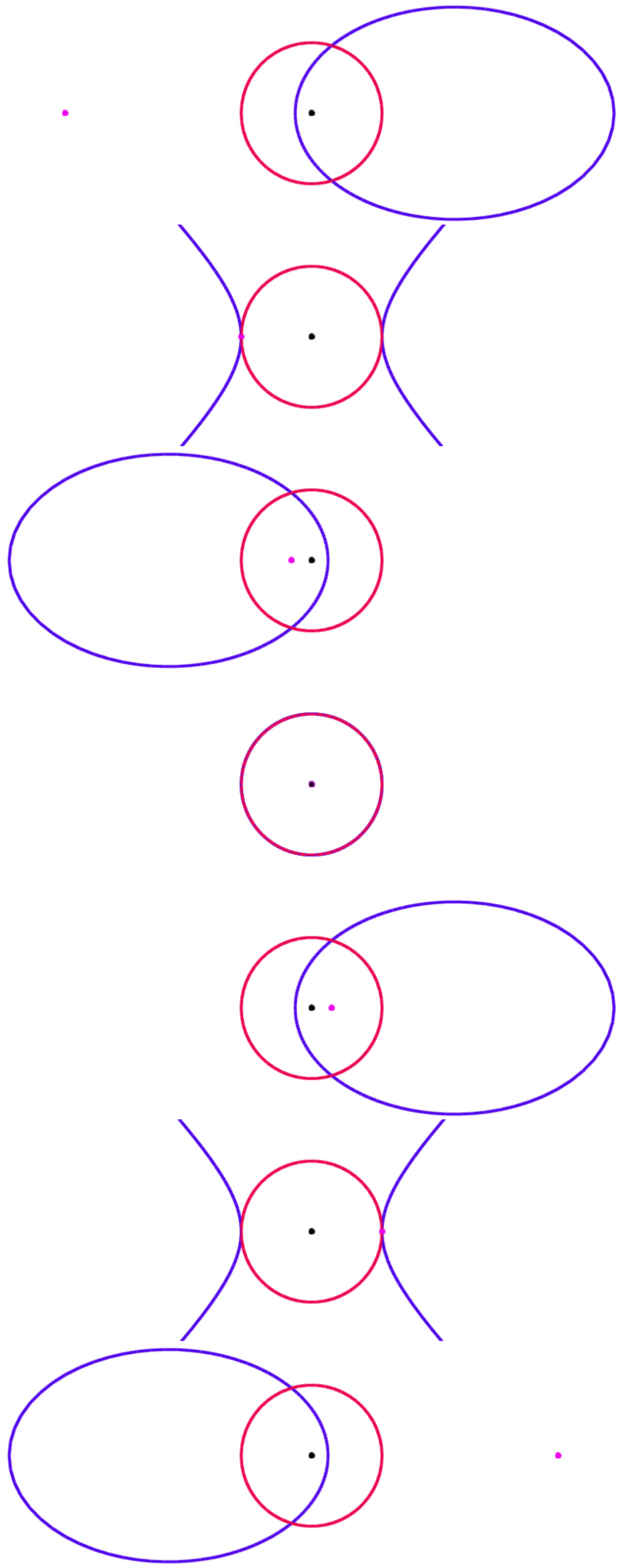


Figure 2.9: Effect of reflecting points on a circle centered on the origin in $\mathbb{E}^2$ on varying homogeneous vectors in $\mathbb{PE}^2$.

2.2 Conformal Space

In this introduction to conformal space we use many of the concepts introduced in the discussion of the projective space. We use again the same trick of embedding Euclidean space in a higher dimensional space, where the extra dimensions have particular meanings (interpretations), such that linear subspaces in conformal space represent particular objects in Euclidean space we are interested in. In projective space the simple ”trick” of adding one dimension and giving it a particular meaning, already enabled us to represent null-dimensional spaces, ie points, in Euclidean space by one dimensional subspaces in projective space. In this way we could also distinguish actual points from directions, which were represented in projective space as elements that project to infinity in Euclidean space.

To introduce conformal space we initially also only add one dimension. However, this time Euclidean space is embedded in a non-linear way in this higher dimensional space. The actual conformal space we will be working with is in fact a special homogenization of the initial conformal space we introduce. That is, when people usually mention the conformal Geometric algebra they actually mean the Geometric algebra over a homogeneous conformal space. We will not break with this tradition here and simply talk of conformal space.

Before we delve into the embedding of Euclidean space in conformal space, we should probably say what conformal actually means. A conformal transformation is one that is locally angle preserving. It turns out that all conformal transformations can be expressed by combinations of inversions. What is an inversion? Well, in $\mathbb{E}^1 \equiv \mathbb{R}$ an inversion of a vector $\mathbf{x} \in \mathbb{R}$ on the unit, one-dimensional sphere centered on the origin is simply $\mathbf{x}^{-1}$. In $\mathbb{E}^3$ the inversion of a plane on the unit sphere centered on the origin is a sphere, as shown in figure 2.10.

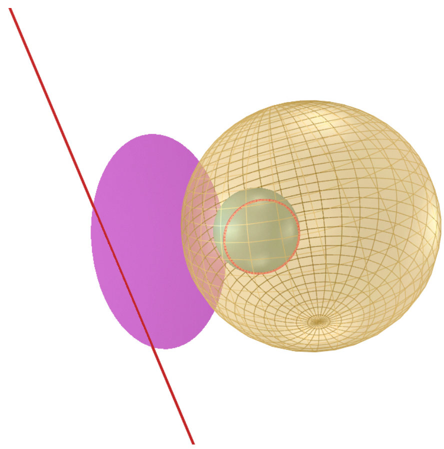


Figure 2.10: Inversion of plane and line on sphere in $\mathbb{E}^3$.

Note that inversions are closely related to reflections in that a reflection is a special case of an inversion. In fact, an inversion on a sphere with infinite radius, ie a plane, is a reflection. Note that all Euclidean transformations can be represented by combinations of reflections. We have already seen this for rotations, which are combinations of two reflections. A translation may be represented by the reflection on two parallel reflection planes. Since all Euclidean transformations can be represented by combinations of reflections and all conformal transformations by combinations of inversions, we see that Euclidean transformations form a subset of conformal transformations.

The actual trick behind the particular embedding of Euclidean space in conformal space is, that a reflection in conformal space represents an inversion in Euclidean space. Here we have to be careful with what we mean when we say reflection. Do we mean a reflection in the space in which the Euclidean space is embedded, or a reflection in Euclidean space itself. In section 2.1.8 we already came across this distinction. A reflection in $\mathbb{PE}^2$ taken as $\mathbb{R}^3$ represented something very unlike a reflection in the corresponding $\mathbb{E}^2$.

2.2.1 Embedding Euclidean Space

We will denote conformal space by $\mathbb{K}^n$ and represent it in $\mathbb{R}^{n+1}$. The additional dimension, however, is this time not a homogeneous dimension. For reasons that will become apparent later on, we will also denote the additional dimension by $\mathsf{e}_+ \equiv \mathsf{e}_{n+1}$. Euclidean space $\mathbb{E}^n$ is embedded in $\mathbb{K}^n$ via a stereographic projection. The embedding function will be denoted by $\mathcal{K}$ and is defined as

$$\mathcal{K} : \mathbf{x} \in \mathbb{E}^n \;\mapsto\; \frac{2}{\mathbf{x}^2 + 1}\,\mathbf{x} \,+\, \frac{\mathbf{x}^2 - 1}{\mathbf{x}^2 + 1}\,\mathsf{e}_+ \;\in \mathbb{K}^n \equiv \mathbb{R}^{n+1}. \tag{2.5}$$

All embedded points lie on a hypersphere of unit radius centered on the origin of $\mathbb{K}^n$. Therefore,

$$\|\mathcal{K}(\mathbf{x} \in \mathbb{E}^n)\| = 1. \tag{2.6}$$

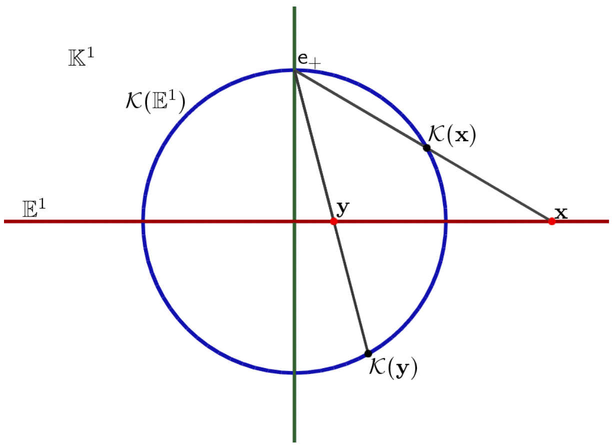


Figure 2.11: Stereographic projection of points $\mathbf{x}, \mathbf{y} \in \mathbb{E}^1$ onto unit circle in $\mathbb{K}^1$.

Figure 2.11 illustrates this embedding for $\mathbb{E}^1$. Note that the point $\mathsf{e}_+ \in \mathbb{K}^1$ represents $\pm\infty \in \mathbb{E}^1$ and $-\mathsf{e}_+$ represents the origin of $\mathbb{E}^1$. Figure 2.12 shows the stereographic projection of a line and a circle from $\mathbb{E}^2$ into $\mathbb{K}^2$. We can see that a line is mapped to a circle on $\mathcal{K}(\mathbb{E}^2)$ that passes through e_+ and a circle in $\mathbb{E}^2$ maps to a circle on $\mathcal{K}(\mathbb{E}^2)$ that does not pass through e_+.

The conformal embedding operator $\mathcal{K}$ transforms the whole of $\mathbb{E}^n$ to a n-dimensional subspace of $\mathbb{K}^n$. This implies that an inverse transformation will only be able to transform points

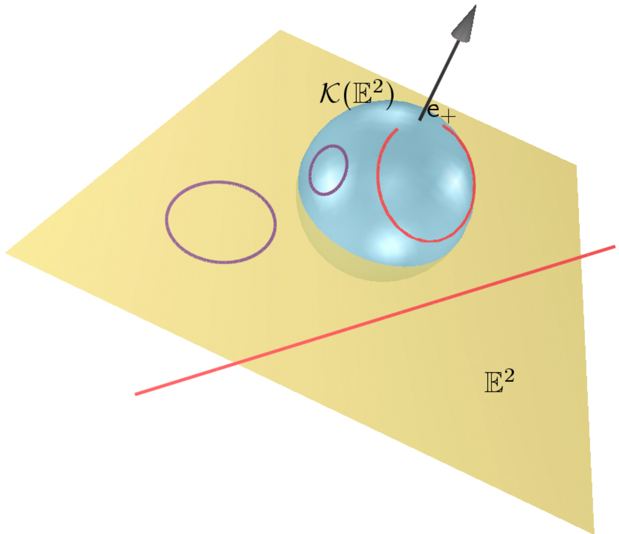


Figure 2.12: Stereographic projection of a line and a circle in $\mathbb{E}^2$ into $\mathbb{K}^2$.

from that subspace of $\mathbb{K}^n$ back to $\mathbb{E}^n$. Recall that for the projective space we also had such a restriction: the plane of homogeneous vectors with a zero e_o component could not be transformed back to Euclidean space.

Mathematically we can express this restriction on the back projection by saying that only vectors $\mathbf{x} \in \mathbb{K}^n$ that satisfy $\|\mathbf{x}\| = 1$ can be projected into $\mathbb{E}^n$. For those vectors the inverse operator $\mathcal{K}^{-1}$ is given by

$$\mathcal{K}^{-1} \, : \, \mathbf{x} \in \mathbb{K}^n, \, \|\mathbf{x}\| = 1 \; \mapsto \; \frac{1}{1 - \mathbf{x} \cdot e_+} \sum_{i=1}^{n} \left(\mathbf{x} \cdot e_i \right) e_i . \tag{2.7}$$

2.2.2 Homogenizing the Embedding of Euclidean Space

Similar to the homogenization of Euclidean space, we will now homogenize conformal space. Specifically, we embed $\mathbb{K}^n$ in a projective space denoted by $\mathbb{PK}^n$, which we will represent by $\mathbb{R}^{n+1,1} \setminus \mathbf{0}$. The space $\mathbb{R}^{n+1,1} \setminus \mathbf{0}$ is of dimension $n+2$, whereby its orthonormal basis contains $n+1$ basis vectors that square to $+1$ and one basis vector that squares to -1. This type of space is also called *Minkowski* space. The effect of using a negatively squaring homogeneous dimension is quite substantial, as we will see throughout the rest of this text.

We will again use the symbol $\mathcal{P}$ to denote the transformation from conformal $\mathbb{K}^n$ to projective conformal $\mathbb{PK}^n$ space. The transformation is defined as

$$\mathcal{P} \, : \, \mathbf{x} \in \mathbb{K}^n \; \mapsto \; \mathbf{X} = \mathbf{x} + e_- \; \in \mathbb{PK}^n , \tag{2.8}$$

where we denoted the homogeneous dimension by e_-, since $e_- \cdot e_- = -1$ by definition. Now it is also clear why we denoted the extra dimension introduced by $\mathbb{K}^n$ as e_+. Figure 2.13 illustrates this embedding for a vector in $\mathbb{E}^1$.

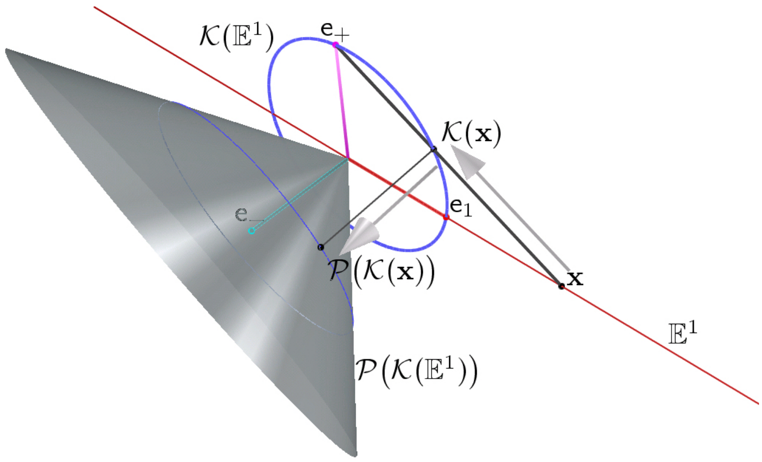


Figure 2.13: Embedding of a vector $\mathbf{x} \in \mathbb{E}^1$ first in $\mathbb{K}^1$ and then in $\mathbb{PK}^1$.

