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P_4-DECOMPOSABILITY IN REGULAR

GRAPHS AND MULTIGRAPHS

David J. Mendell

24 Pages August 2014

The main objective of this thesis is to review and expand the study of graph decomposability. An H-decomposition of a graph $G = (V, E)$ is a partitioning of the edge set, E, into edge-disjoint isomorphic copies of a subgraph H. In particular we focus on the decompositions of graphs into paths. We prove that a 2,4 mutligraph with maximum multiplicity 2 admits a C_2,C_3-free Euler tour (and thus, a P_4-decomposition if it has size a multiple of 3) if and only if it avoids a set of 15 forbidden structures. We also prove that a 4-regular multigraph with maximum multiplicity 2 admits a P_4 decomposition if and only if it has size a multiple of 3 and no three vertices induce more than 4 edges. We go on to outline drafted work reflecting further research into P_4 decomposition problems.

P_4-DECOMPOSABILITY IN REGULAR GRAPHS AND MULTIGRAPHS

DAVID J. MENDELL

A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Degree of

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P_4-DECOMPOSABILITY IN REGULAR GRAPHS AND MULTIGRAPHS

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D. J. M.

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CHAPTER I

INTRODUCTION

Definitions

We include, for the sake of completeness, a thorough list of definitions. Following in the conventions of West [13] we define a graph, G, to be a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation associating with each edge two vertices, which we will call the endpoints of the edge. For simplicity we will often refer to edges and vertices as being in G. If two vertices form the endpoints of some edge in G, we will call them adjacent. For a subset V of $V(G)$ we refer to the subset of $E(G)$ that contains all edges with both endpoints in V as the set of edges induced by V. By the degree of a vertex we will mean the number of edges for which the vertex is an endpoint. If the degree of each vertex in G is precisely r for some positive integer r we say that G is r-regular. If the degree of every vertex in G is either s or t for some positive integers s and t we will refer to G as an s,t graph. In particular we will look closely at $2,4$ graphs which have the quality that every vertex is an endpoint on either 2 or 4 edges.

For our purposes the endpoints of any edge will always be distinct, which is to say that G contains no loops. When G contains at least one pair of vertices that forms the endpoints of at least two edges, we say that G contains multiple edges and we will refer to G as a multigraph. G is called simple if it contains no loops and no multiple edges. We will define maximum multiplicity in a multigraph as the size of the largest subset of $E(G)$ that all share precisely the same endpoints. We shall hereafter deal only with finite graphs, which is to say that the vertex set $V(G)$ and the edge set $E(G)$ are both finite. We refer to the number of edges in G, $|E(G)|$, as the size of G. We also refer to the number of vertices in G, $|V(G)|$, as the order of G. When a graph H has the qualities that: $V(H)$ is a subset of $V(G)$, $E(H)$ is a subset of $E(G)$, and the endpoints associated with the edges in H are the same as those associated to them in G, we refer to H as a subgraph of G and say that G contains H. For our purposes, a decomposition of a graph or multigraph, G, is a parition of the edge set, $E(G)$ into subgraphs H_1, H_2, $\ldots$, H_k which are edge-disjoint. If each such subgraph is isomorphic to a graph H, then we say that the decomposition is an H-decomposition, and say that G admits an H-decomposition.

A trail in G is an alternating sequence of vertices and edges of G, $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ such that for $1 \leq i \leq k$ the edge e_i has as endpoints the vertices v_{i-1} and v_i, and such that no edge appears more than once in the sequence. We say such a trail is closed if $v_0 = v_k$. A graph G is referred to as Eulerian if there exists a closed trail that contains all of the edges in $E(G)$. Such a trail will be referred to as an Eulerian circuit or an Eulerian tour.

By a k-cycle for integers $k \geq 2$, we will mean a graph or subgraph that contains precisely k edges and k vertices, and whose vertices can be drawn on a circle such that they are adjacent if and only if they are drawn consecutively on the circle. We will often refer to a k-cycle by the standard notation C_k. Note that a 2-cycle is necessarily formed by a pair of vertices with two edges connecting them to one another, and as such can only occur in a multigraph. We will refer to an Eulerian circuit as triangle-free or C_3-free if no segment of the sequence, i.e. $\ldots, v_i, e_i, v_{i+1}, e_{i+1}, v_{1i+2}, e_{i+2}, \ldots$ forms a 3-cycle or triangle. Similarly, we will refer to an Eulerian circuit as C_2-free if no two consecutive edges in the tour have precisely the same endpoints. By a d-path or a path of length d, which we will often denote as P_{d+1} we mean a trail with d edges that is such that it can be drawn on a line in a plane such that two vertices are adjacent if and only if they are adjacent on that line. In particular we will be concerned with P_4, a graph with four vertices, two of degree two and two of degree one, along with three edges.

Background

There are many and varied decomposition problems, those that deal with some partition of the set of edges, studied by mathematicians in graph theory. In general, Holyer conjectured and it was shown by Dor and Tarsi [3], that determining if there is an H-decomposition of a graph, G is NP-complete whenever H has size greater than 2. However, many results exist for special cases, regular graphs, etc.

We begin our discussion of decompositions with some classic theorems and motivating conjectures. It is known that the complete graph K_{2n} can be decomposed into n isomorphic copies of P_{2n}. Classically, Lucas showed that the complete graph K_{2n+1} admits a decomposition into n isomorphic copies of C_{2n+1} (see [6]). As mentioned, we look in particular at P_4-decompositions. One famous decomposition conjecture that has driven large quantities of research is due to Ringel [11].

Conjecture 1 (Ringel)**.** *Any fixed tree T with m edges decomposes K_{2m+1}.*

We don't directly address Ringel's conjecture but the results presented here are part of a larger vein of study that is highly related. Ringel's conjecture is a special case of the conjecture by Graham and Häggkvist [5].

