Author(s): PETER WALTERS
Article Title: A natural space of functions for the Ruelle operator theorem
Year of publication: 2007
Link to published version: http://dx.doi.org/10.1017/S0143385707000028
Publisher statement: None

Ergod. Th. & Dynam. Sys. (2007), **27**, 1323–1348 © *2007 Cambridge University Press*
doi:10.1017/S0143385707000028 *Printed in the United Kingdom*

A natural space of functions for the Ruelle operator theorem

PETER WALTERS

Department of Mathematics, University of Warwick, Coventry CV4 7AL, UK
(e-mail: p.walters@warwick.ac.uk)

(*Received* 11 *August* 2006 *and accepted in revised form* 25 *January* 2007)

Abstract. We study a new space, $R(X)$, of real-valued continuous functions on the space X of sequences of zeros and ones. We show exactly when the Ruelle operator theorem holds for such functions. Any g-function in $R(X)$ has a unique g-measure and powers of the corresponding transfer operator converge. We also show $\mathrm{Bow}(X, T) \neq W(X, T)$ and relate this to the existence of bounded measurable coboundaries, which are not continuous coboundaries, for the shift on the space of bi-sequences of zeros and ones.

0. *Introduction*

We study a family of continuous functions on the space, $X = \prod_0^\infty \{0, 1\}$, of sequences $x = (x_n)_0^\infty$ of zeros and ones. This family, $R(X)$, is well behaved with respect to the Ruelle operator theorem (also called the Ruelle–Perron–Frobenius theorem). This theorem concerns the Ruelle transfer operator $\mathcal{L}_\varphi$ on the Banach space $C(X)$ of real-valued continuous functions on X. With suitable assumptions on $\varphi \in C(X)$ there is a number $\lambda > 0$ and some $h \in C(X)$ with $h > 0$ and $\mathcal{L}_\varphi h = \lambda h$, some probability measure ν on X with $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and, for all $f \in C(X)$, $\mathcal{L}_\varphi^n f/\lambda^n$ converges, in the sup norm on $C(X)$, to $\left(\int f \, d\nu\right)h$. Also $\mu_\varphi = h\nu$ turns out to be the unique equilibrium state of φ with respect to the shift transformation T on X. When φ is in our space $R(X) \subset C(X)$ we obtain necessary and sufficient conditions for the existence of such an eigenfunction h, and we show that the existence of h forces the rest of the Ruelle operator theorem to hold. Moreover, if $\varphi \in R(X)$ and an eigenfunction h exist, then $g = e^\varphi h/\lambda h \circ T \in R(X)$ and also $\log g \in R(X)$. This allows us to reduce the study of certain $\varphi \in R(X)$ to that of g-functions in $R(X)$. The space $R(X)$ includes the functions studied by Hofbauer [**Ho**]. These include examples of functions of the type devised by Fisher, without unique equilibrium states [**Fi**].

In §1 we define our space $R(X)$ and obtain necessary and sufficient conditions for a function $\varphi \in R(X)$ to be in the space $\mathrm{Bow}(X, T)$, necessary and sufficient conditions for $\varphi \in R(X)$ to be in $W(X, T)$, and necessary and sufficient conditions for $\varphi \in R(X)$ to be a coboundary. The spaces $\mathrm{Bow}(X, T)$, $W(X, T)$ and $\mathrm{Cob}(X, T)$ are important in the study of transfer operators and equilibrium states. We give examples from $R(X)$ of functions in $\mathrm{Bow}(X, T)$ but not in $W(X, T)$. This type of example can be modified to show that $\mathrm{Bow}(X, T)\backslash W(X, T)$ is non-empty for any non-trivial subshift of finite type $T : X \to X$.

In §2 we study those members of $R(X)$ which are g-functions for the shift T. Each such g has a unique g-measure, which we describe. Also if $\mathcal{L}$ denotes the transfer operator of $\log g$, then, for all $f \in C(X)$, $\mathcal{L}^n f$ converges uniformly on X to a constant $\mu(f)$ as $n \to \infty$. This result had been proved for a smaller class than $R(X)$ as part of the thesis of Hulse [**Hu**].

In §3 we investigate the Ruelle operator theorem for $\varphi \in R(X)$. In Theorem 3.1 we obtain necessary and sufficient conditions for the existence of a positive eigenfunction for $\mathcal{L}_\varphi$. These turn out to be necessary and sufficient for the whole of the conclusion of the Ruelle operator theorem. If $\varphi \in R(X) \cap \mathrm{Bow}(X, T)$ the necessary and sufficient conditions hold. We give examples of $\varphi \in R(X)$ where these conditions do not hold.

In §4 we use $R(X)$ to obtain a class of continuous functions on the two-sided shift space $\widehat{X} = \{0, 1\}^Z$ which are bounded measurable coboundaries but not continuous coboundaries for the shift S on $\hat{X}$.

We now explain our notation and terminology. Let $X = \prod_0^\infty \{0, 1\}$ be the full one-sided shift space with symbols 0 and 1 and let $T : X \to X$ denote the one-sided shift transformation. Points of X are sequences $x = (x_n)_0^\infty$ of zeros and ones. The topology on X is the direct product of the discrete topology on $\{0, 1\}$. If $i \geq 0$, $j \geq 1$ and $a_0, \ldots, a_{j-1} \in \{0, 1\}$ then ${}_i[a_0 \ldots a_{j-1}]_{i+j-1}$ or ${}_i[a_0 \ldots a_{j-1}]$ denote the set $\{x = (x_n)_0^\infty \mid x_{k+i} = a_k, 0 \leq k \leq j - 1\}$. Such a set is called a cylinder set based at coordinate i. All cylinder sets are finite unions of cylinder sets based at coordinate zero, and these form a basis for the topology. Note that $T^{-i}\ {}_0[a_0 \ldots a_{j-1}] = {}_i[a_0 \ldots a_{j-1}]$. A metric on X with this topology is given by: if $x \neq y$, $d(x, y) = 1/(j + 1)$ if j is the smallest non-negative integer with $x_j \neq y_j$.

If $j \geq 1$ and $a_0, \ldots, a_{j-1} \in \{0, 1\}$ then, if $x \in X$, $a_0 \ldots a_{j-1}x$ denotes the point $z = (z_n)_0^\infty$ of X with $z_i = a_i$ for $0 \leq i \leq j - 1$ and $z_{i+j} = x_i$ for $i \geq 0$. If $j \geq 1$ then $0^j x$ is the point $z = (z_n)_0^\infty$ with $z_i = 0$, $0 \leq i \leq j - 1$, and $z_{j+i} = x_i$ for $i \geq 0$. The point 0^∞ is the sequence with all entries zero and if $j \geq 1$ and $a_0, \ldots, a_{j-1} \in \{0, 1\}$ then $a_0 \ldots a_{j-1}0^\infty$ is the point $z = (z_n)$ with $z_n = a_n$, $0 \leq n \leq j - 1$, and $z_{j+i} = 0$ for $i \geq 0$. If $j \geq 1$ and $a_0, \ldots, a_{j-1} \in \{0, 1\}$ then $(a_0 \ldots a_{j-1})^\infty$ is the point $z = (z_n)_0^\infty$ with $z_{mj+i} = a_i$ for $0 \leq i \leq j - 1$ and $m \geq 0$. Such points are exactly the points $z \in X$ with $T^j z = z$.

Let $C(X)$ denote the Banach space of all real-valued continuous functions on X, equipped with the supremum norm. Continuity properties of a function $f : X \to \mathbb{R}$ can often be expressed using the sequence of numbers $\{v_n(f)\}_1^\infty$ defined by

$$v_n(f) = \sup\{f(x) - f(y) \mid x, y \in X \text{ and } x_i = y_i \text{ for } 0 \leq i \leq n - 1\}.$$

For example $f \in C(X)$ if and only if $\mathrm{v}_n(f) \to 0$.

We let $M(X)$ denote the space of all probability measures on the Borel subsets of X, equipped with the weak*-topology, and let $M(X, T)$ denote the non-empty subset of T-invariant members of $M(X)$. We say that $\tau \in M(X)$ has support X if $\tau(U) > 0$ for every non-empty open set U. If $\varphi \in C(X)$ we let $P(T, \varphi)$ denote the pressure of T at φ (see [**W1**]), and let $T_n\varphi$ be the function $\sum_{i=0}^{n-1} \varphi \circ T^i$. The Ruelle operator of $\varphi \in C(X)$ will be denoted by $\mathcal{L}_\varphi : C(X) \to C(X)$, so that $(\mathcal{L}_\varphi f)(x) = \sum e^{\varphi(y)} f(y)$ where the sum is over all $y \in T^{-1}x$. Hence $(\mathcal{L}_\varphi f)(x) = e^{\varphi(0x)} f(0x) + e^{\varphi(1x)} f(1x)$.

The dual operator $\mathcal{L}_\varphi^*$ always has an eigenmeasure in $M(X)$, i.e. there exist $\nu \in M(X)$ and $\lambda > 0$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ (see [**W2**]).

We consider two spaces of functions which are important in studying equilibrium states. These spaces can be defined for a general continuous transformation $T : X \to X$ of a compact metric space. We say that $\varphi \in C(X)$ belongs to $\text{Bow}(X, T)$ if there exist $\delta > 0$, $C > 0$ with the property that whenever $n \geq 1$ and $x, y \in X$ satisfy $d(T^i x, T^i y) < \delta$ for all $0 \leq i \leq n-1$ then $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq C$ (see [**Bow, W4, W5, W6**]). We say that $\varphi \in C(X)$ belongs to $W(X, T)$ if for all $\epsilon > 0$ there exists $\delta > 0$ with the property that whenever $n \geq 1$ and $x, y \in X$ satisfy $d(T^i x, T^i y) < \delta$ for all $0 \leq i \leq n-1$ then $|(T_n\varphi)(x) - (T_n\varphi)(y)| < \epsilon$ (see [**Bou, W5, W6**]). Clearly $W(X, T) \subset \text{Bow}(X, T)$. For the one-sided shift $T : X \to X$ on the space $X = \prod_0^\infty \{0, 1\}$, which we are studying in this paper, we have $\varphi \in \text{Bow}(X, T)$ if and only if $\varphi \in C(X)$ and there exists $p \geq 0$ with $\sup_{n \geq 1} \mathrm{v}_{n+p}(T_n\varphi) < \infty$. This latter condition is equivalent to $\sup_{n\geq 1} \mathrm{v}_n(T_n\varphi) < \infty$. Also $\varphi \in W(X, T)$ if and only if $\sup_{n \geq 1} \mathrm{v}_{n+p}(T_n\varphi) \to 0$ as $p \to \infty$.

In [**W3**] the author showed that, for a topologically mixing subshift of finite type, if $\varphi \in W(X, T)$ then the Ruelle operator theorem holds (that is, there exist $\lambda > 0$, $\nu \in M(X)$, and $h \in C(X)$ with $h > 0$ and $\int h \, d\nu = 1$ such that $\mathcal{L}_\varphi h = \lambda h$, $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and, for all $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \underline{\to} h(x) \int f \, d\nu,$$

where $\underline{\to}$ denotes uniform convergence on X), φ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension. Here $\mu_\varphi = h\nu$, and μ_φ is the unique g-measure for the g-function $g(x) = e^{\varphi(x)} h(x)/\lambda h(Tx)$. In [**W4**], the author considered these questions for $\varphi \in \text{Bow}(X, T)$ and proved a weakened version of the Ruelle operator theorem. Each $\varphi \in \text{Bow}(X, T)$ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension [**W6**].

We shall also use the space of continuous coboundaries. If $T : X \to X$ is any continuous transformation of a compact metric space then the space of continuous coboundaries for T is $\text{Cob}(X, T) = \{f \in C(X) \mid \exists l \in C(X) \text{ with } f = l \circ T - l\}$. Such a function l is called a cobounding function for f. We have $\text{Cob}(X, T) \subset W(X, T)$. Coboundaries are important in the study of equilibrium states.

1. *The space $R(X)$*

We now define the space $R(X)$ of functions on $X = \prod_{n=0}^\infty \{0, 1\}$. A function $\varphi \in C(X)$ is in the space $R(X)$ if it is defined in the following way: there are four convergent sequences of real numbers $(a_n)_2^\infty \to a$, $(b_n)_1^\infty \to b$, $(c_n)_2^\infty \to c$, $(d_n)_1^\infty \to d$ and for all $z \in X$, for all $p \geq 2$, for all $q \geq 1$, $\varphi(0^p 1 z) = a_p$, $\varphi(0 1^q 0 z) = b_q$, $\varphi(1^p 0 z) = c_p$, $\varphi(1 0^q 1 z) = d_q$, $\varphi(0^\infty) = a$, $\varphi(01^\infty) = b$, $\varphi(1^\infty) = c$ and $\varphi(10^\infty) = d$. So at a point with initial symbol 0 the value of φ is a_p if the initial block of zeros has length $p \geq 2$, but if the initial zero is immediately followed by a block of ones of length $q \geq 1$ the value of φ is b_q. Similarly if the initial symbol is 1.

The space $R(X)$ is a vector subspace of $C(X)$ and $\varphi \in R(X)$ if and only if $e^{\varphi} \in R(X)$.

We now characterize the spaces $R(X) \cap \mathrm{Bow}(X, T)$ and $R(X) \cap W(X, T)$ and show that they differ.