One immediate result that follows from the use of a homogeneous dimension with negative signature is that

$$\begin{aligned}\Big(\alpha\,\mathcal{P}\big(\mathcal{K}(\mathbf{x} \in \mathbb{E}^n)\big)\Big)^2 &= \alpha^2\,\big(\mathcal{K}(\mathbf{x}) + e_-\big)^2 \\ &= \alpha^2\,\Big(\big(\mathcal{K}(\mathbf{x})\big)^2 + \big(e_-\big)^2\Big) \\ &= \alpha^2\,(1-1) \\ &= 0,\end{aligned}$$

where $\alpha \in \mathbb{R} \setminus 0$ is some scale. That is, all vectors in $\mathbb{PK}^n$ that resulted from an embedding of a Euclidean vector from $\mathbb{E}^n$, square to zero. For $\mathbb{E}^1$ the set of points in $\mathbb{PK}^1$ that satisfy this condition lie on a cone. Hence, all null vectors in $\mathbb{PK}^n$, ie all vectors that square to zero in $\mathbb{PK}^n$, are said to lie on the *null cone*. This set of vectors will be denoted by $\mathbb{H}^n \subset \mathbb{PK}^n$ and is defined as

$$\mathbb{H}^n := \{\, \mathbf{X} \in \mathbb{PK}^n \;:\; \mathbf{X}^2 = 0 \,\}. \tag{2.9}$$

From our previous considerations it follows that

$$\mathbb{H}^n = \{\, \alpha\,\mathcal{P}\big(\mathcal{K}(\mathbb{E}^n)\big) \;:\; \alpha \in \mathbb{R} \setminus 0 \,\}.$$

The inverse transformation $\mathcal{P}^{-1}$ from $\mathbb{PK}^n$ into $\mathbb{K}^n$ is defined for all elements $\mathbf{X} \in \mathbb{PK}^n$ that satisfy $\mathbf{X} \cdot e_- \neq 0$ as

$$\mathcal{P}^{-1} \;:\; \mathbf{X} \in \mathbb{PK}^n \;\mapsto\; \frac{1}{\mathbf{X} \cdot e_-} \sum_{i=1}^{n+1} (\mathbf{X} \cdot e_i)\, e_i \;\in \mathbb{K}^n. \tag{2.10}$$

To summarize, the embedding of a Euclidean vector $\mathbf{x} \in \mathbb{E}^n$ in (homogeneous) conformal space $\mathbb{PK}^n$ is given by

$$\mathcal{P}\big(\mathcal{K}(\mathbf{x})\big) = \frac{2}{\mathbf{x}^2+1}\,\mathbf{x} \;+\; \frac{\mathbf{x}^2-1}{\mathbf{x}^2+1}\,e_+ \;+\; e_- \;\in \mathbb{PK}^n. \tag{2.11}$$

Since this is an element of a projective space, an overall scale is not important. We may therefore scale the above equation without changing the vector in $\mathbb{E}^n$ it represents. We will in fact do this, in order to get rid of the fractions. Since we represent $\mathbb{PK}^n$ by $\mathbb{R}^{n+1,1}$, the expression $\mathbf{x}^2+1$ can never be zero[1].

$$\begin{aligned}\tfrac{1}{2}(\mathbf{x}^2+1)\,\mathcal{P}\big(\mathcal{K}(\mathbf{x})\big) &= \mathbf{x} \;+\; \tfrac{1}{2}(\mathbf{x}^2-1)\,\mathsf{e}_+ \;+\; \tfrac{1}{2}(\mathbf{x}^2+1)\,\mathsf{e}_- \\ &= \mathbf{x} \;+\; \tfrac{1}{2}\,\mathbf{x}^2\,(\mathsf{e}_- + \mathsf{e}_+) \;+\; \tfrac{1}{2}\,(\mathsf{e}_- - \mathsf{e}_+) \\ &= \mathbf{x} \;+\; \tfrac{1}{2}\,\mathbf{x}^2\,\mathsf{e}_\infty \;+\; \mathsf{e}_o,\end{aligned} \tag{2.12}$$

where we defined

$$\mathsf{e}_\infty := \mathsf{e}_- + \mathsf{e}_+ \quad \text{and} \quad \mathsf{e}_o := \tfrac{1}{2}\,(\mathsf{e}_- - \mathsf{e}_+). \tag{2.13}$$

The embedding of a Euclidean vector in $\mathbb{PK}^n$ will from now on always be given in the form of equation (2.12). We will therefore define a homogeneous conformal embedding operator $\mathcal{C}$ as

$$\mathcal{C}\;:\;\mathbf{x}\in\mathbb{E}^n \;\mapsto\; \tfrac{1}{2}(\mathbf{x}^2+1)\,\mathcal{P}\big(\mathcal{K}(\mathbf{x})\big)\;\in\mathbb{PK}^n, \tag{2.14}$$

such that

$$\mathcal{C}(\mathbf{x}) = \mathbf{x} \;+\; \tfrac{1}{2}\,\mathbf{x}^2\,\mathsf{e}_\infty \;+\; \mathsf{e}_o. \tag{2.15}$$

Figure 2.14 illustrates this type of embedding. The vector $\mathbf{x}$ is embedded in $\mathbb{PK}^1$ just as in figure 2.13. Then it is scaled such that its e_o component is unity. It then lies on the parabola $\mathbb{H}^1_a$. The

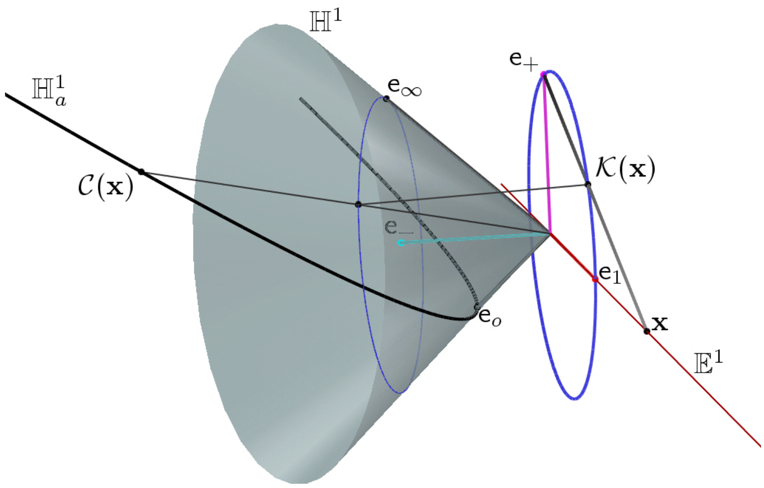


Figure 2.14: Embedding of a vector $\mathbf{x}\in\mathbb{E}^1$ first in $\mathbb{K}^1$ and then in $\mathbb{H}^1_a$.

inverse operator $\mathcal{C}^{-1}$ is only defined for vectors on the null cone $\mathbb{H}^n$.

$$\mathcal{C}^{-1}\;:\;\mathbf{x}\in\mathbb{H}^n \;\mapsto\; \mathcal{K}^{-1}\big(\mathcal{P}^{-1}(\mathbf{x})\big)\;\in\mathbb{E}^n. \tag{2.16}$$

[1]This would be possible if were to regard $\mathbb{PK}^n$ as a vector space over the complex numbers $\mathbb{C}$

The properties of e_∞ and e_o are quite important, so we should state them here. They are easily derived from the properties of e_+ and e_-.

$$\mathsf{e}_\infty^2 = \mathsf{e}_o^2 = 0 \quad \text{and} \quad \mathsf{e}_\infty \cdot \mathsf{e}_o = -1. \tag{2.17}$$

In projective space $\mathbb{PE}^n$ we introduced an operator $\mathcal{A}$ that maps vectors of $\mathbb{PE}^n$ onto the affine plane $\mathcal{P}(\mathbb{E}^n)$. That is, it scales a homogeneous vector such that its component along the homogeneous dimension is unity. The vectors in $\mathbb{PK}^n$ whose component along e_o is unity have a number of useful properties, as will be seen later. Even though e_o is not the homogeneous dimension of $\mathbb{PK}^n$, we will call the set of null vectors that have a unit e_o component the *affine* null cone. The affine null cone is denoted by $\mathbb{H}_a^n$ and defined as

$$\mathbb{H}_a^n := \{\, \mathbf{X} \in \mathbb{H}^n \subset \mathbb{PK}^n \;:\; \mathbf{X} \cdot \mathsf{e}_\infty = -1 \,\}. \tag{2.18}$$

2.2.3 Geometric Algebra on $\mathbb{PK}^n$

Just as for projective space we can form a Geometric algebra over $\mathbb{PK}^n$ denoted by $\mathcal{C}\ell(\mathbb{PK}^n)$. Blades in $\mathcal{C}\ell(\mathbb{PK}^n)$ again represent linear subspaces through their IPNS and OPNS with respect to $\mathbb{PK}^n$ itself. However, we are only interested in the set of *Euclidean* vectors that embedded in conformal space lie in the IPNS or OPNS of a blade of $\mathcal{C}\ell(\mathbb{PK}^n)$. Hence, the Euclidean IPNS and OPNS for $\mathbb{PK}^n$ are defined as

$$\begin{aligned} \mathbb{NO}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &:= \{\, \mathbf{x} \in \mathbb{E}^n \;:\; \mathcal{C}(\mathbf{x}) \wedge A = 0 \,\}, \\ \mathbb{NI}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &:= \{\, \mathbf{x} \in \mathbb{E}^n \;:\; \mathcal{C}(\mathbf{x}) \cdot A = 0 \,\}. \end{aligned} \tag{2.19}$$

Since we know that all vectors on the null cone in $\mathbb{PK}^n$ can be projected into $\mathbb{E}^n$, these sets can also be expressed as

$$\begin{aligned} \mathbb{NO}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \mathcal{C}^{-1}\Big(\{\, \mathbf{X} \in \mathbb{H}^n \;:\; \mathbf{X} \wedge A = 0 \,\}\Big), \\ \mathbb{NI}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \mathcal{C}^{-1}\Big(\{\, \mathbf{X} \in \mathbb{H}^n \;:\; \mathbf{X} \cdot A = 0 \,\}\Big). \end{aligned}$$

In other words, the vectors in $\mathbb{PK}^n$ we are interested in are those that lie on the intersection of the null space represented by $A \in \mathcal{C}\ell(\mathbb{PK}^n)$ and the null cone $\mathbb{H}^n$. That is,

$$\begin{aligned} \mathbb{NO}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \mathcal{C}^{-1}\Big(\mathbb{NO}(A) \cap \mathbb{H}^n\Big), \\ \mathbb{NI}_E\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \mathcal{C}^{-1}\Big(\mathbb{NI}(A) \cap \mathbb{H}^n\Big), \end{aligned}$$

with

$$\begin{aligned} \mathbb{NO}\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \{\, \mathbf{X} \in \mathbb{PK}^n \;:\; \mathbf{X} \wedge A = 0 \,\}, \\ \mathbb{NI}\big(A \in \mathcal{C}\ell(\mathbb{PK}^n)\big) &= \{\, \mathbf{X} \in \mathbb{PK}^n \;:\; \mathbf{X} \cdot A = 0 \,\}. \end{aligned}$$

An example of the OPNS of a bivector in $\mathcal{C}\ell(\mathbb{PK}^1)$ is shown in figure 2.15. Vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{H}_a^1$ span a 2d-subspace in $\mathbb{PK}^1$, the plane $\mathbb{NO}(\mathbf{X} \wedge \mathbf{Y})$. However, the Euclidean OPNS of $\mathbf{X} \wedge \mathbf{Y}$ is the set of points on $\mathbb{H}_a^1$ that lie in $\mathbb{NO}(\mathbf{X} \wedge \mathbf{Y})$. These are simply the points $\mathbf{X}$ and $\mathbf{Y}$. Hence, $\mathbb{NO}_E(\mathbf{X} \wedge \mathbf{Y})$ is the point pair $\mathcal{C}^{-1}(\mathbf{X})$ and $\mathcal{C}^{-1}(\mathbf{Y})$.

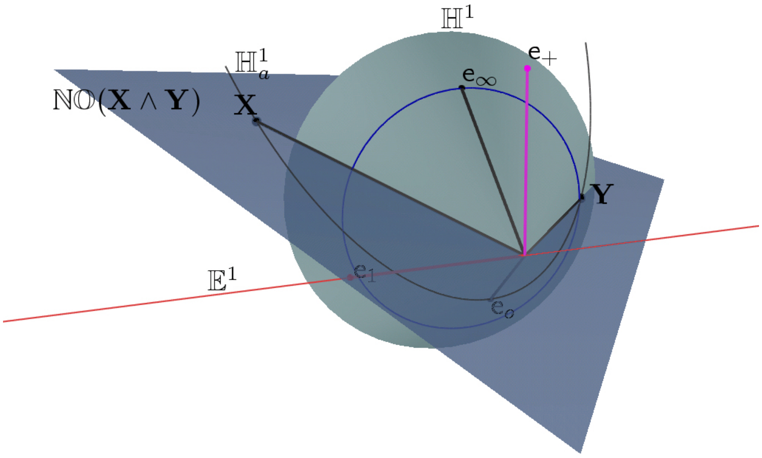


Figure 2.15: OPNS of the outer product of two vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{PK}^1$.

2.2.4 Representation of Geometric Entities in $\mathbb{PK}^3$

It is initially easier to look at the Euclidean IPNS of blades in $\mathbb{PK}^3$. For a start, we will consider a Euclidean vector $\mathbf{a} \in \mathbb{E}^3$ with its conformal embedding

$$\mathbf{A} = \mathcal{C}(\mathbf{a}) = \mathbf{a} + \tfrac{1}{2}\mathbf{a}^2\,\mathrm{e}_\infty + \mathrm{e}_o \;\in \mathbb{H}^3_a.$$

Before we look at the Euclidean IPNS of this vector, we we look at the general inner product of $\mathbf{A}$ with another vector $\mathbf{B} \in \mathbb{H}^3_a$, given by

$$\mathbf{B} = \mathcal{C}(\mathbf{b}) = \mathbf{b} + \tfrac{1}{2}\mathbf{b}^2\,\mathrm{e}_\infty + \mathrm{e}_o \;\in \mathbb{H}^3_a.$$

Using the properties of e_∞ and e_o we find

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (\mathbf{a} + \tfrac{1}{2}\mathbf{a}^2\,\mathrm{e}_\infty + \mathrm{e}_o)\cdot(\mathbf{b} + \tfrac{1}{2}\mathbf{b}^2\,\mathrm{e}_\infty + \mathrm{e}_o)\\ &= \mathbf{a}\cdot\mathbf{b} - \tfrac{1}{2}\mathbf{a}^2 - \tfrac{1}{2}\mathbf{b}^2\\ &= -\tfrac{1}{2}(\mathbf{a}-\mathbf{b})^2\\ &= -\tfrac{1}{2}\|\mathbf{a}-\mathbf{b}\|^2.\end{aligned} \tag{2.20}$$

That is, the inner product of two conformal vectors in $\mathbb{H}^3_a$ gives a measure of the Euclidean distance of their corresponding Euclidean vectors. That's pretty neat and is the fundamental feature of conformal space we will use over and over again.

