GLOBAL INVERTIBILITY THEOREMS AND THEIR APPLICATIONS –A VARIATIONAL APPROACH

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Marek Galewski & Marcin Koniorczyk

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Introduction

This book is concerned with deriving abstract tools which are applicable in solving integro-differential equations posed in a Banach space setting. Our approach relies on deriving conditions which guarantee that certain mapping acting between two Banach spaces is a diffeomorphim thereby obtaining that if this mapping is a solution operator to some equation then this very equation has a unique solution which in fact is well posed in the sense of Hadamard.

Integro-differential and integral operators are usually considered in the space of continuous functions [14, 51], the space of square integrable functions L^2 [43] and recently in AC_0^2 [38, 7]. Thus our approach shows that other setting is also possible therefore extending existing methodology. The application of integral operators can be found in many dispciplines of science and engineering: in biology to investigate the spread of epidemic [29], in mechanics for modelling alloys with a shape memory [60], in nuclear reactor dynamics [14, 16], etc. We believe that readers would find our results motivating. Concerning possible applications, apart from those developed in our presentation, we hope for deriving methodology applicable to the second order problems, possibly posed in a non-variational form. This requires some further research in the topic. Several research suggestions are collected at the end of this book.

Our book is organized as follows. We give some motivation and outline of results firstly. Then we provide necessary functional setting which although seems to be known is not easily found in accessible literature.

The third chapter is concerned with the global version of a diffeomorphism and an implicit function theorems together with their proofs and relevant necessary tools used in the arguments.

The fourth chapter concerns the application of a global diffeomorphism theorem for examining a nonlinear itegro-differential and nonlinear integral equations and related operators. The functionals considered are defined on the Banach space, instead of the setting Hilbert space which is known from the literature, see for example [38, 7]. The most difficult part of the proof is to check, whether the functional satisfies Palais-Smale condition. Based on the Closed Graph Theory and by usage of the so called Bielecki norm, we discuss the alternative formulation of the assumptions.

In the fifth chapter we present the examination of the existence, uniqueness and continuous differentiability of a solution to a nonlinear integro-differential control problem by means of the global implicit function theorem. We use very similar tool as shown in the previous chapter so only the main steps of the proof are presented.

We conclude our presentation in the sixth chapter with some finite dimensional invertibility results which allow us to obtain a global diffeomorphism out of a Fréchet-differentiable, and not necessarily C^1, mapping. It is interesting that we use non-smooth critical point theory in proving the result. We also suggest some application to discrete equations, namely to algebraic equaions which may be viewed as discretizations of second order problem with Dirichlet boundary conditions.

It is our pleasure to thank people who helped us in the preparation of this book. Thanks are due to Professor Stanisław Walczak from Lodz University for suggesting possible research in this area. We would like to thank the student of mathematics at the Institute of Mathematics, Lodz University of Technology, Mr Michał Bełdziński, for helping us with type-

setting of the final version, careful reading of the text and numerous questions which we hope would be fruitful in the future.

This work was prepared in the Institute of Mathematics, Lodz University of Technology where the first author is an associate professor and the second author is a Ph.D. student.

CHAPTER 1

Overview of results

There are two basic concepts of differentiability for operators and functionals, which will be given below. Let X, Y be Banach spaces, and assume that U is an open subset of X. A mapping $f : U \to Y$ is said to be Gâteaux differentiable at $x_0 \in U$ if there exists a continuous linear operator $f'_G(x_0) : X \to Y$ such that for every $h \in X$

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = f'_G(x_0)h.$$

The operator $f'_G(x_0)$ is called the Gâteaux derivative of f at x_0.

An operator $f : U \to Y$ is said to be Fréchet-differentiable at $x_0 \in U$ if there exists a continuous linear operator $f'(x_0) : X \to Y$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - f'(x_0)h\|}{\|h\|} = 0.$$

The operator $f'(x_0)$ is called the Fréchet derivative of operator f at x_0. When F is Fréchet-differentiable it is continuous and Gâteaux differentiable. A mapping f is continuously Fréchet-differentiable if $f' : X \ni x \mapsto f'(x) \in \mathcal{L}(X, Y)$ is continuous in the respective topologies. If f is continuously Gâteaux differentiable then it is also continuously Fréchet-differentiable and thus it is called C^1. It is the most common way to prove the Fréchet -differentiability that one shows that f is continuously

Gâteaux differentiable. In fact for critical point theory tools either the functional usually must be C^1 or locally Lipschitz and so it is no surprise that Gâteaux differentiability is only an auxiliary tool.

Towards Fréchet-differentiability we may adopt another approach: A continuous linear mapping $f'(x_0) : X \to B$, $f'(x_0) \in \mathcal{L}(X, B)$, is a Fréchet derivative of f at $x_0 \in X$ provided that for all $h \in X$ it holds that

$$f(x_0 + h) - f(x_0) = f'(x_0)h + o(h)$$

and where $\lim_{\|h\| \to 0} \frac{\|o(h)\|}{\|h\|} = 0$.

A continuously Fréchet-differentiable map $f : X \to B$ is called a diffeomorphism if it is a bijection and its inverse $f^{-1} : B \to X$ is continuously Fréchet-differentiable as well. Obviously if a mapping f is a diffeomorphism, it is automatically a homeomorphism, while the vice versa is not correct as seen by example of a function $f(x) = x^3$. Recalling the Inverse Function Theorem a continuously Fréchet-differentiable mapping $f : X \to B$ such that for any $x \in X$ the derivative is surjective, i.e. $f'(x)X = B$ and invertible, i.e. there exists a constant $\alpha_x > 0$ such that

$$\left\| f'(x)h \right\| \geq \alpha_x \|h\|$$

defines a local diffeomorphism. This means that for each point x in X, there exists an open set U containing x, such that $f(U)$ is open in B and $f|_U : U \to f(U)$ is a diffeomorphism. If f is a diffeomorphism it obviously defines a local diffeomorphism. Thus the main problem to be overcome is to make a local diffeomorphism a global one. Or in other words:

What assumptions should be imposed on the spaces involved and the mapping f to have a global diffeomorphism from the local one?

This task can be investigated within the critical point theory, or more precisely with mountain geometry and is motivated by a finite dimensional result known as Hadamard's Theorem, see Theorem 5.4 from [40]:

Theorem 1.1. *Let X, B be finite dimensional Euclidean spaces. Assume that $f : X \to B$ is a C^1-mapping such that*

- *$f'(x)$ is invertible for any $x \in X$,*
- *$\|f(x)\| \to \infty$ as $\|x\| \to \infty$,*

then f is a diffeomorphism.

Idczak, Skowron and Walczak [38] using the Mountain Pass Lemma and ideas contained in the proof of Theorem 1.1 (see again [40] for some nice version of the proof) proved the result concerning diffeomorphism between a Banach and a Hilbert space. The result from [38] reads:

Theorem 1.2. *Let X be a real Banach space, H - a real Hilbert space. If $f : X \to H$ is a C^1-mapping such that:*

- *for any $y \in H$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*

  $$\varphi(x) = \frac{1}{2}\|f(x) - y\|^2$$

  satisfies Palais-Smale condition,
- *for any $x \in X$, $f'(x)X = H$ and there exists a constant $\alpha_x > 0$ such that*

  $$\|f'(x)h\| \geq \alpha_x \|h\|,$$

then f is a diffeomorphism.

The question aroused *whether the Hilbert space H in the formulation of the above theorem could be replaced by a Banach space.* This question is of some importance since one would expect diffeomorphism to act between two Hilbert spaces or two Banach spaces rather than between a Hilbert and a Banach space. The applications given in [38] work when both X and H are Hilbert spaces. In our book we provide an affirmative answer to this question. We see that given a Hilbert space H, relation $x \mapsto \frac{1}{2}\|x\|^2$ can be treated as $x \mapsto \frac{1}{2}\langle x, x\rangle$, where $\|\cdot\|$ stands for the norm, $\langle\cdot,\cdot\rangle$ for the scalar product. The other point of view is to treat $x \mapsto \frac{1}{2}\|x\|^2$ as a potential of a duality mapping between H and H^* and finally look at the composition of identity with some C^1 functional which is 0 only at 0 and with derivative sharing the same property. These observations will lead us towards obtaining the counterpart of Theorem 1.2 in a Banach space setting as well as related implicit function results.

We provide also in this short introductory notes with some well known definition of a diffeomorphism between two spaces and a local version of the theorem, giving the sufficient conditions, for its existence [44, 77].

Definition 1.1. Let U and V be nonempty open sets in the Banach spaces X and Y. Let $0 \leq r \leq \infty$. The mapping $f : U \to V$ is called:

- a C^r-diffeomorphism if and only if f is bijective and both f and f^{-1} are C^r-mappings;
- a local C^r-diffeomorphism at the point $u_0 \in X$ is C^r-diffeomorphism from some open neighborhood $U(u_0)$ in X onto some open neighborhood $V(f(u_0))$ in Y.

Obviously, C^0 diffeomorphism is homeomorphism.

Theorem 1.3. *Let $f : U(u_0) \subseteq X \to Y$ be a C^r-mapping on some open neighbourhood of the point u_0, where X and Y are Banach spaces and $0 \leq r \leq \infty$. Then f is a local C^r-diffeomorphism at u_0 (on some neighbourhood of u_0) if and only if $f'(u_0) : X \to Y$ is bijective.*

For background on ingtegral equations we refer to [14], [16], [51], [60]. A good review of global invertibility results is contained in [31].

The methods described there present some overview of results with emphasis on what is known and what has already been obtained. In this book we only concentrate on one of potential methods but on the other hand we supply it with various applications and possible extensions.

CHAPTER 2

Background on space setting

In this section we introduce Sobolev space $\widetilde{W}_0^{1,p}([0,1],\mathbb{R}^n)$ and establish some of its most important properties, which will be extensively used in the sequel. First, let us recall the definition of Sobolev space on the interval $[0,1]$ which we define as follows

$$W^{1,p}([0,1],\mathbb{R}^n) = \{x : [0,1] \to \mathbb{R}^n \text{ is absolutely continuous,}$$
$$x' \in L^p([0,1],\mathbb{R}^n)\}$$

where x' denotes the a.e. derivative of function x, $L^p([0,1],\mathbb{R}^n)$ is the class of all measurable (equal a.e.) functions x defined on $[0,1]$ such that

$$\int_0^1 |x(t)|^p \, dt < \infty$$

with a norm

$$||x||^p_{L^p([0,1],\mathbb{R}^n)} = \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p}.$$

Here and further it the text, $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^n$. We will denote $L^p([0,1],\mathbb{R}^n)$ as L^p and $W^{1,p}([0,1],\mathbb{R}^n)$ as $W^{1,p}$. The $W^{1,p}$ space is equipped with a usual norm

$$||x||^p_{W^{1,p}} = ||x||^p_{L^p} + ||x'||^p_{L^p}.$$

Note that since any $u \in \widetilde{W}$ is continuous, the definition of Sobolev space $\widetilde{W}_0^{1,p}$ is given as follows

$$\widetilde{W}_0^{1,p}([0,1],\mathbb{R}^n) = \{x \in W^{1,p},\ x(0) = 0\}.$$

Again, in order to simplify the notation we will denote $\widetilde{W}_0^{1,p}([0,1],\mathbb{R}^n)$ as $\widetilde{W}_0^{1,p}$. The space $\widetilde{W}_0^{1,p}$ is equipped with the norm

$$||x||_{\widetilde{W}_0^{1,p}} = \left(\int_0^1 |x'(t)|^p \ dt\right)^{\frac{1}{p}} \tag{2.1}$$

for $x \in \widetilde{W}_0^{1,p}$ equivalent to $||x||_{W^{1,p}}$ as we shall show later. By definition, for any $p > 1$, we have the following chain of imbeddings

$$\widetilde{W}_0^{1,p} \hookrightarrow W^{1,p} \hookrightarrow L^p. \tag{2.2}$$

Now, let us introduce some elements from the theory of distributions and a weak or a distributional derivative, which will provide an alternative definition of a space $W^{1,p}$. Let Ω be a domain in $\mathbb{R}^n$. By $C_c^\infty(\Omega)$ denote the space of functions $\phi : \Omega \to \mathbb{R}$ which are infinitely many times differentiable and have a compact support in Ω. Such functions are usualy called test functions. By $L^1_{loc}(\Omega)$ we will denote the space of locally integrable functions. So, let $\phi \in C_c^\infty(\Omega)$ and $u \in L^1_{loc}(\Omega)$. Let us introduce the distribution $T_u : C_c^\infty(\Omega) \to \mathbb{R}$ given by the equation:

$$T_u(\phi) = \int_\Omega u(x)\phi(x)\, dx.$$

Now, we are given $u \in C^1(\Omega)$. Since ϕ has a compact support in Ω (and hence vanishes near $\partial\Omega$) integration by parts leads to

$$\int_\Omega u\phi_{x_i}\, dx = -\int_\Omega u_{x_i}\phi\, dx \quad \text{for} \quad i = 1,..,n. \tag{2.3}$$

Now, let us take $u \in C^k(\Omega)$ for any $k \in \mathbb{N}$ and multiindex $\alpha = (\alpha_1,..,\alpha_n)$ of order $|\alpha| = \alpha_1 + .. + \alpha_n = k$, then integration by parts leads to

$$\int_\Omega uD^\alpha\phi\, dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u\phi\, dx, \tag{2.4}$$

where

$$D^{\alpha}\phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi.$$

It is easy to notice that applying (2.3) $|\alpha|$ times we get (2.4). The right hand side of (2.4) makes sense only if $u \in C^k$, otherwise (2.4) has no obvious meaning. By introducing the definition of a weak derivative we resolve this problem for functions which are not necessarily of C^k class.

Definition 2.1. Suppose that $u, v \in L^1_{loc}(\Omega)$, and α is a multiindex. We say that v is α-weak partial derivative of u, $D^{\alpha}u = v$, provided that for all test functions $\phi \in C_c^{\infty}(\Omega)$ we have

$$\int_{\Omega} u D^{\alpha}\phi \, dx = (-1)^{|\alpha|} \int_{\Omega} v\phi \, dx. \tag{2.5}$$

In other words, if there exists a function v, which verifies (2.5) for all test functions $\phi \in C_c^{\infty}(\Omega)$, then we say that u has an α-weak partial derivative. If such a weak partial derivative exists then it is unique, see [20].

Now let us show that on $W^{1,p}$ the distributional derivative equals to the classical one, which exists almost everywhere on $[0,1]$. Le us take any $u \in W^{1,p}$. By definition, since any function from $\widetilde{W}^{1,p}$ is absolutely continuous, $u' \in L^p$, so the derivative exists almost everywhere on $[0,1]$. Thus the following integral makes sense for all $t \in [0,1]$:

$$U(t) = \int_0^t u'(s) \, ds.$$

The function $U : [0,1] \to \mathbb{R}$ is absolutely continuous on $[0,1]$ and $U'(t) = u'(t)$ almost everywhere on $[0,1]$. Let us take the test function $\phi \in C^{\infty}([0,1])$ and integrate by parts

$$\int_0^1 U(t)\phi'(t) \, dt = [U(1)\phi(1) - U(0)\phi(0)] - \int_0^1 u'(t)\phi(t) \, dt$$

Since $\phi \in C_0^\infty([0,1])$ the first therm on right hand side vanishes and we get the equation

$$\int_0^1 U(t)\phi'(t)\, dt = -\int_0^1 u'(t)\phi(t)\, dt.$$

So function u' is the distributional derivative of function U. Thus the distributional derivative equals to the classical one, which exists almost everywhere in $[0,1]$. Due to the imbedding (2.2), of course the same equality holds for $\widetilde{W}_0^{1,p}$. Having in mind that result, we can formulate the equivalent definition of space $W^{1,p}$ as a space of functions from L^p whose distributional derivatives belong to the L^p.

SECTION 2.1

Auxiliary inequalities

In this section some auxiliary inequalities, which will be exploited in next chapters, are given. Most of them are taken from [1], [8].

Lemma 2.1. *If $1 \le p < \infty$ and $a, b \ge 0$, then*

$$(a+b)^p \le 2^{p-1}(a^p + b^p).$$

Theorem 2.1 (Hölder's Inequality). *Let Ω be a domain in $\mathbb{R}^n$ and let $1 \le p < \infty$ and let q be its conjugate exponent defined by $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ then*

$$\int_\Omega |u(x)v(x)| dx \le ||u||_{L^p} ||v||_{L^q}$$

so $uv \in L^1(\Omega)$.

Let us suppose that $1 \le p < q < \infty$ and $u \in L^p$. Now by Hölder's Inequality, taking $p \ge 1$ and its conjugate exponent q, we have

$$\int_I |u(t)|^p\, dt \le \left(\int_I |u(t)|^q\, dt\right)^{\frac{p}{q}} \left(\int_I 1\, dt\right)^{1-\frac{p}{q}}.$$

Hence

$$||u||_{L^p} \leq ||u||_{L^q}.$$

Therefore

$$L^q \hookrightarrow L^p.$$

Furthermore, we take $u \in L^\infty$ and then

$$\lim_{p \to \infty} ||u||_{L^p} = ||u||_{L^\infty}.$$

If $u \in L^p$ for all $1 \leq p < \infty$ and if there exists a constant M (independent of p) such that

$$||u||_{L^p} \leq M$$

then $u \in L^\infty$ and moreover

$$||u||_{L^\infty} \leq M.$$

The proof of the part concerning L^∞ can be found in [1].

Theorem 2.2 (Poincaré's Inequality in $\widetilde{W}_0^{1,p}$)**.** *There exists a constant C such that for any $u \in \widetilde{W}_0^{1,p}$*

$$||u||_{W^{1,p}} \leq C||u'||_{L^p}.$$

Moreover, the norms $\| \cdot \|_{W^{1,p}}$ and $\| \cdot \|_{\widetilde{W}_0^{1,p}}$ are equivalent on $\widetilde{W}_0^{1,p}$.

Proof. Let $u \in \widetilde{W}_0^{1,p}$ and let p and q be conjugate exponents. By the definition of $\widetilde{W}_0^{1,p}$ and by the Fundamental Theorem of Calculus, we have for any $t \in [0,1]$

$$|u(t)| = |u(t) - u(0)| = \left| \int_0^t u'(s)ds \right| \leq \|u'\|_{L^1}.$$

Taking the supremum and applying Hölder's Inequality we have:

$$\|u\|_{L^\infty} \leq \|u'\|_{L^1} \leq \|u'\|_{L^p}. \tag{2.6}$$

We also have the relation

$$\|u\|_{L^p} \leq \|u\|_{L^\infty}. \tag{2.7}$$

Combining inequalities (2.6) and (2.7) we get

$$||u||_{L^p} \leq ||u'||_{L^p} = ||u||_{\widetilde{W}_0^{1,p}}$$

So

$$||u||_{L^p} \leq ||u||_{\widetilde{W}_0^{1,p}}$$

Now, by the definition of the norm in $\widetilde{W}_0^{1,p}$

$$\begin{aligned} ||u||_{\widetilde{W}_0^{1,p}} = ||u'||_{L^p} \leq ||u||_{L^p} + ||u'||_{L^p} = ||u||_{W^{1,p}} \\ \leq (1+1)\,||u'||_{L^p} = 2||u||_{\widetilde{W}_0^{1,p}} \end{aligned}$$

This finishes the proof. ■

SECTION 2.2

Properties of the space $\widetilde{W}_0^{1,p}$

In this section we would like to give some properties of $\widetilde{W}_0^{1,p}$ space, assuming that $p > 1$. Several of them will also be proven by applying relations presented in the previous section.

Lemma 2.2. *The space $\widetilde{W}_0^{1,p}$ is uniformly convex.*

Proof. Let $2 \leq p < +\infty$. For each $z, w \in \mathbb{R}^n$, it holds the well known Clarkson inequality

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \leq \frac{1}{2}\left(|z|^p + |w|^p\right).$$

Let $u, v \in \widetilde{W}_0^{1,p}$ be such that $||u||_{\widetilde{W}_0^{1,p}} = ||v||_{\widetilde{W}_0^{1,p}} = 1$ and $||u-v||_{\widetilde{W}_0^{1,p}} \geq \varepsilon \in (0,2]$. From the above, we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{\widetilde{W}_0^{1,p}}^p + \left\|\frac{u-v}{2}\right\|_{\widetilde{W}_0^{1,p}}^p &= \int_0^1 \left(\left|\frac{u'(t)+v'(t)}{2}\right|^p + \left|\frac{u'(t)-v'(t)}{2}\right|^p\right) dt \\ &\leq \frac{1}{2}\int_0^1 \left(\left|u'(t)\right|^p + \left|v'(t)\right|^p\right) dt \\ &= \frac{1}{2}\left(||u||_{\widetilde{W}_0^{1,p}}^p + ||v||_{\widetilde{W}_0^{1,p}}^p\right) = 1.\end{aligned}$$

Thus

$$\|u+v\|_{\widetilde{W}_0^{1,p}} \leq 2\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Therefore there exists

$$\delta(\varepsilon) = \left(1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}\right) > 0$$

such that

$$||u+v||_{\widetilde{W}_0^{1,p}} \leq 2(1-\delta(\varepsilon)).$$

If $p \in (1,2)$ then for each $z, w \in \mathbb{R}^n$ it holds:

$$\left|\frac{z+w}{2}\right|^q + \left|\frac{z-w}{2}\right|^q \leq \left(\frac{1}{2}\left(|z|^p + |w|^p\right)\right)^{\frac{1}{p-1}}.$$

for $\frac{1}{p} + \frac{1}{q} = 1$. A simple computation shows that if $v \in \widetilde{W}_0^{1,p}$ then $v' \in L^{p-1}$ and $||v'^q||_{L^{p-1}} = ||v||_{\widetilde{W}_0^{1,p}}^q$.

Let $r, s \in \widetilde{W}_0^{1,p}$. Then $|r'|^q, |s'|^q \in L^{p-1}$, with $0 < p-1 < 1$ and according to [17]

$$\left\||r'|^q + |s'|^q\right\|_{L^{p-1}} \geq \left\||r'|^q\right\|_{L^{p-1}} + \left\||s'|^q\right\|_{L^{p-1}}$$

Consequently,

$$
\begin{aligned}
\left\|\frac{r+s}{2}\right\|^q_{\widetilde{W}_0^{1,p}} + \left\|\frac{r-s}{2}\right\|^q_{\widetilde{W}_0^{1,p}} &= \left\|\left|\left(\frac{r+s}{2}\right)'\right|^q\right\|_{L^{p-1}} + \left\|\left|\left(\frac{r-s}{2}\right)'\right|^q\right\|_{L^{p-1}} \\
&\leq \left\|\left|\left(\frac{r+s}{2}\right)'\right|^q + \left|\left(\frac{r-s}{2}\right)'\right|^q\right\|_{L^{p-1}} \\
&= \left(\int_0^1 \left(\left|\frac{r'+s'}{2}\right|^q + \left|\frac{r'-s'}{2}\right|^q\right)^{p-1}\right)^{\frac{1}{p-1}} \\
&\leq \left(\frac{1}{2}\int_0^1 \left(|r'|^p + |s'|^p\right)\right)^{\frac{1}{p-1}} \\
&= \left(\frac{1}{2}\left\|r\right\|^p_{\widetilde{W}_0^{1,p}} + \frac{1}{2}\left\|s\right\|^p_{\widetilde{W}_0^{1,p}}\right)^{\frac{1}{p-1}}
\end{aligned}
$$

For $u, v \in \widetilde{W}_0^{1,p}$ with $||u||_{\widetilde{W}_0^{1,p}} = ||v||_{\widetilde{W}_0^{1,p}} = 1$ and $||u-v||_{\widetilde{W}_0^{1,p}} \geq \varepsilon \in (0,2]$, we get

$$
\left\|\frac{u+v}{2}\right\|^q_{\widetilde{W}_0^{1,p}} \leq 1 - \left(\frac{\varepsilon}{2}\right)^q.
$$

In either cases there exists $\delta(\varepsilon) > 0$ such that $||u+v||_{\widetilde{W}_0^{1,p}} \leq 2(1-\delta(\varepsilon))$.
■

Let us recall

Theorem 2.3 (Pettis-Milman). *[8] A uniformly convex Banach space is reflexive.*

Theorem 2.4. *$\widetilde{W}_0^{1,p}$ is a reflexive Banach space.*

Proof of Theorem 2.4. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widetilde{W}_0^{1,p}$. By the Poincaré's Inequality $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}$, which is complete [1]. So there exists $x \in W^{1,p}$ such that $\|x_n - x\|_{W^{1,p}} \to 0$ as $n \to \infty$. Of course functions x and x_n for all $n \in \mathbb{N}$ are absolutely continuous. Since for any $x \in W^{1,p}$ we have $\sup_{t\in[0,1]} |x(t)| \leq C\|x\|_{W^{1,p}}$ for some $C > 0$, (see the proof of Poincaré's Inequality) therefore $(x_n)_{n\in\mathbb{N}}$ uniformly converges to x. Moreover $x_n(0) = 0$ for all $n \in \mathbb{N}$, so we have $x(0) = 0$. Consequently, $\widetilde{W}_0^{1,p}$ is complete. By Lemma 2.2 and Theorem 2.3 it is reflexive. ■

CHAPTER 3

Abstract invertibility tools

In this chapter we provide a global diffeomorphism theorem and two global version of implicit function theorem. Let E and B be Banach spaces. We denote by E^* the dual space of E, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between E^* and E.

Let us provide necessary background on critical point theory, see for details [21]. Let $J \in C^1(E, \mathbb{R})$. A point $\overline{u} \in E$ is called a critical point of J if $J'(\overline{u}) = 0$, $J(\overline{u})$ is called the critical value of J.

Definition 3.1. Let $J \in C^1(E, \mathbb{R})$. We say that functional J satisfies Palais-Smale condition, denoted by (PS), if any sequence $(u_n)_{n \in \mathbb{N}} \in E$ such that

- $|J(u_n)| \leq M$ for all $n \in N$ and some $M > 0$,
- $\lim\limits_{n \to \infty} J'(u_n) = 0$ in E^*

admits a convergent subsequence.

Any sequence satisfying the above conditions is called a Palais-Smale sequence.

We say that the functional $J : E \to \mathbb{R}$ is coercive if $J(u) \to \infty$ as $||u|| \to \infty$. It is easy to check that, J is coercive if and only if for any $d \in \mathbb{R}$, the set J^d is bounded, where

$$J^d = \{u \in E : J(u) \leq d\}.$$

The following links the (PS) condition with coercivity [40].

Proposition 3.1. *Let E be a Banach space and assume that $J \in C^1(E, \mathbb{R})$ is bounded from below and satisfies (PS) condition. Then J is coercive.*

Remark 3.1. The converse statement (coercivity implying (PS) condition) is valid in a finite dimensional space only.

When dealing with abstract critical point theorem we need a weaker version of (PS) condition [40].