Conjecture 2 (Graham and Häggkvist)**.** *Every $2n$-regular graph is decomposable by any tree with n edges.*

Bondy [1] first asked which simple graphs admit decompositions into paths of fixed length. It is known from Kotzig [8] that a simple graph admits a decomposition into paths of length 2 if and only if each connected component has even size. Bouchet and Fouquet [2] proved that a 3-regular graph G admits a P_4 decomposition if and only if G has a one-factor (a subgraph containing all vertices of G in which every vertex has degree precisely 1).

We concern ourselves here with decompostition into paths of length 3 (that is, into copies of P_4). In a remarkable result, Thomassen [12] showed that a 171-edge-connected graph (one that remains connected whenever fewer than 171 edges are removed) admits a decomposition into paths of length 3, if and only if it has size equivalent to 0 modulo 3.

In our work we sometimes study multigraphs of limited multiplicity. Mekkia and Mahéo [9] gave necessary and sufficient conditions for the existence of a 2-edge path-decomposition of a multigraph with maximum multiplicty two. Bondy [1] showed that if G is 3-regular, then the multigraph obtained by doubling the edges of G admits a P_4-decomposition.

Here we look to the problem of decomposing 2,4 multigraphs and 4-regular multigraphs with $\mu \leq 2$, into isomorphic copies of P_4, a path of three edges. Heinrich, Liu, and Yu showed [7] that for a 4-regular simple graph to admit a P_4-decomposition the obvious necessary condition that the graph size is a multiple of 3 is also sufficient, in part by showing that a 2,4 simple graph admits a triangle-free Euler tour precisely when it is not a member of the specified set of graphs, depicted in Figure 1. We adopt the conventions of the cited paper, where H refers to the remainder of the graph, which can vary.

We extend this result to multigraphs of maximum multiplicity two, and look towards future work with graphs of higher degree. We first prove that a 2,4 multigraph with maximum multiplicity two admits a C_2, C_3-free Euler tour if and only if it avoids a set of 15 forbidden structures, depicted in Figure 2, which is given in Chapter 2. Second, we show that a 4-regular multigraph with maximum

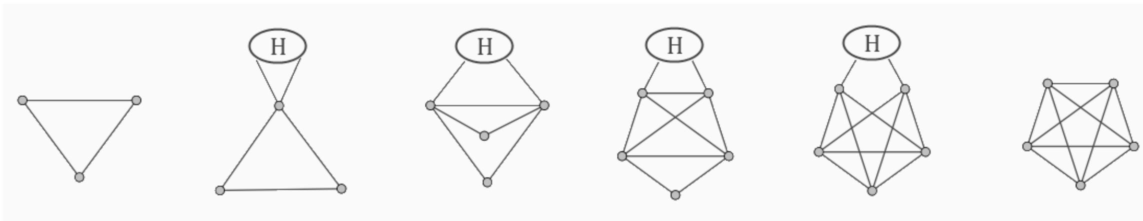

Figure 1: Forbidden structures for triangle-free Euler tours in simple graphs.

multiplicity two admits a P_4 decomposition if and only if no three vertices induce more than four edges.

CHAPTER II

2,4 GRAPHS AND MULTIGRAPHS

This following result provides us with a very convenient method for establishing a sufficiency argument for P_4-decomposability for the stated class of graphs:

Theorem 1 (Heinrich et al.)**.** *A connected* $2,4$ *simple graph admits a triangle-free Euler tour if and only if it avoids the set of structures given in Figure 1*

Clearly, if the number of edges in such a graph is congruent to 0 mod 3, then such a triangle-free Euler tour can be segmented in a straighforward way into isomorphic copies of P_4. We extend that result to multigraphs of multiplicity no more than 2.

Theorem 2. *A connected* $2,4$ *multigraph with maximum multiplicity two admits a triangle-free, C_2-free Euler tour if and only if it avoids the set of 15 structures, F, depicted in Figure 2*

Proof. Let G be a connected $2,4$ multigraph with multiplicity $\mu \leq 2$, such that G is not in F. It is easy to check that for G to admit a C_2,C_3-free Euler tour, G must indeed avoid all 15 structures in F, as any Euler trail through any graph in F must contain some three edges that form a triangle or some two edges that form a 2-cycle. To show sufficiency, we proceed by induction on the number of parallel edge pairs in G. Clearly, the base case when there are no parallel edges follows from Theorem 1.

The general method we would like to use is as follows. Suppose G has precisely one pair of parallel edges, say between vertices u and v. Then we construct $G*$ by

