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Complexity Bounds for Block-IPs

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Technical Report

Abstract

We consider Integer Programs (IPs) with a certain block structure, called two-stage stochastic. A two-stage stochastic IP is an integer program of the form $\min\{c^T x \mid \mathcal{A}x = b, \ell \leq x \leq u, x \in \mathbb{Z}^{s+nt}\}$ where the constraint matrix $\mathcal{A} \in \mathbb{Z}^{rn \times s+tn}$ consists of blocks $A^{(i)} \in \mathbb{Z}^{r \times s}$ on a vertical line and blocks $B^{(i)} \in \mathbb{Z}^{r \times t}$ on the diagonal line aside.

We improve the bound for the Graver complexity of two-stage stochastic IPs. Our bound of $3^{\mathcal{O}(s^s(2r\|A\|_\infty+1)^{rs})}$ reduces the dependency from rs^2 to rs and is asymptotically tight under the exponential time hypothesis in the case that $r = 1$.

The improved Graver complexity bound stems from improved bounds on the intersection for a class of structurally rich integer cones. Our bound of $3^{\mathcal{O}(d\Delta)^d}$ for dimension d and absolute entries bounded by Δ is independent of the number of intersected integer cones. We investigate special properties of this class which is complemented by the fact that these properties do not hold for general integer cones. Moreover, we give structural characterizations of this class that admit their use for two-stage stochastic IPs.

1 Introduction

Every semester most students face a less desirable, yet common, problem. During (and prior to) exam weeks there is often more time required to master every aspect of the course than there is time until the exam. In other words, students can only study a proper subset of topics and thus search an answer to the following question.

What should be studied to achieve the best possible grade?

1.1 Integer Programming

Consider some upcoming exam with n topics. A topic i requires t_i time to study and yields up to p_i points in the exam. If there is t time until the exam, then the best way to study is given by an optimal solution to

$$\begin{aligned} &\text{maximize} \quad && p_1 x_1 + \ldots + p_n x_n \\ &\text{such that} \quad && t_1 x_1 + \ldots + t_n x_n \le t \quad \text{and} \quad x \in \{0,1\}^n. \end{aligned}$$

The variables x_i state whether or not a topic is studied and we require a value of 0 or 1 because fractional knowledge usually does not yield points in an exam and neither does learning topics twice. The constraint states that the time required to study the topics picked by x must not exceed the time left until the exam takes place. Under this restriction the sum of points for topics studied is maximized.

This formulation, which is an instance of the well-known knapsack problem, might require additional constraints. In particular, in exam weeks usually more than one exam is written. This adds one constraint per exam to our formulation. Another type of constraint might include dependencies between topics, i.e. if one topic is required to understand another topic. Hence, the formulation eventually ends up as follows

$$\max\{\, c^T x \;\mid Ax \le b,\, l \le x \le u,\, x \in \mathbb{Z}^n_{\ge 0}\}.$$

This problem is called Integer Program (IP) and the inputs are a matrix $A \in \mathbb{Z}^{m \times n}$, a right-hand side vector $b \in \mathbb{Z}^m$, a cost function $c \in \mathbb{Z}^n$, and bounds for the solution $l, u \in \mathbb{Z}^n$. Consequent chapters focus on an alternative form of IPs, where, instead of inequalities, the constraints are formulated by equations

$$\max\{\, c^T x \;\mid Ax = b,\, l \le x \le u,\, x \in \mathbb{Z}^n_{\ge 0}\}.$$

The student now wants to solve this problem fast to start studying as soon as possible. However, there is little hope for an efficient algorithm, as IPs are NP-complete. In fact, already the restricted version `0-1-`IP, where only solutions $x \in \{0,1\}^n$ are allowed, is shown to be NP-complete by Karp [27].

This motivates the search for algorithms efficient in some parameter. A problem is fixed parameter tractable (fpt) if there exists an algorithm with complexity $f(k) \cdot |I|^{\mathcal{O}(1)}$, where k is some parameter of the problem and $|I|$ the problem encoding length. Parametrized by the number of rows, there exists an algorithm due to Lenstra [24], which was improved by Kannan [25]. Apart from that, there are also algorithms parametrized by the number of columns and largest matrix coefficient [13, 23]. A third parameter considered is the largest absolute determinant of any quadratic submatrix. Although Shevchenko [32] conjectured tractability in this parameter already in 1996, the question whether there exists such an algorithm remains unsolved. However, for unimodular and more recently shown for bimodular matrices, where this parameter is bounded by 1 and 2, respectively, the problem can be solved in polynomial time [1].

1.2 Two-stage stochastic IPs

The previous section showed how the optimal exam preparation problem, where topics, time required per topic, and points per topic are given, can be solved using IPs. A major problem for students is that the points per topic are typically not known before the exam. They have to choose which topics to study under the *uncertainty* of whether the topics will even appear in the exam. The question to solve is

> What should be studied without knowing the exam?

The difficulty is to include uncertainty over all possibly occurring tasks in our model. This can be modelled using a block $A^{(i)}x + B^{(i)}y^{(i)} = b^{(i)}$ to combine the topics to study, stated in $A^{(i)}$, with the possible tasks, stated in $B^{(i)}$, for every task i. If, for example, topics $1, 5, 17$ are required to solve task 3, a constraint could be

$$-x_1 - x_5 - x_{17} + 3y_3 \leq 0$$

to state that all three topics are required to achieve the points for task 3. This leads to the *two-stage* matrix structure shown in Figure 1. The expected

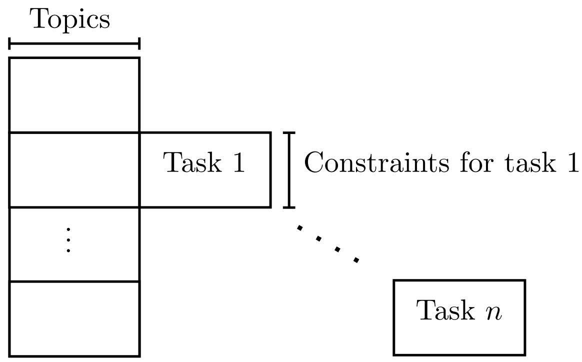


Figure 1: Two-stage matrix

number of points in the exam is given by the sum of points per possible task multiplied by the probability that this task is part of the exam.

This formulation is an instance of *two-stage stochastic IPs*, which considers a class of IPs, where the constraint matrix has non-zero entries only in the following block structure

$$\mathcal{A} = \begin{pmatrix} A^{(1)} & B^{(1)} & & \\ \vdots & & \ddots & \\ A^{(n)} & & & B^{(n)} \end{pmatrix} \tag{1}$$

for submatrices the matrices $A^{(1)}, \ldots, A^{(n)} \in \mathbb{Z}^{r \times s}$ and $B^{(1)} \ldots, B^{(n)} \in \mathbb{Z}^{r \times t}$.

In order to solve this problem, previously stated methods for IPs could be used. However, the number of scenarios n (the number of potential tasks in the exam) is usually large, which carries over to other parameters such as the number of rows and columns. This motivates interest in fpt-algorithms, where n is not beyond the set of parameters.

A common technique are *Graver augmentation algorithms*. Simply put, such algorithms start from a feasible solution which is improved in each augmentation step using a set of minimal, sign compatible kernel elements of the matrix, called Graver elements. The running time of the algorithm highly depends on the size of Graver elements. Using this approach, Hemmecke and Schultz [19] developed the first fpt-algorithm for two-stage stochastic IPs. The algorithm has a running time of $f(r, s, t, \Delta) \cdot poly(n)$, where f is a computable function with no explicit bound. Improving upon the size of the Graver elements, Klein [28] gave an algorithm of complexity $(rs\Delta)^{\mathcal{O}(rs^2(2r\Delta+1)^{rs^2})} \cdot poly(n, t, \varphi)$. The algorithmic framework for augmentation algorithms by Eisenbrand et al. [12] considering IPs with sparse matrices, which especially includes block-structured matrices, improves the running time further.

Complementing these existing algorithms, Klein [28] proved a double exponential lower bound for Graver elements. Recently, Jansen et al. [21] showed a lower bound of $2^{2^{\delta(s+t)}} \cdot |I|^{\mathcal{O}(1)}$ for some $\delta > 0$ assuming the Exponential Time Hypothesis (ETH), which is a conjecture stating that the 3-SAT problem cannot be solved in subexponentially time depending on the number of variables.

We improve upon Klein's bound on Graver elements. section 3 proves that the double exponential dependency is at most rs improving on the previous bound of rs^2. Combined with the augmentation framework due to Eisenbrand et al. [12] the following theorem is obtained.

Theorem 1. *A two-stage stochastic IP can be solved in time*

$$3^{\mathcal{O}((r+s)s^s(2r\Delta+1)^{rs})} \cdot n \log^3 n \cdot \log \|u-l\|_\infty \cdot \log \|c\|_\infty.$$

Simultaneously to this work, Cslovjecsek et al. [10] found a strong fpt algorithm with improved running time, which is subquadratic in the number of rows of the input matrix. Strong fpt algorithms are independent of bit encoding lengths of the numeric input but only depend on the number of rows and the fpt-parameters. Additionally, using similar arguments, they improved upon parametric dependencies regarding the block sizes.

1.3 Further Related Work

1.3.1 Applications

Going beyond the university motivated example, there is a vast area of applications for stochastic programming. This includes any problem involving uncertainty. Typically either there are decisions required before all information is revealed [14, 15] or postponing decisions increases potential costs [33, 26].

The areas of application include for example worker scheduling [26, 4], project planning [29, 35], routing problems [14, 15, 20], and facility location planning [33].