THEOREM 1.1. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^{\infty} \to a$, $(b_q)_1^{\infty} \to b$, $(c_p)_2^{\infty} \to c$, $(d_q)_1^{\infty} \to d$ as above. Then we have the following:*

(i) *$\varphi \in \mathrm{Bow}(X, T)$ if and only if $\sum_{n=2}^{\infty}(a_n - a)$ and $\sum_{n=2}^{\infty}(c_n - c)$ both have bounded sequences of partial sums;*

(ii) *$\varphi \in W(X, T)$ if and only if $\sum_{n=2}^{\infty}(a_n - a)$ and $\sum_{n=2}^{\infty}(c_n - c)$ are both convergent;*

(iii) *$\varphi \in \mathrm{Cob}(X, T)$ if and only if $b_1 + d_1 = 0$ and, for all $p \geq 2$, $b_p + d_1 + \sum_{i=2}^{p} c_i = 0$ and $d_q + b_1 + \sum_{i=2}^{p} a_i = 0$.*

When these conditions hold the cobounding function $k \in C(X)$ has the form $k((0^q 1z)) = \alpha_q$, $q \geq 1$, $z \in X$, $k((1^q 0z)) = \beta_q$, $q \geq 1$, $z \in X$, $k(0^{\infty}) = \alpha$, $k(1^{\infty}) = \beta$ where $\alpha_q \to \alpha$, $\beta_q \to \beta$.

Note that when the equations in (iii) hold then $\sum_{i=2}^{\infty} a_i$ converges so $a = 0$. Similarly $c = 0$ when the equations in (iii) hold.

Note that the conditions for $\varphi \in \mathrm{Bow}(X, T)$ and $\varphi \in W(X, T)$ do not involve the sequences $(b_n)_1^{\infty}$ and $(d_n)_1^{\infty}$. In the condition in (iii) once b_1 is chosen then $(b_i)_{i=2}^{\infty}$ and $(d_j)_{j=1}^{\infty}$ are determined in terms of b_1, $(a_n)_2^{\infty}$ and $(c_n)_2^{\infty}$.

We prove Theorem 1.1 using the following lemma.

LEMMA 1.2. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^{\infty} \to a$, $(b_q)_1^{\infty} \to b$, $(c_p)_2^{\infty} \to c$ and $(d_q)_1^{\infty} \to d$ as in Theorem 1.1. Then we have the following.*

(i) *For $n \geq 2$,*

$$\mathrm{v}_n(\varphi) = \sup\{\max(a_{n+t} - a_{n+s}, b_{n+t-1} - b_{n+s-1}, c_{n+t} - c_{n+s}, d_{n+t-1} - d_{n+s-1}) : s, t \geq 0\}.$$

Hence if

$$\mathcal{C}_n = \sup\{\max(|a_j - a|, |b_{j-1} - b|, |c_j - c|, |d_{j-1} - d|) : j \geq n\}$$

then $\mathcal{C}_n \leq \mathrm{v}_n(\varphi) \leq 2\mathcal{C}_n$.

(ii) *For $n, N \geq 2$,*

$$\mathrm{v}_{n+N}(T_n\varphi) = \max\Bigg(\sup_{i,j \geq N} [(a_{i+1} + \cdots + a_{i+n}) - (a_{j+1} + \cdots + a_{j+n})],$$
$$\sup_{i,j \geq N,\ 1 \leq k \leq n-1} [d_{k+i} - d_{k+j} + (a_{i+1} + \cdots + a_{i+k}) - (a_{j+1} + \cdots + a_{j+k})],$$

$$\sup_{i,j\geq N}(b_i - b_j),\ \sup_{i,j\geq N}[(c_{i+1}+\cdots+c_{i+n})-(c_{j+1}+\cdots+c_{j+n})],$$
$$\sup_{i,j\geq N,\ 1\leq k\leq n-1}[b_{k+i}-b_{k+j}+(c_{i+1}+\cdots+c_{i+k})$$
$$-(c_{j+1}+\cdots+c_{j+k})],\ \sup_{i,j\geq N}(d_i-d_j)\Big).$$

Hence if $D_N = \sup_{i,j\geq N}(d_i - d_j)$, $B_N = \sup_{i,j\geq N}(b_i - b_j)$ *and*

$$A_{n,N} = \max\Big(B_N, D_N, \sup_{i\geq N,\ 1\leq k\leq n}|(a_{i+1}+\cdots+a_{i+k})-ka|,$$
$$\sup_{i\geq N,\ 1\leq k\leq n}(|(c_{i+1}+\cdots+c_{i+k})-kc|)\Big)$$

then for $n, N \geq 2$

$$A_{n,N} - D_N - B_N \leq \mathrm{v}_{n+N}(T_n\varphi) \leq 2A_{n,N} + D_N + B_N.$$

Proof. (i) Let $n \geq 2$ and let $x, y \in X$ have $(x_0,\ldots,x_{n-1}) = (y_0,\ldots,y_{n-1})$.

Suppose $x_0 = y_0 = 0$.

If $x, y \in {}_0[0^p1]$ for some $p \geq 2$ then $\varphi(x) = \varphi(y)$, and if $x, y \in {}_0[01^q0]$ for some $q \geq 1$ then $\varphi(x) = \varphi(y)$.

If $x \in {}_0[0^{n+t}1]$ for some $t \geq 0$ and $y \in {}_0[0^{n+s}1]$ for some $s \geq 0$ then $\varphi(x) - \varphi(y) = a_{n+t} - a_{n+s}$. If $x \in {}_0[0^{n+t}1]$ for some $t \geq 0$ and $y = 0^\infty$ then $\varphi(x) - \varphi(y) = a_{n+t} - a$.

If $x \in {}_0[01^{n-1+t}0]$ for some $t \geq 0$ and $y \in {}_0[01^{n-1+s}0]$ for some $s \geq 0$ then $\varphi(x) - \varphi(y) = b_{n+t-1} - b_{n+s-1}$. If $x \in {}_0[01^{n-1+t}0]$ and $y = (01^\infty)$ then $\varphi(x) - \varphi(y) = b_{n+t-1} - b$.

When $x_0 = y_0 = 1$ we get similar results and hence the expression in (i). The inequality involving $\mathcal{C}_n$ follows from the triangle inequality.

(ii) Let $n, N \geq 2$. Let $x, y \in X$ have $(x_0,\ldots,x_{n+N-1}) = (y_0,\ldots,y_{n+N-1})$.

Consider the case $x_{n-1} = 0 = y_{n-1}$; the case when $x_{n-1} = 1 = y_{n-1}$ is handled in a similar way. Consider firstly when $(x_{n-1}, x_n) = (0, 0) = (y_{n-1}, y_n)$.

Suppose $(x_0,\ldots,x_{n-1}) = 0^n$. If $x \in {}_0[0^{n+i}1]$ for some $i \geq N$ and $y \in {}_0[0^{n+j}1]$ for some $j \geq N$ then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = (a_{n+i}+\cdots+a_{1+i}) - (a_{n+j}+\cdots+a_{1+j}).$$

If $x \in {}_0[0^{n+i}1]$ for some $i \geq N$ and $y = (0^\infty)$ then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = (a_{n+i}+\cdots+a_{1+i}) - na.$$

If $x \in {}_0[0^{n+i}1]$ for some $1 \leq i \leq N-1$ then $y \in {}_0[0^{n+i}1]$ and $(T_n\varphi)(x) = (T_n\varphi)(y)$.

Suppose $x_r = 1$ for some $0 \leq r \leq n-2$, so that $x \in {}_{n-1-k}[10^{k+i}1]$ for some $1 \leq k \leq n-1$ and $i \geq 1$ or $T^{n-1-k}x = (10^\infty)$. If $x \in {}_{n-1-k}[10^{k+i}1]$ for some $1 \leq k \leq n-1$ and $1 \leq i \leq N-1$ then $y \in {}_{n-1-k}[10^{k+i}1]$ and $(T_n\varphi)(x) = (T_n\varphi)(y)$. If $x \in {}_{n-1-k}[10^{k+i}1]$ for some $1 \leq k \leq n-1$ and some $i > N-1$ then either $y \in {}_{n-1-k}[10^{k+j}1]$ for some $j > N-1$ and then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = d_{k+i} - d_{k+j} + (a_{k+i}+\cdots+a_{1+i}) - (a_{k+j}+\cdots+a_{1+j}),$$

or $T^{n-1-k}y = (10^\infty)$ and then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = d_{k+i} + (a_{k+i} + \cdots + a_{1+i}) - d - (n-1)a.$$

If $T^{n-1-k}x = (10^\infty)$ then either $y \in {}_{n-1-k}[10^{k+j}1]$ for some $j > N-1$ and then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = d + (n-1)a - d_{k+j} - (a_{k+j} + \cdots + a_{1+j}),$$

or $x = y$.

Now consider when $(x_{n-1}, x_n) = (0, 1)$. Either $x \in {}_{n-1}[01^i0]$ for some $i \geq 1$, or $T^{n-1}x = (01^\infty)$. Suppose $x \in {}_{n-1}[01^i0]$ for some $i \geq 1$. If $i < N$ then $y \in {}_{n-1}[01^i0]$ and $(T_n\varphi)(x) = (T_n\varphi)(y)$. If $i \geq N$ then either $y \in {}_{n-1}[01^j0]$ for some $j \geq N$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b_i - b_j$, or $T^{n-1}y = (01^\infty)$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b_i - b$. If $T^{n-1}x = (01^\infty)$ then either $y \in {}_{n-1}[01^j0]$ for some $j \geq N$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b - b_j$, or $y = x$.

The corresponding reasoning can be used when $x_{n-1} = 1 = y_{n-1}$ and we get the equality in (ii). The inequalities follow from the triangle inequality. □

Proof of Theorem 1.1. Parts (i) and (ii) follow from Lemma 1.2(ii), since $\varphi \in \mathrm{Bow}(X, T)$ means $\sup_{n\geq 1} \mathrm{v}_{n+N}(T_n\varphi) < \infty$ for some $N \geq 2$ and $\varphi \in W(X, T)$ means $\sup_{n\geq 1} \mathrm{v}_{n+N}(T_n\varphi) \to 0$ as $N \to \infty$.

We turn to the proof of part (iii). Suppose $\varphi \in \mathrm{Cob}(X, T)$. If $T^n(x) = x$ then $T_n\varphi(x) = 0$. If we let $x = (01)^\infty$ then $\varphi((01)^\infty) + \varphi((10)^\infty) = 0$ so $b_1 + d_1 = 0$. Let $p \geq 2$ and let $x = (0^p1)^\infty$. Since $T^{p+1}(x) = x$ we have $(T_{p+1}\varphi)(x) = 0$. Hence $a_p + a_{p-1} + \cdots + a_2 + b_1 + d_p = 0$. Similarly, taking $x = (1^p0)^\infty$ gives $c_p + c_{p-1} + \cdots + c_2 + d_1 + b_p = 0$. Hence we get the equations in (iii).

Now suppose the equations in (iii) hold and we show $\varphi \in \mathrm{Cob}(X, T)$. We have $a = 0 = c$. Let α_1 be any real number. Define α_p for $p \geq 2$ by $\alpha_p = \alpha_1 - \sum_{i=2}^{p} a_i = \alpha_1 + b_1 + d_p$, and define β_q, $q \geq 1$, by $\beta_q = \alpha_1 + b_q$. Then $\alpha_p \to \alpha_1 + b_1 + d$ and $\beta_q \to \alpha_1 + b$.

Define $k : X \to \mathbb{R}$ by $k((0^q1z)) = \alpha_q$, $q \geq 1$, $z \in X$, $k((1^q0z)) = \beta_q$, $k(0^\infty) = \alpha_1 + b_1 + d$, $k(1^\infty) = \alpha_1 + b$. Then $k \in C(X)$ and we show that $k(Tx) - k(x) = \varphi(x)$, $x \in X$.

If $x \in {}_0[0^p1]$ with $p \geq 2$ then $k(Tx) - k(x) = \alpha_{p-1} - \alpha_p = a_p = \varphi(x)$. If $x \in {}_0[01^q0]$ with $q \geq 1$ then $k(Tx) - k(x) = \beta_q - \alpha_1 = b_q = \varphi(x)$.

For $x = (0^\infty)$, $\varphi(0^\infty) = a = 0 = k(Tx) - k(x)$. When $x = (01^\infty)$, $k(Tx) - k(x) = \alpha_1 + b - \alpha_1 = b = \varphi(x)$.

If $x \in {}_0[1^p0]$ with $p \geq 2$ then $k(Tx) - k(x) = \beta_{p-1} - \beta_p = b_{p-1} - b_p = c_p = \varphi(x)$. If $x \in {}_0[10^q1]$ with $q \geq 2$ then $k(Tx) - k(x) = \alpha_q - \beta_1 = \alpha_1 - \beta_1 - \sum_{i=2}^{q} a_i = \alpha_1 - \beta_1 + d_q + b_1 = d_q = \varphi(x)$ by the definition of β_1. If $x \in {}_0[10^q1]$ with $q = 1$ then $k(Tx) - k(x) = \alpha_1 - \beta_1 = -b_1 = d_1 = \varphi(x)$. When $x = (1^\infty)$, $\varphi(x) = c = 0 = k(Tx) - k(x)$, and when $x = (10^\infty)$, $k(Tx) - k(x) = \alpha_1 + b_1 + d - \beta_1 = d = \varphi(x)$ by the definition of β_1. Hence k is a cobounding function for φ.