Definition 3.2. Let E be a Banach space, $J \in C^1(E, \mathbb{R})$, and $c \in \mathbb{R}$. The functional J is said to satisfy the (local, weak) Palais-Smale condition at the level c, denoted by $(PS)_c$ if any sequence $(u_n)_{n\in\mathbb{N}} \in E$ such that

- $\lim\limits_{n\to\infty} J(u_n) = c$,
- $\lim\limits_{n\to\infty} J'(u_n) = 0$ in E^*,

admits a convergent subsequence.

Remark 3.2. When condition (PS) is satisfied, then it is easy to check that $(PS)_c$ holds for all $c \in \mathbb{R}$.

We also apply the weak form of Ekeland's variational principle, see [21].

Theorem 3.1 (Ekeland Variational Principle - weak form)**.** *Let (E, d) be a complete metric space. Let $J : E \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below. Then given $\varepsilon > 0$ there exists $u_\varepsilon \in E$ such that:*

- *$J(u_\varepsilon) \le \inf_{u\in X} J(u) + \varepsilon$;*
- *$J(u_\varepsilon) < J(u) + \varepsilon d(u, u_\varepsilon)$ for all $u \in E$ with $u \neq u_\varepsilon$.*

Assuming that the functional J is defined on Banach space we can use the differential calculus. It allow us, together with the (PS) condition, to formulate a version of the critical point theorem [21] which serves as a counterpart of a direct method in the calculus of variation. Namely, it can be applied when a functional is not weakly l.s.c..

Theorem 3.2 (Proposition 10.1 [40]). *Let E be a Banach space and $J : E \to \mathbb{R}$ be a C^1 functional which satisfies the (PS) condition. Suppose in addition that J is bounded from below. Then the infimum of J is achieved at some point $u_0 \in E$ and u_0 is a critical point of J, i.e. $J'(u_0) = 0$.*

Now we will introduce the Mountain Pass Lemma given firstly by Ambrosetti and Rabinowitz [3].

Theorem 3.3 (Mountain Pass Theorem). *Let E be a Banach space and assume that $J \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition. Assume that*

$$\inf_{\|x\|=r} J(x) \geq \max\{J(0), J(e)\}, \tag{3.1}$$

where $0 < r < \|e\|$ and $e \in E$. Then J has a non-zero critical point x_0. If moreover $\inf_{\|x\|=r} J(x) > \max\{J(0), J(e)\}$, then also $x_0 \neq e$.

We recall from [10], [17] the notion of a duality mapping from E into E^* relative to a normalization function. We shall write simply duality mapping in the sequel with the understanding that we mean a duality mapping relative to some normalization function. Let $p > 1$ be a real number. A duality mapping on E corresponding to a normalization function $\varphi(t) = t^{p-1}$ is an operator $A : E \to 2^{E^*}$ such that for all $u \in E$

$$Au = \{u^* \in E^*, \langle u^*, u\rangle = \|u^*\|_* \|u\|\}, \quad \|A(u)\|_* = \|u\|^{p-1}.$$

The next Lemma provides a sufficient condition for the existence and uniqueness of the duality mapping (which is not continuous in general) and some of its useful properties.

Lemma 3.1. *Assume that E is a reflexive Banach space with a strictly convex dual E^* and that $p > 1$ is fixed. There exists a duality mapping $A : E \to E^*$ corresponding to a normalization function $t \to t^{p-1}$ which is single valued. Operator, A is monotone, i.e.*

$$\langle Au - Av, u - v\rangle \geq \left(\|u\|^{p-1} - \|v\|^{p-1}\right)(\|u\| - \|v\|).$$

for all $u, v \in E$; moreover $\|u\|^{p-1} = \|u^*\|_*$, $u^* \in Au$ and $\|v\|^{p-1} = \|v^*\|_*$, $v^* \in Av$. The functional $u \mapsto \frac{\|u\|^p}{p}$ is Gâteaux differentiable at any u and with Au being a Gâteaux derivative.

Proof. Since E^* is strictly convex, from [17, Proposition 1], it follows that the duality mapping is single valued. Moreover, from [17, Theorem 1], we see that the potential of a duality mapping A, i.e. the functional $u \mapsto \frac{\|u\|^p}{p}$ as a convex functional has a subdifferential in the sense of convex analysis which is single valued by the preceding remarks. By [17, Proposition 3], we see that $A : E \to E^*$ is demicontinuous. Then Proposition 2.8. from [56] suggests that since A is demicontinuous (continuous "norm to weak" in the original terminology in [56]) we obtain that $u \mapsto \frac{\|u\|^p}{p}$ is differentiable in the sense of Gâteaux and operator A provides its derivative. ∎

SECTION 3.1

Global diffeomorphism theorem by an auxiliary functional

Now we will give a first version of a global diffeomorphism theorem which is based on an auxiliary functional and which comes from [23]. This was the first attempt to generalize the global diffeomorphism theorem to the Banach space setting. This version requires no additional assumptions on the spaces.

Theorem 3.4. *Let E, B be Banach spaces. Assume that $f : E \to B$ is a C^1-mapping, $\eta : B \to \mathbb{R}_+$ is a C^1 functional and that the following conditions hold*

- ***1A*** $(\eta(x) = 0 \Longleftrightarrow x = 0)$ *and* $(\eta'(x) = 0 \Longleftrightarrow x = 0)$,
- ***1B*** *for any $y \in B$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \eta(f(x) - y)$$

satisfies the Palais-Smale condition,

- ***1C*** *for any $x \in E$ the Fréchet derivative is surjective, i.e. $f'(x)E = B$, and there exists a constant $\alpha_x > 0$ such that for all $h \in X$*

$$\left\| f'(x)h \right\| \geq \alpha_x \|h\|, \tag{3.2}$$

- ***1D*** *there exist positive constants α, c, M such that*

$$\eta(x) \geq c\|x\|^{\alpha} \text{ for } \|x\| \leq M,$$

then f is a diffeomorphism.

Proof. We follow the ideas used in the proof of Main Theorem in [38] with necessary modifications. In view of the remarks made in Chapter 1 condition **1C** implies that f is a local diffeomorphism. Thus it is sufficient to show that f is "onto" and "one to one".

Firstly we show that f is "onto". Let us fix any point $y \in B$. Observe that φ is a composition of two C^1 mappings, thus $\varphi \in C^1(X, \mathbb{R})$. Moreover, φ is bounded from below and satisfies the Palais-Smale condition. Thus from the Ekeland's Variational Principle it follows that there exists argument of a minimum which we denote by $\overline{x}$, see Theorem 4.7 [21]. We see by the chain rule for Fréchet derivatives and by Fermat's Principle that

$$\varphi'(\overline{x}) = \eta'(f(\overline{x}) - y) \circ f'(\overline{x}) = 0.$$

Since by **1C** mapping $f'(\overline{x})$ is invertible we see that $\eta'(f(\overline{x}) - y) = 0$. Now by **1A** it follows that

$$f(\overline{x}) - y = 0.$$

Thus f is surjective.

Now we argue by contradiction that f is "one to one". Suppose there are x_1 and x_2, $x_1 \neq x_2$, $x_1, x_2 \in X$, such that $f(x_1) = f(x_2) = a \in B$. We will apply Lemma 3.3. Thus we put $e = x_1 - x_2$ and define mapping $g : X \to B$ by the following formula

$$g(x) = f(x + x_2) - a.$$

Observe that $g(0) = g(e) = 0$. We define functional $\psi : X \to \mathbb{R}$ by the following formula

$$\psi(x) = \eta(g(x)).$$

By **1B** functional ψ satisfies the Palais-Smale condition. Next we see that $\psi(e) = \psi(0) = 0$. Using (3.2) we see that there is a number $\rho > 0$ such that

$$\frac{1}{2}\alpha_{\overline{x}} \|x\| \leq \|g(x)\| \text{ for } x \in \overline{B(0, \rho)}. \tag{3.3}$$

Indeed, since $\lim_{\|h\| \to 0} \frac{o(\|h\|)}{\|h\|} = 0$ we see that for $\|h\|$ sufficiently small,say $\|h\| \leq \delta$, it holds that $o(\|h\|) \leq \frac{1}{2}\alpha_{x_2} \|h\|$ and

$$g(0 + h) - g(0) = g'(0)h + o(\|h\|).$$

By definition of g and by **1C** we see for $\|h\| \leq \delta$ that

$$\|g(h)\| + \frac{1}{2}\alpha_{x_2}\|h\| \geq \|g(h) - o(\|h\|)\| = \left\|f'(x_2)h\right\| \geq \alpha_{x_2}\|h\|.$$

We can always assume that $\delta < \rho < \min\{\|e\|, M\}$. Thus (3.3) holds. Take any $0 < r < \rho$. Recall that by **1D** we obtain since (3.3) holds

$$\psi(x) = \eta(g(x)) \geq c\|g(x)\|^{\alpha} \geq c\left(\frac{1}{2}\alpha_{x_2}\right)^{\alpha}\|x\|^{\alpha}.$$

Thus

$$\inf_{\|x\|=r}\psi(x) \geq c\left(\frac{1}{2}\alpha_{x_2}\right)^{\alpha}\|r\|^{\alpha} > 0 = \psi(e) = \psi(0) \tag{3.4}$$

We see that (3.1) is satisfied for $J = \psi$. Thus by Theorem 3.3 we note that ψ has a critical point $v \neq 0$, $v \neq e$ and such that

$$\psi'(v) = \eta'(f(v + x_2) - a) \circ f'(v + x_2) = 0.$$

Since $f'(v + x_2)$ is invertible, we see that $\eta'(f(v + x_2) - a) = 0$. So by the assumption **1A** we calculate $f(v + x_2) - a = 0$ and $\psi(v) > 0$ by (3.4). Thus we obtain a contradiction which shows that f is a "one to one" operator. ∎

We supply our result with a few of remarks.

Remark 3.3. We see that from Theorem 3.4 by putting $\eta(x) = \frac{1}{2}\|x\|^2$ we obtain easily Theorem 1.2. In that case $c = 1$, $M > 0$ is arbitrary, $\alpha = 2$. It seems there is no difference as concerns the finite and infinite dimensional context.

Remark 3.4. Since the deformation lemma is also true with Cerami condition, we can assume that φ satisfies the Cerami condition instead of the Palais-Smale condition. However, in the possible applications, in which the Ambrosetti-Rabinowitz condition could not be assumed, it seems that checking the Palais-Smale condition would be an easier task.

It is of interest if the above proof works also in case of the global diffeomorphism result in [38] for $p \neq 2$. However, we must assume that a functional $u \mapsto \frac{\|u\|^p}{p}$ from Lemma 3.1 is continuously Gâteaux differentiable. This is the case with space $W_0^{1,p}$, see [17] and as well with $\widetilde{W}_0^{1,p}$ which assertion can be proved exactly as in [17].

Therefore we formulate a more subtle version of the above theorem although we believe that it is easier to apply the less refined version.

Theorem 3.5. *Let X and B be real Banach spaces and let $p > 1$ be a real number. Let the potential $u \mapsto \frac{\|u\|^p}{p}$ of a duality mapping $A : B \to B^*$ corresponding to a normalization function $t \to t^{p-1}$ be continuously Gâteaux differentiable. If $f : X \to B$ is a C^1-mapping such that:*

2A *for any $y \in B$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \frac{1}{p} \|f(x) - y\|^p$$

satisfies the Palais-Smale condition,

2B *for any $x \in X$, $f'(x)X = B$ and there exists a constant $\alpha_x > 0$ such that*

$$\left\|f'(x)h\right\| \geq \alpha_x \|h\|, \tag{3.5}$$

then f is a diffeomorphism.

Proof. We follow the ideas used in the proof of Theorem 3.4 with necessary modifications. In view of the remarks made at the beginning of this section, condition **2B** implies that f defines a local diffeomorphism. Thus it is sufficient to show that f is "onto" and "one to one". We will perform this task with the Theorems 3.2 and 3.3.

Firstly we show that f is "onto". Let us fix any point $y \in B$. Observe that φ is a composition of a continuously Gâteaux differentiable (and thus

C^1) functional and a C^1 mapping, so it is C^1 itself. Moreover, φ is bounded from below and it satisfies the Palais-Smale condition.

Thus from Theorem 3.2 it follows that there exists an argument of a minimum which we denote by $\overline{x}$. We see by the chain rule and Fermat's Principle and by Lemma 3.1 that

$$0 = \varphi'(\overline{x}) = A\left(f\left(\overline{x}\right) - y\right) \circ f'(\overline{x}).$$

Since by **2B** the mapping $f'(\overline{x})$ is invertible we see that $A\left(f\left(\overline{x}\right) - y\right) = 0$. Now, by the property that $\|A\left(u\right)\|_* = \|u\|^{p-1}$ we note that

$$\|A\left(f\left(\overline{x}\right) - y\right)\|_* = \|f\left(\overline{x}\right) - y\|^{p-1}.$$

So it follows that

$$f\left(\overline{x}\right) - y = 0.$$

Thus f is surjective.

Now, we argue by contradiction that f is "one to one". Suppose there are x_1 and x_2, $x_1 \neq x_2$, $x_1, x_2 \in X$, such that $f\left(x_1\right) = f\left(x_2\right) = a \in B$. We will apply Theorem 3.3. Thus we put $e = x_1 - x_2$ and define mapping $g : X \to B$ by the following formula

$$g\left(x\right) = f\left(x + x_2\right) - a.$$

Observe that $g\left(0\right) = g\left(e\right) = 0$. We define functional $\psi : X \to \mathbb{R}$ by the following formula

$$\psi\left(x\right) = \frac{1}{p}\|g\left(x\right)\|^p.$$

By **2A** functional ψ satisfies the Palais-Smale condition. Next, we see that $\psi\left(e\right) = \psi\left(0\right) = 0$. Using (3.5) we see that there is a number, $\rho > 0$ such that

$$\frac{1}{2}\alpha_{x_2}\|x\| \leq \|g\left(x\right)\| \text{ for } x \in \overline{B\left(0, \rho\right)}. \tag{3.6}$$

Indeed, since $\lim_{\|h\|\to 0} \frac{o(h)}{\|h\|} = 0$, we see that for $\|h\|$ sufficiently small, say $\|h\| \le \delta$, it holds that $o(h) \le \frac{1}{2}\alpha_{x_2} \|h\|$ and

$$g(h) = g(0+h) - g(0) = g'(0)h + o(h) = f'(x_2)h + o(h).$$

By definition of g and by **2B** we see for $\|h\| \le \delta$ that

$$\|g(h)\| + \frac{1}{2}\alpha_{x_2} \|h\| \ge \|g(h) - o(h)\| = \|f'(x_2)h\| \ge \alpha_{x_2} \|h\|.$$

We can always assume that $0 < \rho < \min(\delta, \|e\|)$. Thus (3.6) holds. Take any $0 < r < \rho$. By definition of ψ we obtain

$$\psi(x) = \frac{1}{p} \|g(x)\|^p \ge \frac{1}{p} \left(\frac{1}{2}\alpha_{x_2}\right)^p \|x\|^p.$$

for any $x \in \overline{B(0,r)}$. Thus we get

$$\inf_{\|x\|=r} \psi(x) \ge \frac{1}{p} \left(\frac{1}{2}\alpha_{x_2}\right)^p r^p > 0 = \psi(e) = \psi(0)$$

We see that (3.1) is satisfied for $J = \psi$. Thus by Theorem 3.3 we note that ψ has a critical point $v \neq 0$, $v \neq e$ and such that

$$0 = \psi'(v) = A(f(v + x_2) - a) \circ f'(v + x_2) \text{ and } \psi(v) > 0.$$

Since $f'(v + x_2)$ is invertible, we see that $A(f(v + x_2) - a) = 0$. Thus $f(v + x_2) - a = 0$. This provides the equality $\psi(v) = 0$ which contradicts $\psi(v) > 0$. Thus we obtain a contradiction which shows that f is a "one to one" operator. ■

Remark 3.5. Based on the Closed Graph Theorem it can be noticed that assumption **2B** of Theorem 3.5 is equivalent to the following: for any $v \in X$, the differential $f'(x) : X \to B$ is "one to one" and "onto".

SECTION 3.2

Global implicit function theorem by a duality mapping

Let us start with the classical local version of the implicit function theorem, to be found in many textbooks for example in [77], which is as follows

Theorem 3.6. *Let X, Y, Z be real Banach spaces. If $U \subset X \times Y$ is an open set, $F : U \ni (x, y) \mapsto F(x, y) \in Z$ is of class C^1, $F(a, b) = 0$ and the differential $F_x(a, b) : X \to Z$ is bijective, then there exist balls $B(a, r), B(b, \rho)$ and a function $f : B(b, \rho) \to B(a, r)$ such that $B(a, r) \times B(b, \rho) \subset U$ and*

- *equations $F(x, y) = 0$ and $f(y) = x$ are equivalent in the set $B(a, r) \times B(b, \rho)$;*
- *function f is of class C^1 with differential $f'(y)$ given by*

$$f'(y) = -\left[F_x(f(y), y)\right]^{-1} \circ F_y(f(y), y) \tag{3.7}$$

for $y \in B(b, \rho)$.

Remark 3.6. If $F = F(x, y) : U \to Z$ is of class C^2 and all conditions of Theorem 3.6 are fulfiled, then additionally f is of class C^2.

In the two following sections we shall provide two versions of a global implicit function theorem. Both depend on whether we apply a duality mapping or else an auxiliary functional. While for the applications which we mean both versions can be used, these differ as far as the assumptions on the underlying spaces are concerned. To prove the new version of global implicit function theorem we apply the variational approach and use Mountain Pass Theorem. A Palais-Smale condition connected with function F, with respect to x, guarantees the existence of an implicit function $f : Y \to X$ described by the equation $F(x, y) = 0$.

Theorem 3.7. *Let X, Y, Z be real Banach spaces. Let $p > 1$ be a real number. Let the potential $u \to \frac{\|u\|^p}{p}$ of a duality mapping $A : Z \to Z^*$ corresponding to a normalization function $t \to t^{p-1}$ be continuously Gâteaux differentiable. Assume that $F : X \times Y \to Z$ is a C^1 mapping such that:*

3A *for any $y \in Y$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \frac{1}{p} \|F(x, y)\|^p \quad \textit{satisfies (PS) condition,}$$

3B *differential $F_x(x, y) : X \to Z$ is bijective for any $(x, y) \in X \times Y$,*

then there exists a unique function $f : Y \to X$ such that equations $F(x, y) = 0$ and $x = f(y)$ are equivalent in the set $X \times Y$. Moreover, $f \in C^1(Y, X)$ with differential given by (3.7).

Proof. The ideas of the proof come from [38] and [24] and our Theorem 3.5. In view of the classical local implicit function theorem it is sufficient to show that for any $y \in Y$ there exists exactly one $x \in X$ such that $F(x, y) = 0$.

Let us fix a point $y \in Y$. Functional φ is a composition of two C^1 mappings, so it is C^1 itself. Moreover, φ is bounded from below and it satisfies the Palais-Smale condition by **3A**. Thus from Theorem 3.2 it follows that there exists an argument of a minimum which we denote by $\overline{x}$. We see by the chain rule and Fermat's Principle and by the assumptions on a duality mapping that

$$0 = \varphi'(\overline{x}) = A(F(\overline{x}, y)) \circ F_x(\overline{x}, y).$$

Since by **3B** the mapping $F_x(\overline{x}, y)$ is invertible we get that $A(F(\overline{x}, y)) = 0$. Now, by the property that $\|A(u)\|_* = \|u\|^{p-1}$ we note that

$$\|A(F(\overline{x}, y))\|_* = \|F(\overline{x}, y)\|^{p-1}.$$

So it follows that

$$F(\overline{x}, y) = 0, \tag{3.8}$$

which proves the existence of $\overline{x} \in X$ for every $y \in Y$, such that (3.8) holds. The uniqueness will be shown by contradiction.

Let y be fixed and let us suppose that there are $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $F(x_1, y) = F(x_2, y) = 0$. Let us put $e = x_2 - x_1$ and define a mapping $g : X \to Z$ by the following formula:

$$g(x) = F(x + x_1, y) \tag{3.9}$$

Observe that $g(0) = g(e) = 0$. Consequently

$$g(0 + h) - g(0) = g'(0)h + o(h) = F_x(x_1, y)h + o(h) \tag{3.10}$$

for $h \in X$, where $\frac{o(h)}{||h||_X} \to 0$ in Z when $h \to 0$ in X. Thus from the bijectivity of $F_x(x, y)$ there exists $\alpha_{x_1} > 0$ that

$$||g(h)||_Z + \frac{1}{2}\alpha_{x_1}||h||_X \geq ||g(h)||_Z + ||o(h)||_Z \geq ||F_x(x_1, y)h||_Z \geq \alpha_{x_1}||h||_X \tag{3.11}$$

for sufficiently small h such that $||o(h)||_Z \leq \frac{1}{2}\alpha_{x_1}||h||$. Thus, there exist $\rho > 0$, such that for all $x \in \overline{B(0, \rho)}$

$$||g(x)|| \geq \frac{1}{2}\alpha_{x_1}||x||. \tag{3.12}$$

Let us define function $\psi : X \to \mathbb{R}$ by the following formula

$$\psi(x) = \frac{1}{p}||g(x)||^p = \frac{1}{p}||F(x + x_1, y)||^p = \varphi(x + x_1)$$

for $x \in X$. Of course, ψ is continuously differentiable in the sense of Gâteaux on X and

$$\psi'(x) = \varphi'(x + x_1)$$

By assumption **3A** functional ψ satisfies the Palais-Smale condition as well. Take $\delta < \min\{\rho, \|e\|\}$. By definition of ψ,

$$\psi(0) = \psi(e) = 0, e \notin \overline{B(0,\delta)}.$$

Moreover

$$\psi(x) \geq \frac{1}{p}\left(\frac{1}{2}\alpha_{x_1}\delta\right)^p$$

for $x \in \partial B(0,\delta)$. Thus, ψ satisfies assumption of the Mountain Pass Theorem, Th. 3.3. Functional ψ has a critical point $v \neq 0$, $v \neq e$ such that $\psi(v) > 0$ and

$$0 = \psi'(v) = A\left(F\left(v + x_1, y\right)\right) \circ F_x(v + x_1, y).$$

Since $F_x(v + x_1, y)$ is invertible, we see that $A\left(F\left(v + x_1, y\right)\right) = 0$ and by the property of duality mapping $F\left(v + x_1, y\right) = 0$. This provides the equality $\psi(v) = 0$ which contradicts $\psi(v) > 0$. The obtained contradiction ends the proof. ∎

We pass to some remarks on a duality mapping which are especially concerning the assumptions on a duality mapping. In Theorem 3.7 we have assumed that the duality mapping has the potential which is continuously differentiable in order to ascertain that the functional $\varphi : X \to \mathbb{R}$ given by the formula

$$\varphi\left(x\right) = \frac{1}{p}\left\|F\left(x,y\right)\right\|^p$$

is continuously differentiable. Using Lemma 3.1 note that the potential of a duality mapping $A : Z \to 2^{Z^*}$ is Gâteaux differentiable in case Z^* is strictly convex and Z is reflexive. Recall that functional

$$x \mapsto \frac{1}{p}\int_0^1 \left|\dot{x}\left(t\right)\right|^p \, dt$$

is C^1 on $\widetilde{W}_0^{1,p}$ as already mentioned, for $2 \leq p < +\infty$.

There is an easy corollary for functions on $\mathbb{R}^n$. It reads as follows. Note that in a finite dimensional space a coercive functional satisfies the Palais-Smale condition.

Theorem 3.8. *Assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1 mapping such that:*

- *for any $y \in \mathbb{R}^m$ the functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ given by the formula*
$$\varphi(x) = \frac{1}{2} \|F(x, y)\|^2$$
is coercive, i.e. $\lim_{\|x\| \to \infty} \varphi(x) = +\infty$,
- *the differential $F_x(x, y) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,*

then there exists a unique function $f : \mathbb{R}^m \to \mathbb{R}^n$ such that equations $F(x, y) = 0$ and $x = f(y)$ are equivalent in the set $\mathbb{R}^n \times \mathbb{R}^m$. Moreover, $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$ with differential given by (3.7).

SECTION 3.3

Global implicit function theorem by an auxiliary functional

Now we will formulate the second version of global implicit function theorem. Here we make use of the auxiliary functional.

Theorem 3.9. *Let X, Y, Z be real Banach spaces. Assume that $F : X \times Y \to Z$ is a C^1-mapping, $\eta : Z \to \mathbb{R}_+$ is a C^1 functional such that the following conditions hold:*

4A $(\eta(z) = 0 \Longleftrightarrow z = 0)$ *and* $(\eta'(z) = 0 \Longleftrightarrow z = 0)$,

4B *for any $y \in Y$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*
$$\varphi(x) = \eta(F(x, y))$$
satisfies the Palais-Smale condition,

4C *for any $(x,y) \in X \times Y$ the differential $F_x(x,y) : X \to Z$ is a bijection,*

4D *there exist positive constants β, c, M such that*

$$\eta(x) \geq c \|x\|^{\beta} \text{ for } \|x\| \leq M,$$

then there exists a unique function $f : Y \to X$ such that equations $F(x,y) = 0$ and $x = f(y)$ are equivalent on $X \times Y$. Moreover, $f \in C^1(Y,X)$ with differential given by (3.7).

Proof. The proof follows in the similar manner to the proof of Theorem 3.7. The differences will be discussed below. For any fixed y functional φ is C^1 and shares the same properties as its counterpart. Hence its argument of a minimum $\overline{x}$ satisfies

$$0 = \varphi'(\overline{x}) = \eta'(F(\overline{x},y)) \circ F_x(\overline{x},y).$$

Since by assumption **4C** mapping $F_x(\overline{x},y)$ is invertible we see that

$$\eta(F(\overline{x},y)) = 0$$

and so $F(\overline{x},y) = 0$ is solvable. Again we argue by contradiction that the solution is unique. Supposing that there are $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $F(x_1,y) = F(x_2,y) = 0$ by formula (3.9) we define function $g : X \to Z$ with properties (3.10)-(3.12). We define functional $\psi : X \to \mathbb{R}$ by

$$\psi(x) = \eta(g(x)) = \eta(F(x + x_1, y)) = \varphi(x + x_1).$$

By assumption **4B** ψ satisfies the Palais-Smale condition as well. Take $\delta < \min\{||e||_X, \rho, M\}$, where $e = x_2 - x_1$. Again $\psi(0) = \psi(e) = 0$, $e \notin \overline{B(0,\delta)}$. By assumption **4D**

$$\psi(x) = \eta(g(x)) \geq c||g(x)||^{\beta} \geq c\left(\frac{1}{2}\alpha_{x_1}\delta\right)^{\beta}$$

for $x \in \partial B(0,\delta)$. Such ψ also satisfies assumption of the Mountain Pass Theorem Th. 3.3, and there exists a point $v \in X$, $v \neq 0$, $v \neq e$ such that $\psi(v) > 0$ and

$$0 = \psi'(v) = \eta'\left(F\left(v + x_1, y\right)\right) \circ F_x(v + x_1, y).$$

Since $F_x(v + x_1, y)$ is invertible, we see that $\eta'\left(F\left(v + x_1, y\right)\right) = 0$. Thus, by assumption **4A** $F\left(v + x_1, y\right) = 0$. This means that $\psi(v) = 0$ holds which contradicts $\psi(v) > 0$. The obtained contradiction ends the proof. ■

Some refinement of results in [35] is given in [36], [37] where some assumptions are weakened. Similar analysis could also be performed in our case. As well we can replace the Palais-Smale condition with its weak version. Moreover, in [77] there are some other versions of the local implicit function theorem. These cannot be successfully applied in our case. Also a different approach towards global implicit function theorem can be investigated with results contained in [66].