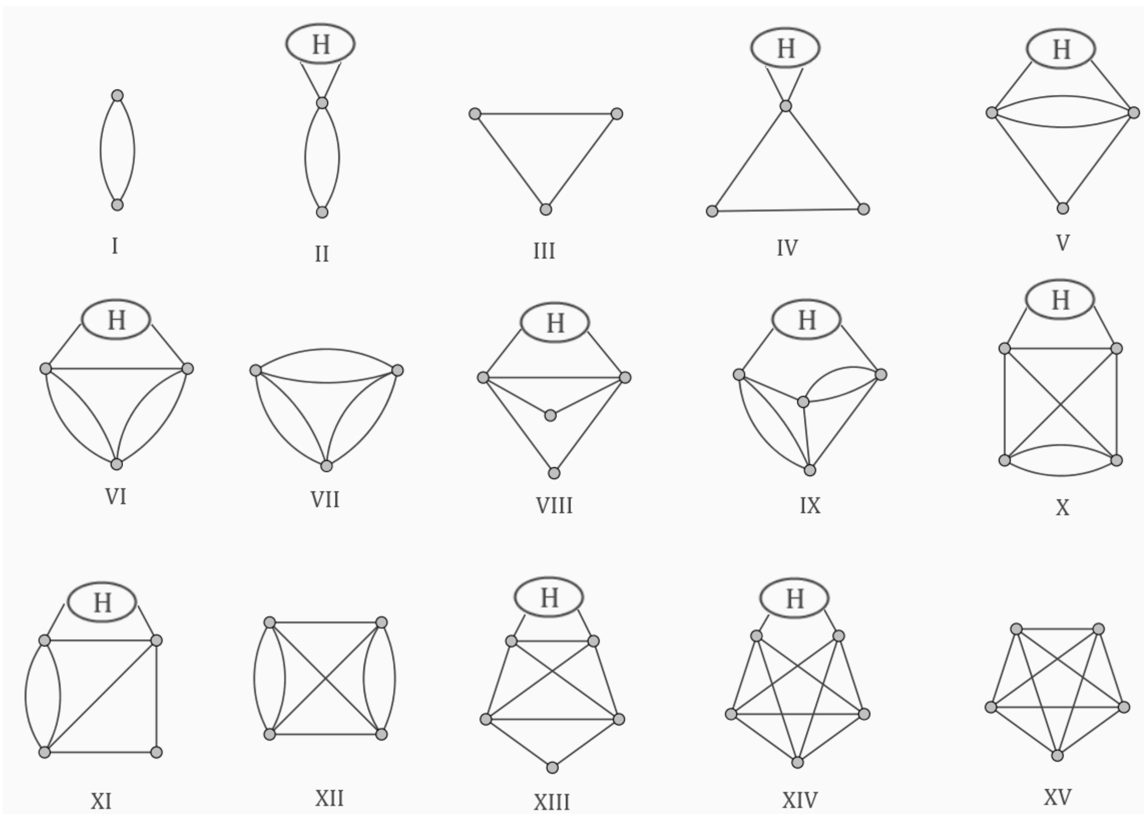

Figure 2: Set F of structures preventing C_2,C_3-free Euler tours.

subdividing one of the u, v edges by adding a vertex, w. Note that doing so cannot result in a $G*$ that is in F. To see this, consider that if $G*$ is in F and G is not, then $G*$ must be a simple graph in F that contains a vertex of degree 2 that is part of a triangle. Moreover, the other two vertices of that triangle must have degree 4 (else G was a type I or II graph in F, see Figure 2). Thus $G*$ must be either a type VIII or XIII graph in F as shown in Figure 2. However, if $G*$ is a type VIII graph in F, then G was a type V graph in F. Similarly, if $G*$ is a type XIII graph in F, it follows that G was a type X graph in F.

Thus, $G*$ clearly has a triangle-free Euler tour, $T*$, by Theorem 1, as it has the quality that every vertex has degree 2 or 4, and is simple (which also implies it is C_2-free). We can then create a C_2-free Euler tour, T, in G, by replacing the uwv (or its reverse) path in $T*$ with the original, subdivided uv edge. Because $T*$ avoided triangles in $G*$, it is easy to confirm that T avoids any C_2. If this tour is also triangle-free, then we have the desired tour. However, it is possible that the pair of parallel edges was in some triangle, in which case it is possible some 4-cycle in $T*$ could have led to the formation of a triangle in T. Generally, more work will be required to ensure that preserve this property.

Next, assume the theorem holds for any such graph with number of parallel edge pairs less than or equal to k for some positive integer k and let G have $k+1$ pairs of parallel edges.

Case 1: G has a pair of parallel edges that is in no triangle.

Proceed as above, creating a 2,4 graph, $G*$, by subdividing one of the edges in that pair. Recalling similar arguments above and noting that if $G*$ is a type V graph in F, that would imply G had multiplicity at least 3, we see that $G*$ must not be in F. Then by our induction hypothesis $G*$ has a C_2,C_3-free Euler tour, $T*$, which we transform into a C_2,C_3-free Euler tour, T, in G by the same process outlined above.

The fact that our chosen pair of parallel edges was in no triangle guranties that the transformation from $T*$ to T does not create a triangle in our tour.

Case 2: Every pair of parallel edges is in some triangle.

Note that it must be the case that no triangle contains more than one such pair, and that all vertices in the triangle must have degree 4, else G would be a type V, VI, or VII graph in F. Thus we have the situation depicted in Figure 3.

The edges $S := \{e_1, e_2, e_3, e_4\}$ are drawn without terminal vertices, which may or may not be common to some of these edges. We explore those options below, taking advantage of symmetry between e_1 and e_2 and between e_3 and e_4 for simplicity.

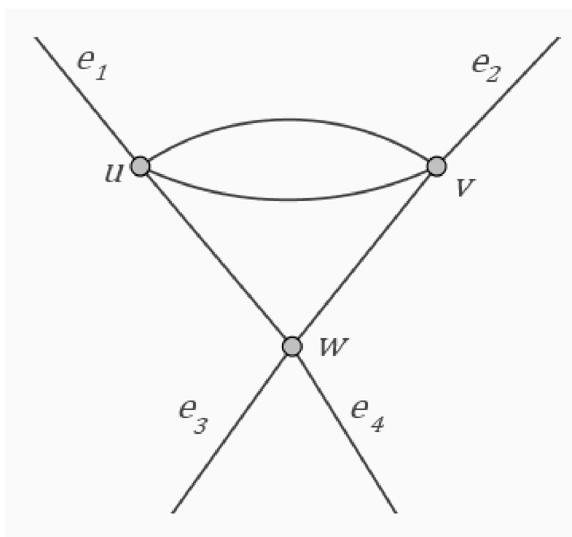

Figure 3: General structure for Case 2.

Subcase 2a: At least 3 of the edges in S are incident on a common vertex, x. Without loss of generality we may assume that one of the following is true:

(a) Of the edges in S, only e_4 is not incident with x,

(b) Of the edges in S, only e_2 is not incident with x, or

(c) All edges in S are incident with x.

So we must have one of the situations depicted in Figure 4.

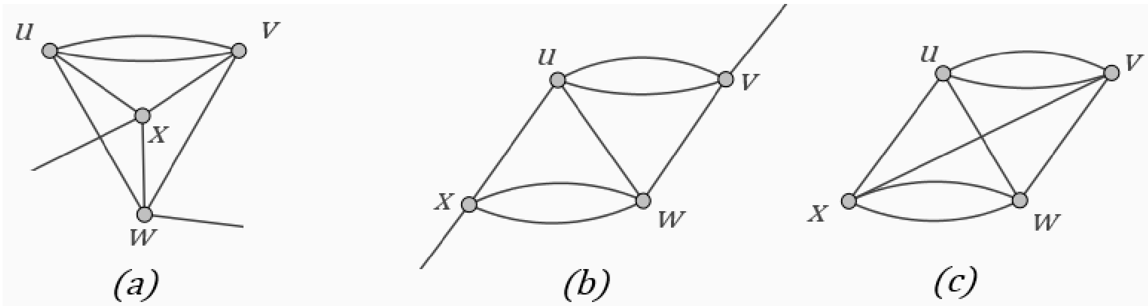

Figure 4: Possible structures in Subcase 2a.

