5-13-2015

Calculating the Shear and Divergence of Light Rays

Matthew Witherell

Calculating the Shear and Divergence of Light Rays

Matthew Witherell

Submitted in Partial Completion of the
Requirements for Departmental Honors in Physics

Bridgewater State University

May 13, 2015

Dr. Thomas Kling, Thesis Director
Dr. Edward Deveney, Committee Member
Dr. Jeffery Williams, Committee Member

Calculating the Shear and Divergence of Light Rays

Matthew Witherell

May 7, 2015

Abstract

The purpose of this research was to calculate the shear and divergence of light rays bundles as they pass black holes. We defined a Lagrangian using the Schwarzchild metric then used the Euler-Lagrangian equation to create 6 first order ODE's for the light rays path. Next we found tangent vectors to the light ray so we could calculate Ψ_0. In order to calculate the shear and divergence the method of calculation required simultaneously solving 16 ordinary differential equations. We used Mathematica to calculate how these light rays act but first we had to use calculus and algebra to derive these equations. We started from the Schwarzchild metric to find the Lagrangian for the motion of the light rays. We then got results which were 6 ODE's that gave velocity and acceleration in (T, r, θ, ϕ) . Then we found the tangent vectors to these paths which allows us to solve for Ψ_0 and finally we solved for the divergence and shear of the light rays.

Introduction

General relativity is the modern description of how gravity works. Since the laws of physics are the same for all inertial frames, space and time can be seen as a combination of both which is the theory of special relativity. Measurements can be converted between inertial frames using coordinate transformations. General relativity generalizes special relativity and relativizes Newton's laws of gravity. The result of this generalization shows us that massive objects such as planets or stars create distortion in space-time, and we see these distortions as gravity. Space-time curvature is negligible in a small enough region. Therefore there is a set of inertial frames that approximates that region.

Albert Einstein started working on general relativity in 1907. Einstein discovered that the force of attraction of two masses is caused by the warping of space and time by those masses. Einstein's theory was able to explain the effects of gravity that Newton's gravitational theory could not, such as small variations in the orbits of the planets. The theory also predicts lesser known effects of gravity such as gravitational lensing, waves, and time dilation.

When bundles of light rotate around a black hole two things tend to happen to them, they diverge and shear. Bundles of light are groups of individual light rays that travel near each other and follow a similar path. Divergence of the light bundles means that the individual light rays will spread out in different directions; they will cease to be parallel. The equation of the divergence is

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}.$$

Shearing light bundles is similar to compressing them, like the compression of a pillow where the sides of it expand outward to deal with the pressure. The light bundles initially start out circular and are then compressed into an elliptical shape. The equation of the shear is

$$D\sigma = (\rho + \bar{\rho})\,\sigma + \Psi_0.$$

In 1907 Einstein realized that his theory of special relativity could be applied to gravitational fields. He argued that free-fall was really inertial motion so special relativity still applied to the one falling. This led to the prediction of the existence of gravitational time dilation. He theorized that a clock in a gravitational field would show time moving faster than one not it a field. This difference was proportional to the gravitational potential. Another of Einstein's thought experiments about a spinning disk led him to conclude that space time could be locally curved. This would push him to study Riemannian geometry in order to develop his full theory of general relativity. He went on to create his gravitational field equations which were named after him. These equations have several solutions, for instance the Schwarzschild and the Kerr solutions for astrophysical black holes.

One application of general reality is gravitational lensing. The gravitational field created by a large mass will bend and refocus the light that passes through or by it. Lensing takes place on different scales from planets and stars or galaxies with the effects of lensing dependent upon the gradient of the gravitational field of the object, where the larger the gradient of the gravitational field, the greater the effect of the lensing.

Einstein's first prediction was a rough approximation that was off by a factor of two but he was able to show that there would be deflection of light by a star. The method he used did not work well near the speed of light. In his full theory he would fix the mistake. This theory would challenge astronomers to try to prove its correctness which they could only do by observing the sun during a solar eclipse. In the late 1910s two groups of astronomers claimed to have taken measurements that show no sign of deflection. But in 1919 a group of British astronomers went to Brazil and Principle, an island off of Africa, and took measurements that confirm that there was deflection of star light by the sun's gravity. Although these measurements were unreliable because the amount of error associated with them was on the scale of the measurement themselves, later more accurate measurements would definitively prove the theory.

The idea of black holes or objects like black holes goes back hundreds of years. In 1783 John Michell created the concept of an object that was so massive that its escape velocity was faster than the speed of light. In 1796 Simon Laplace predicted that the most massive objects might be invisible because light cannot escape them. Then in 1915 Einstein published his theory of general relativity predicting curved space-time, a year later Karl Schwarzschild predicted the radius of black holes later known as the Schwarzschild radius. In 1970 Stephen Hawking developed modern black hole theory where black holes are the aftermath of a stars death. It wasn't till 1994 that real evidence of black hole was found. The Hubble telescope spotted stars orbiting the nucleus of galaxies moving at high velocities suggesting an extremely massive object at the center.

How these black hole's effect the motion of light rays is given equations from general relativity by ordinary differential equations. In order to figure out what happens to these light bundles several ordinary differential equations will have to be solved simultaneously. We begin with the manifold which is a generalized surface that has dimension where points on it can be labeled and any function is differentiable. It is different from a surface because the manifold is an object.

Metrics and Motion

Space is not just a flat surface. Space can be curved in the first place or in the presence of a planet space bends around the surface. Moving along a curved surface is different from a flat one. On a flat surface you can move in a straight line without the line changing. On a curved surface if you attempt to walk in a straight line you will not be successful. Locally, you can walk in a straight line but if you look at the big picture it will show you that the path will bend with the curvature of the surface. The equivalent of a straight line on a curved surface is geodesic.

A manifold is a region of space with a coordinate system on it where points and labels are arbitrary. A manifold is similar to Cartesian coordinates but unlike Cartesian coordinates the coordinates are not necessarily regularly spaced. In the example of a block sliding down a track consisting of rough and smooth patches, it would take more time to pass through the rough sections than the smooth. The block will slide quickly through the smooth patches. Now if you create a one dimensional manifold of this situation where each point is a measurement of time, the points that represent the rough patches are more numerous and bunched together compared to the smooth points. The smooth points are spread apart and infrequent. Since space-time itself is abstract it makes sense to use a system that can also be abstract. One of the other properties manifolds have is that they support a differentiable structure so any function of it has derivatives.