The difference $k_1 - k_2$ of any two cobounding functions for φ is a T-invariant continuous function. Since T is topologically transitive, $k_1 - k_2$ is a constant, so any cobounding function has the form given. □

COROLLARY 1.3. *We have* $W(X, T) \neq \mathrm{Bow}(X, T)$.

Proof. Using Theorem 1.1 we can get examples of $\varphi \in \mathrm{Bow}(X, T)\backslash W(X, T)$. Let $\sum_{n=2}^{\infty} a_n$ be a divergent series with a bounded sequence of partial sums and with $a_n \to 0$. For example we could take $a_n = \sin(\sqrt{n+1}) - \sin\sqrt{n}$. So if we take $\varphi \in R(X)$ to correspond to $(a_n)_2^{\infty}$ as above, $a = 0$, all $c_n = 0$, $c = 0$, and (b_n), (d_n) to be any convergent sequences (say $b_n = 0 = d_n$ for all n), then $\varphi \in \mathrm{Bow}(X, T)$. Clearly $\varphi \notin W(X, T)$ by Theorem 1.1. □

We could choose $\sum_{n=2}^{\infty}(a_n - a)$ and $\sum_{n=2}^{\infty}(c_n - c)$ to be any series with bounded sequences of partial sums and $(b_n)_1^{\infty}$ and $(d_n)_1^{\infty}$ to be any convergent sequences. Then the corresponding $\varphi \in R(T)$ belongs to $\mathrm{Bow}(X, T)\backslash W(X, T)$ as long as one of the above series is not convergent.

The specific example we gave above was an example of the type studied by Hofbauer [**Ho**]. These are given by a sequence $(a_n)_0^{\infty}$ with $a_n \to a$ and we put $b_q = b = a_1$, for all $q \geq 1$, and $c_p = d_q = a_0 = c = d$, for all $p \geq 2$, $q \geq 1$. Hence $\varphi(0^k 1z) = a_k$ for $k \geq 0$, $z \in X$ and $\varphi(0^{\infty}) = a$. For these functions $\varphi \in \mathrm{Bow}(X, T)$ if and only if $\sum_{n=0}^{\infty}(a_n - a)$ has a bounded sequence of partial sums and $\varphi \in W(X, T)$ if and only if $\sum_{n=0}^{\infty}(a_n - a)$ converges. (The condition $\varphi \in \mathrm{Bow}(X, T)$ is the same as φ having a homogeneous measure in the sense of [**Ho**], so the condition above for $\varphi \in \mathrm{Bow}(X, T)$ corrects the theorem of [**Ho**, p. 230] (see [**W4**]).) For such a function $\mathrm{v}_n(\varphi) = \sup_{i,j\geq n}(a_i - a_j)$, $n \geq 2$, and $\sup_{i\geq n}|a_n - a| \leq \mathrm{v}_n(\varphi) \leq 2\sup_{i\geq n}|a_n - a|$ by Lemma 1.2. Note that, for all $f \in C(X)$, $\mathrm{v}_n(f) \geq 0$ and $\mathrm{v}_n(f) \searrow 0$. Given any sequence $(u_n)_1^{\infty}$ with $u_n \geq 0$ and $u_n \searrow 0$ we can get φ of the above type with $\mathrm{v}_n(\varphi) = u_n$ for all $n \geq 1$ by taking $a_n = u_n$, $n \geq 1$ and $a_0 = 0$.

For functions of this Hofbauer type we have $\sum_{n=1}^{\infty}(\mathrm{v}_n(\varphi))^t < \infty$ if and only if $\sum_{n=1}^{\infty}(\sup_{i\geq n}|a_i - a|)^t < \infty$ so we can get for each $t > 0$ a function $\varphi \in W(X, T)$ with $\sum_{n=1}^{\infty}(\mathrm{v}_n(\varphi))^t = \infty$ as follows. Let $a_n = (-1)^{n+1}/n^{1/t}$, $n \geq 1$. Then $a_n \to 0$, so $a = 0$, and $\mathrm{v}_n(\varphi) = \sup_{i\geq n}|a_i| = 1/n^{1/t}$. Hence $\sum_{n=1}^{\infty}(\mathrm{v}_n(\varphi))^t = \infty$. We have that $\sum_{n=1}^{\infty} a_n$ is convergent by the Leibnitz alternating series test, so $\varphi \in W(X, T)$. This shows that the classes studied in [**JO**] do not include all of $W(X, T)$.

The conditions for $\varphi \in R(X)$ to belong to $\mathrm{Bow}(X, T)$ or $W(X, T)$ do not involve $(b_q)_1^{\infty}$ and $(d_q)_1^{\infty}$, whereas $\mathrm{v}_n(\varphi)$ does involve these sequences.

2. *The g-functions in $R(X)$*

A g-function for $T : X \to X$ is a continuous $g : X \to (0, 1)$ satisfying $\sum_{y\in T^{-1}x} g(y) = 1$ for all $x \in X$. We can write this condition as $g(0x) + g(1x) = 1$ for all $x \in X$.

Let $G(X, T)$ denote the set of all g-functions for T. If $g \in G(X, T)$ we can define the continuous operator $\mathcal{L} : C(X) \to C(X)$ by $(\mathcal{L}f)(x) = \sum_{y\in T^{-1}x} g(y)f(y)$. Then $\mathcal{L}1 = 1$, $\|\mathcal{L}\| = 1$, and $\mathcal{L}U_T f = f$ for all $f \in C(X)$ where $U_T f = f \circ T$. We write $\mathcal{L}_{\log g}$ instead of $\mathcal{L}$ to indicate which g is being used, and this fits in with the notation for the Ruelle operator. We say that $\mu \in M(X)$ is a g-measure if $\mathcal{L}^*\mu = \mu$. Such a measure always belongs to $M(X, T)$, and μ is a g-measure if and only if μ is an equilibrium state for $\log g$ (see [**L, W2**]). Since $P(T, \log g) = 0$ for $g \in G(X, T)$, this condition becomes $h_\mu(T) + \int \log g\, d\mu = 0$. All g-measures have support X (see [**W2**]).

We shall see in §3 that $g \in G(X,T) \cap R(X)$ arises naturally from the Ruelle operator theorem applied to certain functions in $R(X)$.

Note that if $g \in G(X,T)$ then $g \in R(X)$ if and only if $\log g \in R(X)$.

We have $g \in G(X,T) \cap R(X)$ if and only if there are sequences $(\gamma_p)_2^\infty \to \gamma$ and $(\delta_p)_2^\infty \to \delta$ for which some $c \in (0,1)$ exists with $c \le \gamma_p,\ \delta_p \le 1-c$ for all $p \ge 2$, and $g(0^p1z) = \gamma_p$, $g(1^p0z) = \delta_p$, for all $p \ge 2$, $z \in X$, $g(01^q0z) = 1-\delta_{q+1}$, $g(10^q1z) = 1-\gamma_{q+1}$ for all $q \ge 1$, $z \in X$, $g(0^\infty) = \gamma$, $g(1^\infty) = \delta$, $g(10^\infty) = 1-\gamma$, and $g(01^\infty) = 1-\delta$.

From Theorem 1.1 we have the following result.

THEOREM 2.1. *Let $g \in G(X,T) \cap R(X)$ be given in terms of $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ as above. Then the following hold:*

(i) $\log g \in \mathrm{Bow}(X,T)$ *if and only if there exists $A > 1$ with $A^{-1} \le \gamma_2 \cdots \gamma_{1+n}/\gamma^n \le A$ and $A^{-1} \le \delta_2 \cdots \delta_{1+n}/\delta^n \le A$ for all $n \ge 1$;*

(ii) $\log g \in W(X,T)$ *if and only if $\sum_{n=2}^\infty \log(\gamma_n/\gamma)$ and $\sum_{n=2}^\infty \log(\delta_n/\delta)$ are both convergent.*

We can get examples of $g \in R(X)$ with $\log g \in \mathrm{Bow}(X,T)\backslash W(X,T)$ as follows. Let $\sum_{i=2}^\infty a_i$ be a non-convergent series with $a_i \to 0$, $|a_i| \le 1$ for all i, and having a bounded sequence of partial sums. Such an example was given in §1. Choose $\gamma \in (0, e^{-1})$ and put $\gamma_p = \gamma e^{a_p}$, $p \ge 2$. Then $\gamma_p \to \gamma$, $\gamma e^{-1} \le \gamma_p \le \gamma e < 1$, for all $p \ge 2$. Since $\log(\gamma_p/\gamma) = a_p$ the series $\sum_{p=2}^\infty \log(\gamma_p/\gamma)$ is not convergent but has a bounded sequence of partial sums. We could choose a similar example for $(\delta_p)_2^\infty$ or we could put $\delta_p = 1/2$ for all $p \ge 2$ and then $\log g \in \mathrm{Bow}(X,T)\backslash W(X,T)$ by Theorem 2.1.

In the proof of the next theorem we often use the following. If $g \in G(X,T)$, μ is a g-measure and ${}_0[a_0,\ldots,a_n]$ is a cylinder set starting at coordinate 0, then

$$\begin{aligned}\mu({}_0[a_0,\ldots,a_n]) &= \int \mathcal{X}_{{}_0[a_0,\ldots,a_n]}\,d\mu = \int \mathcal{L}^n \mathcal{X}_{{}_0[a_0,\ldots,a_n]}\,d\mu \\ &= \int g(a_0\ldots a_n x) g(a_1\ldots a_n x)\cdots g(a_n x)\,d\mu(x).\end{aligned}$$

Note that since $\mu \in M(X,T)$ we have $\mu({}_0[a_0,\ldots,a_n]) = \mu({}_k[a_0,\ldots,a_n])$ for all $k \ge 0$, so we can write $\mu([a_0,\ldots,a_n])$ unambiguously.

We now show that each $g \in G(X,T) \cap R(X)$ has a unique g-measure and we describe this measure.

THEOREM 2.2. *Let $g \in G(X,T) \cap R(X)$ be defined by $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ as above. There is a unique g-measure μ which is given as follows.*

For $k \ge 2$ let $\Gamma_k = \sum_{i=0}^\infty \gamma_k \cdots \gamma_{k+i}$ and $\Delta_k = \sum_{i=0}^\infty \delta_k \cdots \delta_{k+i}$. Then $\mu([0,1]) = \mu([1,0]) = 1/(\Gamma_2 + \Delta_2 + 2)$, $\mu([0,0]) = \Gamma_2/(\Gamma_2 + \Delta_2 + 2)$, and $\mu([1,1]) = \Delta_2/(\Gamma_2 + \Delta_2 + 2)$. For $k \ge 3$, $\mu([0^k]) = \gamma_2 \cdots \gamma_{k-1}\Gamma_k/(\Gamma_2 + \Delta_2 + 2)$ and $\mu([1^k]) = \delta_2 \cdots \delta_{k-1}\Delta_k/(\Gamma_2 + \Delta_2 + 2)$. For $r \ge 1$ and $k_i, l_i \ge 1$ for $1 \le i \le r$,

$\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 0^{k_r}1^{l_r}]) = i_{k_1}d_{l_1}c_{k_2}\cdots c_{k_r}f_{l_r}/(\Gamma_2+\Delta_2+2)$ *where*

$$i_k = \begin{cases} 1 & \text{if } k=1,\\ \gamma_k\cdots\gamma_2 & \text{if } k\ge 2,\end{cases} \quad c_k = \begin{cases} 1-\gamma_2 & \text{if } k=1,\\ (1-\gamma_{k+1})\gamma_k\cdots\gamma_2 & \text{if } k\ge 2,\end{cases}$$

$$d_l = \begin{cases} 1-\delta_2 & \text{if } l=1,\\ (1-\delta_{l+1})\delta_l\cdots\delta_2 & \text{if } l\ge 2,\end{cases} \quad f_l = \begin{cases} 1 & \text{if } l=1,\\ \delta_l\cdots\delta_2 & \text{if } l\ge 2,\end{cases}$$

and $\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 1^{l_{r-1}}0^{k_r}]) = i_{k_1}d_{l_1}c_{k_2}\cdots d_{l_{r-1}}i_{k_r}/(\Gamma_2+\Delta_2+2)$. *The μ-measure of blocks with initial entry* 1 *are given by the corresponding expressions.*

Proof. Since a g-measure has no atoms

$$\mu([0,1]) = \sum_{i=0}^{\infty}\mu([01^{1+i}0]) = (1-\delta_2)\mu([10]) + (1-\delta_3)\delta_2\mu([10]) + \cdots = \mu([10]).$$

Also $\mu([00]) = \sum_{i=0}^{\infty}\mu([0^{2+i}1]) = \Gamma_2\mu([01])$ and, similarly, $\mu([11]) = \Delta_2\mu([01])$. Since $\mu([00])+\mu([01])+\mu([10])+\mu([11]) = 1$ we have $\mu([01]) = 1/(\Gamma_2+\Delta_2+2)$ and we get the expressions for $\mu([00])$ and $\mu([11])$.