Note that if G has one of the structures in Figure 4(a), 4(b), or 4(c), this implies that G was a type X, IX, or XII graph in F, respectively. Therefore we must have no more than two edges in S incident on a single vertex.

Subcase 2b: There is no vertex incident with an edge in $\{e_1, e_2\}$ and an edge in $\{e_3, e_4\}$.

This gives us the case where none of the edges in S are incident on a common vertex (depicted in Figure 3), as well as the case where e_1 and e_2 are incident on a common vertex and e_3 and e_4 are not, the case where e_1 and e_2 are not adjacent and e_3 and e_4 are parallel, and the case in which e_1 and e_2 are incident on a common vertex and e_3 and e_4 are parallel. These situations are depicted in Figures 5(a), 5(b), or 5(c) respectively.

In all cases we construct $G*$ by replacing the structure given in Figure 3, or structures in Figures 5(a), 5(b), or 5(c), respectively, with the structure depicted in Figure 6.

To verify that if G was not in F then $G*$ is not, suppose $G*$ is in F, depicted in Figure 2. Since G was not in F, the replacement structure must be part of a forbidden structure. Then it is clear to see that the triangle in our replacement

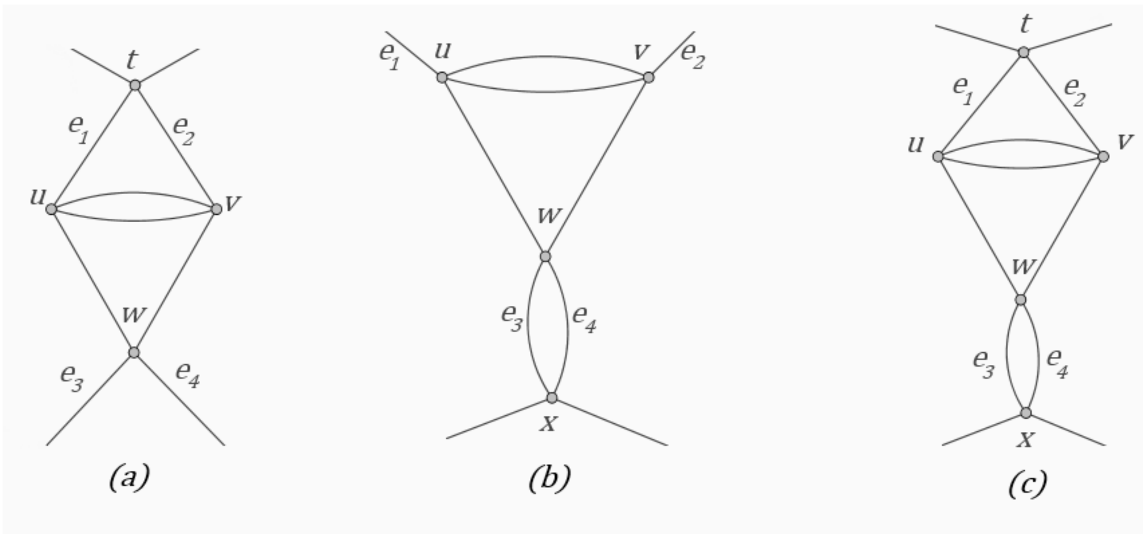

Figure 5: Possible structures for Subcase 2b.

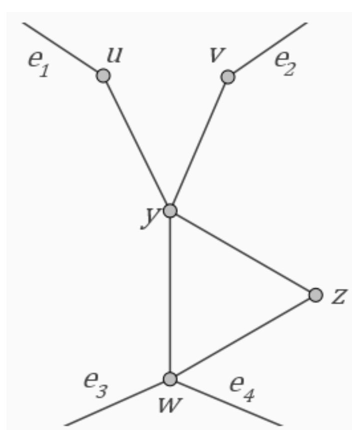

Figure 6: Reduction structure for Subcase 2b.

structure could only be part of a type VIII, XI, or XIII structure. Supposing $G*$ is type VIII, we must have that G contained a vertex incident with an edge in $\{e_1, e_2\}$ and an edge in $\{e_3, e_4\}$, which it was assumed not to have in this case. Supposing instead $G*$ is type XI, we reach a similar conclusion, that G must have contained a vertex incident with an edge in $\{e_1, e_2\}$ and both edges in $\{e_3, e_4\}$. Finally, the type XIII forbidden structure contains only one vertex of degree 2, from which all vertices with distance less than 2 have degree 4, which is impossible with our replacement structure.

Thus by induction hypothesis $G*$ has a triangle-free, C_2-free Euler tour, $T*$. Without loss of generality, we can assume that $T*$ includes two passes through the structure above, one of which must include the path uyw, and the other $vyzw$ (or their reverses). In the case that none of the edges in S were incident on the same vertex, we create a triangle-free, C_2-free Euler tour, T, in G by replacing uyw with uvw and $vyzw$ with vuw.

The arguments for the cases illustrated by Figures 5(a), 5(b), or 5(c) are very similar. In the case of Figure 5(a), we construct T by replacing uyw with $tuvw$, and $vyzw$ with $tvuw$. In the case of Figure 5(b), we construct T by replacing uvw with $uvwx$ and $vyzw$ with $vuwx$. In the case of Figure 5(c) we construct T by replacing uvw with $tuvwx$, and $vyzw$ with $tvuwx$.

Subcase 2c: There are precisely two vertices, t and x, that are incident with an edge in $\{e_1, e_2\}$ and an edge in $\{e_3, e_4\}$.

