INTERNAL BUCKLING IN THE PRESTRAINED LAMINATED MEDIA

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A new mathematical model for the internal stability analysis of multilayered periodic media is proposed. The apprach include the effect of lamina thickness on the global body behaviour. Some special results are discussed.

1. Introduction

The theory of internal stability in homogeneous anisotropic elastic media under initial stress was developed by M. A. Biot, (1, 2). The term *internal* means that we deal with a medium of infinite extent or confined by rigid boundaries. It was shown in (1) that this kind of stability is not possible in isotropic materials. Since the anisotropy can be created artificially by a micro-periodic distribution of material inhomogeneites therefore the results of Biot were also applied to investigate the internal buckling in laminated media composed by stacking alternative soft and hard layers of materials, (1). To do this we have to replace the periodic isotropic inhomogeneous solid by an equivalent homogeneous anisotropic medium. This procedure from the mathematical point of view is well know being referred to as *homogenization*, (3). However, the governing equations of an equivalent homogeneus anisotropic medium (obtained *via* the homogenization procedure) are independent of the lamina thickness. Roughly speaking, using the method proposed by Biot, which is also applied in many recent contibutions, the effect of lamina thickness on the global solid behaviour is neglected. The aim of our contribution is to include this effect into the mathematical description of the internal buckling phenomena. The approach is based on that proposed in (4).

2. Analysis

2. 1 Prerequisites

Let $Ox_1x_2x_3$ be Cartesian coordinate system in the reference space. The subject of analysis is a periodic laminated medium made by stacking alternatively soft and hard perfectly bounded layers of two linear elastic materials with interfaces normal to x_3 - coordinate, cf. Fig. 2.1. Moreover, the coordinate planes x_3 = const. are assumed to be elastic summetry planes. It is assumed that the medium is under constant initial strain with components e_{ij} (Latin and Greek indices take the values 1, 2, 3 and 1, 2, respectively) where $e_{\alpha 3} = e_{33} = 0$. Let l stand for the thickness of two adjacent laminae and hence is the period of inhomogenity in the x_3 - axis direction. We are to show the existence of internal buckling under constant initial strain $e_{\alpha\beta}$ in the framework of a model which takes into account the effect of microstructure size l on a solid behaviour. The analysis will be restricted to the stationary problems.

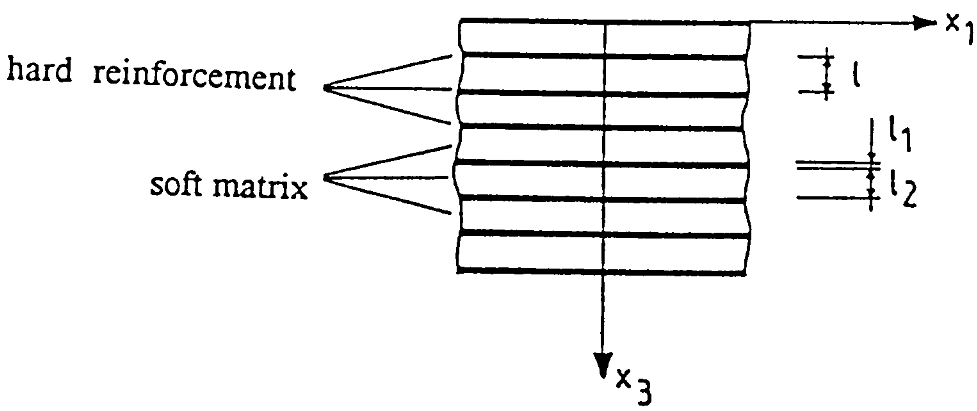

Fig.2.1. The scheme of a laminate

2.2 Assumptions

The strains and displacemants superimposed on the initial strain $e_{\alpha\beta}$ are ε_{ij} and u_i, respectively. Due to the existence of an initial strain in the coordinate planes x_3= const. the posibility of buckling in the x_3 - axis direction will be taken into account. To this end the weak form of the equilibrium equations will be considered under assumption