1.3.2 Block-structured IPs

$$\begin{pmatrix} A^{(1)} & \dots & A^{(n)} \\ B^{(1)} & & \\ & \ddots & \\ & & B^{(n)} \end{pmatrix} \begin{pmatrix} A & B & \dots & B \\ C & D & & \\ \vdots & & \ddots & \\ C & & & D \end{pmatrix} \tag{2}$$

While this work considers only two-stage stochastic IPs, there are more types of block structured matrices. In (2) the left matrix is the transposed of a two-stage matrix structure. The resulting problem type is called n-fold IP. In n-fold matrices all variables share few constraints and each additional constraint involves only a distinct block of variables. After a long line of research (see e.g. [18, 22, 12]), the currently best algorithm by Cslovjecsek et al. [9] has complexity $2^{\mathcal{O}(rs^2)}(rs\Delta)^{\mathcal{O}(r^2s+s^2)}(nt)^{1+o(1)}$. Although the problems are structurally very similar, n-folds require only a single exponential dependency in the block sizes while two-stage stochastic IPs have a double exponential lower bound, unless ETH fails [21].

When the block of shared variables from two-stage stochastic IPs is combined with the shared constraints of n-folds as shown in the right matrix of (2), we obtain a 4-block IP. There is comparably less related work on this block structure. Especially, previous approaches focused on repetitions of the same blocks B, C, and D. Hemmecke et al. [17] found this problem in XP. Chen et al. [7] recently showed matching lower and upper bounds for the size of Graver elements and Chen et al. [6] recently showed NP-hardness for the largest coefficient as a parameter.

2 Preliminaries

We start with bounding on the least common multiple (lcm) of determinants for matrices with bounded entry sizes. It is an essential part in the bound for the intersection of integer cones. The bound comes as a combination of Hadamard's inequality and a bound on the lcm of integral numbers up to Δ by Hanson.

Lemma 1 (Hanson [16]). *The least common multiple of all integral numbers up to Δ is bounded by $lcm(1, \dots, \Delta) \leq 3^{\Delta}$.*

Example 2. *For an even dimension $d \in \mathbb{N}$, $\Delta \in \mathbb{N}$, and $z \in \mathbb{N}$ with $z < \Delta^d$, let $z = \sum_{i=0}^{d-1} z_i \Delta^i$ be the digits of z to base Δ. Consider the matrix*

$$A := \begin{pmatrix} z_{d-1} & -z_{d-2} & \cdots & z_1 & -z_0 \\ 1 & \Delta & & & \\ & & \ddots & & \\ & & & 1 & \Delta \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

When A_{ij} denotes the submatrix of A, where row i and column j are deleted, then by Laplace's expansion along the first row the determinant of A is

$$\det(A) = \sum_{j=1}^{d} (-1)^{j+1} a_{ij} \det(A_{ij}) = \sum_{j=1}^{d} z_{d-j} \Delta^{d-j} = z.$$

When the dimension d is uneven, a matrix A', where compared to A the entry z_1 is changed to $-z_1$ and $-z_0$ is changed to z_0, similarly has $\det(A') = z$.

Lemma 2 (Hadamard's inequality, e.g. in [31])**.** *For a matrix $B \in \mathbb{R}^{m \times n}$ with column vectors $B = (b_1, \ldots, b_n)$ the bound $\sqrt{\det(B^T B)} \leq \prod_{i=1}^n \|b_i\|_2$ holds.*

On the one hand, for every $z \leq \Delta^d - 1$ there exists a matrix $A \in \mathbb{Z}^{d \times d}$ and $\|A\|_\infty \leq \Delta$ and $\det(A) = z$ by Example 2. On the other hand, a consequence of Hadamard's inequality is $|\det(A)| \leq (d\Delta)^d$. The combination with Hanson's bound yields the following corollary.

Corollary 1. *The lcm of the absolute value of determinants for nonsingular $d \times d$ matrices, where the entries are integral and bounded by Δ, is $\leq 3^{(d\Delta)^d}$.*

2.1 Rays

A polyhedron is the set of solutions $x \in \mathbb{R}^d$ to $Ax \leq b$ for a matrix A and a right-hand side b.

Definition 1. *Let P be a polyhedron. A vector $r \in P$ is called* ray *of the polyhedron if $\alpha r \in P$ for every $\alpha \in \mathbb{R}_{\geq 0}$.*

If r, r' are rays and $r = \lambda r'$ for some $\lambda \in \mathbb{R}_{>0}$, they are equivalent in the sense that the same linear subspace is spanned, i.e. $span(r) = span(r')$, and such rays are treated as equal unless stated otherwise.

Definition 2. *Let P be a polyhedron. A ray $r \in P$ is called* extreme ray *of the polyhedron if r cannot be written as a convex combination of two other rays of P. More precisely, if $r = \alpha r^{(1)} + (1-\alpha) r^{(2)}$ for rays $r^{(1)}, r^{(2)} \in P$ and $0 < \alpha < 1$, then $r^{(1)}$ and $r^{(2)}$ are linearly dependent.*

2.2 Basic Feasible Solutions

For vectors $v^{(1)}, \ldots, v^{(t)}$ the combination $\sum_{i=1}^{t} \lambda_i v^{(i)}$ is called *convex* if $\lambda_i \geq 0$ and $\|\lambda\|_1 = 1$. Consider a polyhedron P for $Ax \leq b$. If $x \in P$ cannot be described by a convex combination of other points in P, then x is called a vertex or, in the context of Linear Programs, a basic feasible solution (bfs) of $Ax \leq b$. Linear Programs (LPs) optimize over a polyhedron or, in other words, are IPs where the integrality constraint is dropped.

For vectors x and y write $x \leq y$ if $x_i \leq y_i$ for all i. The following lemma relates an arbitrary LP solution to some bfs and the proof is part of the paper of Klein [28].

Lemma 3. *Consider an LP for a matrix $A \in \mathbb{Z}^{m \times n}$ and a right-hand side $b \in \mathbb{Z}^m$ with solution $\hat{x} \in \mathbb{R}^n_{\geq 0}$, i.e. $A\hat{x} = b$. Then there exists a basic feasible solution $x \in \mathbb{R}^n_{\geq 0}$ with $\frac{1}{\ell} x \leq \hat{x}$, where ℓ is the number of basic feasible solutions.*

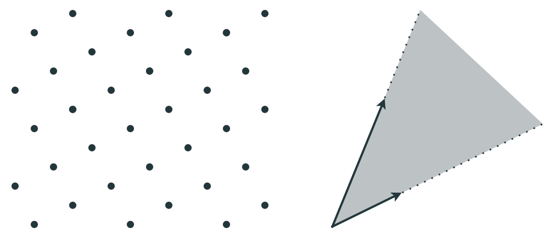
Proof. Let $x^{(1)}, \ldots, x^{(\ell)} \in \mathbb{R}^n_{\geq 0}$ be the basic feasible solutions. Since $\hat{x}$ is a solution of the LP, it is a convex combination of basic feasible solutions [34]. Hence, there exists $\lambda \in \mathbb{R}^{\ell}_{\geq 0}$ with $\|\lambda\|_1 = 1$ such that $\hat{x} = \sum_{i=1}^{\ell} \lambda_i x^{(i)}$. By pigeonhole principle there exists $i \leq \ell$ with $\lambda_i \geq \frac{1}{\ell}$ and thus $\frac{1}{\ell} x^{(i)} \leq \hat{x}$. □

2.3 Lattices

Definition 3. *For $L \subset \mathbb{Z}^d_{\geq 0}$ the* lattice *generated by L is defined by*

$$\mathcal{L}(L) := \{\sum_{b \in L} \lambda_b b \mid \lambda \in \mathbb{Z}^L\}.$$

An example for a two dimensional lattice is shown in Figure 2a. In contrast to other definitions, here linearly dependent vectors are allowed for handier notations. Note that linear independence can be achieved by the Hermite normal form (HNF). A matrix $(B\ \mathbf{0})$ is in HNF if B is a non-singular, lower triangular matrix with non-negative entries and the maximum



(a) Lattice (b) Cone

Figure 2: Examples for a cone and a lattice in the two dimensional space.

entry in each row is on the main diagonal of B. For every matrix A the generated lattice is $\mathcal{L}(A) = \mathcal{L}(\mathrm{HNF}(A))$ [31], where $\mathrm{HNF}(A)$ stands for the Hermite normal form of A. As the Hermite normal form of a matrix is lower triangular, its non-zero columns are linearly independent.

Lemma 4. *Let $L \subset \mathbb{Z}^d_{\geq 0}$. Then for any integral vector in the subspace $z \in \mathbb{Z}^d \cap span(L)$ there exists $\alpha \in \mathbb{Z}_{>0}$, with $\alpha \leq (d \left\|L\right\|_\infty)^d$, such that $\alpha z \in \mathcal{L}(L)$. If $L \in \mathbb{Z}^{d\times d}$ is nonsingular, then $|\det(L)|\, z \in \mathcal{L}(L)$.*

Proof. Assume that the vectors in L are linearly independent, as otherwise the arguments follow similarly using $\mathrm{HNF}(L)$ instead of L as explained above.

Consider the integral elements of the fundamental parallelepiped defined by $P := \{L\lambda \mid 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ and define the remainder for $b \in span(L)$ by $\mathrm{rem}(b) := L(\lambda - \lfloor\lambda\rfloor)$ for $b = L\lambda$. Note that the remainder is well defined since the vectors of L are linearly independent. Then $(P, +_r)$, where $+_r : P^2 \to P$, $x +_r y := \mathrm{rem}(x+y)$, forms a group of order $|P|$. In the case that $L \in \mathbb{Z}^{d\times d}$ is nonsingular, the number of integral points of the parallelepiped is $|P| = |\det(L)|$ [8].