A metric is a rule for giving distances. In two dimensions on a flat surface metrics are relatively simple something like

$$ds^2 = dx^2 + dy^2.$$

If the space is not flat then a simple line element can be used locally but the rest of the space will require more terms in the line element. This line element can be written in matrix form as

$$ds^2 = \begin{pmatrix} dx^2 & dy^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since manifold coordinates are arbitrary, metrics are more complicated and require many more terms. In order to make space-time work in relativity, metrics require four dimensions. This includes the three space coordinates and the fourth is time. Since metrics for complicated manifolds can be very complicated the metric tensor

$$g_{ab}$$

is used to make them more manageable. The metric tensor can be thought of as a matrix that contains all of the variables in the metric. The entries down the diagonal when added together create a line element that gives the distances between

$$ds^2 = \sum_{a,b} g_{ab} dx^a dy^b = \begin{pmatrix} dt & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}.$$

In four dimensions g_{00} is the time part and g_{ij} is the space only part of the whole metric. The metric used in special relativity is,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

This metric can also be written in matrix form like the one above.

One of the steps in this project is to integrate the path of a light ray in a situation where there is curvature. The path will be determined and mathematically represented. This path can be found through the calculus of variations, which is the calculus of finding the extremal solutions of a function, either the maximum or the minimum path. For instance, the way an object rests on a flat surface will always try and minimize the amount of potential energy of the object. If the object were to fall over its orientation will change in a way that will decrease the amount of potential energy.

The path light will take to reach its destination is a similar concept; light follows Fermat's principle "light travels by the path that takes the least amount of time". Then light traveling through curved space will always follow the curves that allow it to reach its destination in the smallest amount of time. Now knowing that information and using the equation of the path of a light ray

$$\mathcal{L} = \frac{1}{2} \mathrm{g_{ab}} \dot{\mathrm{x}}^{\mathrm{a}} \dot{\mathrm{x}}^{\mathrm{b}}.$$

The Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

will define the extremal the path of the light and create ordinary differential equations that will give the path of the light.

Curvature and Parallel Propagation

The Riemann tensor

$$R_{abcd}$$

describes how parallel geodesics fail to remain parallel which is given by the connection. The tensor also directly measures the path dependence of parallel transport which is given by Γ^a_{bc}. The Riemann tensor can be broken up into two components the Ricci tensor

$$R_{ac}$$

and the Weyl tensor

$$C_{abcd}.$$

The Ricci tensor describes how curved space is, due to matter. The Weyl tensor can be used to describe things like gravitational tidal forces and gravitational radiation. Ricci tensor can be considered the trace part and the Weyl tensor can be thought of as the trace free part of the Riemann tensor. The Ricci tensor in our case will be zero. The Weyl tensor will be used to calculate

$$\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d.$$

Parallel propagation is the process of moving a vector so that the vector itself does not change as its location changes. A vector on a curved surface can also be parallel propagated, the equation that defines propagation

$$\ell^a \nabla_a m^b = \ell^a \frac{d}{dx^a} m^b + \ell^a \Gamma^b_{ac} m^c = 0$$

if the vector has been moved correctly. $\ell^a \frac{d}{dx^a} m^b$ is how the vector changes as it moves along the surface, this part is zero if the surface is flat. $\ell^a \Gamma^b_{ac} m^b$ is how the curvature of the space changes.

Parallel transport is the process of moving a vector so that the vector itself does not change. On a flat surface no matter how you move a vector on some path the vector itself will not change.

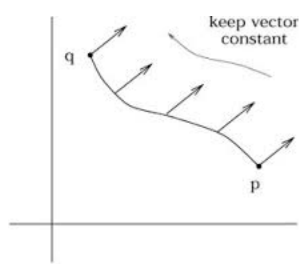


Figure 1: A vector being propagated along a 2-D surface

If a space has curvature then moving a vector in that space will cause the vector to change to some degree. By the time the vector has finished moving it will not be the same anymore.

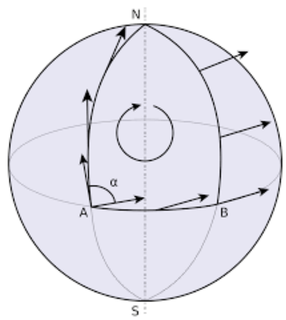


Figure 2: A vector being propagated on a 3-D surface

Parallel transport can be used as a test to see if there is curvature. Mathematically parallel transport is given by.

$$\ell^a \nabla_a m^b = \ell^a \frac{\partial}{\partial x^a} m^b + \Gamma^{\ b}_{ac} \ell^a m^c = 0$$

Where m^b is transported along the curve for which ℓ^a is tangent. m^b is made up of a real and imaginary part $e_1 + ie_2$. The metric connection changes based on how curved the space is. So the parallel transport equation has two parts the first changes based on how m^a changes and the second changes with the curvature of the space. If there is curvature then the second part will be nonzero. The connection is given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(g_{db,c} + g_{cd,b} - g_{bc,d} \right).$$

The comma means take the derivative of the term to the left of the comma with respect to the right side of the comma. So the first term in the parentheses means

$$g_{db,c} = \frac{\partial}{\partial x^c} g_{ab}.$$

When bundles of light rotate around a black hole two things tend to happen to them, they diverge and shear. Bundles of light are groups of individual light rays that travel near each other and follow a similar path. Divergence of the light bundles means that the individual light rays will spread out in different directions; they will cease to be parallel. The equation of the divergence is

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}.$$

Shearing light bundles is similar to compressing them, like the compression of a pillow where the sides of it expand outward to deal with the pressure. The light bundles initially start out circular and are then compressed into an elliptical shape. The equation of the shear is

$$D\sigma = (\rho + \bar{\rho})\, \sigma + \Psi_0.$$

Φ_{00} and Ψ_0 come for the Ricci and Weyl tensor they will have to be calculated in order to solve these equations. Where

$$\Phi_{00} = \frac{1}{2} R_{ac} \ell^a \ell^b$$

and

$$\Psi_0 = C_{abcd} \ell^a m^b \ell^c m^d.$$

The Schwarzchild Metric

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{rc^1}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

is a solution to Einstein's field equations describing the exterior gravitational field of a spherical body. When applied to weak fields the metric reduces to Newton's equations. When applied to large fields there are subtle differences in the motion of the planets. The other predictions are the bending of light, gravitational redshift of light, and time delay effects. When applied to a strong field the metric describes what happens when a large body collapses in on itself due to its own gravity creating a singularity. The Schwarzchild radius

$$r_s = \left(\frac{2GM}{c^2}\right)$$

exist far inside of any spherical body. If a sphere of radius r were to collapse to the to $r \leq r_s$then a singularity would form.

