

Exponential enhancement of field-induced pair creation from the bosonic vacuum

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Using numerical solutions to quantum field theory, the creation of boson-antiboson pairs from the vacuum under a very strong localized external electric field is explored. The simulations reveal that the initial linear increase of the number of particles turns into an exponential growth. This self-amplification can be understood as the result of the interaction of the previously generated particles with the creation process. While the number of particles keeps increasing, the spatial shape of the (normalized) charge density of the created particles reaches a universal form that can be related to the bound states of the supercritical potential well. We accompany the space-time resolved quantum field theoretical simulations with a model calculation that allows us to interpret the numerical simulations in terms of simple classical mechanical concepts.

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The possibility of creating electron-positron pairs directly from the vacuum due to an external electric field is one of the most fascinating predictions of the Dirac equation [1]. While early experiments intended to demonstrate this principle through the collision of highly charged heavy ions have created positrons, it was not conclusively shown that these were exclusively due to the supercritical electric field [2]. There is some hope that very intense laser fields might become available in the next decade [3] to break down the vacuum and to demonstrate for the first time a direct conversion of light into matter. Several authors have computed the minimal required field strength to observe this fascinating process. It ranges from the Schwinger estimate [4] for a static field of about $E = 10^{18}$ V/m to the required laser intensity for electromagnetic fields of the order of 10^{26} W/cm².

In this Letter we present a prediction about the breakdown process of the vacuum for external fields that are so large that even bosonic particle-antiparticle pairs can be created. In contrast to the fermions that obey the Fermi exclusion principle and can completely suppress the pair-creation process after a certain time [5], the initial linear growth of the bosons turns into a self-amplification-like process of exponential growth. The lightest charged bosons are pions with a mass of about 280 times that of the electron. As the critical field strength scales proportionally to the square of the rest mass, one could postulate that the required field for the pion creation is about 10^5 times above the Schwinger limit. This parameter regime is obviously presently out of experimental range with laser sources, but due to the exponential self-amplification described below, further study might be worthwhile.

To compute the pair creation process for bosons, the bosonic quantum field operator $\hat{\Psi}(z, t)$ can be expanded as the sum (integral) over the (two-component) eigenstates with positive and negative energy of the force-free system:

$$\hat{\Psi}(z, t) = \sum_p \hat{b}_p(t) \phi_p(z) + \sum_n \hat{d}_n^\dagger(t) \phi_n(z). \quad (1)$$

Here the \hat{b}_p and \hat{d}_n are the annihilation operators associated with these states and fulfill the commutator relationships, $[\hat{b}_p, \hat{b}_{p'}^\dagger] = \delta_{pp'}$ and $[\hat{d}_n, \hat{d}_n^\dagger] = \delta_{nn'}$. The time dependence of these operators can be obtained from the Heisenberg equation of motion $i\partial\hat{b}_p/\partial t = [\hat{b}_p, \hat{H}]$ and similarly for \hat{d}_n . The

quantum field Hamiltonian \hat{H} is determined from the quantum mechanical Hamiltonian h via $\hat{H} = \int dz \hat{\Psi}^\dagger(z, t) \sigma_3 h \hat{\Psi}(z, t)$. We describe the dynamics with the Klein-Gordon Hamiltonian h , which in the Feshbach-Villars [6] representation $\phi(z, t) = [\phi_1^\dagger]$ takes the form $h = p^2/(2m)\sigma_5 + mc^2\sigma_3 + V(z)$. We focus our analysis on the direction of the force field (z axis). The scalar potential V is associated with the external force and is specified below, c is the speed of light, m is the mass of the boson, and $\sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the usual 2×2 Pauli matrix and the nilpotent matrix is denoted by $\sigma_5 \equiv \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. We note that h is Hermitian with regard to a generalized scalar product [6,7] that involves σ_3 . From now on we use atomic units, for which $q = \hbar = 1$ a.u. and $c = 137.036$ a.u. and denote the boson's reduced Compton wavelength with $\lambda = \hbar/(mc)$.

