DECOMPOSITIONS OF RATIONAL GABOR REPRESENTATIONS

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Abstract. Let $\Gamma = \left\langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \right\rangle$ be a group of unitary operators where T_k is a translation operator and M_l is a modulation operator acting on $L^2\left(\mathbb{R}^d\right)$. Assuming that B is a non-singular rational matrix of order d with at least one entry which is not an integer, we obtain a direct integral irreducible decomposition of the Gabor representation which is defined by the isomorphism $\pi : \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d \to \Gamma$ where $\pi(\theta, l, k) = e^{\frac{2\pi i}{m}\theta} M_l T_k$. We also show that the left regular representation of $\left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d$ which is identified with Γ via π is unitarily equivalent to a direct sum of card $([\Gamma,\Gamma])$ many disjoint subrepresentations of the type: $L_0, L_1, \cdots, L_{\text{card}([\Gamma,\Gamma])-1}$ such that for $k \neq 1$ the subrepresentation L_k of the left regular representation is disjoint from the Gabor representation. Additionally, we compute the central decompositions of the representations π and L_1. These decompositions are then exploited to give a new proof of the Density Condition of Gabor systems (for the rational case). More precisely, we prove that π is equivalent to a subrepresentation of L_1 if and only if $|\det B| \leq 1$. We also derive characteristics of vectors f in $L^2(\mathbb{R})^d$ such that $\pi(\Gamma)f$ is a Parseval frame in $L^2(\mathbb{R})^d$.

1. Introduction

The concept of applying tools of abstract harmonic analysis to time-frequency analysis, and wavelet theory is not a new idea [1, 2, 3, 7, 11]. For example in [1], Larry Baggett gives a direct integral decomposition of the Stone-von Neumann representation of the discrete Heisenberg group acting in $L^2(\mathbb{R})$. Using his decomposition, he was able to provide specific conditions under which this representation is cyclic. In Section 5.5, [7] the author obtains a characterization of tight Weyl-Heisenberg frames in $L^2(\mathbb{R})$ using the Zak transform and a precise computation of the Plancherel measure of a discrete type I group. In [11], the authors present a thorough study of the left regular representations of various subgroups of the reduced Heisenberg groups. Using well-known results of admissibility of unitary representations of locally compact groups, they were able to offer new insights on Gabor theory.

Let B be a non-singular matrix of order d with real entries. For each $k \in \mathbb{Z}^d$ and $l \in B\mathbb{Z}^d$, we define the corresponding unitary operators T_k, M_l such that $T_k f(t) = f(t-k)$ and $M_l f(t) = e^{-2\pi i\langle l,t\rangle} f(t)$ for $f \in L^2\left(\mathbb{R}^d\right)$. The operator T_k is called a shift operator, and the operator M_l is called a modulation operator. Let Γ be a subgroup of the group of unitary operators acting on $L^2\left(\mathbb{R}^d\right)$ which is generated by the set $\left\{T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d\right\}$. We write $\Gamma = \left\langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \right\rangle$. The commutator subgroup of Γ given by $[\Gamma,\Gamma] = \left\{e^{2\pi i\langle l,Bk\rangle} : l, k \in \mathbb{Z}^d\right\}$ is a subgroup of the one-dimensional torus $\mathbb{T}$. Since $[\Gamma,\Gamma]$ is always contained in the center of the group, then Γ is a nilpotent group which is generated by $2d$ elements. Moreover, Γ is given the discrete topology, and as such it is a locally compact group. We observe that if B has at least one irrational entry, then it is a non-abelian group with an infinite center. If B only has rational entries with at least one entry which is not an integer, then $[\Gamma,\Gamma]$ is a finite group, and Γ is a non-abelian group which is regarded as a finite extension of an abelian group. If all entries of B are integers, then Γ is abelian, and clearly $[\Gamma,\Gamma]$ is trivial. Finally, it is worth mentioning that Γ is a type I group if and only if B only has rational entries [12].

It is easily derived from the work in Section 4, [11] that if B is an integral matrix, then the Gabor representation $\pi : B\mathbb{Z}^d \times \mathbb{Z}^d \to \Gamma \subset \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right)$ defined by $\pi(l,k) = M_l T_k$ is equivalent to a

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subrepresentation of the left regular representation of Γ if and only if B is a unimodular matrix. The techniques used by the authors of [11] rely on the decompositions of the left regular representation and the Gabor representation into their irreducible components. The group generated by the operators M_l and T_k is a commutative group which is isomorphic to $\mathbb{Z}_d \times B\mathbb{Z}_d$. The unitary dual and the Plancherel measure for discrete abelian groups are well-known and rather easy to write down. Thus, a precise direct integral decomposition of the left regular representation is easily obtained as well. Next, using the Zak transform, the authors decompose the representation π into a direct integral of its irreducible components. They are then able to compare both representations. As a result, one can derive from the work in the fourth section of [11] that the representation π is equivalent to a subrepresentation of the left regular representation if and only if B is a unimodular matrix. The main objective of this paper is to generalize these ideas to the more difficult case where $B \in GL(d, \mathbb{Q})$ and Γ is not a commutative group.

Let us assume that B is an invertible rational matrix with at least one entry which is not an element of $\mathbb{Z}$. Denoting the inverse transpose of a given matrix M by $M^\star$, it is not too hard to see that there exists a matrix $A \in GL(d, \mathbb{Z})$ such that $\Lambda = B^\star\mathbb{Z}^d \cap \mathbb{Z}^d = A\mathbb{Z}^d$. Indeed, a precise algorithm for the construction of A is described on Page 809 of [4]. Put

$$\Gamma_0 = \left\langle \tau, M_l, T_k : l \in B\mathbb{Z}^d, k \in \Lambda, \tau \in [\Gamma, \Gamma] \ \right\rangle \tag{1}$$

and define $\Gamma_1 = \left\langle \tau, M_l : l \in B\mathbb{Z}^d, \tau \in [\Gamma, \Gamma] \ \right\rangle$. Then Γ_0 is a normal abelian subgroup of Γ. Moreover, we observe that Γ_1 is a subgroup of Γ_0 of infinite index. Let m be the number of elements in $[\Gamma, \Gamma]$. Clearly, since B has at least one rational entry which is not an integer, it must be the case that $m > 1$. Furthermore, it is easy to see that there is an isomorphism $\pi : \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d \to \Gamma \subset \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right)$ defined by $\pi(j, Bl, k) = e^{\frac{2\pi j i}{m}} M_{Bl} T_k$. The multiplication law on the semi-direct product group $\left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d$ is described as follows. Given arbitrary elements

$$(j, Bl, k), (j_1, Bl_1, k_1) \in \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d,$$

we define $(j, Bl, k)(j_1, Bl_1, k_1) = ((j + j_1 + \omega(l_1, k)) \bmod m, B(l + l_1), k + k_1)$ where $\omega(l_1, k) \in \mathbb{Z}_m$, and $\langle Bl_1, k \rangle = \frac{\omega(l_1,k)}{m}$. We call π a rational Gabor representation. It is also worth observing that $\pi^{-1}(\Gamma_0) = \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \times A\mathbb{Z}^d$ and $\pi^{-1}(\Gamma_1) = \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \times \{0\} \simeq \mathbb{Z}_m \times B\mathbb{Z}^d$. Throughout this work, in order to avoid cluster of notations, we will make no distinction between $\left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d$ and Γ and their corresponding subgroups.

The main results of this paper are summarized in the following propositions. Let

$$\Gamma = \left\langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \right\rangle$$

and assume that B is an invertible rational matrix with at least one entry which is not an integer. Let L be the left regular representation of Γ.

Proposition 1. *The left regular representation of* Γ *is decomposed as follows:*

$$L \simeq \oplus_{k=0}^{m-1} \int_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(k,\sigma)} \, d\sigma. \tag{2}$$

Moreover, the measure $d\sigma$ *in (2) is a Lebesgue measure, and (2) is not an irreducible decomposition of* L.

Proposition 2. *The Gabor representation* π *is decomposed as follows:*

$$\pi \simeq \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)} \, d\sigma. \tag{3}$$

Moreover, $d\sigma$ *is a Lebesgue measure defined on the torus* $\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$ *and (3) is an irreducible decomposition of* π.