2.2.4.1 The Representation of Points

The IPNS of a vector $\mathbf{A} \in \mathbb{H}^3$ is given as usual by

$$\mathbb{NI}(\mathbf{A} \in \mathbb{H}^3) := \left\{\, \mathbf{X} \in \mathbb{PK}^3 \;:\; \mathbf{X}\cdot\mathbf{A} = 0 \,\right\}$$

However, we know that vectors on the null cone are null vectors and thus

$$\mathbb{NI}(\mathbf{A} \in \mathbb{H}^3) = \left\{ \alpha\, \mathbf{A} \;:\; \alpha \in \mathbb{R} \setminus 0 \right\},$$

and the corresponding Euclidean IPNS is

$$\mathbb{NI}_E(\mathbf{A} \in \mathbb{H}^3) = \mathcal{C}^{-1}\Big(\mathbb{NI}(\mathbf{A})\Big) = \mathbf{a}.$$

Just as for the projective space, we have again the feature that we can represent null dimensional entities in Euclidean space by one dimensional subspaces in (homogeneous) conformal space.

2.2.4.2 The Representation of Spheres

Now we know that vectors on the null cone in $\mathbb{PK}^3$ represent points in Euclidean space $\mathbb{E}^3$. However, what to vectors in $\mathbb{PK}^3$ off the null cone represent? We will initially only discuss their IPNS representation. Consider the vector $\mathbf{A} \in \mathbb{H}^3_a$ on the affine null cone and the vector $\mathbf{S} \in \mathbb{PK}^3$ off the null cone, given by

$$\mathbf{S} = \mathbf{A} - \tfrac{1}{2}\rho^2\, \mathsf{e}_\infty, \quad \rho \in \mathbb{R}. \tag{2.21}$$

Let $\mathbf{X} \in \mathbb{H}^3_a$, then

$$\begin{aligned} \mathbf{S} \cdot \mathbf{X} &= \mathbf{A} \cdot \mathbf{X} - \tfrac{1}{2}\rho^2\, \mathsf{e}_\infty \cdot \mathbf{X} \\ &= -\tfrac{1}{2}(\mathbf{a} - \mathbf{x})^2 + \tfrac{1}{2}\rho^2. \end{aligned} \tag{2.22}$$

Hence,

$$\mathbf{S} \cdot \mathbf{X} = 0 \iff (\mathbf{a} - \mathbf{x})^2 = \rho^2.$$

That is, the inner product of $\mathbf{S}$ and $\mathbf{X}$ is zero if and only if $\mathbf{x} = \mathcal{C}^{-1}(\mathbf{X})$ lies on a sphere centered on $\mathbf{a} = \mathcal{C}^{-1}(\mathbf{A})$ with radius ρ. Therefore, the Euclidean IPNS of $\mathbf{S}$ is a sphere.

$$\mathbb{NI}_E(\mathbf{S} = \mathbf{A} - \tfrac{1}{2}\rho^2\, \mathsf{e}_\infty) = \left\{ \mathbf{x} \in \mathbb{E}^3 \;:\; \|\mathbf{x} - \mathbf{a}\|^2 = \rho^2 \right\}. \tag{2.23}$$

Note that since we are working in a homogeneous conformal space, also every scaled version of $\mathbf{S}$ represents the same sphere. However, if we use the "affine" form as in equation (2.21), we can also evaluate the radius of the sphere represented by $\mathbf{S}$ quite easily.

$$\mathbf{S}^2 = \mathbf{A}^2 - \rho^2\, \mathbf{A} \cdot \mathsf{e}_\infty = \rho^2. \tag{2.24}$$

For an arbitrarily scaled version of $\mathbf{S}$ we can evaluate the radius via

$$\left(\frac{\mathbf{S}}{-\mathbf{S} \cdot \mathsf{e}_\infty} \right)^2 = \rho^2. \tag{2.25}$$

We can also easily tell whether a point lies inside, on or outside the sphere represented by $\mathbf{S}$. From equation (2.22) it follows that

$$\frac{\mathbf{S} \cdot \mathbf{X}}{(\mathbf{S} \cdot \mathsf{e}_\infty)\,(\mathbf{X} \cdot \mathsf{e}_\infty)} \begin{cases} > 0 & : \ \mathbf{x} \text{ inside sphere} \\ = 0 & : \ \mathbf{x} \text{ on sphere} \\ < 0 & : \ \mathbf{x} \text{ outside sphere} \end{cases} \tag{2.26}$$

This feature also forms that basic idea behind the hypersphere neuron [4, 3]. It may be represented as a perceptron with two "bias" components and allows the separation of the input space of a multi-layer perceptron in terms of hyperspheres and not hyperplanes.

So what about vectors of the form

$$\mathbf{S} = \mathbf{A} + \tfrac{1}{2}\rho^2\,\mathsf{e}_\infty. \tag{2.27}$$

The inner product of $\mathbf{S}$ with some $\mathbf{X} \in \mathbb{H}^3_a$ gives

$$\mathbf{S} \cdot \mathbf{X} = -\tfrac{1}{2}(\mathbf{a} - \mathbf{x})^2 - \tfrac{1}{2}\rho^2,$$

such that

$$\mathbf{S} \cdot \mathbf{X} = 0 \iff (\mathbf{a} - \mathbf{x})^2 = -\rho^2.$$

Since we assumed $\mathbb{E}^3$ to be a vector space over $\mathbb{R}$, this condition is never satisfied for $\rho \neq 0$. However, had we regarded $\mathbb{E}^3$ as a vector space over the complex numbers $\mathbb{C}$, then, together with an appropriate definition of the norm, the solution would be

$$\|\mathbf{a} - \mathbf{x}\| = \mathsf{i}\,\rho,$$

where $\mathsf{i} = \sqrt{-1}$ is the imaginary unit. We may thus say that $\mathbf{S}$ as defined in equation (2.27) represents a sphere with *imaginary* radius in $\mathbb{E}^3$.

Note that any vector in $\mathbb{PK}^3$ may be brought into the form

$$\mathbf{S} = \mathbf{A} \pm \tfrac{1}{2}\rho^2\,\mathsf{e}_\infty,$$

where $\mathbf{A}$ is a vector on the null cone. From a visual point of view, we can say that vectors of the type $\mathbf{S} = \mathbf{A} - \frac{1}{2}\rho^2\,\mathsf{e}_\infty$ lie outside the null cone and vectors of the type $\mathbf{S} = \mathbf{A} + \frac{1}{2}\rho^2\,\mathsf{e}_\infty$ lie inside the null cone. We may thus say that any vector in $\mathbb{PK}^3$ either represents a sphere with positive, zero or imaginary radius. In terms of the Euclidean IPNS, the basic "building blocks" of homogeneous conformal space are therefore spheres.

2.2.4.3 The Representation of Planes

We mentioned earlier that a plane can be regarded as a sphere with infinite radius. Since we are working in a homogeneous space, we can represent infinity by setting the homogeneous component of a vector to zero. And this is also all it takes to make a sphere into a plane, which also becomes clear from equation (2.25). Consider the vector $\mathbf{P} \in \mathbb{H}^3$ given by

$$\mathbf{P} = \mathbf{A} - \mathsf{e}_o - \tfrac{1}{2}\rho^2\,\mathsf{e}_\infty = \mathbf{a} + \tfrac{1}{2}\mathbf{a}^2\,\mathsf{e}_\infty - \tfrac{1}{2}\rho^2\,\mathsf{e}_\infty.$$

The inner product of $\mathbf{P}$ with some vector $\mathbf{X} \in \mathbb{H}^3_a$ gives

$$\begin{aligned} \mathbf{P} \cdot \mathbf{X} &= \mathbf{a} \cdot \mathbf{x} - \tfrac{1}{2}\mathbf{a}^2 + \tfrac{1}{2}\rho^2 \\ &= \|\mathbf{a}\|\,\|\mathbf{x}^{\parallel}\| - \tfrac{1}{2}(\mathbf{a}^2 - \rho^2), \end{aligned}$$

where $\mathbf{x}^{\|}$ is the component of $\mathbf{x}$ parallel to $\mathbf{a}$. Therefore,

$$\mathbf{P} \cdot \mathbf{X} = 0 \iff \|\mathbf{x}^{\|}\| = \frac{\mathbf{a}^2 - \rho^2}{2\,\|\mathbf{a}\|}.$$

Hence, all vectors $\mathbf{x}$ whose component along $\mathbf{a}$ has a fixed length lie in the Euclidean IPNS of $\mathbf{P}$, which thus represents a plane with orthogonal distance $(\mathbf{a}^2 - \rho^2)/(2\|\mathbf{a}\|)$ from the origin and normal $\mathbf{a}$.

A particularly nice representation of planes is the difference of two vectors on the affine null cone. That is, for $\mathbf{A}, \mathbf{B} \in \mathbb{H}^3_a$, we define $\mathbf{P} = \mathbf{A} - \mathbf{B}$. The inner product of $\mathbf{P}$ with a vector $\mathbf{X} \in \mathbb{H}^3_a$ then gives

$$\begin{aligned}\mathbf{P} \cdot \mathbf{X} &= \mathbf{A} \cdot \mathbf{X} - \mathbf{B} \cdot \mathbf{X} \\ &= -\tfrac{1}{2}(\mathbf{a} - \mathbf{x})^2 + \tfrac{1}{2}(\mathbf{b} - \mathbf{x})^2.\end{aligned}$$

It follows that

$$\mathbf{P} \cdot \mathbf{X} = 0 \iff \tfrac{1}{2}(\mathbf{a} - \mathbf{x})^2 = \tfrac{1}{2}(\mathbf{b} - \mathbf{x})^2.$$

This is the case if $\mathbf{x}$ lies on the plane half way between $\mathbf{a}$ and $\mathbf{b}$, with normal $\mathbf{a} - \mathbf{b}$.

2.2.4.4 The Other Entities

We have seen in section 1.1.9 (eqn. (1.23), p. 12), that in terms of the IPNS, the outer product of two vectors represents the intersection of their respective inner product null spaces. This is, of course, still valid here. Hence, the Euclidean IPNS of the outer product of two spheres is their intersection circle or point. If the spheres do not intersect we obtain an intersection circle of imaginary radius. The Euclidean IPNS of the outer product of three spheres is accordingly the intersection of three spheres. This may be a point pair, a single point or an imaginary point pair. The Euclidean IPNS of the intersection of four spheres can at most give a single point. This also works for spheres with infinite radius, ie planes.

Since the OPNS is dual to the IPNS, we find the following representations for blades in terms of their OPNS in $\mathbb{PK}^3$. In the following let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E} \in \mathbb{H}^3_a$ be mutually linearly independent vectors.

$$\begin{aligned}
\mathbb{NO}_E(\mathbf{A}) &: \textbf{Point } \mathbf{a} \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B}) &: \textbf{Point pair } (\mathbf{a},\, \mathbf{b}) \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathsf{e}_\infty) &: \textbf{Point pair } (\mathbf{a},\, \infty) \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}) &: \textbf{Circle through } \mathbf{a},\, \mathbf{b},\, \mathbf{c} \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathsf{e}_\infty) &: \textbf{Line through } \mathbf{a},\, \mathbf{b} \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}) &: \textbf{Sphere through } \mathbf{a},\, \mathbf{b},\, \mathbf{c}, \mathbf{d} \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathsf{e}_\infty) &: \textbf{Plane through } \mathbf{a},\, \mathbf{b},\, \mathbf{c} \\
\mathbb{NO}_E(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D} \wedge \mathbf{E}) &: \textbf{The whole space } \mathbb{E}^3\textbf{.}
\end{aligned} \tag{2.28}$$

It may seem strange that there is a "point pair" object. It clearly has to be there, since otherwise the intersection of, for example, a sphere with a circle could not be expressed. However, a somewhat better explanation is that a point pair is nothing else but a one dimensional sphere: a point pair has a center from which all points on the point pair (the pair itself) have the same distance. This is simply the definition of a sphere applied to one dimension. A circle is thus a two dimensional sphere and a point may in fact be interpreted as a zero dimensional sphere. This shows again that the basic entities of conformal space are spheres.

In equation (2.28) you may have wondered why there is a point pair of a finite point in $\mathbb{E}^3$ and infinity. If we recall the stereographic projection of lines in $\mathbb{E}^2$, it becomes clear that two lines that intersect in a point in $\mathbb{E}^2$ intersect in two points when stereographically projected: the north pole e_+ and another point (see figure 2.16). Two parallel lines in $\mathbb{E}^2$ only intersect in e_+. Since e_+ maps to e_∞ in the homogenization of $\mathbb{K}^3$, it is now clear why we need point pairs of the type $\mathbf{A} \wedge \mathsf{e}_\infty$.

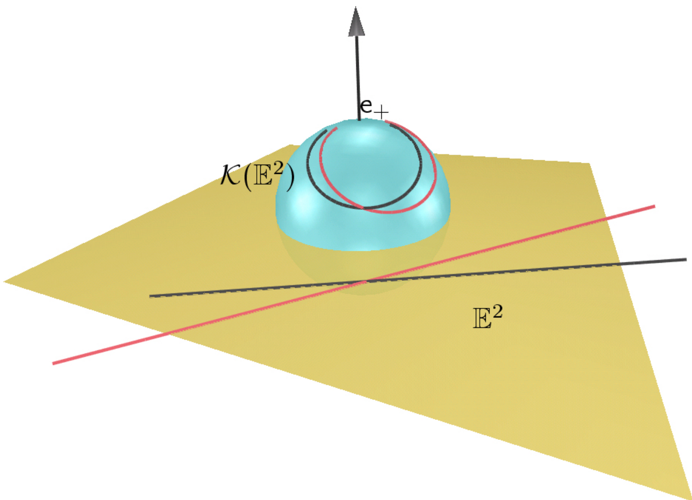


Figure 2.16: Stereographic projection of two intersecting lines.