As a consequence of Fermat's little theorem [3], the order of z, which is the order of the cyclic subgroup generated by z, divides the group order, i.e. $\mathrm{ord}(z) \mid |P|$. Hence there exists $\beta \in \mathbb{Z}_{\geq 0}$ with $\beta\,\mathrm{ord}(z) = |P|$ and

$$|P| \cdot z = \beta\,\mathrm{ord}(z) \cdot z = \beta(\mathrm{ord}(z) \cdot z) = \beta\mathbf{0} = \mathbf{0}.$$

Hence, the remainder is $\mathrm{rem}(|P| \cdot z) = \mathbf{0}$, which is the case if and only if the vector $|P| \cdot z = L\lambda$ for $\lambda \in \mathbb{Z}^d$ and thus $|P| \cdot z \in \mathcal{L}(L)$.

The number of integral points, which are in the fundamental parallelepiped, is bounded by the volume, i.e. $|P| \leq vol(\{L\lambda \mid 0 \leq \lambda_i < 1\})$. By

Hadamard's inequality, see Lemma 2, the volume is bounded by

$$vol(\{L\lambda \mid 0 \leq \lambda_i < 1\}) = \sqrt{\det(L^T L)} \leq \prod_{b \in L} \|b\|_2 \leq (d \|L\|_\infty)^d.$$

□

2.4 Cones

Definition 4. *For $C \subset \mathbb{Z}^d_{\geq 0}$ the* cone *generated by C is defined by*

$$cone(C) := \{\sum_{c \in C} \lambda_c c \mid \lambda \in \mathbb{R}^C_{\geq 0}\}.$$

An example for a two dimensional cone is shown in Figure 2b. The cone $cone(C)$ generated by a set C of finite cardinality is a polyhedron [31]. For any $z \in cone(C)$ with $z = C\lambda$ also $\alpha z = C(\alpha\lambda) \in cone(C)$ for any $\alpha \in \mathbb{R}_{\geq 0}$, which leads to the following observation.

Observation 1. *For $C \subset \mathbb{Z}^d_{\geq 0}$ every vector $r \in cone(C)$ is a ray.*

Remark 3 ([31])*.* For any $C \subset \mathbb{Z}^d_{\geq 0}$ and for each extreme ray r of $cone(C)$ there exists a generating element $c \in C$ on the extreme ray. In particular, there exists $\alpha \in \mathbb{R}_{>0}$ with $r = \alpha c$. Hence define $R(C) \subseteq C$ as the set of generating vectors which lay on some extreme ray of $cone(C)$. More precisely, the *extreme (generating) subset* $R(C)$ is the set of any $c \in C$, where c is an extreme ray of $cone(C)$. As every cone is generated by its extreme rays, this subset suffices to generate $cone(C) = cone(R(C))$.

For the generation of the cone, rays which are not extreme can be omitted. However, the definition allows generating sets including non-extreme elements for a simpler notation. In contrast to lattices, which can be reduced to a linearly independent set of generating vectors, cones might require an arbitrary amount of generating vectors or, stated differently, the amount of extreme rays is not bounded in terms of the dimension. To see this, consider the three dimensional space and any circle in the positive orthant. Any (finite) subset of that circle defines a cone while every element is an extreme ray.

Lemma 5 (Carathéodory's theorem [5])**.** *If $C \subset \mathbb{Z}^n_{\geq 0}$ and $x \in cone(C)$, then $c \in cone\{x_1, \ldots, x_d\}$ for linearly independent vectors $x_1, \ldots, x_d \in C$.*

The support of a vector $x \in \mathbb{R}^n$, defined as $\text{supp}(x) = |\{i \leq n \mid x_i \neq 0\}|$, is the number of non-zero entries of a vector. Carathéodory's theorem bounds the support required for $x \in \mathbb{R}^C_{\geq 0}$ in representations of the form $Cx = b$. Applied to linear programming, a basic feasible solution x can be assumed to have $\text{supp}(x) \leq m$, where m is the number of rows in the constraint matrix.

2.5 Integer Cones

Definition 5. *For $B \subset \mathbb{Z}^d_{\geq 0}$ the* integer cone *generated by B is defined by*

$$int.cone(B) := \{\sum_{b \in B} \lambda_b b \mid \lambda \in \mathbb{Z}^B_{\geq 0}\}.$$

For further discussion the generating set is assumed to be minimal in the sense that for every $b \in B$ removing b changes the generated integer cone, i.e. $int.cone(B \setminus \{b\}) \subsetneq int.cone(B)$, unless stated otherwise. In particular, every $b \in B$ is not decomposable in any $b^{(1)}, b^{(2)} \in int.cone(B)$, where $b^{(1)} \neq \mathbf{0} \neq b^{(2)}$ and $b = b^{(1)} + b^{(2)}$.

By definition $int.cone(B) \subseteq cone(B)$ and $int.cone(B) \subseteq \mathcal{L}(B)$ and thus $int.cone(B) \subset cone(B) \cap \mathcal{L}(B)$. When extreme rays are mentioned in the context of an integer cone, it is referred to the extreme rays of the cone. Similarly, the extreme subset $R(B)$ refers to its definition in the context of cones, see Remark 3.

Subsequent chapters will consider integer cones frequently. Hence, consider the following examples to understand some of the inherent properties of integer cones.

Example 4.

(i) Although for any integer cone $int.cone(B) \subseteq cone(B) \cap \mathcal{L}(B)$ holds, generally $int.cone(B) \not\supseteq cone(B) \cap \mathcal{L}(B)$. Figure 3a shows the example

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 1 \end{pmatrix}.$$

The vector $(1,1)^T$ is in $cone(B)$ and in $\mathcal{L}(B)$ but not in $int.cone(B)$. Another example is shown in Figure 3c.

(ii) The intersection of two integer cones, both generated by a basis, might not be generated by a basis. An example, shown in Figure 3b, is

$$B^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B^{(2)} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

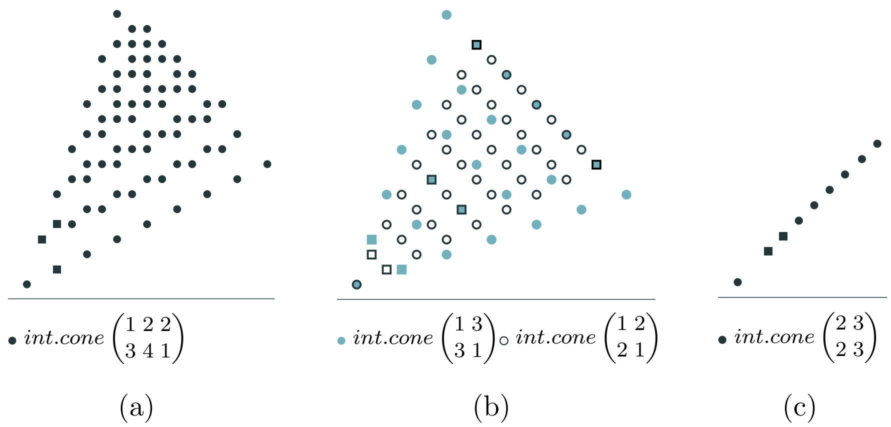


Figure 3: Integer cone examples. Generating elements are represented by squares.

The intersection $int.cone(B^{(1)}) \cap int.cone(B^{(2)})$ *is generated by*

$$\hat{B} = \begin{pmatrix} 5 & 7 & 16 & 8 \\ 7 & 5 & 8 & 16 \end{pmatrix}.$$

(iii) Although on every extreme ray there exists a generating element (see Remark 3), this generating element might not be unique. Figure 3c shows an example, where

$$B = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}.$$

Both generating elements of $int.cone(B)$ *lay on the unique extreme ray of* $int.cone(B)$*.*

In the bound on the Graver elements for two-stage stochastic IPs by Klein [28], the primary complexity stems from the intersection of integer cones and, more precisely, from the size of generating elements for the resulting integer cone.

Lemma 6 (Klein [28])**.** *Let* $B^{(1)}, \ldots, B^{(\ell)} \subset \mathbb{Z}^d_{\geq 0}$ *with* $\left\|B^{(i)}\right\|_\infty \leq \Delta$ *for each* $i \leq \ell$ *and consider the intersection of their integer cones, which is generated by some* $\hat{B} \subset \mathbb{Z}^d_{\geq 0}$*,*

$$int.cone(\hat{B}) = \bigcap_{i=1}^{\ell} int.cone(B^{(i)}).$$

Then for each generating element $b \in \hat{B}$ with $B^{(i)}\lambda^{(i)} = b$ for $\lambda^{(i)} \in \mathbb{Z}_{\geq 0}^{B^{(i)}}$, the bound $\left\|\lambda^{(i)}\right\|_1 \leq \mathcal{O}((d\Delta)^{d(\ell-1)})$ holds for all $1 \leq i \leq \ell$.

2.6 Graver Elements and Augmentation Algorithms

An element $x \in X \subset \mathbb{Z}^n$ is called decomposable in X if there exist $y, z \in X$ with sign-compatible entries $y_i z_i \geq 0$ for every $i \leq n$ and $x = y + z$. If x is not decomposable, it is called indecomposable.

Definition 6. *The* Graver basis *of an integer matrix A is defined as the set $\mathcal{G}(A) \subset \mathbb{Z}^n$ of indecomposable elements in* $\ker_{\mathbb{Z}}(A)$. *Elements of this set are called* Graver elements.

An element of $\ker_{\mathbb{Z}}(A)$ is called a cycle. A fundamental property of the Graver basis is that every cycle is a sign-compatible sum of Graver elements, which enables their use in augmentation algorithms. Given an initial feasible solution, the rough idea of augmentation algorithms is to add Graver elements g of positive cost value $c^T g > 0$ to the solution until no further improvement is possible. In this case an optimal solution was found [30]. The efficiency of augmentation algorithms hence depends on the size of Graver elements.