SECTION 3.4

Conclusion and related results

We would like to mention work [76] for some other approach connected with the nonnegative auxiliary scalar coercive function and the main assumption that for all positive r :

$$\sup_{\|x\|\leq r} \left\| f^{'}(x)^{-1} \right\| < +\infty$$

and $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$. The methods of the proof are quite different as well. One of the results of [76] most closely connected to ours and to those of [38] reads as follows

Theorem 3.10. *Let X, B be real Banach spaces. Assume that $f : X \to B$ is a C^1-mapping, $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$, for all $x \in X$ $f^{'}(x) \in Isom\,(X, B)$ and for all $x \in X$*

$$\sup_{\|x\|\leq r} \left\| f^{'}(x)^{-1} \right\| < +\infty$$

for all $r > 0$. Then f is a diffeomorphism.

The main difference between our results and the existing one is that we do not require condition $\sup_{\|x\|\leqslant r} \left\| f'(x)^{-1} \right\| < +\infty$ for all $r > 0$. We have boundedness of $\left\| f'(x)^{-1} \right\|$ but in a pointwise manner. Recall that

$\varphi(x) = \eta(f(x) - y)$ is bounded from below, C^1 and satisfies the Palais-Smale condition and therefore it is coercive as well. However, coercivity alone does not provide the existence of exactly one minimizer. We would have to add strict convexity to the assumptions. Thus we can obtain easily the following result

Theorem 3.11. *Let X, B be real Banach spaces. Assume that $f : X \to B$ is a C^1-mapping, $\eta : B \to \mathbb{R}_+$ is a C^1 functional and that the following conditions hold*

5A $(\eta(x) = 0 \Longleftrightarrow x = 0)$ *and* $(\eta'(x) = 0 \Longleftrightarrow x = 0)$,

5B *for any $y \in B$ the functional $\varphi : X \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \eta(f(x) - y)$$

is coercive and strictly convex,

5C *for any $x \in X$ the Fréchet derivative is surjective, i.e. $f'(x)X = B$, and there exists a constant $\alpha_x > 0$ such that for all $h \in X$*

$$\left\|f'(x)h\right\| \geq \alpha_x \|h\|,$$

then f is a diffeomorphism.

Proof. Let us fix $y \in B$. Note that by **5B** φ has exactly one minimizer $\overline{x}$. Thus by Fermat's Principle we see that

$$\varphi'(\overline{x}) = \eta'(f(\overline{x}) - y) \circ f'(\overline{x}) = 0.$$

Since by **5C** mapping $f'(\overline{x})$ is invertible we see that $\eta'(f(\overline{x}) - y) = 0$. Now by **5A** it follows that

$$f(\overline{x}) - y = 0.$$

Thus f is surjective and obviously "one to one" since $\overline{x}$ is unique. ■

We believe that checking that φ is strictly convex is still more demanding than proving that φ satisfies the Palais-Smale condition.

It is well-known that the implicit function and inverse theorems are equivalent in the sense that the validity of one implies the validity of the other. The following theorem is known as the Hadamard-Lévy theorem (see [33], [55], [57], [65])

Theorem 3.12 (Hadamard-Lévy)**.** *Let E, F be two Banach spaces and $f : E \to F$ be a local diffeomorphism of class C^1 which satisfies the following integral condition*

$$\int_0^\infty \min_{\|x\|=r} \|f'(x)^{-1}\|^{-1} dr = \infty.$$

Then f is a global diffeomorphism.

One interesting global invertibility result for non-smooth functions was stated in [39]. In order to state the result we shall define the modulus of surjection of a function f at a point x. Let E, F be two Banach spaces, $f : E \to F$ and $x \in E$. We denote by $B[a, r]$ the closed ball of radius r centered at $a \in E$.

$$Sur(f, x)(t) = \sup\{r \geq 0 : B[f(x), r] \subset f[B(x, t)]\} > 0$$

Thus, for any $t > 0$, the value of the modulus of surjection of f at x is the maximal radius of a ball centered at $f(x)$ contained in the f-image of the ball of radius t centered at x. We further introduce the constant of surjection of f at x by

$$sur\,(f, x) = \liminf_{t \to 0} \frac{Sur(f, x)(t)}{t}$$

Obviously, $Sur(f,x) > 0$ is a sufficient condition for f to be surjective at x, that is, for $Sur(f,x)(t)$ to be positive for small t. The following two theorems are taken from [39].

Theorem 3.13. *Let E, F be two Banach spaces and $f : E \to F$. Suppose that the graph of f is closed and there is a positive lower semicontinuous (l.s.c.) function $m : [0,\infty) \to [0,\infty)$ such that*

$$sur\,(f,x) \geq m\,(\|x\|)\,,\ x \in E \tag{3.13}$$

Then

$$Sur(f,x)(r) \geq \int_0^r m\,(s)\ ds \qquad \text{for every } r > 0$$

Theorem 3.14 (Ioffe's)**.** *Let E, F be two Banach spaces and $f : E \to F$ be a continuous mapping that is locally one-to-one (i.e., every $x \in E$ has a neighborhood in which f is one-to-one). Suppose that there is a positive lower semicontinuous (l.s.c.) function $m : [0,\infty) \to [0,\infty)$ such that condition (3.13) and the following condition are satisfied*

$$\int_0^\infty m\,(s)\ ds = +\infty$$

Then f is a global homeomorphism, the inverse mapping f^{-1} is locally Lipschitz, and for every $y \in F$, the Lipschitz constant of f^{-1} at y is not greater than $m\left(\left\|f^{-1}(y)\right\|\right)^{-1}$.

In [41] Katriel proved that from Ioffe's global inversion theorem (that is from Theorem 3.14 stated above) one can obtain the Hadamard-Lévy theorem.

CHAPTER 4

Applications of a diffeomorphism theorem

The results concerning the application of the global diffeomorphism theorem are presented in this chapter. First we prove the existence of a unique solution of an integro-differential equation and its differentiable dependence on a parameter. Consider a space $\widetilde{W}_0^{1,p}([0,1],\mathbb{R}^n)$; with fixed $p \geq 2$. Same as before we denote $L^p([0,1],\mathbb{R}^n)$ by L^p and $\widetilde{W}_0^{1,p}([0,1],\mathbb{R}^n)$ by $\widetilde{W}_0^{1,p}$ for short.

SECTION 4.1

Application to integro-differential system

In this section we will examine the solvability and differentiability of a non-linear operator between two Banach spaces applying a global diffeomorphism theorem, presented in Section 3.1. Usually, in the literature, the existence of the solution to integro-differential equation is obtained by the Banach fixed point theorem or another type of fixed point theorem, see [71], [72]. Let us formulate a nonlinear integro-differential equation with variable integration limit with an initial condition, which reads as follows

$$x'(t) + \int_0^t \Phi(t, \tau, x(\tau))\, d\tau = y(t), \quad \text{for a.e. } t \in [0,1], \tag{4.1}$$

$$x(0) = 0, \tag{4.2}$$

where $y \in L^p$ is fixed for the time being.

Now we impose assumptions on the nonlinear term. These ensure that the problem is well posed in the sense that the solution to (4.1)-(4.2) exists, it is unique and the solution operator depends in a differentiable manner on a parameter y provided we allow it to vary. This implies that problem (4.1)-(4.2) is well posed in the sense of Hadamard.

Let $P_\Delta = \{(t, \tau) \in [0,1] \times [0,1]; \tau \leq t\}$. We assume that function $\Phi : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

6A $\Phi(\cdot, \cdot, x)$ is measurable on P_Δ for any $x \in \mathbb{R}^n$ and $\Phi(t, \tau, \cdot)$ is continuously differentiable on $\mathbb{R}^n$ for a.e. $(t, \tau) \in P_\Delta$;

6B there exist functions $a, b \in L^p(P_\Delta, \mathbb{R}_0^+)$ such that

$$|\Phi(t, \tau, x)| \leq a(t, \tau)|x| + b(t, \tau);$$

for a.e. $(t, \tau) \in P_\Delta$, all $x \in \mathbb{R}^n$ and such that

$$||a||_{L^p(P_\Delta, \mathbb{R})} < 2^{-\frac{(p-1)}{p}}; \tag{4.3}$$

6C there exist functions $c \in L^p(P_\Delta, \mathbb{R}_0^+)$, $\alpha \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and a constant $C > 0$ such that

$$|\Phi_x(t, \tau, x)| \leq c(t, \tau)\alpha(|x|)$$

for a.e. $(t, \tau) \in P_\Delta$ and all $x \in \mathbb{R}^n$, moreover

$$\int_0^t c^q(t, \tau)\, d\tau \leq C, \text{ for } a.e.\ t \in [0,1].$$

We shall further indicate how to weaken the condition (4.3), see Remark 4.1. This is based on work [48] which came to our attention by personal communication of the Author. Work [48] is not already published, neither is posed on any server.

The main result of this section is the following theorem.

Theorem 4.1. *Assume that conditions* ***6A*** *-* ***6C*** *hold. Then for any fixed $y \in L^p$, problem (4.1)-(4.2) has a unique solution $x_y \in \widetilde{W}_0^{1,p}$. Moreover, the operator*

$$L^p \ni y \mapsto x_y \in \widetilde{W}_0^{1,p}$$

which assigns to each $y \in L^p$ a solution to (4.1)-(4.2) is continuously differentiable.

In order to be in the context of Theorem 3.5 we put $X = \widetilde{W}_0^{1,p}$, $B = L^p$ and define an operator

$$f : \widetilde{W}_0^{1,p} \ni x(\cdot) \mapsto x'(\cdot) + \int_0^{\cdot} \Phi(\cdot, \tau, x(\tau))\, d\tau \in L^p \tag{4.4}$$

and a functional $\varphi : \widetilde{W}_0^{1,p} \to \mathbb{R}$

$$\varphi(x) = \frac{1}{p}||f(x) - y||_{L^p}^p = \frac{1}{p}\int_0^1 \left| x'(t) - y(t) + \int_0^t \Phi(t, \tau, x(\tau)) d\tau \right|^p dt. \tag{4.5}$$

By Minkowski's and Hölder's Inequalities we get

$$\begin{aligned}
\int_0^1 \left| \int_0^t \Phi(t, \tau, x(\tau))\, d\tau \right|^p dt &\le \int_0^1 \int_0^t |\Phi(t, \tau, x(\tau))|^p \, d\tau dt \\
&\le \int_0^1 \int_0^t (a(t, \tau)|x(\tau)| + b(t, \tau))^p \, d\tau dt \\
&\le 2^{p-1} \int_0^1 \int_0^t a^p(t, \tau)|x(\tau)|^p \, d\tau dt \\
&+ 2^{p-1} \int_0^1 \int_0^t b^p(t, \tau) \, d\tau dt \\
&\le 2^{p-1} \, \|x\|_{\widetilde{W}_0^{1,p}}^p \, ||a||_{L^p(P_\Delta, \mathbb{R})}^p + 2^{p-1}||b||_{L^p(P_\Delta, \mathbb{R})}^p,
\end{aligned}$$

thus it follows that f is well defined and so is functional φ. Before proving the main theorem we will provide two auxiliary lemmas in which we will show that the assumptions of Theorem 3.5 are satisfied.

Lemma 4.1. *Assume that conditions **6A** - **6C** hold. Then mapping f given by (4.4) is continuously Gâteaux differentiable and its Gâteaux derivative $f'(x)$ at any point $x \in \widetilde{W}_0^{1,p}$ is given by*

$$f'(x(\cdot))\, h(\cdot) = h'(\cdot) + \int_0^t \Phi_x(\cdot, \tau, x(\tau))h(\tau)\, d\tau \tag{4.6}$$

for any $h \in \widetilde{W}_0^{1,p}$. Moreover, for any fixed $x \in \widetilde{W}_0^{1,p}$ operator $f'(x)$ is "one to one" and "onto".

Proof. The first part of the proof is obvious. In order to prove the second part we will show that for any fixed $v \in L^p(0,1)$ the following linear integro-differential equation

$$h'(t) + \int_0^t \Phi_x(t, \tau, x(\tau))h(\tau)\, d\tau = v(t), \quad \text{for a.e. } t \in [0,1] \tag{4.7}$$

has a unique solution in $\widetilde{W}_0^{1,p}$. To prove this property we use some reducing method. First let us fix $v \in L^p$. We consider an equation:

$$h'(t) + u(t) = v(t), \quad \text{for a.e. } t \in [0,1].$$

with any fixed $u \in L^p$. Such equation has a unique solution $h_u \in \widetilde{W}_0^{1,p}$ given by

$$h_u(t) = \int_0^t (-u(s) + v(s))\, ds, \quad \text{for a.e. } t \in [0,1].$$

Introducing the obtained solution into equation (4.7) we get

$$u(t) = (\Gamma u)(t) \quad \text{for a.e. } t \in [0,1], \tag{4.8}$$

where the mapping Γ is given by

$$\Gamma : L^p(0,1) \ni u(\cdot) \to \int_0^{\cdot} \Phi_x(\cdot,\tau,x(\tau))h_u(\tau)\,d\tau \in L^p. \tag{4.9}$$

We will show that Γ is the contraction mapping and thus it has the unique fixed point. We make use of the Bielecki norm in L^p which for the arbitrary $k>0$ is given by

$$||u||_k = \left(\int_0^1 e^{-kt}|u(t)|^p\,dt\right)^{\frac{1}{p}}$$

Let us observe that for any $u \in L^p(0,1)$ the following relation holds

$$e^{-\frac{k}{p}}||u||_{L^p} \le ||u||_k \le ||u||_{L^p} \tag{4.10}$$

so the Bielecki and L^p norms are equivalent. For any $u_1, u_2 \in L^p$ we have

$$\begin{aligned}
\|\Gamma u_1 - \Gamma u_2\|_k^p &= \int_0^1 e^{-kt}\left|\int_0^t \Phi_x(t,\tau,x(\tau))(h_{u_1}(\tau)-h_{u_2}(\tau))\,d\tau\right|^p\,dt \\
&= \int_0^1 e^{-kt}\left|\int_0^t \Phi_x(t,\tau,x(\tau))\int_0^\tau (u_1(s)-u_2(s))\,dsd\tau\right|^p\,dt \\
&\le \int_0^1 e^{-kt}\left(\int_0^t |\Phi_x(t,\tau,x(\tau))|^q\,d\tau\right)^{\frac{p}{q}}\int_0^t\int_0^\tau |u_1(s)-u_2(s)|^p\,dsd\tau dt \\
&\le \int_0^1 e^{-kt}\left(\int_0^t |\Phi_x(t,\tau,x(\tau))|^q\,d\tau\right)^{\frac{p}{q}}\int_0^t |u_1(\tau)-u_2(\tau)|^p\,d\tau dt \\
&\le B^{\frac{p}{q}}\int_0^1\int_0^t e^{-kt}|u_1(s)-u_2(s)|^p\,d\tau dt \\
&\le \frac{B^{\frac{p}{q}}}{k}\int_0^1 e^{-kt}|u_1(t)-u_2(t)|^p dt - e^{-k}\int_0^1 |u_1(\tau)-u_2(\tau)|^p d\tau \\
&\le \frac{B^{\frac{p}{q}}}{k}||u_1-u_2||_k^p,
\end{aligned}$$

where $B>0$ is such that

$$\|x\|_C^q\int_0^t c^q(t,\tau)d\tau \le B, \quad \text{for a.e. } t\in[0,1]$$

(see assumption **6C**). For sufficiently large k we see that $\frac{B^{\frac{p}{q}}}{k} \in (0,1)$, hence the mapping Γ is a contraction with respect to the Bielecki norm. Thus it has a fixed point $u^\star \in L^p$. So, for every $v \in L^p$ there exists unique $u^\star \in L^p$ which solves (4.8). Moreover, they determine unique $h \in \widetilde{W}_0^{1,p}$ according to:

$$h'(t) + u^\star(t) = v(t) \quad \text{for a.e. } t \in [0,1], \tag{4.11}$$

which in fact depends solely on $v \in L^p$. Introducing the definition (4.9) of Γ for $u^\star$ into (4.11) we get that there exists a unique $h \in \widetilde{W}_0^{1,p}$ such that the equation (4.7) holds which finishes the proof. ■

Lemma 4.2. *Assume that conditions **6A** - **6C** hold. Let $y \in L^p$ be fixed. Then functional φ given by (4.5) is continuously Gâteaux differentiable and its Gâteaux derivative at any point $x \in \widetilde{W}_0^{1,p}$ is given by*

$$\begin{aligned}\varphi'(x)h = \int_0^1 & \left| x'(t) - y(t) + \int_0^t \Phi(t,\tau,x(\tau))\, d\tau \right|^{p-2} \\ & \cdot \left(x'(t) - y(t) + \int_0^t \Phi(t,\tau,x(\tau))\, d\tau \right) \\ & \cdot \left(h'(t) + \int_0^t \Phi_x(t,\tau,x(\tau))h(\tau)\, d\tau \right)\, dt\end{aligned}$$

for any $h \in \widetilde{W}_0^{1,p}$. Moreover, functional φ satisfies the (PS) condition for any fixed $y \in L^p$.

Proof. Using (4.6) and the formula for the derivative of $x \mapsto \frac{1}{p}\|x\|^p$ in $\widetilde{W}_0^{1,p}$ together with the theorem on the differentiability of a composition of mappings for Fréchet derivatives we obtain that the differential $\varphi'(x) : \widetilde{W}_0^{1,p} \to \left(\widetilde{W}_0^{1,p}\right)^*$ of φ at any fixed point $x \in \widetilde{W}_0^{1,p}$ is given by the above formula for any $h \in \widetilde{W}_0^{1,p}$.

By Hölder's Inequality for $x \in \widetilde{W}_0^{1,p}$, we have

$$\begin{aligned}
\int_0^t |\Phi(t,\tau,x(\tau))|\, d\tau &\leq \left(\int_0^t |1|^q\, d\tau\right)^{\frac{1}{q}} \left(\int_0^t |\Phi(t,\tau,x(\tau))|^p\, d\tau\right)^{\frac{1}{p}} \\
&= \sqrt[q]{t} \left(\int_0^t |\Phi(t,\tau,x(\tau))|^p\, d\tau\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^t |\Phi(t,\tau,x(\tau))|^p\, d\tau\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, using basic relation between norms (see Poincaré's Inequality) we have

$$\begin{aligned}
\left\|\int_0^{\cdot} \Phi(\cdot,\tau,x(\tau))\, d\tau\right\|_{L^p}^p &= \int_0^1 \left(\int_0^t |\Phi(t,\tau,x(\tau))|\, d\tau\right)^p\, dt \\
&\leq \int_0^1 \left(\int_0^t |\Phi(t,\tau,x(\tau))|^p\, d\tau\right)\, dt \\
&\leq \int_0^1 \left(\int_0^t (a(t,\tau)|x(\tau)| + b(t,\tau))^p\, d\tau\right)\, dt \\
&\leq 2^{p-1} \int_0^1 \left(\int_0^t (a(t,\tau)|x(\tau)|)^p\, d\tau\right)\, dt \\
&\quad + 2^{p-1} \int_0^1 \left(\int_0^t (b(t,\tau))^p\, d\tau\right)\, dt \\
&\leq 2^{p-1} ||a||_{L^p(P_\Delta,\mathbb{R}^n)}^p ||x||_{\widetilde{W}_0^{1,p}}^p + 2^{p-1} ||b||_{L^p(P_\Delta,\mathbb{R})}^p \\
&\leq \left(2^{\frac{p-1}{p}} ||a||_{L^p(P_\Delta,\mathbb{R}^n)} ||x||_{\widetilde{W}_0^{1,p}} + 2^{\frac{p-1}{p}} ||b||_{L^p(P_\Delta,\mathbb{R})}\right)^p.
\end{aligned}$$

Therefore the following inequality might be deduced for any $x \in \widetilde{W}_0^{1,p}$

$$\begin{aligned}
(p\varphi(x))^{\frac{1}{p}} &= \left\| x'(\cdot) - y(\cdot) + \int_0^{\cdot} \Phi(\cdot,\tau,x(\tau))\, d\tau \right\|_{L^p} \qquad (4.12) \\
&\geq \|x'\|_{L^p} - \|y\|_{L^p} - \left\|\int_0^{\cdot} \Phi(\cdot,\tau,x(\tau))\, d\tau\right\|_{L^p} \\
&\geq \left(1 - 2^{\frac{p-1}{p}} ||a||_{L^p(P_\Delta,\mathbb{R}^n)}\right) ||x||_{\widetilde{W}_0^{1,p}} - c,
\end{aligned}$$

where $c = ||y||_{L^p(0,1)} + 2^{\frac{p-1}{p}} ||b||_{L^p(P_\Delta,\mathbb{R}^n)}$.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a (PS) sequence for φ, i.e.

- $\varphi(x_n) \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$,
- $\lim_{n\to\infty} \varphi'(x_n) = 0$ in $\left(\widetilde{W}_0^{1,p}\right)^*$.

We have by (4.12)

$$\sqrt[p]{pM} \geq \sqrt[p]{p\varphi(x_n)} \geq \left(1 - 2^{\frac{p-1}{p}} ||a||_{L^p(P_\Delta,\mathbb{R}^n)}\right) ||x||_{\widetilde{W}_0^{1,p}} - c \tag{4.13}$$

for $n \in \mathbb{N}$. This means that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded because

$$1 - 2^{\frac{p-1}{p}} ||a||_{L^p(P_\Delta,\mathbb{R})} > 0$$

Consequently the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent, possibly after choosing a subsequence which we assume to be chosen, in $\widetilde{W}_0^{1,p}$ to some x_0. Note that (4.13) provides coercivity of φ. Observe also that $(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \to 0$ as $x_n \to x_0$.

From this we see that

$$\begin{aligned}
&(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \\
&= \int_0^1 \left(|\beta(t,x_n)|^{p-2}\beta(t,x_n) - |\lambda(t,x_0)|^{p-2}\lambda(t,x_0)\right) \left(\beta(t,x_n) - \lambda(t,x_0)\right) dt \\
&\quad + \sum_{i=1}^{4} \Psi_i(x_n, x_0),
\end{aligned} \tag{4.14}$$

where

$$\beta(t,x_n) = x_n'(t) - y(t) + \int_0^t \Phi(t,\tau,x_n(\tau))\, d\tau,$$
$$\lambda(t,x_0) = x_0'(t) - y(t) + \int_0^t \Phi(t,\tau,x_0(\tau))\, d\tau$$

and

$$\begin{aligned}\Psi_1(x_n, x_0) = \int_0^1 \Bigg(& \left| x_n'(t) - y(t) + \int_0^t \Phi(t, \tau, x_n(\tau))\, d\tau \right|^{p-2} \\ & \cdot \left(x_n'(t) - y(t) + \int_0^t \Phi(t, \tau, x_n(\tau))\, d\tau \right) \\ & \cdot \int_0^t \Phi(t, \tau, x_n(\tau)) - \Phi(t, \tau, x_0(\tau))\, d\tau \Bigg)\, dt,\end{aligned}$$

$$\begin{aligned}\Psi_2(x_n, x_0) = \int_0^1 \Bigg(& \left| x_0'(t) - y(t) + \int_0^t \Phi(t, \tau, x_0(\tau))\, d\tau \right|^{p-2} \\ & \cdot \left(x_0'(t) - y(t) + \int_0^t \Phi(t, \tau, x_0(\tau))\, d\tau \right) \\ & \cdot \int_0^t \Phi(t, \tau, x_n(\tau)) - \Phi(t, \tau, x_0(\tau))\, d\tau \Bigg)\, dt,\end{aligned}$$

$$\begin{aligned}\Psi_3(x_n, x_0) = \int_0^1 \Bigg(& \left| x_n'(t) - y(t) + \int_0^t \Phi(t, \tau, x_n(\tau))\, d\tau \right|^{p-2} \\ & \cdot \left(x_n'(t) - y(t) + \int_0^t \Phi(t, \tau, x_n(\tau))\, d\tau \right) (x_n(t) - x_0(t)) \\ & \cdot \int_0^t \Phi_x(t, \tau, x_n(\tau))\, d\tau \Bigg)\, dt,\end{aligned}$$

$$\begin{aligned}\Psi_4(x_n, x_0) = \int_0^1 & \left| x_0'(t) - y(t) + \int_0^t \Phi(t, \tau, x_0(\tau))\, d\tau \right|^{p-2} \\ & \cdot \left(x_0'(t) - y(t) + \int_0^t \Phi(t, \tau, x_0(\tau))\, d\tau \right) (x_n(t) - x_0(t)) \\ & \cdot \int_0^t \Phi_x(t, \tau, x_0(\tau))\, d\tau \Bigg)\, dt\end{aligned}$$

We will show that $\Psi_i(x_n, x_0) \to 0$ as $n \to \infty$ for $i = 1, ..., 4$. The weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{W}_0^{1,p}$ to x_0 implies the uniform convergence on $[0,1]$ and the strong convergence in L^p of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0.

Let us consider the term $\Psi_1(x_n, x_0)$. From the Lebesgue Dominated Convergence Theorem it follows that

$$\int_0^t \Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau))\, d\tau \to 0$$

as $n \to \infty$ for all $t \in [0,1]$. Moreover

$$\left| \int_0^t \Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau))\, d\tau \right| \le 2 \int_0^t M_1 a(t,\tau) + b(t,\tau)\, d\tau$$

for $M_1 > 0$ such that

$$|x_n(\tau)| \le M_1$$

for $\tau \in [0,1]$ and $n = 0, 1, ...$ The function

$$[0,1] \ni t \mapsto 2 \int_0^t M_1 a(t,\tau) + b(t,\tau)\, d\tau \in \mathbb{R}$$

belongs to L^p and using the Lebesgue Dominated Convergence Theorem we assert that

$$\int_0^{\cdot} \Phi(\cdot,\tau,x_n(\tau)) - \Phi(\cdot,\tau,x_0(\tau))\, d\tau \to 0$$

in L^p as $n \to \infty$. By Hölder's inequality

$$\begin{aligned}
\Psi_1(x_n, x_0) \leq \int_0^1 & \left(\left| x_n'(t) - y(t) + \int_0^t \Phi(t,\tau,x_n(\tau))\, d\tau \right|^{p-1} \right. \\
& \left. \cdot \int_0^t \Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau))\, d\tau \right) \, dt \\
& \leq \left(\int_0^1 \left| x_n'(t) - y(t) + \int_0^t \Phi(t,\tau,x_n(\tau))\, d\tau \right|^{(p-1)q} \, dt \right)^{\frac{1}{q}} \\
& \cdot \left(\int_0^1 \left| \int_0^t \Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau))\, d\tau \right|^p dt \right)^{\frac{1}{p}} \\
& = \left(\int_0^1 \left| x_n'(t) - y(t) + \int_0^t \Phi(t,\tau,x_n(\tau))\, d\tau \right|^p dt \right)^{\frac{1}{q}} \\
& \cdot \left(\int_0^1 \int_0^t (\Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau)))^p d\tau) dt \right)^{\frac{1}{p}} \\
& \leq \left(\left(1 + ||a||_{L^p(P_\Delta,\mathbb{R}^n)} \right) ||x_n||_{\widetilde{W}_0^{1,p}} + ||y||_{L^p} + ||b||_{L^p(P_\Delta,\mathbb{R}^n)} \right)^{\frac{p}{q}} \\
& \cdot \left(\int_0^1 \int_0^t \left((\Phi(t,\tau,x_n(\tau)) - \Phi(t,\tau,x_0(\tau)))^p \, d\tau \right) \, dt \right)^{\frac{1}{p}} \to 0
\end{aligned}$$

as $n \to \infty$. The same reasoning might be applied to prove that $\Psi_2(x_n, x_0)$ tends to 0 as $n \to \infty$.