There are three ways gravitational collapse can cause holes. The first way Black holes are formed is when a massive star collapses in on itself. Stars keep themselves from collapsing by nuclear fusion, the combining of elements to create heavier elements. Stars contain massive amounts of hydrogen that they burn to create helium. When they run out of hydrogen they start to burn helium. Eventually they reach iron, at that point the star can no longer continue fusion because the energy required to combine iron is too great. When the star runs out of fuel it undergoes gravitational collapse. If the star is sufficiently massive the core can collapse past the Schwarzchild radius creating a singularity hidden inside of a black hole.

The second way is the collapse of the central core of a cluster of stars. Interactions between stars can cause large amounts of energy to be transferred to a single star. The central star will eventually evaporate. Through this process the star will lose large amounts of energy and become more gravitationally bound, the center of the star cluster becomes denser. At this point it is unknown what happens next though there are two likely scenarios. The first is that this dense region of the gas cloud that contains the star cluster will experience tidal distortions and collapse. The other likely scenario is that over time stars within the cluster undergo stellar collapse and gather and combine within the dense region of space.

The universe today is homogenous on a large scale but evidence suggests that regions of the early universe were not. These regions of enhanced density could have produced black holes although it is not possible to know if any black holes were created this way. An interesting feature of these black holes is that they could have masses much smaller than that of a star.

The gravity of a black hole is so strong that nothing can escape its pull, not even light can escape from it. Once something passes the event horizon it can no longer escape. Rotating black holes have an area the surrounds the event horizon called the ergo sphere. This is a region of space where space rotates with the black hole. Objects that fall into this region are forced to rotate with the black hole because of the curvature of the region. Objects can still escape from the black in this area and they can sometimes escape with more energy then they entered with. This process removes energy from the black hole but it gains energy from any object that enters this region. The black hole or the singularity inside it can not be seen. They can only be detected by looking at the motion of objects that surround the black hole.

Calculating The Light Ray Path

In order to integrate light rays first find the path they will take. Fermat's principle says light rays will always take the path of least time. This path can be found by extremizing distance implied by the Schwarzchild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where $G = c = 1$ and using the Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{x}}.$$

This will produce second order differential equations which can then be integrated to find the path of the light ray.

Using the metric the Lagrange is

$$\mathcal{L} = \frac{1}{2}\left[-\left(1 - \frac{2\mathrm{m}}{\mathrm{r}}\right)\dot{\mathrm{t}}^2 + \frac{\dot{\mathrm{r}}^2}{1 - \frac{2\mathrm{m}}{\mathrm{r}}} + \mathrm{r}^2\dot{\theta}^2 + \mathrm{r}^2 \sin^2\theta\dot{\phi}^2\right]$$

Since $\frac{\partial \mathcal{L}}{\partial t} = 0$ and $\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial t}\right) = 0$ then $\frac{d\mathcal{L}}{dt}$ will also equal a constant which will be set to be -1 Then we can say that

$$\dot{t} = \frac{1}{1 - \frac{2m}{r}}.$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

Then $\frac{d\mathcal{L}}{d\dot{\phi}}$ will equal a constant then

$$\dot{\phi} = \frac{b}{r^2 \sin^2 \theta}.$$

Unlike the previous two variables r and θ have components in the Lagrangian. There is no constant of integration. So we solve

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$$

for $\ddot{r}$. First we take the r derivative of the Lagrange which equals

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{m}{r^2}\dot{t}^2 - \frac{m\dot{r}^2}{(r-2m)^2} + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2.$$

Then take the $\dot{r}$ derivative

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{2\dot{r}}{1 - \frac{2m}{r}}$$

and take the total derivative of that using the product rule

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{2m\dot{r}^2}{(r-2m)^2} + \left(1 - \frac{2m}{r}\right)^{-1}\ddot{r}.$$

Now that we have the derivatives set them equal to each other

$$-\frac{m\dot{t}^2}{r^2} - \frac{m\dot{r}^2}{(r-2m)^2} + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 = -\frac{2m\dot{r}^2}{(r-2m)^2} + \left(1 - \frac{2m}{r}\right)^{-1}\ddot{r}$$

and solve for $\ddot{r}$ which results

$$\ddot{r} = \left(1 - \frac{2m}{r}\right)\left(\frac{-m\dot{t}^2}{r^2} + \frac{m\dot{r}^2}{(r-2m)^2} + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2\right).$$

Now examine the θ equation

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$

for $\ddot{\theta}$. First try to solve the θ derivative and get

$$\frac{\partial \mathcal{L}}{\partial \theta} = 2r^2\sin\theta\cos\theta\dot{\phi}^2$$

and take the $\dot{\theta}$ derivative

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta}$$

so that the total derivative is

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 2r\dot{\theta}^2\dot{r} + r^2\ddot{\theta}.$$

Now set the derivatives equal to each other

$$2r^2 \sin\theta\cos\theta\dot{\phi}^2 = 2r\dot{\theta}^2\dot{r} + r^2\ddot{\theta}$$

and solve for $\ddot{\theta}$

$$\ddot{\theta} = 2\sin\theta\cos\theta\left(\frac{b}{r^2\sin^2\theta}\right)^2 - \frac{2\dot{\theta}\dot{r}}{r}$$