Lemma 7 (Eisenbrand, Hunkenschröder, Klein [11])**.** *Let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix with $\|A\|_\infty \leq \Delta$. Let $g \in \mathbb{Z}^n$ be a Graver element of A then $\|g\|_1 \leq (2m\Delta + 1)^m$.*

The augmentation framework of Eisenbrand et al. [12] is parametrized by a certain sparsity measure. For two-stage stochastic IPs the following complexity is derived from Theorem 72 and Corollarie 93.

Corollary 2 (Eisenbrand et al. [12])**.** *Let $g_\infty(A)$ denote a bound on the infinity norm of Graver elements for matrix A. The two-stage stochastic IP can be solved in time*

$$g_\infty(A)^{2(r+s)} \cdot n \log^3 n \cdot \log \|u - l\|_\infty \cdot \log \|c\|_\infty$$

Although, the bound of Eisenbrand et al. [11] could also be applied to two-stage stochastic IPs, there are bounds which only depend on the largest matrix entry and block sizes [2, 28, 19]. The remaining work further improves upon these known bounds. The above stated lemmata then yield more efficient algorithms as stated in Theorem 1 using the improved bounds.

3 Two-Stage Stochastic IPs

This chapter considers two-stage stochastic IPs and gives improved bounds for their Graver elements. A class of integer cones, called *regular* integer cones, is introduced. This class of integer cones is closed under intersection and yields a bound on the generating set for the intersection independent of the number of intersected integer cones. When bounding the Graver elements, the intersection of integer cones is in turn the dominating factor.

3.1 Regular Integer Cones

Definition 7. *Let $B \subset \mathbb{Z}^d_{\geq 0}$. An integer cone $int.cone(B)$ is called* regular *if $int.cone(B) = cone(B) \cap \mathcal{L}(B)$ holds.*

Example 4.i shows that not every integer cone is regular. However, an integer cone is regular whenever it is generated by linearly independent vectors.

Lemma 8. *An integer cone, which is generated by linearly independent vectors $B \subset \mathbb{Z}^d_{\geq 0}$, is regular.*

Proof. As $int.cone(B) \subseteq cone(B) \cap \mathcal{L}(B)$ holds by definition, it remains to show that $cone(B) \cap \mathcal{L}(B) \subseteq int.cone(B)$. Hence, consider some vector $b \in cone(B) \cap \mathcal{L}(B)$. By the definition of cones and the definition of lattices, there exist $y \in \mathbb{R}^d_{\geq 0}$ and $z \in \mathbb{Z}^d$ with $By = b = Bz$. Because the generating elements in B are linearly independent, this yields $y = z \in \mathbb{Z}^d_{\geq 0}$, where the vector is non-negative because y is non-negative, and it is integral because z is integral. Hence, the vector $b \in int.cone(B)$ and $cone(B) \cap \mathcal{L}(B) \subseteq int.cone(B)$. □

Scaling an integral vector eventually yields some point of any lattice by Lemma 4. Using the above lemma, this scaling argument can be transferred to integer cones.

Lemma 9. *For any generating set $B \subseteq \mathbb{Z}^d_{\geq 0}$ and $z \in \mathbb{Z}^d \cap cone(B)$ there exists $\alpha \in \mathbb{Z}_{>0}$ and $\alpha \leq (d \, \|B\|_\infty)^d$ with $\alpha z \in int.cone(B)$. If B is a basis, this holds for $\alpha = |\det(B)|$.*

Proof. Let $d' := \dim(B)$ be the number linearly independent vectors in B. By Carathéodory's theorem, see Lemma 5, there exist d' linear independent vectors $\bar{B} := \{x_1, \ldots, x_{d'}\} \subseteq B$ such that $z \in cone(\bar{B}) \subset span(\bar{B})$. As $\bar{B} \subseteq B$, the integer cone is also a subset $int.cone(\bar{B}) \subseteq int.cone(B)$.

Since the vectors in $\bar{B}$ are linearly independent, Lemma 8 can be applied and the integer cone $int.cone(\bar{B})$ is regular. Hence, Lemma 4 can be applied and there exists $\alpha \in \mathbb{Z}_{>0}$ with $\alpha \leq (d \left\|\bar{B}\right\|_\infty)^d \leq (d \left\|B\right\|_\infty)^d$ such that $\alpha z \in \mathcal{L}(\bar{B})$ and if B is a basis, then $\bar{B} = B$ and this holds for $\alpha = |\det(B)|$. Due to Observation 1 every vector of the cone is a ray and $\alpha z \in cone(\bar{B})$. Hence, the scaled vector is in both the lattice and in the cone and thus $\alpha z \in cone(\bar{B}) \cap \mathcal{L}(\bar{B}) = int.cone(\bar{B}) \subseteq int.cone(B)$. □

3.1.1 Intersection Maintains Regularity

When integer cones generated by bases are intersected, the resulting integer cone is in general not generated by a basis, see Example 4.ii. However, basis generated integer cones are regular and the intersection of regular integer cones is again regular, which is proven in this section.

Lemma 10. *For $i \in \{1, 2\}$ and $B^{(i)}, C^{(i)}, L^{(i)} \subset \mathbb{Z}^d_{\geq 0}$, consider integer cones with $int.cone(B^{(i)}) = cone(C^{(i)}) \cap \mathcal{L}(L^{(i)})$. Then the intersection integer cone is of similar structure, i.e.*

$$int.cone(\hat{B}) = int.cone(B^{(1)}) \cap int.cone(B^{(2)}) = cone(\hat{C}) \cap \mathcal{L}(\hat{L}),$$

for some generating sets $\hat{B}, \hat{C}, \hat{L} \subset \mathbb{Z}^d_{\geq 0}$.

Proof. Commutativity of the intersection yields

$$\begin{aligned} int.cone(\hat{B}) &= (cone(C^{(1)}) \cap \mathcal{L}(L^{(1)})) \cap (cone(C^{(2)}) \cap \mathcal{L}(L^{(2)})) \\ &= (cone(C^{(1)}) \cap cone(C^{(2)})) \cap (\mathcal{L}(L^{(1)}) \cap \mathcal{L}(L^{(2)})). \end{aligned}$$

Therefore, it remains to show that $cone(C^{(1)}) \cap cone(C^{(2)})$ is a cone and that $\mathcal{L}(L^{(1)}) \cap \mathcal{L}(L^{(2)})$ is a lattice.

Claim: The intersection of two cones is again a cone. More precisely, $\hat{C} := cone(C^{(1)}) \cap cone(C^{(2)})$ is a cone.

Proof of the claim: Consider the cone $cone(\hat{C})$. It is sufficient to prove $\hat{C} = cone(\hat{C})$. For every $c \in \hat{C}$ the vector can be written as $c = \hat{C} e_c$ and $\hat{C} \subseteq cone(\hat{C})$.

Consider $c \in cone(\hat{C})$ with $\sum_{\hat{b} \in \hat{C}} \lambda_{\hat{b}} \hat{b} = c$ for some $\lambda \in \mathbb{R}^{\hat{C}}_{\geq 0}$. For every $\hat{b} \in \hat{C}$ there exist $\lambda^{(1,\hat{b})} \in \mathbb{R}^{C^{(1)}}_{\geq 0}$ and $\lambda^{(2,\hat{b})} \in \mathbb{R}^{C^{(2)}}_{\geq 0}$ such that $\sum_{b \in C^{(1)}} \lambda^{(1)}_b b =$

$\hat{b} = \sum_{b \in C^{(2)}} \lambda_b^{(2)} b$. This yields

$$c = \sum_{\hat{b} \in \hat{C}} \lambda_{\hat{b}} \hat{b} = \sum_{b \in C^{(1)}} (\sum_{\hat{b} \in \hat{C}} \lambda_{\hat{b}} \lambda_b^{(1,\hat{b})}) b = \sum_{b \in C^{(2)}} (\sum_{\hat{b} \in \hat{C}} \lambda_{\hat{b}} \lambda_b^{(2,\hat{b})}) b.$$

The vector can be written as a conical combination of generating vectors for $C^{(1)}$ and $C^{(2)}$ and hence $c \in \mathit{cone}(C^{(1)}) \cap \mathit{cone}(C^{(2)}) = \hat{C}$. $\lhd$

Claim: The intersection of two lattices is again a lattice. More precisely, $\hat{L} := \mathcal{L}(L^{(1)}) \cap \mathcal{L}(L^{(2)})$ is a lattice.

Proof of the claim: Consider $\mathcal{L}(\hat{L})$, it is sufficient to prove $\hat{L} = \mathcal{L}(\hat{L})$. For every $b \in \hat{L}$ the vector can be written as $b = \hat{L} e_b$ and hence $\hat{L} \subseteq \mathcal{L}(\hat{L})$.