As a hybrid between a quantum mechanical state and an operator, the field $\hat{\Psi}$ fulfills both the Heisenberg as well as Schrödinger-like equation of motion, $i\partial\hat{\Psi}/\partial t = [\hat{\Psi}, \hat{H}(t)] = h\hat{\Psi}$. As a consequence, the time evolution for $\hat{\Psi}$ in Eq. (1) can be rewritten as $\hat{\Psi}(z, t) = \sum_p \hat{b}_p \phi_p(z, t) + \sum_n \hat{d}_n^\dagger \phi_n(z, t)$ where the states are the solutions to $i\partial\phi(z, t)/\partial t = h\phi(z, t)$. The latter can be obtained numerically [8] on a grid with N_z spatial and N_t temporal grid points using sophisticated FFT split-operator techniques with a typical CPU time of several days. Using the Bogoliubov transformations $\hat{b}_p(t) = \sum_{p'} \hat{b}_{p'} \langle p|\sigma_3|p'(t) \rangle + \sum_n \hat{d}_n^\dagger \langle p|\sigma_3|n(t) \rangle$ and similarly for $\hat{d}_n^\dagger(t)$ one can construct the quantum field operator from the complete set of all single-particle solutions $|p(t)\rangle$ and $|n(t)\rangle$.

A quantity of interest is the spatial density of the (positively charged) bosons, $\rho(z, t) \equiv \langle \text{vac} | \hat{\Psi}_+^\dagger(z, t) \sigma_3 \hat{\Psi}_+(z, t) | \text{vac} \rangle$, where $\hat{\Psi}_+$ is the positive frequency part of $\hat{\Psi}$ and $|\text{vac}\rangle$ is the initial (force-free) vacuum state. This density can be used to compute the total number of boson-antiboson pairs, defined as $N(t) \equiv \int dz \langle \text{vac} | \hat{\Psi}_+^\dagger(z, t) \sigma_3 \hat{\Psi}_+(z, t) | \text{vac} \rangle$. If one inserts the solution for $\hat{\Psi}(z, t)$ into this expression it simplifies to $N(t) = \sum_{n,p} |\langle p|\sigma_3|n(t) \rangle|^2$.

To analyze a specific external field, we chose the Sauter [9] potential well $V(z) = V \{ \tanh[(z - D/2)/W] - \tanh[(z + D/2)/W] \} / 2$ where D denotes the separation of both barriers and W is the width of each barrier. If the strength of the potential V exceeds $2mc^2$ for a sufficiently large D , this potential can create boson-antiboson pairs from the vacuum.

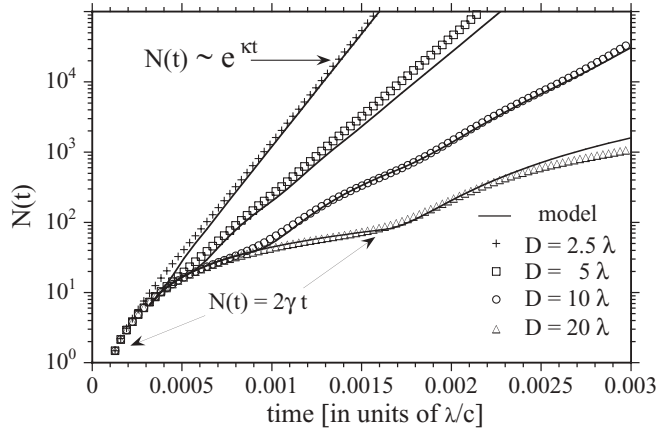


FIG. 1. The number of created boson-antiboson pairs $N(t)$ as a function of time for external potentials with four barrier separations D . The continuous line is the prediction from a simplified model based on Eq. (2). ($V = 3mc^2$, $W = 0.1\lambda$, $N_z = 1024$ numerical grid points for a numerical box $L = 140\lambda$, temporal step size $\Delta t = 0.06\lambda/c$.)

In Fig. 1 we show (on a logarithmic scale) the temporal growth of the particle number $N(t)$ for four potential wells that differ by their barrier separation D . Each graph is characterized by three regimes. For early times ($t \ll \lambda/c$) $N(t)$ grows quadratically associated with the abrupt turn-on of the force at $t = 0$. In the next regime ($\lambda/c < t < D/v$), $N(t)$ increases linearly $N(t) = 2\gamma t$. With v we denote a typical escape velocity of the created particles and γ is a constant to be determined below. For larger times ($t \gg D/v$), the growth finally becomes exponential.

The three observed temporal regimes have a direct interpretation. The particle pairs can be created only in those spatial regions where the external force $V'(z)$ is sufficiently large. For our Sauter potential well, these regions are centered around $z = \pm D/2$. The negatively charged bosons are ejected to $\pm\infty$ while the positive particles move inwards. If the spacing D between the two sides of the potential is large, the linear growth regime (characteristic of the mutually independent vacuum decays at each side [10]) can be established. Once the bosons that were created at the left side of the well have reached the right side of the well, the pair-creation rate at the right barrier becomes enhanced. Once this enhanced particle flux reaches the left barrier, it will increase the production there as well, leading after a certain time to a particle explosion characterized by an exponential growth.

