PHYSICAL REVIEW A **75**, 013425 (2007)

Unitary and nonunitary approaches in quantum field theory

K. D. Lamb, C. C. Gerry,* Q. Su, and R. Grobe
Intense Laser Physics Theory Unit and Department of Physics, Illinois State University, Normal, Illinois 61790-4560, USA
(Received 21 September 2006; published 31 January 2007)

We use a simplified essential state model to compare two quantum field theoretical approaches to study the creation of electron-positron pairs from the vacuum. In the unitary approach the system is characterized by a state with different numbers of particles that is described by occupation numbers and evolves with conserved norm. The nonunitary approach can predict the evolution of wave functions and density operators with a fixed number of particles but time-dependent norms. As an example to illustrate the differences between both approaches, we examine the degree of entanglement for the Klein paradox, which describes the creation of an electron-positron pair from vacuum in the presence of an initial electron. We demonstrate how the Pauli blocking by the initial electron comes at the expense of a gain in entanglement of this electron with the created electron as well as with the created positron.

DOI: 10.1103/PhysRevA.75.013425 PACS number(s): 03.65.Pm

I. INTRODUCTION

The possibility to create matter directly from a very intense and focused laser beam has recently led to several theoretical investigations. Traditionally, the creation of electron-positron pairs from vacuum has been studied theoretically and also experimentally for heavy-ion collisions [1,2], in which the combined Coulombic fields of highly charged ions can exceed the threshold of supercriticality when the ions are accelerated towards each other.

Most of the theoretical investigations are based on the evaluation of various transition matrix elements in order to provide differential and total cross sections to accompany the experimental data [3]. About 5 years ago we became interested in exploring the pair-creation process from a more fundamental point of view, addressing questions such as where and when the pair-creation process takes place. These challenging questions require a quantum field theory that can provide full temporal and spatial resolution. It turns out that if the spatial dimensions are reduced to only one, forcing all involved particles onto a single line, the Dirac equation for the electron-positron quantum field operator can be solved numerically. This approach has helped to illuminate a wide range of controversial questions. Some of these questions arose for complicated physical situations such as how an electron scatters off a supercritical potential barrier (Klein paradox) [4–8]. This situation required the application of quantum field theory to study the combined effect of pair production due to the supercriticality of the potential together with the scattering at the barrier involving the Pauli principle. Other questions dealt with less complicated systems such as force-free environments. Two good examples are the mathematical phenomenon of Schrödinger's Zitterbewegung and the relativistic localization problem for a free electron [9]. This computational approach also permitted a first space-time resolved study of how a bound state is formed in a supercritical and localized force field [10].

Recently it was announced [11–13] that within the next few decades new laser sources could become available that—when sufficiently focused—have intensities large enough to "break down" the vacuum and to spontaneously produce electron-positron pairs. As this advancement would be a first demonstration of how light can be converted directly into matter, one can expect that this program would trigger new developments in quantum field theory.

The work reported here is part of a major project devoted to obtaining a better insight into the pair-creation process with full temporal and spatial resolution. The system we will analyze is purposely reduced to its largest possible degree of simplicity. It does not attempt to make any quantitative predictions for a real experiment, but it can help to explore various fundamental questions within the simplest possible context. It compares the predictions and the relationship among two—at least in principle—equivalent approaches to quantum field theory. In its unitary version, the total norm of the quantum states containing all occupation numbers involved is conserved, the generator of the time evolution is known, and the vacuum, single particle, and multiparticle states are treated in the same way.

The nonunitary version is more complicated, but at the same time it is the key to a spatially resolved analysis of the pair-creation process involving electron-electron entanglement. For a situation in which the total number of particles can change, the norm of the underlying single or multiparticle states must change in time, and the precise form of the corresponding (nonunitary) time evolution operator is usually not known. Here the electronic or positronic portions of the electron-positron field operator or various direct products of it need to be projected on the initial quantum state and the vacuum to compute the corresponding density operators or wave functions.

To make this comparison computationally feasible, in this work we introduce an oversimplified model system where the number of electronic and positronic states is only two each. It is similar in spirit to the essential state approximation used so successfully in quantum optics [14] and also in the 1980s in ionization physics [15–18], in which only the relevant electronic states were allowed to interact. However,

*Permanent address: Department of Physics and Astronomy, Lehman College, The City University of New York, 250 Bedford Park Boulevard West, Bronx, NY 10468-1589.

our essential state model also has positronic states in its manifold. As mentioned above, this approach cannot claim any quantitative predictions for real experiments; however, it has several advantages: first, it permits a direct and possibly even fully analytical comparison between unitary and nonunitary approaches to quantum field theory. Second, due to an absence of the unnecessary complexity of a real system, the model can serve as an alternative approach illustrating main principles of quantum field theory involving renormalization, charge conjugation, and fermionic noncommuting algebras. Third, it more fully illuminates certain aspects, such as the usefulness of the hole theory and the growth of entanglement associated with Pauli blocking in the Klein paradox. We will also discuss how this model system could provide an efficient description of fermion-fermion interactions on a nonperturbative basis, for the first time permitting the observation of the birth of a force field between two created particles.

II. ESSENTIAL STATE MODELS IN QUANTUM FIELD THEORY

The interaction of electrons and positrons with time-dependent external force fields modeled by the vector and scalar potentials $A(t)$ and $V(t)$ is given by the Dirac-Hamilton operator (in atomic units) [19]

$$h(t) \equiv c\alpha(p - A_0/c) + \beta c^2 + V_0 - \alpha A(t) + V(t), \quad (2.1)$$

where α and β are the 4×4 Pauli matrices and c (=137.036 a.u.) is the speed of light. We assume that $h_0 \equiv c\alpha[p - A_0/c] + \beta c^2 + V_0$ is subcritical, which permits an unambiguous separation of electronic states $|p\rangle$ (with energy $e_p > -c^2$), with $h_0|p\rangle = e_p|p\rangle$, and states associated with the negative energy continuum, $h_0|n\rangle = e_n|n\rangle$ and $e_n < -c^2$. For simplicity we assume that V_0 is negative, making it attractive for the electronic states.

As we will need it below, let us mention the charge conjugation operation associated with the operator $C \equiv \kappa w$. Here κ is the (antiunitary) complex conjugation and $w = -i\alpha_y\beta$ is a real unitary operator. If we were to interpret $h(t)$ as an operator predicting the time evolution for a single particle wave function, it would describe the dynamics for a negative charge. The operator $h_+(t)$, obtained from $h(t)$ by reversing the signs of the vector and scalar potentials, would describe the evolution of a positively charged particle. It turns out that these two generators $h(t)$ and $h_+(t)$ can be related to each other via $Ch(t)C^{-1} = -h_+(t)$. As a consequence the negative energy eigenstates of the time-independent part obtain a nice interpretation. If we assume $h_0|n\rangle = e_n|n\rangle$, then we immediately obtain $h_{+0}C|n\rangle = -e_nC|n\rangle$; in other words, we can identify the charge conjugated states of the negative energy continuum, $C|n\rangle$ (which we abbreviate as $|Cn\rangle$ from now on), as eigenstates of h_{+0} for a positron with *positive* energy $-e_n$.

The operator $h(t)$ can also be expressed in terms of the transition elements between the energy eigenstates of the system h_0, denoted by $|p\rangle$ and $|n\rangle$. As a result, $h(t)$ becomes

$$h(t) = \sum_{pp'} \langle p|h(t)|p'\rangle |p\rangle\langle p'| + \sum_{nn'} \langle n|h(t)|n'\rangle |n\rangle\langle n'| + \sum_{np} \langle n|h(t)|p\rangle |n\rangle\langle p| + \sum_{pn} \langle p|h(t)|n\rangle |p\rangle\langle n|, \quad (2.2)$$

where the double summation (integration) $\sum$ is performed over the states as indicated by the subscript.