$$\varepsilon_{ij} = u_{(i,j)} + \frac{1}{2} u^3_{,i}\, u_{3,j} \tag{1}$$

Following the approach proposed in (4) we also assume

$$u_i = U_i + h\, V_i \tag{2}$$

where h= h(x_3) is the *l*-periodic saw-like function which is continuous, linear across every lamina and takes the values $\sqrt{3}\ l$ and - $\sqrt{3}\ l$ alternatively on the interfaces. Functions U_i , V_i are the new basic unknowns and together with their derivatives, from the computational viewpoint, in every interval (x_3, x_3 + l) and any (x_1, x_2) can be treafed as constant modulo a certain tolerance parameter ε, cf. (4). Functions of this kind are called *ε - macrofunctions* (related to the microstructure length parameter l). Bearing in mind formulae (1) and (2) we shall postulate that the principle of virtual work is assumed to hold for every test function of the form $\bar{u}_i$= $\bar{U}_i$ + h $\bar{V}_i$, where $\bar{U}_i$, $\bar{V}_i$ are arbitrary sufficiently regular ε - macrofunctions. Taking into account the stress-strain relations of the linear elasticity and using the averaging procedure explained in (4) we arrive finally at the system of equations for unknowns functions U_i and V_i . These functions will be called *macrodisplacements* and *quasi-internal variables*, respectively.

2.3 Results

In order to simplity the analytical description of the problem under consideration in the aforementioned equations all non-linear terms involving U_i and V_i will be neglected. At the some time we neglect the effect of body forces. Let $A^{ijkl} = A^{ijkl}(x_3)$ stand for components of the elastic modulae tensor which are l–periodic piecewise constant functions i.e. constant in every lamina. Define by

$$< f > \equiv \frac{1}{l}\int_0^l f(x_3)dx_3$$

the averaged value of an arbitrary integrable l-periodic function f(·). Introducing the following system of internal avaraged forces

$$S^{ij} = < A^{ijkl} > U_{k,l} + < A^{ijk3}\, h_{,3} > V_k \,;$$

$$H^{i} = < A^{i3kl}\, h_{,3} > U_{k,l} + < A^{i3k3}\, (h_{,3})^2 > V_k \,;$$

$$R^{i\alpha} = l^2 < A^{\alpha i\beta k} > V_{k,\beta} \tag{3}$$

we obtain the governing equations for the investigation of *internal buckling in the prestrained laminated medium* :

$$S^{\alpha j}{}_{,j} = 0 \, ;$$

$$S^{3j}{}_{,j} + e_{\alpha\beta} < A^{\alpha\beta\gamma\delta} > U^{3}{}_{,\gamma\delta} = 0 \, ;$$

$$R^{\alpha\beta}{}_{,\beta} - H^{\alpha} = 0 \, ;$$

$$R^{3\beta}{}_{,\beta} - H^{3} + l^{2} e_{\alpha\beta} < A^{\alpha\beta\gamma\delta} > V^{3}{}_{,\gamma\delta} = 0 \, ; \qquad (4)$$

The above equations have to be considered together with homogeneous boundary conditions, cf (4). Substituting the right-hand sides of Eqs (3) into Eqs (4) we arrive at the system of three equations for macrodisplacements U_i coupled with three equations for unknowns V_i . It can be seen that the equations for V_i do not involve derivatives of V_i with respect to x_3 ; that in why V_i were called quasi-internal variables, (4). It has to be emphasized that every solution to equations (3), (4) has a physical meaning only if $U(x_1,x_2,\cdot)$, $V(x_1,x_2,\cdot)$ are ε-macro functions for every (x_1,x_2). The model obtained depends on l and hence it is able to describe the effect of lamina thickness on the internal buckling of the laminated solid under consideration.
Neglecting terms depending on l we can eliminate V_i from the above equations. To this end we have to observe that $< A^{i3k3} (h_{,3})^2 >$ is a non-singular 3x3 matrix; denoting by C_{mn} the components of the inverse matrix, after some simple manipulations, instead of Eqs.(3) we obtain

$$S^{ij} = A_0{}^{ijkl} \; U_{k,l}$$

where

$$A_0{}^{ijkl} \equiv < A^{ijkl} > - < A^{ijm3} > C_{mn} < A^{kln3} > .$$

It can be proved that $A_0{}^{ijkl} \varepsilon_{ij} \varepsilon_{kl}$ is a positive definite form for every elastic modulae tensor A^{ijkl}. The above equation has to be considered together with the first and second from equations (4). In this case we deal with a certain equivalent homogeneous anisotropic continuum constituting the homogenized model of a laminated solid.