We see that we have two first order ODE's and two second order ODE's. For numerical integration we simply uncouple them to 6 first order OED's. We substitute $\dot{t}$ and $\dot{\phi}$ into the equations to uncouple them to be able to solve these ODE's.

$$\dot{t} = \frac{1}{1-\frac{2m}{r}},$$

$$\dot{\phi} = \frac{b}{r^2\sin^2\theta},$$

$$\dot{r} = v_r,$$

$$\dot{\theta} = v_\theta,$$

$$\dot{v}_r = \left(1-\frac{2m}{r}\right)\left(\frac{-m}{r^2}\left(\frac{1}{1-\frac{2m}{r}}\right)^2 + \frac{mv_r^2}{(r-2m)^2} + rv_\theta^2 + \frac{b^2}{r^3\sin^2\theta}\right),$$

and

$$\dot{v}_\theta = 2\sin\theta\cos\theta\left(\frac{b}{r^2\sin^2\theta}\right)^2 - \frac{2v_\theta\dot{r}}{r}.$$

Four of the equations define the velocity in the (t, r, θ, ϕ) directions. While two of them define the acceleration in the r and θ directions.

Now that we have these first order ODE's we have to set boundary conditions. These conditions will give a starting point for integrating the rays. First start off by setting $\mathcal{L} = 0$ from there we can solve for $\dot{r}$. Doing so gives

$$\dot{r} = \pm\left[\left(1-\frac{2m}{r}\right)\left(\left(1-\frac{2m}{r}\right)\dot{t}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right)\right]^{\frac{1}{2}}.$$

The positive component of $\dot{r}$ heads away from the black hole so it is not useful therefore we will be using the negative part which approaches the black hole. Now the terms under the square root must not be negative because the path can not be imaginary. So set those terms equal to Q then equal to zero

$$Q = \left(1-\frac{2m}{r}\right)\left[\left(1-\frac{2m}{r}\right)\dot{t}^2 - r^2v_\theta^2 - r^2\sin^2\theta\dot{\phi}^2\right] = 0.$$

Having $Q=0$ will allow us to solve for $\dot{\theta}$ which equals

$$v_\theta = \pm\sqrt{\left(1-\frac{2m}{r}\right)\frac{\dot{t}^2}{r^2} - \sin^2\theta\dot{\phi}^2}.$$

Solving for $\dot{\theta}$ gives a range of values

$$-\sqrt{\left(1-\frac{2m}{r}\right)\frac{\dot{t}^2}{r^2} - \sin^2\theta\dot{\phi}^2} < v_\theta < +\sqrt{\left(1-\frac{2m}{r}\right)\frac{\dot{t}^2}{r^2} - \sin^2\theta\dot{\phi}^2}.$$

Setting boundary conditions is important now that have BC's for $\dot{\theta}$ we should find them for $\dot{\phi}$ as well. For v_θ to be real

$$\left(1-\frac{2m}{r}\right)\dot{t}^2 - r^2\sin^2\theta\dot{\phi} > 0$$

$$\dot{\phi}^2 < \frac{\left(1-\frac{2m}{r}\right)\dot{t}^2}{r^2\sin^2\theta}$$

$$b^2 = \frac{r^2\sin^2\theta}{\left(1-\frac{2m}{r}\right)}$$

$$\frac{-r^2\sin^2\theta}{\sqrt{1-\frac{2m}{r}}} < b < +\frac{r^2\sin^2\theta}{\sqrt{1-\frac{2m}{r}}}$$

Therefore b is the constant that controls the initial direction in which the ray travels. Any larger or smaller then the conditions in the above then the light ray will not be a possible light ray.

Surface Vectors

Now that we have the path of these light rays we have to find the suface vectors. To do that we take the parallel propagation equation

$$\ell^a\nabla_a e^b = \ell^a\frac{\partial}{\partial x^a}e^b + \Gamma^b_{ac}\ell^a e^c = 0$$

and solve it for $\dot{e}^b$ is a simplification of $\frac{\partial}{\partial x^a}e^b$ and $\ell^a\frac{\partial}{\partial x^a}e^b$. We have to solve for this vector for each of the coordinates. Γ^b_{ac} can be found in appendix A. Substituting in terms for the connection and solving for $\dot{e}^b$ gives

$$\dot{e}^t = -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\left(\dot{t}e^r + \dot{r}e^t\right),$$

$$\dot{e}^r = \frac{-m}{r^2}\left(1-\frac{2m}{r}\right)\dot{t}e^t + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)\dot{r}e^r + r\left(1-\frac{2m}{r}\right)\dot{\theta}e^\theta + r\sin^2\theta\left(1-\frac{2m}{r}\right)\dot{\phi}e^\phi,$$

$$\dot{e}^\theta = -\frac{1}{r}\left(\dot{r}e^\theta + \dot{\theta}e^r\right) + \sin\theta\cos\theta\dot{\phi}e^\phi,$$

and

$$\dot{e}^\phi = -\frac{1}{r}\left(\dot{r}e^\phi + \dot{\phi}e^r\right) - \cot\theta\left(\dot{\theta}e^\phi + \dot{\phi}e^\theta\right).$$

Now we should consider boundary conditions for these tangent vectors. m^a is complex and it is given by $m^a = e_1 + ie_2$. For the $e_1$which is initially pointed in the ϕ direction we set $e_1^t = 0$, $e_1^\theta = 0$, and $e_1^\phi = 1$. Using

$$g_{ab}\ell^a e_1^b = 0$$

we can find the BC's for e_1^r therefore

$$0 = \frac{v_r e^r}{1-\frac{2m}{r}} + be^\phi$$

$$e_1^r = \frac{-b}{v_r}\left(1-\frac{2m}{r}\right).$$

Next for the vector in the θ direction we set $e_2^t = 0$, $e_2^\theta = 1$, and $e_2^\phi = 0$ and using

$$g_{ab}\ell^a e_2^b = 0$$

we can find e_2^r.

