

PAIRS OF FULL-RANK LATTICES WITH PARALLELEPIPED-SHAPED COMMON FUNDAMENTAL DOMAINS

HEIDI BURGIEL AND VIGNON OUSSA

ABSTRACT. We provide a complete characterization of pairs of full-rank lattices in \mathbb{R}^d which admit common connected fundamental domains of the type $N[0,1]^d$ where N is an invertible matrix of order d . Using our characterization, we construct several pairs of lattices of the type $(M\mathbb{Z}^d, \mathbb{Z}^d)$ which admit a common fundamental domain of the type $N[0,1]^d$. Moreover, we show that for $d = 2$, there exists an uncountable family of pairs of lattices of the same volume which do not admit a common connected fundamental domain of the type $N[0,1]^2$.

1. INTRODUCTION

Let $\{e_k : 1 \leq k \leq d\}$ be a basis for the vector space \mathbb{R}^d . A full-rank lattice Γ is a discrete subgroup of \mathbb{R}^d which is generated by the set $\{e_k : 1 \leq k \leq d\}$. The number of generators of the lattice is called the rank of the lattice, and the set $\{e_k : 1 \leq k \leq d\}$ is called a basis for the lattice. Adopting the convention that vectors in \mathbb{R}^d are written as $d \times 1$ matrices, it is convenient to describe the lattice Γ as $\Gamma = M\mathbb{Z}^d = \{Mk : k \in \mathbb{Z}^d\}$ where the j^{th} column of the matrix M corresponds to the vector e_j . For a full-rank lattice $M\mathbb{Z}^d$, the positive number $|\det M|$ which is equal to the volume of the parallelepiped $M[0,1]^d$ is conveniently called the volume of the lattice $M\mathbb{Z}^d$.

Let E be a Lebesgue measurable subset of \mathbb{R}^d . We say that E packs \mathbb{R}^d by Γ if and only if for any $\lambda, \gamma \in \Gamma, \lambda \neq \gamma$, $(E + \lambda) \cap (E + \gamma) = \emptyset$. Moreover, we say that E is a measurable fundamental domain of Γ if and only if for any $\lambda, \gamma \in \Gamma, \lambda \neq \gamma$, $(E + \lambda) \cap (E + \gamma) = \emptyset$ and $\cup_{\gamma \in \Gamma} (E + \gamma) = \mathbb{R}^d$. It is worth noticing that if E packs \mathbb{R}^d by Γ and if the Lebesgue measure of E is equal to the volume of Γ then E is a fundamental domain of Γ .

According to a remarkable result of Deguang and Wang (Theorem 1.1 of [2]), it is known that two full-rank lattices in \mathbb{R}^d of the same volume have a common fundamental domain. This result has profound applications in time-frequency analysis [4, 7, 1]. In [2], the authors provide a general procedure for constructing a fundamental domain for any given pair of lattices of the same volume. However, it is often the case that the fundamental domains obtained in [2] are disconnected, unbounded and difficult to describe. It is therefore natural to ask if it is possible to characterize pairs of lattices which admit ‘simple’ common fundamental domains.

Let us be more precise about what we mean by a ‘simple’ fundamental domain for a lattice. Let Γ_1 , and Γ_2 be two full-rank lattices of the same volume. We say that the pair (Γ_1, Γ_2) is a **good pair of lattices** if and only if there exists an invertible matrix N of order d such that the parallelepiped $N[0,1]^d$ is a common fundamental domain for Γ_1, Γ_2 . Clearly, such a fundamental domain is a simple set in the sense that it is connected, star-shaped, convex and is easily described. Although the investigation of good pairs of lattices is an interesting problem on its own right, it is also worth noting that common fundamental domains for pairs of lattices which are bounded and star-shaped are of central importance in the construction of smooth frames which are compactly supported [7].

Date: July 17, 2015.

2000 Mathematics Subject Classification. 52C22, 52C17, 42B99, 42C30.

Key words and phrases. Lattices, Fundamental Domains, Tiling, Packing, Orthonormal bases.

1.1. Short overview of the paper. Our main objective in this paper is to provide solutions to the following problems:

Problem 1. *Is it possible to obtain a simple characterization of good pair of lattices?*

Problem 2. *For which unimodular matrices M is $(M\mathbb{Z}^d, \mathbb{Z}^d)$ a good pair of lattices (or not)?*

On one hand, we are able to address Problem 1 in a way that we judge is satisfactory. On the other, while we are able to describe several non-trivial families of good pairs of lattices of the type $(M\mathbb{Z}^d, \mathbb{Z}^d)$, to the best of our knowledge Problem 2 is still open.

Here is a summary of the results obtained in this paper:

- We present a simple yet powerful characterization of good pairs of lattices in Proposition 3, and we describe various properties (Proposition 9) of good pairs of lattices.
- Addressing Problem 2, in Proposition 4, and Proposition 5 we construct several non-trivial families of good pairs of lattices of the type $(M\mathbb{Z}^d, \mathbb{Z}^d)$ in any given dimension. Moreover, in Proposition 6 we establish the existence of an uncountable collection of pairs of lattices in dimension 2 which have the same volume and are not good pairs.
- We provide methods that can be exploited to construct good pairs of lattices in higher dimensions from good pairs of lattices in lower dimensions (Proposition 16.)

Among several results obtained in this work, here are the main ones.