It is worth pointing out here that the decomposition of π given in Proposition 2 is consistent with the decomposition obtained in Lemma 5.39, [7] for the specific case where $d=1$ and B is the inverse of a natural number larger than one.

Proposition 3. *Let m be the number of elements in the commutator subgroup $[\Gamma,\Gamma]$. There exists a decomposition of the left regular representation of Γ such that*

$$L \simeq \bigoplus_{k=0}^{m-1} L_k \tag{4}$$

and for each $k \in \{0,1,\cdots,m-1\}$, the representation L_k is disjoint from π whenever $k \neq 1$. Moreover, the Gabor representation π is equivalent to a subrepresentation of the subrepresentation L_1 of L if and only if $|\det B| \leq 1$.

Although this problem of decomposing the representations π and L into their irreducible components is interesting in its own right, we shall also address how these decompositions can be exploited to derive interesting and relevant results in time-frequency analysis. The proof of Proposition 3 allows us to state the following:

(1) There exists a measurable set $\mathbf{E} \subset \mathbb{R}^d$ which is a subset of a fundamental domain for the lattice $B^\star\mathbb{Z}^d \times A^\star\mathbb{Z}^d$, satisfying

$$\mu(\mathbf{E}) = \frac{1}{|\det(B)\det(A)|\dim\left(l^2\left(\Gamma/\Gamma_0\right)\right)}$$

where μ is the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$.

(2) There exists a unitary map

$$\mathfrak{A} : \int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} l^2\left(\frac{\Gamma}{\Gamma_0}\right)\right) d\sigma \to L^2\left(\mathbb{R}^d\right) \tag{5}$$

which intertwines $\int_{\mathbf{E}}^{\oplus}\left(\oplus_{k=1}^{\ell(\sigma)}\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\right) d\sigma$ with π such that, the multiplicity function ℓ is bounded, the representations $\chi_{(1,\sigma)}$ are characters of the abelian subgroup Γ_0 and

$$\int_{\mathbf{E}}^{\oplus}\left(\oplus_{k=1}^{\ell(\sigma)}\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\right) d\sigma \tag{6}$$

is the central decomposition of π (Section 3.4.2, [7]).

Moreover, for the case where $|\det(B)| \leq 1$, the multiplicity function ℓ is bounded above by the number of cosets in Γ/Γ_0 while if $|\det(B)| > 1$, then the multiplicity function ℓ is bounded but is greater than the number of cosets in Γ/Γ_0 on a set $\mathbf{E}' \subseteq \mathbf{E}$ of positive Lebesgue measure. This observation that the upper-bound of the multiplicity function behaves differently in each situation may mistakenly appear to be of limited importance. However, at the heart of this observation, lies a new justification of the well-known Density Condition of Gabor systems for the rational case (Theorem 1.3, [8]). In fact, we shall offer a new proof of a rational version of the Density Condition for Gabor systems in Proposition 9.

It is also worth pointing out that the central decomposition of π as described above has several useful implications. Following the discussion on Pages 74-75, [7], the decomposition given in (6) may be used to:

(1) Characterize the commuting algebra of the representation π and its center.
(2) Characterize representations which are either quasi-equivalent or disjoint from π (see [7] Theorem 3.17 and Corollary 3.18).

Additionally, using the central decomposition of π, in the case where the absolute value of the determinant of B is less or equal to one, we obtain a complete characterization of vectors f such that $\pi(\Gamma) f$ is a Parseval frame in $L^2(\mathbb{R}^d)$.

Proposition 4. *Let us suppose that* $|\det B| \leq 1$. *Then*

$$\pi \simeq \int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\right) d\sigma \tag{7}$$

with $\ell(\sigma) \leq \operatorname{card}(\Gamma/\Gamma_0)$. *Moreover,* $\pi(\Gamma) f$ *is a Parseval frame in* $L^2(\mathbb{R}^d)$ *if and only if* $f = \mathfrak{A}(a(\sigma))_{\sigma\in\mathbf{E}}$ *such that for* $d\sigma$*-almost every* $\sigma \in \mathbf{E}$, $\|a(\sigma)(k)\|^2_{l^2\left(\frac{\Gamma}{\Gamma_0}\right)} = 1$ *for* $1 \leq k \leq \ell(\sigma)$ *and for distinct* $k, j \in \{1, \cdots, \ell(\sigma)\}$, $\langle a(\sigma)(k), a(\sigma)(j)\rangle = 0$.

This paper is organized around the proofs of the propositions mentioned above. In Section 2, we fix notations and we revisit well-known concepts such as induced representations and direct integrals which are of central importance. The proof of Proposition 1 is obtained in the third section. The proofs of Propositions 2, 3 and examples are given in the fourth section. Finally, the last section contains the proof of Proposition 4.

2. Preliminaries

Let us start by fixing our notations and conventions. All representations in this paper are assumed to be unitary representations. Given two equivalent representations π and ρ, we write that $\pi \simeq \rho$. We use the same notation for isomorphic groups. That is, if G and H are isomorphic groups, we write that $G \simeq H$. All sets considered in this paper will be assumed to be measurable. Given two disjoint sets A and B, the disjoint union of the sets is denoted $A \dot{\cup} B$. Let $\mathbf{H}$ be a Hilbert space. The identity operator acting on $\mathbf{H}$ is denoted $1_{\mathbf{H}}$. The unitary equivalence classes of irreducible unitary representations of G is called the unitary dual of G and is denoted $\widehat{G}$.

Several of the proofs presented in this work rely on basic properties of induced representations and direct integrals. The following discussion is mainly taken from Chapter 6, [6]. Let G be a locally compact group, and let K be a closed subgroup of G. Let us define $q : G \to G/K$ to be the canonical quotient map and let φ be a unitary representation of the group K acting in some Hilbert space which we call $\mathbf{H}$. Next, let $\mathbf{K}_1$ be the set of continuous $\mathbf{H}$-valued functions f defined over G satisfying the following properties:

(1) $q(\text{support } (f))$ is compact,
(2) $f(gk) = \varphi(k)^{-1} f(g)$ for $g \in G$ and $k \in K$.

Clearly, G acts on the set $\mathbf{K}_1$ by left translation. Now, to simplify the presentation, let us suppose that G/K admits a G-invariant measure. We remark that in general, this is not always the case. However, the assumption that G/K admits a G-invariant measure holds for the class of groups considered in this paper. We construct a unitary representation of G by endowing $\mathbf{K}_1$ with the following inner product:

$$\langle f, f'\rangle = \int_{G/K} \langle f(g), f'(g)\rangle_{\mathbf{H}} \, d(gK) \text{ for } f, f' \in \mathbf{K}_1.$$

Now, let $\mathbf{K}$ be the Hilbert completion of the space $\mathbf{K}_1$ with respect to this inner product. The translation operators extend to unitary operators on $\mathbf{K}$ inducing a unitary representation $\mathrm{Ind}_K^G(\varphi)$ which is defined as follows:

$$\mathrm{Ind}_K^G(\varphi)(x) f(g) = f\left(x^{-1}g\right) \text{ for } f \in \mathbf{K}.$$

We notice that if φ is a character, then the Hilbert space $\mathbf{K}$ can be naturally identified with $L^2(G/H)$. Induced representations are natural ways to construct unitary representations. For example, it is easy to prove that if e is the identity element of G and if 1 is the trivial representation of $\{e\}$ then the representation $\mathrm{Ind}_{\{e\}}^{G}(1)$ is equivalent to the left regular representation of G. Other properties of induction such as induction in stages will be very useful for us. The reader who is not familiar with these notions is invited to refer to Chapter 6 of the book of Folland [6] for a thorough presentation.

We will now present a short introduction to direct integrals; which are heavily used in this paper. For a complete presentation, the reader is referred to Section 7.4, [6]. Let $\{H_\alpha\}_{\alpha\in A}$ be a family of separable Hilbert spaces indexed by a set A. Let μ be a measure defined on A. We define the direct integral of this family of Hilbert spaces with respect to μ as the space which consists of vectors φ defined on the parameter space A such that $\varphi(\alpha)$ is an element of H_α for each $\alpha \in A$ and $\int_A \|\varphi(\alpha)\|_{H_\alpha}^2 d\mu(\alpha) < \infty$ with some additional measurability conditions which we will clarify. A family of separable Hilbert spaces $\{H_\alpha\}_{\alpha\in A}$ indexed by a Borel set A is called a field of Hilbert spaces over A. A map $\varphi : A \to \bigcup_{\alpha\in A} H_\alpha$ such that $\varphi(\alpha) \in H_\alpha$ is called a vector field on A. A measurable field of Hilbert spaces over the indexing set A is a field of Hilbert spaces $\{H_\alpha\}_{\alpha\in A}$ together with a countable set $\{e_j\}_j$ of vector fields such that

(1) the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle_{H_\alpha}$ are measurable for all j, k,
(2) the linear span of $\{e_k(\alpha)\}_k$ is dense in H_α for each $\alpha \in A$.