Now let $k\ge 3$. Then

$$\mu([0^k]) = \sum_{i=0}^{\infty}\mu([0^{k+i}1]) = \sum_{i=0}^{\infty}\gamma_{k+i}\cdots\gamma_2\mu([01]) = \frac{\gamma_2\cdots\gamma_{k-1}\Gamma_k}{\Gamma_2+\Delta_2+2}.$$

We get the corresponding expressions for $\mu([1^k])$.

To prove the expression for $\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 0^{k_r}1^{l_r}])$ we use induction on r. Consider the case $r=1$. We study $\mu([0^k1^l])$. If $k=1=l$ we know that the stated expression is true. Let $k=1$ and $l\ge 2$. Then

$$\mu([01^l]) = \sum_{i=0}^{\infty}\mu([01^{l+i}0]) = \sum_{i=0}^{\infty}(1-\delta_{l+i+1})\delta_{l+i}\cdots\delta_2\mu([10]) = \delta_l\cdots\delta_2\mu([10]).$$

Now let $k\ge 2$, $l=1$. Then $\mu([0^k1]) = \gamma_k\cdots\gamma_2\mu([01])$. Now if $k,l\ge 2$,

$$\begin{aligned}\mu([0^k1^l]) &= \sum_{i=0}^{\infty}\mu([0^k1^{l+i}0]) = \gamma_k\cdots\gamma_2\sum_{i=0}^{\infty}(1-\delta_{l+i+1})\delta_{l+i}\cdots\delta_2\mu([10])\\ &= \gamma_k\cdots\gamma_2\delta_l\cdots\delta_2\mu([10]).\end{aligned}$$

Hence the statement holds for $r=1$.

Now assume that the stated equalities hold for the natural number r and we shall show that they hold for $r+1$.

Let $k_i, l_i\ge 1$ be given for $1\le i\le r+1$. If $k_1, l_1\ge 2$ then

$$\begin{aligned}&\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}])\\ &\quad = \gamma_{k_1}\cdots\gamma_2(1-\delta_{l_1+1})\delta_{l_1}\cdots\delta_2(1-\gamma_{k_2+1})\mu([0^{k_2}1^{l_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}])\end{aligned}$$

and the required result follows by the induction assumption.

If $k_1\ge 2$ and $l_1=1$ then

$$\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}]) = \gamma_{k_1}\cdots\gamma_2(1-\delta_2)(1-\gamma_{k_2+1})\mu([0^{k_2}1^{l_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

If $k_1 = 1$ and $l_1 \geq 2$ then

$$\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}]) = (1-\delta_{l_1+1})\delta_{l_1}\cdots\delta_2(1-\gamma_{k_2+1})\mu([0^{k_2}1^{l_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

If $k_1 = 1 = l_1$ then

$$\mu([010^{k_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}]) = (1-\delta_2)(1-\gamma_{k_2+1})\mu([0^{k_2}1^{l_2}\ldots 0^{k_{r+1}}1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

The formula for $\mu([0^{k_1}1^{l_1}0^{k_2}\ldots 1^{l_{r-1}}0^{k_r}])$ can be proved by induction in a similar way. □

COROLLARY 2.3. *For $g \in G(X,T)\cap R(X)$ the unique g-measure μ is reversible, i.e.*

$$\mu([a_0, a_1, \ldots, a_{n-1}]) = \mu([a_{n-1}, a_{n-2}, \ldots, a_0])$$

for all $a_0, a_1, \ldots, a_{n-1} \in \{0,1\}$, $n \geq 1$.

We can state this in terms of the natural extension $\hat{\mu}$ of μ to the two-sided shift space $\hat{X} = \prod_{-\infty}^{\infty}\{0,1\}$. The measure $\hat{\mu}$ is determined by requiring that $\hat{\mu}({}_l[a_0, a_1, \ldots, a_n]) = \mu({}_0[a_0, a_1, \ldots, a_n])$ for all $l \in Z$, $n \geq 0$, $a_0, a_1, \ldots, a_n \in \{0,1\}$. Here

$$({}_l[a_0, a_1, \ldots, a_n]) = \{(x_i)_{-\infty}^{\infty} \in \hat{X} \mid x_{k+l} = a_k \ 0 \leq k \leq n\}.$$

If $\Phi : \hat{X} \to \hat{X}$ is the reversal map, defined by

$$\Phi(\ldots, x_{-2}, x_{-1}, \overset{*}{x_0}, x_1, x_2, \ldots) = (\ldots, x_2, x_1, \overset{*}{x_0}, x_{-1}, x_{-2}, \ldots)$$

then Corollary 2.3 means that $\hat{\mu}\circ\Phi = \hat{\mu}$. Here $*$ indicates the entry in the 0th position.

We now show that if $g \in G(X,T)\cap R(X)$ then, for all $f \in C(X)$, $\mathcal{L}^n_{\log g} f \underset{\to}{\to} \int f\,d\mu$, where μ is the unique g-measure. This has been proved in the cases when $\delta_p = \delta$ for all $p \geq 2$ by Hulse [**Hu**]. Here the symbol $\underset{\to}{\to}$ denotes that the convergence is uniform on X.

THEOREM 2.4. *Let $g \in G(X,T)\cap R(X)$. For every $f \in C(X)$ there exists $c(f) \in \mathbb{R}$ with $\mathcal{L}^n_{\log g} \underset{\to}{\to} c(f)$. In fact, $c(f) = \int f\,d\mu$ where μ is the unique g-measure.*

Proof. We write $\mathcal{L}$ instead of $\mathcal{L}_{\log g}$. Let g be defined using the sequences $(\gamma_n)_2^{\infty}$ and $(\delta_n)_2^{\infty}$. Since linear combinations of characteristic functions of cylinders based at coordinate zero, $\mathcal{X}_{0[w_0,w_1,\ldots,w_{k-1}]}$, are dense in $C(X)$, it suffices to consider $f = \mathcal{X}_{0[w_0,w_1,\ldots,w_{k-1}]}$, where $w = (w_0, w_1, \ldots) \in X$.

Fix $w \in X$ and $k \geq 1$ and let $f = \mathcal{X}_{0[w_0,w_1,\ldots,w_{k-1}]}$. For $n \geq 1$

$$\begin{aligned}(\mathcal{L}^{n+k} f)(x) &= \sum_{z\in T^{-(n+k)}x} g(z)g(Tz)\cdots g(T^{n+k-1}z)f(z)\\ &= \sum_{y_0,\ldots,y_{n-1}} [g(y_0\ldots y_{n-1}x)\cdots g(y_{n-1}x)\\ &\qquad \times g(w_0\ldots w_{k-1}y_0\ldots y_{n-1}x)\cdots g(w_{k-1}y_0\ldots y_{n-1}x)].\end{aligned}$$

We first show that it suffices to consider only the two cases $w_0 = w_1 = \cdots = w_{k-1}$.

Assume that $w_{k-1} = 0$. If $w_0 = w_1 = \cdots = w_{k-1} = 0$ then we need not consider further. So let $w_i = 1$ for some $i < k-1$, and choose $i < k-1$ so that $w_i = 1$ and $w_{i+1} = 0 = w_{i+2} = \cdots = w_{k-1}$. Hence

$$[w_0, w_1, \ldots, w_{k-1}] = [w_0, w_1, \ldots, w_{i-1}10^{k-i-1}].$$

If $0 \le j < i$ then, by the definition of g, $g(w_j \ldots w_i \ldots w_{k-1}y_0 \ldots y_{n-1}x)$ does not depend on $(y_0 \ldots y_{n-1}x)$. Hence $\prod_{j=0}^{i-1} g(w_j \ldots w_{k-1}y_0 \ldots y_{n-1}x) = C$, a constant. Then,

$$(\mathcal{L}^{n+k} f)(x) = C\bigg[\sum_{y_0,\ldots,y_{n-1}} g(y_0 \ldots y_{n-1}x) \cdots g(y_{n-1}x) \times g(10^{k-i-1}y_0 \ldots y_{n-1}x) \cdots g(0y_0 \ldots y_{n-1}x)\bigg].$$

But $g(10^{k-i-1}y_0 \ldots y_{n-1}x) = 1 - g(0^{k-i}y_0 \ldots y_{n-1}x)$ so

$$(\mathcal{L}^{n+k} f)(x) = C[(\mathcal{L}^{n+k-i-1}\mathcal{X}_{[0^{k-i-1}]})(x) - (\mathcal{L}^{n+k-i}\mathcal{X}_{[0^{k-i}]})(x)].$$

So when $w_{k-1} = 0$ it suffices to consider $(\mathcal{L}^{n+k-i-1}\mathcal{X}_{[0^{k-i-1}]})(x)$ and $(\mathcal{L}^{n+k-i}\mathcal{X}_{[0^{k-i}]})(x)$.

Now assume that $w_{k-1} = 1$. The corresponding argument shows that the convergence of $(\mathcal{L}^{n+k} f)(x)$ depends on that of $(\mathcal{L}^{n+k-i-1}\mathcal{X}_{[1^{k-i-1}]})(x)$ and $(\mathcal{L}^{n+k-i}\mathcal{X}_{[1^{k-i}]})(x)$.

So we only need to consider the cases when $f = \mathcal{X}_{{}_0[0^k]}$ and $f = \mathcal{X}_{{}_0[1^k]}$.

So now assume that $f = \mathcal{X}_{{}_0[0^k]}$. The case when $f = \mathcal{X}_{{}_0[1^k]}$ follows by symmetry.

Let $l \ge 1$, and we now show that $\mathcal{L}^n f$ is constant on ${}_0[0^l1]$. Let $x \in {}_0[0^l1]$. Then

$$(\mathcal{L}^n f)(x) = \sum_{y_0,\ldots,y_{n-1}} g(y_0 \ldots y_{n-1}x) \cdots g(y_{n-1}x) f(y_0 \ldots y_{n-1}x).$$

If $n + l < k$ then $(y_0 \ldots y_{n-1}x) \notin_0 [0^k]$ so $(\mathcal{L}^n f)(x) = 0$.

If $k \le n$ then $f(y_0 \ldots y_{n-1}x) = 1$ if and only if $y_0 = 0 = \cdots = y_{k-1}$ and then $(\mathcal{L}^n f)(x) = \sum_{y_k,\ldots,y_{n-1}} g(0^k y_k \ldots y_{n-1}x) \cdots g(y_{n-1}x)$ which is constant on ${}_0[0^l1]$.

If $n < k \le n + l$ then $f(y_0 \ldots y_{n-1}x) = 1$ if and only if $y_0 = 0 = \cdots = y_{n-1}$ and then $(\mathcal{L}^n f)(x) = \gamma_{n+l} \ldots \gamma_{1+l}$.

Hence $(\mathcal{L}^n f)$ is constant on ${}_0[0^l1]$ and we denote this value by $(\mathcal{L}^n f)([0^l1])$.

Again let $l \ge 1$ and we now show that $\mathcal{L}^n f$ is constant on ${}_0[1^l0]$:

$$(\mathcal{L}^n f)(x) = \sum_{y_0,\ldots,y_{n-1}} g(y_0 \ldots y_{n-1}x)g(y_1 \ldots y_{n-1}x) \cdots g(y_{n-1}x) f(y_0 \ldots y_{n-1}x).$$

If $n < k$ then $f(y_0 \ldots y_{n-1}x) = 0$ so $(\mathcal{L}^n f)(x) = 0$.

If $n = k$ then $f(y_0 \ldots y_{n-1}x) = 1$ if and only if $y_0 = 0 = \cdots = y_{n-1}$ so $(\mathcal{L}^n f)(x) = \gamma_k \ldots \gamma_2(1 - \delta_{l+1})$.

If $k < n$ then $f(y_0 \ldots y_{n-1}x) = 1$ if and only if $y_0 = 0 = \cdots = y_{k-1}$ so $(\mathcal{L}^n f)(x) = \sum_{y_k,\ldots,y_{n-1}} g(0^k y_k \ldots y_{n-1}x) \cdots g(y_{n-1}x)$ which is constant on ${}_0[1^l0]$.

Hence $(\mathcal{L}^n f)$ is constant on ${}_0[1^l0]$ and we denote this value by $(\mathcal{L}^n f)([1^l0])$.

We now show that if $x_0 = 0$ then for all $n \geq 1$

$$(\mathcal{L}^{n+k} f)(x) = \left(\prod_{i=1}^{n} g(0^i x)\right)[(\mathcal{L}^n f)(0^n x) - (\mathcal{L}^n f)([10])] + (\mathcal{L}^{n+k-1} f)([10])$$
$$+ \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x)\right)[(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])], \tag{1}$$

where the final term is absent if $n = 1$.

We use induction on n. When $n = 1$ the right side of (1) becomes

$$g(0x)[(\mathcal{L}^k f)(0x) - (\mathcal{L}^k f)([10])] + (\mathcal{L}^k f)([10])$$
$$= g(0x)(\mathcal{L}^k f)(0x) + g(1x)(\mathcal{L}^k f)(1x),$$

which equals $(\mathcal{L}^{1+k} f)(x)$. Hence (1) holds for $n = 1$.