Proposition 3. *Let $\mathbf{0}$ be the zero vector in \mathbb{R}^d . Let $\Gamma_1 = M_1\mathbb{Z}^d$ and $\Gamma_2 = M_2\mathbb{Z}^d$ be two full-rank lattices of the same volume. (Γ_1, Γ_2) is a good pair of lattices if and only if there exists a unimodular matrix N ($|\det N| = 1$) such that $N(-1, 1)^d \cap (M_1^{-1}M_2)\mathbb{Z}^d = \{\mathbf{0}\}$ and $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$.*

Notice that for any given invertible matrix M of order d , the zero vector is always an element of the set $((M_1^{-1}M_2)\mathbb{Z}^d \cup \mathbb{Z}^d) \cap M(-1, 1)^d$. Thus, $(M_1\mathbb{Z}^d, M_2\mathbb{Z}^d)$ is a good pair of full-rank lattices if and only if there exists a matrix N of order d such that $|\det N| = 1$ and the set $((M_1^{-1}M_2)\mathbb{Z}^d \cup \mathbb{Z}^d) \cap (N(-1, 1)^d)$ is a singleton. We also observe that the condition described in Proposition 3 is easily checked (especially in lower dimensional vector spaces) and will be exploited to derive other results. Additionally, we would like to point out that since the volume of the set $N(-1, 1)^d$ must be equal to 2^d , according to a famous theorem of Minkowski (Theorem 2, [6]) the closure of the set $N(-1, 1)^d$ must contain points of the lattices $(M_1^{-1}M_2)\mathbb{Z}^d, \mathbb{Z}^d$ other than the zero vector. Thus, $(M_1\mathbb{Z}^d, M_2\mathbb{Z}^d)$ is a good pair of lattices if and only if there exists a unimodular matrix N such that the only nonzero elements of $(M_1^{-1}M_2)\mathbb{Z}^d$ and \mathbb{Z}^d which belong to the closure of the set $N(-1, 1)^d$ are on the boundary of the $N(-1, 1)^d$.

Proposition 4. *Let M be a triangular matrix of order d with ones on the diagonal and let P, Q be unimodular integral matrices of order d . Then $(PMQ\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices with common fundamental domain $PM[0, 1)^d$.*

Put $\mathbf{p} = (p_1, \dots, p_{d-1}) \in \mathbb{R}^{d-1}$ and define the matrix-valued functions $\mathbf{p} \mapsto M(\mathbf{p})$ and $\mathbf{p} \mapsto N(\mathbf{p})$ as follows:

$$M(\mathbf{p}) = \begin{bmatrix} p_1 & 1 & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & p_{d-1} & \\ & & & & 1 \\ & & & & \prod_{k=1}^{d-1} \frac{1}{p_k} \end{bmatrix}, \quad N(\mathbf{p}) = \begin{bmatrix} & & & & 1 \\ & & & & 1 & p_2 \\ & & & & \ddots & \\ & & & & \ddots & \\ & & & & & p_{d-1} \\ 1 & & & & & \prod_{k=1}^{d-1} \frac{1}{p_k} \end{bmatrix}.$$

Furthermore, given $\mathbf{m} = (m_1, m_2, \dots, m_{d-1}) \in \mathbb{Z}^{d-1}$, we define the matrix-valued function:

$$\mathbf{m} \mapsto D(\mathbf{m}) = \begin{bmatrix} \frac{1}{m_1} & & & & & \\ & \frac{1}{m_2} & & & & \\ & & \ddots & & & \\ & & & \frac{1}{m_{d-1}} & & \\ & & & & \prod_{k=1}^{d-1} m_k & \end{bmatrix}.$$

Proposition 5. *Let P, Q be unimodular integral matrices of order d , and let U, V be unimodular integral matrices of order 2.*

- (1) *If $p_1, \dots, p_{d-1} \neq 0$ then $((PM(\mathbf{p})Q)\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices with common fundamental domain $PN(\mathbf{p})[0, 1)^d$.*
- (2) *If $m_1 m_2 \dots m_{d-1} \neq 0$ then $((PD(\mathbf{m})Q)\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices.*
- (3) *If m, n are nonzero integers such that $\gcd(m, n) = 1$ then*

$$\left(\left(U \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} V \right) \mathbb{Z}^2, \mathbb{Z}^2 \right)$$

is a good pair of lattices.

It is worth mentioning that Part 3 of Proposition 5 has also been proved in [7], Proposition 5.3. However, the novelty here lies in our proof.

Next, for any real number r , we define the matrix-valued function

$$r \mapsto R(r) = \begin{bmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{bmatrix}.$$

Proposition 6. *For any unimodular integral matrices P, Q , if r is a natural number such that \sqrt{r} is irrational then $((PR(r)Q)\mathbb{Z}^2, \mathbb{Z}^2)$ is not a good pair of lattices.*

We remark that Proposition 6 is consistent with Proposition 5.3, [7] where it is proved that it is not possible to find a star-shaped common fundamental domain for the lattices $R(2)\mathbb{Z}^2, \mathbb{Z}^2$.

The present work is organized around the proofs of the results mentioned above. In the second section we fix notations and present several results crucial to the third section of the paper, in which we prove our main propositions.

2. GENERALITIES AND INTERMEDIATE RESULTS

We remark that the investigation of good pairs of lattices in \mathbb{R}^d where $d = 1$ is not interesting. In fact, let us suppose that Γ_1, Γ_2 are two full-rank lattices of the same volume in \mathbb{R} . Then there exist nonzero real numbers a, b such that $\Gamma_1 = a\mathbb{Z}$ and $\Gamma_2 = b\mathbb{Z}$ and $|a| = |b|$. Thus, the half-open interval $|a|[0, 1)$ is a common fundamental domain for the pair (Γ_1, Γ_2) , and clearly (Γ_1, Γ_2) is a good pair of lattices. As such, in the one-dimensional case every pair of lattices of the same volume is a good pair. However, as we shall see in Proposition 6, there exist lattices of the same volume in dimension two which are not good pairs.

2.1. Notation and Terminology. Throughout this paper, we shall assume that d is a natural number strictly greater than one. Let M be a matrix. The transpose of the matrix M is denoted M^{tr} . Let v be a vector (in column form) in \mathbb{R}^d . The Euclidean norm of v is given by $\|v\|_2 = (\sum_{k=1}^d v_k^2)^{1/2}$, where

$$v = \begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix}^{tr}.$$

Given two vectors $v, w \in \mathbb{R}^d$, the inner product of v and w is $\langle v, w \rangle = \sum_{k=1}^d v_k w_k$.