3. Applications

In order to evaluate the effect of lamina thickness on the internal stability we restrict considerations to the initial strain given by one non-zero component $e \equiv e_{11}$, and to the one-dimensional problem by assuming $U_i = U_i(x_1)$, $V_i = V_i(x_1)$. In this case from the general equations (3) and (4) two independent systems of equations for U_3, V_1 and U_1, V_3, respectively, can be derived. In the analysis of internal buckling outlined below only sinusoidal deformations are considered. Substituting

$$U_3 = A_3 \sin\frac{\pi x_1}{L} \; ; \qquad V_1 = A_1 \cos\frac{\pi x_1}{L}$$

into differential equations for U_3, V_1 we obtain nontrivial solutions A_1, A_3 to the resulting linear algebraic equations only for some critical values of $e \equiv e_{11}$ for which the internal instability takes place. For the laminate made of two isotropic materials with Lame modulae given by the l-periodic piecewise constant functions $\lambda = \lambda(x_3)$, $\mu = \mu(x_3)$, the critical value of initial strain is equal to

$$e = -\frac{2 < \mu >}{< \lambda + 2\mu >}\left[1 - \frac{< \mu h_{,3} >^2}{(1+\eta) < \mu >< \mu (h_{,3})^2 >}\right] \tag{5}$$

where we have denoted

$$\eta \equiv \frac{< (\lambda + 2\mu) h^2 >}{2 < \mu (h_{,3})^2 >}\left(\frac{\pi}{L}\right)^2 \tag{6}$$

It can be shown that in any case condition $e<0$ holds. The effect of lamina thickness on the critical value of initial strain is described by the nondimensional constant η and has to be taken into account only if the wavelength L of a deformation pattern is not small compared to the thickness l of two adjacent laminae. This situation takes place e.g. for a layer bounded by planes $x_1=0$, $x_1=L$, where L is of an order of the lamina thickness l. Since in the framework of the linearized theory the absolute value of e is restricted by $|e| << 1$, therefore formula (5) has the physical meaning only if the term in the square brackets (which is always positive) is sufficienthy small compared to 1. Such situation takes place if $l_1 << l_2$, $\mu_1 >> \mu_2$, $\lambda_1 >> \lambda_2$ where l_1, μ_1, λ_1 and l_2, μ_2, λ_2 stand for the thickness and Lame modulae of the adjacent laminae. Roughly speaking, the internal instability under initial strain is possible only if we deal with a hard reinforcement embeded in a soft matrix material. Some numerical results are shown in Fig.3.1 under assumption that $\mu_2/\mu_1=\lambda_2/\lambda_1$, where the ratio $\xi\equiv\mu_2/\mu_1$ is used as a parameter and $\delta\equiv l_1/l_2$ is a structural constant. The diagrams in Fig 3.1 are restricted to the cases in which $1+\eta\approx 1$ i.e, the effect of lamina thickness on the internal buckling can be neglected. More general discussion of the problem under consideration will be investigated in the lecture.

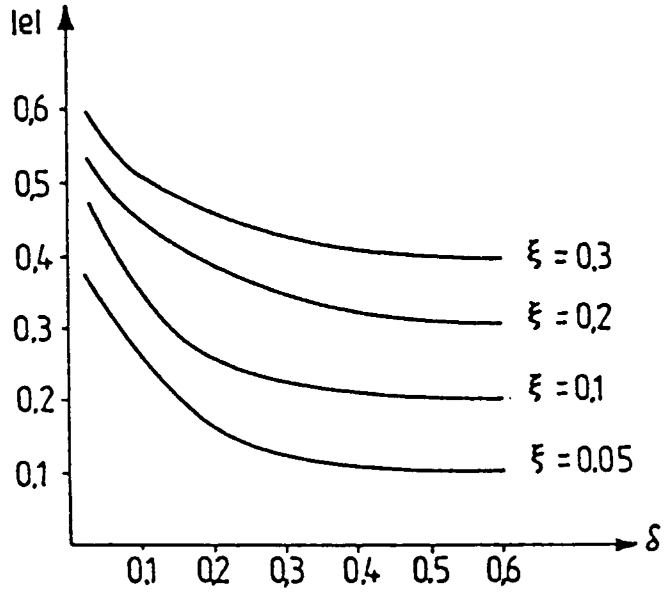

Fig.3.1. Critical values of the initial strain versus structural constant δ

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