The direct integral of the spaces H_α with respect to the measure μ is denoted by $\int_A^\oplus H_\alpha d\mu(\alpha)$ and is the space of measurable vector fields φ on A such that $\int_A \|\varphi(\alpha)\|_{H_\alpha}^2 d\mu(\alpha) < \infty$. The inner product for this Hilbert space is: $\langle \varphi_1, \varphi_2\rangle = \int_A \langle \varphi_1(\alpha), \varphi_2(\alpha)\rangle_{H_\alpha} d\mu(\alpha)$ for $\varphi_1, \varphi_2 \in \int_A^\oplus H_\alpha d\mu(\alpha)$.

3. The regular representation and its decompositions

In this section, we will discuss the Plancherel theory for Γ. For this purpose, we will need a complete description of the unitary dual of Γ. This will allow us to obtain a central decomposition of the left regular representation of Γ. Also, a proof of Proposition 1 will be given in this section.

Let L be the left regular representation of Γ. Suppose that Γ is not commutative and that B is a rational matrix. We have shown that Γ has an abelian normal subgroup of finite index which we call Γ_0. Moreover, there is a canonical action (74-79, [10]) of Γ on the group $\widehat{\Gamma}_0$ (the unitary dual of Γ_0) such that for each $P \in \Gamma$ and $\chi \in \widehat{\Gamma}_0$, $P \cdot \chi(Q) = \chi(P^{-1}QP)$. Let us suppose that $\chi = \chi_{(\lambda_1,\lambda_2,\lambda_3)}$ is a character in the unitary dual $\widehat{\Gamma}_0$ where $(\lambda_1, \lambda_2, \lambda_3) \in \{0, 1, \cdots, m-1\} \times \frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d} \simeq \widehat{\Gamma}_0$, and $\chi_{(\lambda_1,\lambda_2,\lambda_3)}\left(e^{\frac{2\pi ik}{m}} M_{Bl} T_{Aj}\right) = e^{\frac{2\pi i\lambda_1 k}{m}} e^{2\pi i\langle\lambda_2, Bl\rangle} e^{2\pi i\langle\lambda_3, Aj\rangle}$. We observe that $\mathbb{R}^d$ is identified with its dual $\widehat{\mathbb{R}^d}$ and $\{0, 1, \cdots, m-1\}$ parametrizes the unitary dual of the commutator subgroup $[\Gamma, \Gamma]$ which is isomorphic to the cyclic group $\mathbb{Z}_m$. For any $\tau \in [\Gamma, \Gamma]$, we have $\tau \cdot \chi_{(\lambda_1,\lambda_2,\lambda_3)} = \chi_{(\lambda_1,\lambda_2,\lambda_3)}$. Moreover, given $M_l, T_k \in \Gamma$,

$$M_l \cdot \chi_{(\lambda_1,\lambda_2,\lambda_3)} = \chi_{(\lambda_1,\lambda_2,\lambda_3)}, \text{ and } T_k \cdot \chi_{(\lambda_1,\lambda_2,\lambda_3)} = \chi_{(\lambda_1,\lambda_2-k\lambda_1,\lambda_3)}.$$

Next, let $\Gamma_\chi = \{P \in \Gamma : P \cdot \chi = \chi\}$. It is easy to see that the stability subgroup of the character $\chi_{(\lambda_1,\lambda_2,\lambda_3)}$ is described as follows:

$$\Gamma_{\chi_{(\lambda_1,\lambda_2,\lambda_3)}} = \left\{\tau M_l T_k \in \Gamma : \tau \in [\Gamma, \Gamma], l \in B\mathbb{Z}^d, k\lambda_1 \in B^\star\mathbb{Z}^d\right\}. \tag{8}$$

It follows from (8) that the stability group $\Gamma_{\chi_{(\lambda_1,\lambda_2,\lambda_3)}}$ contains the normal subgroup Γ_0. Indeed, if $\lambda_1 = 0$ then $\Gamma_{\chi_{(\lambda_1,\lambda_2,\lambda_3)}} = \Gamma$. Otherwise,

$$\Gamma_{\chi_{(\lambda_1,\lambda_2,\lambda_3)}} = \left\{\tau M_l T_k \in \Gamma : \tau \in [\Gamma, \Gamma],\ l \in B\mathbb{Z}^d, k \in \left(\frac{1}{\lambda_1} B^\star\mathbb{Z}^d\right) \cap \mathbb{Z}^d\right\}. \tag{9}$$

According to Mackey theory (see Page 76, [10]) and well-known results of Kleppner and Lipsman (Page 460, [10]), every irreducible representation of Γ is of the type $\mathrm{Ind}_{\Gamma_\chi}^{\Gamma}(\widetilde{\chi} \otimes \widetilde{\sigma})$ where $\widetilde{\chi}$ is an extension of a character χ of Γ_0 to Γ_χ, and $\widetilde{\sigma}$ is the lift of an irreducible representation σ of Γ_χ/Γ_0

to Γ_χ. Also two irreducible representations $\mathrm{Ind}_{\Gamma_\chi}^{\Gamma}(\widetilde{\chi}\otimes\widetilde{\sigma})$ and $\mathrm{Ind}_{\Gamma_{\chi_1}}^{\Gamma}(\widetilde{\chi_1}\otimes\widetilde{\sigma})$ are equivalent if and only if the characters χ and χ_1 of Γ_0 belong to the same Γ-orbit. Since Γ is a finite extension of its subgroup Γ_0, then it is well known that there is a measurable set which is a cross-section for the Γ-orbits in $\widehat{\Gamma}_0$. Now, let Σ be a measurable subset of $\widehat{\Gamma}_0$ which is a cross-section for the Γ-orbits in $\widehat{\Gamma}_0$. The unitary dual of Γ is a fiber space which is described as follows:

$$\widehat{\Gamma}=\bigcup_{\chi\in\Sigma}\left\{\pi_{\chi,\sigma}=\mathrm{Ind}_{\Gamma_\chi}^{\Gamma}(\widetilde{\chi}\otimes\widetilde{\sigma})\ :\ \sigma\in\widehat{\frac{\Gamma_\chi}{\Gamma_0}}\right\}.$$

Finally, since Γ is a type I group, there exists a unique standard Borel measure on $\widehat{\Gamma}$ such that the left regular representation of the group Γ is equivalent to a direct integral of all elements in the unitary dual of Γ, and the multiplicity of each irreducible representation occurring is equal to the dimension of the corresponding Hilbert space. So, we obtain a decomposition of the representation L into a direct integral decomposition of its irreducible representations as follows (see Theorem 3.31, [7] and Theorem 5.12, [11])

$$L\simeq\int_\Sigma^\oplus\int_{\widehat{\frac{\Gamma_\chi}{\Gamma_0}}}^\oplus\oplus_{k=1}^{\dim\left(l^2\left(\frac{\Gamma}{\Gamma_\chi}\right)\right)}\pi_{\chi,\sigma}\,d\omega_\chi(\sigma)\,d\chi \tag{10}$$

and $\dim\left(l^2\left(\frac{\Gamma}{\Gamma_\chi}\right)\right)\le\mathrm{card}\left(\Gamma/\Gamma_0\right)$. The fact that $\dim\left(l^2\left(\frac{\Gamma}{\Gamma_\chi}\right)\right)\le\mathrm{card}\left(\Gamma/\Gamma_0\right)$ is justified because the number of representative elements of the quotient group $\frac{\Gamma}{\Gamma_\chi}$ is bounded above by the number of elements in $\frac{\Gamma}{\Gamma_0}$. Moreover the direct integral representation in (10) is realized as acting in the Hilbert space

$$\int_\Sigma^\oplus\int_{\widehat{\frac{\Gamma_\chi}{\Gamma_0}}}^\oplus\oplus_{k=1}^{\dim\left(l^2\left(\frac{\Gamma}{\Gamma_\chi}\right)\right)}l^2\left(\frac{\Gamma}{\Gamma_\chi}\right)d\omega_\chi(\sigma)\,d\chi. \tag{11}$$

Although the decomposition in (10) is canonical, the decomposition provided by Proposition 1 will be more convenient for us.