To introduce the essential state model, we restrict the available levels to only $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$. The first two states can be electronic bound or free states, and the second pair is related to positron states $|C3\rangle$ and $|C4\rangle$. In this four-dimensional finite Hilbert space, the Hamiltonian $h(t)$ takes the form

$$\begin{aligned} h(t) = {} & e_1|1\rangle\langle 1| + e_2|2\rangle\langle 2| + e_3|3\rangle\langle 3| + e_4|4\rangle\langle 4| + h_{12}(t)|1\rangle\langle 2| \\ & + h_{13}(t)|1\rangle\langle 3| + h_{14}(t)|1\rangle\langle 4| + h_{21}(t)|2\rangle\langle 1| + h_{23}(t)|2\rangle\langle 3| \\ & + h_{24}(t)|2\rangle\langle 4| + h_{31}(t)|3\rangle\langle 1| + h_{32}(t)|3\rangle\langle 2| + h_{34}(t)|3\rangle\langle 4| \\ & + h_{41}(t)|4\rangle\langle 1| + h_{42}(t)|4\rangle\langle 2| + h_{43}(t)|4\rangle\langle 3|, \end{aligned} \quad (2.3)$$

where the matrix elements are defined as $h_{ij}(t) \equiv \langle i|h(t)|j\rangle$. For simplicity we have assumed that the time-dependent perturbation does not affect the size of the energy of each level, $h_{ii} \equiv \langle i|h(t)|i\rangle \equiv \langle i|h_0|i\rangle = e_i$. Due to the assumption that the Hamiltonian is subcritical at time $t=0$, where $h(t=0) = h_0$, the two electronic energies e_1 and e_2 are larger than $-c^2$, whereas the other two states have negative energies less than $-c^2$. Also note that the time-dependent couplings h_{12} and h_{34} describe ordinary quantum mechanical transitions within the electron or positron manifolds, whereas the group of h_{13}, h_{14}, h_{23}, and h_{24} involves intermanifold transitions that—as we discuss below—are associated with a change in the total number of particles.

The Hamiltonian for the unitary quantum field theory (U-QFT), denoted by $H(t)$, can be obtained from $h(t)$ if we introduce the quantum field operator $\Psi(t)$. It is defined as a hybrid of single-particle states and fermionic operators

$$\Psi(t) \equiv b_1(t)|1\rangle + b_2(t)|2\rangle + d_3^\dagger(t)|3\rangle + d_4^\dagger(t)|4\rangle. \quad (2.4)$$

Here $b_i(t)$ and $d_i^\dagger(t)$ are the fermionic annihilation and creation operators. It is important to note that $d_3^\dagger$, e.g., does not create the true positron state $|C3\rangle$, but only its charge conjugated form $|3\rangle$. These operators fulfill the following anticommutation relationships $[b_i(t), b_j^\dagger(t)]_+ = \delta_{ij}$ and $[b_i(t), b_j(t)]_+ = 0$ at all times, where the anticommutator is defined as $[A,B]_+ \equiv AB + BA$. If we exclude the time dependence in our notation below, we refer to the operators at the initial time $t=0$. The Hamiltonian for the U-QFT is then defined as $H'(t) \equiv \Psi^\dagger(t=0)h(t)\Psi(t=0)$ and takes the form $H'(t)$

$$\begin{aligned} H'(t) = {} & e_1 b_1^\dagger b_1 + e_2 b_2^\dagger b_2 + (-e_3) d_3^\dagger d_3 + (-e_4) d_4^\dagger d_4 + e_3 + e_4 \\ & + h_{12}(t) b_1^\dagger b_2 + h_{13}(t) b_1^\dagger d_3^\dagger + h_{14}(t) b_1^\dagger d_4^\dagger + h_{21}(t) b_2^\dagger b_1 \\ & + h_{23}(t) b_2^\dagger d_3^\dagger + h_{24}(t) b_2^\dagger d_4^\dagger + h_{31}(t) d_3 b_1 + h_{32}(t) d_3 b_2 \\ & + h_{34}(t) d_3 d_4^\dagger + h_{41}(t) d_4 b_1 + h_{42}(t) d_4 b_2 + h_{43}(t) d_4 d_3^\dagger, \end{aligned} \quad (2.5)$$

where we have replaced $d_i d_i^\dagger$ by $-d_i^\dagger d_i+1$ for $i=3,4$. The constant negative energy term e_3+e_4 is dynamically irrelevant and can be subtracted out (renormalized). As a result, we obtain $H(t) \equiv H'(t)-(e_3+e_4)$. Please note that as e_3 and e_4 are negative, all diagonal terms (bare energies) in $H(t)$ are positive.

The time evolution of the fermionic operators b_i and d_i is given by the Heisenberg equations of motion, $i\partial A/\partial t = [A,H(t)]_-$, where $A \in \{b_i,\ d_i,\ b_i^\dagger,$ and $d_i^\dagger\}$ using the usual commutator defined as $[A,B]_- \equiv AB-BA$. We note that also the electron-positron field operator $\Psi(t)$ fulfills the same relationship, $i\partial\Psi(t)/\partial t=[\Psi(t),H(t)]_-$. As a side remark, we should note that this operator also satisfies the original Dirac equation $i\partial\Psi(t)/\partial t=h(t)\Psi(t)$. Using either equation, the time evolution of $\Psi(t)$ can be obtained. We obtain two fully equivalent solutions,

$$\Psi(t) = U^{-1}(t)\Psi(t=0)U(t) = u(t)\Psi(t=0), \tag{2.6}$$

where we have used the notation for the time-ordered unitary propagation operators, $U(t) \equiv \exp[-i\int^t d\tau H(\tau)]$ and $u(t) \equiv \exp[-i\int^t d\tau\, h(\tau)]$. If we enter the Dirac equation with the ansatz $\Psi(t)=b_1|1(t)\rangle+b_2|2(t)\rangle+d_3^\dagger|3(t)\rangle+d_4^\dagger|4(t)\rangle$, we can obtain the time evolution of the states as $|i(t)\rangle=\Sigma_j\langle j|u(t)|i\rangle|j\rangle$, where $u_{ij}(t)$ is the matrix element of the time-ordered unitary propagation operator $u(t)$, which is the key building block of nonunitary quantum field theory of noninteracting fermions. The same solution can be obtained if we use the Heisenberg equation of motion for $\Psi(t)$, using the solutions for the time-dependent annihilation and creation operators,

$$b_i(t) = u_{i1}(t)b_1 + u_{i2}(t)b_2 + u_{i3}(t)d_3^\dagger + u_{i4}(t)d_4^\dagger, \tag{2.7a}$$

$$d_j^\dagger(t) = u_{j1}(t)b_1 + u_{j2}(t)b_2 + u_{j3}(t)d_3^\dagger + u_{j4}(t)d_4^\dagger, \tag{2.7b}$$

where $i=1,2$ and $j=3,4$.