Thus, consider $\bar{b} \in \mathcal{L}(\hat{L})$ with $\sum_{\hat{b} \in \hat{L}} \lambda_{\hat{b}} \hat{b} = \bar{b}$ for some $\lambda \in \mathbb{Z}^{\hat{L}}$. For every $\hat{b} \in \hat{L}$ there exist $\lambda^{(1,\hat{b})} \in \mathbb{Z}^{L^{(1)}}$ and $\lambda^{(2,\hat{b})} \in \mathbb{Z}^{L^{(2)}}$ such that $\sum_{b \in L^{(1)}} \lambda_b^{(1)} b = \hat{b} = \sum_{b \in L^{(2)}} \lambda_b^{(2)} b$. This yields

$$\bar{b} = \sum_{\hat{b} \in \hat{L}} \lambda_{\hat{b}} \hat{b} = \sum_{b \in L^{(1)}} (\sum_{\hat{b} \in \hat{L}} \lambda_{\hat{b}} \lambda_b^{(1,\hat{b})}) b = \sum_{b \in L^{(2)}} (\sum_{\hat{b} \in \hat{L}} \lambda_{\hat{b}} \lambda_b^{(2,\hat{b})}) b.$$

Thus, the vector can be written as an integral combination of generating vectors for both $L^{(1)}$ and $L^{(2)}$ and hence $\bar{b} \in \mathcal{L}(L^{(1)}) \cap \mathcal{L}(L^{(2)}) = \hat{L}$. $\lhd$

By the first claim $\mathit{cone}(C^{(1)}) \cap \mathit{cone}(C^{(2)})$ is a cone and by the second claim $\mathcal{L}(L^{(1)}) \cap \mathcal{L}(L^{(2)})$ is a lattice which proves the lemma. $\square$

In the above lemma the generating sets for the integer cone, cone and lattice are each independent of each other. For regularity each is generated by the same set. The following observation deals with the exchange of the generating set regarding the cone and the observation thereafter regarding the lattice.

Observation 2. *Let $B \subset \mathbb{Z}^d_{\geq 0}$. If there exist sets $C, L \subset \mathbb{Z}^d_{\geq 0}$ such that $\mathit{int.cone}(B) = \mathit{cone}(C) \cap \mathcal{L}(L)$, then $\mathit{int.cone}(B) = \mathit{cone}(B) \cap \mathcal{L}(L)$.*

Proof. As $\mathit{int.cone}(B) \subseteq \mathcal{L}(L)$ and $\mathit{int.cone}(B) \subseteq \mathit{cone}(B)$, it remains to show $\mathit{cone}(B) \cap \mathcal{L}(L) \subseteq \mathit{int.cone}(B)$.

Consider $b \in \mathit{cone}(B) \cap \mathbb{Z}^d$. There exists $\lambda \in \mathbb{R}^d_{\geq 0}$ with $B\lambda = b$ and there exists $\alpha \in \mathbb{Z}_{>0}$ with $\alpha b \in \mathit{int.cone}(B) = \mathit{cone}(C) \cap \mathcal{L}(L)$ due to Lemma 9. As αb is in the cone of C, there exists $\bar{\lambda} \in \mathbb{R}_{\geq 0}$ with $C\bar{\lambda} = \alpha b$. Thus,

$b = C(\frac{1}{\alpha}\bar{\lambda}) \in cone(C) \cap \mathbb{Z}^d$ and $cone(B) \cap \mathbb{Z}^d \subseteq cone(C) \cap \mathbb{Z}^d$. Therefore, also $cone(B) \cap \mathcal{L}(L) \subseteq cone(C) \cap \mathcal{L}(L) = int.cone(B)$. $\square$

Observation 3. *Let $B \subset \mathbb{Z}^d_{\geq 0}$. If there exists a set $L \subset \mathbb{Z}^d_{\geq 0}$ such that $int.cone(B) = cone(B) \cap \mathcal{L}(L)$, then the integer cone is regular.*

Proof. By definition $int.cone(B) \subseteq cone(B) \cap \mathcal{L}(B)$ and it remains to show $cone(B) \cap \mathcal{L}(B) \subseteq int.cone(B)$.

Therefore, let $b \in \mathcal{L}(B)$. There exists $\lambda \in \mathbb{Z}^d$ with $b = B\lambda$. Note that $B \subset \mathcal{L}(L)$ since $B \subset int.cone(B)$. Thus, for every $\hat{b} \in B$ exists $\lambda^{(\hat{b})} \in \mathbb{Z}^d$ with $\hat{b} = L\lambda^{(\hat{b})}$. Therefore, the vector is in the lattice as

$$b = B\lambda = \sum_{\hat{b}\in B} \lambda_{\hat{b}} b = \sum_{\hat{b}\in B} \lambda_{\hat{b}} L\lambda^{(\hat{b})} = L(\sum_{\hat{b}\in B} \lambda_{\hat{b}}\lambda^{(\hat{b})}) \in \mathcal{L}(L).$$

Therefore, the lattice of B is a subset of the lattice of L, $\mathcal{L}(B) \subseteq \mathcal{L}(L)$, and thus $cone(B) \cap \mathcal{L}(B) \subseteq cone(B) \cap \mathcal{L}(L) = int.cone(B)$. $\square$

Using Lemma 10 together with Observation 2 and Observation 3, the intersection of two regular integer cones is regular. When applied inductively, the following corollary is obtained.

Corollary 3. *For integer cones $int.cone(B^{(1)}), \dots, int.cone(B^{(\ell)})$, generated by bases $B^{(1)}, \dots, B^{(\ell)} \in \mathbb{Z}^{d\times d}_{\geq 0}$, the intersection is regular.*

3.1.2 Properties of Regular Integer Cones

In general, there might be multiple generating elements on the same extreme ray as shown in Example 4.iii. However, for regular integer cones the generating element on each extreme ray is unique.

Lemma 11. *Consider a generating set $B \subset \mathbb{Z}^d_{\geq 0}$, where $int.cone(B)$ is regular. For any extreme ray r of the integer cone, there exists a unique generating element on the extreme ray. More precisely, there exists exactly one $b \in B$ where there exists $\lambda \in \mathbb{R}_{>0}$ with $\lambda r = b$.*

Proof. The existence follows by Remark 3. Assume there exist multiple distinct elements of the generating set on the extreme ray. Let $b_1, b_2 \in B$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ and $\lambda_1 r = b_1$ and $\lambda_2 r = b_2$. Additionally, let b_1 be minimal in the sense that every $b \in B$ with $\lambda r = b$ for some $\lambda \in \mathbb{R}_{>0}$ has $\lambda_1 \leq \lambda$.

If $\lambda_2 \equiv 0 \pmod{\lambda_1}$, then there exists $k \in \mathbb{Z}_{\geq 0}$ with $k\lambda_1 = \lambda_2$. Hence, b_2 is decomposable by $kb_1 = k\lambda_1 r = \lambda_2 r = b_2$ which contradicts to $b_2 \in B$ and that B is assumed to be minimal.

If otherwise $\lambda_2 \not\equiv 0 \pmod{\lambda_1}$, consider $\hat{b} := (\lambda_2 - \kappa\lambda_1)r$ with $\kappa = \lfloor \frac{\lambda_2}{\lambda_1} \rfloor$. Since $\lambda_2 \bmod \lambda_1 > 0$, the vector is non-zero and $\hat{b} \in cone(B)$. Additionally, both $b_2 = \lambda_2 r \in \mathcal{L}(B)$ and $\kappa\lambda_1 \cdot r \in \mathcal{L}(B)$ are in the lattice. Hence, $\hat{b} = b_2 - \kappa\lambda_1 \cdot r \in \mathcal{L}(B)$. Hence, the vector is in $\hat{b} \in int.cone(B)$ and there exists $\hat{\lambda} \in \mathbb{Z}^d_{\geq 0}$ with $B\hat{\lambda} = \hat{b}$. As r is extreme, it is not a convex combination of two other rays in $cone(B)$ and every $b \in \operatorname{supp}(\hat{\lambda})$ lays on the extreme ray. This contradicts the minimality of b_1 as $\hat{b} \in int.cone(B)$ and $\lambda_b \leq \lambda_2 - \kappa\lambda_1 < \lambda_1$ for any $b \in \operatorname{supp}(\hat{\lambda})$ and $b = \lambda_b r$. □

Let $\Delta \in \mathbb{Z}_{\geq 0}$ and consider the integer cone $int.cone(B)$ for

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & \Delta \end{pmatrix}.$$

Then b_1 and b_2 are extreme rays and b_3 is not. For any Δ the vector b_3 is not decomposable due to the first entry. Hence, in general the norm of generating elements B can not be bounded by the norm of extreme generating elements $R(B)$ as Δ can be chosen arbitrarily large. However, for the case of regular integer cones the following lemma gives such a bound.

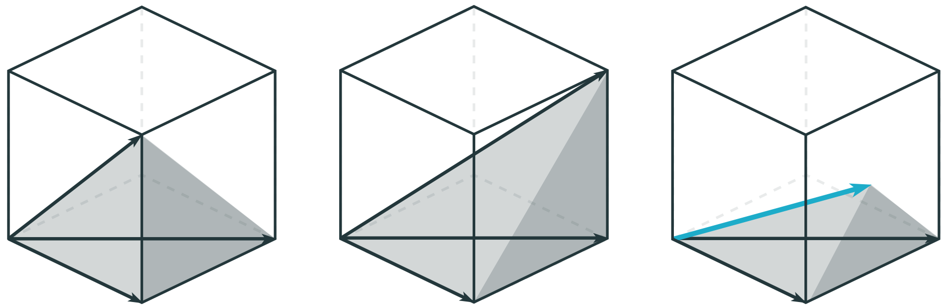
Lemma 12. *Consider a regular integer cone $int.cone(B)$ for some generating set $B \subset \mathbb{Z}^d_{\geq 0}$. Then the size of the generating elements is bounded by the size of extreme generating elements by $\|B\|_\infty \leq d\,\|R(B)\|_\infty$.*

Proof. Consider for any $v \in int.cone(B)$ the LP

$$\begin{aligned} R(B)x &= v \\ x &\in \mathbb{R}^{R(B)}_{\geq 0}. \end{aligned} \qquad \text{LP (1)}$$

LP (1) is feasible because $int.cone(B) \subset cone(B) = cone(R(B))$, see Remark 3. Consider any basic feasible solution $x \in \mathbb{R}^{R(B)}_{\geq 0}$ and assume that there exists $r \in R(B)$ with $x_r > 1$. Note that $r = R(B)e_r \in int.cone(B)$. Because $x_j - (e_r)_j \geq 0$ for every $j \in R(B)$, this yields

$$v - r = R(B)(x - e_r) \in cone(B).$$



(a) $cone(C_1)$ (b) $cone(C_2)$ (c) $cone(C_1) \cap cone(C_2)$

Figure 4: Each figure shows a cone in the unit cube. The vector $(1, \frac{1}{2}, \frac{1}{2})^T$ is an extreme ray of the intersection cone $cone(C_1) \cap cone(C_2)$ but not an extreme ray of either $cone(C_1)$ or $cone(C_2)$.