All subsets of \mathbb{R}^d that we are concerned with in this paper will be assumed to be Lebesgue measurable. Let E be a subset of \mathbb{R}^d . Then χ_E stands for the **indicator function** of the set E . That is, $\chi_E : \mathbb{R}^d \rightarrow \mathbb{R}$ is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

For any subset $E \subseteq \mathbb{R}^d$ we define the set $E - E$ as follows:

$$E - E = \{x - y \in \mathbb{R}^d : x, y \in E\}.$$

Throughout this paper, $\mathbf{0}$ stands for the zero vector in \mathbb{R}^d , and we recall that M is a **unimodular matrix** if and only if $\det M = \pm 1$.

2.2. General Facts about Lattices and Good Pairs of Lattices.

Lemma 7. *Let P, M be two matrices of the same order such that $|\det P| = |\det M|$. Then $(P\mathbb{Z}^d, M\mathbb{Z}^d)$ is a good pair of lattices if and only if for any invertible matrix N of order d , $((NP)\mathbb{Z}^d, (NM)\mathbb{Z}^d)$ is a good pair of lattices.*

Proof. Assume that $(P\mathbb{Z}^d, M\mathbb{Z}^d)$ is a good pair of lattices. Then from [7], Page 3 we know that $(P\mathbb{Z}^d, M\mathbb{Z}^d)$ is a good pair of lattices if and only if there exists a set $E = Z[0, 1)^d$ such that $\sum_{k \in \mathbb{Z}^d} \chi_E(x + Pk) = \sum_{k \in \mathbb{Z}^d} \chi_E(x + Mk) = 1$ for all $x \in \mathbb{R}^d$, where Z is a matrix of order d and $|\det Z| = |\det P| = |\det M|$. We shall show that the functions

$$\mathbb{R} \ni x \mapsto \sum_{k \in \mathbb{Z}^d} \chi_{NE}(x + NPk) \quad \text{and} \quad \mathbb{R} \ni x \mapsto \sum_{k \in \mathbb{Z}^d} \chi_{NE}(x + NMk)$$

are each equal to the constant function $\mathbb{R} \ni x \mapsto 1$.

Indeed, given any $x \in \mathbb{R}^d$, since $\sum_{k \in \mathbb{Z}^d} \chi_E(x + Pk)$ is equal to one, it follows that

$$\sum_{k \in \mathbb{Z}^d} \chi_{NE}(x + NPk) = \sum_{k \in \mathbb{Z}^d} \chi_{NE}(NN^{-1}x + NPk) = \sum_{k \in \mathbb{Z}^d} \chi_E(N^{-1}x + Pk) = 1.$$

Similarly, using the fact that $\sum_{k \in \mathbb{Z}^d} \chi_E(x + Mk) = 1$ for all $x \in \mathbb{R}^d$, we obtain:

$$\sum_{k \in \mathbb{Z}^d} \chi_{NE}(x + NMk) = \sum_{k \in \mathbb{Z}^d} \chi_{NE}(N(N^{-1}x + Mk)) = \sum_{k \in \mathbb{Z}^d} \chi_E(N^{-1}x + Mk) = 1.$$

Therefore, NE is a fundamental domain for $NP\mathbb{Z}^d$ and for $NM\mathbb{Z}^d$ as well.

Now, let us assume that $(NP\mathbb{Z}^d, NM\mathbb{Z}^d)$ is a good pair of lattices. That is, there is a set $E = Z[0, 1)^d$ for some matrix Z such that

$$\sum_{k \in \mathbb{Z}^d} \chi_E(x + NPk) = \sum_{k \in \mathbb{Z}^d} \chi_E(x + NMk) = 1$$

for all $x \in \mathbb{R}^d$. Next,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \chi_E(x + NPk) &= \sum_{k \in \mathbb{Z}^d} \chi_{N^{-1}E}(N^{-1}x + Pk) = 1 \\ \sum_{k \in \mathbb{Z}^d} \chi_E(x + NMk) &= \sum_{k \in \mathbb{Z}^d} \chi_{N^{-1}E}(N^{-1}x + Mk) = 1 \end{aligned}$$

and $N^{-1}E$ is a common fundamental domain for $M\mathbb{Z}^d$ and $P\mathbb{Z}^d$. □

Lemma 8. *The following holds true:*

- (1) Let $\Gamma = M\mathbb{Z}^d$ where M is an invertible matrix with entries in \mathbb{Z} . If $|\det M| = 1$ then $\Gamma = M\mathbb{Z}^d = \mathbb{Z}^d$.
- (2) Let $\Gamma_1 = M_1\mathbb{Z}^d$, $\Gamma_2 = M_2\mathbb{Z}^d$ be two full-rank lattices of the same volume. Then $\Gamma_1 = \Gamma_2$ if and only if $M_1 = M_2U$ for some integral unimodular matrix U .

Proof. For the first part, if M is an integral matrix, then clearly, Γ is a subgroup of \mathbb{Z}^d . In order to prove that \mathbb{Z}^d is a subgroup of Γ , it is enough to show that the canonical basis elements of the lattice \mathbb{Z}^d are also elements of $M\mathbb{Z}^d$. Let $\{e_1, \dots, e_d\}$ be the canonical basis for the lattice \mathbb{Z}^d . That is, the matrix $\begin{bmatrix} e_1 & \dots & e_d \end{bmatrix}$ is the identity matrix of order d . Now, let $b_j = M^{-1}e_j$ for $j \in \{1, \dots, d\}$. Since $|\det M| = 1$ and M^{-1} is an integral matrix, it is clear that each b_j is an integral vector and $Mb_j = e_j$. Thus the set containing vectors e_1, \dots, e_d is a subset of $M\mathbb{Z}^d$ and \mathbb{Z}^d is a subgroup of Γ .