$$0 = \left(1-\frac{2m}{r}\right)^{-1} v_r e_2^r + r^2 v_\theta e_2^\theta$$

$$e_2^r = \frac{-r^2 v_\theta}{v_r}\left(1-\frac{2m}{r}\right).$$

We define Ψ_0 as

$$\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d$$

most of the parts of it are zero. Ψ_0 is the summation of all non-zero parts. The surviving C_{abcd} are

$$C_{0101} = \frac{2m}{r^2}$$

$$C_{0202} = \frac{m\,(2m-r)}{r^2}$$

$$C_{0303} = \frac{m\,(2m-r)}{r^2}\sin^2\theta$$

$$C_{1212} = \frac{-m}{2m-r}$$

$$C_{1313} = \frac{-m \sin^2 \theta}{2m - r}$$

$$C_{2323} = -2mr \sin^2 \theta$$

As an example the first part of Ψ_0 is

$$\frac{2m}{r^2} \left[\ell^t m^r \ell^t m^r - 2\ell^r m^t \ell^t m^r + \ell^r m^t \ell^r m^t\right] + ...$$

Results

Solving the Lagrangian gives the path of the light rays passing the black hole. We were able to vary the path of the ray by changing the initial conditions.

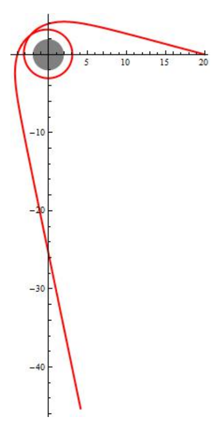


Figure 3: Light ray path

Here we see a light ray in red passing a black hole (the Black circle) at the origin with a radius of 2. It starts at around $(20, 0)$ then heads towards the black hole wraps around once and flies off to negative infinity in the y-direction.

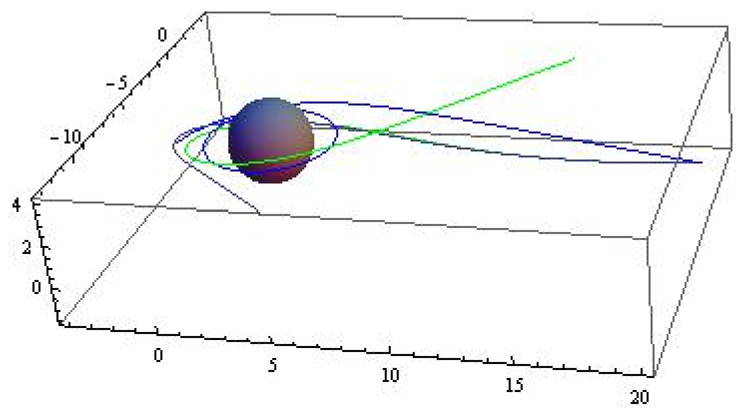


Figure 4: Multiple Light Rays

In the diagram above we see that we can model several light rays in three dimension. Two of the rays start in the plane and circle the black hole at the origin represented by the sphere. The third in green starts above the black hole circles it and heads off the graph. Now that we have the paths we can find out what happens to the vectors tangent to the path.

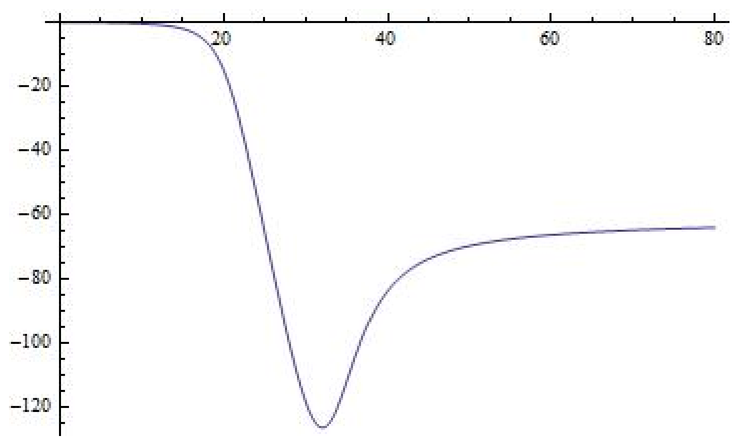


Figure 5: The time component of the e_1 vector

This graph shows how fast the time component of the tangent vector changes. It starts at zero and drops down toward negative 120 then begins to increase and starts to asymptote as it approaches negative 70. The region of drastic change from around 20 to 40 in x represents when the ray is passing near the black hole. Once the ray passes the black hole its effect diminishes and the time component ceases changing

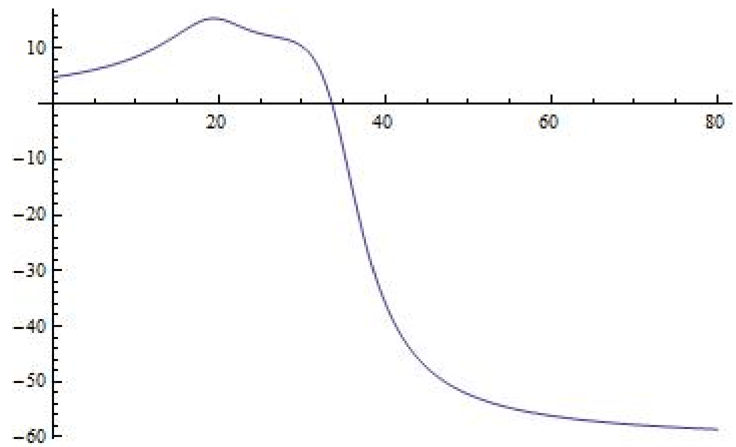


Figure 6: The radial component

In this diagram we see the radial component increase as it approaches the black hole at 20 in x. It then starts to decrease slowly but right before it passes the black it begins to drop drastically only to become stable after it is well away from the black hole.