3.1. **Proof of Proposition 1.** Let e be the identity element in Γ, and let 1 be the trivial representation of the trivial group $\{e\}$. We observe that $L\simeq\mathrm{Ind}_{\{e\}}^{\Gamma}(1)$. It follows that

$$L\simeq\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\mathrm{Ind}_{\{e\}}^{\Gamma_0}(1)\right) \tag{12}$$

$$\simeq\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\int_{\widehat{\Gamma}_0}^\oplus\chi_t\,dt\right)$$

$$\simeq\int_{\widehat{\Gamma}_0}^\oplus\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\chi_t)\,\,dt. \tag{13}$$

The second equivalence given above is coming from the fact that $\mathrm{Ind}_{\{e\}}^{\Gamma_0}(1)$ is equivalent to the left regular representation of the group Γ_0. Since Γ_0 is abelian, its left regular representation admits a direct integral decomposition into elements in the unitary dual of Γ_0, each occurring once. Moreover, the measure dt is a Lebesgue measure (also a Haar measure) supported on the unitary dual of the group, and the Plancherel transform is the unitary operator which is intertwining the representations $\mathrm{Ind}_{\{e\}}^{\Gamma_0}(1)$ and $\int_{\widehat{\Gamma}_0}^\oplus\chi_t\,dt$. Based on the discussion above, it is worth mentioning that the representations occurring in (13) are generally reducible since it is not always the case that $\Gamma_0=\Gamma_{\chi_t}$. We observe that $\widehat{\Gamma}_0$ is parametrized by the group $\mathbb{Z}_m\times\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d}\times\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$. Thus, identifying $\widehat{\Gamma}_0$ with $\mathbb{Z}_m\times\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d}\times\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$, we reach the desired result: $L\simeq\oplus_{k=0}^{m-1}\int_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d}\times\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^\oplus\mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(k,t)}\,dt$.

Remark 5. *Referring to (9), we remark that* $\Gamma_{\chi(1,t_2,t_3)} = \Gamma_0$, *and in this case* $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,t_2,t_3)}\right)$ *is an irreducible representation of the group* Γ.

4. Decomposition of π

In this section, we will provide a decomposition of the Gabor representation π. For this purpose, it is convenient to regard the set $\mathbb{R}^d$ as a fiber space, with base space the d-dimensional torus. Next, for any element t in the torus, the corresponding fiber is the set $t+\mathbb{Z}^d$. With this concept in mind, let us define the periodization map $\mathfrak{R} : L^2\left(\mathbb{R}^d\right) \to \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}^{\oplus} l^2\left(\mathbb{Z}^d\right) dt$ such that $\mathfrak{R}f(t) = (f(t+k))_{k\in\mathbb{Z}^d}$. We remark here that we clearly abuse notation by making no distinction between $\frac{\mathbb{R}^d}{\mathbb{Z}^d}$ and a choice of a measurable subset of $\mathbb{R}^d$ which is a fundamental domain for $\frac{\mathbb{R}^d}{\mathbb{Z}^d}$. Next, the inner product which we endow the direct integral Hilbert space $\int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}^{\oplus} l^2\left(\mathbb{Z}^d\right) dt$ with is defined as follows. For any vectors

$$f \text{ and } h \in \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}^{\oplus} l^2\left(\mathbb{Z}^d\right) dt$$

the inner product of f and g is equal to $\langle f,h\rangle_{\mathfrak{R}(L^2(\mathbb{R}^d))} = \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}\left(\sum_{k\in\mathbb{Z}^d} f(t+k)\overline{h(t+k)}\right) dt$, and it is easy to check that $\mathfrak{R}$ is a unitary map.

4.1. **Proof of Proposition 2.** For $t\in\mathbb{R}^d$, we consider the unitary character $\chi_{(1,-t)} : \Gamma_1 \to \mathbb{T}$ which is defined by $\chi_{(1,-t)}\left(e^{2\pi i z}M_l\right) = e^{2\pi i z}e^{-2\pi i\langle t,l\rangle}$. Next, we compute the action of the unitary representation $\mathrm{Ind}_{\Gamma_1}^{\Gamma}\chi_{(1,-t)}$ of Γ which is acting in the Hilbert space

$$\mathbf{H}_t = \left\{\begin{array}{c} f:\Gamma\to\mathbb{C} : f(PQ) = \chi_{(1,-t)}(Q)^{-1} f(P), Q\in\Gamma_1 \\ \text{and } \sum_{P\Gamma_1\in\frac{\Gamma}{\Gamma_1}} |f(P)|^2 < \infty \end{array}\right\}.$$

Let Θ be a cross-section for Γ/Γ_1 in Γ. The Hilbert space $\mathbf{H}_t$ is naturally identified with $l^2(\Theta)$ since for any $Q\in\Gamma_1$, we have $|f(PQ)| = |f(P)|$. Via this identification, we may realize the representation $\mathrm{Ind}_{\Gamma_1}^{\Gamma}\chi_{(1,-t)}$ as acting in $l^2(\Theta)$. More precisely, for $a\in l^2(\Theta)$ and $\rho_t = \mathrm{Ind}_{\Gamma_1}^{\Gamma}\chi_{(1,-t)}$ we have

$$(\rho_t(X)a)(T_j) = \left\{\begin{array}{cc} a(T_{j-k}) & \text{if } X = T_k \\ e^{-2\pi i\langle j,l\rangle}e^{-2\pi i\langle t,l\rangle}a(T_j) & \text{if } X = M_l \\ e^{2\pi i\theta}a(T_j) & \text{if } X = e^{2\pi i\theta} \end{array}\right. .$$

Defining the unitary operator $\mathfrak{J} : l^2(\Theta) \to l^2\left(\mathbb{Z}^d\right)$ such that $(\mathfrak{J}a)(j) = a(T_j)$, it is easily checked that:

$$\mathfrak{J}^{-1}\left[(\mathfrak{R}Xf)(t)\right] = \left\{\begin{array}{cc} \rho_t(T_k)\left[\mathfrak{J}^{-1}(\mathfrak{R}f(t))\right] & \text{if } X = T_k \\ \rho_t(M_l)\left[\mathfrak{J}^{-1}(\mathfrak{R}f(t))\right] & \text{if } X = M_l \\ \rho_t\left(e^{2\pi i\theta}\right)\left[\mathfrak{J}^{-1}(\mathfrak{R}f(t))\right] & \text{if } X = e^{2\pi i\theta} \end{array}\right. .$$

Thus, the representation π is unitarily equivalent to

$$\int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}^{\oplus} \rho_t\, dt. \tag{14}$$

Now, we remark that ρ_t is not an irreducible representation of the group Γ. Indeed, by inducing in stages (see Page 166, [6]), we obtain that the representation ρ_t is unitarily equivalent to $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\left(\chi_{(1,-t)}\right)\right)$ and $\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\chi_{(1,-t)}$ acts in the Hilbert space

$$\mathbf{K}_t = \left\{\begin{array}{c} f:\Gamma_0\to\mathbb{C} : f(PQ) = \chi_{(1,-t)}\left(Q^{-1}\right) f(P), Q\in\Gamma_1 \\ \text{and } \sum_{P\Gamma_1\in\frac{\Gamma_0}{\Gamma_1}} |f(P)|^2 < \infty \end{array}\right\}. \tag{15}$$