The four continuous graphs in Fig. 1 are the predictions of a simple classical model for this process. One can show [10] that the pair-creation rate γ for a single barrier can be obtained from the energy integral over the corresponding quantum mechanical transmission coefficient $T(E)$ that would describe the supercritical scattering of an incoming boson off the potential barrier, $\gamma \equiv \int dE |T(E)| / (2\pi)$. The integration limits for the energy are between mc^2 and $V - mc^2$. It turns out that the same coefficient γ also describes the amount by which the pair creation process can be enhanced due to the already present bosons.

To introduce the model, let us define the particle numbers that are associated with the left and right barriers ($z = \pm D/2$) as $N_L(t)$ and $N_R(t)$. In the absence of the other barrier,

both numbers grow linearly in time for $t > \lambda/c$ according to $dN_{R,L}(t)/dt = \gamma$ with $N_{R,L}(t = 0) = 0$. As the rates of change of each density mutually enhance each other in a time-delayed way, we could model the time evolution by the differential equations $dN_R(t)/dt = \gamma + \gamma N_L(t - D/v)$ and $dN_L(t)/dt = \gamma + \gamma N_R(t - D/v)$, where D/v is again the characteristic time it takes the bosons to pass from one side of the well to the other to contribute to the enhancement. The resulting equation for the total number of particles $N \equiv N_L + N_R$ to model the time-delayed enhancement mechanism becomes

$$dN(t)/dt = 2\gamma + \gamma N(t - D/v). \quad (2)$$

This unusual differential equation is valid for $t > D/v$ and needs to be solved under the initial condition that $N(t) = 2\gamma t$ for $0 < t < D/v$, corresponding to the linear regime before the bosons can mutually enhance the growth rate. Solving this recursion-type differential equation we obtain

$$N(t) = \sum_{n=0}^{\infty} \theta(t; nD/v, (n+1)D/v) f_n(t), \quad (3a)$$

with

$$f_n(t) = \frac{2}{(n+1)!} \gamma^{n+1} (t - D/v)^{n+1} - 2 + \sum_{j=0}^n \frac{2\gamma^{n-j}}{(n-j)!} \times \sum_{m=0}^j \frac{(j-m+1)^m}{m!} (\gamma D/v)^m (t - nD/v)^{n-j}, \quad (3b)$$

where the switch function $\theta(t; \tau_1, \tau_2)$ ($\theta \equiv 1$ if $\tau_1 \leq t < \tau_2$ and $\theta \equiv 0$ otherwise) reflects the quasistepwise nature of the solution. The integer n reflects how often the bosons have traveled the distance D . We note that each of the expansion coefficients in Eq. (3b) is positive. For short times ($t < D/v$), the solution reproduces correctly the linear regime $N(t) = f_0(t) = 2\gamma t$, reflecting the two independent growth processes at $z = \pm D/2$. During the next time interval ($D/v \leq t \leq 2D/v$), we see the beginning of the self-amplification process, $N(t) = f_1(t) = 2\gamma t + \gamma(t - D/v)^2$.

The most interesting case is the long-time limit, characteristic of an exponential growth. To find the scaling of this exponent (which we denote by κ) with the parameters that characterize the potential, however, is nontrivial. As n plays the role of a discretized time [$n \approx tv/D$], we can insert $t = nD/v$ into Eq. (3) and obtain the simpler form, $f_n = -2 + 2 \sum_{m=0}^n \frac{(n-m+1)^m}{m!} (\gamma D/v)^m$. If we assume that $f_n = A \exp[\kappa nD/v]$ we can find the following solution for the exponent κ :

$$\kappa = \lim_{n \rightarrow \infty} v/(nD) \ln \left\{ \sum_{m=0}^n \frac{(n-m+1)^m}{m!} (\gamma D/v)^m \right\}. \quad (4)$$

If we try to solve our original time-delayed differential equation with the ansatz characteristic of the long-time behavior, $N(t) = A \exp(\kappa t)$, we would obtain a transcendental equation for the exponent $\kappa = \gamma \exp(-\kappa D/v)$. It turns out that Eq. (4) is the exact solution to this transcendental equation thus verifying it from an independent approach.

For spatially narrow quantum wells ($D/v \ll \gamma^{-1}$), we obtain $\kappa \approx \gamma \gamma^2 D/v$. In the opposite limit, ($D/v \gg \gamma^{-1}$), the exponent approaches zero very slowly, reflecting the fact that with increasing delay time the self-amplification is reduced.