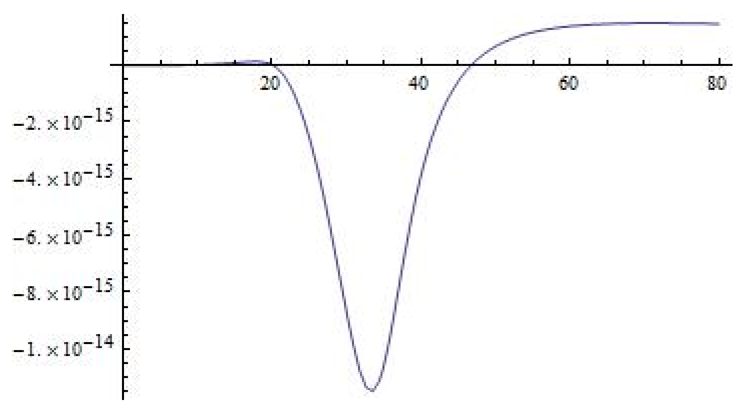


Figure 7: The theta component

While at first glance this graph appears to do something it really does not, the graph is on the scale of 10^{-15} meaning that the theta component it zero which is what we set it to be.

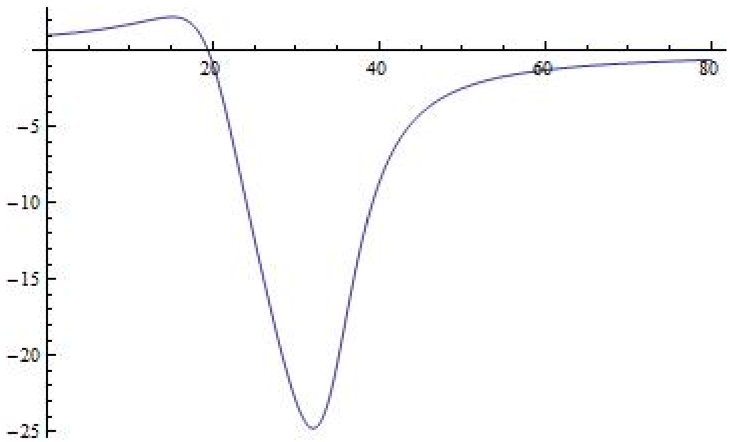


Figure 8: The phi component

This graph show us how the phi component acts as it approaches the black hole. Initially starting at one it then increases as it approaches the black hole at 20 then drops to around negative 25 in y. As it moves away it increases and asymptotes to negative one after it has passed the black hole.

To test if the equations were coded properly we plotted

$$g_{ab}\ell^a e_1^b = 0$$

This will show us whether or not this vector is changing which it should not.

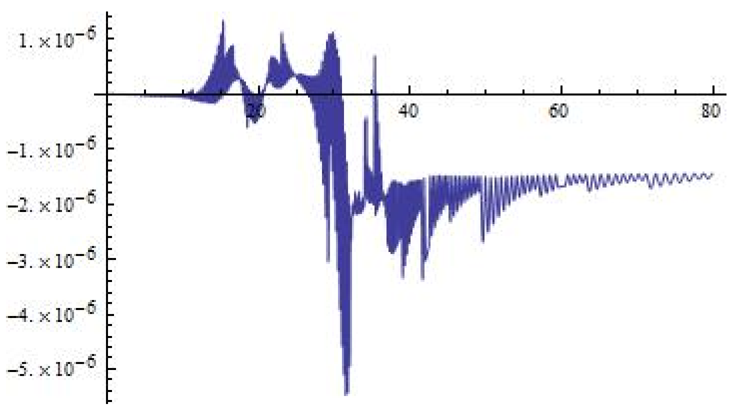


Figure 9: coding test one

This graph shows that the code is working the way it should. The values on the graph are numerical error on the order of 10^{-6}. To test if the path was coded correctly we plotted

$$g_{ab}\ell^a\ell^b = 0.$$

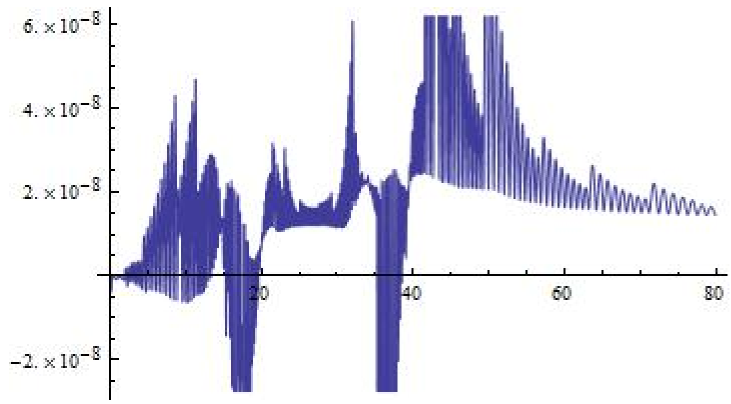


Figure 10: coding test two

Which is zero again with an error on the scale of 10^{-8}.
The end result is that we were able to plot the divergence ρ

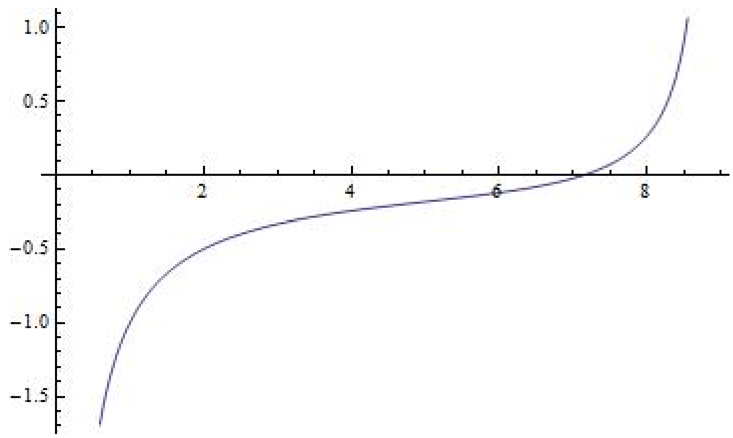


Figure 11: ρ as a function of s

It does what we expected it to do but not where we expected it to happen. ρ is initially at negative infinity and increases with our parameters as it approaches the black it blows up to positive infinity. The problem is that the black hole is around 24-25 in s. As the graph shows us ρ blows up around 7 which is nowhere near the black hole. We are not really sure why this happened there could be errors in our code or math, a problem with our boundary conditions, or we are just not understanding what is actually happening to the divergence.