Let us consider $\Psi_3(x_n, x_0)$. From the Dominated Convergence Theorem it follows that

$$\int_0^{\cdot} \Phi_x(\cdot,\tau,x_0(\tau))(x_n(\tau) - x_0(\tau))\, d\tau \to 0$$

as $n \to \infty$. As in the previous case applying Hölder's inequality and **6C**

$$\begin{aligned}
\Psi_3(x_n, x_0) \leq & \left(\left(1 + ||a||_{L^p(P_\Delta,\mathbb{R}^n)} \right) ||x_n||_{\widetilde{W}_0^{1,p}} + ||y||_{L^p} + ||b||_{L^p(P_\Delta,\mathbb{R}^n)} \right)^{\frac{p}{q}} \\
& \cdot \left(\int_0^1 \left(\int_0^t |\Phi_x(t,\tau,x_0(\tau))(x_n(\tau) - x_0(\tau))|^p \, d\tau \right) \, dt \right)^{\frac{1}{p}} \to 0
\end{aligned}$$

as $n \to \infty$. Convergence of $\Psi_4(x_n, x_0)$ to 0 as $n \to \infty$ follows from the reasoning presented above.

Now, the following well known inequality will be used: For all $p \geq 2$ it holds, see [10], p. 3

$$\left\langle\, |a|^{p-2}a - |b|^{p-2}b \,\middle|\, a-b \,\right\rangle \geq c_p|a-b|^p \tag{4.15}$$

for all $a, b \in \mathbb{R}^n$, $n \in \mathbb{N}$, where $c_p = \frac{2}{p(2^{p-1}-1)}$ and $\langle \cdot \mid \cdot \rangle$ denotes an inner product on $\mathbb{R}^n$.

Using relation (4.15) and substituting a and b with β and λ defined in (4.14) we conclude that

$$\begin{aligned}
&\sqrt[p]{c_p}||x_n - x_0||_{\widetilde{W}_0^{1,p}} \\
&\leq \sqrt[p]{c_p}\left(\int_0^1 |x_n'(t) - x_0'(t) - \int_0^t (\Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau)))d\tau|^p dt\right)^{\frac{1}{p}} \\
&\quad + \sqrt[p]{c_p}\left(\int_0^1\int_0^t |\Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau))|^p d\tau dt\right)^{\frac{1}{p}},
\end{aligned}$$

where c_p is given above. Considering equality (4.14) and taking the above relation into account

$$\begin{aligned}
&(\varphi'(x_n) - \varphi'(x_0)(x_n - x_0) \\
&\quad\geq c_p \int_0^1 \left| x_n'(t) - x_0'(t) - \int_0^t (\Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau)))d\tau \right|^p dt \\
&\qquad + \sum_{i=1}^4 \Psi_i(x_n, x_0)
\end{aligned}$$

so

$$\begin{aligned}
&c_p \int_0^1 \left| x_n'(t) - x_0'(t) - \int_0^t \Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau))\, d\tau \right|^p dt \\
&\qquad\leq |(\varphi'(x_n) - \varphi'(x_0)(x_n - x_0)| + \left|\sum_{i=1}^4 \Psi_i(x_n, x_0)\right|.
\end{aligned}$$

On the other hand one can observe that

$$|\varphi'(x_n)(x_n - x_0)| \leq ||\varphi'(x_n)||_{\left(\widetilde{W}_0^{1,p}\right)^*} ||x_n - x_0||_{\widetilde{W}_0^{1,p}}$$

Therefore

$$\varphi'(x_n)(x_n - x_0) \to 0$$

as $n \to \infty$ due to the fact that $(x_n)_{n\in\mathbb{N}}$ is bounded and $\varphi'(x_n) \to 0$ as $n \to \infty$. From the weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0 in $\widetilde{W}_0^{1,p}$ it follows that

$$\varphi'(x_0)(x_n - x_0) \to 0$$

as $n \to \infty$. So

$$(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \to 0.$$

as $n \to \infty$. Thus

$$\int_0^1 \left| x_n'(t) - x_0'(t) - \int_0^t \Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau))\, d\tau \right|^p \, dt \to 0$$

as $n \to \infty$. Of course

$$\int_0^1 \left(\int_0^t |\Phi(t,\tau,x_0(\tau)) - \Phi(t,\tau,x_n(\tau))|^p \, d\tau \right) \, dt \to 0$$

as $n \to \infty$. Consequently, $||x_n - x_0||_{\widetilde{W}_0^{1,p}} \to 0$ as $n \to \infty$, i.e. the function φ satisfies the (PS) condition. ■

Now we can proceed with the proof of the main result.

Proof of Theorem 4.1. We will use Theorem 3.5 with $E = \widetilde{W}_0^{1,p}$ and $B = L^p(0,1)$. Condition **2B** follows from Lemma 6.3.

Based on the Closed Graph Theorem one can conclude that the assumption (b2) from Theorem 3.5 is equivalent to the following one: for any $x \in E$, the differential $f'(x) : E \to B$ is "onto" and "one to one". Thus it is sufficient to use Lemma 4.1 to get the assertion. ■

Remark 4.1. Applying the similar reasoning as in [48] the assumption **6B** of Theorem 4.1 might be reduced to a slightly weaker form:

F2′ there exist functions $a, b \in L^p(P_\Delta, \mathbb{R}_0^+)$ such that

$$|\Phi(t,\tau,x)| \leq a(t,\tau)|x| + b(t,\tau)$$

for a.e. $(t,\tau) \in P_\Delta$, all $x \in \mathbb{R}^n$ and there exist a constant $\overline{a} > 0$ such that

$$\int_0^t a^p(t,\tau)d\tau \leq \overline{a}^p$$

for a.e. $t \in [0,1]$.

For any $k > 0$ let us define another form of the Bielecki's type norm

$$||x||_{\widetilde{W}_0^{1,p},k} = \left(\int_0^1 e^{-kt}|x'(t)|^p\, dt\right)^{\frac{1}{p}}.$$

For $k = 0$ the above function defines a norm introduced by (2.1) and therefore further we will skip index 0. It is easy to notice that:

$$e^{\frac{-k}{p}}||x||_{\widetilde{W}_0^{1,p}} \leq ||x||_{\widetilde{W}_0^{1,p},k} \leq ||x||_{\widetilde{W}_0^{1,p}}.$$

For any $k > 0$ and $x \in \widetilde{W}_0^{1,p}$ we state the following relations:

$$||x||_k \leq \frac{||x||_{\widetilde{W}_0^{1,p},k}}{k^{\frac{1}{p}}} \tag{4.16}$$

and

$$\left\|\int_0^{\cdot} |x(\tau)|\, d\tau\right\|_k = \left(\int_0^1 e^{-kt}\left(\int_0^t |x(\tau)|\, d\tau\right)^p dt\right)^{\frac{1}{p}} \leq \frac{||x||_{\widetilde{W}_0^{1,p},k}}{k^{\frac{2}{p}}}. \tag{4.17}$$

Now let us prove the stated relations starting with (4.16). Fix $k > 0$ and $x \in \widetilde{W}_0^{1,p}$. Then

$$\begin{aligned}
||x||_k^p &= \int_0^1 e^{-kt}|x(t)|^p dt = \int_0^1 e^{-kt}\left|\int_0^t x'(\tau)d\tau\right|^p dt \\
&\leq \int_0^1 \left(e^{-kt}\int_0^t \left|x'(\tau)\right|^p\, d\tau\right)\, dt \\
&= \int_0^1 \left|x'(\tau)\right|^p \left(\int_\tau^1 e^{-kt}\, dt\right)\, d\tau \\
&= \frac{1}{k}\int_0^1 e^{-kt}|x'(t)|^p\, dt - \frac{e^{-k}}{k}\int_0^1 |x'(t)|^p\, dt \\
&\leq \frac{1-e^{-k}}{k}\int_0^1 e^{-kt}|x'(t)|^p\, dt \leq \frac{||x||^p_{\widetilde{W}_0^{1,p},k}}{k}.
\end{aligned}$$

Now let us turn to the relation (4.17).

$$\begin{aligned}
\left\|\int_0^{\cdot} |x(\tau)|d\tau\right\|_k^p &= \int_0^1 e^{-kt}\left(\int_0^t |x(\tau)|d\tau\right)^p dt \\
&\leq \int_0^1 e^{-kt}\left(\int_0^t |x(\tau)|^p d\tau\right) dt \\
&= \int_0^1 |x(\tau)|^p \left(\int_\tau^1 e^{-kt}dt\right) d\tau \\
&= \frac{1}{k}\int_0^1 e^{-kt}|x(t)|^p dt - \frac{e^{-k}}{k}\int_0^1 |x(t)|^p dt \\
&\leq \frac{||x||_k^p}{k} \leq \frac{||x||^p_{\widetilde{W}_0^{1,p},k}}{k^2}.
\end{aligned}$$

Having in mind the relation (4.10), which states that L^p norm $||\cdot||_{L^p}$ and Bielecki norm $||\cdot||_k$ are equivalent we can redefine functional $\varphi : \widetilde{W}_0^{1,p} \to \mathbb{R}$ in the form:

$$\begin{aligned}
\varphi(x) &= \frac{1}{p}\,\|f(x)-y\|_k^p \\
&= \frac{1}{p}\int_0^1 e^{-kt}\left|x'(t)-y(t)+\int_0^t \Phi(t,\tau,x(\tau))d\tau\right|^p dt.
\end{aligned}$$

Therefore the following inequality might be deduced for any $x \in \widetilde{W}_0^{1,p}$

$$
\begin{aligned}
(p\varphi(x))^{\frac{1}{p}} &= \left\| x'(\cdot) - y(\cdot) + \int_0^{\cdot} \Phi(\cdot,\tau,x(\tau))d\tau \right\|_k \\
&\geq ||x'||_k - ||y||_k - \left\| \int_0^{\cdot} \Phi(\cdot,\tau,x(\tau))d\tau \right\|_k \\
&\geq ||x'||_k - ||y||_k - \overline{a} \left\| \int_0^{\cdot} x(\tau)d\tau \right\|_k - \left\| \int_0^{\cdot} b(\cdot,\tau)d\tau \right\|_k \\
&\geq ||x||_{\widetilde{W}_0^{1,p},k} - \frac{\overline{a}}{k^{\frac{2}{p}}} ||x||_{\widetilde{W}_0^{1,p},k} + d,
\end{aligned}
$$

where $d = ||y||_k - \left\| \int_0^{\cdot} b(\cdot,\tau)d\tau \right\|_k$. For sufficiently large $k > 0$, namely for $k > \max\left\{1, \overline{a}^{\frac{p}{2}}\right\}$ we have the coercivity of functional φ.

4.1.1 Example

We finish this section with an example of a nonlinear term satisfying our assumptions **6A** - **6C**. Let us consider the function

$$\Phi : P_\Delta \times \mathbb{R} \to \mathbb{R}$$

given by

$$\Phi(t,\tau,x) = 2^{1-p}(t-\tau)^{\frac{5}{2}} \ln\left(1 + (t-\tau)^2 x^2\right)$$

for $t,\tau \in [0,1]$, $t > \tau$, $x \in \mathbb{R}$. Since

$$\ln\left(1+s^2z^2\right) \leq \ln\left(\left(1+s^2\right)\left(1+z^2\right)\right) = \ln\left(1+s^2\right) + \ln\left(1+z^2\right) \leq |s| + |z|$$

for $s,z \in \mathbb{R}$, therefore

$$|\Phi(t,\tau,x)| \leq 2^{1-p}(t-\tau)^{\frac{5}{2}}|x| + 2^{1-p}(t-\tau)^{\frac{5}{2}}$$

Let us put

$$a(t,\tau) = 2^{\frac{1-p}{2p}}(t-\tau)^{\frac{5}{2}}$$

for $t,\tau \in [0,1]$, $t > \tau$.

$$\begin{aligned} ||a||^p_{L^p(P_\Delta,\mathbb{R})} &= \int_0^1 \left(\int_0^t \left(2^{1-p} (t-\tau)^{\frac{5}{2}} \right)^p \, d\tau \right) \, dt \\ &= 2^{p(1-p)} \frac{4}{(5p+2)(5p+4)} \leq 2^{p(1-p)}. \end{aligned}$$

Consequently, $||a||_{L^p(P_\Delta,\mathbb{R})} \leq 2^{-(p-1)} < 2^{\frac{-(p-1)}{p}}$. Moreover

$$|\Phi_x(t,\tau,x)| \leq 2^{1-p}(t-\tau)^{\frac{5}{2}}|x|$$

and

$$\int_0^t c(t,\tau)^q \, d\tau = 2^{-p} \int_0^t (t-\tau)^{\frac{5q}{2}} \, d\tau = \frac{2^{1-p}}{5q+2} t^{\frac{5q}{2}+1} \leq \frac{2^{1-p}}{5q+2}, t \in [0,1].$$

Hence, Φ satisfies assumptions **6A** - **6C**. Theorem 4.1 shows that the initial value problem

$$x'(t) + \int_0^t 2^{1-p}(t-\tau)^{\frac{1}{2}} \ln\left(1 + (t-\tau)^2 x^2\right) \, d\tau = y(t), \quad t \in [0,1] \; a.e.$$

with

$$x(0) = 0$$

has a unique solution $x_y \in \widetilde{W}^{1,p}_0$ for any fixed $y \in L^p$. Moreover, the solution mapping

$$L^p \ni y \mapsto x_y \in \widetilde{W}^{1,p}_0$$

is continuously differentiable.

CHAPTER 5

Application of a global implicit function theorem

The results of this Chapter are based on [26]. Let us consider the following integro-differential equation

$$x'(t) + \int_0^t \Phi(t, \tau, x(\tau), z(\tau))d\tau = y(t), \text{for } a.e.\ t \in [0,1],\ x(0) = 0, \quad (5.1)$$

where $y \in L^p$, $z \in L^p([0,1], \mathbb{R}^m)$, $P_\Delta = \{(t, \tau) \in [0,1] \times [0,1]; \tau \leq t\}$ and $p \geq 2$. The solutions belong to a $\widetilde{W}_0^{1,p}$. On the function

$$\Phi : P_\Delta \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

we assume that

7A $\Phi(\cdot, \cdot, x, z)$ is measurable on P_Δ for any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$; $\Phi(t, \tau, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ for a.e. $(t, \tau) \in P_\Delta$

7B there exist functions $a \in L^p(P_\Delta, \mathbb{R}_0^+)$, $b \in L^\infty(P_\Delta, \mathbb{R}_0^+)$ such that

$$|\Phi(t, \tau, x, z)| \leq a(t, \tau)|x| + b(t, \tau)|z|;$$

for a.e. $(t, \tau) \in P_\Delta$, all $x \in \mathbb{R}^n$, all $z \in \mathbb{R}^m$ and also

$$||a||_{L^p(P_\Delta, \mathbb{R})} < 2^{\frac{1-p}{p}}$$

7C there exist functions $c, e \in L^p(P_\Delta, \mathbb{R}_0^+)$, $\alpha, \beta \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and constants $C, k, p > 0$ such that

$$|\Phi_x(t,\tau,x,z)| \leq c(t,\tau)\alpha(|x|) + k|z|$$

$$|\Phi_z(t,\tau,x,z)| \leq e(t,\tau)\beta(|x|) + p|z|$$

for a.e. $(t,\tau) \in P_\Delta$, all $x \in \mathbb{R}^n$, all $z \in \mathbb{R}^m$ and

$$\int_0^t c^q(t,\tau)d\tau \leq C, \text{ for } a.e.\ t \in [0,1].$$

In what follows we shall assume that **7A**-**7C** hold.

We should show that the mapping

$$F : \widetilde{W}_0^{1,p} \times L^p \times L^p([0,1], \mathbb{R}^m) \to L^p$$

given by the formula

$$F(x,y,z) = x'(t) + \int_0^t \Phi(t,\tau,x(\tau),z(\tau))d\tau - y(t)$$

satisfies assumption of the global implicit function theorem 3.7 with

$$X = \widetilde{W}_0^{1,p},\ Y = L^p \times L^p([0,1], \mathbb{R}^m) \text{ and } Z = L^p.$$

We mentioned already that space L^p is a uniformly convex Banach space and so is its dual. Hence we can use the first version of global implicit function theorem. By a direct calculation we get the following (see [70] for the excellent background on calculation of derivatives in Banach spaces)

Lemma 5.1. *Assume that **7A**-**7C** hold. Then mapping F is of class C^1 and its differentials in x and in (y,z), respectively, read as follows*

$$F_x(x,y,z) : \widetilde{W}_0^{1,p} \to L^p,$$

$$F_x(x,y,z)h = h'(t) + \int_0^t \Phi_x(t,\tau,x(\tau),z(\tau))h(\tau)d\tau,$$

$$F_{(y,z)}(x,y,z) : L^p \times L^p([0,1],\mathbb{R}^m) \to L^p,$$

$$F_{(y,z)}(x,y,z)(u,v) = \int_0^t \Phi_z(t,\tau,x(\tau),z(\tau))v(\tau)d\tau - u(t).$$

Lemma 5.2. *Assume that* ***7A-7C*** *hold. Fix functions $y \in L^p$ and $z \in L^p([0,1],\mathbb{R}^m)$. Define the mapping $\overline{\Phi} : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ by formula $\overline{\Phi}(t,\tau,x) = \Phi(t,\tau,x,z(t))$. Then the functional $\varphi : \widetilde{W}_0^{1,p} \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \frac{1}{p}\|F(x,y)\|^p = \frac{1}{p}\int_0^1 \left| x'(t) + \int_0^t \overline{\Phi}(t,\tau,x(\tau))d\tau - y(t) \right|^p dt$$

satisfies the (PS) condition.

Proof. Observe φ as a composition of two C^1 mappings is in fact C^1. By Hölder's Inequality and classical embedding results we have for any $x \in \widetilde{W}_0^{1,p}$

$$\begin{aligned}
\left\| \int_0^t \overline{\Phi}(t,\tau,x(\tau))d\tau \right\|_{L^p}^p &\le \int_0^1 \left(\int_0^t |\overline{\Phi}(t,\tau,x(\tau))|^p d\tau \right) dt \\
&\le \int_0^1 \left(\int_0^t \left(a(t,\tau)|x(\tau)| + \widetilde{b}(t,\tau) \right)^p d\tau \right) dt \\
&\le 2^{p-1}||a||_{L^p(P_\Delta,\mathbb{R}^n)}^p ||x||_{\widetilde{W}_0^{1,p}}^p + 2^{p-1}||\widetilde{b}||_{L^p(P_\Delta,\mathbb{R})}^p.
\end{aligned}$$

with $\widetilde{b}(t,\tau) = b(t,\tau)|z(\tau)|$. Let

$$c = ||y||_{L^p} + 2^{\frac{p-1}{p}}||\widetilde{b}||_{L^p(P_\Delta,\mathbb{R}^n)}.$$

We see that for any $x \in \widetilde{W}_0^{1,p}$

$$\begin{aligned}
(p\varphi(x))^{\frac{1}{p}} &= \left\| x'(\cdot) - y(\cdot) + \int_0^{\cdot} \overline{\Phi}(\cdot,\tau,x(\tau))\, d\tau \right\|_{L^p} \\
&\ge ||x'||_{L^p} - ||y||_{L^p} - \left\| \int_0^{\cdot} \overline{\Phi}(\cdot,\tau,x(\tau))\, d\tau \right\|_{L^p} \qquad (5.2) \\
&\ge \left(1 - 2^{\frac{(p-1)}{p}}||a||_{L^p(P_\Delta,\mathbb{R}^n)} \right) ||x||_{\widetilde{W}_0^{1,p}} - c
\end{aligned}$$

Note that $1 - 2^{\frac{(p-1)}{p}} ||a||_{L^p(P_\Delta,\mathbb{R})} > 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a (PS) sequence for φ, so $|\varphi(x_n)| \le M$ for some $M > 0$. We have by (5.2)

$$\sqrt[p]{pM} \ge \sqrt[p]{p\varphi(x_n)} \ge \left(1 - 2^{\frac{(p-1)}{p}} ||a||_{L^p(P_\Delta,\mathbb{R}^n)}\right) ||x_n||_{\widetilde{W}_0^{1,p}} - c$$

for $n \in \mathbb{N}$ which means that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, so it may be assumed to be weakly convergent in $\widetilde{W}_0^{1,p}$ to some x_0.

Let us calculate

$$\begin{aligned}
&(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0)\\
&\quad = \int_0^1 (|\beta(t,x_n)|^{p-2}\beta(t,x_n) - |\lambda(t,x_0)|^{p-2}\lambda(t,x_0))(\beta(t,x_n) - \lambda(t,x_0))\,dt\\
&\qquad + \sum_{i=1}^{4} \Psi_i(x_n,x_0),
\end{aligned}$$

where

$$\begin{aligned}
\beta(t,x_n) &= x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_n(\tau))\,d\tau\\
\lambda(t,x_0) &= x_0'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_0(\tau))\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
\Psi_1(x_n,x_0) = \int_0^1 \Bigg(&\left| x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_n(\tau))\,d\tau \right|^{p-2}\\
&\cdot \left(x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_n(\tau))\,d\tau \right)\\
&\cdot \int_0^t \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\,d\tau \Bigg)\,dt,
\end{aligned}$$

$$\begin{aligned}
\Psi_2(x_n,x_0) = \int_0^1 \Bigg(&\left| x_0'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_0(\tau))d\tau \right|^{p-2}\\
&\cdot \left(x_0'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_0(\tau))\,d\tau \right)\\
&\cdot \int_0^t \overline{\Phi}(t,\tau,x_n(\tau)) - \overline{\Phi}(t,\tau,x_0(\tau))\,d\tau \Bigg)\,dt,
\end{aligned}$$

$$\Psi_3(x_n, x_0) = \int_0^1 \left(\left| x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t, \tau, x_n(\tau))\, d\tau \right|^{p-2} \right.$$
$$\cdot \left(x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t, \tau, x_n(\tau))\, d\tau \right)$$
$$\left. \cdot (x_n(t) - x_0(t)) \int_0^t \overline{\Phi}_x(t, \tau, x_n(\tau))\, d\tau \right) dt,$$

$$\Psi_4(x_n, x_0) = \int_0^1 \left(\left| x_0'(t) - y(t) + \int_0^t \overline{\Phi}(t, \tau, x_0(\tau))\, d\tau \right|^{p-2} \right.$$
$$\cdot \left(x_0'(t) - y(t) + \int_0^t \overline{\Phi}(t, \tau, x_0(\tau))\, d\tau \right)$$
$$\left. \cdot\ (x_0(t) - x_n(t)) \int_0^t \overline{\Phi}_x(t, \tau, x_0(\tau))\, d\tau \right) dt$$

We will show that $\Psi_i(x_n, x_0) \to 0$ as $n \to \infty$ for $i = 1, .., 4$. The weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{W}_0^{1,p}$ to x_0 implies the uniform convergence on $[0, 1]$ and the strong convergence in L^p of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0.

Let us consider the term $\Psi_1(x_n, x_0)$. From the Lebesgue Dominated Convergence Theorem it follows that

$$\int_0^t \overline{\Phi}(t, \tau, x_n(\tau)) - \overline{\Phi}(t, \tau, x_n(\tau))\, d\tau \to 0$$

as $n \to \infty$ for $t \in [0, 1]$. Moreover

$$\left| \int_0^t \overline{\Phi}(t, \tau, x_n(\tau)) - \overline{\Phi}(t, \tau, x_0(\tau))\, d\tau \right| \le 2 \int_0^t \left(M_1 a(t, \tau) + \widetilde{b}(t, \tau) \right) d\tau$$

for $M_1 > 0$ such that

$$|x_n(\tau))| \le M_1, \quad \text{for } \tau \in [0, 1] \text{ and } n = 0, 1, \ldots .$$

Since the function

$$[0,1] \ni t \to 2\int_0^t \left(a(t,\tau)M_1 + \widetilde{b}(t,\tau)\right)\, d\tau \in \mathbb{R}$$

belongs to L^p and using the Lebesgue Dominated Convergence Theorem we assert that

$$\int_0^{\cdot} \overline{\Phi}(\cdot,\tau,x_n(\tau)) - \overline{\Phi}(\cdot,\tau,x_0(\tau))\, d\tau \to 0$$

as $n \to \infty$ in L^p. By Hölder's inequality

$$\Psi_1(x_n,x_0) \le \left(\int_0^1 \left| x_n'(t) - y(t) + \int_0^t \overline{\Phi}(t,\tau,x_n(\tau))\, d\tau\right|^{(p-1)q} dt\right)^{\frac{1}{q}}$$

$$\cdot \left(\int_0^1 \left(\int_0^t \overline{\Phi}(t,\tau,x_n(\tau)) - \overline{\Phi}(t,\tau,x_0(\tau))\, d\tau\right)^p dt\right)^{\frac{1}{p}}$$

$$\le \left(\left(1 + 2^{\frac{1}{q}}||a||_{L^p(P_\Delta,\mathbb{R}^n)}\right)||x_n||_{\widetilde{W}_0^{1,p}} + ||y||_{L^p} + 2^{\frac{1}{q}}||\widetilde{b}||_{L^p(P_\Delta,\mathbb{R}^n)}\right)^{\frac{p}{q}}$$

$$\cdot \left(\int_0^1 \left(\int_0^t \left(\overline{\Phi}(t,\tau,x_n(\tau)) - \overline{\Phi}(t,\tau,x_0(\tau)\right)^p\, d\tau\right)\, dt\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$. The same reasoning might be applied to prove that $\Psi_2(x_n,x_0)$ tends to 0 as $n \to \infty$.