For the second part, assume that $\Gamma_1 = \Gamma_2$. For each $k \in \{1, 2, \dots, d\}$ there exists $\ell_k \in \mathbb{Z}^d$ such that $M_1 e_k = M_2 \ell_k$. Next, let $U = \begin{bmatrix} \ell_1 & \dots & \ell_d \end{bmatrix}$ be a matrix of order d . By assumption, $M_1 = M_2 U$. Moreover, since $U = M_2^{-1} M_1$ then $|\det U| = 1$. Next, let us suppose that $M_1 = M_2 U$ for some integral matrix U where $|\det U| = 1$. For any $\ell \in \mathbb{Z}^d$, $M_1 \ell = M_2 (U\ell) \in M_2 \mathbb{Z}^d$. It follows that for any $\ell \in \mathbb{Z}^d$, $M_2 \ell = M_1 (U^{-1}\ell) \in M_1 \mathbb{Z}^d$. Therefore, $\Gamma_1 = \Gamma_2$. This completes the proof. \square

Proposition 9. *Let M, M_1, M_2 be invertible matrices of order d . Then*

- (1) $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices if and only if for any unimodular integral matrices P and T , $(PMT\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices.
- (2) $(M_1\mathbb{Z}^d, M_2\mathbb{Z}^d)$ is a good pair of lattices if and only if $(\mathbb{Z}^d, M_1^{-1}M_2\mathbb{Z}^d)$ is a good pair of lattices.
- (3) $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices if and only if $(M^{-1}\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices.

Proof. For Part 1, assume that $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. Let T, P be two unimodular integral matrices. Then $(M\mathbb{Z}^d, \mathbb{Z}^d) = (M(T\mathbb{Z}^d), \mathbb{Z}^d)$. By applying Lemma 7 we see that $(PMT\mathbb{Z}^d, P\mathbb{Z}^d)$ is a good pair of lattices. However, according to Lemma 8 Part 1, we have $P\mathbb{Z}^d = \mathbb{Z}^d$. Therefore, $(PMT\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. Now, for the converse, let us assume that $(PMT\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. Since the inverse of P is an integral unimodular matrix, then

$$(MT\mathbb{Z}^d, P^{-1}\mathbb{Z}^d) = (MT\mathbb{Z}^d, \mathbb{Z}^d) = (M\mathbb{Z}^d, \mathbb{Z}^d)$$

is a good pair of lattices. This completes the proof of Part 1.

Part 2 follows from Lemma 7; indeed, $(M_1\mathbb{Z}^d, M_2\mathbb{Z}^d)$ is a good pair of lattices if and only if

$$(M_1^{-1}(M_1\mathbb{Z}^d), M_1^{-1}(M_2\mathbb{Z}^d)) = (\mathbb{Z}^d, (M_1^{-1}M_2)\mathbb{Z}^d)$$

is a good pair of lattices. Similarly, Part 3 follows from Lemma 7 as well and is simply due to the fact that

$$(M^{-1}M\mathbb{Z}^d, M^{-1}\mathbb{Z}^d) = (\mathbb{Z}^d, M^{-1}\mathbb{Z}^d).$$

\square

The following lemmas play a central role in the proof of our main results.

Lemma 10. *Let $\Gamma_1 = M\mathbb{Z}^d$ such that $|\det M| = 1$. E is a common fundamental domain for Γ_1 and $\Gamma_2 = \mathbb{Z}^d$ if and only if $(E - E) \cap M\mathbb{Z}^d = \{\mathbf{0}\}$, $(E - E) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ and the volume of E is equal to one.*

Proof. Assume that E is a common fundamental domain for Γ_1 and $\Gamma_2 = \mathbb{Z}^d$. Then clearly, the volume of the set E must be equal to one. Next, given distinct $k, l \in \mathbb{Z}^d$, it is clear that $(E + Mk) \cap (E + Ml)$ is an empty set. Therefore, given any $x, y \in E$, it must be true that $x - y$ can never be equal to Mn for some $n \in \mathbb{Z}^d$ unless $n = \mathbf{0}$. Therefore, $(E - E) \cap M\mathbb{Z}^d = \{\mathbf{0}\}$. If M is the identity matrix, a similar argument allows us to derive that $(E - E) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ as well.

Next, assuming that $(E - E) \cap M\mathbb{Z}^d = \{\mathbf{0}\}$ and $(E - E) \cap \mathbb{Z}^d = \{\mathbf{0}\}$, a calculation similar to that found in the proof of Lemma 13 shows that $(E + Mk) \cap (E + Ml)$ is an empty set for l not equal to k . Finally, since it is assumed that the volume of E is equal to one then E is a common fundamental domain for Γ_1 and $\Gamma_2 = \mathbb{Z}^d$. This completes the proof. \square

Lemma 11. *Assume that $|\det M| = 1$. Then $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair if and only if there exists a unimodular matrix N such that $N(-1, 1)^d \cap M\mathbb{Z}^d = \{\mathbf{0}\}$, and $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$.*

Proof. $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair if and only if there exists a common fundamental domain $E = N[0, 1)^d$ for the lattices $M\mathbb{Z}^d, \mathbb{Z}^d$ where N is a unimodular matrix. Now, appealing to Lemma 10, this holds if and only if $(E - E) \cap M\mathbb{Z}^d = \{\mathbf{0}\}$ and $(E - E) \cap \mathbb{Z}^d = \{\mathbf{0}\}$. Finally, the proof is completed by observing that $(E - E) = N[0, 1)^d - N[0, 1)^d = N(-1, 1)^d$. \square

Appealing to Lemma 10, the following is immediate:

Lemma 12. *Let $\Gamma_1 = M\mathbb{Z}^d$ such that $|\det M| = 1$. Then $M[0, 1)^d$ is a common fundamental domain for $M\mathbb{Z}^d$ and \mathbb{Z}^d if and only if $M(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$.*

2.3. Constructing Good Pairs from Known Good Pairs.

Lemma 13. *Let $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 is a full-rank lattice in \mathbb{R}^n and Γ_2 is a full-rank lattice in \mathbb{R}^m . If E_1 is a common fundamental domain for Γ_1 and \mathbb{Z}^n in \mathbb{R}^n and E_2 is a common fundamental domain for Γ_2 and \mathbb{Z}^m in \mathbb{R}^m then $E = E_1 \times E_2$ is a common fundamental domain for Γ and $\mathbb{Z}^n \times \mathbb{Z}^m$ in $\mathbb{R}^n \times \mathbb{R}^m$.*