Since $\chi_{(1,-t)}$ is a character, for any $f \in \mathbf{K}_t$, we notice that $|f(PQ)| = |f(P)|$ for $Q \in \Gamma_1$. Thus, the Hilbert space $\mathbf{K}_t$ is naturally identified with $l^2\left(\frac{\Gamma_0}{\Gamma_1}\right) \simeq l^2\left(A\mathbb{Z}^d\right)$ where $A\mathbb{Z}^d$ is a parametrizing set for the quotient group $\frac{\Gamma_0}{\Gamma_1}$. Via this identification, we may realize $\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\chi_{(1,-t)}$ as acting in $l^2\left(A\mathbb{Z}^d\right)$. More precisely, for $T_j, j \in A\mathbb{Z}^d$, we compute the action of $\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\chi_{(1,-t)}$ as follows:

$$\left[\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\chi_{(1,-t)}(X) f\right](T_j) = \begin{cases} f(T_{j-k}) & \text{if } X = T_k \\ e^{-2\pi i\langle t,l\rangle} f(T_j) & \text{if } X = M_l \\ e^{2\pi i\theta} f(T_j) & \text{if } X = e^{2\pi i\theta} \end{cases}.$$

Now, let $\mathbf{F}_{A\mathbb{Z}^d}$ be the Fourier transform defined on $l^2\left(A\mathbb{Z}^d\right)$. Given a vector f in $l^2\left(A\mathbb{Z}^d\right)$, it is not too hard to see that

$$\left[\mathbf{F}_{A\mathbb{Z}^d}\left(\mathrm{Ind}_{\Gamma_1}^{\Gamma_0}\chi_{(1,-t)}(X) f\right)\right](\xi) = \begin{cases} \chi_\xi(T_k)\,\mathbf{F}_{A\mathbb{Z}^d} f(\xi) & \text{if } X = T_k \\ e^{-2\pi i\langle t,l\rangle}\mathbf{F}_{A\mathbb{Z}^d} f(\xi) & \text{if } X = M_l \\ e^{2\pi i\theta}\mathbf{F}_{A\mathbb{Z}^d} f(\xi) & \text{if } X = e^{2\pi i\theta} \end{cases}$$

where χ_ξ is a character of the discrete group $A\mathbb{Z}^d$. As a result, given $X \in \Gamma$ we obtain:

$$\mathbf{F}_{A\mathbb{Z}^d} \circ \rho_t(X) \circ \mathbf{F}_{A\mathbb{Z}^d}^{-1} = \int_{\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \chi_{(1,-t,\xi)}(X)\; d\xi, \tag{16}$$

where $\chi_{(1,-t,\xi)}$ is a character of Γ_0 defined as follows:

$$\chi_{(1,-t,\xi)}(X) = \begin{cases} \chi_\xi(T_k) & \text{if } X = T_k \\ e^{-2\pi i\langle t,l\rangle} & \text{if } X = M_l \\ e^{2\pi i\theta} & \text{if } X = e^{2\pi i\theta} \end{cases}.$$

Using the fact that induction commutes with direct integral decomposition (see Page 41,[5]) we have

$$\rho_t \simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\int_{\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \chi_{(1,-t,\xi)}\; d\xi\right) \simeq \int_{\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \left(\mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,-t,\xi)}\right)\; d\xi. \tag{17}$$

Putting (14) and (17) together, we arrive at: $\pi \simeq \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}\times\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,\sigma)} d\sigma$. Finally, the fact that $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,\sigma)}$ is an irreducible representation is due to Remark 5. This completes the proof.

Remark 6. *We shall offer here a different proof of Proposition 2 by exhibiting an explicit intertwining operator which is a version of the Zak transform for the representation π and the direct integral representation described given in (3). Let $C_c\left(\mathbb{R}^d\right)$ be the space of continuous functions on $\mathbb{R}^d$ which are compactly supported. Let $\mathcal{Z}$ be the operator which maps each $f \in C_c\left(\mathbb{R}^d\right)$ to the function*

$$(\mathcal{Z}f)\left(x, w, j + A\mathbb{Z}^d\right) = \sum_{k\in\mathbb{Z}^d} f(x + Ak + j)\, e^{2\pi i\langle w, Ak\rangle} \equiv \phi\left(x, w, j + A\mathbb{Z}^d\right) \tag{18}$$

where $x, w \in \mathbb{R}^d$ and j is an element of a cross-section for $\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}$ in the lattice $\mathbb{Z}^d$. Given arbitrary $m' \in \mathbb{Z}^d$, we may write $m' = Ak' + j'$ where $k' \in \mathbb{Z}^d$ and j' is an element of a cross-section for $\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}$. Next, it is worth observing that given $m, m' \in \mathbb{Z}^d$, $\phi\left(x, w + A^\star m, j + A\mathbb{Z}^d\right) = \phi\left(x, w, j + A\mathbb{Z}^d\right)$ and $\phi\left(x + m', w, j + A\mathbb{Z}^d\right)$ is equal to $e^{-2\pi i\langle w, Ak'\rangle}\phi\left(x, w, j + j' + A\mathbb{Z}^d\right)$. This observation will later help us explain the meaning of Equality (18). Let Σ_1 and Σ_2 be measurable cross-sections for $\frac{\mathbb{R}^d}{\mathbb{Z}^d}$ and $\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$ respectively in $\mathbb{R}^d$. For example, we may pick $\Sigma_1 = [0,1)^d$ and $\Sigma_2 = A^\star[0,1)^d$. Since f is square-integrable, by a periodization argument it is easy to see that

$$\|f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\Sigma_1} \sum_{m\in\mathbb{Z}^d} |f(x+m)|^2\, dx. \tag{19}$$

Therefore, the integral on the right of (19) is finite. It immediately follows that

$$\int_{\Sigma_1} \sum_{j+A\mathbb{Z}^d \in \mathbb{Z}^d/A\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |f(x+Ak+j)|^2 \, dt < \infty.$$

Therefore, the sum $\sum_{j+A\mathbb{Z}^d\in\mathbb{Z}^d/A\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|f(x+Ak+j)|^2$ *is finite for almost every* $x\in\Sigma_1$ *and a fixed* j *which is a cross-section for* $\mathbb{Z}^d/A\mathbb{Z}^d$ *in* $\mathbb{Z}^d$*. Next, observe that*

$$\sum_{k\in\mathbb{Z}^d} f(x+Ak+j)\, e^{2\pi i\langle w,Ak\rangle} \tag{20}$$

is a Fourier series of the sequence $(f(x+Ak+j))_{Ak\in A\mathbb{Z}^d}\in l^2\left(A\mathbb{Z}^d\right)$*. So, for almost every* x *and for a fixed* j*, the function* $\phi\left(x,\cdot,j+A\mathbb{Z}^d\right)$ *is regarded as a function of* $L^2\left(\mathbb{R}^d\right)$ *which is* $A^\star\mathbb{Z}^d$*-periodic (it is an* $L^2(\Sigma_2)$ *function). In summary, we may regard the function* $(\mathcal{Z}f)\left(x,w,j+A\mathbb{Z}^d\right)$ *as being defined over the set* $\Sigma_1\times\Sigma_2\times\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}$*. Let us now show that* $\mathcal{Z}$ *maps* $C_c\left(\mathbb{R}^d\right)$ *isometrically into the Hilbert space* $L^2\left(\Sigma_1\times\Sigma_2\times\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}\right)$*. Given any square-summable function* f *in* $L^2\left(\mathbb{R}^d\right)$*, we have*

$$\int_{\mathbb{R}^d}|f(t)|^2\,dt=\int_{\Sigma_1}\sum_{k\in\mathbb{Z}^d}|f(t+k)|^2\,dt=\int_{\Sigma_1}\sum_{j+A\mathbb{Z}^d\in\mathbb{Z}^d/A\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|f(t+Ak+j)|^2\,dt.$$

Regarding $(f(t+Ak+j))_{Ak\in A\mathbb{Z}^d}$ *as a square-summable sequence in* $l^2\left(A\mathbb{Z}^d\right)$*, the function*

$$w\mapsto\sum_{k\in\mathbb{Z}^d}f(t+Ak+j)\,e^{2\pi i\langle w,Ak\rangle}$$

is the Fourier transform of the sequence $(f(t+Ak+j))_{Ak\in A\mathbb{Z}^d}$*. The Plancherel theorem being a unitary operator, we have*

$$\sum_{k\in\mathbb{Z}^d}|f(t+Ak+j)|^2=\int_{\Sigma_2}\left|\sum_{k\in\mathbb{Z}^d}f(t+Ak+j)\,e^{2\pi i\langle w,Ak\rangle}\right|^2dw.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^d}|f(t)|^2\,dt&=\int_{\Sigma_1}\sum_{j+A\mathbb{Z}^d\in\mathbb{Z}^d/A\mathbb{Z}^d}\int_{\Sigma_2}\left|\sum_{k\in\mathbb{Z}^d}f(t+Ak+j)\,e^{2\pi i\langle w,Ak\rangle}\right|^2dwdt\\&=\int_{\Sigma_1}\int_{\Sigma_2}\sum_{j+A\mathbb{Z}^d\in\mathbb{Z}^d/A\mathbb{Z}^d}\left|\mathcal{Z}f\left(x,w,j+A\mathbb{Z}^d\right)\right|^2dwdt\\&=\|\mathcal{Z}f\|^2_{L^2\left(\Sigma_1\times\Sigma_2\times\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}\right)}.\end{aligned}$$