Let us consider $\Psi_3(x_n,x_0)$. Firstly, we see that for any $t \in [0,1]$ the following estimation holds

$$\left|\int_0^t \overline{\Phi}_x(t,\tau,x_0(\tau)\, d\tau\, (x_n(t) - x_0(t))\right| \le \int_0^1 \left|\overline{\Phi}_x(t,\tau,x_0(\tau))\right|\, d\tau\, \|x_n - x_0\|_C.$$

Since sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly convergent in C to x_0, we see that

$$\int_0^{\cdot} \overline{\Phi}_x(\cdot,\tau,x_0(\tau))\, d\tau(x_n(\cdot) - x_0(\cdot)) \to 0$$

as $n \to \infty$ on $[0,1]$. As in the previous case we see that

$$\Psi_3(x_n,x_0) \le \left(\left(1 + ||a||_{L^p(P_\Delta,\mathbb{R}^n)}\right)||x_n||_{\widetilde{W}_0^{1,p}} + ||y||_{L^p} + ||\widetilde{b}||_{L^p(P_\Delta,\mathbb{R}^n)}\right)^{\frac{p}{q}}$$

$$\cdot \left(\int_0^1 \left(\int_0^t \left|\overline{\Phi}_x(t,\tau,x_0(\tau))(x_n(\tau) - x_0(\tau))\right|^p\, d\tau\right)\, dt\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$. Convergence of $\Psi_4(x_n, x_0)$ to 0 as $n \to \infty$ follows from the reasoning presented above.

Using relation (4.15) and the Lebesgue Dominated Convergence Theorem we conclude that

$$\begin{aligned}
&\sqrt[p]{c_p}||x_n - x_0||_{\widetilde{W}_0^{1,p}} \\
&\quad \leq \sqrt[p]{c_p}\left(\int_0^1 \left|x_n'(t) - x_0'(t) - \int_0^t \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\, d\tau\right|^p \, dt\right)^{\frac{1}{p}} \\
&\qquad + \sqrt[p]{c_p}\left(\int_0^1 \left(\int_0^t \left|\overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\right|^p \, d\tau\right) \, dt\right)^{\frac{1}{p}}.
\end{aligned}$$

From the formula for a derivative and from (4.15) we have

$$\begin{aligned}
&(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \\
&\quad \geq c_p \int_0^1 \left|x_n'(t) - x_0'(t) - \int_0^t \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\, d\tau\right|^p \, dt \\
&\qquad + \sum_{i=1}^4 \Psi(x_n, x_0).
\end{aligned}$$

So,

$$\begin{aligned}
&c_p \int_0^1 \left|x_n'(t) - x_0'(t) - \int_0^t \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\, d\tau\right|^p \, dt \\
&\qquad \leq \left|(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0)\right| + \left|\sum_{i=1}^4 \Psi(x_n, x_0)\right|.
\end{aligned}$$

On the other hand one can observe that

$$|\varphi'(x_n)(x_n - x_0)| \leq ||\varphi'(x_n)||_{\left(\widetilde{W}_0^{1,p}\right)^*}||x_n - x_0||_{\widetilde{W}_0^{1,p}}$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded and $\varphi'(x_n) \to 0$ as $n \to \infty$ we see that

$$\varphi'(x_n)(x_n - x_0) \to 0$$

as $n \to \infty$. From the weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0 in $\widetilde{W}_0^{1,p}$ it follows that

$$\varphi'(x_0)(x_n - x_0) \to 0$$

as $n \to \infty$. So

$$(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \to 0$$

as $n \to \infty$. Thus

$$\int_0^1 \left| x_n'(t) - x_0'(t) - \int_0^t \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau))\, d\tau \right|^p dt \to 0$$

as $n \to \infty$. Of course,

$$\int_0^1 \left(\int_0^t \left| \overline{\Phi}(t,\tau,x_0(\tau)) - \overline{\Phi}(t,\tau,x_n(\tau)) \right|^p d\tau \right) dt \to 0$$

as $n \to \infty$. Consequently,

$$||x_n - x_0||_{\widetilde{W}_0^{1,p}} \to 0$$

as $n \to \infty$. ■

We recall that the Bielecki norm in L^p for the arbitrary $k > 0$ is given by

$$||u||_k = \left(\int_0^1 e^{-kt} |u(t)|^p\, dt \right)^{\frac{1}{p}}.$$

Let us observe that for any $u \in L^p$ the following relation holds

$$e^{-\frac{k}{p}} ||u||_{L^p} \le ||u||_k \le ||u||_{L^p}$$

so the Bielecki and L^p norms are equivalent.

Lemma 5.3. *Assume that* **7A-7C** *hold. Fix functions $y \in L^p$ and $z \in L^p([0,1], \mathbb{R}^m)$. Then for any admissible $h \in \widetilde{W}_0^{1,p}$ we see that*

$$h'(t) + \int_0^t \Phi_x(t, \tau, x(\tau), z(\tau))h(\tau)\, d\tau = h'(t) + \int_0^t \overline{\Phi}_x(t, \tau, x(\tau))h(\tau)\, d\tau$$

and $F_x(x, y, z)$ is bijective for any $x \in \widetilde{W}_0^{1,p}$.

Proof. We must prove that for any fixed $v \in L^p$ the following linear integro-differential equation

$$h'(t) + \int_0^t \overline{\Phi}_x(t, \tau, x(\tau))h(\tau)\, d\tau = v(t), \quad \text{for a.e. } t \in [0,1] \tag{5.3}$$

has a unique solution $x \in \widetilde{W}_0^{1,p}$. We consider an auxiliary equation

$$h'(t) + u(t) = v(t), \quad \text{for a.e. } t \in [0,1],$$

where $u \in L^p$ and which is uniquely solvable by $h_u \in \widetilde{W}_0^{1,p}$ given by

$$h_u(t) = \int_0^t (-u(s) + v(s))\, ds, \quad \text{for } t \in [0,1].$$

Now, consider the mapping

$$\Gamma : L^p \ni u(\cdot) \mapsto \int_0^{\cdot} \overline{\Phi}_x(\cdot, \tau, x(\tau))h_u(\tau)\, d\tau \in L^p.$$

In order to show Γ is the contraction mapping and thus it has the unique fixed point, we make use of the Bielecki norm. Let

$$d(t, \tau) = (c(t, \tau) + k|z(\tau)|) \max\{\alpha(|x(\tau)|) + 1, \tau \in [0,1]\}$$

and let $B > 0$ be such that (see assumption **7C**).

$$\int_0^t d^q(t, \tau)\, d\tau \leq B, \quad \text{for a.e. } t \in [0,1].$$

For any $u_1, u_2 \in L^p$ we have using suitable change of variables for double integral applied to $\int_0^1 \int_0^t e^{-kt}|u_1(\tau) - u_2(\tau)|^p \, d\tau dt$ that

$$
\begin{aligned}
&\|\Gamma u_1 - \Gamma u_2\|_k^p \\
&\quad \leq \int_0^1 \left(e^{-kt} \left(\int_0^t |\overline{\Phi}_x(t,\tau,x(\tau))|^q \, d\tau \right)^{\frac{p}{q}} \int_0^t \int_0^\tau |u_1(s) - u_2(s)|^p \, ds d\tau \right) dt \\
&\quad \leq \int_0^1 \left(e^{-kt} \left(\int_0^t |\overline{\Phi}_x(t,\tau,x(\tau))|^q \, d\tau \right)^{\frac{p}{q}} \int_0^t |u_1(\tau) - u_2(\tau)|^p \, d\tau \right) dt \\
&\quad \leq B^{\frac{p}{q}} \int_0^1 \left(\int_0^t e^{-kt}|u_1(\tau) - u_2(\tau)|^p \, d\tau \right) dt \\
&\quad \leq \frac{B^{\frac{p}{q}}}{k} \int_0^1 e^{-kt}|u_1(t) - u_2(t)|^p \, dt - e^{-k} \int_0^1 |u_1(\tau) - u_2(\tau)|^p \, d\tau \\
&\quad \leq \frac{B^{\frac{p}{q}}}{k} ||u_1 - u_2||_k^p
\end{aligned}
$$

For sufficiently large k we see that $\frac{B^{\frac{p}{q}}}{k} \in (0,1)$, hence the mapping Γ is a contraction with respect to the Bielecki norm. Thus it has a fixed point which solves uniquely (5.3). ■

The above Lemmas show that all assumptions of Theorem 3.7 are satisfied. Thus we can formulate the following

Theorem 5.1. *Assume that **7A-7C** hold. Fix functions $y \in L^p$ and $z \in L^p([0,1], \mathbb{R}^m)$. Then there exists a unique solution $x_{y,z} \in \widetilde{W}_0^{1,p}$ of equation (5.1) and a C^1 mapping*

$$f : L^p \times L^p([0,1], \mathbb{R}^m) \ni (y,z) \mapsto x_{y,z} \in \widetilde{W}_0^{1,p}$$

with the differential $f'(y,z)$ at point $(y,z) \in L^p \times L^p([0,1], \mathbb{R}^m)$

$$L^p \times L^p([0,1], \mathbb{R}^m) \ni (u,v) \mapsto g_{u,v} \in \widetilde{W}_0^{1,p},$$

where $g_{u,v}$ is such that

$$
\begin{aligned}
g'_{u,v}(t) &+ \int_0^t \Phi_x(t,\tau,x_{y,z}(\tau),z(\tau))g_{u,v}\,d\tau \\
&= -\int_0^t \Phi_y(t,\tau,x_{y,z}(\tau))v(\tau)\,d\tau + u(\tau)
\end{aligned}
$$

for a.e. t in $[0,1]$.

Now we provide an example of a function satisfying conditions **7A-7C**. Let $\Phi : P_\Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$\Phi(t,\tau,x,z) = 2^{1-p}(t-\tau)^{\frac{1}{2}}\ln(1+x^2) + \tau^{\frac{1}{3}}t^4(\sin^2 x)\ln(1+z^2)$$

It might be noticed that

$$\Phi(t,\tau,x,z) \le 2^{1-p}(t-\tau)^{\frac{1}{2}}|x| + \tau^{\frac{1}{3}}t^4|z|$$

Let us put

$$a(t,\tau) = 2^{1-p}(t-\tau)^{\frac{1}{2}}$$

Consequently

$$
\begin{aligned}
||a||_{L^p}^p &= \int_0^1 \left(\int_0^t (2^{1-p}(t-\tau)^{\frac{1}{2}})^p\,d\tau\right)\,dt \\
&= 2^{p(1-p)}\frac{4}{(p+2)(p+4)} \le 2^{p(1-p)} \le 2^{(1-p)}
\end{aligned}
$$

for $t,\tau \in [0,1]$, $t > \tau$, $p \ge 2$. Moreover

$$
\begin{aligned}
&|\Phi_x(t,\tau,x,z)| \le 2^{2-p}(t-\tau)^{\frac{1}{2}}|x| + \tau^{\frac{1}{3}}t^4|z|, \\
&|\Phi_z(t,\tau,x,z)| \le 2\tau^{\frac{1}{3}}t^4|z|
\end{aligned}
$$

and

$$\int_0^t c(t,\tau)^q\,d\tau = 2^{(2-p)q}\int_0^t (t-\tau)^{\frac{q}{2}}\,d\tau = \frac{2^{(2-p)q+1}}{q+2}t^{\frac{q}{2}+1} \le \frac{2^{(2-p)q+1}}{q+2}$$

for $t \in [0,1]$. Hence conditions **7A-7C** are satisfied.

CHAPTER 6

An application of Diffeomorphism Theorem to Volterra integral operator

SECTION 6.1
Introduction

We will denote space $\widetilde{W}_0^{1,2}$ by $\widetilde{H}_0^1$ and recall that it consists of absolutely continuous functions $x\colon [0,1] \to \mathbb{R}^n$ that $x(0) = 0$, $\dot{x} \in L^2([0,1], \mathbb{R}^n)$. We define

$$V(x)(t) = x(t) + \int_0^t v(t, \tau, x(\tau))\, d\tau. \tag{6.1}$$

In this chapter we shall investigate the nonlinear integral operator $V\colon \widetilde{H}_0^1 \to \widetilde{H}_0^1$ defined pointwisely for all $t \in [0,1]$ by (6.1). Thus V is considered with an initial condition

$$x(0) = 0. \tag{6.2}$$

We focus on showing that V is a diffeomorphism under some conditions imposed on the nonlinear term v. This in turn ensures that the associated Volterra integral equation

$$\begin{cases} x(t) + \int_0^t v(t, \tau, x(\tau))\, d\tau = y\,(t) \text{ for } t \in [0, 1], \\ x(0) = 0 \end{cases} \tag{6.3}$$

is solvable for any $y \in \widetilde{H}_0^1$ and that the solution operator which assigns to each y the unique solution to (6.3) is of class C^1. In other words, we can say that solution to (6.3) depends in a C^1 manner on a functional parameter y. The proof relies on a global diffeomorphism theorem. We are inspired by [7] which contains similar approach in Hilbert spaces. However it is not shown there that the mapping is C^1. The Authors require only differentiability which is not sufficient in order to obtain a diffeomorphism. We fill this gap. Our calculations are based on the work mentioned however, we use the scheme which we developed already.

In this chapter we investigate the solvability of Volterra equations by variational methods, since the main theorem on which we base our investigation is proved with the mountain pass geometry. This again is not very common since for the unique solvability of Volterra equations researchers used to apply a fixed point approach based on the Banach fixed point theorem, or else the successive approximations, the Schauder and Schauder-Tikhonov Theorem together with some other tools, see for example [5], [9], [54], [67]. We also use as a technical tool the method of successive approximations for auxiliary linear problem. Integro-differential and integral operators are usually considered in the space of continuous functions [14, 51], the space of square integrable functions L^2 [43]. The application of numerical methods such as Wavelet-Galerkin Method (WGM), Lagrange interpolation method, Tau method, Adomian's decomposition method and Taylor polynomials [4, 64, 34, 19, 49], for solving the nonlinear integro-differential equations is rather common as they are hard to solve analytically and exact solutions are scarce. The application of integral operators can be found in many discipline of science and engineering: in biology to investigate the spread of epidemic [29], in mechanics for modelling alloys with a shape memory [11, 60], in nuclear reactor dynamics [14], [15].

The chapter is organized as follows. We formulate assumptions and main results pertaining to the properties of the Volterra operator V defined above and the associated initial value problem. For the proof of main results, we investigate the associated Volterra equation. We use methods typical in L^2 setting in the case of equations. Then, we prove the main result which requires construction of a suitable action functional and demonstrating that it has a mountain geometry. An example of a nonlinear Volterra equation satisfying our assumptions finishes the paper.

SECTION 6.2

Assumptions and main results

Let

$$P_\Delta = \{(t, \tau) \in [0, 1] \times [0, 1] : \tau \leq t\}.$$

We assume that function $v \colon P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions

8A **(i)** the function $v(\cdot, \tau, \cdot)$ is continuous on the set $G := [0, 1] \times \mathbb{R}^n$ for a.e. $\tau \in [0, 1]$,

(ii) there exists $v_t(\cdot, \tau, \cdot)$ continuous on G for a.e. $\tau \in [0, 1]$,

(iii) there exists $v_x(\cdot, \tau, \cdot)$ continuous on G for a.e. $\tau \in [0, 1]$,

(iv) there exists $v_{xt}(\cdot, \tau, \cdot)$ continuous on G for a.e. $\tau \in [0, 1]$;

8B **(i)** the function $v(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$, there exists functions $c_1, d_1 \in L^2(P_\Delta, \mathbb{R}^+)$ such that

$$|v(t,\tau,x)| \leq c_1(t,\tau)|x| + d_1(t,\tau)$$

for a.e $(t,\tau) \in P_\Delta$, $x \in \mathbb{R}^n$ and

$$\left(1 - \sqrt{2}||c_1||_{L^2(P_\Delta,\mathbb{R})}\right) > 0, \tag{6.4}$$

(ii) the function $v_t(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and there exist functions $c_2, d_2 \in L^2(P_\Delta, \mathbb{R}^+)$ such that

$$|v_t(t,\tau,x)| \leq c_2(t,\tau)|x| + d_2(t,\tau)$$

for a.e. $(t,\tau) \in P_\Delta$, $x \in \mathbb{R}^n$,

(iii) the function $v_x(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and locally bounded with respect to x,

(iv) the function $v_{xt}(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and locally bounded with respect to x.

Assumption **8B(i)**. means that v is locally bounded with respect to x, i.e. for every $\rho > 0$ there exists $k_\rho > 0$ such that for $(t,\tau) \in P_\Delta$ and $x \in B_\rho = \{x \in \mathbb{R}^n \colon |x| \leq \rho\}$ we have $|v(t,\tau,x| \leq k_\rho$. This follows by the growth condition and since x is absolutely continuous. The same comment concerns the other assumptions. Assumption **8B(i)** may seem a strong one, but it is required if one wants to prove that operator V defined by (6.1)-(6.2) is well defined. Any weaker integrability condition assumed on function c_1 and in a consequence on d_1 would provide that the operator V would act into a space different than $\widetilde{H}^1_0$.

Our main results read as follows

Theorem 6.1. *Assume that conditions **8A**-**8B** hold. Then operator V defined by (6.1)-(6.2) is a diffeomorphism.*

Theorem 6.1 can be restated as follows.

Theorem 6.2. *Assume that conditions **8A-8B** hold. Then for any $y \in \widetilde{H}_0^1$ problem (6.3) has a unique solution which depends in a continuously differentiable manner on the parameter y or in other words, the solution operator is a diffeomorphism.*

Theorem 6.2 admits immediately the following

Corollary 6.1. *Assume that conditions **8A-8B** hold. Let $w \in L^2$. Then the initial value problem*

$$\dot{x}(t) + v(t,\tau,x(\tau)) + \int_0^t v_t(t,\tau,x(\tau))\,d\tau = w\,(t)$$

with an initial condition

$$x(0) = 0$$

has exactly one solution x which is defined on $[0,1]$. Moreover, $x \in \widetilde{H}_0^1$.

Proof. For the proof it suffice to use Theorem 6.2. Indeed, define

$$y\,(t) = \int_0^t w(\tau)\,d\tau.$$

Note that $y \in \widetilde{H}_0^1$ and then we find such a unique x that $V\,(x)\,(t) = y\,(t)$. The direct differentiation finishes the proof while the fact that x satisfies the boundary condition follows from the definition of the space which we consider. ■

The remaining part of the chapter is devoted to the proofs of the above results preceded by some properties of the Volterra operator considered in the setting of space $\widetilde{H}_0^1$ which we provide.

SECTION 6.3

On the Volterra operator

This section is concerned with some Volterra equation in the setting of space $\widetilde{H}_0^1$. We think that such results are of independent interest. We start with showing that in our case, operator V is well defined.

Lemma 6.1. *Assume that conditions* ***8A*(i)**, ***8A*(ii)**, ***8B*(i)** *and* ***8B*(ii)** *are satisfied. Then the operator $V\colon \widetilde{H}_0^1 \to \widetilde{H}_0^1$ given by (6.1) is well defined.*

Proof. Let $x \in \widetilde{H}_0^1$. It is enough to show that the function

$$u(t) = \int_0^t v(t, \tau, x_0(\tau))\, d\tau$$

is absolutely continuos and its derivative is p-integrable since $u(0) = 0$ by definition. Note that u is already continuous as a function of an upper integration limit. Take any system of points $t_1 < t_2 < ... < t_{N+1}$ from $[0, 1]$. By **8B(ii)** using Lagrange Mean Value Theorem applied to the function v with respect to first variable we have

$$\begin{aligned}
\sum_{i=1}^{N} |u(t_{i+1}) - u(t_i)| &= \sum_{i=1}^{N} \left| \int_0^{t_i} v(t_{i+1}, \tau, x_0(\tau)) - v(t_i, \tau, x_0(\tau))\, d\tau \right. \\
&\quad \left. + \int_{t_i}^{t_{i+1}} v(t_{i+1}, \tau, x_0(\tau)) d\tau \right| \\
&\le \int_0^1 k_\rho\, d\tau \sum_{i=1}^{N} |t_{i+1} - t_i| + k_\rho \sum_{i=1}^{N} |t_{i+1} - t_i| \\
&= 2k_\rho \sum_{i=1}^{N} |t_{i+1} - t_i|.
\end{aligned}$$

where $0 \le t_1 < t_2 < .. < t_N < t_{N+1} \le 1$. Therefore u is absolutely continuous. Hence for almost every $t \in [0, 1]$ there exists the derivative of u which is an L^1 function. Thus we must show that $\dot{u}$ is integrable with power 2. Now, by Hölder's inequality we see that

$$\begin{aligned}\int_0^1 |\dot{u}(t)|^2\, dt &\leq 2\int_0^1 |v(t,t,x_0(t)|^2\, dt + 2\int_0^1 \left|\int_0^t v_t(t,\tau,x_0(t)\, d\tau\right|^2\, dt \\ &\leq 2\int_0^1 |v(t,t,x_0(t)|^2\, dt + 2\int_0^1\int_0^t |v_t(t,\tau,x_0(t)|^2\, d\tau\, dt.\end{aligned} \tag{6.5}$$

Consequently applying **8B(i)**, **8B(ii)** we have the more general estimate

$$\begin{aligned}||v\,(t,\tau,x\,(\tau))\;d\tau||^2_{L^2(P_\Delta,\mathbb{R}^n)} &= \int_0^1\left(\int_0^t |v\,(t,\tau,x\,(\tau))\,|\,d\tau\right)^2 dt \\ &\leq \int_0^1\left(\int_0^t |v\,(t,\tau,x\,(\tau))\,|^2\,d\tau\right)\,dt \\ &\leq \int_0^1\left(\int_0^t (c_1\,(t,\tau)\,|x\,(\tau)\,|+d_1\,(t,\tau))^2\,d\tau\right)dt \\ &\leq 2\int_0^1\left(\int_0^t (c_1\,(t,\tau)\,|x\,(\tau)\,|)^2\,d\tau\right)dt \qquad (6.6)\\ &\quad+2\int_0^1\left(\int_0^t (d_1\,(t,\tau))^2\,d\tau\right)dt \\ &\leq 2||c_1||^2_{L^2(P_\Delta,\mathbb{R}^n)}||x||^2_{\widetilde{H}^1_0}+2||d_1||^2_{L^2(P_\Delta,\mathbb{R})} \\ &\leq \left(\sqrt{2}||c_1||_{L^2(P_\Delta,\mathbb{R}^n)}||x||_{\widetilde{H}^1_0}+\sqrt{2}||d_1||_{L^2(P_\Delta,\mathbb{R})}\right)^2,\end{aligned}$$

and analogously

$$\begin{aligned}||v_t(t,\tau,x(\tau))\;d\tau||^2_{L^2(P_\Delta,\mathbb{R}^n)} &\leq \int_0^1\int_0^t [c_2(t,\tau)|x(\tau)|+d_2(t,\tau)]^2\,d\tau\,dt \\ &\leq 2||x||^2_C\int_0^1\int_0^t c_2^2(t,\tau)\,d\tau\,dt \\ &\quad+2\int_0^1\int_0^t d_2^2(t,\tau)\,d\tau\,dt \\ &\leq 2||c_2||^2_{L^2(P_\Delta,\mathbb{R}^n)}||x||^2_{\widetilde{H}^1_0}+2||d_2||^2_{L^2(P_\Delta,\mathbb{R}^+)} \\ &\leq \left(\sqrt{2}||c_2||_{L^2(P_\Delta,\mathbb{R}^+)}||x||_{\widetilde{H}^1_0}+\sqrt{2}||d_2||_{L^2(P_\Delta,\mathbb{R}^+)}\right)^2.\end{aligned} \tag{6.7}$$

We have thus proved that V is well defined in $\widetilde{H}_0^1$. ■

Let us now consider for any $t \in [0,1]$ and arbitrary $x, g \in \widetilde{H}_0^1$ the linear integral equation of the form

$$h(t) + \int_0^t v_x(t,\tau,x(\tau))h(\tau)\, d\tau = g(t) \tag{6.8}$$

for $t \in [0,1]$. The solvability of this equation is crucial in the application of the Global Diffeomorphism Theorem since it provides the invertibility of the derivative of the Volterra operator. In case of Volterra operators the question of uniqueness requires some different means then in previous chapters due to the nonlinear structure of this operator. The technique which we present is commonly used for such kind of operators.

For every fixed $\rho > 0$ define l_ρ such that

$$l_\rho = \max\left\{ \sup_{(t,\tau)\in P_\Delta, |x|\le\rho} |v_x(t,\tau,x)|, \sup_{(t,\tau)\in P_\Delta, |x|\le\rho} |v_{xt}(t,\tau,x)| \right\} \tag{6.9}$$

and fix M such that

$$\sup_{t\in[0,1]} |g(t)| \le M.$$

Constant l_ρ is well defined by remarks following the assumptions **8A-8B**, while M is finite by continuity.