In order to check the validity of our simple model of Eq. (2) to describe the pair-creation process, we have to calculate the most probable value for the velocity v . The energy spectrum of the created bosons for a single barrier is proportional to the transmission coefficient. $|T(E)|$ takes its maximum value at $E = E/2$. The corresponding velocity v associated with this energy amounts to $v = 2c[(c/2)^2 - m^2c^4]^{1/2}/V$. The four continuous lines in Fig. 1 represent the predictions of Eqs. (3). Except the negligible temporal shift associated with the turn-on time of the potential, there are no adjustable parameters. The form of the potential well $V(z)$ leads to its unique quantum mechanical transmission coefficient such that the parameters V and γ follow. They amount to $v = 0.74c$ and $\gamma = 1.5c/\lambda$ for our potential with $V = 3mc^2$ and $W = 0.1\lambda$. In view of the simplistic nature of our model, the agreement with the result of the full quantum field theoretical calculations is quite remarkable and shows that the differential equation (2) captures the essential physics of the self-amplification process.

In order to find any aspects in the dynamics that cannot be modeled by classical mechanical means we have graphed in Fig. 2 the growth of the spatial charge density of the bosons. For comparison, we have included at the bottom the shape of the potential $V(z)$ which shows that at early times the density grows at those locations ($z = \pm 5\lambda$) at which the force $V'(z)$ is maximum. The bosons are then accelerated toward the center of the potential while the oppositely charged boson (not shown) escape to $\pm\infty$. When the two portions that were created on opposite sides of the potential begin to cross each other ($t \approx 17 \times 0.3\lambda/c$), there is no interference due to the entanglement with their environment [5]. Once the particles

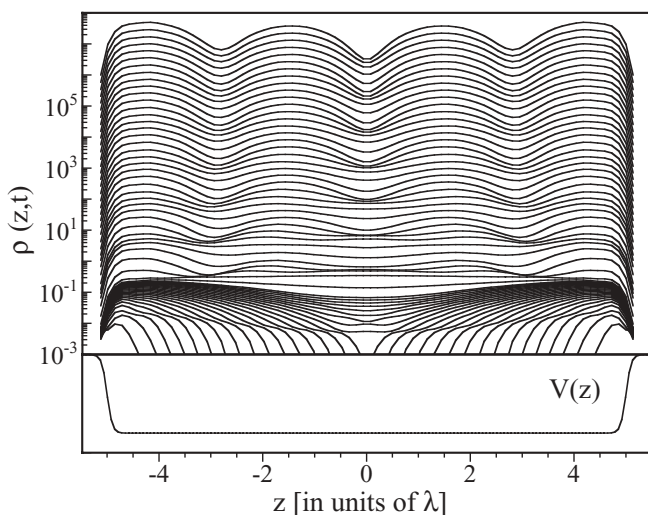


FIG. 2. Forty-eight snapshots of the spatial density for the bosons $\rho(z,t)$ at times $t_j = j \cdot 0.3\lambda/c$ with $j = 1, 2, \dots, 25$ and $j = 30, 40, \dots, 250$ as created by the potential $V(z)$ shown on the bottom. The data have not been shifted in the vertical direction. ($V = 3mc^2$, $W = 0.1\lambda$, $D = 10\lambda$, $N_z = 1024$ grid points in a numerical box $L = 140\lambda$, temporal step size $\Delta t = 0.06\lambda/c$.)

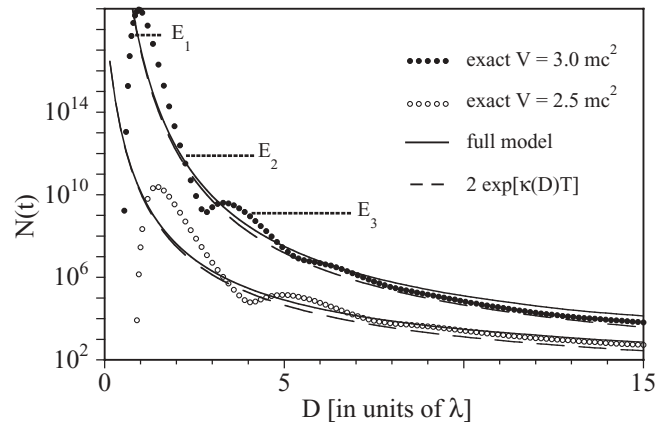


FIG. 3. The final number of created boson pairs after time $T = 60\lambda/c$ for two potential strengths $V = 2.5mc^2$ and $3mc^2$ as a function of the spatial width D of the potential. The continuous line is the prediction for $N(T)$ according to Eq. (3) and the dashed line is just the exponentiated long-time rate κ from Eq. (4). The left value of each horizontal dotted line indicates the characteristic width D when the bound state dives into the negative energy continuum ($E_j \approx -mc^2$) while the right most value characterizes D when the corresponding bound state is under the threshold at $E_j \approx -mc^2 - V/2$. The data correspond to the lowest three bound states. ($W = 0.1\lambda$, $N_z = 1024$ numerical grid points for a numerical box $L = 140\lambda$, temporal step size $\Delta t = 0.06\lambda/c$.)

reach the other pair-creation zone ($t \approx 30 \times 0.3\lambda/c$), they enhance this process as is seen by the logarithmic scale.