2.2.5 Discovering $\mathcal{C}\ell(\mathbb{E}^3)$ and $\mathcal{C}\ell(\mathbb{PE}^3)$ in $\mathcal{C}\ell(\mathbb{PK}^3)$

When we look again at equation (2.28), it is interesting to see that those geometric entities that can also be represented in $\mathcal{C}\ell(\mathbb{PE}^3)$ are represented by the outer product of a blade of null vectors and e_∞. It therefore seems as if $\mathcal{C}\ell(\mathbb{PE}^3)$ is a subalgebra of $\mathcal{C}\ell(\mathbb{PK}^3)$. Even though the operators do not carry over immediately, there is an isomorphism between the algebraic entities of the two spaces. We will not give a proof here but a motivation.

Consider again a vector $\mathbf{A} \in \mathbb{H}^3_a$ with

$$\mathbf{A} = \mathbf{a} + \tfrac{1}{2}\mathbf{a}^2\, \mathsf{e}_\infty + \mathsf{e}_o.$$

If we take the outer product of $\mathbf{A}$ with e_∞ we obtain

$$\mathbf{A} \wedge \mathsf{e}_\infty = \mathbf{a} \wedge \mathsf{e}_\infty + \mathsf{e}_o \wedge \mathsf{e}_\infty.$$

If we identify $\mathsf{e}_o \wedge \mathsf{e}_\infty$ with the homogeneous dimension and the bivectors $\{\mathsf{e}_i \wedge \mathsf{e}_\infty\}_{i=1}^3$ with the orthonormal basis vectors of a vector space, then we do obtain an element of $\mathbb{PE}^3$. This also carries over to blades of the type $A_{\langle k \rangle} \wedge \mathsf{e}_\infty$, where $A_{\langle k \rangle}$ is a blade of null vectors excluding e_∞.

In a similar way we can also rediscover $\mathcal{C}\ell(\mathbb{E}^3)$. This time we take the outer product of $\mathbf{A}$ with $\mathsf{e}_\infty \wedge \mathsf{e}_o$,

$$\mathbf{A} \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o = \mathbf{a} \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o.$$

Now we could identify the $\{\mathsf{e}_i \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o\}_{i=1}^3$ with the orthonormal basis of a Euclidean space $\mathbb{E}^3$. In fact, $\mathbb{NO}_E(\mathbf{a} \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o)$ is a line through the origin with the direction of $\mathbf{a}$, that is exactly the same as $\mathbb{NO}(\mathbf{a} \in \mathcal{C}\ell(\mathbb{E}^3))$. Similarly $\mathbb{NO}_E(\mathbf{a} \wedge \mathbf{b} \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o)$ is the same plane through the origin as $\mathbb{NO}(\mathbf{a} \wedge \mathbf{b} \in \mathcal{C}\ell(\mathbb{E}^3))$.

This shows that when we are working in conformal space, we have all the features of Euclidean and projective space combined. This also carries over to the operators, as we will see in the next section. This embedding of Euclidean and projective space in a single framework, offers immediately the possibility to implement the ideas laid out in the well known paper "Stratification of Three Dimensional Vision: Projective, Affine and Metric Representations" by Olivier Faugeras [11], without changing spaces or representations. This has, for example, been used quite successfully in [33, 34].

2.2.6 Inversions in $\mathbb{PK}^n$

When we introduced conformal space initially, we said that it takes its name from the conformal mappings that are possible within it. We carried on to say that a conformal transformation can be expressed by a combination of inversions. However, so far we have not shown how an inversion may be expressed in $\mathbb{PK}^n$.

Before we go for the full monty, let us take a look what effect an inversion has in $\mathbb{K}^n$. Recall that the embedding of a vector $\mathbf{x} \in \mathbb{E}^n$ in $\mathbb{K}^n$ was a stereographic projection defined as

$$\mathcal{K}(\mathbf{x} \in \mathbb{E}^n) = \frac{2}{\mathbf{x}^2 + 1}\,\mathbf{x} + \frac{\mathbf{x}^2 - 1}{\mathbf{x}^2 + 1}\,\mathsf{e}_+.$$

The inverse of $\mathbf{x}$ can be written as

$$\mathbf{x}^{-1} = \frac{\mathbf{x}}{\mathbf{x}^2},$$

which is the same as the inversion of $\mathbf{x}$ on the unit sphere centered at the origin. The embedding of $\mathbf{x}^{-1}$ in $\mathbb{K}^n$ gives

$$\begin{aligned}
\mathcal{K}(\mathbf{x}^{-1}) &= \frac{2}{\frac{\mathbf{x}^2}{\mathbf{x}^4} + 1}\,\frac{\mathbf{x}}{\mathbf{x}^2} + \frac{\frac{\mathbf{x}^2}{\mathbf{x}^4} - 1}{\frac{\mathbf{x}^2}{\mathbf{x}^4} + 1}\,\mathsf{e}_+ \\
&= \frac{2}{\frac{1}{\mathbf{x}^2} + 1}\,\frac{\mathbf{x}}{\mathbf{x}^2} + \frac{\frac{1}{\mathbf{x}^2} - 1}{\frac{1}{\mathbf{x}^2} + 1}\,\mathsf{e}_+ \\
&= \mathbf{x}^2\,\frac{2}{1 + \mathbf{x}^2}\,\frac{\mathbf{x}}{\mathbf{x}^2} + \frac{\frac{1}{\mathbf{x}^2}}{\frac{1}{\mathbf{x}^2}}\,\frac{1 - \mathbf{x}^2}{1 + \mathbf{x}^2}\,\mathsf{e}_+ \\
&= \frac{2}{\mathbf{x}^2 + 1}\,\mathbf{x} - \frac{\mathbf{x}^2 - 1}{\mathbf{x}^2 + 1}\,\mathsf{e}_+.
\end{aligned}$$

This shows that in order to invert a vector in $\mathbb{E}^n$, we only have to negate its e_+ component in its embedding in $\mathbb{K}^n$. That's quite neat. Especially since we can express this negation by a reflection in $\mathbb{K}^n$ on the Euclidean subspace. For example, in $\mathbb{K}^1$, vector $\mathbf{x} = \alpha \mathsf{e}_1$ becomes

$$\mathcal{K}(\mathbf{x}) = \frac{2\alpha}{\alpha^2+1}\,\mathsf{e}_1 + \frac{\alpha^2-1}{\alpha^2+1}\,\mathsf{e}_+.$$

The inverse of $\mathbf{x}$ is then given by

$$\begin{aligned}
\mathbf{x}^{-1} &= \mathcal{K}^{-1}\big(\mathsf{e}_1\,\mathcal{K}(\mathbf{x})\,\mathsf{e}_1\big) \\
&= \mathcal{K}^{-1}\Big(\frac{2\alpha}{\alpha^2+1}\,\mathsf{e}_1 - \frac{\alpha^2-1}{\alpha^2+1}\,\mathsf{e}_+\Big) \\
&= \frac{1}{1+\frac{\alpha^2-1}{\alpha^2+1}}\,\frac{2\alpha}{\alpha^2+1}\,\mathsf{e}_1 \\
&= \frac{2\alpha}{(\alpha^2+1)+(\alpha^2-1)}\,\mathsf{e}_1 \\
&= \alpha^{-1}\,\mathsf{e}_1,
\end{aligned}$$

where we used equation (2.7) to evaluate $\mathcal{K}^{-1}$. Figure 2.17 illustrates this example.

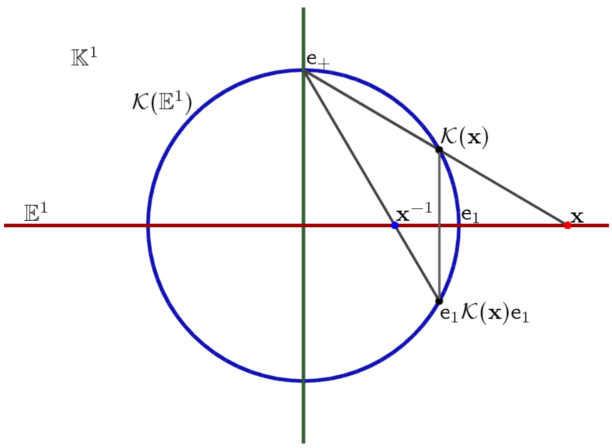


Figure 2.17: Inversion of vector $\mathbf{x}$ in $\mathbb{K}^1$.

Let us now return to $\mathbb{PK}^n$. Here it turns out that an inversion of a vector on the unit sphere centered at the origin is given by a reflection on e_+. Mathematically we find for a vector $\mathbf{X} \in \mathbb{H}^n_a$,

$$\begin{aligned}
\mathsf{e}_+\,\mathbf{X}\,\mathsf{e}_+ &= \mathsf{e}_+\,(\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2\,\mathsf{e}_\infty + \mathsf{e}_o)\,\mathsf{e}_+ \\
&= \ldots\ \text{exercise :-)} \\
&= -1\,(\mathbf{x} + \mathbf{x}^2\,\mathsf{e}_o + \tfrac{1}{2}\,\mathsf{e}_\infty) \\
&= -\mathbf{x}^2\,(\mathbf{x}^{-1} + \tfrac{1}{2}\mathbf{x}^{-2}\,\mathsf{e}_\infty + \mathsf{e}_o).
\end{aligned}$$

Projecting this vector back into $\mathbb{E}^n$ then clearly gives

$$\mathcal{C}^{-1}(\mathsf{e}_+ \, \mathcal{C}(\mathbf{x}) \, \mathsf{e}_+) = \mathbf{x}^{-1}.$$

This is visualized in figure 2.18. Vector $\mathbf{X} \in \mathbb{H}^1_a$ on the affine null cone is reflected on e_+ which gives $\mathsf{e}_+ \mathbf{X} \mathsf{e}_+$. However, if we scale the latter vector such that its e_o component is unity, we obtain $\mathbf{Y}$ which lies again on the affine null cone. Projecting $\mathbf{Y}$ back into $\mathbb{E}^1$ then gives the inverse of $\mathcal{C}^{-1}(\mathbf{X})$.

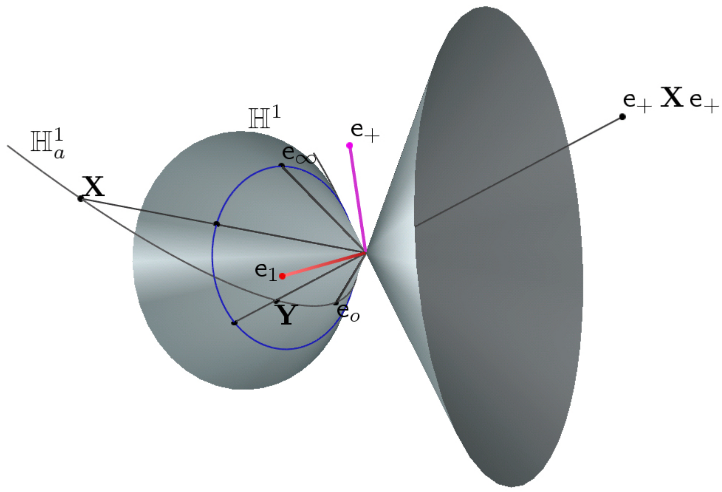


Figure 2.18: Inversion of vector $\mathbf{X} \in \mathbb{H}^1_a$ via reflection on e_+.

Note that using the definitions of $\mathsf{e}_\infty = \mathsf{e}_- + \mathsf{e}_+$ and $\mathsf{e}_o = \frac{1}{2}(\mathsf{e}_- - \mathsf{e}_+)$, we find for the unit sphere $\mathbf{S}$ centered at the origin

$$\begin{aligned} \mathbf{S} &= \underbrace{\mathsf{e}_o}_{\text{origin}} - \underbrace{\tfrac{1}{2}\mathsf{e}_\infty}_{\text{radius 1}} \\ &= \tfrac{1}{2}(\mathsf{e}_- - \mathsf{e}_+) - \tfrac{1}{2}(\mathsf{e}_- + \mathsf{e}_+) \\ &= -\mathsf{e}_+. \end{aligned}$$

Hence, we seem to be able to use vectors in $\mathbb{PK}^n$ representing spheres in their Euclidean IPNS to invert vectors in $\mathbb{H}^n$ on them. In fact, it turns out that we can indeed use any sphere vector to invert *any other blade* in $\mathbb{PK}^n$. In this way we can invert points, lines, circles, planes and spheres on spheres. Note that inversions on circles and point pairs are also possible, but we will not discuss this any further here.

2.2.7 Rotations in $\mathbb{PK}^n$

We said before that the group of Euclidean transformation is a subgroup of the conformal group. Since the conformal group can be created by combinations of inversions and we can express inversion in $\mathcal{C}\ell(\mathbb{PK}^n)$, we should also be able to find operators for the group of Euclidean transformations.

It turns out that in $\mathcal{C}\ell(\mathbb{PK}^n)$ we can not just express reflections on planes that pass through the origin but on arbitrary planes. Therefore, we can also reflect consecutively on two arbitrary planes. The intersection line of two such planes then gives the rotation axis. If the two planes are parallel, ie the rotation axis lies at infinity, we obtain a translation. This, however, will be discussed in the next section.

It is interesting to see how, by enlarging the embedding space of Euclidean space, we get more and more freedom of expression. In Euclidean space we could only express planes that pass through the origin and reflections on planes through the origin. In projective space $\mathbb{PE}^n$, we managed to "free" planes from the origin and place them anywhere. Reflections, however, were still confined to planes through the origin. In conformal space $\mathbb{PK}^n$ we finally also managed to place reflection planes arbitrarily in space.

In order to achieve the reflection of a vector $\mathbf{x} \in \mathbb{E}^n$ on a line with direction $\mathbf{n} \in \mathbb{E}^n$, in $\mathbb{PK}^n$ we have to reflect $\mathcal{C}(\mathbf{x})$ on

$$\mathbf{n} \wedge \mathsf{e}_\infty \wedge \mathsf{e}_o = \mathbf{n} \wedge \mathsf{e}_+ \wedge \mathsf{e}_- .$$

This is similar to what we found for the projective space and is also in accordance with what we said about the embedding of $\mathcal{C}\ell(\mathbb{E}^n)$ in $\mathcal{C}\ell(\mathbb{PK}^n)$ in section 2.2.5. A rotor R expressed as two consecutive reflections on $\mathbf{n} \in \mathbb{E}^n$ and $\mathbf{m} \in \mathbb{E}^n$, will thus take the following form in $\mathcal{C}\ell(\mathbb{PK}^n)$.

$$R = (\mathbf{n} \wedge \mathsf{e}_+ \wedge \mathsf{e}_-)\,(\mathbf{m} \wedge \mathsf{e}_+ \wedge \mathsf{e}_-) = \mathbf{n}\,\mathbf{m}\,\mathsf{e}_+\,\mathsf{e}_-\,\mathsf{e}_+\,\mathsf{e}_- = \mathbf{n}\,\mathbf{m}. \tag{2.29}$$

Therefore, a rotor in $\mathcal{C}\ell(\mathbb{PK}^n)$ expressing a rotation about an axis through the origin, takes again the same form as for $\mathcal{C}\ell(\mathbb{E}^n)$.