Let us now introduce the Hilbert space associated with $H(t)$. This space is 16 dimensional, corresponding to all possible combinations of the four (single particle) states mentioned above. To distinguish these states from the single particle states $|i\rangle$, $i=1,2,3,4$, we use the notation $\|bbbb\rangle\rangle$, where each of the four bits denotes the occupation number (0 or 1) of the corresponding single particle state. They are the fermionic analog of the Fock states for bosons. For example, the book keeping symbol $\|0101\rangle\rangle$ represents an occupied state $|2\rangle$ and $|4\rangle$, describing an electron in state $|2\rangle$ and a positron in state $|C4\rangle$.

The (unperturbed) vacuum state is the one with lowest energy, denoted by $\|0000\rangle\rangle$. All of the other 15 states are defined by the action of the creation and annihilation operators on this vacuum. We use here the convention based on an ascending ordering in which the left-most creation operator acts on the left-most occupation bit [20,21]. For example, the fully occupied state $\|1111\rangle\rangle$ can be generated from the vacuum via the definition $\|1111\rangle\rangle \equiv b_1^\dagger b_2^\dagger d_3^\dagger d_4^\dagger \|0000\rangle\rangle$, or the two-particle state $\|0101\rangle\rangle$ is defined as $\|0101\rangle\rangle \equiv b_2^\dagger d_4^\dagger \|0000\rangle\rangle$.

We should also point out that these book-keeping symbols merely represent the occupation of our original levels and therefore do not allow any direct association with particular particles. As a consequence the states denoted by the double bars $\|bbbb\rangle\rangle$ do not have any symmetry property that one usually associates with an exchange of particles. However, these states $\|bbbb\rangle\rangle$ can be converted into true wave function states with N electrons and M positrons. These wave functions are then given by scalar products of the vacuum state and $\|bbbb\rangle\rangle$ acted on by sequences of direct products of the electronic and positronic parts of the field operator [22],

$$|\phi\rangle = \langle\langle 0000\|\Psi_e \otimes \cdots \Psi_e \otimes \Psi_p \otimes \cdots \Psi_p\|bbbb\rangle\rangle / \sqrt{(N!M!)}. \tag{2.8}$$

As an example, N and M can be 0, 1 or 2 for our system, where the subscript $e(p)$ is the electronic (positronic) part of $\Psi(t)$, defined as $\Psi_e \equiv [|1\rangle\langle 1|+|2\rangle\langle 2|]\Psi$ and $\Psi_p \equiv C[|3\rangle\langle 3| +|4\rangle\langle 4|]\Psi$. Please note that due to the charge conjugation operation C, we have $\Psi=\Psi_e+C^{-1}\Psi_p$. To give a few more examples, we obtain $\langle\langle 0000\|\Psi_e\|0100\rangle\rangle=|2\rangle$, $\langle\langle 0000\|\Psi_p\|0001\rangle\rangle=|C4\rangle$, $\langle\langle 0000\|\Psi_e\otimes\Psi_p\|0101\rangle\rangle=|2\rangle|C4\rangle$, and $\langle\langle 0000\|\Psi_e\otimes\Psi_e\|1100\rangle\rangle/\sqrt{2}=(|1\rangle|2\rangle-|2\rangle|1\rangle)/\sqrt{2}$. The latter state describes two indistinguishable electrons and is therefore antisymmetric under a particle exchange.

We should finish with a short note on how to express the annihilation and creation operators in this basis. The book-keeping symbols $\|bbbb\rangle\rangle$ naturally factor into electronic (e) and positronic (p) groups, such that we can use the notation $\|bbbb\rangle\rangle \equiv \|bb\rangle\rangle_e \otimes \|bb\rangle\rangle_p$, and I_p for the unit operator in positronic space can be expressed as $I_p \equiv \|00\rangle\rangle_p\langle\langle 00\|_p +\|10\rangle\rangle_p\langle\langle 10\|_p+\|01\rangle\rangle_p\langle\langle 01\|_p+\|11\rangle\rangle_p\langle\langle 11\|_p$. For each operator we obtain the sum over eight projectors:

$$b_1 = [\|00\rangle\rangle_e\langle\langle 10\|_e + \|01\rangle\rangle_e\langle\langle 11\|_e] \otimes I_p, \tag{2.9a}$$

$$b_2 = [\|00\rangle\rangle_e\langle\langle 01\|_e - \|10\rangle\rangle_e\langle\langle 11\|_e] \otimes I_p, \tag{2.9b}$$

$$d_3^\dagger = I_e \otimes [\|10\rangle\rangle_p\langle\langle 00\|_p + \|11\rangle\rangle_p\langle\langle 01\|_p], \tag{2.9c}$$

$$d_4^\dagger = I_e \otimes [\|01\rangle\rangle_p\langle\langle 00\|_p - \|11\rangle\rangle_p\langle\langle 10\|_p]. \tag{2.9d}$$

III. TIME EVOLUTION OF THE HILBERT STATES

Let us now analyze the time evolution for the states $\|bbbb\rangle\rangle$. It turns out that due to the special functional form of our Dirac Hamiltonian $h(t)$ from Eq. (2.1) and correspondingly the bilinear form for $H(t)$ in Eq. (2.5), not all of these 16 states are coupled. In fact, there are five invariant subspaces. The largest one (the vacuum space) is six dimensional and contains the vacuum $\|0000\rangle\rangle$ (which we denote from now on by the short-hand notation $\|\alpha_1\rangle\rangle$), four electron-positron pair states $\|1010\rangle\rangle$ $(\equiv\|\alpha_2\rangle\rangle)$, $\|1001\rangle\rangle$ $(\equiv\|\alpha_3\rangle\rangle)$, $\|0110\rangle\rangle$ $(\equiv\|\alpha_4\rangle\rangle)$, and $\|0101\rangle\rangle$ $(\equiv\|\alpha_5\rangle\rangle)$ as well as the two-pair state $\|1111\rangle\rangle$ $(\equiv\|\alpha_6\rangle\rangle)$. Its six energies are 0, e_1-e_3, e_1-e_4, e_2-e_3, e_2-e_4, and $e_1+e_2-e_3-e_4$. The Klein submanifold of states (called Ke) consists of the four states that are coupled to the state with a single electron $\|1000\rangle\rangle$ (which we denote as $\|\beta_1\rangle\rangle$). The other three states are $\|0100\rangle\rangle$

$(\equiv\|\beta_2\rangle\rangle)$, $\|1110\rangle\rangle$ $(\equiv\|\beta_3\rangle\rangle)$, and $\|1101\rangle\rangle$ $(\equiv\|\beta_4\rangle\rangle)$. Its four energies are e_1, e_2, $e_1+e_2-e_3$, and $e_1+e_2-e_4$. The corresponding Klein manifold for the positron (labeled Kp) is spanned by $\|0001\rangle\rangle$, $\|0010\rangle\rangle$, $\|1011\rangle\rangle$, and $\|0111\rangle\rangle$. Due to the constraint of finitude for the essential state model, the two-particle states $\|1100\rangle\rangle$ and $\|0011\rangle\rangle$ are each invariant submanifolds. In this notation the vacuum state associated with h_0 satisfies $h_0\,\|\alpha_1\rangle=0\,\|\alpha_1\rangle\rangle$.