Proof. Indeed, let us assume that E_1 is a common fundamental domain for Γ_1 and \mathbb{Z}^n in \mathbb{R}^n , E_2 is a common fundamental domain for Γ_2 and \mathbb{Z}^m in \mathbb{R}^m , and there exist distinct $\gamma, k \in \mathbb{Z}^n \times \mathbb{Z}^m$ such that the set $(E + \gamma) \cap (E + k)$ is not empty ($E = E_1 \times E_2$). Then there exist $z, z' \in E$ such that $z + \gamma = z' + k$. Now, we write $z = (x, y)$, $z' = (x', y')$, $\gamma = (\gamma_1, \gamma_2)$ and $k = (k_1, k_2)$. Thus,

$$(x, y) + \gamma = (x + \gamma_1, y + \gamma_2) = (x' + k_1, y' + k_2).$$

As a result, $x + \gamma_1 = x' + k_1$ and $y + \gamma_2 = y' + k_2$. Since $\gamma \neq k$ then either $\gamma_1 \neq k_1$ or $\gamma_2 \neq k_2$. So, we obtain that either $\gamma_1 \neq k_1$ and $x + \gamma_1 = x' + k_1$, or $\gamma_2 \neq k_2$ and $y + \gamma_2 = y' + k_2$. This contradicts our assumption that E_1 is a common fundamental domain for Γ_1 and \mathbb{Z}^n in \mathbb{R}^n and E_2 is a common fundamental domain for Γ_2 and \mathbb{Z}^m in \mathbb{R}^m . \square

Appealing to Lemma 13, the following is immediate.

Lemma 14. *Let M_1 and M_2 be two invertible matrices of order d . Assume that $(M_1\mathbb{Z}^n, \mathbb{Z}^n)$ and $(M_2\mathbb{Z}^m, \mathbb{Z}^m)$ are good pairs of lattices. If*

$$M = M_1 \oplus M_2 = \begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix}$$

then $(M(\mathbb{Z}^n \times \mathbb{Z}^m), \mathbb{Z}^n \times \mathbb{Z}^m)$ is a good pair of lattices.

Given two matrices A, B of order a and b respectively, such that $A = [A_{i,j}]_{1 \leq i, j \leq a}$ the **tensor product** (or **Kronecker product**) of the matrices $A \otimes B$ is a matrix of order ab given by

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1d}B \\ \vdots & \cdots & \vdots \\ A_{d1}B & \cdots & A_{dd}B \end{bmatrix}.$$

Lemma 15. *Let I_p be the identity matrix of order p , and let M be an invertible matrix of order d .*

- (1) *If $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices then $((I_p \otimes M)\mathbb{Z}^{pd}, \mathbb{Z}^{pd})$ is a good pair of lattices.*
- (2) *If $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices then $((M \otimes I_p)\mathbb{Z}^{pd}, \mathbb{Z}^{pd})$ is a good pair of lattices.*

Proof. For Part 1, we observe that

$$I_p \otimes M = \begin{bmatrix} M & & \\ & \ddots & \\ & & M \end{bmatrix}.$$

Applying Lemma 14 an appropriate number of times gives us the desired result. For Part 2, let

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1d} \\ \vdots & \cdots & \vdots \\ m_{d1} & \cdots & m_{dd} \end{bmatrix} \text{ and } I_p = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

Then

$$M \otimes I_p = \begin{bmatrix} m_{11}I_p & \cdots & m_{1d}I_p \\ \vdots & \cdots & \vdots \\ m_{d1}I_p & \cdots & m_{dd}I_p \end{bmatrix}.$$

We shall show that $M \otimes I_p$ and $I_p \otimes M$ are similar matrices. In other words, there exists an integral unimodular matrix P such that:

$$(1) \quad P(M \otimes I_p)P^{-1} = (I_p \otimes M).$$

Indeed, let $\{e_i \otimes e_j : 1 \leq i \leq d, 1 \leq j \leq p\}$ be a basis for $\mathbb{R}^d \otimes \mathbb{R}^p$. Define $Q : \mathbb{R}^d \otimes \mathbb{R}^p \rightarrow \mathbb{R}^p \otimes \mathbb{R}^d$ such that $Q(e_i \otimes e_j) = e_j \otimes e_i$. It is easy to see that Q is a linear isomorphism whose matrix is an integral unimodular matrix. Moreover,

$$\begin{aligned} Q^{-1}(M \otimes I_p)Q(e_i \otimes e_j) &= Q^{-1}(M \otimes I_p(e_j \otimes e_i)) \\ &= Q^{-1}(Me_j \otimes e_i) \\ &= e_i \otimes Me_j \\ &= (I_p \otimes M)(e_i \otimes e_j). \end{aligned}$$

Formula (1) is finally obtained by setting $Q = P^{-1}$. Now, since $((I_p \otimes M)\mathbb{Z}^{pd}, \mathbb{Z}^{pd})$ is a good pair of lattices by Part 1, it follows that $((M \otimes I_p)\mathbb{Z}^{pd}, \mathbb{Z}^{pd})$ is a good pair of lattices. \square

Proposition 16. *Let M be an invertible matrix of order d . If N is a unimodular integral matrix of order n and if $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices then $((M \otimes N)\mathbb{Z}^{dn}, \mathbb{Z}^{dn})$ is a good pair of lattices.*

Proof. We observe that $M \otimes N = (M \otimes I_n)(I_d \otimes N)$. If N is a unimodular integral matrix of order n then $I_d \otimes N$ is a unimodular integral matrix of order dn . Next, since $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices then it follows from Lemma 15 Part 2, that $((M \otimes I_p)\mathbb{Z}^{pd}, \mathbb{Z}^{pd})$ is a good pair. Now, since $I_d \otimes N$ is a unimodular integral matrix, appealing to Lemma 8 we obtain the desired result: $(M \otimes I_p)\mathbb{Z}^{pd} = (M \otimes N)\mathbb{Z}^{pd}$. \square

3. PROOFS OF MAIN RESULTS

3.1. Proof of Proposition 3. The fact that (Γ_1, Γ_2) is a good pair of lattices if and only if $((M_1^{-1}M_2)\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair is due to Part 2 of Proposition 9. The fact that (Γ_1, Γ_2) is a good pair of lattices is equivalent to the statement that there exists a unimodular matrix N such that $N(-1, 1)^d \cap (M_1^{-1}M_2)\mathbb{Z}^d = \{\mathbf{0}\}$, and $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$ is due to Lemma 11.