2.2.8 Translations in $\mathbb{PK}^n$

It may be shown that a translation in $\mathbb{E}^n$ can also be expressed by two consecutive reflections on two parallel lines. In $\mathbb{PK}^n$ the appropriate operator does take on a similar form as that of a rotor. To cut a long story short, the translation operator, also called *translator*, for a translation by a Euclidean vector $\mathbf{t}$, is given by

$$T = 1 - \tfrac{1}{2}\mathbf{t}\,\mathsf{e}_\infty. \tag{2.30}$$

That is, a translator also has a scalar and a bivector part, just like a rotor. In fact, in terms of the representation of $\mathbb{PK}^n$ as $\mathbb{R}^{n+1,1}$, a translator expresses a rotation. However, the rotation plane does not lie in the Euclidean subspace but in a mixed subspace.

It may be shown that since

$$(\mathbf{t}\,\mathsf{e}_\infty)^2 = \mathbf{t}\,\mathsf{e}_\infty\,\mathbf{t}\,\mathsf{e}_\infty = -\mathbf{t}\,\mathbf{t}\,\mathsf{e}_\infty\,\mathsf{e}_\infty = 0,$$

the operator T can be expressed in exponential form as

$$T = \exp\big(-\tfrac{1}{2}\mathbf{t}\,\mathsf{e}_\infty\big). \tag{2.31}$$

Furthermore, $T\tilde{T} = 1$. If we apply T to the origin e_o we get

$$\begin{aligned} T\,\mathsf{e}_o\tilde{T} &= (1 - \tfrac{1}{2}\mathbf{t}\,\mathsf{e}_\infty)\,\mathsf{e}_o\,(1 + \tfrac{1}{2}\mathbf{t}\,\mathsf{e}_\infty) \\ &= \ldots \text{ difficult exercise :-(} \\ &= \mathbf{t} + \tfrac{1}{2}\mathbf{t}^2\,\mathsf{e}_\infty + \mathsf{e}_o. \end{aligned}$$

If we translate the point at infinity, e_∞, it remains the point at infinity. That is,

$$T\,\mathsf{e}_\infty\,\tilde{T} = \mathsf{e}_\infty.$$

With a translator we can again translate any blade in $\mathcal{C}\ell(\mathbb{PK}^n)$. That is, we can use the operator to translate points, lines, planes, circles and spheres. We can even translate a rotor with a translator, which then results in a rotation about an arbitrary axis in space. Such a general rotation operator may simply be given by

$$M = T\,R\,\tilde{T}. \tag{2.32}$$

If we apply M to a vector $\mathbf{X} \in \mathbb{PK}^n$ we get

$$M\,\mathbf{X}\,M = T\,\underbrace{\overbrace{R\;\underbrace{\tilde{T}\,X\,T}_{\text{translation by }-\mathbf{t}}\;\tilde{R}}^{\text{rotation}}\,\tilde{T}}_{\text{translation by }\mathbf{t}}.$$

One very nice effect of having a translator available is that for many properties it is enough to show that they are valid at the origin. Applying the translation operator it is then possible to show that this property holds everywhere in space. A simple example may elucidate this. For a sphere centered at the origin of radius ρ, we know that the expression in $\mathbb{PK}^n$ is

$$\mathbf{S} = \mathsf{e}_0 - \tfrac{1}{2}\rho^2\,\mathsf{e}_\infty.$$

It is easily shown that

$$\mathbf{S}\cdot\mathbf{S} = \rho^2,$$

in this case. But is this true for any sphere? Suppose now $\mathbf{S}'$ is a sphere with center $\mathbf{t}$ in Euclidean space and let T denote a translator representing a translation by $\mathbf{t}$, then $\mathbf{S}' = \tilde{T}\,\mathbf{S}\,T$, if $\mathbf{S}$ denotes a sphere of the same radius as $\mathbf{S}'$ at the origin. We then find that

$$\mathbf{S}'\,\mathbf{S}' = T\,\mathbf{S}\,\tilde{T}\,T\,\mathbf{S}\,\tilde{T} = T\,\mathbf{S}\,\mathbf{S}\,\tilde{T} = \rho^2\,T\,\tilde{T} = \rho^2.$$

Thus we can relate a property that is valid at the origin to any point in space.

Chapter 3

An Interactive Introduction to Geometric Algebra

by Dietmar Hildenbrand

Geometric Algebra promises to stimulate new methods and insights in all areas of science dealing with geometric properties. It has a lot of advantages, e. g. it allows simple, compact, coordinate-free and dimensionally fluid formulations. It comprises a lot of mathematical systems like

- Clifford Algebra
- Vector Algebra
- Grassmann Algebra
- Complex Numbers
- Quaternions

3.1 Introduction to this interactive Tutorial

In this tutorial we use the **CLUCalc** software to **calculate with Geometric Algebra** and to **visualize the results** of these calculations. CluCalc is available for download at [27]. With help of the CLUCalc Software you are able to edit and run Scripts called **CLUScripts**.

CluCalc offers the following three windows

- editor window
- visualization window (results can be arranged with help of the left mouse button)
- output window

The following CLUScript example ”BaseVectorsE3.clu” draws the 3 base vectors of the 3-dimensional Euclidean space.

```
DefVarsE3();
_BGColor = Color(1,1,1);    // Background white

:Red;
:a=e1;
:b=e2;
:c=e3;
```

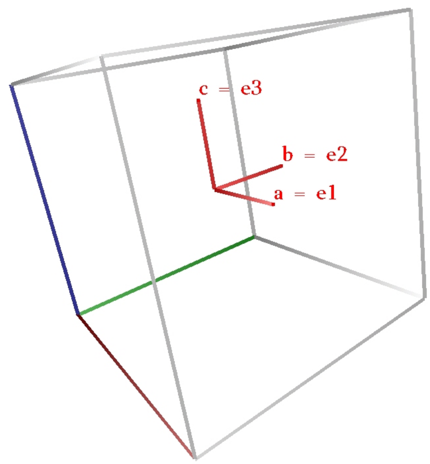


Figure 3.1: BaseVectorsE3.clu

DefVarsE3(); in this CLUScript indicates that we are working in the 3-dimensional Euclidean space E3.
:Red; means that the succeeding geometric objects will be drawn in red.
:a=e1; assigns the base vector e_1 to the variable a and visualize it (Note : without the leading colon it would not be visualized).

Figures generated by CLUScripts are labeled by the name of the script.
These CLUScripts can be downloaded at

```
http://www.dgm.informatik.tu-darmstadt.de/research/gat.zip
```

For details regarding CLUScript please refer to the CLUScript Manual [27].

3.2 Blades and Vectors

Blades are the basic computational elements of the Geometric Algebra.
The Geometric Algebra of the Euclidean 3D space consists of blades with dimension (usually called grade) 0, 1, 2 and 3.
A scalar is a **0-blade** (blade of grade 0).
1-blades are the 3 base vectors e_1, e_2, e_3.
2-blades are plane elements spanned by 2 base vectors.

In the following CLUScript "plane_element.clu" the 2-blade $e_1 \wedge e_2$ (spanned by the 2 base vectors e_1 and e_2) is drawn in red.

```
DefVarsE3();              // 3D Euclidean space

:Blue;
:a=e1;
:b=e2;
:c=e3;

:Red;
:PE = e1^e2;
```

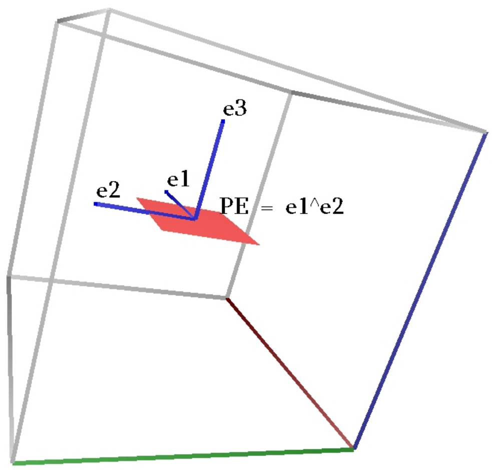


Figure 3.2: plane_element.clu

The Geometric Algebra of the Euclidean 3D space also consists of a **3-blade** $e_1 \wedge e_2 \wedge e_3$ spanned by all the 3 base vectors.

A linear combination of k-blades is called a **k-vector** (also called vectors, bivectors, trivectors ...).

Table 3.1 lists the 8 blades of the Geometric Algebra of the Euclidean 3D space.

Table 3.1: list of blades of the 3D Euclidean space

	blade	**grade**	**abbreviation**
1.	1	0	1
2.	e_1	1	e1
3.	e_2	1	e2
4.	e_3	1	e3
5.	$e_2 \wedge e_3$	2	e23
6.	$e_3 \wedge e_1$	2	e31
7.	$e_1 \wedge e_2$	2	e12
8.	$e_1 \wedge e_2 \wedge e_3$	3	I

3.3 The products of the Geometric Algebra

The Geometric Algebra offers 3 products

- outer product
- inner product
- geometric product

3.3.1 The Outer Product and Parallelness

Geometric Algebra provides an outer product $\wedge$ with the following properties

	Property	Meaning
1.	anti-symmetry	$a \wedge b = -(b \wedge a)$
2.	linearity	$a \wedge (b + c) = a \wedge b + a \wedge c$
3.	associativity	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

What is $\mathbf{a} \wedge \mathbf{a}$ then ?

As you can easily see, the outer product of a vector with itself is always 0.
$a \wedge a = -(a \wedge a) = 0$.

The outer product of parallel vectors is 0. This is why the outer product can be used as a measure for **parallelness**.

3.3.1.1 Bivectors

A **bivector** is a plane element spanned by two vectors. It is the result of the outer product of the vectors.

The following CLUScript bivectorE3.clu computes and draws a simple bivector

```
DefVarsE3();
:Blue;
:a = e1 + e2;
:b = e1 - e2;
:Red;
:c = a ^ b;
?c;  // output in separate window
```

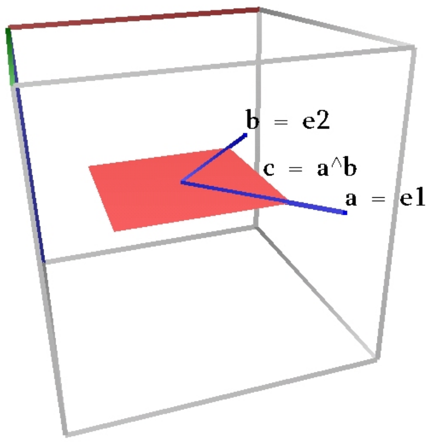


Figure 3.3: bivectorE3.clu

The 2 vectors $a = \mathrm{e}_1 + \mathrm{e}_2$ and $b = \mathrm{e}_1 - \mathrm{e}_2$ are drawn in blue. The result c of their outer product c is a bivector. It is visualized as a plane element in red color.

The algebraic representation of the bivector c is shown in a separate window (see the question mark in front of the variable c) as

```
c  = -2 e12
```

According to table 3.1 this is the same as $-2\ (\mathrm{e}_1 \wedge \mathrm{e}_2)$.

We compute the above mentioned example in order to better understand its geometrical meaning.

$$c = a \wedge b$$

$$= (\mathsf{e}_1 + \mathsf{e}_2) \wedge (\mathsf{e}_1 - \mathsf{e}_2)$$
$$= (\mathsf{e}_1 \wedge \mathsf{e}_1) - (\mathsf{e}_1 \wedge \mathsf{e}_2) + (\mathsf{e}_2 \wedge \mathsf{e}_1) - (\mathsf{e}_2 \wedge \mathsf{e}_2)$$

since $a \wedge a = 0$

$$c = -(\mathsf{e}_1 \wedge \mathsf{e}_2) + (\mathsf{e}_2 \wedge \mathsf{e}_1)$$

because of anti-symmetry

$$c = -(\mathsf{e}_1 \wedge \mathsf{e}_2) - (\mathsf{e}_1 \wedge \mathsf{e}_2)$$

$$= -2(\mathsf{e}_1 \wedge \mathsf{e}_2)$$

because of anti-symmetry

$$c = 2(\mathsf{e}_2 \wedge \mathsf{e}_1)$$

We see that the resulting plane element is

- twice the plane element spanned by the base vectors e_2 and e_1, or
- twice the plane element spanned by the base vectors e_1 and e_2 and inverted **orientation**

3.3.1.2 Trivectors

A trivector is a volume element resulting from the outer product of three vectors. The following CLUScript computes and draws a simple trivector in E3

```
DefVarsE3();

:Blue;
:a = e1 + e2;
:b = e1 - e2;
:c = e3;

:Red;
:d = a ^ b ^ c;
?d;
```

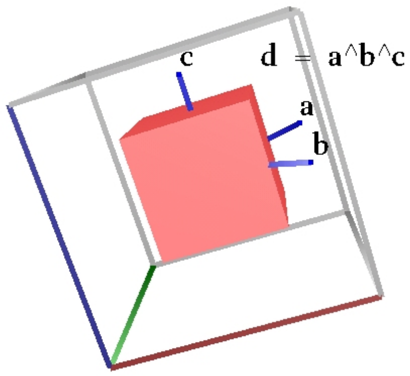


Figure 3.4: trivectorE3.clu

The 3 vectors a, b, c are drawn in blue and their outer product d in red color.

We compute the above mentioned example in order to better understand its geometrical meaning.

$$d = a \wedge b \wedge c = (\mathrm{e}_1 + \mathrm{e}_2) \wedge (\mathrm{e}_1 - \mathrm{e}_2) \wedge \mathrm{e}_3$$
$$= ((\mathrm{e}_1 \wedge \mathrm{e}_1) - (\mathrm{e}_1 \wedge \mathrm{e}_2) + (\mathrm{e}_2 \wedge \mathrm{e}_1) - (\mathrm{e}_2 \wedge \mathrm{e}_2)) \wedge \mathrm{e}_3$$
$$= (-(\mathrm{e}_1 \wedge \mathrm{e}_2) + (\mathrm{e}_2 \wedge \mathrm{e}_1)) \wedge \mathrm{e}_3$$
$$= (-(\mathrm{e}_1 \wedge \mathrm{e}_2) - (\mathrm{e}_1 \wedge \mathrm{e}_2)) \wedge \mathrm{e}_3$$
$$= (-2(\mathrm{e}_1 \wedge \mathrm{e}_2)) \wedge \mathrm{e}_3$$
$$= -2(\mathrm{e}_1 \wedge \mathrm{e}_2 \wedge \mathrm{e}_3)$$
$$= -2I$$

This means, the resulting geometric object $a \wedge b \wedge c$ is equal to -2 multiplied by the volume element spanned by the 3 base vectors $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$. This is often denoted as I, the so-called **pseudoscalar**.