Theorem 6.3. *Assume that conditions **8A**, **8B** hold. Fix $x, g \in \widetilde{H}_0^1$ and define $\rho = \sup_{t\in[0,1]} |x(t)|$ and l_ρ by (6.9). Then the equation (6.8) has a unique solution $h \in \widetilde{H}_0^1$.*

Proof. Let us define the bounded linear operator $T\colon \widetilde{H}_0^1 \to \widetilde{H}_0^1$ pointwisely for all $t \in [0,1]$

$$Th(t) = \int_0^t v_x(t,\tau,x(\tau))h(\tau)\, d\tau.$$

Note that T is linear bounded due to assumptions on v_x. For any $n \in N_0$, $t \in [0,1]$ and $g \in \widetilde{H}_0^1$ we define the following sequence of iterations

$$T^n h(t) = \int_0^t v_x(t,\tau,x(\tau)) T^{n-1} h(\tau)\, d\tau.$$

First, we will estimate $|T^n h(t)|$ for $t \in [0,1]$ with some fixed n. It is easy to observe that the first iterate can be estimated by

$$|T^1 h(t)| = \left| \int_0^t v_x(t,\tau,x(\tau)) h(\tau)\, d\tau \right| \le t l_\rho M$$

for all $t \in [0,1]$. Similarly, the second iterate might be estimated by

$$\begin{aligned} |T^2 h(t)| &= \left| \int_0^t v_x(t,\tau,x(\tau)) (T^1 h)(\tau)\, d\tau \right| \\ &\le \left| \int_0^t v_x(t,\tau,x(\tau)) \tau l_\rho M\, d\tau \right| \le \frac{t^2}{2} l_\rho^2 M \end{aligned}$$

for all $t \in [0,1]$. The third one for all $t \in [0,1]$ is estimated by

$$\begin{aligned} |T^3 h(t)| &= \left| \int_0^t v_x(t,\tau,x(\tau)) (T^2 h)(\tau)\, d\tau \right| \\ &\le \left| \int_0^t v_x(t,\tau,x(\tau)) \frac{\tau^2}{2} l_\rho^2 M\, d\tau \right| \le \frac{t^3}{3!} l_\rho^3 M. \end{aligned}$$

Therefore, we assert that

$$|T^n h(t)| \le \frac{t^n}{n!} l_\rho^n M$$

for all $t \in [0,1]$. Assume that

$$|T^{(n-1)} h(t)| \le \frac{t^{(n-1)}}{(n-1)!} l_\rho^{(n-1)} M$$

for all $t \in [0,1]$. Now by induction let us observe that

$$\begin{aligned}
|T^n h(t)| &= \left| \int_0^t v_x(t,\tau,x(\tau))(T^{(n-1)}h)(\tau)\, d\tau \right| \\
&\leq \left| \int_0^t v_x(t,\tau,x(\tau)) \frac{\tau^{(n-1)}}{(n-1)!} l_\rho^{(n-1)} M \, d\tau \right| \\
&\leq \frac{1}{(n-1)!} l_\rho^n M \left| \int_0^t \tau^{(n-1)}\, d\tau \right| = \frac{t^n}{n!} l_\rho^n M \qquad (6.10)
\end{aligned}$$

for all $t \in [0,1]$. Taking into account the definition of the operator T, we see that equation (6.8) might be rewritten in the following form

$$h + Th = g.$$

Let us consider the sequence $(h_n)_{n \in \mathbb{N}_0}$ given by the formula

$$h_{n+1} = g - Th_n$$

for all $n \in \mathbb{N}_0$ with $h_0 = 0$. Note that as is the case with Volterra equation in L^2, h_0 can be chosen arbitrarily and the choice $h_0 = 0$ is just for convenience. It might be observed that

$$\begin{aligned}
h_{n+1} &= g - Tg + T^2 g - T^3 g + .. + (-1)^n T^n g \\
&= g + \sum_{i=1}^{n} (-1)^i T^i g \qquad (6.11)
\end{aligned}$$

for all $n \in \mathbb{N}_0$ with $T^0 g = g$. Again considering the estimate (6.10), applying the induction and the fact that by Lemma 6.1 for all $k \in \mathbb{N}_0$ $T^k g \in \widetilde{H}_0^1$, we get for any $n \in \mathbb{N}_0$

$$\begin{aligned}
||T^n g||_{\widetilde{H}_0^1}^p &= \int_0^1 \left| v_x(t,t,x(t)) T^{n-1} g(t) + \int_0^t v_{xt}(t,\tau,x(\tau) T^{n-1} g(\tau)\, d\tau \right|^p dt \\
&\leq \int_0^1 \left(l_\rho^n M \frac{t^{n-1}}{(n-1)!} + \int_0^t l_\rho^n M \frac{\tau^{n-1}}{(n-1)!}\, d\tau \right)^p dt \\
&\leq \int_0^1 \left(\frac{l_\rho^n M}{(n-1)!} + \frac{l_\rho^n M}{(n-1)!} \int_0^1 d\tau \right)^p dt \leq 2^p \left(\frac{l_\rho^n M}{(n-1)!} \right)^p . \qquad (6.12)
\end{aligned}$$

Consequently, for any $n \in \mathbb{N}_0$, we get

$$||T^n g||_{\widetilde{H}^1_0} \leq 2\frac{l_\rho^n M}{(n-1)!}.$$

By the classical d'Alembert's criterion series $\sum\limits_{n=1}^{\infty} 2\frac{l_\rho^n M}{(n-1)!}$ converges. Moreover, it provides majorant for the series $\sum\limits_{i=1}^{n}(-1)^i T^i g$. Consequently, the sequence $(h_{n+1})_{n\in\mathbb{N}_0}$ defined by (6.11) is a Cauchy sequence in $\widetilde{H}^1_0$, which converges to some $h \in \widetilde{H}^1_0$. The operator $T\colon \widetilde{H}^1_0 \to \widetilde{H}^1_0$ is continuous. Therefore, h is a solution to equation (6.8).

Now we will show that h is a unique solution to equation (6.8). By contradiction, let us assume that there exists $h_1 \in \widetilde{H}^1_0$, which satisfies (6.8) and $h \neq h_1$. For $h^* = h - h_1$ the following equation holds

$$h^* - Th^* = 0.$$

Applying operator T on the above equation n times we obtain

$$(-1)^n h^* + T^n h^* = 0.$$

Note that $T^n h^* \to 0$ as $n \to \infty$ by estimate (6.12). Then also $T^{2n} h^* \to 0$ as $n \to \infty$ and so consequently $h^* = 0$. Therefore, equation (6.8) possesses one unique solution in $\widetilde{H}^1_0$, what completes the proof. ■

SECTION 6.4

Proofs of main results

The proof of our main results relies on the application of Theorem 1.2 with $X = B = \widetilde{H}^1_0$. Thus we must show that V is continuously Fréchet differentiable on X, next we should properly define functional φ and what is the most difficult task we must show that φ satisfies the PS-condition. We consider the functional $\varphi\colon \widetilde{H}_0^1 \to \mathbb{R}^+$ defined by the following formula

$$\begin{aligned}\varphi(x) &= \frac{1}{2}||V(x)-y||^2_{\widetilde{H}_0^1}\\ &= \frac{1}{2}\int_0^1 \left| x(t)-y(t)+v(t,t,x(t))+\int_0^t v_t(t,\tau,x(\tau))\,d\tau\right|^2 dt. \quad (6.13)\end{aligned}$$

By (6.6) and (6.7) the functional φ is well defined which is very easy to be verified.

Before we proof the main results we will introduce some auxiliary lemmas.

Lemma 6.2. *Assume that* ***8A*(i)**, ***8A*(ii)**, ***8B*(i)** *and* ***8B*(iii)** *are satisfied. Then operator V defined by (6.1) is continuously Fréchet differentiable at every point $\hat{x} \in \widetilde{H}_0^1$ and its derivative reads*

$$V'(\hat{x})h(t) = h(t) + \int_0^t v_x(t,\tau,\hat{x}(\tau))h(\tau)\,d\tau \quad (6.14)$$

for $h \in \widetilde{H}_0^1$ and for any $t \in [0,1]$.

Proof. It is sufficient to show that the operator

$$\widetilde{V}(\hat{x})(t) = \int_0^t v(t,\tau,\hat{x}(\tau))\,d\tau$$

is continuously Fréchet differentiable. Applying Mean Value Theorem 3.2.6. p. 119 [18] for any $t \in [0,1]$, any $h \in \widetilde{H}_0^1$ and some $\theta \in [0,1]$ we can write

$$\begin{aligned}\widetilde{V}(\hat{x}+h)(t)-\widetilde{V}(\hat{x})(t) &= \int_0^t (v(t,\tau,\hat{x}(\tau)+h(\tau))-v(t,\tau,\hat{x}(\tau)))\,d\tau\\ &= \int_0^1\int_0^t v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau))h(\tau)\,d\tau\,d\theta\\ &= \int_0^t [v_x(t,\tau,\hat{x}(\tau))h(\tau)\,d\tau\\ &\quad -\int_0^1\int_0^t (v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau))-v_x(t,\tau,\hat{x}(\tau)))\,h(\tau)\,d\tau\,d\theta.\end{aligned}$$

Moreover

$$\int_0^1 \int_0^t [v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau)) - v_x(t,\tau,\hat{x}(\tau))]h(\tau)\, d\tau\, d\theta$$
$$\leq ||h||_{W_0^{1,p}} \int_0^1 \int_0^t |v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau)) - v_x(t,\tau,\hat{x}(\tau))|\, d\tau\, d\theta.$$

Since the norm convergence in $\widetilde{H}_0^1$ implies the uniform convergence in $C\,(0,1)$, by the assumption of the lemma and by the Lebesgue Dominated Convergence Theorem we obtain that for all $t \in [0,1]$

$$\int_0^t |v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau)) - v_x(t,\tau,\hat{x}(\tau))|\ d\tau \to 0$$

when $||h||_{\widetilde{H}_0^1} \to 0$. Thus

$$||h||_{\widetilde{H}_0^1} \int_0^1 \int_0^t |v_x(t,\tau,\hat{x}(\tau)+\theta h(\tau)) - v_x(t,\tau,\hat{x}(\tau))|\, d\tau\, d\theta = o(h),$$

where $\frac{o(h)}{||h||_{\widetilde{H}_0^1}} \to 0$ as $||h||_{\widetilde{H}_0^1} \to 0$. In a consequence

$$\widetilde{V}(\hat{x}+h)(t) - \widetilde{V}(\hat{x})(t) = \int_0^t v_x(t,\tau,\hat{x}(\tau))h(\tau)\, d\tau + o(h)$$

which means that $\widetilde{V}$ is Fréchet differentiable. In order to prove that $\widetilde{V}$ is continuously Fréchet differentiable one needs to show that

$$\widetilde{H}_0^1 \ni x \to \int_0^{\cdot} v_x(\cdot,\tau,x(\tau))h(\tau)\, d\tau \in \left(\widetilde{H}_0^1\right)^*$$

is continuous in x uniformly in h from the unit sphere in $\widetilde{H}_0^1$ (see Theorem 5.9. p.119 from [70]). Let us take a sequence $(x_n)_{n\in\mathbb{N}}$ convergent in $\widetilde{H}_0^1$ to some x_0. This sequence is also convergent in L^2 and in $C\,(0,1)$. Fix $h \in \widetilde{H}_0^1$ such that $\|h\| = 1$. Applying the same arguments as in obtaining (6.5) we see that

$$\int_0^1 \left| \frac{d}{dt} \int_0^t \left(v_x(t,\tau,x_n(\tau)) - v_x(t,\tau,x_0(\tau))\right) h(\tau)\, d\tau \right|^2$$
$$\leq ||h||^2_{\widetilde{H}_0^1} 2 \int_0^1 |v_x(t,t,x_n(t)) - v_x(t,t,x_0(t))|^2\, dt$$
$$+ ||h||^2_{\widetilde{H}_0^1} 2 \int_0^1 \int_0^t |v_{xt}(t,\tau,x_n(\tau)) - v_{xt}(t,\tau,x_0(\tau))|^2\, d\tau\, dt.$$

Again by the Lebesgue Dominated Convergence Theorem we get the following

$$\int_0^1 |v_x(t,t,x_n(t) - v_x(t,t,x_0(t)|^2\, dt \to 0$$

and

$$\int_0^1 \int_0^t |v_{xt}(t,\tau,x_n(\tau)) - v_{xt}(t,\tau,x_0(\tau))|^2\, d\tau\, dt \to 0$$

as $n \to \infty$. This finishes the proof. ■

Lemma 6.3. *Assume that conditions* ***8A*** *and* ***8B*** *hold. Then functional φ given by (6.13) is continuously Gâteaux differentiable and a Gâteaux derivative at any point $x \in \widetilde{H}_0^1$ is given by*

$$\varphi'(x)h = \int_0^1 \left(x(t) - y(t) + v(t,t,x(t)) + \int_0^t v_t(t,\tau,x(\tau))\, d\tau \right) \cdot \left(h(t) + v_x(t,t,x(t))h(t) + \int_0^t v_{xt}(t,\tau,x(\tau))h(\tau)\, d\tau \right) dt \tag{6.15}$$

for any $h \in \widetilde{H}_0^1$. Moreover, functional φ satisfies the PS-condition for any fixed $y \in \widetilde{H}_0^1$.

Proof. Using (6.14) and the formula for the derivative of $x \mapsto \frac{1}{2}\|x\|^2$ in $\widetilde{H}_0^1$ together with the chain rule for Fréchet derivatives we obtain that the differential $\varphi'(x)$ of φ at any fixed point $x \in \widetilde{H}_0^1$ is given by (6.15) for any $h \in \widetilde{H}_0^1$.

Therefore the following inequality might be easily deduced for any $x \in \widetilde{H}_0^1$

$$\begin{aligned}\sqrt{2\varphi(x)} &= \left\| x(\cdot) - y(\cdot) + \int_0^{\cdot} v(\cdot,\tau,x(\tau))\, d\tau \right\|_{\widetilde{H}_0^1} \\ &\geq ||x||_{\widetilde{H}_0^1} - ||y||_{\widetilde{H}_0^1} - \left\| \int_0^{\cdot} v(\cdot,\tau,x(\tau))\, d\tau \right\|_{\widetilde{H}_0^1} \\ &\geq \left(1 - \sqrt{2}||c_1||_{L^2(P_\Delta,\mathbb{R}^n)}\right) ||x||_{\widetilde{H}_0^1([0,1],\mathbb{R}^n)} - \widetilde{c} \end{aligned} \tag{6.16}$$

where $\widetilde{c} = ||y||_{\widetilde{H}_0^1} + \sqrt{2}||d_1||_{L^2(P_\Delta,\mathbb{R}^n)}$. This means that φ is coercive.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a (PS) sequence for φ, i.e.

- $\varphi(x_n) \leq M$ for all $n \in N$ and some $M > 0$,
- $\lim\limits_{n\to\infty} \varphi'(x_n) = 0.$

We have by (6.16)

$$\sqrt{2M} \geq \sqrt{2\varphi(x_n)} \geq \left(1 - \sqrt{2}||c_1||_{L^2(P_\Delta,\mathbb{R}^n)}\right) ||x||_{\widetilde{H}_0^1} - \widetilde{c}$$

for $n \in N$. This means that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded because of condition (6.4).

Consequently the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent in $\widetilde{H}_0^1$ to some x_0, and uniformly convergent on $[0,1]$. Moreover, we get the weak convergence of derivatives of $(\dot{x}_n)_{n\in\mathbb{N}}$ in L^2 of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0'.

Let us calculate

$$2(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) = \int_0^1 |\eta(t,x_n) - \eta(t,x_0)|^2\, dt + \sum_{i=1}^{6} \Psi_i(x_n,x_0)$$

where

$$\eta(t,x_n) = \dot{x}_n(t) + v(t,t,x_n(t)) + \int_0^t v_t(t,\tau,x_n(\tau))\,d\tau$$
$$\eta(t,x_0) = \dot{x}_0(t) + v(t,t,x_0(t)) + \int_0^t v_t(t,\tau,x_0(\tau))\,d\tau$$

and

$$\begin{aligned}
\Psi_1(x_n,x_0) &= \int_0^1 (\eta(t,x_n) - \eta(t,x_0)) \\
&\quad \cdot \left(\int_0^t (v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau)))\; d\tau \right)\, dt, \\
\Psi_2(x_n,x_0) &= \int_0^1 (\eta(t,x_n) - \eta(t,x_0)) \cdot (v(t,t,x_0(t)) - v(t,t,x_n(t)))\,dt, \\
\Psi_3(x_n,x_0) &= \int_0^1 \left(\eta(t,x_n) \int_0^t v_{xt}(t,\tau,x_n(\tau))(x_n(\tau) - x_0(\tau))\,d\tau \right)\, dt, \\
\Psi_4(x_n,x_0) &= \int_0^1 \eta(t,x_n) v_x(t,t,x_n(t))(x_n(t) - x_0(t))\,dt, \\
\Psi_5(x_n,x_0) &= -\int_0^1 \left(\eta(t,x_0) \int_0^t v_{xt}(t,\tau,x_0(\tau))(x_n(\tau) - x_0(\tau))\,d\tau \right)\, dt, \\
\Psi_6(x_n,x_0) &= -\int_0^1 \eta(t,x_0) v_x(t,t,x_0(t))(x_n(t) - x_0(t))\,dt.
\end{aligned}$$

We will show that $\Psi_i(x_n,x_0) \to 0$ as $n \to \infty$ for $i = 1,\ldots,6$. First let us consider $\Psi_1(x_n,x_0)$. Form the Lebesgue Dominated Convergence Theorem it follows that for all fixed $t \in [0,1]$

$$\int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\,d\tau \to 0$$

as $n \to \infty$. From the properties of the sequence $(x_n)_{n\in\mathbb{N}}$ we see that there is $M_1 > 0$ for which

$$|x_n(\tau))| \le M_1$$

for all $\tau \in [0,1]$ and $n = 0,1,\ldots$. Let us consider the term $\Psi_1(x_n, x_0)$. Note that for any fixed $t \in [0,1]$

$$\left|\int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\,d\tau\right| \leq 2\int_0^t (M_1 c_2(t,\tau) + d_2(t,\tau))\,d\tau$$

and that the function

$$[0,1] \ni t \to 2\int_0^t c_2(t,\tau)M_1 + d_2(t,\tau)d\tau \in \mathbb{R}$$

belongs to $L^2([0,1],\mathbb{R})$. Using the Lebesgue Dominated Convergence Theorem we assert that

$$\int_0^{\cdot} (v_t(\cdot,\tau,x_n(\tau)) - v_t(\cdot,\tau,x_0(\tau)))\,d\tau \to 0$$

as $n \to \infty$ in L^2. Let us observe that

$$\begin{aligned}\Psi_1(x_n,x_0) \leq \int_0^1 &\left(\left|\int_0^t (v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau)))\ d\tau\right|\right.\\ &\left. + \int_0^t (v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau)))\ d\tau\right)\ dt. \qquad (6.17)\end{aligned}$$

By Hölder's inequality and (6.6), (6.7) the first term of (6.17) might be estimated

$$\begin{aligned}&\int_0^1 \left(|\eta(t,x_n(t))| \int_0^t (v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau)))\ d\tau\right)\ dt\\ &\leq \left(\int_0^1 |\eta(t,x_n(t))|^2\,dt\right)^{\frac{1}{2}}\\ &\quad\cdot\left(\int_0^1\int_0^t |v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau))|^2\,d\tau\,dt\right)^{\frac{1}{2}}\\ &= \left(\int_0^1 |\eta(t,x_n(t))|^2\,dt\right)^{\frac{1}{2}}\\ &\quad\cdot\left(\int_0^1\int_0^t |v_t(t,\tau,x_0(\tau)) - v_t(t,\tau,x_n(\tau))|^2\,d\tau\,dt\right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $n \to \infty$. Note that by definition and since $(x_n)_{n\in\mathbb{N}}$ is bounded it follows that

$$\left(\left(\int_0^1 |\eta(t,x_n(t))|^2\,dt\right)^{\frac{1}{2}}\right)_{n\in\mathbb{N}}$$

is bounded. The second term of (6.17) tends to 0 as $n \to \infty$ as well. Consequently $\Psi_1(x_n,x_0) \to 0$ as $n \to \infty$.

The similar reasoning might be applied to prove that $\Psi_2(x_n,x_0)$ tends to 0 as $n \to \infty$. Let us consider $\Psi_3(x_n,x_0)$. Firstly, we see that for any $t \in [0,1]$ the following estimation holds

$$\left|\int_n^t (v_{xt}(t,\tau,x_n(\tau))(x_n(\tau)-x_0(\tau))\,d\tau\right| \\ \leq \int_0^1 |(v_{xt}(t,\tau,x_0(\tau))\,d\tau|\,\|x_n - x_0\|_C\,.$$

Since the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly convergent in $C\,(0,1)$ to x_0, we see that

$$\int_0^{\cdot} v_{xt}(\cdot,\tau,x_n(\tau))\,d\tau(x_n(\cdot)-x_0(\cdot)) \to 0$$

as $n \to \infty$ on $[0,1]$. As in the previous case applying Hölder's inequality

$$\Psi_3(x_n,x_0) \leq \left(\int_0^1 |\eta(t,x_n)|^2 dt\right)^{\frac{1}{2}} \\ \cdot \left(\int_0^1 \int_0^t |v_{xt}(t,\tau,x_n(\tau))(x_n(\tau)-x_0(\tau))|^2 d\tau\,dt\right)^{\frac{1}{2}} \to 0.$$

Convergence of $\Psi_i(x_n,x_0)$ to 0 for $i = 4,5,6$ as $n \to \infty$ follows from the reasoning presented above.

Using Lebesgue Dominated Convergence Theorem we conclude that

$$
\begin{aligned}
||x_n - x_0||_{\widetilde{H}_0^1} \leq \Bigg(\int_0^1 \Big| & \dot{x}_n(t) - \dot{x}_0(t) + v(t,t,x_n(t)) - v(t,t,x_0(t)) \\
& + \int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\, d\tau \Big|^2 dt \Bigg)^{\frac{1}{2}} \\
& - \Bigg(\int_0^1 \Big| v(t,t,x_n(t)) - v(t,t,x_0(t)) \\
& + \int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\, d\tau \Big|^2 dt \Bigg)^{\frac{1}{2}} .
\end{aligned}
$$

From the equality (6.14) we have

$$
\begin{aligned}
& (\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \\
& \qquad \geq \int_0^1 \Big| \dot{x}_n(t) - \dot{x}_0(t) + v(t,t,x_n(t)) - v(t,t,x_0(t)) \\
& \qquad + \int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\, d\tau \Big|^2 dt + \sum_{i=1}^{6} \Psi_i(x_n, x_0).
\end{aligned}
$$

So,

$$
\begin{aligned}
& \int_0^1 \Big| \dot{x}_n(t) - \dot{x}_0(t) + v(t,t,x_n(t)) - v(t,t,x_0(t)) \\
& \qquad + \int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\, d\tau \Big|^2 dt \\
& \qquad\qquad \leq \Big| (\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \Big| + \Big| \sum_{i=1}^{6} \Psi_i(x_n, x_0) \Big| .
\end{aligned}
$$

On the other hand one can observe that

$$
|\varphi'(x_n)(x_n - x_0)| \leq ||\varphi'(x_n)||_{(\widetilde{H}_0^1)^*} ||x_n - x_0||_{\widetilde{H}_0^1}.
$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded and $\varphi'(x_n) \to 0$ as $n \to \infty$ we see that

$$
\varphi'(x_n)(x_n - x_0) \to 0.
$$

as $n \to \infty$. From the weak convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0 in $\widetilde{H}_0^1$ it follows that

$$\varphi'(x_0)(x_n - x_0) \to 0$$

as $n \to \infty$. So,

$$(\varphi'(x_n) - \varphi'(x_0))(x_n - x_0) \to 0$$

as $n \to \infty$. Thus

$$\int_0^1 \Big| \dot{x}_n(t) - \dot{x}_0(t) + v(t,t,x_n(t)) - v(t,t,x_0(t)) \\ + \int_0^t (v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))\, d\tau \Big|^2 dt \to 0$$

as $n \to \infty$. Of course,

$$\int_0^1 |v(t,t,x_n(t)) - v(t,t,x_0(t))|^2 \; dt \to 0$$

and

$$\int_0^1 \int_0^t |(v_t(t,\tau,x_n(\tau)) - v_t(t,\tau,x_0(\tau)))|^2 \; d\tau\, dt \to 0$$

as $n \to \infty$. Consequently,

$$||x_n - x_0||_{\widetilde{H}_0^1} \to 0$$

i.e. the function φ satisfies (PS) condition. ■

Now we are in position to prove our main result.

The proof of Theorem 6.1. Set $X = B = \widetilde{H}_0^1$. From Lemma 6.3 we conclude that for any $y \in \widetilde{H}_0^1$ the functional $\varphi(x) = \frac{1}{2}\|V(x) - y\|^2_{\widetilde{H}_0^1}$ satisfies (PS) condition, i.e. the first assumption of Theorem 1.2 is fulfilled. Theorem 6.3 provides sufficient requirements for equation $V'(x) = g$ to possesses a unique solution in $\widetilde{H}_0^1$ for any $g \in \widetilde{H}_0^1$ which is equivalent to the second assumption of Theorem 1.2. Therefore, $V\colon \widetilde{H}_0^1 \to \widetilde{H}_0^1$ is a diffeomorphism and the theorem is proved. ■

SECTION 6.5

Example

We finish the paper with an example of function satisfying assumptions **8A** and **8B**. Let us assume $v\colon P_\Delta \times \mathbb{R} \to \mathbb{R}$ is given by

$$v(t,\tau,x) = \frac{1}{2}(t-\tau)^{\frac{1}{2}} \ln(1+(t-\tau)^4 x^2)$$

for $t,\tau \in [0,1]$, $t>\tau$, $x \in R$. Since

$$\ln(1+s^2z^2) \leq \ln((1+s^2)(1+z^2)) = \ln(1+s^2) + \ln(1+z^2) \leq |s| + |z|$$

for $s,z \in R$, therefore

$$|v(t,\tau,x)| \leq \frac{1}{2}(t-\tau)^{\frac{1}{2}}|x| + \frac{1}{2}(t-\tau)^{5/2}.$$

Let us put

$$c_1(t,\tau) = \frac{1}{2}(t-\tau)^{\frac{1}{2}}$$

for $t,\tau \in [0,1]$, $t>\tau$. Then

$$||c_1||^2_{L^2(P_\Delta,\mathbb{R})} = \int_0^1 \int_0^t \left(\frac{1}{2}(t-\tau)^{\frac{1}{2}}\right)^2 d\tau dt \leq \frac{1}{24}$$

for all $t,\tau \in [0,1]$, $t>\tau$, $p \geq 2$. Consequently,

$$||c_1||_{L^2(P_\Delta,\mathbb{R})} \leq \frac{1}{2},$$

hence condition (6.4) is satisfied. We see that

$$v_t(t,\tau,x) = \frac{1}{2}(t-\tau)^{-\frac{1}{2}} \ln(1+(t-\tau)^4x^2) + \frac{1}{2}(t-\tau)^{\frac{1}{2}} \frac{4(t-\tau)^3x^2}{1+(t-\tau)^4x^2}.$$

Since $\ln\left(1+a^2\right) \leq |a|$ for $a \in \mathbb{R}$ and

$$\frac{4(t-\tau)^3x^2}{1+(t-\tau)^4x^2} \leq 4$$

for all $t, \tau \in [0,1]$, $t > \tau$ and any $x \in \mathbb{R}$ we see that the following inequality holds

$$|v_t(t,\tau,x)| \leq \frac{1}{2}(t-\tau)^{3/2}|x| + 2^{3-p}(t-\tau)^{\frac{1}{2}}.$$

Let us put

$$c_2(t,\tau) = \frac{1}{2}(t-\tau)^{3/2}$$

and

$$d_2(t,\tau) = 2(t-\tau)^{\frac{1}{2}}.$$

In an elementary way it can be checked that $c_2, d_2 \in L^2(P_\Delta, \mathbb{R}^+)$. Let us note that

$$v_x(t,\tau,x) = \frac{1}{2}(t-\tau)^{\frac{1}{2}} \frac{2(t-\tau)^4 x}{1+(t-\tau)^4 x^2}$$

and

$$v_{xt}(t,\tau,x) = \frac{1}{2}(t-\tau)^{7/2} x \frac{9+17(t-\tau)^4 x^2}{\left(1+(t-\tau)^4 x^2\right)^2}.$$

Observe

$$|v_x(t,\tau,x)| \leq \frac{1}{2}(t-\tau)^{5/2}$$

and

$$|v_{xt}(t,\tau,x)| \leq 13(t-\tau)^{7/2}\,|x|\,.$$

Thus we conclude that v satisfies assumptions **8A** and **8B**.

Then we can formulate the following result. Note the existence of the initial value problem for the related first order differential equation is defined on the whole interval $[0,1]$.

Proposition 6.1. *The integral Volterra operator $V\colon \widetilde{H}_0^1 \to \widetilde{H}_0^1$ defined pointwisely for all $t \in [0,1]$ by*

$$\begin{cases} V(x)(t) = x(t) + \displaystyle\int_0^t \frac{1}{2}(t-\tau)^{\frac{1}{2}} \ln(1+(t-\tau)^4 x(\tau)^2)\, d\tau, \\ x(0) = 0 \end{cases}$$

defines a global diffeomorphism. Moreover, problem the initial value problem

$$\begin{aligned} -\dot{x} =& (t-\tau)^{\frac{1}{2}} \ln(1+(t-\tau)^4 x^2) \\ &+ \frac{1}{2}\int_0^t \Bigg((t-\tau)^{-\frac{1}{2}} \ln(1+(t-\tau)^4 x^2) \\ &+ \frac{1}{2}(t-\tau)^{\frac{1}{2}} \frac{4(t-\tau)^3 x^2}{1+(t-\tau)^4 x^2} \Bigg)\, d\tau = w(t) \end{aligned}$$

with $x(0) = 0$ has exactly one solution for any fixed of parameter $w \in L^2$ and the solution depends in a C^1 manner in parameter w.