Now, by density, we may extend the operator $\mathcal{Z}$ *to* $L^2\left(\mathbb{R}^d\right)$*, and we shall next show that the extension*

$$\mathcal{Z}:L^2\left(\mathbb{R}^d\right)\to L^2\left(\Sigma_1\times\Sigma_2\times\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}\right)$$

is unitary. At this point, we only need to show that $\mathcal{Z}$ *is surjective. Let* φ *be any vector in the Hilbert space* $L^2\left(\Sigma_1\times\Sigma_2\times\frac{\mathbb{Z}^d}{A\mathbb{Z}^d}\right)$*. Clearly for almost every* x *and given any fixed* j*, we have* $\varphi(x,\cdot,j)\in L^2(\Sigma_2)$*. For such* x *and* j*, let* $(c_\ell(x,j))_{\ell\in A\mathbb{Z}^d}$ *be the Fourier transform of* $\varphi(x,\cdot,j)$*. Next, define* $f_\varphi\in L^2\left(\mathbb{R}^d\right)$ *such that for almost every* $x\in\Sigma_1$*,*

$$f_\varphi(x+A\ell+j)=c_\ell(x,j).$$

Now, for almost every $w \in \Sigma_2$,

$$\begin{aligned} Zf_{\varphi}(x,w) &= \sum_{\ell\in\mathbb{Z}^d} f_{\varphi}(x+A\ell+j)\, e^{2\pi i\langle w, A\ell\rangle} \\ &= \sum_{\ell\in\mathbb{Z}^d} c_{\ell}(x,j)\, e^{2\pi i\langle w, A\ell\rangle} \\ &= \varphi\left(x,w,j+A\mathbb{Z}^d\right). \end{aligned}$$

It remains to show that our version of Zak transform intertwines the representation π with $\int_{\Sigma_1\times\Sigma_2}^{\oplus}\rho_{(1,x,w)}$ $dxdw$ such that $\rho_{(1,x,w)}$ is equivalent to the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,-x,w)}$. Let $\mathcal{R}: l^2(\Gamma/\Gamma_0)\to l^2\left(\mathbb{Z}^d/A\mathbb{Z}^d\right)$ be a unitary map defined by

$$\mathcal{R}\left(f\left(j+A\mathbb{Z}^d\right)_{j+A\mathbb{Z}^d}\right) = \left(f\left(T_j\Gamma_0\right)\right)_{T_j\Gamma_0}.$$

Put

$$\rho_{(1,x,w)}(X) = \mathcal{R}\circ \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,-x,w)}(X)\circ\mathcal{R}^{-1} \text{ for every } X\in\Gamma.$$

It is straightforward to check that

$$\begin{aligned} (\mathcal{Z}T_j f)(x,w,\cdot) &= \sum_{k\in\mathbb{Z}^d} T_j f(x+\cdot)\, e^{2\pi i\langle w, Ak\rangle} \\ &= \sum_{k\in\mathbb{Z}^d} T_j f(x+(\cdot-j))\, e^{2\pi i\langle w, Ak\rangle} \\ &= \left[\rho_{(1,x,w)}(T_j)\right](\mathcal{Z}f)(x,w,\cdot) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{Z}M_{Bl}f)(x,w,\cdot) &= \sum_{k\in\mathbb{Z}^d} e^{-2\pi i\langle Bl, x+Ak+j\rangle} f(x+\cdot)\, e^{2\pi i\langle w, Ak\rangle} \\ &= e^{-2\pi i\langle Bl, x+j\rangle}\sum_{k\in\mathbb{Z}^d} e^{-2\pi i\langle Bl, Ak\rangle} f(x+\cdot)\, e^{2\pi i\langle w, Ak\rangle} \\ &= e^{-2\pi i\langle Bl, x+j\rangle} f(x+\cdot)\, e^{2\pi i\langle w, Ak\rangle} \\ &= e^{-2\pi i\langle Bl, x+j\rangle}(\mathcal{Z}f)(x,w,\cdot) \\ &= \rho_{(1,x,w)}(M_{Bl})(\mathcal{Z}f)(x,w,\cdot). \end{aligned}$$

In summary, given any $X\in\Gamma$,

$$\mathcal{Z}\circ\pi(X)\circ\mathcal{Z}^{-1} = \int_{\Sigma_1\times\Sigma_2}^{\oplus}\rho_{(1,x,w)}(X)\ dxdw.$$

Lemma 7. *Let $\Gamma_1 = A_1\mathbb{Z}^d$ and $\Gamma_2 = A_2\mathbb{Z}^d$ be two lattices of $\mathbb{R}^d$ such that A_1 and A_2 are non-singular matrices and $|\det A_1|\le|\det A_2|$. Then there exist measurable sets Σ_1,Σ_2 such that Σ_1 is a fundamental domain for $\frac{\mathbb{R}^d}{A_1\mathbb{Z}^d}$ and Σ_2 is a fundamental domain for $\frac{\mathbb{R}^d}{A_2\mathbb{Z}^d}$ and $\Sigma_1\subseteq\Sigma_2\subset\mathbb{R}^d$.*

Proof. According to Theorem 1.2, [8], there exists a measurable set Σ_1 such that Σ_1 tiles $\mathbb{R}^d$ by the lattice $A_1\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $A_2\mathbb{Z}^d$. By packing, we mean that given any distinct $\gamma,\kappa\in A_2\mathbb{Z}^d$, the set $(\Sigma_1+\gamma)\cap(\Sigma_1+\kappa)$ is an empty set and $\sum_{\lambda\in A_2\mathbb{Z}^d}1_{\Sigma_1}(x+\lambda)\le 1$ for $x\in\mathbb{R}^d$ where 1_{Σ_1} denotes the characteristic function of the set Σ_1. We would like to construct a set Σ_2 which tiles $\mathbb{R}^d$ by $A_2\mathbb{Z}^d$ such that $\Sigma_1\subseteq\Sigma_2$. To construct such a set, let Ω be a fundamental domain for $\frac{\mathbb{R}^d}{A_2\mathbb{Z}^d}$. It follows that, there exists a subset I of $A_2\mathbb{Z}^d$ such that $\Sigma_1\subseteq\dot{\bigcup_{k\in I}}(\Omega+k)$ and each $(\Omega+k)\cap\Sigma_1$ is a set of positive Lebesgue measure. Next, for each $k \in I$, we define $\Omega_k = (\Omega + k) \cap \Sigma_1$. We observe that $\Sigma_1 = \dot{\bigcup_{k\in I}} ((\Omega + k) \cap \Sigma_1) = \dot{\bigcup_{k\in I}} \Omega_k$ where $\Omega_k = (\Omega + k) \cap \Sigma_1$. Put

$$\Sigma_2 = \left(\Omega - \dot{\bigcup_{k\in I}} (\Omega_k - k) \right) \dot{\cup}\, \Sigma_1.$$

The disjoint union in the equality above is due to the fact that for distinct $k, j \in I$, the set $(\Omega_k - k) \cap (\Omega_j - j)$ is a null set. This holds because, Σ_1 packs $\mathbb{R}^d$ by $A_2\mathbb{Z}^d$. Finally, we observe that

$$\Sigma_2 = \left(\Omega - \dot{\bigcup_{k\in I}} (\Omega_k - k) \right) \dot{\cup} \left(\dot{\bigcup_{k\in I}} \Omega_k \right)$$

and $\Omega = \left(\Omega - \dot{\bigcup_{k\in I}} (\Omega_k - k) \right) \dot{\cup} \left(\dot{\bigcup_{k\in I}} \Omega'_k \right)$ where each Ω'_k is $A_2\mathbb{Z}^d$-congruent with Ω_k. Therefore Σ_2 is a fundamental domain for $\frac{\mathbb{R}^d}{A_2\mathbb{Z}^d}$ which contains Σ_1. This completes the proof. □

Now, we are ready to prove Proposition 3. Part of the proof of Proposition 3 relies on some technical facts related to central decompositions of unitary representations. A good presentation of this theory is found in Section 3.4.2, [7].