Conclusion

The purpose of this research was to calculate the shear and divergence of light rays bundles as they pass black holes. We defined a Lagrangian using the Schwarzchild metric then used the Euler-Lagrangian equation to create 6 first order ODE's for the light rays path. Next we found tangent vectors to the light ray so we could calculate Ψ_0 in order to calculate the shear and divergence.

Using mathematica we were able to integrate the path of the light rays using our Lagrangian, allowing us to solve for the tangent vectors which were made of real and imaginary parts $e_1 + ie_2$, having these we were able to calculate Ψ_0. We have not had much success in calculating ρ and σ, they are supposed to blow up when they reach the black hole. This corresponds to the light refocusing due to gravity, essentially it is compressed into a point. We are seeing this refocusing but it happens before the light even reaches the black hole. We do not understand why this is happening although it could be one of three issues. One it could be an error in our code or math. Two, we may have our boundary condition incorrect. Finally, we may not have a proper understanding of what is actually happening when the light refocuses.

Going forward if we manage to solve these errors someone else will be able to use my work to solve real problems. This work was a proof of a concept, using it as a basis someone could use the same process for different metrics allowing for the shear and divergence to be solved for in a variety of different situations. The problem that we have might just mean that this method just does not work as we intended. This research involved much time and effort and would require additional time to complete in a satisfactory manner.

Appendix A

Γ^b_{ac} also known as the connection is used in calculating the parallel propagation vectors to the path of the light rays. For a Schwarzchild metric, the connection components are

$$\Gamma^t_{ij} = \begin{pmatrix} 0 & \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^\phi_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix}$$

$$\Gamma^\theta_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma^r_{ij} = \begin{pmatrix} \frac{m}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{2m}{r}\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{2m}{r}\right) \end{pmatrix}$$

The e^a equations for the vectors from parallel transport are given by

$$\dot{e}^t = -\Gamma^t_{ij}\ell^a m^b$$

$$\dot{e}^r = -\Gamma^r_{ij}\ell^a m^b$$

$$\dot{e}^\theta = -\Gamma^\theta_{ij}\ell^a m^b$$

$$\dot{e}^\phi = -\Gamma^\phi_{ij}\ell^a m^b.$$

Multiplying these equations through and simplifying gives the results in the text above.

Appendix B

```
(* Version of April 17 2015 *)
vrdot = (1 - 2 m / r[s]) ((-1 * m * tdot ^ 2 / (r[s]) ^ 2 + (m * (vr[s]) ^ 2) / (r[s] - 2 m) ^ 2) +
     r[s] * (vtheta[s]) ^ 2 + r[s] * (Sin[theta[s]]) ^ 2 * (phidot) ^ 2)
```

$$\left(1-\frac{2\,\mathrm{m}}{\mathrm{r[s]}}\right)\left(-\frac{\mathrm{m\,tdot}^2}{\mathrm{r[s]}^2}+\mathrm{phidot}^2\,\mathrm{r[s]\,Sin[theta[s]]}^2+\frac{\mathrm{m\,vr[s]}^2}{(-2\,\mathrm{m}+\mathrm{r[s]})^2}+\mathrm{r[s]\,vtheta[s]}^2\right)$$

```
vthetadot = Sin[theta[s]] * Cos[theta[s]] * phidot ^ 2 - 2 * vtheta[s] * vr[s] / r[s]
```

$$\mathrm{phidot}^2\,\mathrm{Cos[theta[s]]\,Sin[theta[s]]}-\frac{2\,\mathrm{vr[s]\,vtheta[s]}}{\mathrm{r[s]}}$$

```
tdot = 1 / (1 - 2 m / r[s])
```

$$\frac{1}{1-\frac{2\,\mathrm{m}}{\mathrm{r[s]}}}$$

```
phidot = b / ((r[s]) ^ 2 * (Sin[theta[s]]) ^ 2)
```

$$\frac{\mathrm{b\,Csc[theta[s]]}^2}{\mathrm{r[s]}^2}$$

```
tdot
phidot
vrdot
vthetadot
```

$$\frac{1}{1-\frac{2\,\mathrm{m}}{\mathrm{r[s]}}}$$

$$\frac{\mathrm{b\,Csc[theta[s]]}^2}{\mathrm{r[s]}^2}$$

$$\left(1-\frac{2\,\mathrm{m}}{\mathrm{r[s]}}\right)\left(\frac{\mathrm{b}^2\,\mathrm{Csc[theta[s]]}^2}{\mathrm{r[s]}^3}-\frac{\mathrm{m}}{\left(1-\frac{2\,\mathrm{m}}{\mathrm{r[s]}}\right)^2\mathrm{r[s]}^2}+\frac{\mathrm{m\,vr[s]}^2}{(-2\,\mathrm{m}+\mathrm{r[s]})^2}+\mathrm{r[s]\,vtheta[s]}^2\right)$$

$$\frac{\mathrm{b}^2\,\mathrm{Cot[theta[s]]\,Csc[theta[s]]}^2}{\mathrm{r[s]}^4}-\frac{2\,\mathrm{vr[s]\,vtheta[s]}}{\mathrm{r[s]}}$$