As we mentioned, in this basis not all states are coupled and the matrix representation of $H(t)$ is block diagonal in these five subspaces, $H(t)=H^{\mathrm{vac}}(t)+H^{Ke}(t)+H^{ee}(t)+H^{pp}(t)+H^{Kp}(t)$, which take the form

$$H^{\mathrm{vac}}(t)=\begin{pmatrix} 0 & h_{31} & h_{41} & h_{32} & h_{42} & 0 \\ h_{13} & e_1-e_3 & -h_{43} & h_{12} & 0 & -h_{42} \\ h_{14} & -h_{34} & e_1-e_4 & 0 & h_{12} & h_{32} \\ h_{23} & h_{21} & 0 & e_2-e_3 & -h_{43} & h_{41} \\ h_{24} & 0 & h_{21} & -h_{34} & e_2-e_4 & -h_{31} \\ 0 & -h_{24} & h_{23} & h_{14} & -h_{13} & e_1+e_2-e_3-e_4 \end{pmatrix}, \tag{3.1a}$$

$$H^{Ke}(t)=\begin{pmatrix} e_1 & h_{12} & h_{32} & h_{42} \\ h_{21} & e_2 & -h_{31} & -h_{41} \\ h_{23} & -h_{13} & e_1+e_2-e_3 & -h_{43} \\ h_{24} & -h_{14} & -h_{34} & e_1+e_2-e_4 \end{pmatrix}, \tag{3.1b}$$

$$H^{ee}(t)=(e_1+e_2), \tag{3.1c}$$

$$H^{pp}(t)=(-e_3-e_4), \tag{3.1d}$$

$$H^{Kp}(t)=\begin{pmatrix} -e_3 & -h_{43} & -h_{41} & -h_{42} \\ -h_{34} & -e_4 & h_{31} & h_{32} \\ -h_{14} & h_{13} & e_1-e_3-e_4 & h_{12} \\ -h_{24} & h_{23} & h_{21} & e_2-e_3-e_4 \end{pmatrix}. \tag{3.1e}$$

As mentioned above the evolution of the state $\|\Phi(t)\rangle\rangle$ is given by the "Dirac-like" equation $i\partial/\partial t\|\Phi(t)\rangle\rangle=H(t)\|\Phi(t)\rangle\rangle$. Let us assume an initial state inside the vacuum space is given by $\|\Phi(t=0)\rangle\rangle=\Sigma_j\alpha_j(t=0)\|\alpha_j\rangle\rangle$. The general time evolution is given by

$$\begin{aligned}\|\Phi(t)\rangle\rangle &= \alpha_1(t)\|0000\rangle\rangle+\alpha_2(t)\|1010\rangle\rangle+\alpha_3(t)\|1001\rangle\rangle \\ &\quad+\alpha_4(t)\|0110\rangle\rangle+\alpha_5(t)\|0101\rangle\rangle+\alpha_6(t)\|1111\rangle\rangle,\end{aligned} \tag{3.2}$$

where the time-dependent expansion coefficients $\alpha_i(t)$ are given by the sum, $\alpha_i(t)=\Sigma_j\alpha_j(t=0)U_{ij}(t)$, where $U_{ij}(t)\equiv\langle\langle\alpha_i\|U(t)\|\alpha_j\rangle\rangle$ is the matrix element of the time-ordered unitary propagation operator $U(t)\equiv\exp[-\int^t d\tau\,H(\tau)]$ for a given set of coupling parameters h_{ij}. For the special case of $\alpha_i(t=0)=\delta_{1i}$, the decay of the vacuum can be obtained. This vacuum decay amplitude is an important measure for the amount of spontaneous creation of supercritical fields.

Due to the fermionic anticommutation relationships we have not been able to find a closed form simplification of the operator $U(t)$ in terms of the creation and annihilation operators. The exponentiated form of the operator $H(t)$ can be found numerically by diagonalization and exponentiation of the diagonal matrix. Even though the matrix elements of the 4×4 matrix h_{ij} and the 16×16 matrix H_{ij} are easily related as shown in Eqs. (3.1), the relationship between the exponentiated forms u_{ij} and U_{ij} is nontrivial. The most general form to relate these coefficients is given by the equality (2.6) $U(t)^{-1}\Psi(t=0)U(t)=u(t)\Psi(t=0)$, based on the fact that the quantum field operator Ψ solves simultaneously the Dirac equation, $i\partial/\partial t\Psi(t)=h(t)\Psi(t)$, as well as the Heisenberg equation of motion, $i\partial/\partial t\Psi(t)=[\Psi(t),H(t)]_-$. The general relationship between the matrix elements of u and U is not straightforward.

We will need two specific solutions in Sec. V. If we start with the vacuum as our initial state $\|\Phi(t=0)\rangle\rangle=\|\alpha_1\rangle\rangle$, then we obtain the state $\|\Phi(t=0)\rangle\rangle=\Sigma_j U^{\mathrm{vac}}_{j,1}(t)\|\alpha_j\rangle\rangle$, where the 6×6 matrix U^{vac}_{ji} is associated with H^{vac} defined in Eq. (3.1a). Since the total norm $\langle\langle\Phi(t)\|\Phi(t)\rangle\rangle=1$ is conserved in this unitary framework, it is a little bit misleading to view the state $\|\alpha_1\rangle\rangle$ as an infinite reservoir for particles. In fact, once a single pair is created, the amplitude of $\|\alpha_1\rangle\rangle$ in $\|\Phi(t)\rangle\rangle$ becomes zero. Similarly, if we start with the single electron state $|1\rangle$ as our initial state $\|\Phi(t=0)\rangle\rangle=\|\beta_1\rangle\rangle$, then we obtain the state $\|\Phi(t)\rangle\rangle=\Sigma_j U^{Ke}_{j,1}(t)\|\beta_j\rangle\rangle$, where the 4×4 matrix U^{Ke}_{ji} is associated with H^{Ke} as defined in Eq. (3.1b).

IV. ELECTRON-POSITRON AND ELECTRON-ELECTRON ENTANGLEMENT

Since we have two types of particles, we need to define two different types of degrees of entanglement between the corresponding states. In order to compute the entanglement between groups of electrons and positrons, we must compute the total density operator, or $\|\Phi(t)\rangle\rangle\langle\langle\Phi(t)\|$, whereas the de-

termination of the electron-electron entanglement requires the computation of the electronic density operator, $\langle\langle\Phi(t=0)\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle$. In Sec. V we will study the time evolution of the corresponding degrees of entanglement.

A. Electron-positron entanglement

Let us first describe how one could characterize a degree of entanglement between two different groups of particles, such as electrons and positrons [23,24]. In order to characterize the amount of entanglement between a group of electrons with a group of positrons, one strategy would be to eliminate all positronic degrees of freedom from the total density operator $\|\Phi(t)\rangle\rangle\langle\langle\Phi(t)\|$ and explore the resulting electronic density operator, $P_e(t)\equiv \mathrm{Tr}_p\ [\|\Phi(t)\rangle\rangle\langle\langle\Phi(t)\|]$, to determine whether it describes a pure or mixed state. The tracing over the positronic states is possible, as the bookkeeping symbols $\|bbbb\rangle\rangle$ naturally factor into electronic (e) and positronic (p) groups, $\|bbbb\rangle\rangle=\|bb\rangle\rangle_e\otimes\|bb\rangle\rangle_p$ as discussed above. In other words, the Tr_p operator is defined as $\mathrm{Tr}_p\cdots\equiv\Sigma_{b'b}\langle\langle b'b\|_p\cdots\|b'b\rangle\rangle_p$. If $P_e(t)$ describes a pure state [equivalent to $\mathrm{Tr}_e\ P_e^2(t)=1$], then the original state $\|\Phi(t)\rangle\rangle$ did not entangle the electrons with their environment, the positrons. On the other hand, if $\mathrm{Tr}_e\ P_e^2(t)<1$, it is not possible to describe the group of electrons by wave functions. In 1994 it was shown [25] that the numerical degree of entanglement, measured by the numerical parameter $K_{ep}\equiv 1/[\mathrm{Tr}_e\ P_e^2(t)]$, describes the number of single particle states necessary to synthesize the original multiparticle states, sometimes also called the Everett-Schmidt decomposition [26,27]. $K_{ep}=1$ corresponds to no entanglement and the larger the value for K_{ep}, the larger is the degree of entanglement,

$$K_{ep}=1/\mathrm{Tr}_e\ P_e^2=1/\mathrm{Tr}_e\{\mathrm{Tr}_p\|\Phi(t)\rangle\rangle\langle\langle\Phi(t)\|\}^2. \quad (4.1a)$$