4.2. **Proof of Proposition 3.** From Proposition 2, we know that the representation π is unitarily equivalent to

$$\int^{\oplus}_{\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)}\, d\sigma. \tag{21}$$

We recall that Γ_0 is isomorphic to the discrete group $\mathbb{Z}_m \times B\mathbb{Z}^d \times A\mathbb{Z}^d$ and that Γ_1 is isomorphic to $\mathbb{Z}_m \times B\mathbb{Z}^d$ where m is the number of elements in the commutator group of Γ which is a discrete subgroup of the torus. From Proposition 1, we have

$$L \simeq \oplus_{k=0}^{m-1} \int^{\oplus}_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(k,\sigma)} \right)\, d\sigma. \tag{22}$$

Now, put

$$L_k = \int^{\oplus}_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(k,\sigma)}\, d\sigma. \tag{23}$$

From (22), it is clear that $L = L_0 \oplus \cdots \oplus L_{m-1}$. Next, for distinct i and j, the representations L_i and L_j described above are disjoint representations. This is due to the fact that if $i \neq j$ then the Γ-orbits of $\chi_{(i,\sigma)}$ and $\chi_{(j,\sigma)}$ are disjoint sets and therefore the induced representations $\mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(i,\sigma)}$ and $\mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(j,\sigma)}$ are disjoint representations. Thus, for $k \neq 1$ the representation L_k must be disjoint from π. Let us assume for now that $|\det B| > 1$ (or $|\det (B^\star)| < 1$.) According to Lemma 7, there exist measurable cross-sections Σ_1, Σ_2 for $\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$ and $\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$ respectively such that $\Sigma_1, \Sigma_2 \subset \mathbb{R}^2$, $\Sigma_1 \supset \Sigma_2$ and $\Sigma_1 - \Sigma_2$ is a set of positive Lebesgue measure. Therefore,

$$\pi \simeq \int^{\oplus}_{\Sigma_1} \left(\mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right)\, d\sigma \text{ and } L_1 \simeq \int^{\oplus}_{\Sigma_2} \left(\mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right)\, d\sigma \tag{24}$$

and the representations above are realized as acting in the direct integrals of finite dimensional vector spaces: $\int^{\oplus}_{\Sigma_1} l^2 (\Gamma/\Gamma_0)\, d\sigma$ and $\int^{\oplus}_{\Sigma_2} l^2 (\Gamma/\Gamma_0)\, d\sigma$ respectively. We remark that the direct integrals described in (24) are irreducible decompositions of π and L_1. Now, referring to the central decomposition of the left regular representation which is described in (10) there exists a measurable subset $\mathbf{E}$ of Σ_2 such that the central decomposition of L_1 is given by (see Theorem 3.26, [7])

$$\int_{\mathbf{E}}^{\oplus} \oplus_{k=1}^{\dim\left(l^2(\Gamma/\Gamma_0)\right)} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma.$$

Furthermore, recalling that $L_1 \simeq \int_{\Sigma_2}^{\oplus}\left(\mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\right)\ d\sigma$ and letting μ be the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$, it is necessarily the case that

$$\mu(\mathbf{E}) = \frac{\mu(\Sigma_2)}{\dim\left(l^2(\Gamma/\Gamma_0)\right)} = \frac{1}{\left|\det(B)\det(A)\right| \dim\left(l^2(\Gamma/\Gamma_0)\right)}.$$

From the discussion provided at the beginning of the third section, the set $\mathbf{E}$ is obtained by taking a cross-section for the Γ-orbits in $\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$. Moreover, since $\Sigma_1 \supset \Sigma_2$ and since $\Sigma_1 - \Sigma_2$ is a set of positive Lebesgue measure then $\pi \simeq \int_{\Sigma_1}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma \simeq L_1 \oplus \int_{\Sigma_1-\Sigma_2}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma$ and $\int_{\Sigma_1-\Sigma_2}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma$ is a subrepresentation of π. Thus a central decomposition of π is given by

$$\int_{\mathbf{E}}^{\oplus} \oplus_{k=1}^{u(\sigma)\dim\left(l^2(\Gamma/\Gamma_0)\right)} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma$$

and the function $u : \mathbf{E} \to \mathbb{N}$ is greater than one on a subset of positive measure of $\mathbf{E}$. Therefore, according to Theorem 3.26, [7], it is not possible for π to be equivalent to a subrepresentation of the left regular representation of Γ if $|\det B| > 1$. Now, let us suppose that $|\det B| \leq 1$. Then $|\det B^\star| \geq 1$. Appealing to Lemma 7, there exist measurable sets Σ_1 and Σ_2 which are measurable fundamental domains for $\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$, and $\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$ respectively, such that $\Sigma_1, \Sigma_2 \subset \mathbb{R}^d$ and $\Sigma_1 \subseteq \Sigma_2$. Next,

$$\begin{aligned} L_1 &\simeq \int_{\Sigma_2}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma \\ &\simeq \left(\int_{\Sigma_1}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma\right) \oplus \left(\int_{\Sigma_2-\Sigma_1}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma\right) \\ &\simeq \pi \oplus \left(\int_{\Sigma_2-\Sigma_1}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\chi_{(1,\sigma)}\right)\ d\sigma\right). \end{aligned}$$

Finally, π is equivalent to a subrepresentation of L_1 and is equivalent to a subrepresentation of the left regular representation L.

4.3. **Examples.** In this subsection, we shall present a few examples to illustrate the results obtained in Propositions 1, 2 and 3.

(1) Let us start with a trivial example. Let $d = 1$ and $B = \frac{2}{3}$. Then $B^\star = \frac{3}{2}, A = 3,$ and $A^\star = \frac{1}{3}$. Next, $L \simeq \oplus_{k=0}^{2} \int_{\left[0,\frac{3}{2}\right)\times\left[0,\frac{1}{3}\right)}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(k,\sigma)}\ d\sigma$ and $\pi \simeq \int_{[0,1)\times\left[0,\frac{1}{3}\right)}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,\sigma)}\ d\sigma$. Now, the central decomposition of L_1 is given by

$$\int_{\left[0,\frac{1}{2}\right)\times\left[0,\frac{1}{3}\right)}^{\oplus} \oplus_{j=1}^{3} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,\sigma)}\ d\sigma$$

and the central decomposition of the rational Gabor representation π is

$$\int_{\left[0,\frac{1}{2}\right)\times\left[0,\frac{1}{3}\right)}^{\oplus} \oplus_{j=1}^{2} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(1,\sigma)}\ d\sigma.$$

From these decompositions, it is obvious that the rational Gabor representation π is equivalent to a subrepresentation of the left regular representation L.

(2) If we define $B = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$, then $B^{\star} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$, $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^{\star} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Next, the left regular representation of Γ can be decomposed into a direct integral of representations as follows:

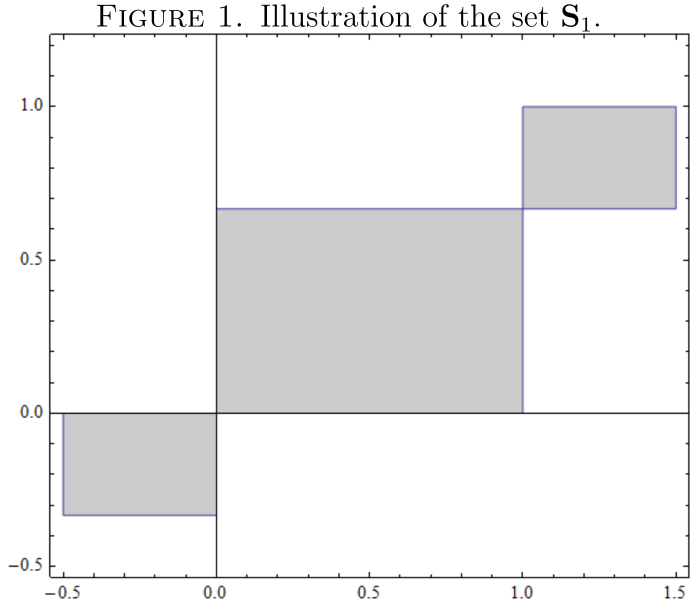
$$L \simeq \oplus_{k=0}^{5} \int_{\mathbf{S}}^{\oplus} \operatorname{Ind}_{\Gamma_0}^{\Gamma} \chi_{(k,\sigma)} \; d\sigma$$

where $\mathbf{S} = \mathbf{S}_1 \times A^{\star}\,[0,1)^2$ and

$$\mathbf{S}_1 = \left([0,1) \times \left[0, \frac{2}{3}\right)\right) \cup \left(\left[1, \frac{3}{2}\right) \times \left[\frac{2}{3}, 1\right)\right) \cup \left(\left[-\frac{1}{2}, 0\right) \times \left[-\frac{1}{3}, 0\right)\right)$$

is a common connected fundamental domain for the lattices $B^{\star}\mathbb{Z}^2$ and $\mathbb{Z}^2$.