CHAPTER 7

Finite dimensional results - invertibility without continuous differentiability

In order to present our results we start with some preliminaries.

A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is called locally Lipschitz continuous, if to every $u \in \mathbb{R}^n$ there corresponds a neighbourhood V_u of u and a constant $L_u \geq 0$ such that

$$\|f(z) - f(w)\| \leq L_u \|z - w\| \quad \text{for all } z, w \in V_u.$$

If $k = 1$ ($f : \mathbb{R}^n \to \mathbb{R}$) and $u, z \in \mathbb{R}^n$, we write $f^0(u; z)$ for the generalized directional derivative of f at the point u along the direction z, i.e.,

$$f^0(u; z) := \limsup_{w \to u,\, t \to 0^+} \frac{f(w + tz) - f(w)}{t}.$$

The generalized gradient of the function f at u, denoted by $\partial f(u)$, is the set

$$\partial f(u) := \{\xi \in L(\mathbb{R}^n, \mathbb{R}) : \langle \xi, z \rangle \leq f^0(u; z), \text{ for all } z \in \mathbb{R}^n\}.$$

For the definition of a generalized Jacobian of a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ we refer to [13] p. 69. We denote the generalized Jacobian at

x again by $\partial f(x)$. For a fixed x the set $\partial f(x)$ being of maximal rank means that all matrices in $\partial f(x)$ are nonsingular. This assumption is equivalent, when f is smooth, with the assumption that $\det f'(x) \neq 0$ for every $x \in D$ where $D \subset \mathbb{R}^n$ is some open set. Compare with [66] where this condition provides local diffeomorphism for a differentiable mapping. Note that it is not enough to assume that $\det[f'(x)] \neq 0$ whenever it exists, which happens a.e. for a locally Lipschitz function.

A point u is called a (generalized) critical point of the locally Lipschitz continuous functional $J : \mathbb{R}^n \to \mathbb{R}$ if $0 \in \partial J(u)$. In this case we identify $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ with $\mathbb{R}^n$ so that $\partial J(u) \subset \mathbb{R}^n$. J is said to fulfill the non-smooth Palais-Smale condition, see [53], if every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $(J(u_n))_{n\in\mathbb{N}}$ is bounded and

$$J^0(u_n; u - u_n) \geq -\varepsilon_n \|u - u_n\|$$

for all $u \in \mathbb{R}^n$, where $\varepsilon_n \to 0^+$, admits a convergent subsequence. Our main tool will be the following result based on the zero-altitude version of Mountain Pass Theorem from [53], where we replace non-smooth (PS) condition with coercivity which we require and which guarantees that (PS) condition holds.

Theorem 7.1. *[52] Let $J : \mathbb{R}^n \to \mathbb{R}$ be a coercive locally Lipschitz continuous functional. If there exist $u_1, u_2 \in \mathbb{R}^n$, $u_1 \neq u_2$ and $r \in (0, \|u_2 - u_1\|)$ such that*

$$\inf\{J(u) : \|u - u_1\| = r\} \geq \max\{J(u_1), J(u_2)\}$$

and we denote by Γ the family of continuous paths $\gamma : [0,1] \to \mathbb{R}^n$ joining u_1 and u_2, then

$$c := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} J(\gamma(s)) \geq \max\{J(u_1), J(u_2)\}$$

is a critical value for J on $\mathbb{R}^n$ and $K_c \backslash \{u_1, u_2\} \neq \emptyset$, where K_c is the set of critical points at the level c, i.e.

$$K_c = \{u \in \mathbb{R}^n : J(u) = c \text{ and } 0 \in \partial J(u)\}.$$

The basic properties of generalized directional derivative and generalized gradient were studied in [13] and later in [50].

We consider locally invertible mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ that are Fréchet-differentiable and which need not be continuously Fréchet-differentiable. Additionally, we assume that f is strictly (Hadamard-like) differentiable. Let us recall that a function $f : D \to \mathbb{R}^n$ defined on a open subset D of $\mathbb{R}^n$ is strictly differentiable at $x_0 \in D$, see [12] p. 30, if there exists an element $f'(x_0) \in \mathbb{R}^n$ (called the strict derivative) such that

$$\lim_{w \to x_0, t \to 0^+} \frac{f(w + tz) - f(w)}{t} = \langle f'(x_0), z \rangle \text{ for all } z \in \mathbb{R}^n$$

provided the convergence is uniform for z in compact sets. We will denote any derivative at x_0 by $f'(x_0)$ and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$ and also for the action of linear mappings on $\mathbb{R}^n$. A continuously Gâteaux differentiable, thus a continuously Fréchet-differentiable functional has necessarily the strict derivative which coincides with the Fréchet derivative, see Corollary and its proof, p 32. [12]. On the other hand a Fréchet-differentiable functional $f : \mathbb{R}^n \to \mathbb{R}^n$ need not be strictly differentiable, see Example 2.2.3 p. 33 [12] in case $n = 1$. However, if f is Fréchet-differentiable and locally Lipschitz and the generalized gradient reduces to a singleton, then both differentiability notions mentioned coincide, see Propositions 2.2.1 and 2.2.2 from [12].

The methods which we apply are the known result on local diffeomorphism in case of Fréchet-differentiable mappings contained in [66] and the Mountain Pass Theorem (MPT for short). Since the MPT works either for C^1 functionals or for locally Lipschitz ones, we must use its locally Lipschitz counterpart. Using non-smooth critical point theory applied to a functional $x \mapsto \frac{1}{2} \|f(x)\|^2$ we provide sufficient conditions for f to be global diffeomorphism. The local invertibility results we base on is as follows, [66], see also [6] for a result concerning homeomorphism only.

Lemma 7.1. *Let D be an open subset of $\mathbb{R}^n$. Assume that $f : D \to \mathbb{R}^n$ is a Fréchet-differentiable map and the following condition holds:*

- *$\det f'(x) \neq 0$ for every $x \in D$.*

Then f is a local diffeomorphism.

If f is strictly differentiable the Clarke subdifferential reduces to a singleton, i.e. its strict derivative, see Proposition 2.2.4 page 33 [12]. Note that a Fréchet-differentiable function f need not necessarily yield that the Clarke derivative at x reduces to a singleton, namely to $\{f'(x)\}$, see the mentioned Example 2.2.3 p. 33 [12].

With the above, see [25].

Theorem 7.2. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Fréchet-differentiable mapping such that*

9A *for any $y \in \mathbb{R}^n$ the functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\varphi(x) = \frac{1}{2} \|f(x) - y\|^2$$

is coercive,

9B *for any $x \in \mathbb{R}^n$ we have* $\det f'(x) \neq 0$,

9C *$f : \mathbb{R}^n \to \mathbb{R}^n$ is strictly differentiable,*

then f is a diffeomorphism.

Proof. By Lemma 7.1 condition **9B** implies that f defines a local diffeomorphism. Thus it is sufficient to show that f is "onto" and "one to one".

Let us fix any point $y \in \mathbb{R}^n$. Since φ is a composition of a C^1 mapping and a strictly-differentiable mapping it is locally Lipschitz continuous by Theorem 2.3.10 (Chain rule) p. 45 [12]. Moreover, by the mentioned Chain rule and condition **9B** the Clarke subdifferential $\partial f(x)$ is equal to $\{(f(x) - y) \circ f'(x)\}$ for any $x \in \mathbb{R}^n$. Since φ is continuous and coercive it has an argument of a minimum $\overline{x}$, which, since φ is Fréchet-differentiable as a composition of a C^1 functional and a Fréchet-differentiable mapping, it satisfies the classical Fermat's rule, i.e.

$$(f(\overline{x}) - y) \circ f'(\overline{x}) = 0$$

which means that $0 = f'(\overline{x})^T \circ f(\overline{x}) - y)$, where ξ^T denotes the transpose of the matrix ξ. Since by **9B** $\det f'(\overline{x}) \neq 0$, we see that $f(\overline{x}) - y = 0$. Thus f is surjective.

Now we argue by contradiction that f is "one to one". Suppose there are x_1 and x_2, $x_1 \neq x_2$, $x_1, x_2 \in \mathbb{R}^n$, such that $f(x_1) = f(x_2) = a \in \mathbb{R}^n$. We will apply Theorem 7.1. We put $e = x_1 - x_2$ and define mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ and a locally Lipschitz functional $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = f(x + x_2) - a \text{ and } \psi(x) = \frac{1}{2} \|g(x)\|^2 .$$

Indeed, g is strictly differentiable and ψ is locally Lipschitz by the same arguments as φ is. Note that $\psi(e) = \psi(0) = 0$. By **9A** φ is coercive, so it satisfies the non-smooth Palais-Smale condition. The same conclusion holds for functional ψ.

Observe that $g(0) = g(e) = 0$. Consequently, since g is Fréchet-differentiable

$$g(0 + h) - g(0) = g'(0)h + o(h) = f'(x_2)h + o(h) \tag{7.1}$$

for $h \in \mathbb{R}^n$, where $\frac{o(h)}{||h||} \to 0$ when $h \to 0$. From **9B** there exists $\alpha_{x_2} > 0$ that $||f'(x_2)h|| \geq \alpha_{x_2}||h||$. Thus from (7.1) we see that

$$||g(h)|| + \frac{1}{2}\alpha_{x_2}||h|| \geq ||g(h)|| + ||o(h)|| \geq ||f'(x_2)h|| \geq \alpha_{x_2}||h||$$

for sufficiently small h such that $||o(h)|| \leq \frac{1}{2}\alpha_{x_2}||h||$. Thus, there exist $\rho \in (0, \|e\|)$, such that for all $x \in \overline{B(0,\rho)}$

$$||g(x)|| \geq \frac{1}{2}\alpha_{x_2}||x||. \tag{7.2}$$

By the classical Weierstrass Theorem ψ has an argument of a minimum over $\partial\overline{B(0,\rho)}$ which we denote by w and which is non-zero and satisfies

$$\psi(w) \geq \frac{1}{2}\left(\frac{1}{2}\alpha_{x_2}\rho\right)^2$$

by definition of ψ and by (7.2). Therefore

$$\inf_{\|x\|=\rho} \psi(x) \geq \psi(w) > 0 = \psi(e) = \psi(0). \tag{7.3}$$

Thus by Theorem 7.1 applied to $J = \psi$ we note that ψ has a critical point $v \neq 0$, $v \neq e$ and such that

$$\psi'(v) = (f(v + x_2) - a) \circ f'(v + x_2) = 0.$$

Since $\det f'(v + x_2) \neq 0$ we see that $f(v + x_2) - a = 0$. This means that $\psi(v) = 0$. By (7.3) we obtain that $\psi(v) = c \geq \psi(w) > 0$. The obtained contradiction shows that f is a "one to one" operator. ■

Now we provide some result which extends a bit a result of Katriel. Namely, see [25],

Corollary 7.1. *Let X, B be finite dimensional spaces. Assume that $f : X \to B$ is a C^1-mapping, $\eta : B \to \mathbb{R}_+$ is a C^1 functional and that the following conditions hold*

10A $(\eta(x) = 0 \Longleftrightarrow x = 0)$ *and* $(\eta'(x) = 0 \Longleftrightarrow x = 0)$*;*

10B *for any* $y \in B$ *the functional* $\varphi : X \to \mathbb{R}$ *given by the formula*

$$\varphi(x) = \eta(f(x) - y)$$

is coercive;

10C $\det f'(x) \neq 0$ *for any* $x \in X$*;*

10D *there exist positive constants* α*,* c*,* M *such that*

$$\eta(x) \geq c\,\|x\|^{\alpha} \text{ for } \|x\| \leq M.$$

Then f *is a diffeomorphism from* X *onto* B*.*

Remark 7.1. The notion of strict derivative (Hadamard-like) is not to be confused with the notion of Hadamard derivative which reads

$$\lim_{t\to 0^+} \frac{f(x_0 + tz) - f(x_0)}{t} = \langle f'(x_0), z\rangle \text{ for all } z \in \mathbb{R}^n$$

provided the convergence is uniform for z in compact sets. This notion coincides with the Fréchet-differentiability in finite dimensional spaces but it is irrelevant to our considerations, see remarks in [12] p. 30 concerning classical derivatives.

SECTION 7.1
Applications to algebraic equations

We conclude this section with some applications of Theorem 7.2 to the unique solvability of nonlinear equations of the form $Ax = F(x)$ where A is a nonsingular matrix and F is a C^1 nonlinear operator. We mention papers [75], [74] which concern existence and multiplicity of solutions to such problems by variational and also monotonicity tools.

In the following we consider the problem

$$Ax = F(x) + \xi, \tag{7.4}$$

where $\xi \in \mathbb{R}^n$ is fixed, A is an $n \times n$ matrix which is not positive definite, negative definite or symmetric; $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. We consider $\mathbb{R}^n$ with Euclidean norm in both theoretical results and the example which follows.

Note that when A is such as above one cannot apply even the simplest variational approach, i.e. the direct method relying on minimizing the Euler action functional

$$J(x) = \langle Ax, x \rangle - \mathcal{F}(x) - \langle \xi, x \rangle,$$

and where $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is the potential of F. The difficulties are due to the fact that term $\langle Ax, x \rangle$ need not be coercive nor anti-coercive. Moreover, uniqueness which we achieve, in the classical approach requires strict convexity of the action functional which is again an assumption rather demanding.

In order apply Theorem 7.2 to the solvability of (7.4) we need some assumptions. Let us recall that if A^* denotes the transpose of matrix A, then A^*A is symmetric and positive semidefinite. However, A^*A being positive semidefinite is not sufficient for our purposes. We assume what follows

11A Matrix A is nonsingular.

By assumption **11A** we see that matrix A^*A is positive definite with eigenvalues ordered as

$$0 < \lambda_1 \leq \cdots \leq \lambda_n.$$

Now we can state the following existence theorems.

Theorem 7.3. *Assume that **11A** holds, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function and that the following conditions hold:*

12A *There exists a constant $0 < a < \sqrt{\lambda_1}$ such that*

$$\|F(x)\| \le a\|x\|$$

for all sufficiently large $x \in \mathbb{R}^n$,

12B $\det(A - F'(x)) \neq 0$ *for every every $x \in \mathbb{R}^n$.*

Then problem (7.4) has exactly one solution for any $\xi \in \mathbb{R}^n$.

Proof. We need to show that assumptions of Theorem 7.2 are satisfied. We put $\varphi(x) = Ax - F(x)$. In order to demonstrate **9A** we see that for sufficiently large $x \in \mathbb{R}^n$

$$\begin{aligned}\|\varphi(x)\| &= \|Ax - F(x)\| \ge \|Ax\| - \|F(x)\| \\ &\ge \sqrt{\langle A^*Ax, x\rangle} - a\|x\| \ge \left(\sqrt{\lambda_1} - a\right)\|x\|.\end{aligned}$$

Hence the function φ is coercive. From **12B** it follows that condition **9B** is satisfied and **9C** is obviously satisfied. From Theorem 7.2 it follows that φ is a global homeomorphism and equation (7.4) has exactly one solution for any $\xi \in \mathbb{R}^n$. ■

Theorem 7.4. *Assume that **11A** holds, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function and that the following conditions hold:*

13A *There exists a constant $b > \sqrt{\lambda_n}$ such that*

$$\|F(x)\| \ge b\|x\|$$

for all sufficiently large $x \in \mathbb{R}^n$,

13B $\det(A - F'(x)) \neq 0$ *for every every $x \in \mathbb{R}^n$.*

Then problem (7.4) has exactly one solution for any $\xi \in \mathbb{R}^n$.

Proof. We put $\varphi_1(x) = F(x) - Ax$ and we observe that for sufficiently large $x \in \mathbb{R}^n$

$$\|\varphi_1(x)\| = \|F(x) - Ax\| \geq \|F(x)\| - \|Ax\| \geq \left(b - \sqrt{\lambda_n}\right) \|x\|.$$

Hence the function φ_1 is coercive and the assertion follows as in the proof of the above result. ■

Remark 7.2. We note that in order to get coercivity of function φ in Theorem 7.3 we can replace condition **12A** with the following assumption:

14A exist constants $\alpha > 0$, $0 < \gamma < 1$ such that

$$\|F(x)\| \leq \alpha \|x\|^\gamma$$

for all sufficiently large $x \in \mathbb{R}^n$.

Concerning Theorem 7.4 we can replace condition **13A** with the following assumption:

15A there exist constants $\beta > 0$, $\theta > 1$ such that

$$\|F(x)\| \geq \beta \|x\|^\theta$$

for all sufficiently large $x \in \mathbb{R}^n$.

Example 7.1. Consider an indefinite matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}$ and function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = \left(x^3 + y, 4x + y + y^3\right)$$

Consider on $\mathbb{R}^2$ the Euclidean norm, that is $\|(x, y)\| = \sqrt{x^2 + y^2}$. We recall that

$$\|(x, y)\| \leq 2^{\frac{1}{3}} \sqrt[6]{x^6 + y^6}$$

Note that

$$F(x,y) = \left(x^3, y^3\right) + (0, 4x) + (y, y).$$

Hence

$$\begin{aligned}\|F(x,y)\| &\geq \left\|\left(x^3, y^3\right)\right\| - \|(0,4x)\| - \|(y,y)\| = \sqrt{x^6 + y^6} - 4 \cdot |x| - \sqrt{2}\,|y| \\ &\geq \frac{1}{2}\,\|(x,y)\|^3 - 4\,(|x| + |y|) \geq \frac{1}{2}\,\|(x,y)\|^3 - 4\sqrt{2}\,\|(x,y)\|.\end{aligned}$$

Let

$$\varphi(x,y) = F(x,y) - A(x,y), \quad (x,y) \in \mathbb{R}^2.$$

Note that

$$\begin{aligned}\|\varphi(x,y)\| &\geq \|F(x,y)\| - \|A(x,y)\| \\ &\geq \frac{1}{2}\,\|(x,y)\|^3 - 4\sqrt{2}\,\|(x,y)\| - \|A\|\,\|(x,y)\| \\ &= \|(x,y)\|\left(\frac{1}{2}\,\|(x,y)\|^2 - \left(4\sqrt{2} + \|A\|\right)\right).\end{aligned}$$

From the last sequence of inequalities it results that φ is coercive.

One can easily see that F' has the following form

$$F'(x,y) = \begin{bmatrix} 3x^2+1 & 0 \\ 4 & 3y^2+1 \end{bmatrix},$$

for any $(x,y) \in \mathbb{R}^2$. Note also that

$$F'(x,y) - A = \begin{bmatrix} 3x^2+2 & 0 \\ 0 & 3y^2+3 \end{bmatrix}.$$

One can easily see that

$$\det\left(F'(x,y) - A\right) > 0.$$

Sometimes it is easier to prove coercivity of $\varphi(x,y) = F(x,y) - A(x,y)$ directly than to use the growth conditions on the nonlinear term. Moreover, when we prove the coercivity directly, there is no need to assume that A^*A is positive definite. Thus from the proof of Theorems 7.3 and 7.4 it follows that

Corollary 7.2. *Assume that*

- $\|Ax - F(x)\| \to \infty$ *as* $\|x\| \to \infty$*;*
- $\det(A - F'(x)) \neq 0$ *for every* $x \in \mathbb{R}^n$*.*

Then (7.4) has exactly one solution for any fixed $\xi \in \mathbb{R}^n$*.*

There is some easy motivation to consider the algebraic equations. Since some discrete problems can be written in a form of a nonlinear system, see for example [2], [74], we shall undertake the following problem

$$Au = f(u), \quad u \in \mathbb{R}^n \tag{7.5}$$

in case when the necessarily symmetric $n \times n$ matrix A need not be positive definite. We always assume that f has the following form $f = (f_1, f_2, ..., f_n)$ and

16A $f_i : \mathbb{R}^n \to R$ is continuous for $k = 1, 2, ..., n$ and $f_i(0) \neq 0$ for at least one $i = 1, 2, ..., n$.

We recall that a column of vector $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ is a solution if substitution of u into (7.5) renders it an identity. Moreover, 0 is not a solution to (7.5) due to **16A**.

System (7.5) can be treated as a representation of some discrete boundary value problem which in turn arises as discretization of some continuous models. Let us take for example the Emden-Fowler equation

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right) + t^{\delta}u^{\gamma} = 0$$

which originated in the gaseous dynamics in astrophysics and further was used in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemically reacting systems, see [73]. The discrete version of the generalized Emden-Fowler equation $(p(t)y')' + q(t)y = f(t,y)$ received some considerable interest lately mainly by the use of critical point theory, see for example [46], [30]. The discretization of the generalized Emden-Fowler type boundary value problem can be put as follows

$$\Delta\left(p\left(k-1\right)\Delta x\left(k-1\right)\right)+q\left(k\right)x\left(k\right)+f\left(k,x\left(k\right)\right)=0 \tag{7.6}$$

with boundary conditions

$$x\left(0\right)=x\left(n\right),\quad p\left(0\right)\Delta x\left(0\right)=p\left(n\right)\Delta x\left(n\right) \tag{7.7}$$

and where

$$f\in C\left(\left[1,n\right]\times\mathbb{R}\to\mathbb{R}\right),\quad p\in C\left(\left[0,n+1\right],\mathbb{R}\right),\quad q\in C\left(\left[1,n\right],\mathbb{R}\right)$$

$p\left(n\right)\neq 0$; $[a,b]$ for $a<b$, $a,b\in\mathbb{Z}$ denotes a discrete interval $\{a,a+1,...,b\}$; Δ is the forward difference operator defined by $\Delta u(k)=u(k+1)-u(k)$. The realization of the form of (7.5) requires the following matrices

$$M=\begin{bmatrix} p\left(0\right)+p\left(1\right) & -p\left(1\right) & 0 & \dots & -p\left(0\right) \\ -p\left(1\right) & p\left(1\right)+p\left(2\right) & -p\left(2\right) & \dots & 0 \\ 0 & -p\left(2\right) & p\left(2\right)+p\left(3\right) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -p\left(n-1\right) \\ -p\left(0\right) & 0 & 0 & \dots & p\left(n-1\right)+p\left(0\right) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -q(1) & 0 & 0 & \dots & 0 & 0 \\ 0 & -q(2) & 0 & \dots & 0 & 0 \\ 0 & 0 & -q(3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -q(n-1) & 0 \\ 0 & 0 & 0 & \dots & 0 & -q(n) \end{bmatrix}.$$

Setting $A = M + Q$, $f_k(x) = f(k, x)$ and using the assumption that $p(n) \neq 0$ we see that problem (7.6)-(7.7) has a form of a nonlinear system (7.5). Indeed, in this case there is a "one to one" correspondence between solutions to (7.5) and solutions to (7.6)-(7.7).

CHAPTER 8

Future research directions

It seems that our methods and research approaches are applicable also to some other problems. We may observe from what has been before that the scheme which is applied in this book can be used for solving nonlinear equations which are in some sense locally solvable. Then we can define a suitable functional and if we can determine that it has some mountain geometry then we can solve the equation globally. We would suggest three possible directions:

- Volterra Integral Equations;
- Second Order Dirichlet Problem for ODE together with their approximation;
- Invertibility of locally Lipschitz mappings and related implicit function theorem.

Now we describe in some detail possible problems and motivations for the above tasks.

SECTION 8.1

On Volterra equations

It is possible to investigate the nonlinear integral operator $V : \widetilde{W}_0^{1,p} \to \widetilde{W}_0^{1,p}$ defined pointwisely for all $t \in [0,1]$ by

$$V(x)(t) = x(t) + \int_0^t v(t,\tau,x(\tau))\, d\tau. \tag{8.1}$$

Thus V is considered with an initial condition

$$x(0) = 0. \tag{8.2}$$

We would focus on showing that V is a diffeomorphism under some conditions imposed on the nonlinear term v. This in turn ensures that the associated Volterra integral equation

$$x(t) + \int_0^t v(t,\tau,x(\tau))\, d\tau = y\,(t) \quad \text{for } t \in [0,1], \quad x(0) = 0, \tag{8.3}$$

is solvable for any $y \in \widetilde{W}_0^{1,p}$ and that the solution operator which assigns to each y the unique solution to (8.3) is of class C^1. In other words, we can say that solution to (8.3) depends in a C^1 manner on a functional parameter y. The proof would rely on a global diffeomorphism theorem 3.5 and on some ideas contained in Chapter 6 which followed [7] where spaces of functions integrable with square are considered. Since such spaces are Hilbert ones, the reasoning is of course much simpler. This is not the case with $p > 2$ and therefore several technical problems have to be overcome. Moreover, the global diffeomorphism theorem is more involving since it now uses a duality mapping relative to a normalization function t^{p-1} and not a square of a norm as is the case in the Hilbert space setting. The main technical difficulty is to demonstrate the Palais-Smale condition which is required by the global diffeomorphism theorem, even continuous differentiability of the functional under consideration is more demanding without a Hilbert space structure.

8.1.1 Suggested assumptions and foreseen main results

Let $P_\Delta = \{(t,\tau) \in [0,1] \times [0,1]; \tau \leq t\}$. We assume, following suggestions in [7], that function $v : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

17A **(i)** the function $v(\cdot,\tau,\cdot)$ is continuous on the set $G := [0,1] \times \mathbb{R}^n$ for a.e. $\tau \in [0,1]$;

(ii) there exists $v_t(\cdot,\tau,\cdot)$ continuous on G for a.e. $\tau \in [0,1]$;

(iii) there exists $v_x(\cdot,\tau,\cdot)$ continuous on G for a.e. $\tau \in [0,1]$;

(iv) there exists $v_{xt}(\cdot,\tau,\cdot)$ continuous on G for a.e. $\tau \in [0,1]$;

17B **(i)** the function $v(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$, there exist functions $c_1, d_1 \in L^p(P_\Delta, \mathbb{R}^+)$ such that

$$|v(t,\tau,x)| \leq c_1(t,\tau)|x| + d_1(t,\tau)$$

for a.e $(t,\tau) \in P_\Delta$, $x \in \mathbb{R}^n$ and

$$\left(1 - 2^{\frac{(p-1)}{p}} ||c_1||_{L^p(P_\Delta,\mathbb{R})}\right) > 0;$$

(ii) the function $v_t(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and there exist functions $c_2, d_2 \in L^p(P_\Delta, \mathbb{R}^+)$ such that

$$|v_t(t,\tau,x)| \leq c_2(t,\tau)|x| + d_2(t,\tau)$$

for a.e. $(t,\tau) \in P_\Delta$, $x \in \mathbb{R}^n$;

(iii) the function $v_x(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and locally bounded with respect to x;

(iv) the function $v_{xt}(t,\cdot,x)$ is measurable on $[0,1]$ for all $(t,x) \in G$ and locally bounded with respect to x.