3.3.2 The Inner Product and Perpendicularity

Geometric Algebra offers a so-called inner product denoted by $A \cdot B$
(in CLUScript A.B).

3.3.2.1 The Inner Product of vectors

For Euclidean spaces, the inner product of 2 vectors is the same as the well known Euclidean **scalar product** of 2 vectors.

The result of the following CLUScript

```
DefVarsE3();
B = e1+e2;
? length = sqrt(B.B);
```

is

```
length = 1.41421,
```

the length of the vector $e_1 + e_2$.

For **perpendicular** vectors the inner product is 0.

The result of the following CLUScript

```
DefVarsE3();
? norm = e1.e2;
```

is

```
norm = 0,
```

since the two base vectors are perpendicular.

3.3.2.2 The general Inner Product

In Geometric Algebra, the inner product is not only defined for vectors.

The following CLUScript innerProductE3.clu computes and draws inner product calculations of
- 2 bivectors
- a vector and a bivector

```
DefVarsE3();

:Red;
:B = e1 ^ e2;
? norm = B.B;

:Green;
:x = e1+e3;

:Blue;
// xiB is a vector in the B-plane perpendicular to x
:xiB = x.B;
```

The surprising result of the square product B^2 of the bivector $B = \mathsf{e}_1 \wedge \mathsf{e}_2$ is -1.

The result of the inner product of the vector $x = \mathsf{e}_1 + \mathsf{e}_3$ and the bivector B is a vector in the plane (represented by the bivector B).
The resulting vector is **perpendicular** to x.

Remark : the inner product is grade decreasing, e. g. in the previous example the result of the inner product of an element with grade 2 and grade 1 is an element of grade 2-1 =1.

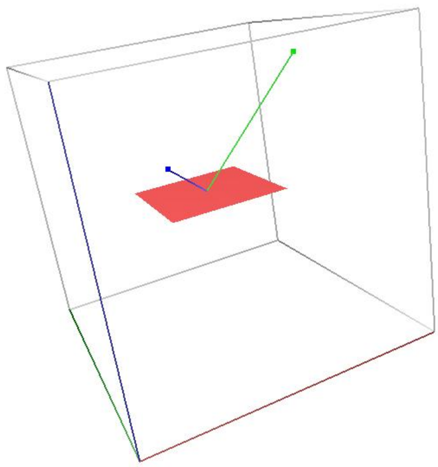


Figure 3.5: innerProductE3.clu

3.3.3 The Geometric Product and Duality

The geometric product is a combination of the outer product and the inner product. The Geometric Product of u and v is denoted by uv (in CLUScript u*v).
As we will see, it is an amazingly powerful operation.

3.3.3.1 The Geometric Product of Vectors

For vectors u and v the geometric product uv is defined as

$$uv = u \wedge v + u \cdot v \tag{3.1}$$

We derive for the inner and the outer product

$$u \cdot v = \frac{1}{2}(uv + vu) \tag{3.2}$$

$$u \wedge v = \frac{1}{2}(uv - vu) \tag{3.3}$$

Example 1 : What is the **square of a vector** ?

$$a^2 = aa = a \wedge a + a \cdot a = a \cdot a$$

for example

$$\mathsf{e}_1\mathsf{e}_1 = \mathsf{e}_1 \cdot \mathsf{e}_1 = 1$$

Example 2 : What is $(\mathsf{e}_1 + \mathsf{e}_2)(\mathsf{e}_1 + \mathsf{e}_2)$?

```
DefVarsE3();
?(e1+e2)*(e1+e2);
```

results in

```
Constant = 2
```

$$(\mathsf{e}_1 + \mathsf{e}_2)(\mathsf{e}_1 + \mathsf{e}_2) = (\mathsf{e}_1 + \mathsf{e}_2) \cdot (\mathsf{e}_1 + \mathsf{e}_2) = \mathsf{e}_1 \cdot \mathsf{e}_1 + \mathsf{e}_1 \cdot \mathsf{e}_2 + \mathsf{e}_2 \cdot \mathsf{e}_1 + \mathsf{e}_2 \cdot \mathsf{e}_2 = \mathsf{e}_1 \cdot \mathsf{e}_1 + \mathsf{e}_2 \cdot \mathsf{e}_2 = 2$$

Example 3 : What is $\mathsf{e}_1\mathsf{e}_2$?

```
DefVarsE3();
?e1*e2;
```

results in

```
Constant = e12
```

$$e_1e_2 = e_1 \wedge e_2 + e_1 \cdot e_2 = e_1 \wedge e_2$$

Example 4 : What is $e_1(e_1 + e_2)$?

```
DefVarsE3();
?e1*(e1+e2);
```

results in

```
Constant = e12 +1
```

$$e_1(e_1 + e_2) = e_1e_1 + e_1e_2 = 1 + e_1 \wedge e_2$$

Note : The result of this calculation is a linear combination of different types of blades (in this example of a scalar and a bivector). These kind of expressions are called **multivectors**.

3.3.3.2 Extension of the Geometric Product to general multivectors

The geometric product is not only defined for vectors but also for all kind of multivectors.
Let us for example calculate the geometric product of 2 bivectors:

```
DefVarsE3();
?(e1^e2)*((e1+e2)^e3);
```

The result is

```
Constant = - e23 - e31
```

Proof

$$(e_1 \wedge e_2)((e_1 + e_2) \wedge e_3)$$

$$= (e_1e_2)(e_1 \wedge e_3 + e_2 \wedge e_3)$$

$$= e_1e_2(e_1e_3 + e_2e_3)$$

$$= e_1e_2e_1e_3 + e_1e_2e_2e_3$$

$$= -e_2e_1e_1e_3 + e_1e_3$$

$$= -e_2e_3 + e_1e_3$$

$$= -(e_2 \wedge e_3) + e_1 \wedge e_3$$

3.3.3.3 Invertibility

The **invertibility** of a blade A is defined by

$$AA^{-1} = 1$$

The inverse of a vector v is

$$v^{-1} = \frac{v}{v \cdot v}$$

Proof

$$v\frac{v}{v \cdot v} = \frac{v \cdot v}{v \cdot v} = 1$$

Example 1 : What is the inverse of the vector $v = 2\mathsf{e}_1$

```
DefVarsE3();
:v=2*e1;
? 1/v;
```

results in $0.5e1$

Example 2 : What is the **inverse of the pseudoscalar** ?

```
DefVarsE3();
? 1/I;
```

results in the negative of the pseudoscalar ($-I$)

```
Constant = - I;
```

Proof

$$II = (\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_3)(\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \mathsf{e}_3) = (\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3)(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3)$$

$$= \underbrace{\mathsf{e}_1\mathsf{e}_2}_{-\mathsf{e}_2\mathsf{e}_1}\mathsf{e}_3\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3 = -\mathsf{e}_2\mathsf{e}_1\mathsf{e}_3\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3 = \mathsf{e}_2\mathsf{e}_3\underbrace{\mathsf{e}_1\mathsf{e}_1}_{1}\mathsf{e}_2\mathsf{e}_3$$

$$= \mathsf{e}_2\mathsf{e}_3\mathsf{e}_2\mathsf{e}_3 = -\mathsf{e}_3\mathsf{e}_2\mathsf{e}_2\mathsf{e}_3 = -\mathsf{e}_3\mathsf{e}_3 = -1$$

$$\rightarrow II = -1$$

$$\rightarrow II(I^{-1}) = -I^{-1}$$

$$\rightarrow I^{-1} = -I$$

3.3.3.4 Duality

Since the geometric product is **invertible**, divisions by geometric objects are possible. The **dual** of a geometric object is calculated by its division by the pseudoscalar I.

In the following CLUScript DualE3.clu the dual of the plane A is calculated.

```
DefVarsE3();

:Blue;
:A= e2 ^ (e1+e3);

:Green;
:b= A/I;
?b;
```

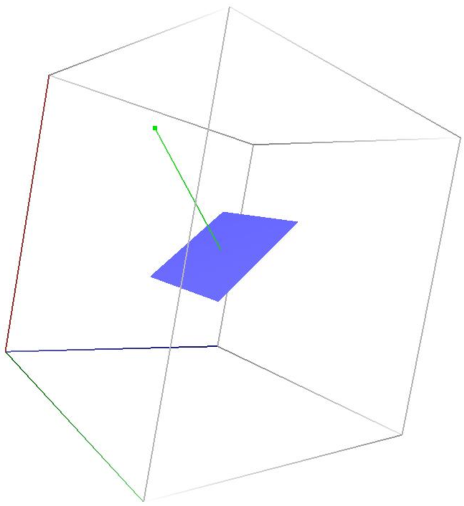


Figure 3.6: DualE3.clu

The resulting vector b

```
b = e1 - e3
```

corresponds to the normal vector of the plane.

Let us verify the result.
A superscript "*" means the dual operator. In CLUScript this is denoted by a leading "*".

$$(e_2 \wedge (e_1 + e_3))^* = (e_2 \wedge (e_1 + e_3))(e_1e_2e_3)^{-1}$$
$$= (e_2 \wedge (e_1 + e_3))(-e_1e_2e_3) = -(e_2(e_1 + e_3))e_1e_2e_3$$
$$= -e_2e_1e_1e_2e_3 - e_2e_3e_1e_2e_3 = -e_2e_2e_3 + e_3e_2e_1e_2e_3$$
$$= -e_3 - e_3e_1e_2e_2e_3 = -e_3 - e_3e_1e_3$$
$$= -e_3 + e_1e_3e_3 = -e_3 + e_1$$

3.4 Geometric Properties

The products of the Geometric Algebra already have some geometric meaning. We will now see some additional geometric properties.

3.4.1 Projection and Rejection

In the following example ProjectE3.clu we compute and draw the projection and rejection of a vector v to a plane B.
The projection is calculated with help of

$$v_{par} = (v \cdot B)/B$$

and the rejection with help of

$$v_{perp} = (v \wedge B)/B$$

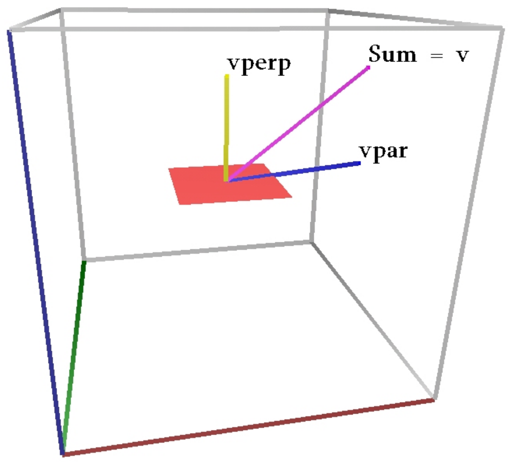


Figure 3.7: ProjectE3.clu

```
DefVarsE3();
:Red;
:B = e1^(e1+e2);
v = 1.5*e1 + e2/3 +e3;
```

The plane B and the vector v are computed. The plane B is drawn in red color.
Remark : the vector v is only computed but not drawn because of the missing colon.

```
:Blue;
:vpar = (v.B)/B;
?vpar;
```

v_{par} is computed as $v_{par} = (v \cdot B)/B$ and drawn in blue color.
It is the **part of v parallel to B** .

```
:Yellow;
:vperp= (v ^ B)/B;
?vperp;
```

v_{perp} is computed as $v_{perp} = (v \wedge B)/B$ and drawn in yellow color.
It is the **part of v perpendicular to B** .

```
:Magenta;
:Sum = vpar + vperp;
?Sum;
```

Sum (as the sum of the 2 vectors v_{par} and v_{perp}) results in the original vector v, since
$vB = v \cdot B + v \wedge B$ and therefore
$(vB)/B = (v \cdot B)/B + (v \wedge B)/B = v$

3.4.2 Reflection

The reflection of a vector v from a plane M is defined by

$$v_{refl} = MvM$$

In the following example ReflectE3.clu we reflect a vector from a plane.

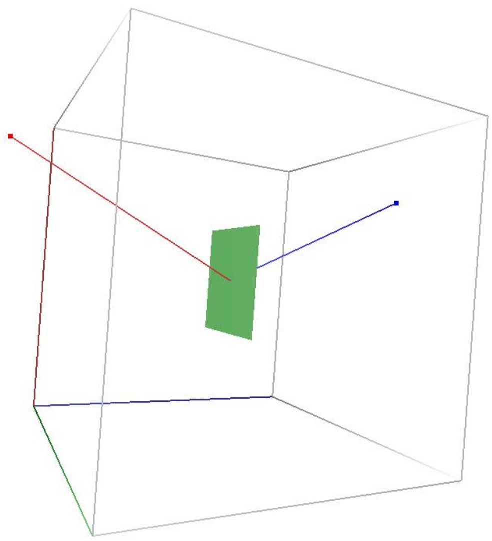


Figure 3.8: ReflectE3.clu

```
DefVarsE3();
:Blue;
:v=e1+2*e3;
:Green;
:M = e1 ^ e2;
```

The vector v is drawn in blue color, the plane M in green.

```
:Red;
:vrefl = M*v*M;
? vrefl;
```

With help of the geometric product MvM the reflected vector v_{refl} is calculated, drawn and printed.

3.4.3 Rotation

In geometric algebra, the ratio $\mathbf{b}/\mathbf{a}$ of two vectors defines the rotation/scaling between these two vectors. In the following example `Rotor2d.clu` we rotate a vector with help of the geometric quotient of 2 vectors.

```
DefVarsE3();

:Blue;
:a = e1;

:Green;
:b = 1/sqrt(2)*(e1+e2);
```

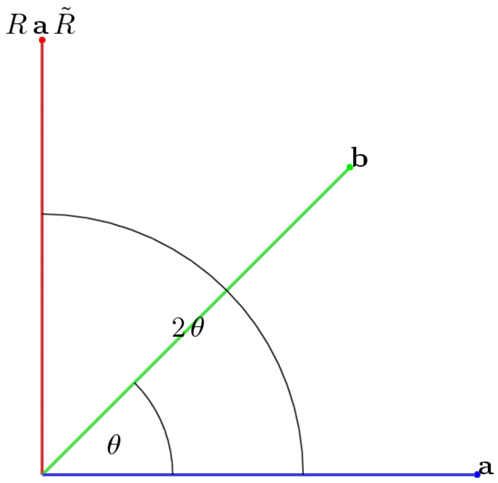


Figure 3.9: Rotor2d.clu

The vector a is drawn in blue color, the vector b in green.

```
?R = b/a;
```

The rotation operator R is calculated as quotient of the vector b and the vector a.

```
:Red;
:c = R*a*~R;
```

The rotated vector c is calculated and drawn in red color. We see that R rotates a by twice the angle between a and b.