As x might be fractional, it is not immediately clear whether $v - r$ is in the lattice $\mathcal{L}(B)$. This can be seen as both $v, r \in int.cone(B) \subseteq \mathcal{L}(B)$. Hence, there exist $y, z \in \mathbb{Z}^d$ with $v = By$ and $r = Bz$. This yields

$$v - r = By - Bz = B(y - z) \in \mathcal{L}(B).$$

Thus, $v - r \in cone(B) \cap \mathcal{L}(B) = int.cone(B)$ and $v = r + (v - r)$ is not an element of the generating set, which was assumed to be minimal.

Therefore, for every $b \in B$ and every basic feasible solution $x \in \mathbb{R}^{R(B)}_{\geq 0}$ of $R(B)x = b$ has $\|x\|_\infty \leq 1$. This yields

$$\|B\|_\infty \leq \sum_{r \in R(B)} \|R(B)\|_\infty x_r \leq d \|R(B)\|_\infty .$$

The last inequality follows by $\|x\|_\infty \leq 1$ and that by Carathéodory's theorem, Lemma 5, each basic feasible solution x has $\text{supp}(x) \leq d$. □

3.2 A Bound for the Intersection of Integer Cones

In order to bound the size of generating elements for the intersection of integer cones, the extreme generating elements rays are bounded and Lemma 12 is used to transfer the bound to remaining generating elements.

Example 5. *Consider the cones in Figures 4a and 4b, where*

$$C_1 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad C_2 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The cone of the intersection, shown in Figure 4c, is generated by

$$cone(C_1) \cap cone(C_2) = cone \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Hence, the extreme rays of a cone, which is the intersection of cones, are not given by extreme rays of the intersected cones. In order to bound the size of generating elements on the extreme rays, a first step is to bound the number of cones required to obtain an extreme ray.

Lemma 13. *Consider cones $cone(B^{(1)}), \dots, cone(B^{(\ell)})$ for some generating sets $B^{(1)}, \dots, B^{(\ell)} \subset \mathbb{Z}^d_{\geq 0}$. For any extreme ray r of the intersection $cone(\hat{B}) = \bigcap_{i=1}^{\ell} cone(B^i)$ there exists a subset S with $|S| < d$ such that r is an extreme ray of $cone(\hat{B}) := \bigcap_{i \in S} cone(B^{(i)})$.*

Proof. Let $F_1, \dots, F_k$ be the facets of $cone(B^{(1)}), \dots, cone(B^{(\ell)})$ containing r. Let $a_1^{(i)}, \dots, a_d^{(i)}$ be a set of affinely independent points for facet F_i and consider the subspace V_i of the affine hull $\mathrm{aff}(a_1^{(i)}, \dots, a_d^{(i)})$. Note that for all $i \leq k$ the facet is a subset $F_i \subset V_i$.

Define $\hat{V}^{(i)} = \bigcap_{j=1}^{i} V_j$. Since r is an extreme ray of $cone(\hat{B})$, $\hat{V}_k$ has dimension 1 and is spanned by r. If $k \geq d$, then by pigeonhole principle there exists $1 < i \leq k$ with $\dim(\hat{V}_i) = \dim(\hat{V}_{i-1})$. Since $\hat{V}_{i-1} \subseteq \hat{V}_i$, this implies $\hat{V}_{i-1} = \hat{V}_i$. This implies

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{k} V_j \subseteq \hat{V}^{(i-1)} = \hat{V}^{(i)} \subseteq V_i$$

Due to the subset relation, the intersection does not change the set, i.e.

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{k} V_j = \bigcap_{\substack{j=1 \\ j \neq i}}^{k} V_j \cap V_i = \hat{V}_k.$$

Using $\dim(\hat{V}_k) = 1$ and that $\hat{V}_k$ is spanned by r yields

$$\dim(\bigcap_{\substack{j=1 \\ j \neq i}}^{k} F_j) = 1 \quad \text{and} \quad r \in \bigcap_{\substack{j=1 \\ j \neq i}}^{k} F_j.$$

Therefore, facets can be eliminated up to a set S' with $|S'| \leq d$ and the intersection of these facets is one-dimensional and spanned by the extreme ray. For the set S, $|S| \leq |S'| \leq d$, of the cones corresponding to the set of facets S', then r is an extreme ray of $\bigcap_{j \in S} cone(B^{(j)})$. □

Using Lemma 13 an integral vector of bounded size on each extreme ray can be obtained. Combined with Lemma 9 the following lemma obtains elements on each extreme ray, which are in the integer cone and of bounded size. Lemma 11 excludes larger extreme generating elements on the respective ray.

Lemma 14. *For bases $B^{(1)}, \ldots, B^{(\ell)} \in \mathbb{Z}_{\geq 0}^{d \times d}$ with bounded entry sizes $\left\|B^{(i)}\right\|_\infty \leq \Delta$ for all $i \leq \ell$, consider $int.cone(B^{(1)}), \ldots, int.cone(B^{(\ell)})$. Consider the intersection $int.cone(\hat{B}) = \bigcap_{i=1}^{\ell} int.cone(B^{(i)})$ for some set of generating elements $\hat{B} \subset \mathbb{Z}_{\geq 0}^d$.*

The extreme elements are bounded by $\left\|R(\hat{B})\right\|_\infty \leq \gamma \cdot \mathcal{O}(\Delta(d\Delta)^{d(d-2)})$ with $\gamma = lcm(\left|\det(B^{(1)})\right|, \ldots, \left|\det(B^{(\ell)})\right|)$.

Proof. Consider some extreme element $r \in R(\hat{B})$. Due to Lemma 13 there exists a subset S with $|S| < d$, where r is an extreme ray of the integer cone $int.cone(\bar{B}) := \bigcap_{i \in S} int.cone(B^{(i)})$. From Lemma 6 follows that generating elements $b \in \bar{B}$ with $\lambda^{(i)} \in \mathbb{Z}_{\geq 0}^{B^{(i)}}$ such that $B^{(i)}\lambda^{(i)} = b$ satisfy $\left\|\lambda^{(i)}\right\|_1 \leq \mathcal{O}((d\Delta)^{d(d-2)})$. Because r is an extreme ray of $int.cone(\bar{B})$, there exists a generating element on the extreme ray r by Remark 3. Thus, there exists an integer point $z \in \mathbb{Z}_{\geq 0}^d$ on the extreme ray r with $\|z\|_\infty \leq \mathcal{O}(\Delta(d\Delta)^{d(d-2)})$.

Due to Lemma 9, the scaled vector is in $\left|\det(B^{(i)})\right| z \in int.cone(B^{(i)})$ for every $i \leq \ell$. For $\gamma := \mathrm{lcm}(\left|\det(B^{(1)})\right|, \ldots, \left|\det(B^{(\ell)})\right|)$ the scaled vector $\gamma z \in int.cone(\hat{B})$. Due to Corollary 3 the intersection integer cone is regular and by Lemma 11 there exists an unique extreme generating element $\hat{r} \in R(\hat{B})$ of $int.cone(\hat{B})$ on r. Then there exists $\alpha \in \mathbb{Z}_{>0}^d$ with $\gamma z = \alpha \hat{r}$ since r is extreme. Therefore, the extreme element is bounded by

$$\|\hat{r}\|_\infty \leq \gamma \cdot \|z\|_\infty \leq \gamma \cdot \mathcal{O}(\Delta(d\Delta)^{d(d-2)}).$$

□

A part of the preceding proof used Lemma 13 to bound the size of an integral element on each extreme ray.

Corollary 4. *In the setting of Lemma 14 there exists an integer point* $z \in \mathbb{Z}^d_{\geq 0}$ *on any extreme ray of* $int.cone(\hat{B})$ *with* $\|z\|_\infty \leq \mathcal{O}(\Delta(d\Delta)^{d(d-2)})$.

Lemma 15. *Consider again the setting of Lemma 14. For each generating element* $b \in \hat{B}$ *with* $B^{(i)}\lambda^{(i)} = b$ *for some* $\lambda^{(i)} \in \mathbb{Z}^d_{\geq 0}$*, the bound*

$$\left\|\lambda^{(i)}\right\|_1 \leq \gamma \cdot \mathcal{O}((d\Delta)^{d(d-2)+1})$$

holds for all $1 \leq i \leq \ell$ *and with* $\gamma := lcm(\left|\det(B^{(1)})\right|, \ldots, \left|\det(B^{(\ell)})\right|)$*. Moreover, for every* $1 \leq i \leq \ell$ *each* $\lambda^{(i)}$ *is bounded by*

$$\left\|\lambda^{(i)}\right\|_1 \leq 3^{\mathcal{O}((d\Delta)^d)}. \tag{3}$$

Proof. By Lemma 14 the basis elements on the extreme rays can be bounded by $\left\|R(\hat{B})\right\|_\infty \leq \gamma \cdot \mathcal{O}(\Delta(d\Delta)^{d(d-2)})$. Using Corollary 3 the intersection integer cone can be written as $int.cone(\hat{B}) = cone(\hat{B}) \cap \mathcal{L}(\hat{B})$. Hence, Lemma 12 can be applied to $int.cone(\hat{B})$ which yields

$$\left\|\hat{B}\right\|_\infty \leq d\left\|R(\hat{B})\right\|_\infty \leq \gamma \cdot \mathcal{O}((d\Delta)^{d(d-2)+1}).$$

Hence, for $b \in \hat{B}$ and $\lambda^{(i)} \in \mathbb{Z}^d_{\geq 0}$ with $B^{(i)}\lambda^{(i)} = b$ this yields

$$\left\|\lambda^{(i)}\right\|_1 \leq \|b\|_1 \leq \gamma \cdot \mathcal{O}((d\Delta)^{d(d-2)+1}).$$

The combination with Corollary 1 implies

$$\left\|\lambda^{(i)}\right\|_1 \leq 3^{(d\Delta)^d} \cdot \mathcal{O}((d\Delta)^{d(d-2)+1}) \leq 3^{\mathcal{O}((d\Delta)^d)}.$$

□

3.3 A Bound on Graver Elements

This section follows the proof of Klein [28] but the improved bound for the intersection of integer cones from the preceding section is used instead of Lemma 6. First, the structural result on the intersection of paths is revisited. Each path is represented by a multiset of vectors, which sum up to the same vector. The lemma shows that there exist a subpaths (submultisets) of bounded size that also sum up to the same vector.