Assume that (1) holds for $n - 1$ and we shall prove it for n. Let $x_0 = 0$. Then

$$(\mathcal{L}^{n+k} f)(x) = g(0x)(\mathcal{L}^{n+k-1} f)(0x) + g(1x)(\mathcal{L}^{n+k-1} f)(1x)$$
$$= g(0x)[(\mathcal{L}^{n+k-1} f)(0x) - (\mathcal{L}^{n+k-1} f)([10])] + (\mathcal{L}^{n+k-1} f)([10])$$
$$= g(0x)\Bigg[\left(\prod_{i=1}^{n-1} g(0^{i+1} x)\right)\{(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])\} + (\mathcal{L}^{n+k-2} f)([10])$$
$$+ \sum_{i=1}^{n-2} \left(\prod_{j=1}^{n-1-i} g(0^{j+1} x)\right)\{(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])\}$$
$$- (\mathcal{L}^{n+k-1} f)([10])\Bigg] + (\mathcal{L}^{n+k-1} f)([10])$$

using the induction assumption. Hence

$$(\mathcal{L}^{n+k} f)(x) = \left(\prod_{i=1}^{n} g(0^i x)\right)[(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])] + (\mathcal{L}^{n+k-1} f)([10])$$
$$+ \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x)\right)[(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])].$$

Hence (1) holds for all $n \geq 1$ and all $x \in {}_0[0]$.

We next show that if $x_0 = 1$ then for all $n \geq 1$

$$(\mathcal{L}^{n+k} f)(x) = \left(\prod_{i=1}^{n} g(1^i x)\right)[(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01])$$
$$+ \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(1^j x)\right)[(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])], \tag{2}$$

where the last term is absent if $n = 1$.

We use induction on n. When $n = 1$ the right side of (2) becomes

$$g(1x)[(\mathcal{L}^k f)(1x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^k f)([01])$$
$$= g(1x)(\mathcal{L}^k f)(1x) + g(0x)(\mathcal{L}^k f)(0x),$$

which equals $(\mathcal{L}^{1+k} f)(x)$ and so (2) holds for $n = 1$.

Assume that (2) holds for $n - 1$ and we shall prove it for n. Let $x_0 = 1$. Then

$$\begin{aligned}(\mathcal{L}^{n+k} f)(x) &= g(1x)(\mathcal{L}^{n+k-1} f)(1x) + (1 - g(1x))(\mathcal{L}^{n+k-1} f)(0x) \\ &= g(1x)[(\mathcal{L}^{n+k-1} f)(1x) - (\mathcal{L}^{n+k-1} f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &= g(1x)\Bigg[\bigg(\prod_{i=1}^{n-1} g(1^{i+1}x)\bigg)\{(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])\} + (\mathcal{L}^{n+k-2} f)([01]) \\ &\quad + \sum_{i=1}^{n-2}\bigg(\prod_{j=1}^{n-1-i} g(1^{j+1}x)\bigg)\{(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])\} \\ &\quad - (\mathcal{L}^{n+k-1} f)([01])\Bigg] + (\mathcal{L}^{n+k-1} f)([01])\end{aligned}$$

using the induction assumption. Hence

$$\begin{aligned}(\mathcal{L}^{n+k} f)(x) = \bigg(\prod_{i=1}^{n} g(1^i x)\bigg)&[(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &+ \sum_{i=1}^{n-1}\bigg(\prod_{j=1}^{n-i} g(1^j x)\bigg)[(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])].\end{aligned}$$

Hence (2) holds for all $n \geq 1$ and all $x \in {}_0[1]$.

We use (1) to show that if $(\mathcal{L}^n f)([10]) \to c(f)$ then $(\mathcal{L}^n f)(x) \to c(f)$ uniformly for $x \in {}_0[0]$. Assume that $(\mathcal{L}^n f)([10]) \to c(f)$.

By (1) we have

$$\begin{aligned}&(\mathcal{L}^{n+k} f)(x) - (\mathcal{L}^{n+k-1} f)([10]) \\ &\quad = \bigg(\prod_{i=1}^{n} g(0^i x)\bigg)[(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])] \\ &\qquad + \sum_{i=1}^{n-1}\bigg(\prod_{j=1}^{n-i} g(0^j x)\bigg)[(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])].\end{aligned}$$

Note that $\big|\big(\prod_{j=1}^{n} g(0^j x)\big)[(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])]\big| \leq 2(\sup g)^n \to 0$ as $n \to \infty$.

Given $\epsilon > 0$ choose N so that $\sum_{i=N}^{\infty} (\sup g)^i < \epsilon$ and so that $n \geq N$ implies $|(\mathcal{L}^{n+k-1} f)([10]) - (\mathcal{L}^{n+k} f)([10])| < \epsilon$.

For all $n \geq 2N$

$$\begin{aligned}&\bigg|\sum_{i=1}^{n-1}\bigg(\prod_{j=1}^{n-i} g(0^j x)\bigg)[(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])]\bigg| \\ &\quad \leq 2\sum_{i=1}^{N} (\sup g)^{n-i} + \epsilon \sum_{i=N+1}^{n-1} (\sup g)^{n-i}\end{aligned}$$

$$\leq 2\sum_{q=N}^{\infty}(\sup g)^q + \epsilon\sum_{p=1}^{\infty}(\sup g)^p$$
$$< \epsilon\left(2+\sum_{p=1}^{\infty}(\sup g)^p\right).$$

Therefore $|(\mathcal{L}^{n+k}f)(x)-(\mathcal{L}^{n+k-1}f)([10])| \to 0$ as $n\to\infty$, uniformly on ${}_0[0]$.

Similarly (2) implies that if $(\mathcal{L}^{n+k-1}f)([01])$ converges then$(\mathcal{L}^{n+k}f)(x)$ converges to the same limit uniformly for $x \in {}_0[1]$.

So consider $(\mathcal{L}^{n+k}f)([10])$.

By (2) we have

$$\begin{aligned}
&(\mathcal{L}^{n+k}f)([10])\\
&\quad=\left(\prod_{i=1}^{n} g(1^{i+1}0)\right)[(\mathcal{L}^k f)([1^{n+1}0]) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1}f)([01])\\
&\qquad+\sum_{i=1}^{n-1}\left(\prod_{j=1}^{n-i} g(1^{1+j}0)\right)[(\mathcal{L}^{k+i-1}f)([01]) - (\mathcal{L}^{k+i}f)([01])]\\
&\quad=\left(\prod_{i=1}^{n}\gamma_{i+1}\right)[(\mathcal{L}^k f)([1^{n+1}0]) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1}f)([01])\\
&\qquad+\sum_{i=1}^{n-1}\left(\prod_{j=1}^{n-i}\gamma_{j+1}\right)[(\mathcal{L}^{k+i-1}f)([01]) - (\mathcal{L}^{k+i}f)([01])]\\
&\quad=\left(\prod_{j=2}^{n+1}\gamma_j\right)(\mathcal{L}^k f)([1^{n+1}0]) + \sum_{i=0}^{n-2}(\mathcal{L}^{k+i}f)([01])\left(\prod_{j=2}^{n-i}\gamma_j\right)(1-\gamma_{n+1-i})\\
&\qquad+(\mathcal{L}^{k+n-1}f)([01])(1-\gamma_2).
\end{aligned}$$

Similarly, using (1) we have

$$\begin{aligned}
(\mathcal{L}^{n+k}f)([01]) &= \left(\prod_{j=2}^{n+1}\delta_j\right)(\mathcal{L}^k f)([0^{n+1}1]) + \sum_{i=0}^{n-2}(\mathcal{L}^{k+i}f)([10])\left(\prod_{j=2}^{n-i}\delta_j\right)(1-\delta_{n+1-i})\\
&\quad+(\mathcal{L}^{k+n-1}f)([10])(1-\delta_2).
\end{aligned}$$

For $n\geq 0$ put $u_n = (\mathcal{L}^{n+k}f)([01])$ and $v_n = (\mathcal{L}^{n+k}f)([10])$. Then

$$v_n = \beta_n + \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \cdots + \alpha_n u_0 \quad \text{for } n\geq 1,$$

where $\beta_n = \left(\prod_{j=2}^{n+1}\gamma_j\right)(\mathcal{L}^k f)([1^{n+1}0]) > 0$ for $n\geq 1$, $\alpha_1 = 1-\gamma_2 > 0$ and for $n\geq 2$, $\alpha_n = \left(\prod_{j=2}^{n}\gamma_j\right)(1-\gamma_{n+1})$.

Note that $\sum_{n=1}^{\infty}\alpha_n = 1$ and $0<\beta_n\leq(\sup_j\gamma_j)^{n-1}$ so $\sum\beta_n<\infty$.

If we let $\alpha_n' = 1-\delta_2$, $\alpha_n' = \left(\prod_{j=2}^{n}\delta_j\right)(1-\delta_{n+1})$ for $n\geq 2$, and $\beta_n' = \left(\prod_{j=2}^{n+1}\delta_j\right)(\mathcal{L}^k f)([0^{n+1}1]) > 0$ then

$$u_n = \beta_n' + \alpha_1' v_{n-1} + \cdots + \alpha_n' v_0 \quad \text{for } n\geq 1.$$

If we put $\beta_0 = v_0$, $\alpha_0 = 0$ and if we let $A(s) = \sum_{n=0}^{\infty} \alpha_n s^n$, $B(s) = \sum_{n=0}^{\infty} \beta_n s^n$, $U(s) = \sum_{n=0}^{\infty} u_n s^n$, $V(s) = \sum_{n=0}^{\infty} v_n s^n$ then we have $V(s) = B(s) + A(s)U(s)$. Note that $A(1) = \sum_{n=0}^{\infty} \alpha_n = 1$ and $B(1) = \sum_{n=0}^{\infty} \beta_n < \infty$.

Similarly $U(s) = B'(s) + A'(s)V(s)$ where $\beta'_0 = u_0$, $\alpha'_0 = 0$, $A'(s) = \sum_{n=0}^{\infty} \alpha'_n s^n$ and $B'(s) = \sum_{n=0}^{\infty} \beta'_n s^n$.

Then we have

$$\begin{aligned} U(s) &= B'(s) + A'(s)[B(s) + A(s)U(s)] \\ &= (B'(s) + A'(s)B(s)) + A'(s)A(s)U(s). \end{aligned}$$

This gives a renewal equation for (u_n) of the form

$$u_n = b_n + a_0 u_n + a_1 u_{n-1} + \cdots + a_n u_0 \quad \text{for } n \geq 0,$$

where b_n is the coefficient of s^n in $B'(s) + A'(s)B(s)$ and a_n is the coefficient of s^n in $A'(s)A(s)$. Hence $\sum_{n=0}^{\infty} b_n = B'(1) + A'(1)B(1) = B'(1) < \infty$ and $\sum_{n=0}^{\infty} a_n = A'(1)A(1) = 1$ so by the renewal theorem [**Fe**, p. 291] we have $u_n \to \sum_{i=0}^{\infty} b_i / \sum_{i=0}^{\infty} i a_i$.

Similarly

$$V(s) = (B(s) + A(s)B'(s)) + A(s)A'(s)V(s)$$

so

$$v_n = b'_n + a_0 v_n + a_1 v_{n-1} + \cdots + a_n v_0 \quad \text{for } n \geq 0,$$

where b'_n is the coefficient of s^n in $B(s) + A(s)B'(s)$. Hence

$$\sum_{i=0}^{\infty} b'_i = B(1) + A(1)B'(1) = B(1) + B'(1) = \sum_{i=0}^{\infty} b_i$$

and the renewal theorem gives $v_n \to \sum_{i=0}^{\infty} b_i / \sum_{i=0}^{\infty} i a_i$.

Hence $(\mathcal{L}^{n+k} f)([01])$ and $(\mathcal{L}^{n+k} f)([10])$ converge to the same limit, $c(f)$, so $(\mathcal{L}^{n+k} f)(x)$ converges uniformly to $c(f)$. Therefore $(\mathcal{L}^n f)(x)$ converges uniformly to $c(f)$.

If μ is a g-measure then integrating $\mathcal{L}^n f \underset{\to}{\to} c(f)$ with respect to μ gives $c(f) = \int f \, d\mu$ for all $f \in C(X)$. This gives another way of showing that there is a unique g-measure. □

The convergence $\mathcal{L}^n f \underset{\to}{\to} \int f \, d\mu$ gives several properties of μ. One is that T is an exact endomorphism with respect to μ (i.e. all sets in the σ-algebra $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}(X)$ have μ-measure 0 or 1, where $\mathcal{B}(X)$ is the σ-algebra of Borel subsets of X) [**W3**].

One can obtain examples of g-functions with $\mathcal{L}^n f$ converging uniformly to a constant but $\log g \notin \mathrm{Bow}(X, T)$ as follows. Let $\gamma, \delta \in (0, 1)$ and for $p \geq 2$ put $\gamma_p = p\gamma/(p+1)$, $\delta_p = \delta$. The corresponding g is in $R(X)$ so we get the convergence by Theorem 2.4. However $\log g \notin \mathrm{Bow}(X, T)$ by Theorem 2.1 since $\gamma_2 \cdots \gamma_{1+n}/\gamma^n = 2/(n+2)$.

3. *Ruelle operator theorem for functions in $R(X)$*

In this section we investigate exactly when $\varphi \in R(X)$ satisfies the Ruelle operator theorem for $T : X \to X$.