Quite remarkably, it turns out that the same parameter K_{ep} can also be obtained directly from the expansion coefficients $\|\Phi(t)\rangle\rangle$ *without any tracing* over positronic degrees of freedom. To demonstrate this method with a simple example, let us analyze the vacuum space and neglect the vacuum state $\|\alpha_1\rangle$ and the two-pair state $\|\alpha_6\rangle\rangle$ for better illustration. In this four-dimensional basis, each state describes a single electron-positron pair, $\|\Phi(t)\rangle\rangle=\Sigma_j\alpha_j(t)\|\alpha_j\rangle\rangle$. There is an immediate way to determine *directly* from the four coefficients $\alpha_j(t)$ whether the underlying electron is entangled with the positron: if $\alpha_2(t)\alpha_4(t)-\alpha_3(t)\alpha_5(t)=0$, then the electron is not entangled. This determinant condition can be easily understood, since we can write down the most general form of an unentangled electron-positron state, $|\phi_{ep}\rangle=(c_1|1\rangle+c_2|2\rangle)(c_3|C3\rangle+c_4|C4\rangle)$. Because the superposition of the electronic and positronic states is factorized, the two particles are unentangled. This state translates directly into the special functional form of the four coefficients, $\alpha_2=c_1c_2$, $\alpha_3=c_1c_4$, $\alpha_4=c_2c_3$, and $\alpha_5=c_2c_4$ (associated with $\|1010\rangle\rangle$, $\|1001\rangle\rangle$, $\|0110\rangle\rangle$, and $\|0101\rangle\rangle$), thereby proving the above given determinant condition.

As a side remark, we should point out that this condition generalizes to higher dimensional essential state models. If the electron is permitted N states and the positron M states described by state $\|\Phi(t)\rangle\rangle$ then the two particles are not entangled if any subdeterminant formed by the $N\times M$ matrix (with elements $\alpha_i, i=1,NM$) vanishes. It turns out that the degree of entanglement [also characterized by the trace over all electronic states of the square of the reduced density matrix in Eq. (4.1a)] can be expressed in terms of the sums of $\binom{N}{2}\binom{M}{2}$ determinants associated with all possible 2×2 submatrices, denoted by $\det S_i$ associated with $\|\Phi(t)\rangle\rangle$,

$$K_{ep}=1\Big/\left[1-2\sum_i|\det S_i|^2\right]. \quad (4.1b)$$

As K_{ep} describes the entanglement of a group of electron states with a group of positron states, it is interesting to note that each 2×2 submatrix describes the entanglement between an individual electron and positron states. In other words, the total entanglement among groups is identical to the sum of the degree of entanglements within each single pair of particles.

B. Electron-electron entanglement

So far our discussion used the unitary approach to quantum field theory based on $\|\Phi(t)\rangle\rangle$ and $P_e(t)$. In its nonunitary version, the electronic density operator can be defined as

$$\rho_e(t)\equiv\langle\langle\Phi(t=0)\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle. \quad (4.2)$$

Because the total number of electrons changes in time, the trace of the operator $\rho_e(t)$ changes in time. This approach is perfectly suited to calculating the degree of entanglement between two individual electrons. If we were to normalize $\rho_e(t)$ to fulfill $\mathrm{tr}_e\ \rho_e(t)=1$, then $K_{ee}\equiv 1/\mathrm{tr}\ \rho_e(t)^2$ would serve as an unambiguous measure for the degree of electron-electron entanglement [25]. Here we define tr_e as the summation over the electron states, e.g., $\mathrm{tr}_e\cdots\equiv\langle 1|\cdots|1\rangle+\langle 2|\cdots|2\rangle$. The definition of the electron-electron entanglement is

$$K_{ee}\equiv[\mathrm{tr}_e\langle\langle\Phi(t=0)\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle]^2/\mathrm{tr}_e\langle\langle\Phi(t=0)\times\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle^2. \quad (4.3)$$

We will examine a specific example in the next section. Please note that the two electronic density operators $\rho_e(t)\equiv\langle\langle\Phi(t=0)\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle$ and $P_e(t)\equiv\mathrm{Tr}_p[\|\Phi(t)\rangle\rangle\langle\langle\Phi(t)\|]$ act in different spaces: a matrix representation of $\rho_e(t)$ requires two states $|1\rangle$ and $|2\rangle$, whereas $P_e(t)$ would be represented by a 4×4 matrix in general, based on $\|10\rangle\rangle_e$, $\|01\rangle\rangle_e$, $\|11\rangle\rangle_e$, and $\|00\rangle\rangle_e$.

V. TIME EVOLUTION OF THE ELECTRON-ELECTRON AND ELECTRON-POSITRON ENTANGLEMENT FOR THE KLEIN SPACE

Analyzing the time evolution in Klein space can also be illustrative in demonstrating how the Pauli-blocking mechanism for the pair-creation process in the presence of an existing electron in state $|1\rangle$ contributes to a change in the single-particle wave function character of the initial electron,

as well as a growth in entanglement with the created positron and the other electron. Once again we assume the Hamiltonian h is subcritical. This assumption permits a unique identification of the states with electronic and positronic properties. As the mechanism to create pairs, we use the time dependence of an external force, which can be represented by the instantaneous turn-on of the off-diagonal coupling parameters h_{ij}. For simplicity we assume the electron is initially in state $|1\rangle$, which in the U-QFT framework corresponds to the mode $\|1000\rangle\rangle$.

As outlined above, the single electron wave function can be obtained from the projected electronic part of the quantum field operator. We should note that in the context of a supercritical field, the correct form of this projection operator is not yet known. Also, for a time-dependent system such as ours, this projection operator is itself time dependent [28,29], since the instantaneous energy eigenstates of $h(t)$ change in time. In our case, the correct electronic projection operator should be $\Pi_{e(t)}=|1_{(t)}\rangle\langle 1_{(t)}|+|2_{(t)}\rangle\langle 2_{(t)}|$. Here it is important to remark that the notation $|1_{(t)}\rangle$ and $|2_{(t)}\rangle$ does not represent the time-evolved states $|1(t)\rangle$ and $|2(t)\rangle$ discussed in the last paragraph of Sec. II, but merely the instantaneous energy eigenstates defined as $h(t)\ |1_{(t)}\rangle=e_{1(t)}|1_{(t)}\rangle$, and similarly for $|2_{(t)}\rangle$. In order to get a simpler final expression, we approximate this projection operator here by $\Pi_e\approx|1\rangle\langle 1|+|2\rangle\langle 2|$. Even though the individual states change, it seems the total upper subspace projector should change less. The interpretation for using Π_e instead of $\Pi_{e(t)}$ is clear: while $\Pi_{e(t)}\Psi(t)$ describes the true electronic portion of the field operator during the interaction, the field $\Pi_e\Psi(t)$ is the electronic property that the system would take if the external time-dependent field (given by h_{ij}) were turned off instantly [28,29]. If we are willing to restrict our discussion based on the latter interpretation, the replacement of $\Pi_{e(t)}$ by Π_e is merely a simplification and not an approximation at all.