Note that both of these vertices must be degree 4, else we have that G is a type XI graph in F. By symmetry we can assume we have the structure given in Figure 7(a).

In this case we construct $G*$ by replacing in G the structure in Figure 7(a) with the structure in Figure 7(b). It is again easy to check that this does not create a $G*$

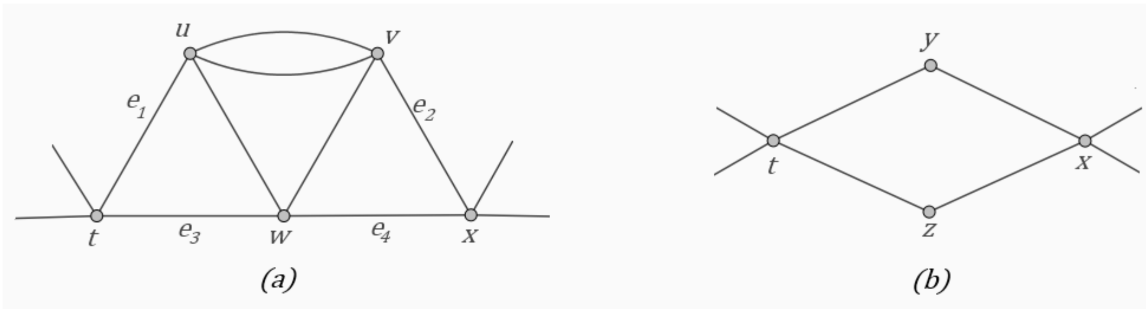

Figure 7: Relevant structures for Subcase 2c.

that is in F, and so $G*$ has a triangle-free, C_2-free Euler tour, $T*$, by our induction hypothesis. We can assume that $T*$ passes through the structure in Figure 7(b) twice, and without loss of generality, assume it contains the segments tyx and tzx (or their reverses). We then construct a triangle-free, C_2-free Euler tour in G by replacing tyx with $twuvx$, and tzx with $tuvwx$.

Subcase 2d: There is precisely one vertex that is incident with an edge in $\{e_1, e_2\}$ and an edge in $\{e_3, e_4\}$. By symmetry, we can assume that e_1 and e_3 are adjacent but e_2 and e_4 are not.

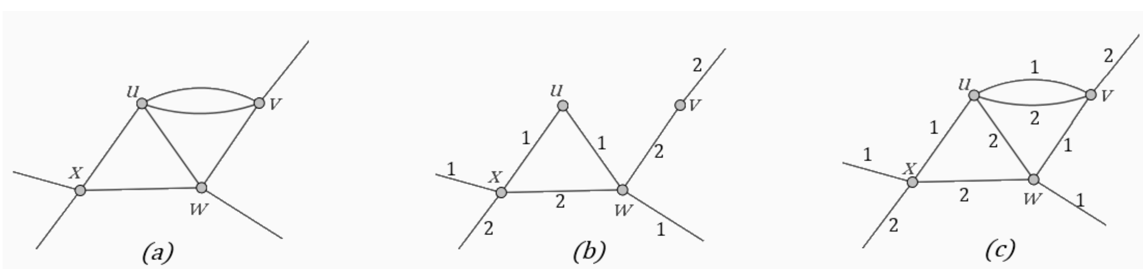

Figure 8: Relevant structures for Subcase 2d.

Then we have the structure depicted in Figure 8(a). We construct $G*$ by replacing this structure with the structure in Figure 8(b). It is again easy to check that this does not create a $G*$ that is in F, and that $G*$ has a triangle-free, C_2-free Euler tour, $T*$, by our induction hypothesis. Without loss of generality we may assume that $T*$ passes through the structure depicted in $7(b)$ twice, and contains the segments $xuwv$ and xwv (or their reverses). We then construct our triangle-free, C_2-free Euler tour, T, in G by replacing $xuwv$ with $xuvw$ and xwv with $xwuv$. For the sake of clarity, these trails are indicated with the edge labels 1 and 2 in both Figure $8(b)$ and Figure $8(c)$. The result follows by principal of mathematical induction. □

The gives us the obvious corollary, obtained by naturally dividing the established C_2,C_3-free Euler tour into 3-edge segments.

Corollary 1. *A connected 2,4 multigraph G with maximum multiplicity 2 admits a P_4 decomposition if G is not in F and the size of G is a multiple of 3.*

CHAPTER III

4-REGULAR GRAPHS AND MULTIGRAPHS

Heinrich et al. [7] expanded on Theorem 1 to show that all 4-regular simple graphs admit P_4 decompositions when their size is a multiple of 3, we will state the theorem formally for later reference.

Theorem 3 (Heinrich et al.)**.** *A connected 4-regular simple graph G admits a P_4 decomposition if and only if the size of G is a multiple of 3.*

We proceed in a similar direction, and give a necessary and sufficient condition for the P_4-decomposability of a 4-regular multigraph of maximum multiplicity 2.

Theorem 4. *A connected 4-regular multigraph with maximum multiplicity 2 admits a P_4 decomposition if and only if no 3 vertices of G induce more than 4 edges and G has size equivalent to 0 mod 3 .*

Proof. Clearly if G has such a decomposition then G must have size equivalent to 0 mod 3. Moreover, no three vertices can induce more than 4 edges because in a 4-regular graph, this ensures that G belongs to the set F of graphs depicted in Figure 2. In particular, G must be a type VI or type VII graph in F. It is easy to confirm that any attempt to decompose these structures into isomorphic copies of P_4 fails, as one is forced to create a C_2 or C_3. Therefore the conditions of the theorem are necessary, it remains to show they are sufficient.

Let G be a connected 4-regular multigraph of maximum multiplicity 2, with size equivalent to 0 mod 3, such that no three vertices of G induce more than 4 edges.

Then if G is not in F, the result follows by virtue of the fact that Theorem 2 ensures a C_2,C_3-free Euler tour which, because G has size equivalent to 0 mod 3, implies a natural P_4-decomposition. So assume G is in F. Thus we can assume G must be a type IX, X, or XIV graph (and perhaps, all three) in F.