The rotation operator can also be calculated with help of an exponential function.

```
DefVarsE3();
:Green;
:i = e1 ^ e2;
```

The plane i is drawn in green.

```
R = exp(-i*(1/2)*(Pi/4));
```

The rotation operator is calculated with help of i and the specific angle `Pi/4` which results in $\pi/4$.

```
:Blue;
:a = e1;
:Red;
:b = R*a*~R;
```

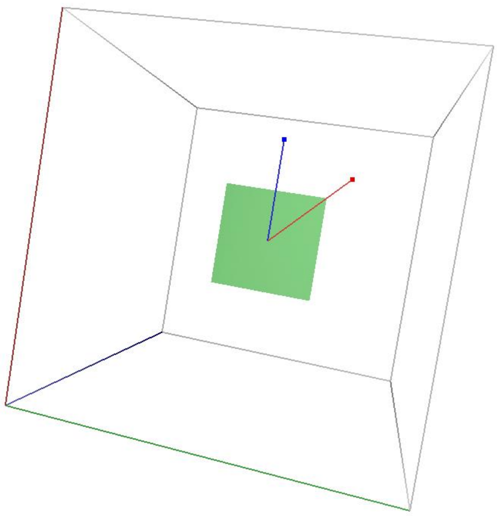


Figure 3.10: Rotate _ EXP _ E3.clu

The operator $R = e^{-i\phi}$ with $i = e1 \wedge e2$ can be decomposed as follows :
With help of the Taylor series and the fact that $i^2 = -1$

$$R = e^{-i\phi} = 1 + \frac{-i\phi}{1!} + \frac{(-i\phi)^2}{2!} + \frac{(-i\phi)^3}{3!} + \frac{(-i\phi)^4}{4!} + \frac{(-i\phi)^5}{5!} + \frac{(-i\phi)^6}{6!}...$$
$$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!}...$$
$$+ - i\frac{\phi}{1!} + -i\frac{(\phi)^3}{3!} - -i\frac{(\phi)^5}{5!}...$$
$$= cos(\phi) - isin(\phi)$$

3.4.4 Intersection

In the geometric algebra, there is a powerful meet operation to calculate the intersection between geometric objects.
The meet operation between two blades A and B is given by

$$A \vee B = A^* \cdot B,$$

if the direct sum of the OPNS of A and B is the whole vector space. In the following example `meetE3.clu` we intersect two planes.

```
DefVarsE3();
:Blue;
:A = e2 ^ (e1 + e3);
:Green;
:B = e1 ^(e2 + e3/2);
```

The two planes A and B are calculated and drawn in blue and green color.

```
:Red;
:mAB = *A.B;
```

The meet operation between A and B is in this case given by $A^* \cdot B$, since the two planes together span the whole 3d-Euclidean space. The result of this intersection is drawn in red.

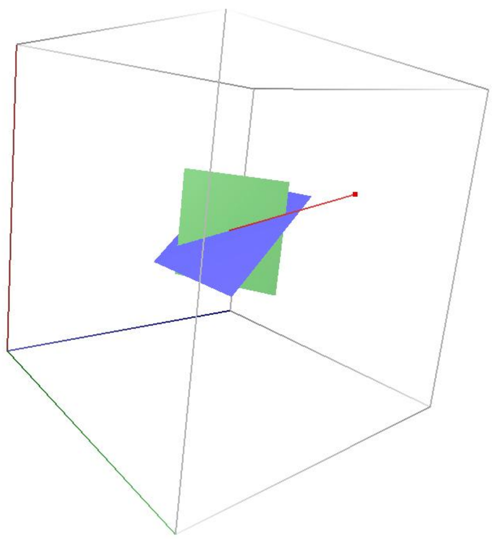


Figure 3.11: meetE3.clu

3.5 The Conformal Geometric Algebra

Up to now we have dealt with the well known Euclidean space.
In the this section we will extend our investigations to one specific non-Euclidean space, the so-called conformal space.
The Conformal Geometric Algebra is a 5-dimensional Geometric Algebra. For details please refer to [38]. In this Algebra, points, spheres and planes are easily represented as vectors (grade 1 blades).

3.5.1 The two additional base vectors

The Conformal Geometric Algebra uses 2 additional base vectors ($\mathsf{e}_+, \mathsf{e}_-$) with the following properties.

$$\mathsf{e}_+^2 = 1 \qquad \mathsf{e}_-^2 = -1 \qquad \mathsf{e}_+ \cdot \mathsf{e}_- = 0 \tag{3.4}$$

Another base ($\mathsf{e}_\infty, \mathsf{e}_o$) can be defined with the following relations

$$\mathsf{e}_o = \frac{1}{2}(\mathsf{e}_- - \mathsf{e}_+) \qquad \mathsf{e}_\infty = \mathsf{e}_- + \mathsf{e}_+$$

The reader is encouraged to verify the following equations.

$$\mathsf{e}_o^2 = e^2 = 0, \qquad \mathsf{e}_\infty \cdot \mathsf{e}_o = -1$$

$$\mathsf{e}_- = \mathsf{e}_o + \frac{1}{2}\mathsf{e}_\infty \qquad \mathsf{e}_+ = \frac{1}{2}\mathsf{e}_\infty - \mathsf{e}_o$$

The outer product $\mathsf{e}_\infty \wedge \mathsf{e}_o$ is often abbreviated by E.

3.5.2 Vectors in Conformal Geometric Algebra

A vector can be written as

$$S = s_1\mathsf{e}_1 + s_2\mathsf{e}_2 + s_3\mathsf{e}_3 + s_4\mathsf{e}_\infty + s_5\mathsf{e}_o \tag{3.5}$$

The point $\mathbf{s} = s_1\mathsf{e}_1 + s_2\mathsf{e}_2 + s_3\mathsf{e}_3$ is denoted as inhomogenous point of the Euclidean space.
Note : bold points $\mathbf{s}$ in this document mean $\mathbf{s} \in \mathbb{R}^3$.

The meaning of the two additional coordinates of the Conformal Geometric Algebra is as follows :

	$s_5 = 0$	$s_5 \neq 0$
$s_4 = 0$	plane through origin	sphere/point through origin
$s_4 \neq 0$	plane	sphere/point

3.5.2.1 Spheres

The sphere S with inhomogenous center point $\mathbf{s}$ and radius r is represented as

$$S = \mathbf{s} + s_4 \mathsf{e}_\infty + \mathsf{e}_o \tag{3.6}$$

with

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2) = \frac{1}{2}(\mathbf{s}^2 - r^2)$$

The radius of the sphere results in

$$r^2 = \mathbf{s}^2 - 2s_4 = s_1^2 + s_2^2 + s_3^2 - 2s_4$$

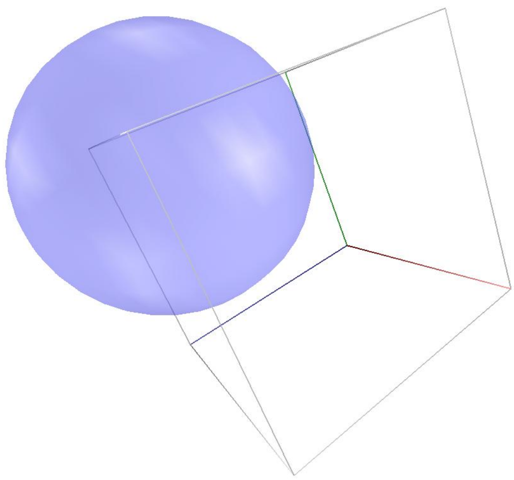


Figure 3.12: OneSphereN3.clu

In the example OneSphereN3.clu

```
DefVarsN3();
:IPNS;
:N3_SOLID;
:S = e2 +e3 - e +e0;
```

the radius the radius of the sphere $S = \mathsf{e}_2 + \mathsf{e}_3 - \mathsf{e}_\infty + \mathsf{e}_o$ results in

$$r^2 = 1 + 1 - 2 * (-1) = 4$$

:DefVarsN3(); is needed because of our conformal space calculations.
:IPNS; means that we describe the sphere with help of the inner product null space (IPNS). OPNS would be used if we would like to describe the sphere with help of its dual representation (quadvector instead of vector).
:N3_SOLID; is needed in order to visualize the sphere solid instead of a wired (N3_WIRED).

3.5.2.2 Points

Points are degenerate spheres with radius $r = 0$. The inhomogenous point $\mathbf{p}$ is represented as

$$X = \mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_\infty + e_o \tag{3.7}$$

3.5.2.3 Planes

Planes are degenerate spheres with infinite radius. They are represented as a vector with $s_5 = 0$.

$$Plane = n_1 e_1 + n_2 e_2 + n_3 e_3 + d e_\infty \tag{3.8}$$

with the normal vector (n_1, n_2, n_3)

$$n_1^2 + n_2^2 + n_3^2 = 1$$

and d as the distance of the plane from the origin.

In the following CLUScript PlaneN3.clu the plane $e_2 + e_\infty$ is drawn in red.

```
DefVarsN3();
:N3_IPNS;

:Red;
:a=VecN3(0,0,0);
:Plane=e2+\be_\infty;
:Green;
:b=VecN3(0,1,0);
```

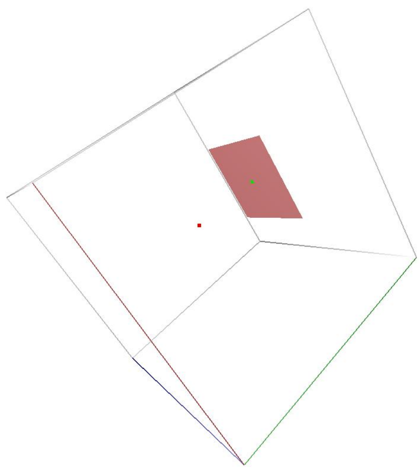


Figure 3.13: PlaneN3.clu

Its normal vector is $(n_1, n_2, n_3) = (0, 1, 0)$ and the distance is 1 (indicated in the picture by the red point a at the origin and the green point b). The points in conformal space are generated by the function **VecN3()**.

3.5.3 Bivectors in Conformal Geometric Algebra

The representation of bivectors of Conformal Geometric Algebra are circles and lines.
Lines are degenerate circles with infinite radius.

3.5.3.1 Circles

A circle can be defined by 3 points. Its algebraic description in Conformal Geometric Algebra is the dual of the outer product of these 3 points.

In the following CLUScript CircleN3.clu a circle is shown in green based on the red points a, b, c.

```
DefVarsN3();
:IPNS;

:Red;
:a=VecN3(0,-0.5,-0.5);
:b=VecN3(0,0.5,0.5);
:c=VecN3(0.5, 0.5, 0.5);

:Green;
:Circle=*(a^b ^ c);

?Circle;
```

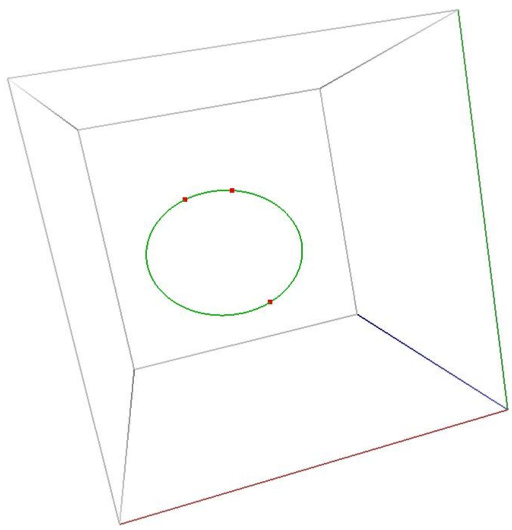


Figure 3.14: CircleN3.clu

The resulting bivector is calculated and printed.

3.5.3.2 Lines

A line as a degenerate circle with infinite radius can be defined by 2 points and the point at infinity.
Its algebraic description in Conformal Geometric Algebra is the dual of the outer product of these 3 points.

In the following CLUScript LineN3.clu a line is shown in green based on the red points a, b.

```
DefVarsN3();
:IPNS;

:Red;
:a=VecN3(0,-0.5,-0.5);
:b=VecN3(0,0.5,0.5);

:Green;
:c=* (a ^ b ^ n );

?c;
```

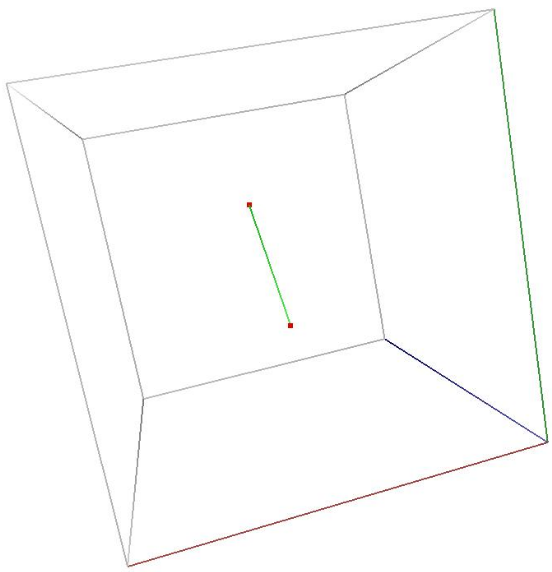


Figure 3.15: LineN3.clu

The point at infinity is indicated by the predefined value n. The resulting bivector is calculated and printed.

3.5.4 Dual Vectors in Conformal Geometric Algebra

In the previous section we already saw circles and lines as the dual of trivectors based on the outer product of three points.
In the same way we are able to define spheres and planes as the dual of the outer product of four points (IPNS) or as the outer product of four points (OPNS).

The dual of vectors in conformal geometric algebra are 4-vectors (or quadvectors).