3.2. Proof of Proposition 4. It suffices to show that $M[0, 1)^d$ is a common fundamental domain for $M\mathbb{Z}^d$ and \mathbb{Z}^d . First, let us assume that M is an upper triangular unipotent matrix. We will offer a proof by induction on d . For the base case, let us assume that $d = 2$. We define

$$M_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

for some $s \in \mathbb{R}$.

The fact that $M_s[0,1]^2$ is a fundamental domain for the lattice $M_s\mathbb{Z}^2$ is obvious. Now, let $z = \begin{bmatrix} x & y \end{bmatrix}^{tr} \in (-1,1)^2$ such that $M_s z = k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^{tr} \in \mathbb{Z}^2$. We would like to show that $k_1 = k_2 = 0$. First, we observe that

$$z = M_s^{-1}k = \begin{bmatrix} k_1 - sk_2 & k_2 \end{bmatrix}^{tr} \in (-1,1)^2.$$

This is only possible if $k_2 = k_1 = 0$. Therefore $M_s(-1,1)^2 \cap \mathbb{Z}^2 = \{\mathbf{0}\}$ and by Lemma 12, $M_s[0,1]^2$ is a common fundamental domain for $M_s\mathbb{Z}^2$ and \mathbb{Z}^2 .

Now, let us suppose that for all $d \leq m-1 \in \mathbb{N}$ we have that $M[0,1]^d$ is a common fundamental domain for $M\mathbb{Z}^d$ and \mathbb{Z}^d whenever M is a unipotent matrix. More precisely, let

$$M = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{m-1} \\ & 1 & a_m & \cdots & a_{2m-3} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & a_{\frac{m^2-m}{2}} \\ & & & & 1 \end{bmatrix}$$

be an arbitrary unipotent matrix of order m with real entries. Let

$$v = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & a_m & \cdots & a_{2m-3} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{\frac{m^2-m}{2}} \\ & & & 1 \end{bmatrix}$$

so that

$$M = \begin{bmatrix} 1 & v \\ 0 & M_1 \end{bmatrix}.$$

Next, assume that for any given $z \in (-1,1)^m$ we have that $Mz \in \mathbb{Z}^m$. We want to show that z is the zero vector. Writing

$$Mz = \begin{bmatrix} 1 & v \\ 0 & M_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 + \langle v, z_2 \rangle \\ M_1 z_2 \end{bmatrix},$$

where $\langle v, z_2 \rangle$ is the dot product of the vectors v, z_2 , it follows that $M_1 z_2 \in \mathbb{Z}^{m-1}$. By the assumption of the induction, then $z_2 = 0$ and it follows that $z_1 + \langle v, z_2 \rangle = z_1 \in \mathbb{Z}$. Since $z \in (-1,1)^m$ then $z_1 = 0$ and this completes the induction.

Now, let us suppose that M is a lower triangular unipotent matrix. Put

$$J = \begin{bmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{bmatrix}.$$

Notice that JMJ^{-1} is an upper triangular matrix. Since $(JMJ^{-1}\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices, using the fact that J is a unimodular integral matrix together with Proposition 9, Part 1, it follows that $(M\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices as well. This completes the proof of the first part.

3.3. Proof of Proposition 5. Put

$$M = M(\mathbf{p}) = \begin{bmatrix} p_1 & 1 & & \\ & \ddots & \ddots & \\ & & p_{d-1} & \\ & & & \frac{1}{p_1 \cdots p_{d-1}} \end{bmatrix},$$

and

$$N = N(\mathbf{p}) = \begin{bmatrix} & & & & 1 \\ & & & 1 & p_2 \\ & & \ddots & \ddots & \\ & & & p_{d-1} & \\ 1 & \frac{1}{p_1 \cdots p_{d-1}} & & & \end{bmatrix}.$$

We would like to show that $N[0, 1]^d$ is a common fundamental domain for the pair $(M\mathbb{Z}^d, \mathbb{Z}^d)$. For this purpose, it is enough to show (see Lemma 10) that $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$ and

$$N(-1, 1)^d \cap M\mathbb{Z}^d = \{\mathbf{0}\}.$$

In order to prove that $N(-1, 1)^d \cap M\mathbb{Z}^d = \{\mathbf{0}\}$, let us suppose that $Nv = Mk$, $v \in (-1, 1)^d$ and $k \in \mathbb{Z}^d$. It follows that $M^{-1}Nv \in \mathbb{Z}^d$. Computing the inverse of M , we obtain

$$(2) \quad M^{-1} = \begin{bmatrix} \frac{1}{p_1} & \frac{(-1)^1}{p_1 p_2} & \frac{(-1)^2}{p_1 p_2 p_3} & \cdots & \frac{(-1)^{d-2}}{p_1 p_2 \cdots p_{d-1}} & (-1)^{d-1} \\ & \frac{1}{p_2} & \frac{(-1)^1}{p_2 p_3} & \cdots & \frac{(-1)^{d-3}}{p_2 \cdots p_{d-1}} & (-1)^{d-2} p_1 \\ & & \frac{1}{p_3} & \cdots & \frac{(-1)^{d-2}}{p_3 \cdots p_{d-1}} & (-1)^{d-3} p_1 p_2 \\ & & & \ddots & \frac{1}{p_{d-1}} & \vdots \\ & & & & & (-1)^{d-(d-1)} p_1 \cdots p_{d-2} \\ & & & & & p_1 \cdots p_{d-1} \end{bmatrix}.$$

Next, with some formal calculations we obtain that

$$M^{-1}Nv = \begin{bmatrix} (-1)^{d-1} & & & & & \\ (-1)^{d-2} p_1 & & & & & 1 \\ (-1)^{d-3} p_1 p_2 & & & & \ddots & \\ \vdots & & & & 1 & \\ (-1)^{d-(d-1)} p_1 \cdots p_{d-2} & & & & 1 & \\ p_1 \cdots p_{d-1} & & & & 1 & \end{bmatrix} v = k \in \mathbb{Z}^d.$$

Therefore, v must be the zero vector.