For $\varphi \in C(X)$ the Ruelle operator $\mathcal{L}_\varphi : C(X) \to C(X)$ is defined by

$$(\mathcal{L}_\varphi f)(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) = e^{\varphi(0x)} f(0x) + e^{\varphi(1x)} f(1x).$$

To say the Ruelle operator theorem holds for φ means that there exist $\lambda \in \mathbb{R}, \lambda > 0$, $h \in C(X), h > 0, v \in M(X)$ with $\mathcal{L}_\varphi h = \lambda h$ and $\mathcal{L}_\varphi^* v = \lambda v$, and if we normalize h so that $v(h) = 1$ then for all $f \in C(X)$,

$$\frac{\mathcal{L}_\varphi^n f}{\lambda^n} \underset{\to}{\to} v(f)h.$$

We shall give necessary and sufficient conditions for $\varphi \in R(X)$ to satisfy the Ruelle operator theorem. This turns out to be equivalent to the existence of a positive eigenfunction h. When these conditions hold then

$$g = \frac{e^\varphi h}{\lambda h \circ T} \in G(X, T) \cap R(X),$$

and since

$$\varphi - \log g = \log \lambda + \log h \circ T - \log h$$

the unique equilibrium state for φ is the unique g-measure for g. Also λ is given as the solution to an equation.

THEOREM 3.1. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^\infty \to a$, $(b_q)_1^\infty \to b$, $(c_p)_2^\infty \to c$ and $(d_q)_1^\infty \to d$ as in §1. The following statements are pairwise equivalent.*

(i) *There exists $h \in C(X)$, $h > 0$, and a real number $\lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$.*

(ii) *We have*

$$\frac{1}{e^{2\max(a,c)}}\left[e^{d_1} + \sum_{j=1}^\infty e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}}\right]\left[e^{b_1} + \sum_{j=1}^\infty e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}\right] > 1,$$

where the left side could be ∞.

(iii) *There exists $h \in C(X)$, $h > 0$, and a real number $\lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$ and h has the following form: there exist sequences $(\alpha_q)_1^\infty$ and $(\beta_q)_1^\infty$ with $\alpha_q \to \alpha$, $\beta_q \to \beta$, $h(0^q1z) = \alpha_q$, $q \geq 1$, $h(1^q0w) = \beta_q$, $q \geq 1$, $h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$.*

(iv) *There exists $h \in C(X)$, $h > 0$, $\lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$ and there exists $v \in M(X)$ with $\mathcal{L}_\varphi^* v = \lambda v$ and, for all $f \in C(X)$, $(\mathcal{L}_\varphi^n f)(x)/\lambda^n \underset{\to}{\to} h(x)v(f)$ as $n \to \infty$.*

When φ satisfies the statements above and h is given in (iii) then $g = e^\varphi h/\lambda h \circ T$ is a g-function for T and $g \in R(X)$. Hence φ has a unique equilibrium state which is the unique g-measure.

Note that (iv) says that the Ruelle operator theorem holds for φ.

We shall use the following lemmas in the proof of Theorem 3.1. We use the notation from Theorem 3.1.

LEMMA 3.2. *The power series $\sum_{j=1}^\infty e^{d_{1+j}} e^{a_2+\cdots+a_{1+j}} x^j$ has radius of convergence e^{-a}.*

Proof. We have $\sqrt[n]{e^{d_1+n} e^{a_2+\cdots+a_{1+n}}} \to e^a$ since $d_{1+n}/n \to 0$ and $(a_2 + \cdots + a_{1+n}/n) \to a$. □

LEMMA 3.3. *Let $\varphi \in R(X)$. We can find $\rho > \max(e^a, e^c)$ with*

$$\rho^{-2}\bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\rho^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\rho^j}\bigg] < 1.$$

Proof. Let

$$F(\rho) = \bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\rho^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\rho^j}\bigg].$$

By Lemma 3.2 if $\rho_0 > \max(e^a, e^c)$ then $F(\rho) < \infty$. But $\rho > \rho_0$ implies that $F(\rho) < F(\rho_0)$ so $\rho^{-2}F(\rho) < \rho^{-2}F(\rho_0) < 1$ for large enough ρ. □

LEMMA 3.4. *Statement (ii) in Theorem 3.1 is equivalent to the existence of $\lambda > \max(e^a, e^c)$ with*

$$\frac{1}{\lambda^2}\bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\bigg] = 1.$$

Proof. Let $G(\rho) = \rho^{-2}F(\rho)$, where F is defined in the proof of Lemma 3.3. By Lemma 3.3 there is $\rho_0 > \max(e^a, e^c)$ with $G(\rho_0) < 1$.

If statement (ii) holds then $G(\max(e^a, e^c)) > 1$. If $G(\max(e^a, e^c)) < \infty$ then on the interval $[\max(e^a, e^c), \rho_0]$ G is continuous and, by the intermediate value theorem, there is some $\lambda \in (\max(e^a, e^c), \rho_0)$ with $G(\lambda) = 1$.

Suppose $G(\max(e^a, e^c)) = \infty$. By Lemma 3.2, $G(\rho) < \infty$ for all $\rho > \max(e^a, e^c)$. If $G(\rho) \le 1$ for all $\rho > \max(e^a, e^c)$ then, for all $J \ge 1$,

$$\rho^{-2}\bigg[e^{d_1} + \sum_{j=1}^{J} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\rho^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{J} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\rho^j}\bigg] \le 1$$

for all $\rho > \max(e^a, e^c)$. Then

$$e^{-2\max(a,c)}\bigg[e^{d_1} + \sum_{j=1}^{J} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{J} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}\bigg] \le 1$$

for all $J \ge 1$ so $G(\max(e^a, e^c)) \le 1$, a contradiction. So we can choose $\rho_1 \in (\max(e^a, e^c), \rho_0)$ with $1 < G(\rho_1) < \infty$ and the intermediate value theorem, applied to G restricted to $[\rho_1, \rho_0]$, gives some $\lambda \in (\rho_1, \rho_0)$ with $G(\lambda) = 1$.

If there exists $\lambda > \max(e^a, e^c)$ with $G(\lambda) = 1$ then $G(\max(e^a, e^c)) > G(\lambda) = 1$ so statement (ii) of Theorem 3.1 holds. □

We now turn to the proof of the theorem.

Proof of Theorem 3.1. (i) ⇒ (ii) Let $h \in C(X)$, $h > 0$, and let $\lambda > 0$ satisfy $\mathcal{L}_\varphi h = \lambda h$. We shall show that

$$1 \le \frac{1}{\lambda^2}\bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\bigg]$$

and $\lambda > \max(e^a, e^c)$.

We have $e^{\varphi(0x)}h(0x) + e^{\varphi(1x)}h(1x) = \lambda h(x)$. Put $x = (0^{q+j}1z)$, $q \geq 1$, $j \geq 0$, $z \in X$ to get

$$e^{a_{q+j+1}}h(0^{q+j+1}1z) + e^{d_{q+j}}h(10^{q+j}1z) = \lambda h(0^{q+j}1z).$$

Multiply this equation by $e^{a_{q+1}+\cdots+a_{q+j}}/\lambda^j$ if $j \geq 1$, and by 1 if $j = 0$, and sum over j from 0 to n to get

$$\frac{e^{a_{q+1}+\cdots+a_{q+n+1}}}{\lambda^n}h(0^{q+n+1}1z) + e^{d_q}h(10^q1z) + \sum_{j=1}^{n} e^{d_{q+j}}\frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) = \lambda h(0^q1z).$$

The right side of this equation is independent of n and both terms on the left side are non-negative. Therefore

$$\sum_{j=1}^{\infty} e^{d_{j+q}}\frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) < \infty$$

and since $\inf h > 0$ we have

$$\sum_{j=1}^{\infty} e^{d_{j+q}}\frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j} < \infty.$$

Hence $e^{a_{q+1}+\cdots+a_{q+j}}/\lambda^j \to 0$ as $j \to \infty$. Therefore

$$e^{d_q}h(10^q1z) + \sum_{j=1}^{\infty} e^{d_{q+j}}\frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) = \lambda h(0^q1z), \tag{3}$$

$q \geq 1, z \in X$.

By Lemma 3.2 we have $\lambda \geq e^a$. From $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$ with $x = 0^\infty$ we have $e^a h(0^\infty) + e^d h(10^\infty) = \lambda h(0^\infty)$, so $e^a < \lambda$ since $h > 0$. Similarly we have

$$e^{b_q}h(01^q0w) + \sum_{j=1}^{\infty} e^{b_{q+j}}\frac{e^{c_{q+1}+\cdots+c_{q+j}}}{\lambda^j}h(01^{q+j}0w) = \lambda h(1^q0w) \tag{4}$$

and $\lambda > e^c$.

By (3) and (4) with $q = 1$ we have

$$\lambda^2 h(01z)h(10w) = \left[e^{d_1}h(101z) + \sum_{j=1}^{\infty} e^{d_{1+j}}\frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}h(10^{1+j}1z)\right] \times \left[e^{b_1}h(010w) + \sum_{j=1}^{\infty} e^{b_{1+j}}\frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}h(01^{1+j}0w)\right].$$

Choose z, w so that $h(01z) = \sup_{y\in X} h(01y)$ and $h(10w) = \sup_{x\in X} h(10x)$. Then

$$\lambda^2 \leq \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}}\frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\right]\left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}}\frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\right].$$

Since $\lambda > \max(e^a, e^c)$ this implies (ii).

(ii) ⇒ (iii) By Lemma 3.4 choose $\lambda > \max(e^a, e^c)$ with

$$\frac{1}{\lambda^2}\bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\bigg]\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\bigg] = 1.$$

Let $\alpha > 0$ and define β by

$$\beta = \frac{\alpha e^b(\lambda - e^a)}{e^d(\lambda - e^c)\lambda}\bigg[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\bigg].$$

For $q \geq 1$ define α_q and β_q by

$$\alpha_q = \frac{\alpha(\lambda - e^a)}{\lambda e^d}\bigg[e^{d_q} + \sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j}\bigg],$$

$$\beta_q = \frac{\beta(\lambda - e^c)}{\lambda e^b}\bigg[e^{b_q} + \sum_{j=1}^{\infty} e^{b_{q+j}} \frac{e^{c_{q+1}+\cdots+c_{q+j}}}{\lambda^j}\bigg].$$

We show that $\alpha_q \to \alpha$ as $q \to \infty$. Let

$$u_q = \sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\cdots+a_{q+j}}}{\lambda^j}$$

which is finite since $\lambda > e^a$. Since $a_n \to a$ we have $a_n < a + \epsilon$ for n sufficiently large, so for q sufficiently large

$$u_q \leq e^{\sup(d_n)} \sum_{j=1}^{\infty} \bigg(\frac{e^{a+\epsilon}}{\lambda}\bigg)^j.$$

Hence $\bar{u} = \limsup_{n\to\infty}(u_n) < \infty$ and since $u_q = (e^{a_{q+1}}/\lambda)[e^{d_{q+1}} + u_{q+1}]$ we have $\bar{u} = (e^a/\lambda)[e^d + \bar{u}]$ so that $\bar{u} = e^{a+d}/(\lambda - e^a)$.

Similarly $\underline{u} = \liminf_{n\to\infty}(u_n) = e^{a+d}/(\lambda - e^a)$ so $u_q \to e^{a+d}/(\lambda - e^a)$ and $\alpha_q \to \alpha$.

Similarly $\beta_q \to \beta$.

Define $h : X \to \mathbb{R}$ by $h(0^q 1z) = \alpha_q$, $q \geq 1$, $z \in X$, $h(1^q 0z) = \beta_q$, $q \geq 1$, $z \in X$, $h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$. Then $h > 0$ and $h \in C(X)$.

We shall now show that $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$.

Note that $\beta_1 = \alpha(\lambda - e^a)/e^d$ since

$$\begin{aligned}\beta_1 &= \frac{\beta(\lambda - e^c)}{\lambda e^b}\bigg[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\bigg] \\ &= \frac{\beta(\lambda - e^c)}{e^b} \frac{\lambda}{[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} e^{a_2+\cdots+a_{1+j}}/\lambda^j]} \\ &= \frac{\alpha(\lambda - e^a)}{e^d}\end{aligned}$$

by the definitions of λ and β.

When $x = 0^\infty$,

$$(\mathcal{L}_\varphi h)(0^\infty) = e^{\varphi(0^\infty)} h(0^\infty) + e^{\varphi(10^\infty)} h(10^\infty) = e^a \alpha + e^d \beta_1 = \lambda\alpha = \lambda h(0^\infty).$$

Note that, for $q \geq 1$, $e^{a_{q+1}}\alpha_{q+1} + e^{d_q}\beta_1 = \lambda\alpha_q$, since

$$\lambda\alpha_q = \frac{\alpha(\lambda - e^a)}{e^d}\left[e^{d_q} + \frac{e^{a_{q+1}}}{\lambda}\left\{e^{d_{q+1}} + \sum_{j=1}^{\infty} e^{d_{q+1+j}}\frac{e^{a_{q+2}+\cdots+a_{q+1+j}}}{\lambda^j}\right\}\right]$$
$$= \beta_1 e^{d_q} + e^{a_{q+1}}\alpha_{q+1}.$$

Now when $x = (0^q 1z)$, $q \geq 1$, $z \in X$,

$$(\mathcal{L}_\varphi h)(0^q 1z) = e^{a_{q+1}}\alpha_{q+1} + e^{d_q}\beta_1 = \lambda\alpha_q = \lambda h(0^q 1z).$$

Similarly $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$ when $x = 1^\infty$ and $x = (1^q 0w)$, $q \geq 1$, $w \in X$.