Assumption **17A(i)** means that v is locally bounded with respect to x, i.e. for every $\rho > 0$ there exists $k_\rho > 0$ such that for $(t,\tau) \in P_\Delta$ and $x \in B_\rho = \{x \in \mathbb{R}^n : |x| \leq \rho\}$ we have $|v(t,\tau,x)| \leq k_\rho$. This follows by the growth condition and since x is absolutely continuous.

Our main results considering the existence and differentiability of the solution to Volterra operator would read as follows

Theorem 8.1. *Assume that conditions **17A**-**17B** hold. Then operator V defined by (8.1)-(8.2) is a diffeomorphism.*

Theorem 8.1 can be restated as follows.

Theorem 8.2. *Assume that conditions **17A**-**17B** hold. Then for any $y \in \widetilde{W}_0^{1,p}$ problem (8.3) has a unique solution which depends in a continuously differentiable manner on the parameter y or in other words, the solution operator is a diffeomorphism.*

Main problems to be overcome here are as follows:

- Then the operator $V : \widetilde{W}_0^{1,p} \to \widetilde{W}_0^{1,p}$ given by (8.1) is well defined.
- Fix $x, g \in \widetilde{W}_0^{1,p}$. Then the equation

$$h(t) + \int_0^t v_x(t,\tau,x(\tau))h(\tau)d\tau = g(t)$$

  for $t \in [0,1]$ has a unique solution $h \in \widetilde{W}_0^{1,p}$.
- V defined by (8.1) is continuously Fréchet-differentiable on $\widetilde{W}_0^{1,p}$ and its derivative reads

$$V'(\hat{x})h(t) = h(t) + \int_0^t v_x(t,\tau,\hat{x}(\tau))h(\tau)d\tau$$

  for $h \in \widetilde{W}_0^{1,p}$ for any $t \in [0,1]$.

- Continuous differentiability of

$$\begin{aligned}\varphi(x) &= \frac{1}{p}\|V(x)-y\|^p_{\widetilde{W}_0^{1,p}}\\ &= \frac{1}{p}\int_0^1 \left| x(t)-y(t)+\int_0^t v(t,\tau,x(\tau))\,d\tau \right|^p\,dt.\end{aligned}$$

- Functional φ satisfies the PS-condition for any fixed $y \in \widetilde{W}_0^{1,p}$.

SECTION 8.2

On a second order Dirichlet problem

We consider in $H_0^1(0,1) \cap H^2(0,1)$ solvability of the following Dirichlet problem

$$\begin{cases} \ddot{x}(t) = f(t, x(t)) + v(t), \\ x(0) = x(1) = 0, \end{cases} \tag{8.4}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function and $v \in L^2(0,1)$, together with its standard discretization suggested in [22] and in [42]. Such problems are well described in [47]. The idea of solving (8.4) is as expected via a global invertibility result and thus we investigate the classical solution operator T given (pointwisely) a.e. on $[0,1]$ by

$$(Tx)(\cdot) := \ddot{x}(\cdot) - f(\cdot, x(\cdot)),$$

acting from $H_0^1(0,1) \cap H^2(0,1)$ to $L^2(0,1)$.

As mentioned, we shall consider discretization also of (8.4) as follows. For a, b such that $a < b < \infty$, $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$ we denote $\mathbb{N}(a,b) = \{a, a+1, ..., b-1, b\}$. For a fixed $N \in \mathbb{N}$, $N \geqslant 2$, the non-linear difference equation with Dirichlet boundary conditions is given as follows

$$\begin{cases} \Delta^2 x(k-1) = \frac{1}{N^2} f\left(\frac{k}{N}, x(k)\right) + \frac{1}{N^2} v\left(\frac{k}{N}\right), \\ x(0) = x(N) = 0, \end{cases} \tag{8.5}$$

for $k \in \mathbb{N}(1, N-1)$. Here Δ is the forward difference operator, i.e. $\Delta x\,(k-1) = x\,(k) - x\,(k-1)$ and we see that $\Delta^2 x\,(k-1) = x\,(k+1) - 2x\,(k) + x\,(k-1)$.

Now we introduce the idea of a non-spurious solution. This reads as follows. Assume that both, continuous boundary value problem (8.4) and for each fixed $N \in \mathbb{N}$, $N \geqslant 2$, discrete boundary value problem (8.5), are uniquely solvable by, respectively $x^\star$ and $x_N = (x_N(k))_{k=0}^{N}$. Then, if v is at least continuous, solutions x_N of (8.5) converges to solution $x^\star$ of (8.4) in following sense

$$\lim_{N\to\infty} \max_{k\in\mathbb{N}(0,N)} \left| x^\star\left(\frac{k}{N}\right) - x_N(k) \right| = 0$$

Such solutions to discrete BVPs are called non-spurious. The spurious solutions may diverge or else may converge to anything else but the solution to a given continuous Dirichlet problem.

Example 8.1. The continuous problem

$$\begin{cases} \ddot{x}(t) + \frac{\pi^2}{n^2} x(t) = 0, \\ x(0) = x(n) = 0 \end{cases}$$

has an infinite number of solutions $x(t) = c \sin \frac{\pi t}{n}$ (c is arbitrary) whereas its discrete analogue $\Delta^2 x(k) + \frac{\pi^2}{n^2} x(k) = 0$, $x(0) = x(n) = 0$ has only one solution $x(k) \equiv 0$. The problem $\ddot{x}(t) + \frac{\pi^2}{4n^2} x(t) = 0$, $x(0) = 0$, $x(n) = 1$ has only one solution $x(t) = \sin \frac{\pi t}{2n}$, and its discrete analogue $\Delta^2 x(k) + \frac{\pi^2}{4n^2} x(k) = 0$, $x(0) = 0$, $x(n) = 1$ also has one solution. The continuous problem $\ddot{x}(t) + 4\sin^2 \frac{\pi}{2n} x(t) = 0$, $x(0) = 0$, $x(n) = \varepsilon \neq 0$ has only one solution

$$x(t) = \varepsilon \frac{\sin\left(\left(2\sin\frac{\pi}{2n}\right) t\right)}{\sin\left(\left(2\sin\frac{\pi}{2n}\right) n\right)},$$

whereas its discrete analogue $\Delta^2 x(k) + 4\sin^2 \frac{\pi}{2n} x(k) = 0$, $x(0) = 0$, $x(n) = \varepsilon \neq 0$ has no solution.

The definition of a non-spurious solution which we employ follows from paper [62] and is given as in [27]. The existence of a non-spurious solutions have been considered by variational methods in [27] while previously there had been some research in this case addressing mainly problems whose solutions where obtained by the fixed point theorems and the method of lower and upper solutions, [63], [69].

Main problems to be overcome here are as follows:

- Proper functional setting: typically variational second order problems are considered in H_0^1.
- Denote $L^2 := L^2(0,1)$. H^2 denotes space of those functions form H^1 for which $\dot{x} \in H^1$. We define $H_0^1 := \{x \in H^1 : x(0) = x(1) = 0\}$. The following inequalities hold for any $x \in H_0^1$, see [32],

  $$\|x\|_\infty \leq \|\dot{x}\|_{L^2}, \qquad \|x\|_{L^2} \leq \tfrac{1}{\pi}\|\dot{x}\|_{L^2}.$$

  What is the relation between norms in $H^2 \cap H_0^1$ and in H_0^1
- What are the properties of the following functional $\varphi : H^2 \cap H_0^1 \to \mathbb{R}$ by

  $$\varphi(x) := \frac{1}{2}\|Tx - y\|_{L^2}^2 = \frac{1}{2}\int_0^1 |\ddot{x}(t) - f(t, x(t)) - y(t)|^2 dt.$$
- Solvability of the related discrete boundary value problem and the bound on its solution in a uniform manner.

SECTION 8.3

On invertibility of locally Lipschitz mappings

The finite dimensional version of the Hadamard-Lévy theorem, recalling the following result

Let E, F be two Banach spaces and $f : E \to F$ be a local diffeomorphism of class C^1 which satisfies the following integral condition

$$\int_0^\infty \min_{\|x\|=r} \left\| f'(x)^{-1} \right\|^{-1} dr = \infty.$$

Then f is a global diffeomorphism

was extended to locally Lipschitz functions by Pourciau, see [58], [59]. This extension as expected involves a finite dimensional setting since as mentioned before, one cannot expect any relevant results in infinite dimensional Banach spaces. If A is a square matrix we denote $[A] = \inf_{\|u\|=1} \|Au\|$.

Theorem 8.3 (Pourciau's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz function and suppose that the generalized Jacobian $\partial f(x)$ is of full rank for every $x \in \mathbb{R}^n$. Let $m(t) = \inf_{\|z\| \leq t} [\partial f(z)] = \inf_{\|z\| \leq t} \inf_{A \in \partial f(z)} [A]$ and suppose that*

$$\int_0^\infty m(t)\, dt = +\infty$$

Then f is a bijective function and the inverse of f, that is f^{-1}, is a locally Lipschitz function.

Based on our earlier motivation we are concerned with generalization of our previous results to the case of a locally Lipschitz setting as suggested by our preliminary results contained in [28]. Due to explanation contained in [61] we must resort to finite dimensional setting since there are no contained local results in the infinite one. We will provide conditions for the existence of a global implicit function for the equation $F(x, y) = 0$, where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a locally Lipschitz functions. The following theorem is a finite dimensional counterpart of the main result given in [35].

Theorem 8.4. *Assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1 mapping such that:*

18A *for any $y \in \mathbb{R}^m$ the functional $\varphi_y : \mathbb{R}^n \to \mathbb{R}$ given by the formula*

$$\varphi_y(x) = \frac{1}{2}\|F(x,y)\|^2$$

is coercive, i.e. $\lim_{\|x\| \to \infty} \varphi_y(x) = +\infty$;

18B *the Jacobian matrix $F_x(x,y)$ is bijective for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$.*

Then there exists a unique function $f : \mathbb{R}^m \to \mathbb{R}^n$ such that equations $F(x,y) = 0$ and $x = f(y)$ are equivalent in the set $\mathbb{R}^n \times \mathbb{R}^m$, in other words $F(f(y), y) = 0$ for any $y \in \mathbb{R}^m$. Moreover, $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$.

We see that using the following local result which is a recent one and which shows (in the application contained in the source mentioned) that such results are of some interest.

Theorem 8.5. *[68]Assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz mapping in a neigbourhood of a point (x_0, y_0) such that $F(x_0, y_0) = 0$. Assume further that $\partial_x F(x_0, y_0)$ is of maximal rank. Then there exists a neighborhood $V \subset \mathbb{R}^m$ of y_0 and a Lipschitz function $G : V \to \mathbb{R}^n$ such that for every y in V it holds $F(G(y), y) = 0$ and $G(y_0) = x_0$.*

one can obtain the locally Lipschitz counterpart of the above result. Problems appearing here are as follows:

- Finding proper chain formula. There is one chain formula which is commonly used for differentiation in the sense of Clarke. But it requires that the outer function is Clarke differentiable and the inner function is continuously differentiable.
- Clarke differentiability of

  $$\varphi_y(x) = \frac{1}{2}\|F(x,y)\|^2$$

  when F is locally Lipschitz.

- Verify the mountain geometry for a locally Lipschitz functional. It is much more difficult in this case due to fact that the intermediate value theorem works differently and also one lacks the nice Taylor expansion methods.
- Extending application to algebraic equation to this setting.
- Relation with existing results.

Appendix

In this appendix we remark on solving by using a critical point theory to Dirichlet problem. We will show how classical method works and describe possible problems which would appear in the application of Theorem 1.2 This shows that method provided in Theorem 1.2 must be suitably amended in the future in order to make it more applicable.

Problem on the finite interval

We firstly consider

$$\begin{cases} \ddot{x}(t) = f(t, x(t)), \\ x(0) = x(1) = 0, \end{cases} \tag{A.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function. We formulate such a problem in order to compare with solutions obtained with the aid of Theorem 1.2 and to describe how variational methods work in the classical setting. We describe at the end difficulties which would appear in investigating this problem within the global invertibility framework. This means that with problem under consideration we must associate the Euler action functional, prove that this functional is weakly lower semicontinuous in a suitable function space, coercive and at least Gâteaux differentiable. Given this three conditions one knows that at least a weak solution to problem under consideration exists whose regularity can further be improved with known tools. Such a scheme, commonly used within the critical point theory is well described in the first chapters of [47].

The solutions to (A.1) will be investigated in the space $H_0^1([0,1],\mathbb{R})$ consisting, as we recall, of absolutely continuous functions satisfying the boundary conditions and with a.e. derivative being integrable with square. Such a $H_0^1([0,1],\mathbb{R})$ solution is called a weak one, see [47], i.e. a function $x \in H_0^1([0,1],\mathbb{R})$ is a weak $H_0^1([0,1],\mathbb{R})$ solution to (A.1), if

$$\int_0^1 \dot{x}(t)\dot{v}(t)\ dt + \int_0^1 f(t,x(t))\,v(t)\ dt = 0 \tag{A.2}$$

for all $v \in H_0^1([0,1],\mathbb{R})$. In order to obtain (A.2) one multiplies the given equation by a test function and next integration is performed.

The application of variational methods allows one to obtain only weak solutions which are easy to obtain with classical tools. The question arises what is the relation between equation (A.1) and its weak solution. It can be described as follows by introducing the notion of a classical solution.

The classical solution to (A.1) is then defined as a function $x : [0,1] \to \mathbb{R}$ belonging to $H_0^1([0,1],\mathbb{R})$ such that $\ddot{x}$ exists a.e. and $\ddot{x} \in L^1([0,1],\mathbb{R})$. Since f is jointly continuous, then it is known from the Fundamental Theorem of the Calculus of Variations, see [47], that x is in fact twice differentiable with classical continuous second derivative. Thus $x \in H_0^1([0,1],\mathbb{R}) \cap C^2([0,1])$. This means that any weak solution is in a fact a classical one and that is why we look only for weak solutions getting at the same time classical ones.

Let

$$F(t,x) = \int_0^x f(t,s)\ ds$$

for $(t,x) \in [0,1] \times \mathbb{R}$. We introduce the following action functional $J : H_0^1([0,1],\mathbb{R}) \to \mathbb{R}$ given by

$$J(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2\,dt + \int_0^1 F(t,x(t))\ dt. \tag{A.3}$$

Let us examine J for a while. Due to the continuity of f functional J is well defined which means that $J(x) < +\infty$ for any $x \in H_0^1([0,1],\mathbb{R})$. Recall that the norm in $H_0^1([0,1],\mathbb{R})$ reads

$$\|x\| = \sqrt{\int_0^1 |\dot{x}(t)|^2\, dt}.$$

Then we see that

$$x \mapsto \frac{1}{2}\int_0^1 |\dot{x}(t)|^2 dt = \frac{1}{2}\|x\|^2$$

is a C^1 functional by standard facts. Its derivative is a functional on $H_0^1([0,1],\mathbb{R})$ which reads at a fixed $x \in H_0^1([0,1],\mathbb{R})$

$$v \mapsto \int_0^1 \dot{x}(t)\,\dot{v}(t)\, dt$$

for any $v \in H_0^1([0,1],\mathbb{R})$. Concerning the nonlinear part we see that for any fixed $v \in H_0^1([0,1],\mathbb{R})$ (which is continuous of course) function

$$\varepsilon \mapsto \int_0^1 F(t, x(t) + \varepsilon v(t))\ dt$$

considered on $(0,1)$ (where the integral we can treat as the Riemann one) due to the Leibnitz differentiation formula under integral sign is C^1 and the derivative of

$$x \mapsto \int_0^1 F(t, x(t))\ dt$$

is a functional on $H_0^1([0,1],\mathbb{R})$ which reads

$$v \mapsto \int_0^1 f(t, x(t))\, v(t)\ dt$$

if we recall that

$$F(t,x) = \int_0^x f(t,s)\ ds,$$

compare this with equation (A.2). Since the above is obviously continuous in x uniformly in v form unit sphere, we see that J given by (A.3) is in fact C^1. This procedure is common in obtaining derivatives of integral functionals, see [70]. Now we describe how to link solutions to (A.1) with critical points to J. We see that a derivative of J calculated at any point $x \in H_0^1([0,1],\mathbb{R})$ reads

$$J'(x)\,v = \int_0^1 \dot{x}(t)\,\dot{v}(t)\;dt + \int_0^1 f(t,x(t))\,v(t)\;dt$$

for all $v \in H_0^1([0,1],\mathbb{R})$. Thus equating $J'(x)\,v = 0$ we obtain that a critical point to J, i.e. a point satisfying $J'(x)\,v = 0$ for all $v \in H_0^1([0,1],\mathbb{R})$ is a weak solution to (A.1) and thus a classical one.

Recall also Poincaré inequality

$$\int_0^1 |x(t)|^2\,dt \le \frac{1}{\pi^2}\int_0^1 \dot{x}^2(t)\;dt$$

and Sobolev's one

$$\max_{t\in[0,1]} |x(t)| \le \int_0^1 |\dot{x}(t)|^2\,dt.$$

We sum up the assumptions on the nonlinear term in (A.1) since in order to get the above mentioned observations continuity of f is sufficient. We assume that

A1 $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t,0) \neq 0$ for $t \in [0,1]$;

A2 f is nondecreasing in x for all $t \in [0,1]$.

We recall here a bit about convex functions, see [70]. Since f is nondecreasing in x by **A1** and since f is a derivative of F it follows that F is convex in x for all $t \in [0,1]$. This means by the definition of convexity that for all $u, v \in \mathbb{R}$ and all $t \in [0,1]$ we have

$$F(t,v) - F(t,u) \ge F'(t,u)\,(v-u)\,.$$

Since $F'(t,u) = f(t,u)$ we see that the above inequality reads

$$F(t,v) - F(t,u) \geq f(t,u)(v-u) \tag{A.4}$$

for all $u, v \in \mathbb{R}$ and all $t \in [0,1]$. Note also that this an important property of convex functionals that pointwise convexity of an integrand provides convexity of the integral functional. This means that inequality (A.4) provides at once its integral counterpart and convexity of F implies that of

$$x \mapsto \int_0^1 F(t, x(t))\ dt.$$

Proposition A.1. *Assume that* ***A1*** *and* ***A2*** *are satisfied. Then problem (A.1) has exactly one nontrivial solution.*

Proof. Firstly, we consider the existence part. Note that by the classical Weierstrass Theorem there exists $c > 0$ such that

$$|f(t,0)| \leq c \tag{A.5}$$

for all $t \in [0,1]$. Since $F(t,0) = 0$ for all $t \in [0,1]$, $F'(t,u) = f(t,u)$ we obtain taking $v = x$, $u = 0$ from (A.4) and from estimation (A.5) the following inequality

$$F(t,x) = F(t,x) - F(t,0) \geq f(t,0)\,x \geq -|f(t,0)\,x| \geq -c\,|x| \tag{A.6}$$

which is valid for any x and all for all $t \in [0,1]$. We observe that from (A.6) we get

$$F(t,x) \geq -c\,|x| \tag{A.7}$$

for all $t \in [0,1]$ and all $x \in \mathbb{R}$. We see by Schwartz and Poincaré inequalities that

$$\int_0^1 |x(t)|\ dt \leq \sqrt{\int_0^1 |x(t)|^2\,dt} \leq \frac{1}{\pi}\,\|x\|\,.$$

Integrated the both sides of (A.7) for any $x \in H_0^1([0,1],\mathbb{R})$ we obtain

$$\int_0^1 F(t,x(t))\ dt \geq -c\int_0^1 |x(t)|\ dt \geq -\frac{c}{\pi}\|x\|.$$

Therefore

$$J(x) \geq \frac{1}{2}\|x\|^2 - \frac{c}{\pi}\|x\|. \tag{A.8}$$

Hence from (A.8) we obtain that J is coercive. Note that $x \mapsto \frac{1}{2}\|x\|^2$ is obviously w.l.s.c. on $H_0^1([0,1],\mathbb{R})$. Next, by the Arzela-Ascoli Theorem and Lebesgue Dominated Convergence, see these arguments in full detail in the proof of [47, Theorem 1.1], we see that

$$x \mapsto \int_0^1 F(t,x(t))\ dt$$

is weakly continuous. Thus J is weakly l.s.c. as a sum of a w.l.s.c. and weakly continuous functionals. Since J is C^1 and convex functional it has exactly one argument of a minimum which is necessarily a critical point and thus a solution to (A.1). Putting $x = 0$ in (A.1) one see that we have a contradiction, so any solution is nontrivial. ■

Remark A.1. In order to get the existence of nontrivial solution to (A.1) it would suffice to assume that $f(t_0,0) \neq 0$ for some $t_0 \in [0,1]$. Moreover, there is another way to prove the weak lower semincontinuity of J, namely show that J is continuous. Then it is weakly l.s.c. since it is convex. However, in proving continuity of J on $H_0^1([0,1],\mathbb{R})$ one uses the same arguments.

Remark A.2. As we saw from the previous consideration the application of Theorem 1.2 would require the investigation of the following functional $\varphi : H^2([0,1],\mathbb{R}) \cap H_0^1([0,1],\mathbb{R}) \to \mathbb{R}$ by

$$\varphi(x) := \frac{1}{2}\|Tx\|_{L^2}^2 = \frac{1}{2}\int_0^1 |\ddot{x}(t) - f(t,x(t))|^2\, dt.$$

In contrast to the already described method we would need to look for solutions in $H^2([0,1],\mathbb{R}) \cap H_0^1([0,1],\mathbb{R})$, thus we already obtain the existence of a classical solution by the properties of the space in which it is considered. However, assumptions leading to the existence of solution seem insufficient for our purposes. However, as asserted before, it would be possible to use Theorem 1.2 for our case also with some additional assumptions. Indeed, concerning the examples of nonlinear terms any nondecreasing f is of order bounded or unbounded, see

(i) $f(t,x) = g(t)\exp\left(x - t^2\right)$,

(ii) $f(t,x) = g(t)\arctan(x)$,

(iii) $f(t,x) = g(t)\,x^3 + \exp\left(x - t^2\right)$,

where g is any lower bounded continuous function with positive values. Thus we see that with examples (i) and (iii) would not get coercivity of φ. Therefore there is no direct link between coercivity of J and φ. However, restricting the growth would help us obtain coercivity of φ. The main problem here is the convexity of F. If only F is bounded from below then by coercivity of the norm, we get coercivity of J. As concerns φ coercivity would rather be reached by investigating coercivity of the term

$$\frac{1}{2}\sqrt{\int_0^1 |\ddot{x}(t)|^2\,dt} - \frac{1}{2}\sqrt{\int_0^1 |f(t,x(t))|^2\,dt}$$

which obviously requires that f is restricted in growth.

Problem on the infinite interval

Symbol $L^p([0,+\infty),\mathbb{R})$ for $p \geq 1$ means the space of such measurable real valued functions defined on $[0,+\infty)$ that

$$\int_0^\infty |u(t)|^p\ dt < +\infty.$$

We say that $u \in H_0^1([0,+\infty),\mathbb{R})$ if $u \in L^2([0,+\infty),\mathbb{R})$ and if there exists a function $g \in L^2([0,+\infty),\mathbb{R})$, called a weak derivative and such that

$$\int_0^{+\infty} u(t)\dot{\varphi}(t)\,dt = -\int_0^{+\infty} g(t)\varphi(t)\,dt$$

for all $\varphi \in C_c^\infty([0,+\infty),\mathbb{R})$, where $C_c^\infty([0,+\infty),\mathbb{R})$ is the space of compactly supported functions from $C^\infty([0,\ +\infty),\mathbb{R})$. We denote $\dot{u} := g$. We endow the space $H_0^1([0,+\infty),\mathbb{R})$ with its natural norm

$$\|u\| = \left(\int_0^{+\infty} |u(t)|^2\,dt + \int_0^{+\infty} |\dot{u}(t)|^2\ dt\right)^{\frac{1}{2}},$$

associated with the scalar product

$$\langle\, u \,|\, v \,\rangle = \int_0^{+\infty} u(t)v(t)\,dt + \int_0^{+\infty} \dot{u}(t)\dot{v}(t)\,dt.$$

Assume that $f : [0,\ +\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. In the space $H_0^1([0,+\infty),\mathbb{R})$ we consider the following Dirichlet problem

$$\begin{cases} -\ddot{u}(t) + u(t) = f(t,u(t)), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{A.9}$$

We investigate solutions to (A.9) as critical points to the Euler action functional $J : H_0^1([0,+\infty),\mathbb{R}) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_0^{+\infty} |\dot{u}(t)|^2\ dt + \frac{1}{2}\int_0^{+\infty} |u(t)|^2\,dt - \lambda\int_0^{+\infty} F(t,u(t))\,dt \tag{A.10}$$

where as always

$$F(t,u) = \int_0^u f(t,s)ds.$$

Let $p : [0,+\infty) \to (0,+\infty)$ be a continuously differentiable and bounded function such that $M = 2\max\left(\|p\|_{L^2}, \|\dot{p}\|_{L^2}\right) < +\infty$.

In order to have term

$$\int_0^{+\infty} F(t, u(t))\, dt$$

well defined we assume that

A3 for any constant $r > 0$ there exists a nonnegative function h_r for which $\frac{h_r}{p} \in L^1([0, +\infty, \mathbb{R})$ such that

$$\sup_{|y| \leq r} \left| f\left(t, \tfrac{y}{p(t)}\right) \right| \leq h_r(t)$$

for a.e. $t \in [0, \infty)$.

As it is common with variational problems for O.D.E. (A.9) admits two types of solutions, namely a weak and a classical one. Function $u \in H_0^1([0, +\infty), \mathbb{R})$ is a weak solution of (A.9) if

$$\int_0^{+\infty} \dot{u}(t)\dot{v}(t)\, dt + \int_0^{+\infty} u(t)v(t)\, dt - \int_0^{+\infty} f(t, u(t))v(t)\, dt = 0$$

for all $v \in H_0^1([0, +\infty), \mathbb{R})$ Function $u \in H_0^1([0, +\infty), \mathbb{R})$ is a classical solution to (A.9) if both u and $\dot{u}$ are locally absolutely continuous functions on $[0, +\infty)$,

$$-\ddot{u}(t) + u(t) = f(t, u(t))$$

for a.e. $t \in [0, \infty)$ and the boundary conditions $u(0) = u(+\infty)$ are satisfied. We would like to recall that any function $u \in H_0^1([0, +\infty), \mathbb{R})$ is locally absolutely continuous, i.e. absolutely continuous on any closed bounded interval contained in $[0, +\infty)$ however it is not in general absolutely continuous on the whole half line which makes the problem different from the classical bounded one. Another difference is that we lack now the Poincaré inequality, which means that the term

$$\frac{1}{2}\int_0^{+\infty} |\dot{u}(t)|^2\, dt + \frac{1}{2}\int_0^{+\infty} |u(t)|^2\, dt$$

is responsible for coercivity of the norm. Thus whatever assumptions we impose on f, we would never get coercivity of the corresponding functional φ in an easy and direct manner. Thus we think that in order to apply Theorem 1.2 one need to have two factors: Poincaré inequality and some regularity of solutions.

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