To show that $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$, let $z \in (-1, 1)^d$ such that $Nz = k \in \mathbb{Z}^d$. More precisely, we have

$$\begin{bmatrix} & & & & 1 \\ & & & 1 & p_2 \\ & & \ddots & \ddots & \\ & & & p_{d-1} & \\ 1 & \frac{1}{p_1 \cdots p_{d-1}} & & & \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{d-1} \\ z_d \end{bmatrix} = \begin{bmatrix} z_d \\ z_{d-1} + z_d \\ \vdots \\ z_2 + z_3 \\ z_1 + \frac{z_2}{p_1 \cdots p_{d-1}} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{d-1} \\ k_d \end{bmatrix}.$$

Now using the fact that $k \in \mathbb{Z}^d$ together with $z \in (-1, 1)^d$ gives us that z must be equal to the zero vector. Therefore, $N(-1, 1)^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$. In light of Proposition 9 Part 1, $((PM(\mathbf{p})Q)\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices with common fundamental domain $(PN(\mathbf{p})) [0, 1]^d$ whenever P, Q are integral unimodular matrices.

For Part 2, appealing again to Proposition 9 Part 1 it is enough to show that $(D(\mathbf{m})\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. First, let

$$Z = \begin{bmatrix} \frac{1}{m_1} & & & & & \\ & 1 & & & & \\ & & \frac{1}{m_2} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{m_{d-1}} & \\ & & & & & 1 \\ & & & & & & \prod_{i=1}^{d-1} m_i \end{bmatrix}.$$

Next, applying the first part of the proposition it is clear that $(Z\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. Now, put

$$U = \begin{bmatrix} 1 & (-1)^1 m_1 & (-1)^2 m_1 m_2 & (-1)^3 m_1 m_2 m_3 & \cdots & (-1)^{d-1} \prod_{i=1}^{d-1} m_i \\ & 1 & (-1)^1 m_2 & (-1)^2 m_2 m_3 & \cdots & (-1)^{d-2} \prod_{i=2}^{d-1} m_i \\ & & 1 & (-1)^1 m_3 & \cdots & (-1)^{d-3} \prod_{i=3}^{d-1} m_i \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & (-1)^1 m_{d-1} \\ & & & & & 1 \end{bmatrix}.$$

Since U is a unimodular integral matrix, then $Z\mathbb{Z}^d = ZU\mathbb{Z}^d$ (Lemma 8). It is easy to check that ZU is equal to the diagonal matrix

$$\begin{bmatrix} \frac{1}{m_1} & & & & & \\ & \frac{1}{m_2} & & & & \\ & & \ddots & & & \\ & & & \frac{1}{m_{d-1}} & & \\ & & & & \prod_{i=1}^{d-1} m_i & \end{bmatrix} = D(\mathbf{m}).$$

Therefore $(ZU\mathbb{Z}^d, \mathbb{Z}^d) = (D(\mathbf{m})\mathbb{Z}^d, \mathbb{Z}^d)$ is a good pair of lattices. For the last part of the proposition, put

$$S = \begin{bmatrix} 0 & 1 \\ 1 & \frac{n}{m} \end{bmatrix}, S' = \begin{bmatrix} \frac{m}{n} & 1 \\ 0 & \frac{n}{m} \end{bmatrix}.$$

We claim that $S[0,1]^2$ is a common fundamental domain for the lattices $S'\mathbb{Z}^2$ and \mathbb{Z}^2 . To see this, it suffices (see Proposition 3) to check that $S(-1,1)^2 \cap \mathbb{Z}^2 = \{\mathbf{0}\}$ and $S(-1,1)^2 \cap S'\mathbb{Z}^2 = \{\mathbf{0}\}$. Let $z = \begin{bmatrix} x & y \end{bmatrix}^{tr} \in (-1,1)^2$ such that $Sz \in \mathbb{Z}^2$. Then $z = S^{-1}k = \begin{bmatrix} k_2 - \frac{nk_1}{m} & k_1 \end{bmatrix}^{tr}$ where $k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^{tr} \in \mathbb{Z}^2$. So, $z = \mathbf{0}$. Next, let us assume that $Sz \in S'\mathbb{Z}^2$. That is, $Sz = S'k$ for some $k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^{tr} \in \mathbb{Z}^2$. It follows that

$$z = \begin{bmatrix} x & y \end{bmatrix}^{tr} = S^{-1}S' \begin{bmatrix} k_1 & k_2 \end{bmatrix}^{tr} = \begin{bmatrix} -k_1 & \frac{mk_1+nk_2}{n} \end{bmatrix}^{tr}.$$

Thus $z = \mathbf{0}$ as well.

Next, since $\gcd(m, n) = 1$ there exist $\ell_1, \ell_2 \in \mathbb{Z}$ such that $1 - n\ell_2 - m\ell_1 = 0$. As such, it follows that

$$\begin{aligned} \begin{bmatrix} 1 & -m\ell_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{m}{n} & 1 \\ 0 & \frac{n}{m} \end{bmatrix} \begin{bmatrix} 1 & -\ell_1 n \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} \frac{m}{n} & 1 - n\ell_2 - m\ell_1 \\ 0 & \frac{1}{m}n \end{bmatrix} \\ &= \begin{bmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{bmatrix}. \end{aligned}$$

By Proposition 9, Part 1,

$$(3) \quad \left(\begin{bmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{bmatrix} \mathbb{Z}^2, \mathbb{Z}^2 \right).$$

Using the fact that

$$\begin{bmatrix} 1 & -\ell_1 n \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2 = \mathbb{Z}^2$$

together with Lemma 7, we conclude that

$$\begin{bmatrix} 1 & -m\ell_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \frac{n}{m} \end{bmatrix} [0, 1)^2 = \begin{bmatrix} -m\ell_2 & 1 - n\ell_2 \\ 1 & \frac{n}{m} \end{bmatrix} [0, 1)^2$$

is a common connected fundamental domain for the pair (3). Finally, the first part of Proposition 9 gives the desired result.