(iii) $\Rightarrow$ (iv) Let h be as in (iii) and put $g = e^\varphi h/\lambda h \circ T$. Then $g \in G(X,T) \cap R(X)$.

By Theorem 2.4, $(\mathcal{L}^n_{\log g} f)(x) \underset{\rightarrow}{\rightarrow} \mu(f)$ for all $f \in C(X)$ where μ is the unique g-measure. Hence for all $f \in C(X)$

$$\frac{(\mathcal{L}^n_\varphi f)(x)}{\lambda^n} \underset{\rightarrow}{\rightarrow} h(x)\mu(f/h).$$

Let $\nu(f) = \mu(f/h)$ and we have $\mathcal{L}^*_\varphi \nu = \lambda\nu$.

Clearly (iv) implies (i).

This completes the proof of Theorem 3.1 □

COROLLARY 3.5. *Let $\varphi \in R(X)$ satisfy the statements in Theorem 3.1. There is only one number $\lambda > 0$ that satisfies statement (i) and it is that number $\lambda > \max(e^a, e^c)$ satisfying*

$$\frac{1}{\lambda^2}\left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}}\frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\right]\left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}}\frac{e^{c_2+\cdots+c_{1+j}}}{\lambda^j}\right] = 1.$$

We have $\lambda = e^{P(T,\varphi)}$. The function h satisfying statement (i) is unique up to scalar multiples. There is a unique $\nu \in M(X)$ with $\mathcal{L}^*_\varphi \nu = \lambda\nu$.

Proof. In the proof of Theorem 3.1 we showed that the number λ given above satisfies $\mathcal{L}_\varphi h = \lambda h$ for a certain continuous $h > 0$, and that, for all $f \in C(X)$, $(\mathcal{L}^n_\varphi f)(x)/\lambda^n \underset{\rightarrow}{\rightarrow} h(x)\nu(f)$. If also $\mathcal{L}_\varphi l = \tau l$ for some number $\tau > 0$ and some $l \in C(X)$ with $l > 0$ then $(\tau/\lambda)^n l(x) \underset{\rightarrow}{\rightarrow} h(x)\nu(l)$. Since $h(x)\nu(l) > 0$ we have $\tau = \lambda$ and $l(x) = h(x)\nu(l)$. If $\sigma \in M(X)$ satisfies $\mathcal{L}^*_\varphi\sigma = \lambda\sigma$ then integrating $(\mathcal{L}^n_\varphi f)(x)/\lambda^n \underset{\rightarrow}{\rightarrow} h(x)\nu(f)$ with respect to σ gives $\sigma(f) = \sigma(h)\nu(f)$ for all $f \in C(X)$. Putting $f = 1$ gives $\sigma(h) = 1$ and $\sigma = \nu$.

Since $(1/n)\log(\mathcal{L}^n_\varphi 1)(x) \underset{\rightarrow}{\rightarrow} P(T,\varphi)$ (see [**W4**, Theorem 1.3]) we have $P(T,\varphi) = \log\lambda$. □

We now show that if $\varphi \in R(X) \cap \mathrm{Bow}(X,T)$ then the Ruelle operator theorem holds for φ.

COROLLARY 3.6. *Let $\varphi \in R(X) \cap \mathrm{Bow}(X,T)$. Then statement (ii) of Theorem 3.1 holds so there exists $h \in C(X)$, $h > 0$ with $\mathcal{L}_\varphi h = \lambda h$, where $\lambda = e^{P(X,\varphi)}$, and $\nu \in M(X)$ with $\mathcal{L}^*_\varphi \nu = \lambda\nu$ and, for all $f \in C(X)$, $(\mathcal{L}^n_\varphi f)(x)/\lambda^n \underset{\rightarrow}{\rightarrow} h(x)\nu(f)$.*

The measure μ given by $\mu(f) = \nu(hf)$ is the unique equilibrium state for φ.

Proof. From Theorem 1.1 there exists $K > 0$ so that

$$|a_2 + \cdots + a_{1+j} - ja| \leq K \quad \text{and} \quad |c_2 + \cdots + c_{1+j} - jc| \leq K$$

for all $j \geq 1$. Therefore $e^{-K} e^{aj} \leq e^{a_2+\cdots+a_{1+j}}$ and $e^{-K} e^{cj} \leq e^{c_2+\cdots+c_{1+j}}$ for all $j \geq 1$. Hence

$$\frac{1}{e^{2\max(a,c)}}\left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}}\right]\left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}\right]$$
$$\geq \frac{e^{\inf d_i} e^{\inf b_i}}{e^{2\max(a,c)}}\left[1 + e^{-K}\sum_{j=1}^{\infty}\left(\frac{e^{a}}{e^{\max(a,c)}}\right)^j\right]\left[1 + e^{-K}\sum_{j=1}^{\infty}\left(\frac{e^{c}}{e^{\max(a,c)}}\right)^j\right] = \infty.$$

Hence statement (ii) of Theorem 3.1 holds. □

COROLLARY 3.7. *Let $\varphi \in R(X)$ be defined using the sequences $(a_p)_2^\infty$, $(b_q)_1^\infty$, $(c_p)_2^\infty$ and $(d_q)_1^\infty$ as in §1. If $(a_p)_2^\infty$, $(b_q)_2^\infty$, $(c_p)_2^\infty$ and $(d_q)_2^\infty$ satisfy*

$$\left[\sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}}\right]\left[\sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}\right] \geq e^{2\max(a,c)},$$

then for all choices of b_1 and d_1 an eigenfunction $h > 0$ exists. If

$$\left[\sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}}\right]\left[\sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}\right] < e^{2\max(a,c)},$$

then for some choices of b_1 and d_1 an eigenfunction $h > 0$ exists and for the other choices of b_1 and d_1 no positive eigenfunction exists.

Note that one or both of the sums above could be ∞. This is the case when $\varphi \in \mathrm{Bow}(X, T)$.

Proof. Statement (ii) of Theorem 3.1 says

$$[e^{d_1} + S_1][e^{b_1} + S_2] > e^{2\max(a,c)}, \tag{5}$$

where

$$S_1 = \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,c)}} \quad \text{and} \quad S_2 = \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\cdots+c_{1+j}}}{e^{j\max(a,c)}}.$$

If $S_1 S_2 \geq e^{2\max(a,c)}$ then (5) is true for all choices of b_1 and d_1.

If $S_1 S_2 < e^{2\max(a,c)}$ then (5) holds for some choices of b_1 and d_1 and fails for other choices. □

The following result deals with the class of functions studied by Hofbauer [**Ho**]. He studied the case when $a = 0$.

THEOREM 3.8. *Let $(a_n)_0^\infty$ be a convergent sequence of real numbers with $(a_n) \to a$, and let $\varphi \in C(X)$ be defined by $\varphi(0^k 1 z) = a_k$ for $k \geq 0$, $z \in X$ and $\varphi(0^\infty) = a$. Then there exist $h \in C(X)$ with $h > 0$ and $\mathcal{L}_\varphi h = \lambda h$ for some real number $\lambda > 0$ if and only if $\sum_{i=0}^{\infty} e^{a_0+a_1+\cdots+a_i-(i+1)a} > 1$.*

When this holds $\lambda = e^{P(T,\varphi)} > \max(a, a_0)$ *and is given by*

$$\sum_{j=0}^{\infty} \frac{e^{a_0+a_1+\cdots+a_j}}{\lambda^{1+j}} = 1.$$

When $\sum_{i=0}^{\infty} e^{a_0+a_1+\cdots+a_i-(i+1)a} > 1$ *the unique equilibrium state for* φ *is the unique g-measure for the g-function given by:* $g(01^q0z) = 1 - e^{a_0}/\lambda$, *for all* $q \geq 1$, $z \in X$, *and* $g(0^p1z) = D_p/(1 + D_p)$ *for* $p \geq 2$, $z \in X$ *where*

$$D_p = \sum_{i=0}^{\infty} \frac{e^{a_p+\cdots+a_{p+i}}}{\lambda^{i+1}},$$

$g(0^\infty) = e^a/\lambda$ *and* $g(01^\infty) = 1 - e^{a_0}/\lambda$.

When $\sum_{i=0}^{\infty} e^{a_0+a_1+\cdots+a_i-(i+1)a} > 1$ *we have, for all* $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \underset{\to}{\to} h(x)\nu(f)$$

where ν *is the unique member of* $M(X)$ *with* $\mathcal{L}_\varphi^* \nu = \lambda\nu$.

Proof. In the notation of Theorem 3.1 $b_q = b = a_1$ for all $q \geq 1$ and $c_p = c = d_q = d = a_0$ for all $p \geq 2$, $q \geq 1$. Statement (ii) of Theorem 3.1 becomes

$$\frac{e^{a_0+a_1}}{e^{2\max(a,a_0)}}\left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{j\max(a,a_0)}}\right]\left[1 + \sum_{j=1}^{\infty}\left(\frac{e^{a_0}}{e^{\max(a,a_0)}}\right)^j\right] > 1.$$

If $a_0 \geq a$ the second series diverges to ∞ so the above inequality holds.

If $a_0 < a$ the above inequality becomes

$$e^{a_0+a_1-2a}\left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\cdots+a_{1+j}}}{e^{ja}}\right]\frac{1}{1 - e^{a_0-a}} > 1.$$

This is equivalent to

$$e^{a_0-a} + e^{a_0+a_1-2a} + \sum_{j=1}^{\infty} e^{a_0+a_1+a_2+\cdots+a_{1+j}-(2+j)a} > 1.$$

Therefore, by Theorem 3.1, a positive continuous eigenfunction h exists for $\mathcal{L}_\varphi$ if and only if

$$\sum_{i=0}^{\infty} e^{a_0+\cdots+a_i-(i+1)a} > 1.$$

When this condition holds Corollary 3.5 shows that $\lambda = e^{P(T,\varphi)} > \max(e^a, e^{a_0})$ and

$$\frac{e^{a_0}}{\lambda^2}\left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\cdots+a_{1+j}}}{\lambda^j}\right]e^{a_1}\left[\frac{\lambda}{(\lambda - e^a)}\right] = 1.$$

The last equation becomes

$$\frac{e^{a_0}}{\lambda} + \frac{e^{a_0+a_1}}{\lambda} + \sum_{j=1}^{\infty} \frac{e^{a_0+\cdots+a_{1+j}}}{\lambda^{2+j}} = 1.$$

From the proof of Theorem 3.1 the eigenfunction, h, for $\mathcal{L}_\varphi$ has the following form. Let $\alpha > 0$. Let $\beta = \alpha(\lambda - e^a)/e^{a_0}$. For $q \geq 1$ let $\alpha_q = (\alpha(\lambda - e^a)/\lambda)[1 + D_{q+1}]$ and $\beta_q = \beta$.

Then $h(0^q 1z) = \alpha_q$, $h(1^q 0z) = \beta$, $q \geq 1$, $z \in X$, and $h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$. Then the corresponding g-function is $g = e^\varphi h/\lambda h \circ T$ so $g(0^p 1z) = e^{a_p}\alpha_p/\lambda\alpha_{p-1} = D_p/(1 + D_p)$ for all $p \geq 2, z \in X$, $g(01^q 0z) = e^{a_1}\alpha_1/\lambda\beta$ for all $q \geq 1, z \in X$, $g(1^p 0z) = a_0/\lambda$ for all $p \geq 2$, $z \in X$, and $g(10^q 1z) = 1/(1 + D_{q+1})$ for all $q \geq 1, z \in X$. □

We can get functions of Hofbauer type for which $\mathcal{L}_\varphi$ has no continuous eigenfunction $h > 0$ as follows. Suppose $a_1, a_2, \ldots$ satisfy $a_n \to a$ and $\sum_{j=1}^\infty e^{a_1+\cdots+a_j-ja} < \infty$. Then choose a_0 so that

$$e^{a_0-a}\left(1 + \sum_{j=1}^\infty e^{a_1+\cdots+a_j-ja}\right) \leq 1.$$

Examples are given by choosing $s > 1$ and, for $n \geq 1$,

$$a_n = s\log\left(\frac{n}{n+1}\right).$$

Then

$$1 + \sum_{j=1}^\infty e^{a_1+\cdots+a_j-ja} = \sum_{i=1}^\infty \frac{1}{i^s}.$$

4. *Coboundaries for the two-sided shift*

We can use the space $R(X)$ to obtain examples of functions on the two-sided shift space $\hat{X} = \prod_{-\infty}^\infty\{0, 1\}$ which are not continuous coboundaries, with respect to the shift $S : \hat{X} \to \hat{X}$, but are bounded measurable coboundaries. Points of $\hat{X}$ are bisequences $\hat{x} = (x_n)_{-\infty}^\infty$ of zeros and ones and the homomorphism S is defined by $S\hat{x} = (y_n)_{-\infty}^\infty$ where $y_n = x_{n+1}$ for all $n \in Z$.