The dynamics of the Klein space is described by the time evolution of the initial state $\|1000\rangle$ under the Dirac Hamiltonian of Eq. (3.1b). The total state is given by a superposition of states with one and three particles,

$$\|\Phi(t)\rangle\rangle = \beta_1(t)\|1000\rangle\rangle + \beta_2(t)\|0100\rangle\rangle + \beta_3(t)\|1110\rangle\rangle + \beta_4(t)\|1101\rangle\rangle. \quad (5.1)$$

The four expansion coefficients $\beta_i(t)$ carry all information about the system and are identical to the matrix elements of the propagator for the electronic Klein space, $\beta_i(t)=U^{Ke}{}_{i,1}(t)$. Due to the unitarity of the time-evolution operator U^{Ke}, the total norm is conserved, $\Sigma_i|\beta_i(t)|^2=1$. For instance, the corresponding single and three particle states can be obtained via

$$\begin{aligned}|\phi_e(t)\rangle &\equiv \langle\langle 0000\|\Psi_e(t=0)\|\Phi(t)\rangle\rangle \\ &= u_{11}(t)|1\rangle + u_{21}(t)|2\rangle \\ &= \beta_1(t)|1\rangle + \beta_2(t)|2\rangle, \end{aligned} \quad (5.2)$$

$$|\phi_p(t)\rangle \equiv \langle\langle 0000\|\Psi_p(t=0)\|\Phi(t)=0\rangle\rangle, \quad (5.3)$$

$$\begin{aligned}|\phi_{eep}(t)\rangle &\equiv \langle\langle 0000\|\Psi_e(t=0)\otimes\Psi_e(t=0) \\ &\quad\otimes\Psi_p(t=0)\|\Phi(t)\rangle\rangle/\sqrt{2} \\ &= \beta_3(t)(|2\rangle|1\rangle-|1\rangle|2\rangle)|C3\rangle/\sqrt{2} \\ &\quad+\beta_4(t)(|2\rangle|1\rangle-|1\rangle|2\rangle)|C4\rangle/\sqrt{2} \\ &= u^*_{32}(t)(|1\rangle|2\rangle-|2\rangle|1\rangle)|C3\rangle/\sqrt{2} \\ &\quad+u^*_{42}(t)(|1\rangle|2\rangle-|2\rangle|1\rangle)|C4\rangle/\sqrt{2}. \end{aligned} \quad (5.4)$$

Even though in principle a superposition of entangled states could become unentangled, the position wave function $|\phi_p(t)\rangle$ in Eq. (5.3) vanishes at all times, as apparently the positron is being created fully entangled with another electron.

We should also comment on another interesting property concerning the conservation of particles in this Klein space. Complementary to the wave function states $|\phi_e(t)\rangle$ and $|\phi_{eep}(t)\rangle$ we can also compute the corresponding electronic and positronic density operators:

$$\rho_e(t) \equiv \langle\langle\Phi(t=0)\|\Psi_e^\dagger(t)\Psi_e(t)\|\Phi(t=0)\rangle\rangle, \quad (5.5a)$$

$$\rho_p(t) \equiv \langle\langle\Phi(t=0)\|\Psi_p^\dagger(t)\Psi_p(t)\|\Phi(t=0)\rangle\rangle. \quad (5.5b)$$

As the created position must be produced together with another electron, the average number of positrons is given by the (time-dependent) norm of the three-particle wave function $\langle\phi_{eep}(t)|\phi_{eep}(t)\rangle$. This must be identical to the total "norm" contained in $\rho_p(t)$ and we obtain

$$\mathrm{tr}_p\rho_p(t) = \langle\phi_{eep}(t)|\phi_{eep}(t)\rangle = |\beta_3(t)|^2+|\beta_4(t)|^2. \quad (5.6)$$

The latter equality makes sense as only the states $\|\beta_3\rangle\rangle$ and $\|\beta_4\rangle\rangle$ contain an occupied positron bit. On the other hand, whenever a positron is created, another electron is created as well, whose average number is measured by $\mathrm{tr}_e\,\rho_e(t)$; in our case a value between 1 and 2. As a result, we obtain $\mathrm{tr}_e\,\rho_e(t)-\mathrm{tr}_p\,\rho_p(t)=1$, reflecting the conserved total amount of charge, equivalent to a constant difference in the two average particle numbers.

The relationships from Eq. (5.2), $u_{11}=\beta_1$ and $u_{21}=\beta_2$ are interesting as only the amplitude $\beta_1(t)$ (associated with state $\|1000\rangle\rangle$) determines the amplitude for state $|1\rangle$. In principle, the amplitudes of the following three states could contribute to the single particle amplitude for state $|1\rangle$, $\|1000\rangle$, $\|1110\rangle\rangle$, and $\|1101\rangle\rangle$.

Let us now demonstrate the impact of the Pauli blocking. Let us assume for the moment that the couplings h_{ij} are such that the state $\|0100\rangle\rangle$ remains unpopulated corresponding to $h_{4j}=h_{j4}=0$. Note that in each of the three remaining states ($\|1000\rangle$, $\|1110\rangle\rangle$, and $\|1101\rangle\rangle$) the level $|1\rangle$ is constantly occupied. One could (incorrectly) interpret the constant occupation of this level with a "spectator electron" that simply occupies its level $|1\rangle$ and passively observes how an independent electron-positron pair is created. Depending on the final state of the created positron, the created states would be $\|1110\rangle\rangle$ and $\|1101\rangle\rangle$.

It is important to note that this passive spectator view of the initial electron is incorrect. As the dynamics evolves into the states $\|1110\rangle\rangle$ and $\|1101\rangle\rangle$, the norm of its state

$\langle \phi_e(t) | \phi_e(t) \rangle = |u_{11}(t)|^2 + |u_{21}(t)|^2$ changes in time and the electron can no longer be described by a single-electron wave function as the initial electron becomes entangled with the other created electron as well as with the other positron. Following the general description of Sec. IV we have computed the amount of electron-position entanglement via the reduced density operator for the electrons,

$$P_e = |\beta_1(t)|^2 \|10\rangle\rangle_e \langle\langle 10\|_e + |\beta_2(t)|^2 \|01\rangle\rangle_e \langle\langle 01\|_e + (|\beta_3(t)|^2 + |\beta_4(t)|^2) \|11\rangle\rangle_e \langle\langle 11\|_e + \beta_1(t)\beta_2(t)^* \|10\rangle\rangle_e \langle\langle 01\|_e + \beta_2(t)\beta_1(t)^* \|01\rangle\rangle_e \langle\langle 10\|_e. \quad (5.7)$$

The resulting degree of electron-positron entanglement is

$$K_{ep} = 1/[\mathrm{Tr}_e\, P_e^2] = 1/[1 - 2(|\beta_1(t)|^2 + |\beta_2(t)|^2)(|\beta_3(t)|^2 + |\beta_4(t)|^2)]. \quad (5.8)$$

This expression for K_{ep} is just a special case of the general expression Eq. (4.1a) and (4.1b) applied to the electronic Klein subspace in which, among the possible $\binom{N}{2}^2 = 16$ submatrices, only four are nonzero. We note that the degree of entanglement is $K_{ep}=1$ for the initial state $[\beta_1^2(t=0)=1]$. In other words, in this simplified situation the electron becomes entangled with the positron as time goes on, but at a later time it can become unentangled again, corresponding to $|\beta_1(t)|^2=0$ and $|\beta_2(t)|^2=0$.