Lemma 16. *Given multisets $T_1, \dots, T_n \subset \mathbb{Z}^d_{\geq 0}$ where all elements $t \in T_i$ have bounded size $\|t\|_\infty \leq \Delta$. Assuming that the sum of all elements in each set is equal, i.e.*

$$b = \sum_{t \in T_1} t = \dots = \sum_{t \in T_n} t$$

then there exist nonempty submultisets $S_1 \subseteq T_1, \dots, S_n \subseteq T_n$ of bounded size $|S_i| \leq 3^{\mathcal{O}((d\Delta)^d)}$ such that

$$\sum_{s \in S_1} s = \dots = \sum_{s \in S_n} s.$$

Proof. Let $P \subset \mathbb{Z}^d_{\geq 0}$ be the set of all non-negative integer points p with $\|p\|_\infty \leq \Delta$. Describe every multiset $T_1, \dots, T_n \subset \mathbb{Z}^d_{\geq 0}$ by a multiplicity vector $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{Z}^P_{\geq 0}$, where $\lambda^{(i)}_p$ states the multiplicity of $p \in P$ in multiset T_i yielding the representation $\sum_{p \in P} \lambda^{(i)}_p p = \sum_{t \in T_i} t = b$.

Consider the linear program

$$\begin{aligned} \sum_{p \in P} x_p p &= b \\ x &\in \mathbb{R}^P_{\geq 0}. \end{aligned} \qquad \text{LP (2)}$$

Let $x^{(1)}, \dots, x^{(\ell)} \in \mathbb{R}^d_{\geq 0}$ be all basic feasible solutions of LP (2) corresponding to bases $B^{(1)}, \dots, B^{(\ell)} \in \mathbb{Z}^{d \times d}_{\geq 0}$, i.e. $B^{(i)} x^{(i)} = b$. Given a basis B the vector x with $Bx = b$ is unique and hence the number of basic feasible solutions is bounded by $\ell \leq \Delta^{d^2}$ the number of non-negative $d \times d$ matrices with maximal entry Δ.

By Lemma 3 for every multiplicity vector $\lambda^{(i)}$ there exists a basic feasible solution $x^{(j)}$ with $\frac{1}{\ell} x^{(j)} \leq \lambda^{(i)}$. Hence, in the case that any given multiplicity vector has $\|\lambda^{(i)}\|_1 > 3^{\mathcal{O}((d\Delta)^d)}$, it is sufficient to find new multiplicity vectors for each basic feasible solution.

As $\sum_{p \in P} \lambda^{(i)}_p p = b = \sum_{p \in B^{(j)}} x^{(j)}_p p$ and every $p \in P$ is bounded by $\|p\|_\infty \leq \Delta$, the multiplicity vectors are bounded by

$$\|\lambda^{(i)}\|_1 \leq \left\| \sum_{p \in B^{(j)}} x^{(j)}_p p \right\|_1 \leq \sum_{p \in B^{(j)}} x^{(j)}_p \|p\|_1 \leq d\Delta \sum_{p \in B^{(j)}} x^{(j)}_p = d\Delta \|x^{(j)}\|_1$$

for every $i \leq n$ and every $j \leq \ell$. Let K denote the constant in the $\mathcal{O}$-notation of inequality (3). If $\|\lambda^{(i)}\|_1 > d^2 \Delta \ell \cdot 3^{K(d\Delta)^d}$ holds for some $i \leq n$, then for every $j \leq \ell$

$$\left\| \frac{1}{\ell} x^{(j)} \right\|_1 \geq \frac{1}{d\Delta\ell} \left\| \lambda^{(i)} \right\|_1 > d \cdot 3^{K(d\Delta)^d}.$$

Claim: If for all $i \leq \ell$ the bound $\left\| \frac{1}{\ell} x^{(i)} \right\|_1 > d \cdot 3^{K(d\Delta)^d}$ holds, then there exist non-zero vectors $y^{(1)}, \ldots, y^{(\ell)} \in \mathbb{Z}^d_{\geq 0}$ with $y^{(i)} \leq \frac{1}{\ell} x^{(i)}$ and $\left\| y^{(i)} \right\|_1 \leq d \cdot 3^{K(d\Delta)^d}$ such that $B^{(1)} y^{(1)} = \ldots = B^{(\ell)} y^{(\ell)}$.

Proof of the claim: For every $i \leq \ell$ it holds that $b \in cone(B^{(i)})$. Thus, there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that $\alpha b \in int.cone(\hat{B}) = \bigcap_{i=1}^{n} int.cone(B^{(i)})$ by Lemma 4 for some generating set $\hat{B} \subset \mathbb{Z}^d_{\geq 0}$ and thus for the intersection integer cone $int.cone(\hat{B}) \neq \{0\}$ holds. There exists $\hat{x} \in \mathbb{R}^{\hat{B}}_{\geq 0}$ with $\hat{B}\hat{x} = \frac{1}{\ell} b$ as $b \in cone(\hat{B})$.

Assume that $\hat{x}_p < 1$ for every $p \in \hat{B}$. Then for every $i \leq n$ and $p \in \hat{B}$ there exists $y^{(i,p)} \in \mathbb{Z}^{B^{(i)}}_{\geq 0}$ with $p = B^{(i)} y^{(i,p)}$ and $\left\| y^{(i,p)} \right\|_1 \leq 3^{K(d\Delta)^d}$ by Lemma 15. As $B^{(i)}(\frac{1}{\ell} x^{(i)}) = \frac{1}{\ell} b = \sum_{p \in \hat{B}} \hat{x}_p p = \sum_{p \in \hat{B}} \hat{x}_p (B^{(i)} y^{(i,p)})$, the equality $\frac{1}{\ell} x^{(i)} = \sum_{p \in \hat{B}} y^{(i,p)} \hat{x}_p$ holds since $B^{(i)}$ is a basis. This yields

$$\left\| \frac{1}{\ell} x^{(i)} \right\|_1 \leq \sum_{p \in \hat{B}} \left\| y^{(i,p)} \hat{x}_p \right\|_1 \overset{\hat{x}_p < 1}{<} \sum_{p \in \hat{B}} \left\| y^{(i,p)} \right\|_1 \leq d \cdot 3^{K(d\Delta)^d},$$

which is a contradiction. Thus, if $\left\| \frac{1}{\ell} x^{(i)} \right\|_1 > d \cdot 3^{K(d\Delta)^d}$, there exists $\hat{x}_q \geq 1$ for some $q \in \hat{B}$.

Then $q' := \frac{1}{\ell} b - q = \hat{B}(\hat{x} - e_q) \in cone(\hat{B})$ because $\hat{x}_q \geq 1$. Thus, there exists $z^{(j)} \in \mathbb{R}^{B^{(j)}}_{\geq 0}$ with $B^{(j)} z^{(j)} = q'$. The vector $\frac{1}{\ell} b$ can be written as

$$\frac{1}{\ell} b = B^{(j)} x^{(j)} = q + q' = B^{(j)}(y^{(j,q)} + z^{(j)})$$

implying $\frac{1}{\ell} x^{(j)} = y^{(j,q)} + z^{(j)}$ because $B^{(j)}$ is a basis.

In total, this proofs the claim as $y^{(j,q)} \leq \frac{1}{\ell} x^{(j)}$ is integral, bounded by $\left\| y^{(j,q)} \right\|_1 \leq d \cdot 3^{K(d\Delta)^d}$, and yields $q = B^{(1)} y^{(1,q)} = \cdots = B^{(\ell)} y^{(\ell,q)}$. $\triangleleft$

Hence, if $\left\| \lambda^{(i)} \right\|_1 > d^2 \Delta \ell \cdot 3^{K(d\Delta)^d}$ holds for some $i \leq n$, the second claim can be applied to find $y^{(j)} \leq \frac{1}{\ell} x^{(j)}$ with $\left\| y^{(j)} \right\|_1 \leq d \cdot 3^{K(d\Delta)^d}$ and $B^{(1)} y^{(1)} = \ldots = B^{(\ell)} y^{(\ell)}$. Hence, for every multiplicity vector $\lambda^{(i)}$ there exists $j \leq \ell$ with $y^{(j)} \leq \frac{1}{\ell} x^{(j)} \leq \lambda^{(i)}$.