In the following CLUScript DualSphereN3.clu a sphere generated by four points is visualized.

```
DefVarsN3();
:OPNS;
:N3_SOLID;

:Red;
:A=VecN3(-0.5,0,1);

:Blue;
:B=VecN3(1,-0.5,2);

:Green;
:C=VecN3(0,1.5,3);

:Black;
:D=VecN3(0,2,2);

:Yellow;
:Sphere=A^B^C^D;

?Sphere;
```

The sphere is generated by the outer product of the four points A, B, C, D. These points are indicated by different colors. The resulting quadvector is shown in the output window.

3.5.5 Distances

In the Conformal Geometric Algebra points, planes and spheres are represented as vectors. The inner product of this kind of objects is a scalar and can be used as measure for distances.

In the following examples we will see that the inner product $P \cdot S$ of two vectors P and S can be used for tasks like
- the Euclidean distance between two points
- the distance between one point and one plane
- the decision whether a point is inside or outside of a sphere

Let us first translate the inner product to an expression in Euclidean space.

The inner product between a vector P and a vector S is defined by

$$P \cdot S = (\mathbf{p} + p_4 \mathrm{e}_\infty + p_5 \mathrm{e}_o) \cdot (\mathbf{s} + s_4 \mathrm{e}_\infty + s_5 \mathrm{e}_o)$$

$$= \mathbf{p} \cdot \mathbf{s} + s_4 \underbrace{\mathbf{p} \cdot \mathrm{e}_\infty}_{0} + s_5 \underbrace{\mathbf{p} \cdot \mathrm{e}_o}_{0}$$

$$+ p_4 \underbrace{\mathrm{e}_\infty \cdot \mathbf{s}}_{0} + p_4 s_4 \underbrace{e^2}_{0} + p_4 s_5 \underbrace{\mathrm{e}_\infty \cdot \mathrm{e}_o}_{-1}$$

$$+ p_5 \underbrace{\mathrm{e}_o \cdot \mathbf{s}}_{0} + p_5 s_4 \underbrace{\mathrm{e}_o \cdot \mathrm{e}_\infty}_{-1} + p_5 s_5 \underbrace{\mathrm{e}_o^2}_{0}$$

It results in

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - p_5 s_4 - p_4 s_5 \tag{3.9}$$

or

$$P \cdot S = p_1 s_1 + p_2 s_2 + p_3 s_3 - p_5 s_4 - p_4 s_5$$

3.5.5.1 Distances between points

In the case of P and S being points we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, p_5 = 1$$

$$s_4 = \frac{1}{2}\mathbf{s}^2, s_5 = 1$$

The inner product of these points is

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 - \frac{1}{2}\mathbf{p}^2$$

$$= p_1 s_1 + p_2 s_2 + p_3 s_3 - \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) - \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$$

$$= -\frac{1}{2}(s_1^2 + s_2^2 + s_3^2 + p_1^2 + p_2^2 + p_3^2 - 2p_1 s_1 - 2p_2 s_2 - 2p_3 s_3)$$

$$= -\frac{1}{2}((s_1 - p_1)^2 + (s_2 - p_2)^2 + (s_3 - p_3)^2)$$

$$= -\frac{1}{2}(\mathbf{s} - \mathbf{p})^2$$

We recognize that the square of the Euclidean distance of the inhomogenous points corresponds to the inner product of the homogenous points multiplied by -2.

$$(\mathbf{s} - \mathbf{p})^2 = -2(P \cdot S)$$

3.5.5.2 Distance between points and planes

For a vector P representing a point we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, p_5 = 1$$

For a vector S representing a plane with normal vector $\mathbf{n}$ and distance d we get

$$\mathbf{s} = \mathbf{n}, s_4 = d, s_5 = 0$$

The inner product of point and plane is

$$P \cdot S = \mathbf{p} \cdot \mathbf{n} - d$$

representing the Euclidean distance of a point and a plane.

3.5.5.3 is a point inside or outside of a sphere ?

We will see now that the inner product of a point and a sphere can be used for the decision of whether a point is inside of a sphere or not.

For a vector P representing a point we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, p_5 = 1$$

For a vector S representing a sphere we get

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2), s_5 = 1$$

The inner product of point and sphere is

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}(\mathbf{s}^2 - r^2) - \frac{1}{2}\mathbf{p}^2$$

$$= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 + \frac{1}{2}r^2 - \frac{1}{2}\mathbf{p}^2$$

$$= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s}^2 - 2\mathbf{p} \cdot \mathbf{s} - \mathbf{p}^2)$$

$$= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s} - \mathbf{p})^2$$

We get

$$2(P \cdot S) = r^2 - (\mathbf{s} - \mathbf{p})^2$$

With help of

$$\mathbf{p} = \mathbf{s} \rightarrow P \cdot S = \frac{1}{2}r^2 > 0$$

$$P \cdot S = 0 \rightarrow r^2 = (\mathbf{s} - \mathbf{p})^2$$

we can see that

$r^2 - (\mathbf{s} - \mathbf{p})^2 > 0 :$ $\mathbf{p}$ is inside of the sphere
$r^2 - (\mathbf{s} - \mathbf{p})^2 = 0 :$ $\mathbf{p}$ is on the sphere
$r^2 - (\mathbf{s} - \mathbf{p})^2 < 0 :$ $\mathbf{p}$ is outside of the sphere

3.5.5.4 is a point inside or outside of a circumcircle of a triangle ?

The reader is encouraged to verify that the following CLUScript PointInsideCircleN3.clu is able to decide whether a point is inside or outside of a circumcircle of a triangle.

```
DefVarsN3();

:IPNS;
:N3_SOLID;

:Red;
:A=VecN3(-0.5,0,1);
:Blue;
:B=VecN3(1,-0.5,2);
:Green;
:C=VecN3(0,1.5,3);
:Black;
:X=VecN3(0,4,4);

:Magenta;
:Circle=*(A^B^C);
Plane=*(A^B^C^e);

:Yellow;
:Sphere=Circle*Plane;
?Distance=Sphere.X;
```

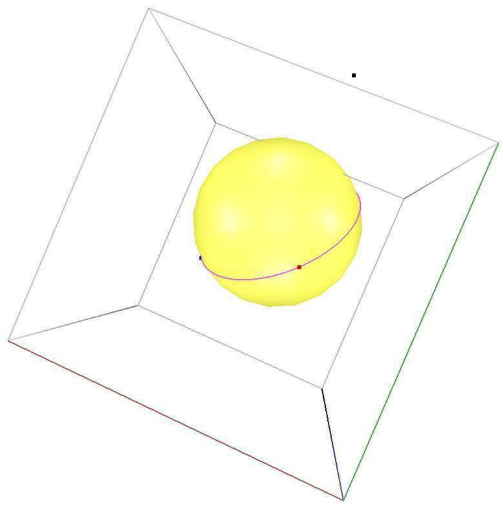


Figure 3.16: PointInsideCircleN3.clu

3.5.6 Intersections

As already mentioned for the 3D Euclidean space the meet operation between two blades A and B may be given by

$$A \vee B = A^* \cdot B,$$

if A and B together span the whole vector space. In the following examples we will compute intersections between different objects like spheres, lines and planes.

3.5.6.1 Intersection of two spheres

In the following CLUScript meetSphereSphereN3.clu the intersection of two spheres is calculated with help of the meet operation.

```
DefVarsN3();
:OPNS;
:N3_SOLID;

:Red;
:a=*(VecN3(0,-0.5,-0.5)-0.5*e);
:b=*(VecN3(0,0.5,0.5)-0.5*e);

:Blue;
:M=*a.b;
?M;
```

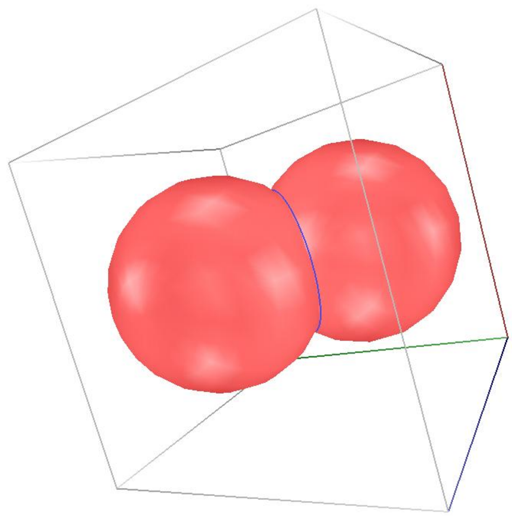


Figure 3.17: meetSphereSphereN3.clu

Two spheres, defined as dual vectors in OPNS are drawn in red color. The intersection of theses spheres is calculated with help of the meet operation. The resulting circle is drawn in blue.

3.5.6.2 Intersection of a line and a sphere

In the following CLUScript meetSphereLineN3.clu the intersection of one sphere and one line is calculated with help of the meet operation.

```
DefVarsN3();
:OPNS;
:N3_SOLID;

:Red;
:a=VecN3(0,-0.5,-0.5);
:b=VecN3(0,0.5,0.5);

:Green;
:l=a^b^n;
?l;

:Yellow;
s=VecN3(0,1,1) -0.1*e;
:s=*s;

:Magenta;
:r=*s.l;
```

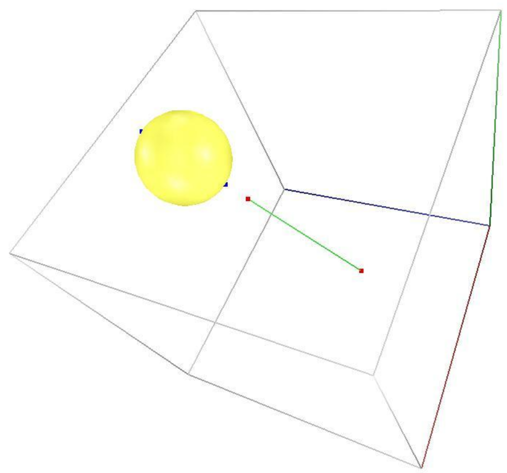


Figure 3.18: meetSphereLineN3.clu

The intersection of the line l (defined by the points a and b) and the sphere s is a **point pair**. This geometric object is visualized in magenta.
A point pair is a trivector in Conformal Geometric Algebra.

3.5.6.3 Intersection of a line and a plane

In the following CLUScript meetPlaneLineN3.clu the intersection of one plane and one line is calculated with help of the meet operation.

```
DefVarsN3();
:OPNS;

:Red;
:a=VecN3(0,-0.5,-0.5);
:b=VecN3(0,0.5,0.5);

:Green;
:l=a^b^n;
?l;

:c=VecN3(2,1,2);
:d=VecN3(1,-1,1);
:e=VecN3(-1,-2,-1);

:Yellow;
:p=c^d^e^n;

:Magenta;
:r=*p.l;
?r;
```

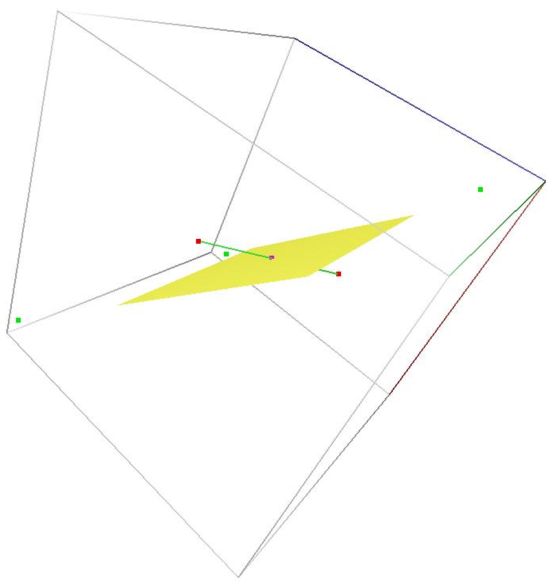


Figure 3.19: meetPlaneLineN3.clu

The plane p is defined with help of the three points c, d, e and the point at infinity n. The intersection with the line l (defined with help of a, b, n) is visualized in magenta.

3.5.7 Reflection

In the following CLUScript ReflectN3.clu we visualize the reflection of one line from one plane.

```
DefVarsN3();
:OPNS;

a=VecN3(0,-0.5,-0.5);
b=VecN3(0,2,2);

:Green;
:l=a^b^n;
?l;

c=VecN3(2,1,2);
d=VecN3(1,-1,1);
e=VecN3(-1.5,-2,-1);

:Yellow;
:p=c^d^e^n;

:Magenta;
:r=(p*l)/p;
?r;
```

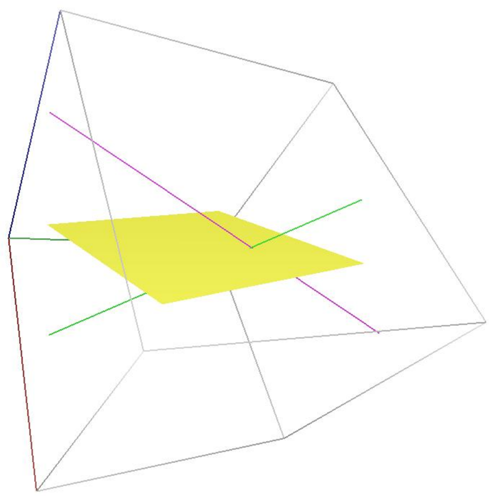


Figure 3.20: ReflectN3.clu

The result is one reflected line drawn in magenta.

3.5.8 Projection

In the following CLUScript ProjectN3.clu we visualize the projection of one line to one plane.

```
DefVarsN3();
:OPNS;

a=VecN3(0,-0.5,-0.5);
b=VecN3(0,2,2);

:Green;
:l=a^b^n;
?l;

c=VecN3(2,1,2);
d=VecN3(1,-1,1);
e=VecN3(-1.5,-2,-1);

:Yellow;
:p=c^d^e^n;

:Magenta;
:r=(p.l)/p;
?r;
```

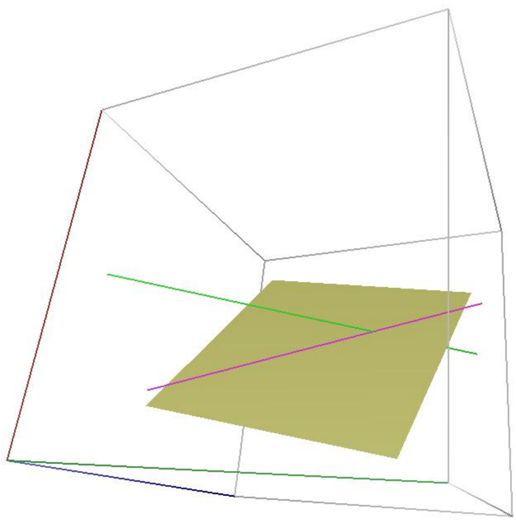


Figure 3.21: ProjectN3.clu

The result is the projected line drawn in magenta.

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