Let us also show how the electron becomes entangled with the other electron. As two electrons can only be described by an antisymmetric state, one could (incorrectly) expect that a minimal degree of entanglement of $K=2$ cannot be avoided once the second electron is created completely. However, one can expect even higher numbers for K if the dimension of the subspace were extended to include more states. While the computation of the degree of electron-electron entanglement from two-electron states is trivial [30], the possible presence of the positron makes a computation conceptually more challenging.

If we insert the time evolution of $\Psi_e(t)$ from Eq. (2.4) into this expression, we obtain $\rho_e(t) = \langle\langle 1000 \| \Psi_e^\dagger(t) \Psi_e(t) \| 1000 \rangle\rangle \equiv \rho_{ij}(t) |i\rangle\langle j|$, where

$$\rho_{11}(t) = |u_{11}(t)|^2 + |u_{13}(t)|^2 + |u_{14}(t)|^2, \quad (5.9a)$$

$$\rho_{21}(t) = u_{21}(t)^* u_{11}(t) + u_{23}(t)^* u_{13}(t) + u_{24}(t)^* u_{14}(t), \quad (5.9b)$$

$$\rho_{22}(t) = |u_{21}(t)|^2 + |u_{23}(t)|^2 + |u_{24}(t)|^2. \quad (5.9c)$$

Using $\rho_{ij}(t)$ we can compute the electron-electron entanglement K_{ee} defined in Eq. (4.3).

VI. NUMERICAL EXAMPLES FOR THE TIME DEPENDENCES

In order to show a specific example, we must introduce numerical values for the original coupling coefficients. Let us begin with the simplest nontrivial situation, a system in which the electron can be in levels $|1\rangle$ and $|2\rangle$ and the positron has only the single state $|C3\rangle$. We have chosen arbitrarily the coefficients ($e_1=0.1$, $e_2=0.2$, $e_3=-0.1$, $h_{12}=0.3$, $h_{13}=0.3$, $h_{14}=0.5$). Using numerical diagonalization we have computed the average number of positrons, denoted by $\langle \phi_{eep} | \phi_{eep} \rangle = \mathrm{tr}_p\, P_p$ [see Eq. (3.1)], the degree of entanglement between the groups of electrons and the positron $K_{ep}(t)$ [see Eq. (5.6)], the entanglement between two electrons $K_{ee}(t)$ [see Eq. (4.3)], the time-dependent norm of the single-electron state $\langle \phi_e | \phi_e \rangle$ [see Eq. (5.2)], and the norm of the vacuum state $|\alpha_1(t)|^2$.

The top graph in Fig. 1(a) shows how the positron is created from vacuum. The norm of $|\phi_{eep}\rangle$ rises to nearly 0.8 after which the population shrinks to almost zero. At time around $t=5$ the number of positrons reaches its largest value close to 1. We believe that any oscillatory behavior is due to the finiteness of the Hilbert space and the associated discrete number of frequencies. The second graph from the top shows the growth of the electron-positron correlation from $K_{ep}(t=0)=1$ to its largest value, $K_{ep}(t)=2$. It is interesting that shortly before time $t=1.7$, when the positron population approaches its first maximum, the entanglement $K_{ep}(t)$ actually decreases. This clearly shows that the entanglement cannot be a trivial function of the amount of available positrons. The third graph [Fig. 1(c)] displays the electron-electron entanglement $K_{ee}(t)$. It follows the positronic population curve relatively closely, suggesting that the created electron becomes immediately entangled with the initial electron.

The bottom graph (dashed line) shows the decay probability of the vacuum state $\|0000\rangle$. It decays first to nearly zero but then recovers back to nearly one, once again as a consequence of the truncated Hilbert space. As one might expect, the norm of the single-particle electron wave function cannot exceed its initial value of 1, and even as the vacuum decays, its norm cannot grow beyond bound.

In order to estimate how the results generalize if more than just three levels are permitted, we have chosen ($h_{23}=0.8$, $h_{24}=0.4$, and $h_{34}=0.1$) to also permit the positron to have two states. The resulting data are shown in the sequence of graphs in Fig. 2. There are several differences. First, unlike a real system (with infinitely many levels), where the vacuum can decay permanently to zero, after some delay it begins to grow again. Second, also the three-particle norm $\langle \phi_{eep} | \phi_{eep} \rangle$ does not return to zero, which is again more typical of an irreversible growth pattern. Third, the electron-electron entanglement K_{ee} is also no longer able to return to a value close to 1; the gain in entanglement appears more permanent as the created positron has more degrees of freedom.

VII. OUTLOOK

We have shown the essential state model is pedagogically illustrative for the analysis of the complicated electron-positron creation process. In future studies we plan to include the nonperturbative interaction with a second quantized photon field, with the goal of computing the fermion-fermion interaction. The dream is to start with two electrons

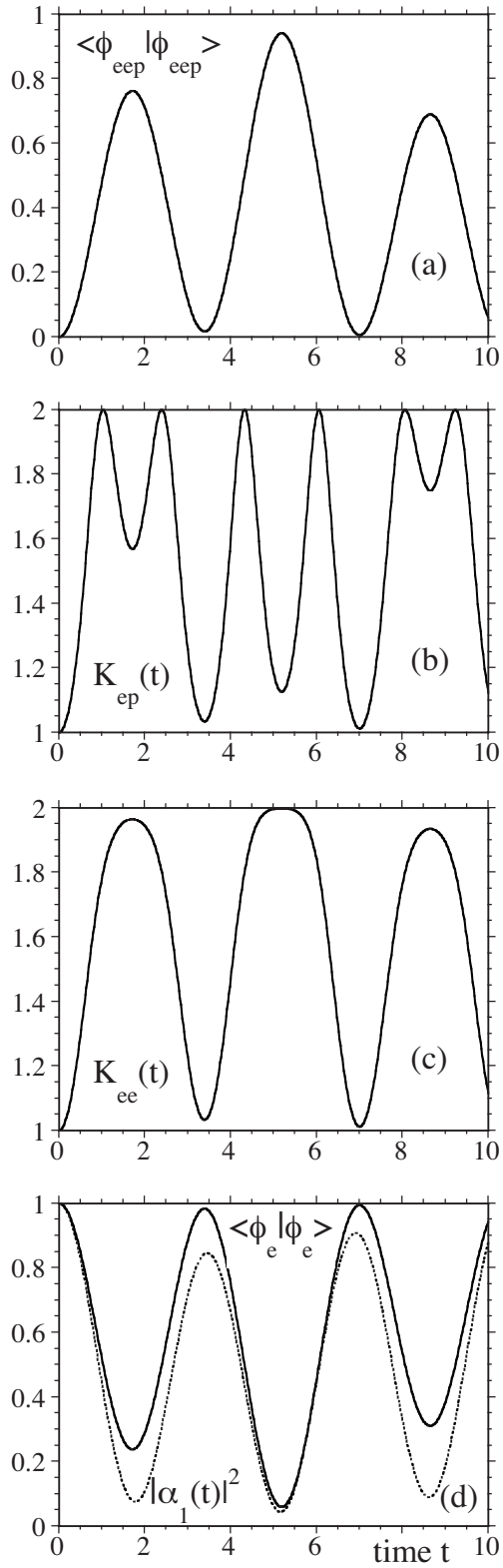

FIG. 1. The time dependence of various quantities for the creation of an electron-positron pair from vacuum in the presence of an initial electron. The positron has the state $|C3\rangle$ available, whereas the electron has $|1\rangle$ and $|2\rangle$. (a) The number of positrons. (b) Degree of entanglement between the groups of electrons and the positron. (c) Degree of entanglement between the two electrons. (d) Norm of the single-electronic state, the dashed line is the vacuum probability [e_1=0.1, e_2=0.2, e_3=−0.1, h_{12}=0.3, h_{13}=0.3, h_{23}=0.8].