Let $\text{Cob}(\hat{X}, S) = \{F \in C(\hat{X}) \mid \exists H \in C(\hat{X}) \text{ with } F = H \circ S - H\}$ be the space of continuous coboundaries, and let $\text{Cob}_{\text{BM}}(\hat{X}, S) = \{F \in C(\hat{X}) \mid \exists H : \hat{X} \to \mathbb{R}$ which is bounded and Borel measurable with $F = H \circ S - H\}$ be the space of bounded measurable coboundaries. If $F = HS - H$ then H is called a cobounding function for F. Similarly we can define $\text{Cob}(X, T)$ and $\text{Cob}_{\text{BM}}(X, T)$.

We have $\text{Cob}(\hat{X}, S) \subset \text{Cob}_{\text{BM}}(\hat{X}, S)$ and $\text{Cob}(X, T) \subset \text{Cob}_{\text{BM}}(X, T)$, and for the one-sided shift $T : X \to X$ Quas [**Q**] has shown that $\text{Cob}(X, T) = \text{Cob}_{\text{BM}}(X, T)$ but $\text{Cob}(\hat{X}, S) \neq \text{Cob}_{\text{BM}}(\hat{X}, S)$.

We show how we can use $\varphi \in R(X) \cap (\text{Bow}(X, T)\backslash W(X, T))$ to get members of $\text{Cob}_{\text{BM}}(\hat{X}, S)\backslash\text{Cob}(\hat{X}, S)$.

We use the following well-known characterization of the members of $\text{Cob}_{\text{BM}}(X, T)$ for a continuous transformation $T : X \to X$ of a compact metric space (see [**KH**, p. 102] where sup should be replaced by lim sup or lim inf).

THEOREM 4.1. *Let T be a continuous transformation of a compact metric space X. Let $f \in C(X)$. Then $f \in \text{Cob}_{\text{BM}}(X, T)$ if and only if there exists $K > 0$ such that $|(T_n f)(x)| \leq K$ for all $x \in X$, for all $n \geq 1$. When this condition holds $l(x) = -\limsup_{n\to\infty}(T_n f)(x)$ is a cobounding function.*

We now return to the shift maps $T : X \to X$ and $S : \hat{X} \to \hat{X}$.

LEMMA 4.2. *Let* $\varphi \in R(X)$*, let* $n \geq 1$ *and choose* $x_i \in \{0, 1\}$ *for* $0 \leq i \leq n-1$*. Then* $(T_n\varphi)((x_0 \ldots x_{n-1})^\infty) = (T_n\varphi)((x_{n-1} \ldots x_0)^\infty)$.

Proof. Let φ be defined by the sequences $(a_p)_2^\infty$, $(b_q)_1^\infty$, $(c_p)_2^\infty$ and $(d_q)_1^\infty$ as in §1. Let

$$A_k = \begin{cases} 1 & \text{if } k = 1, \\ a_k a_{k-1} \ldots a_2 & \text{if } k \geq 2, \end{cases} \quad \text{and} \quad C_l = \begin{cases} 1 & \text{if } l = 1, \\ c_l c_{l-1} \ldots c_2 & \text{if } l \geq 2. \end{cases}$$

Let $x_0 = 0$.

If $x_0 \ldots x_{n-1} = 0^{k_1} 1^{l_1} \ldots 0^{k_r} 1^{l_r}$ with $k_i, l_i \geq 1$, $1 \leq i \leq r$, then

$$(T_n\varphi)((x_0 \ldots x_{n-1})^\infty) = A_{k_1} b_{l_1} C_{l_1} d_{k_2} \ldots C_{l_r} d_{k_1}$$

and

$$(T_n\varphi)((x_{n-1} \ldots x_0)^\infty) = C_{l_1} d_{k_{r-1}} A_{k_{r-1}} \ldots A_{k_1} b_{l_r},$$

so the result holds.

If $x_0 \ldots x_{n-1} = 0^{k_1} 1^{l_1} \ldots 0^{k_r} 1^{l_r} 0^{k_{r+1}}$ then

$$(T_n\varphi)((x_0 \ldots x_{n-1})^\infty) = A_{k_1} b_{l_1} C_{l_1} d_{k_2} \ldots C_{l_r} d_{k_1 + k_{r+1}} a_{k_1 + k_{r+1}} \ldots a_{1+k_1}$$

and

$$(T_n\varphi)((x_{n-1} \ldots x_0)^\infty) = A_{k_{r+1}} b_{l_r} C_{l_r} \ldots d_{k_1 + k_r} a_{k_1 + k_{r+1}} \ldots a_{1+k_{r+1}},$$

so the result holds. Similar calculations deal with the cases when $x_0 = 1$. □

Let $\Phi : \hat{X} \to \hat{X}$ be the reversal map of $\hat{X}$, defined by $\Phi(\hat{x}) = \hat{y}$ where $y_n = x_{-n}$ for all $n \in Z$. Let $\pi : \hat{X} \to X$ be the natural projection, given by $\pi((x_n)_{-\infty}^\infty) = (x_j)_0^\infty$.

THEOREM 4.3. *Let* $\varphi \in R(X)$*. Then the following hold:*
(i) $\varphi \in \text{Bow}(X, T)$ *if and only if* $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$*;*
(ii) $\varphi \in W(X, T)$ *if and only if* $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}(\hat{X}, S)$.

Proof. Let $\varphi \in R(X)$.

(i) Let $\varphi \in R(X) \cap \text{Bow}(X, T)$. We want to find a constant K so that $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq K$ for all $n \geq 1$, $\hat{x} \in \hat{X}$, and then we can use Theorem 4.1.

Let C be the constant occurring in the Bowen condition so that if $x, y \in X$, $n \geq 1$, and $x_i = y_i$, $0 \leq i \leq n-1$, then $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq C$.

Let $\hat{x} = (x_j)_{-\infty}^\infty \in \hat{X}$. Let $n \geq 1$. Then we have

$$\begin{aligned} S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) &= (T_n\varphi)(x_0 x_1 x_2 \ldots) - (T_n\varphi)(x_n x_{n-1} \ldots x_1 x_0 x_{-1} x_{-2} \ldots) \\ &= (T_n\varphi)(x_0 x_1 x_2 \ldots) - (T_n\varphi)((x_0 \ldots x_{n-1})^\infty) \\ &\quad + (T_n\varphi)((x_0 \ldots x_{n-1})^\infty) - (T_n\varphi)((x_{n-1} \ldots x_0)^\infty) \\ &\quad + (T_n\varphi)((x_{n-1} \ldots x_0)^\infty) - (T_n\varphi)(x_{n-1} \ldots x_1 x_0 x_{-1} x_{-2} \ldots) \end{aligned}$$

so $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq 2C$ by Lemma 4.2. Hence $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$ by Theorem 4.1.

Now let $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \mathrm{Cob}_{\mathrm{BM}}(\hat{X}, S)$. Then there exists K such that $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq K$ for all $n \geq 1$, $\hat{x} \in \hat{X}$. Let $x, y \in X$ and $x_i = y_i, 0 \leq i \leq n-1$. Choose $y_j = 0 = x_j$ for all $j < 0$ to form $\hat{x} = (x_i)_{-\infty}^{\infty}$ and $\hat{y} = (y_i)_{-\infty}^{\infty} \in \hat{X}$. Then we have

$$\begin{aligned}(T_n\varphi)(x) - (T_n\varphi)(y) &= (T_n\varphi)(x) - (T_n\varphi)(x_{n-1}\ldots x_1x_0x_{-1}x_{-2}\ldots) \\ &\quad + (T_n\varphi)(x_{n-1}\ldots x_1x_0x_{-1}x_{-2}\ldots) - (T_n\varphi)(y) \\ &= S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) - S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{y}).\end{aligned}$$

Hence $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq 2K$, and $\varphi \in \mathrm{Bow}(X, T)$.

(ii) Let $\varphi \in R(X) \cap W(X, T)$. Since

$$(S_n(\varphi \circ \pi \circ \Phi))(\hat{x}) = (T_n\varphi)(x_nx_{n-1}\ldots x_1x_0x_{-1}x_{-2}\ldots)$$

we have $\varphi \circ \pi \circ \Phi \in W(X, T)$ so there exists $\varphi_+ \in C(X)$ such that $\varphi \circ \pi \circ \Phi - \varphi_+ \circ \pi \in \mathrm{Cob}(\hat{X}, S)$ (see [**Bou**]). By (i) $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \mathrm{Cob}_{\mathrm{BM}}(\hat{X}, S)$ so $\varphi \circ \pi - \varphi_+ \circ \pi \in \mathrm{Cob}_{\mathrm{BM}}(\hat{X}, S)$. By Theorem 4.1 applied to S and T we have $\varphi - \varphi_+ \in \mathrm{Cob}_{\mathrm{BM}}(X, T)$, so $\varphi - \varphi_+ \in \mathrm{Cob}(X, T)$ by [**Q**]. Hence $\varphi \circ \pi \circ \Phi - \varphi \circ \pi \in \mathrm{Cob}(\hat{X}, S)$.

Now let $\varphi \circ \pi - \varphi \circ \pi \circ \Phi = FS - F$ where $F \in C(\hat{X})$. We show that $\sup_{n\geq 1} \mathrm{v}_{n+N}(T_n\varphi) \to 0$ as $N \to \infty$.

Let $n \geq 1$ and $N \geq 1$ and let $x = (x_j)_0^{\infty}$, $y = (y_j)_0^{\infty} \in X$ have $x_j = y_j, 0 \leq j \leq n+N-1$. Let $x_i = 0 = y_i$ for all $i \leq -1$ to obtain $\hat{x} = (x_j)_{i-\infty}^{\infty}$ and $\hat{y} = (y_j)_{-\infty}^{\infty} \in \hat{X}$. Then

$$\begin{aligned}&(T_n\varphi)(x) - (T_n\varphi)(y) \\ &\quad = S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) - S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{y}) \\ &\quad = F(S^n\hat{x}) - F(\hat{x}) - F(S^n\hat{y}) + F(\hat{y}) \\ &\quad = F(\ldots \overset{*}{x}_n \ldots x_{n+N-1}x_{n+N}\ldots) - F(\ldots \overset{*}{y}_n \ldots y_{n+N-1}y_{n+N}\ldots) \\ &\quad\quad - [F(\ldots \overset{*}{x}_0 \ldots x_{n+N-1}x_{n+N}\ldots) - F(\ldots \overset{*}{y}_0 \ldots y_{n+N-1}y_{n+N}\ldots)] \\ &\quad \leq \mathrm{v}_N(F) + \mathrm{v}_{n+N}(F) \leq 2\mathrm{v}_N(F).\end{aligned}$$

Hence $\sup_{n\geq 1} \mathrm{v}_{n+N}(T_n\varphi) \leq 2\mathrm{v}_N(F)$ so $\varphi \in W(X, T)$.

This completes the proof of Theorem 4.3. □

We can get members of $\mathrm{Cob}_{\mathrm{BM}}(\hat{X}, S)\backslash\mathrm{Cob}(X, S)$ as follows.

COROLLARY 4.4. *Let $\varphi \in R(X)$. Then $\varphi \in \mathrm{Bow}(X, T)\backslash W(X, T)$ if and only if $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \mathrm{Cob}_{\mathrm{BM}}(\hat{X}, S)\backslash\mathrm{Cob}(\hat{X}, S)$.*

Examples of functions in $R(X) \cap (\mathrm{Bow}(X, T)\backslash W(X, T))$ are given in §1.

Results of this type, in a more general setting, will appear in another paper.

REFERENCES

[**Bou**] T. Bousch. La condition de Walters. *Ann. Sci. École Norm. Sup. (4)* **34** (2001), 287–311.
[**Bow**] R. Bowen. Some systems with unique equilibrium states. *Math. Syst. Theory* **8** (1974), 193–202.

[Fe] W. Feller. *An Introduction to Probability Theory and Its Applications*, 2nd edn. Vol. 1. Wiley, New York, 1962.

[Fi] M. E. Fisher. *Physica* **3** (1967), 255–283.

[Ho] F. Hofbauer. Examples for the nonuniqueness of the equilibrium state. *Trans. Amer. Math. Soc.* **228** (1977), 223–241.

[Hu] P. Hulse. *PhD Thesis*, University of Warwick, 1980.

[JO] A. Johansson and A. Oberg. Square summability of variations of g-functions and uniqueness of g-measures. *Math. Res. Lett.* **10** (2003), 1–15.

[KH] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.

[L] F. Ledrappier. Principe variationnel et systèmes symboliques. *Z. Wahr. Verw. Gebiete* **30** (1974), 185–202.

[Q] A. Quas. Rigidity of continuous coboundaries. *Bull. London Math. Soc.* **29** (1997), 595–600.

[W1] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, Berlin, 1982.

[W2] P. Walters. Ruelle's operator theorem and g-measures. *Trans. Amer. Math. Soc*. **214** (1975), 375–387.

[W3] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. *Trans. Amer. Math. Soc.* **236** (1978), 121–153.

[W4] P. Walters. Convergence of the Ruelle operator for a function satisfying Bowen's condition. *Trans. Amer. Math. Soc.* **353** (2001), 327–347.

[W5] P. Walters. A necessary and sufficient condition for a two-sided continuous function to be cohomologous to a one-sided continuous function. *Dyn. Sys.* **18** (2003), 131–138.

[W6] P. Walters. Regularity conditions and Bernoulli properties of equilibrium states and g-measures. *J. London Math. Soc (2)* **71** (2005), 379–396.