FIGURE 1. Illustration of the set $\mathbf{S}_1$.



Moreover, we decompose the rational Gabor representation as follows: $\pi \simeq \int_{\mathbf{S}}^{\oplus} \operatorname{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)} \; d\sigma$. One interesting fact to notice here is that: the rational Gabor representation π is actually equivalent to L_1 and

$$L = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5.$$

(3) Let $\Gamma = \langle T_k, M_{Bl} : k, l \in \mathbb{Z}^3 \rangle$ where $B = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ 1 & -1 & 5 \end{bmatrix}$. The inverse transpose of the matrix B is given by $B^{\star} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$. Next, we may choose the matrix A such that

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A^{\star} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Finally, we observe that $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & \frac{1}{5} & 0 \end{bmatrix} [0,1)^3$ is a common fundamental domain for both $B^\star\mathbb{Z}^3$ and $\mathbb{Z}^3$. Put

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & \frac{1}{5} & 0 \end{bmatrix} [0,1)^3 \times \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ 1 & -1 & 1 \end{bmatrix} [0,1)^3 .$$

Then $L \simeq \oplus_{k=0}^{4} \int_{\mathbf{S}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(k,t)} dt$ and $\pi \simeq \int_{\mathbf{S}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,t)} dt$.

5. Application to time-frequency analysis

Let π be a unitary representation of a locally compact group X, acting in some Hilbert space $\mathcal{H}$. We say that π is admissible, if and only if there exists some vector $\phi \in \mathcal{H}$ such that the operator W_ϕ^π defined by $W_\phi^\pi : \mathcal{H} \to L^2(X)$, $W_\phi^\pi \psi(x) = \langle \psi, \pi(x)\phi \rangle$ is an isometry of $\mathcal{H}$ into $L^2(X)$.

We continue to assume that B is an invertible rational matrix with at least one entry which is not an integer. Following Proposition 2.14 and Theorem 2.42 of [7], the following is immediate.

Lemma 8. *A representation of Γ is admissible if and only if the representation is equivalent to a subrepresentation of the left regular representation of Γ.*

Given a countable sequence $\{f_i\}_{i\in I}$ of vectors in a Hilbert space $\mathbf{H}$, we say $\{f_i\}_{i\in I}$ forms a frame if and only if there exist strictly positive real numbers A, B such that for any vector $f \in \mathbf{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 .$$

In the case where $A = B$, the sequence of vectors $\{f_i\}_{i\in I}$ forms a tight frame, and if $A = B = 1$, $\{f_i\}_{i\in I}$ is called a Parseval frame. We remark that an admissible vector for the left regular representation of Γ is a Parseval frame by definition.

The following proposition is well-known for the more general case where B is any invertible matrix (not necessarily a rational matrix.) Although this result is not new, the proof of Proposition 9 is new, and worth presenting in our opinion.

Proposition 9. *Let B be a rational matrix. There exists a vector $g \in L^2(\mathbb{R}^d)$ such that the system $\{M_l T_k g : l \in B\mathbb{Z}^d, k \in \mathbb{Z}^d\}$ is a Parseval frame in $L^2(\mathbb{R}^d)$ if and only if $|\det B| \leq 1$.*

Proof. The case where B is an element of $GL(d,\mathbb{Z})$ is easily derived from [11], Section 4. We shall thus skip this case. So let us assume that B is a rational matrix with at least one entry not in $\mathbb{Z}$. We have shown that the representation π is equivalent to a subrepresentation of the left regular representation of L if and only if $|\det B| \leq 1$. Since Γ is a discrete group, then its left regular representation is admissible if and only if $|\det B| \leq 1$. Thus, the representation π of Γ is admissible if and only if $|\det B| \leq 1$. Suppose that $|\det B| \leq 1$. Then π is admissible and there exists a vector $f \in L^2(\mathbb{R}^d)$ such that the map W_f^π defined by $W_f^\pi h\left(e^{2\pi i\theta} M_l T_k\right) = \left\langle h, e^{2\pi i\theta} M_l T_k f\right\rangle$ is an isometry. As a result, for any vector $h \in L^2(\mathbb{R}^d)$, we have

$$\left(\sum_{\theta\in[\Gamma,\Gamma]} \sum_{l\in B\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \left|\left\langle h, e^{2\pi i\theta} M_l T_k f\right\rangle\right|^2 \right)^{1/2} = \|h\|_{L^2(\mathbb{R}^d)} .$$

Next, for $m = \mathrm{card}([\Gamma,\Gamma])$,

$$\sum_{\theta\in[\Gamma,\Gamma]} \sum_{l\in B\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \left|\left\langle h, e^{2\pi i\theta} M_l T_k f\right\rangle\right|^2 = \sum_{l\in B\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \left|\left\langle h, M_l T_k \left(m^{1/2} f\right)\right\rangle\right|^2 .$$

Therefore, if $g = m^{1/2} f$ then

$$\left(\sum_{l \in B\mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \left| \langle h, M_l T_k g \rangle \right|^2 \right)^{1/2} = \|h\|_{L^2\left(\mathbb{R}^d\right)}.$$

For the converse, if we assume that there exists a vector $g \in L^2\left(\mathbb{R}^d\right)$ such that the system

$$\left\{ M_l T_k g : l \in B\mathbb{Z}^d, k \in \mathbb{Z}^d \right\}$$

is a Parseval frame in $L^2\left(\mathbb{R}^d\right)$ then it is easy to see that π must be admissible. As a result, it must be the case that $|\det B| \leq 1$. $\square$

5.1. **Proof of Proposition 4.** Let us suppose that $|\det B| \leq 1$. From the proof of Proposition 3, we recall that there exists a unitary map $\mathfrak{A} : \int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} l^2 \left(\frac{\Gamma}{\Gamma_0} \right) \right) d\sigma \to L^2\left(\mathbb{R}^d\right)$ which intertwines the representations $\int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right) d\sigma$ with π such that $\int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right) d\sigma$ is the central decomposition of π, and $\mathbf{E} \subset \mathbb{R}^d$ is a measurable subset of a fundamental domain for the lattice $B^{\star}\mathbb{Z}^d \times A^{\star}\mathbb{Z}^d$ and the multiplicity function ℓ satisfies the condition: $\ell(\sigma) \leq \dim l^2 \left(\frac{\Gamma}{\Gamma_0} \right)$. Next, according to the discussion on Page 126, [7] the vector a is an admissible vector for the representation $\tau = \int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right) d\sigma$ if and only if $a \in \int_{\mathbf{E}}^{\oplus} \left(\oplus_{k=1}^{\ell(\sigma)} l^2 \left(\frac{\Gamma}{\Gamma_0} \right) \right) d\sigma$ such that for $d\sigma$-almost every $\sigma \in \mathbf{E}$, $\|a(\sigma)(k)\|^2_{l^2\left(\frac{\Gamma}{\Gamma_0}\right)} = 1$ for $1 \leq k \leq \ell(\sigma)$ and for distinct $k, j \in \{1, \cdots, \ell(\sigma)\}$ we have $\langle a(\sigma)(k), a(\sigma)(j) \rangle = 0$. Finally, the desired result is obtained by using the fact that $\mathfrak{A}$ intertwines the representations τ with π.

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