We will create from G a multigraph $G*$ that contains no structures that are in F, and so by Theorem 2 will admit a C_2,C_3-free Euler tour, and so a P_4 decomposition, because we will ensure that $G*$ will have size equivalent mod 3 to the size of G. We will then show how this gives a P_4 decomposition of G.

To avoid creating any unwanted structures, we create $G*$ by replacing these forbidden structures in G with paths of length 4, in the case of a type IX or type X structure, or a path of length 5, in the case of a type XIV structure (paths with length equivalent to the number of edges in the forbidden structures they are replacing, mod 3).

Then, $G*$ has a triangle-free, C_2-free Euler tour, $T*$, by Theorem 1. We give an iterative algorithm to create an Euler tour, T, in G.

Begin by traversing $T*$ until the next edge to traverse is part of a replacement structure, letting m be the number of edges we have traversed until now (reduced modulo 3), removing the replacement structure added in the creation of $G*$, and returning the original vertices and edges of G that were replaced by it. In Figures 9, 10, and 11, the directions in which these edges should be traveled are indicated with arrows, and the order in which the edges will be traveled have been indicated unambiguously by labels that occur in sets of three, which also indicate which edges will be partitioned together in the final decomposition of G. Note that we lose no generality by assuming the structures are entered from the left-hand side of the Figures. Even in the case of a type IX graph in F, which is not strictly symmetric, it is easy to confirm a symmetry of argument between the labelings given in Figures

$9(a),(b),and(c)$, $10(a),(b),and(c)$, and $11(a),(b),and(c)$. For each structure returned, the labeling chosen depends on m, as well as the type of structure replaced. If m is equivalent to 0 modulo 3, the order is given by Figure $9(a)$, $10(a)$, or $11(a)$. If m is equivalent to 1 modulo 3, the order is given by Figure $9(b)$, $10(b)$, or $11(b)$. If m is equivalent to 2 modulo 3, the order is given by Figure $9(c)$, $10(c)$, or $11(c)$. It is easy also to see that edges with matching labels, in any case, will form copies of P_4 and include no triangles or structures containing a C_2.

For example, in Figure 9(b) the original structure was to have been type IX. There are only 2 edges labeled "1" because it depicts a scenario in which the Euler tour $T*$ in $G*$ contains a group of three edges, making up a P_4 in the decomposition of $G*$, of which one edge lies outside of the replacement structure and two edges lie inside of it (i.e. m is equivalent to 1 mod 3). The one of these three edges that lies outside the replacement structure will be grouped with the two edges labeled "1" in the original structure in the final P_4 decomposition of G. Then, edges labeled "2" and "3" constitute a second and third isomorphic copy of P_4 respectively. Similarly, there is depicted only one edge labeled "4" because $T*$ contains a group of three edges, making up a P_4 in the decomposition of $G*$, of which one edge lies inside of the replacement structure and two edges lie outside of it. The edge labeled "4" will be grouped with the two of these edges that lie outside of the replacement structure in the final P_4 decomposition of G.

Continue traversing the edges of $T*$ after leaving the returned structure and iterate this process until all structures that were replaced when $G*$ was created have been returned, and all edges of G have been traversed. This process must terminate because G was assumed to be a finite graph, and must have finitely many edges.

This completes our construction of T, necessarily not triangle-free and C_2-free, but having the quality that if each edge is labeled with non-negative integers p and

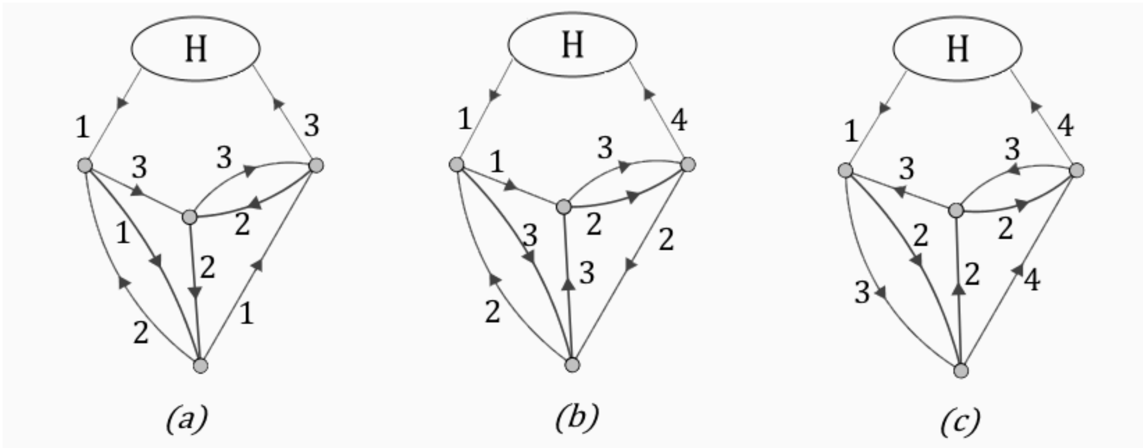

Figure 9: Labelings for type IX structures being returned.

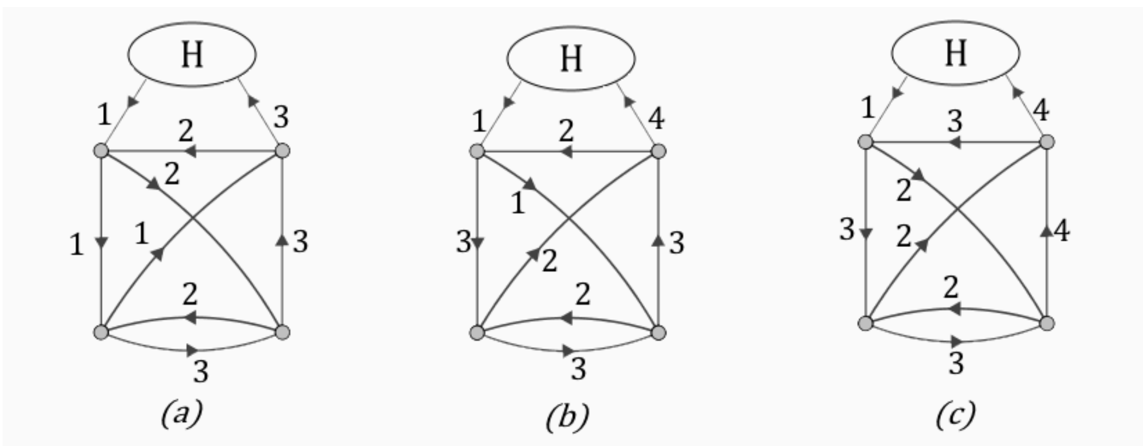

Figure 10: Labelings for type X structures being returned.