In total, either the given multisets were already bounded by

$$|T_i| = \left\|\lambda^{(i)}\right\|_1 \leq d^2\Delta\ell \cdot 3^{K(d\Delta)^d} \leq d^2\Delta^{d^2+1} \cdot 3^{K(d\Delta)^d} = 3^{\mathcal{O}((d\Delta)^d)}$$

for every $i \leq n$ or there exist subsets $S_i \subseteq T_i$ defined by multiplicity vectors $y^{(j)} \leq \lambda^{(i)}$ with $|S_i| = \|y^j\|_1 \leq d \cdot 3^{K(d\Delta)^d} = 3^{\mathcal{O}((d\Delta)^d)}$ and equality $B^{(1)}y^{(1)} = \ldots = B^{(\ell)}y^{(\ell)}$. □

For the bound on Graver elements it remains to revisit the proof for Graver elements [28] but the improved structural result is used to obtain improved bounds.

Theorem 6. *Let y be a Graver element of a two-stage matrix $\mathcal{A}$. Then $\|y\|_\infty$ is bounded by $3^{\mathcal{O}(s^s(2r\Delta+1)^{rs})}$.*

Proof. Let $y = (y^{(0)}, y^{(1)}, \ldots, y^{(n)})^T \in \mathbb{Z}^{s+nt}_{\geq 0}$ be a cycle, i.e. $\mathcal{A}y = 0$ and in particular

$$\begin{pmatrix} A^{(i)} & B^{(i)} \end{pmatrix} \begin{pmatrix} y^{(0)} \\ y^{(i)} \end{pmatrix} = \mathbf{0}.$$

Let $v^{(i)} := (y^{(0)}y^{(i)})^T$ denote the subcycle of each block row-wise. Then $v^{(i)}$ can be written as the sum of Graver elements C_i and in particular $\sum_{c \in C_i} c = v^{(i)}$. Note that every $c \in C_i$ has only non-negative components as y is non-negative and the decomposition is sign compatible. By Lemma 7 the 1-norm can be bounded by $\|c\|_1 \leq (2r\Delta + 1)^r$.

Define two projections p to the first block and $\bar{p}$ to the second block by $p((y^{(0)}y^{(i)})^T) := y^{(0)}$ and $\bar{p}((y^{(0)}y^{(i)})^T) := y^{(i)}$. Additionally, let $p(C_i) := \{p(c) \mid c \in C_i\}$ and similarly let $\bar{p}(C_i) := \{\bar{p}(c) \mid c \in C_i\}$. The projected sums of the first component are identical, i.e.

$$y^{(0)} = \sum_{c \in p(C_1)} c = \ldots = \sum_{c \in p(C_n)} c.$$

By Lemma 16 there exist subsets $S_1 \subseteq p(C_1), \ldots, S_n \subseteq p(C_n)$ such that $|S_i| \leq 3^{\mathcal{O}(s^s(2r\Delta+1)^{rs})}$ and $\sum_{s \in S_1} s = \ldots = \sum_{s \in S_n} s$.

As $S_i \subseteq p(C_i)$ there exist $\bar{C}_i \subseteq C_i$ with $p(\bar{C}_i) = S_i$ for all $i \leq n$. Define

$$\bar{y}^{(0)} := \sum_{c \in \bar{C}_i} p(c) = \sum_{s \in S_i} s \quad \text{and} \quad \bar{y}^{(i)} := \sum_{c \in \bar{C}_i} \bar{p}(c) \quad \text{for all } 1 \leq i \leq n.$$

Then $\bar{y} = (\bar{y}^{(0)}, \bar{y}^{(1)}, \dots, \bar{y}^{(n)})$ is a cycle of $\mathcal{A}$ with $\bar{y} \leq y$. Additionally, its norm is bounded by

$$\left\|\bar{y}^{(i)}\right\|_1 \leq (2r\Delta + 1)^r \cdot 3^{\mathcal{O}(s^s(2r\Delta+1)^{rs})} = 3^{\mathcal{O}(s^s(2r\Delta+1)^{rs})}.$$

□

For the sake of completeness, the following theorem combines the running time of Corollary 2 with the new bound for Graver elements. This yields the result as stated in section 1.

Theorem 1. *A two-stage stochastic IP can be solved in time*

$$3^{\mathcal{O}((r+s)s^s(2r\Delta+1)^{rs})} \cdot n \log^3 n \cdot \log \|u - l\|_\infty \cdot \log \|c\|_\infty.$$

References

[1] S. Artmann, R. Weismantel, and R. Zenklusen. A strongly polynomial algorithm for bimodular integer linear programming. In *STOC*, pages 1206–1219. ACM, 2017.

[2] M. Aschenbrenner and R. Hemmecke. Finiteness theorems in stochastic integer programming. *Found. Comput. Math.*, pages 183–227, 2007.

[3] S. Bosch. *Algebra*. Springer, 2009.

[4] G. M. Campbell. A two-stage stochastic program for scheduling and allocating cross-trained workers. *Journal of the Operational Research Society*, pages 1038–1047, 2011.

[5] C. Carathéodory. Über den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen. *Rendiconti Del Circolo Matematico di Palermo*, pages 193–217, 1911.

[6] L. Chen, H. Chen, and G. Zhang. Block-structured integer programming: Can we parameterize without the largest coefficient?, 2020.

[7] L. Chen, L. Xu, W. Shi, and M. Koutecký. New bounds on augmenting steps of block-structured integer programs, 2019.

[8] M. Conforti, G. Cornuéjols, and G. Zambelli. *Integer programming*, volume 271. Springer, 2014.

[9] J. Cslovjecsek, F. Eisenbrand, C. Hunkenschröder, L. Rohwedder, and R. Weismantel. Block-structured integer and linear programming in strongly polynomial and near linear time. In *Proc. SODA*, pages 1666–1681, 2021.

[10] J. Cslovjecsek, F. Eisenbrand, M. Pilipczuk, M. Venzin, and R. Weismantel. Efficient sequential and parallel algorithms for multistage stochastic integer programming using proximity. *CoRR*, abs/2012.11742, 2020.

[11] F. Eisenbrand, C. Hunkenschröder, and K.-M. Klein. Faster algorithms for integer programs with block structure. In *ICALP 2018*, pages pp. 49:1–49:13.

[12] F. Eisenbrand, C. Hunkenschröder, K.-M. Klein, M. Koutecký, A. Levin, and S. Onn. An algorithmic theory of integer programming. *CoRR*, abs/1904.01361, 2019.

[13] F. Eisenbrand and R. Weismantel. Proximity results and faster algorithms for integer programming using the steinitz lemma. *ACM Trans. Algorithms*, pages pp. 5:1–5:14, 2020.

[14] A. R. Ferguson and G. B. Dantzig. The allocation of aircraft to routes—an example of linear programming under uncertain demand. *Management science*, pages 45–73, 1956.

[15] M. Gendreau, G. Laporte, and R. Séguin. An exact algorithm for the vehicle routing problem with stochastic demands and customers. *Transportation science*, pages 143–155, 1995.

[16] D. Hanson. On the product of the primes. *Canadian Mathematical Bulletin*, pages 33–37, 1972.

[17] R. Hemmecke, M. Köppe, and R. Weismantel. A polynomial-time algorithm for optimizing over n-fold 4-block decomposable integer programs. *Lecture Notes in Computer Science*, page pp. 219–229, 2010.

[18] R. Hemmecke, S. Onn, and L. Romanchuk. n-fold integer programming in cubic time. *Math. Program.*, pages 325–341, 2013.

[19] R. Hemmecke and R. Schultz. Decomposition of test sets in stochastic integer programming. *Math. Program.*, pages 323–341, 2003.

[20] T.-W. Huang and T.-Y. Ho. A two-stage integer linear programming-based droplet routing algorithm for pin-constrained digital microfluidic biochips. *IEEE transactions on computer-aided design of integrated circuits and systems*, pages 215–228, 2011.

[21] K. Jansen, K.-M. Klein, and A. Lassota. The double exponential runtime is tight for 2-stage stochastic ilps. *CoRR*, abs/2008.12928, 2020.

[22] K. Jansen, A. Lassota, and L. Rohwedder. Near-linear time algorithm for n-fold ilps via color coding. *SIAM J. Discret. Math.*, pages 2282–2299, 2020.

[23] K. Jansen and L. Rohwedder. On integer programming and convolution. In *Proc. ITCS 2019*, pages pp. 43:1–43:17.

[24] H. W. Lenstra Jr. Integer programming with a fixed number of variables. *Math. Oper. Res.*, pages 538–548, 1983.

[25] R. Kannan. Minkowski's convex body theorem and integer programming. *Math. Oper. Res.*, pages 415–440, 1987.

[26] E. P. C. Kao and M. Queyranne. Budgeting costs of nursing in a hospital. *Management Science*, pages 608–621, 1985.

[27] R. M. Karp. Reducibility among combinatorial problems. In *Proc. Complexity of Computer Computations, 1972*, pages 85–103. Plenum Press, New York.

[28] K.-M. Klein. About the complexity of two-stage stochastic ips. In *Proc. IPCO 2020*, pages 252–265. Springer.

[29] V. I. Norkin, Y. M. Ermoliev, and A. Ruszczyński. On optimal allocation of indivisibles under uncertainty. *Operations Research*, pages 381–395, 1998.

[30] S. Onn. Nonlinear discrete optimization. *Zurich Lectures in Advanced Mathematics, European Mathematical Society*, 2010.

[31] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998.

[32] V. N. Shevchenko. *Qualitative topics in integer linear programming*. American Mathematical Soc., 1996.

[33] D. B. Shmoys and C. Swamy. An approximation scheme for stochastic linear programming and its application to stochastic integer programs. *Proc. JACM*, pages 978–1012, 2006.

[34] G. Sierksma and Y. Zwols. *Linear and integer optimization: theory and practice*. CRC Press, 2015.

[35] G. Zhu, J. F. Bard, and G. Yu. A two-stage stochastic programming approach for project planning with uncertain activity durations. *Journal of Scheduling*, pages 167–180, 2007.