3.4. Proof of Proposition 6. According to Proposition 9, it is enough to show that if r is a natural number such that \sqrt{r} is irrational then $(R(r)\mathbb{Z}^2, \mathbb{Z}^2)$ is not a good pair of lattices. Put

$N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $|\det N| = 1$. Let us suppose that $\Omega = N[0, 1)^2$ is a common fundamental domain for \mathbb{Z}^2 and $R\mathbb{Z}^2 = R(r)\mathbb{Z}^2$. There must exist a non-zero element k in \mathbb{Z}^2 such that one corner of the closure of the set $\Omega + k$ meet Ω at the origin. Similarly, since Ω tiles the plane by $R(r)\mathbb{Z}^2$, there must exist a non-trivial element k of $R(r)\mathbb{Z}^2$ such that one corner of the closure of $\Omega + k$ intersects Ω at the origin as well (see Figure below)

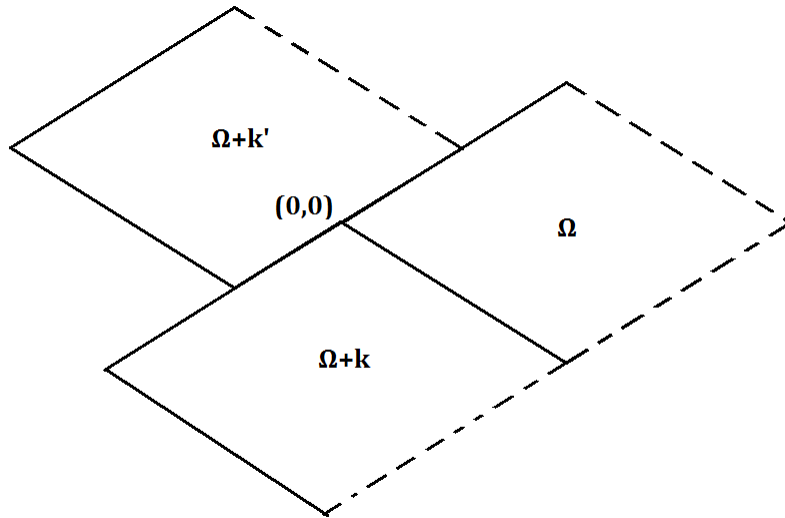


FIGURE 1. Behavior of a tiling around the origin of the plane

Hence, there exist

$$p, q \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

such that

$$(4) \quad \begin{cases} Np + k = \mathbf{0} \\ Nq + j = \mathbf{0} \end{cases}$$

for some $k \in \mathbb{Z}^2 - \{\mathbf{0}\}$ and $j \in R\mathbb{Z}^2 - \{\mathbf{0}\}$. By assumption, N is a unimodular matrix. However, without loss of generality, we may assume that $\det N = 1$. Indeed, if $\det N = -1$, then (4) is equivalent to

$$\begin{cases} JNp + Jk = \mathbf{0} \\ JNq + Jj = \mathbf{0} \end{cases} \text{ where } \det(JN) = 1, Jk \in \mathbb{Z}^2 - \{\mathbf{0}\}, Jj \in R\mathbb{Z}^2 - \{\mathbf{0}\} \text{ and } J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We shall prove that if $\sqrt{r} \notin \mathbb{Q}$ then (4) has no solution. There are several possible cases that may arise from all the possible choices for p, q . First of all, since $R\mathbb{Z}^2 \cap \mathbb{Z}^2 = \{\mathbf{0}\}$, it is easy to see that (4) has no solution whenever $p = q$. Therefore, we should only focus on the cases where p is not equal to q . Put

$$k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^{tr} \text{ and } j = \begin{bmatrix} \sqrt{r}j_1 & \frac{1}{\sqrt{r}}j_2 \end{bmatrix}^{tr} \text{ where } k_1, k_2, j_1, j_2 \in \mathbb{Z}.$$

Case 1.1 If

$$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, N = \begin{bmatrix} \frac{bc+1}{d} & b \\ c & d \end{bmatrix}$$

and $d \neq 0$ then

$$k_1 = \frac{j_1 k_2 r + \sqrt{r}}{j_2}, k_2 = -c, j_1 = \frac{-b}{\sqrt{r}}, j_2 = -d\sqrt{r}$$

and $k_1 j_2 - j_1 k_2 r = \sqrt{r}$. Thus, System (4) has no solution since \sqrt{r} is irrational.

Case 1.2 If

$$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, N = \begin{bmatrix} a & b \\ -\frac{1}{b} & 0 \end{bmatrix}$$

and $b \neq 0$ then

$$k_1 = a, k_2 = -\frac{1}{b}, j_1 = \frac{b}{\sqrt{r}}, j_2 = 0$$

and $j_1 = -\frac{1}{k_2 \sqrt{r}}, k_2 \neq 0$. This is absurd since j_1 is an integer.

Case 2.1 If

$$p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, N = \begin{bmatrix} \frac{bc+1}{d} & b \\ c & d \end{bmatrix}, d \neq 0$$

then

$$k_1 = -b, k_2 = -d, j_1 = \frac{-1 - bc}{d\sqrt{r}}, j_2 = -c\sqrt{r}$$

and $k_2 j_1 r - j_2 k_1 = \sqrt{r}$ which is absurd.

Case 2.2 If

$$p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, N = \begin{bmatrix} a & b \\ -\frac{1}{b} & 0 \end{bmatrix}, b \neq 0$$

then

$$k_1 = b, k_2 = 0, j_1 = \frac{a}{\sqrt{r}}, j_2 = -\frac{\sqrt{r}}{b}.$$

Therefore, $j_2 = -\frac{\sqrt{r}}{k_1}$ and this is absurd.

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DEPT. OF MATHEMATICS & COMPUTER SCIENCE, BRIDGEWATER STATE UNIVERSITY, BRIDGEWATER, MA 02325 U.S.A.,

E-mail address: vignon.oussa@bridgew.edu

DEPT. OF MATHEMATICS, BRIDGEWATER STATE UNIVERSITY, BRIDGEWATER, MA 02325 U.S.A.,