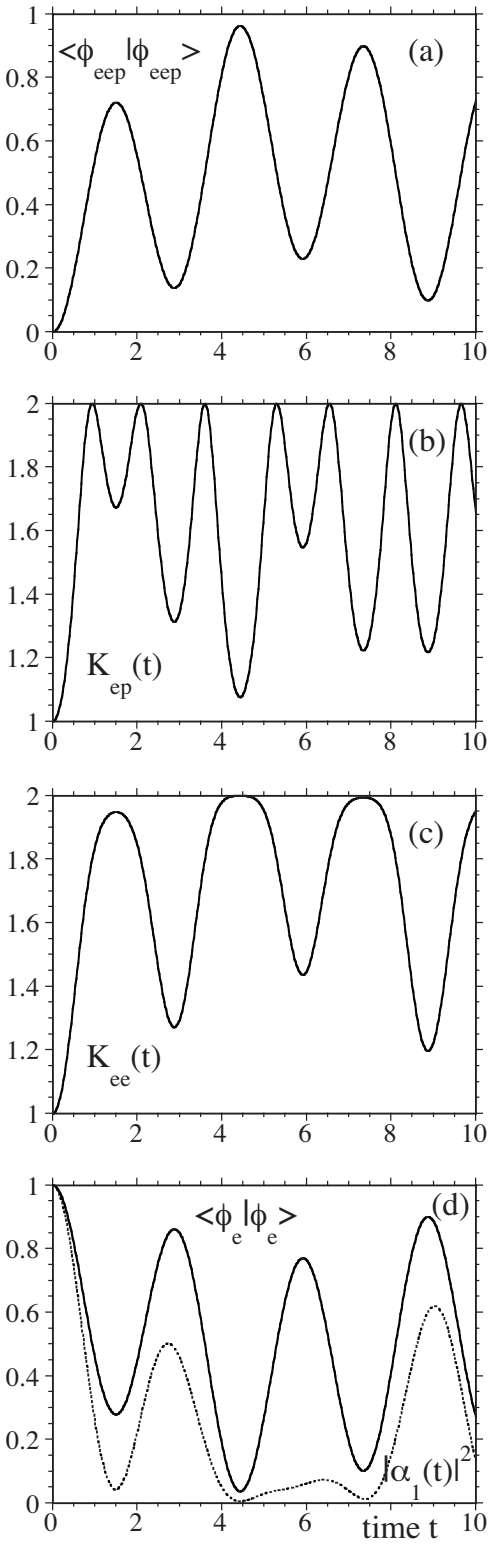

FIG. 2. The time dependence of various quantities for the creation of an electron-positron pair from vacuum in the presence of an initial electron. The positron has the states $|C3\rangle$ and $|C4\rangle$ available, whereas the electron has $|1\rangle$ and $|2\rangle$. (a) The number of positrons. (b) Degree of entanglement between the groups of electrons and the positron. (c) Degree of entanglement between the two electrons. (d) Norm of the single-electronic state, the dashed line is the vacuum probability [e_1=0.1, e_2=0.2, e_3=−0.1, h_{12}=0.3, h_{13}=0.3, h_{14}=0.5, h_{23}=0.8, h_{24}=0.4, h_{34}=0.1].

and observe how these two electrons begin to repel each other due to the mutual force between them, which is usually pictured based on an exchange of photons. In certain limits, this force is approximated by the Coulomb $1/r^2$ force. In this study, it would be computed without any approximations. Due to the increased number of dynamical degrees of freedom required in such a calculation, an *ab initio* reduction of the number of dynamically accessible states would be very beneficial.

Another challenge concerns the computation of true entanglements involving more than just pairwise couplings. The unitary quantum field theory provides a good framework to determine the degree of entanglement of groups of electrons with groups of positrons as the bits in the occupation number representation allow for an unambiguous separation of electronic and positronic states. In case of electron-electron entanglement, however, the unitary approach is no longer suitable; one must rely on nonunitary quantum field theory. The latter, however, is difficult to apply for states that contain a mixture of states with different numbers of particles.

ACKNOWLEDGMENTS

This work has been supported by the NSF. The authors have enjoyed several discussions with I. Bialynicki-Birula, S. Bowen, M. V. Fedorov, and H. R. Reiss. The authors also acknowledge support from the Research Corporation and NCSA for supercomputing time.

[1] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985).
[2] J. Eichler and W. Meyerhof, *Relativistic Collisions* (Academic, San Diego, 1995).
[3] M. E. Peskin and D. V. Schroeder, *Introduction to Quantum Field Theory* (ABP Westview Press, New York, 1995).
[4] O. Klein, Z. Phys. **53**, 157 (1929); **41**, 407 (1927).
[5] N. Dombey and A. Calogeracos, Phys. Rep. **315**, 41 (1999).
[6] A. Hansen and F. Ravndal, Phys. Scr. **23**, 1036 (1981).
[7] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. Lett. **93**, 043004 (2004).
[8] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. A **72**, 064103 (2005).
[9] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. Lett. **93**, 043004 (2004).
[10] P. Krekora, K. Cooley, Q. Su, and R. Grobe, Phys. Rev. Lett. **95**, 070403 (2005).
[11] D. P. Umstadter, C. Barty, M. Perry, and G. A. Mourou, Opt. Photonics News **9**(1), 41 (1998).
[12] G. A. Mourou, C. P. J. Barty, and M. D. Perry, Phys. Today **51**(1), 22 (1998).
[13] G. A. Mourou and V. Yanovsky, Opt. Photonics News **15**(5), 40 (2004).
[14] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover, New York, 1975).
[15] M. Crance and M. Aymar, J. Phys. B **13**, L421 (1980).
[16] Z. Bialynicka-Birula, J. Phys. B **17**, 3091 (1984).
[17] M. Edwards, L. Pan, and L. Armstrong, Jr., J. Phys. B **17**, L515 (1984).
[18] Z. Deng and J. H. Eberly, Phys. Rev. Lett. **53**, 1810 (1984).
[19] B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).
[20] R. H. Landau, *Quantum Mechanics II* (Wiley, New York, 1990).
[21] D. S. Koltun and J. M. Eisenberg, *Quantum Mechanics of Many Degrees of Freedom* (Wiley, New York, 1988).
[22] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1962).
[23] P. Krekora, Q. Su, and R. Grobe, J. Mod. Opt. **20**, 489 (2005).
[24] M. V. Fedorov, M. A. Efremov, and P. A. Volkov, Opt. Commun. **264**, 413 (2006).
[25] R. Grobe, K. Rzazewski, and J. H. Eberly, J. Phys. B **27**, L503 (1994).
[26] E. Schmidt, Math. Ann. **63**, 433 (1906).
[27] H. Everett, Rev. Mod. Phys. **29**, 454 (1957).
[28] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. A **73**, 022114 (2006).
[29] C. C. Gerry, Q. Su, and R. Grobe, Phys. Rev. A **74**, 044103 (2006).
[30] W.-C. Liu, J. H. Eberly, S. L. Haan, and R. Grobe, Phys. Rev. Lett. **83**, 520 (1999).