Even though the total number of particles grows exponentially in the long-time limit, the spatial charge density (when renormalized) approaches a universal shape. For our parameters in the figure ($D = 10\lambda$ and $V = 3mc^2$) this distribution is four-peaked. In separate simulations for which D was increased we note that the number of peaks increased as well. The densities resemble the structures of the bound states that can be associated with the potential $V(z)$. Due to the supercritical nature of the potential, the relevant bound states have dived into the negative energy continuum and are therefore manifest only in the form of the avoid level crossings [1,5].

Let us now focus our attention on the sensitivity of the exponential growth with the width D of the potential. In Fig. 3 we show the final number of created pairs $N(T)$ as a function of the width. For small values of D , the exponential regime is easily being reached at the final time ($T = 60\lambda/c$) whereas for $D > vT$ the bosons did not even have enough time to reach the opposite side of the potential. It is thus in the linear growth regime. The continuous lines are again the predictions by the model solution of Eq. (3). For comparison, we have also graphed $N(T) = 2 \exp[\kappa(D)T]$ based on Eq. (4) showing the differences from a purely exponential decay with increasing D . The qualitative agreement between our model predictions and the exact data is again astonishing.

The differences might be due to the energies of the associated quantum mechanical bound states. The overall trend of a decreasing final number of boson pairs with increasing D is interrupted with the occurrence of bound states that have dived into the negative energy continuum. The critical value (often

associated with the onset of supercritical behavior for the corresponding fermionic systems) of D is indicated by the left value of each dashed horizontal line. The discrepancy between our model and the quantum field theoretical data is largest for small D , for which even the lowest lying bound states has not yet dived into the negative energy continuum. Analyzing the decay at each barrier separately, we have argued that most bosons are being created with an energy corresponding to $E = V/2$. We believe that if this particular energy can be matched with the energy difference $-mc^2 - E_j$ (associated with the location of the bound states E_j embedded into the negative continuum), a maximum amplification should occur. This requires the (formerly) bound state to take the energy $E_j = -mc^2 - V/2$. We have marked the corresponding value for D by the right most value of the dotted horizontal line. The locations of the first three characteristic regions in D agree with the local maxima of $N(T)$. This illustrates that not every detail of the bosonic self-amplification process can be explained solely in terms of classical mechanical concepts.

It is quite interesting to compare the exponential bosonic pair-creation process with the corresponding fermionic process that occurs at smaller electric fields, which has been studied extensively recently. Here the electrons are ejected to infinity, while the created positrons are bound by the potential. Once the created positrons reach the opposite side of the potential barrier, however, they suppress the pair-creation process as a direct consequence of the Pauli-exclusion principle until it comes to a complete halt. This situation is also related to the Klein-paradox in which an incoming fermion suppresses the pair-creation process at the barrier [11,12]. This behavior is opposite to the bosonic enhancement discussed in this work. As the bosons can multiply occupy each mode one might also investigate the statistical properties of the created states,

similarly as has been done in quantum optical systems for photons [13].

Like many self-amplifying processes in nature there are also here additional mechanisms that would eventually stop the creation of an infinite number of particles. Probably most importantly, the present theory does not include the exchange of additional force-intermediating bosons such as photons, which would lead to a Coulomb-like repulsion between the created bosons inside the well. Due to computational limitations, this is a general problem and has not been addressed even within the context of electron-positron pair creation. While the predicted effect might have a direct relevance to several astrophysical pair-creation processes such as Hawking radiation or neutron stars, on earth the required field strength could be generated by heavy-ion collisions. Also, sufficiently focused laser fields might become available in the future such that any standing wave patterns could provide the required high-intensity quantum wells for the predicted effect. As the breakdown process of the fermionic vacuum is expected to occur at much lower fields, the predicted cloud of created electron-positron pairs might screen off a slowly ramped-up force field. As a result of this reduction in effective field strength the required threshold for the production of bosons might be difficult to reach. But all of these numerous possibilities are just speculations at the moment and much more systematic studies are required.

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