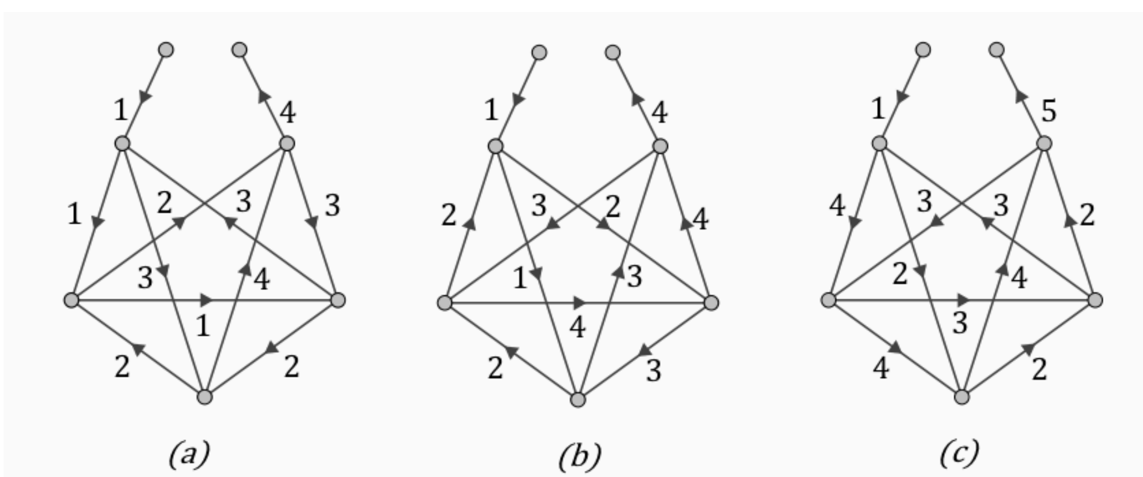

Figure 11: Labelings for type XIV structures being returned.

r $(0 \leq r \leq 2)$ such that the quantity $3p + r$ is equal to that edges order in T, the partition of the edge set $E(G)$ into sets of edges for which the integer p is the same gives a P_4 decomposition of G. $\square$

CHAPTER IV

FUTURE WORK AND CONJECTURES

Two natural ways to extend this work are to consider graphs and multigraphs of higher degree, or to allow higher multiplicity in multigraphs. However, recent communications from Diwan and Tipnis show complexity limits on these veins of research via the following two theorems.

Theorem 5. *Determining whether a 5-regular simple graph has a P_4-decomposition is NP-complete. In fact, for $n > 1$, determining whether a $(2n+1)$-regular simple graph has a P_4-decomposition is NP-complete.*

Theorem 6. *Determing whether a 6-regular multigraph of maximum multiplicity 3 has a P_4-decomposition is NP-complete.*

So it seems natural to think that further results along these lines are limited. However, if we restrict attention to regular graphs of even degree, it is possible to obtain further results. Examples are given by the following result from El-Zanati, Plantholt, and Tipnis [4] as well as Oksimets' [10] dissertation:

Theorem 7 (El-Zanati, Plantholt, and Tipnis)**.** *Every 6-regular multigraph with edge multiplicity at most 2 has a P_4-decomposition.*

Theorem 8 (Oksimets)**.** *Any connected k-regular simple graph with k even and $k \geq 6$ has a triangle-free Euler tour. This implies a P_4-decomposition if the graph has size equal to a multiple of three. In fact, the result holds for any Eulerian graph with minimum degree at least 6 and size a multiple of 3.*

We are motivated to extend the result from Oksimets to multigraphs, but at present have no way to do so, as the proof is very involved. We do make the following natural conjectures:

Conjecture 3. *Any connected k-regular multigraph with k even, $k \geq 6$, and multiplicity at most 2, has a C_2,C_3-free Euler tour, and therefore a P_4-decomposition if the size is a multiple of 3.*

And more generally ...

Conjecture 4. *Any connected mk-regular multigraph, $k \geq 3$, multiplicity at most m with m even, has a C_2,C_3-free Euler tour, and therefore a P_4-decomposition if the size is a multiple of 3.*

Because a full extension of the Oksimet result to more general multigraphs is currently beyond our reach, since establishing a proof of Theorem 4, the focus of our research has been on searching for a simpler, algorithmic proof of Theorem 8. We have explored a method for splitting 8-regular simple graphs into two components, each 4-regular, in such a way that each avoids the structures shown in Figure 1. Drawing on Theorem 1, we can thusly argue that each component of the resulting 4-regular graphs are constructed such that they admit triangle-free Euler tours. We have drafted a proof for the following theorem:

Theorem 9. *Any 8-regular simple graph has a decomposition into a pair of 4-regular simple graphs, not necessarily connected, both of which admit a component-wise triangle-free Euler tour.*

This implies the following corollary:

Corollary 2. *If G is a simple, $4k$-regular graph with $k \geq 2$, G has a decomposition into k 4-regular graphs, each component in the decomposition having a triangle-free Euler tour.*

We have also drafted a proof for the following lemma, necessary to proceed in our desired direction.

Lemma 1. *Let G be a connected, simple graph with a decomposition of its edges into two graphs, G_1 and G_2. Further suppose that G_1 and G_2 both admit triangle-free Euler tours. If v is a vertex in G having degree at least 4 in each of G_1 and G_2, then G has a triangle-free Euler tour.*

Using this, we arrive at the following corollary to Theorem 9:

Corollary 3. *For any integer $k \geq 2$, any connected $4k$-regular simple graph has a triangle-free Euler tour and thus a P_4-decomposition if the graph has size equal to a multiple of three.*

Finally, we mention one potential branch of further work which had held our interest. If one was to consider extending the methods used in the proof of Theorem 1 (see [7]) towards results on P_5 decomposition problems, it is natural to ask whether or not one can find a finite list of structures that prevent C_4-free Euler tours in bipartite graphs. However, we have set this particular question aside after considering the graphs shown in Figures 12 and 13, which appear to establish an infinite series of graphs that we suspect all disallow the existence of a C_4-free Euler tour.

It is interesting to note that a similar construction can be used to generate other series of graphs that we believe prevent C_k-free Euler tours when k is even and at least 4.

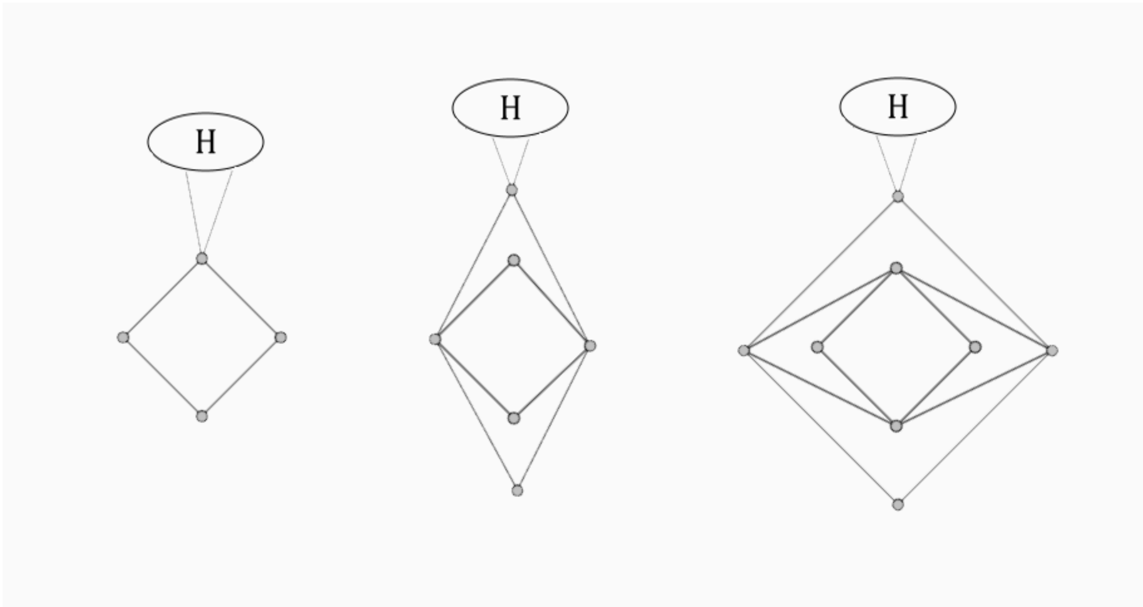

Figure 12: First three graphs in the series believed to prevent C_4-free Euler tours.

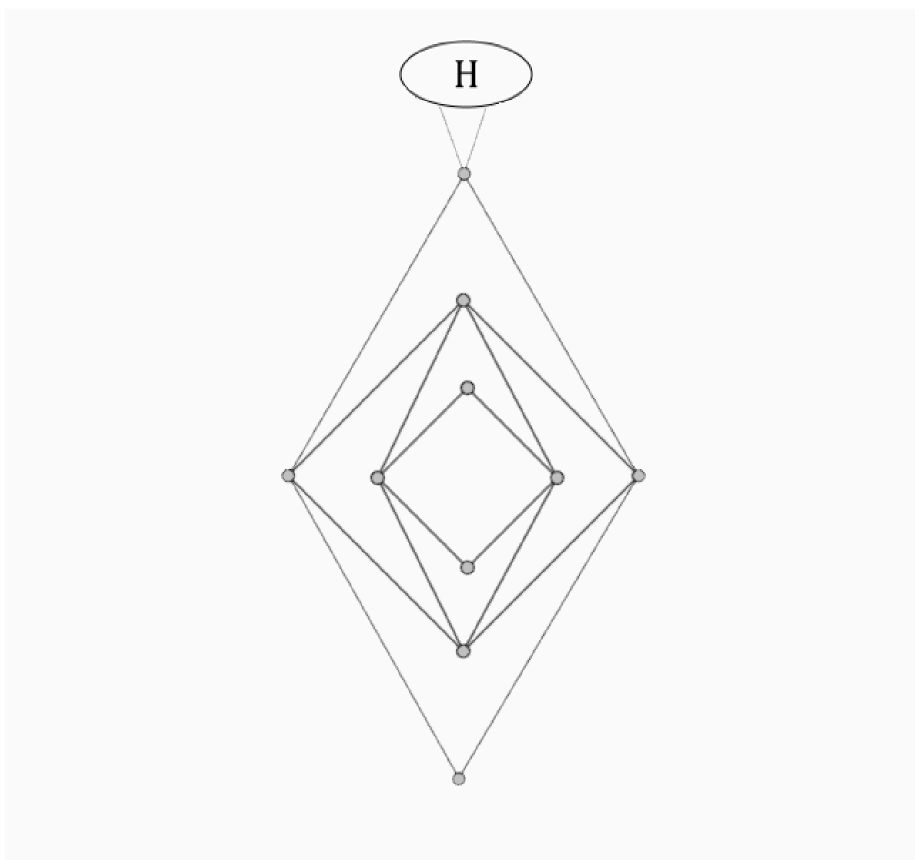

Figure 13: Fourth graph in the series believed to prevent C_4-free Euler tours.

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