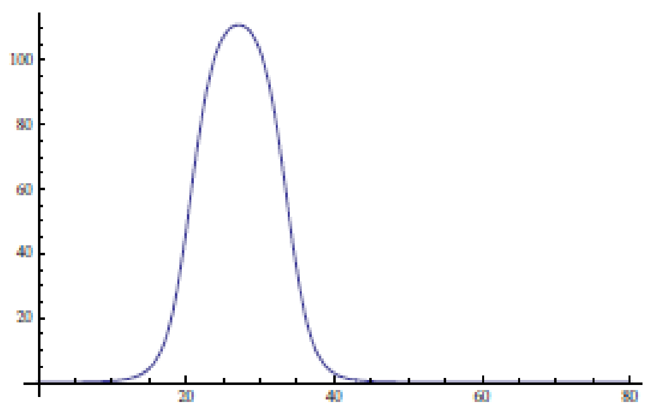
```
(* Setting initial conditions *)

edott = (m / r[s]^2) * (1 / (1 - 2 m / r[s])) * (-tdot * er[s] - vr[s] * et[s]);
edotr = - (m / r[s]^2) * (1 - 2 m / r[s]) * tdot * et[s] +
    (m / r[s]^2) * (1 / (1 - 2 m / r[s])) * vr[s] * er[s] + r[s] * (1 - 2 m / r[s]) * vtheta[s] *
     etheta[s] + r[s] * ((Sin[theta[s]])^2) * (1 - 2 m / r[s]) * phidot * ephi[s];
edottheta = - (1 / r[s]) * (vr[s] * etheta[s] + vtheta[s] * er[s]) +
    (Sin[theta[s]]) * (Cos[theta[s]]) * phidot * ephi[s];
edotphi = - (1 / r[s]) * (vr[s] * ephi[s] + phidot * er[s]) -
    (Cot[theta[s]]) * (phidot * etheta[s] + vtheta[s] * ephi[s]);

edott2 = (m / r[s]^2) * (1 / (1 - 2 m / r[s])) * (-tdot * er2[s] - vr[s] * et2[s]);
edotr2 = - (m / r[s]^2) * (1 - 2 m / r[s]) * tdot * et2[s] +
    (m / r[s]^2) * (1 / (1 - 2 m / r[s])) * vr[s] * er2[s] + r[s] * (1 - 2 m / r[s]) * vtheta[s] *
     etheta2[s] + r[s] * ((Sin[theta[s]])^2) * (1 - 2 m / r[s]) * phidot * ephi2[s];
edottheta2 = - (1 / r[s]) * (vr[s] * etheta2[s] + vtheta[s] * er2[s]) +
    (Sin[theta[s]]) * (Cos[theta[s]]) * phidot * ephi2[s];
edotphi2 = - (1 / r[s]) * (vr[s] * ephi2[s] + phidot * er2[s]) -
    (Cot[theta[s]]) * (phidot * etheta2[s] + vtheta[s] * ephi2[s]);

Q = (1 - 2 m / rinit) ((tdotinit^2) * (1 - 2 m / rinit) -
      (rinit^2) * (vthetainit^2) - (rinit^2 * ((Sin[(thetainit)]^2) * phidotinit^2)) );
F = ((1 - 2 m / rinit) ((tdotinit^2) / (rinit^2) - (Sin[thetainit]^2) * phidotinit^2));

m = 1;
b = 5.197;
rinit = 20;
tinit = 1;
phiinit = 0;
thetainit = Pi / 2;
phidotinit = b / ((rinit)^2 * (Sin[thetainit])^2);
tdotinit = 1 / (1 - 2 / rinit)
(*vthetainit = 0.2*Sqrt[F]*)
vthetainit = 0;
vrinit = -Sqrt[Q]
```

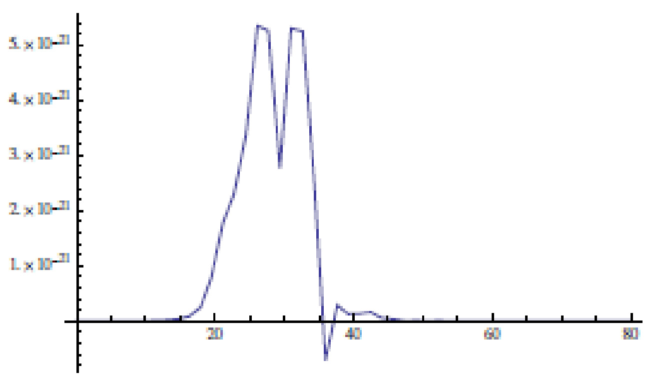
$\frac{10}{9}$

-0.969139

```
solution = NDSolve[{t'[s] == tdot, phi'[s] == phidot, r'[s] == vr[s],
   theta'[s] == vtheta[s], vr'[s] == vrdot, vtheta'[s] == vthetadot,
   et'[s] == edott, er'[s] == edotr, etheta'[s] == edottheta, ephi'[s] == edotphi,
    et2'[s] == edott2, er2'[s] == edotr2, etheta2'[s] == edottheta2, ephi2'[s] == edotphi2,

   t[0] == tinit, phi[0] == phiinit, r[0] == rinit,
   theta[0] == thetainit, vr[0] == vrinit, vtheta[0] == vthetainit, et[0] == 0,
   er[0] == -b / vrinit * (1 - 2 * m / rinit), ephi[0] == 1, etheta[0] == 0, et2[0] == 0, er2[0] ==
    (-((r[s]^2) * vthetainit) / vrinit) * (1 - 2 * m / rinit), ephi2[0] == 0, etheta2[0] == 1},

  {t[s], phi[s], r[s], theta[s], vr[s], vtheta[s], et[s], er[s],
   etheta[s], ephi[s], et2[s], er2[s], etheta2[s], ephi2[s]}, {s, 0, 80}]
{{t[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  phi[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  r[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  theta[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  vr[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  vtheta[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  et[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  er[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  etheta[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  ephi[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  et2[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  er2[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  etheta2[s] -> InterpolatingFunction[{{0., 80.}}, <>][s],
  ephi2[s] -> InterpolatingFunction[{{0., 80.}}, <>][s]}}
sfin = 80;
rsol = First[r[s] /. solution];
phisol = First[phi[s] /. solution];
thetasol = First[theta[s] /. solution];
vrsol = First[vr[s] /. solution];
x = rsol * Sin[thetasol] * Cos[phisol];
y = rsol * Sin[thetasol] * Sin[phisol];
z = rsol * Cos[thetasol];
etsol = First[et[s] /. solution];
ersol = First[er[s] /. solution];
ethetasol = First[etheta[s] /. solution];
ephisol = First[ephi[s] /. solution];
vthetasol = First[vtheta[s] /. solution];
etsol2 = First[et2[s] /. solution];
ersol2 = First[er2[s] /. solution];
ethetasol2 = First[etheta2[s] /. solution];
ephisol2 = First[ephi2[s] /. solution];
mt = etsol + I * etsol2;
mr = ersol + I * ersol2;
mtheta = ethetasol + I * ethetasol2;
mphi = ephisol + I * ephisol2;

tdotsol = 1 / (1 - 2 * m / rsol);
phidotsol = b / ((rsol)^2 * (Sin[thetasol])^2);
```

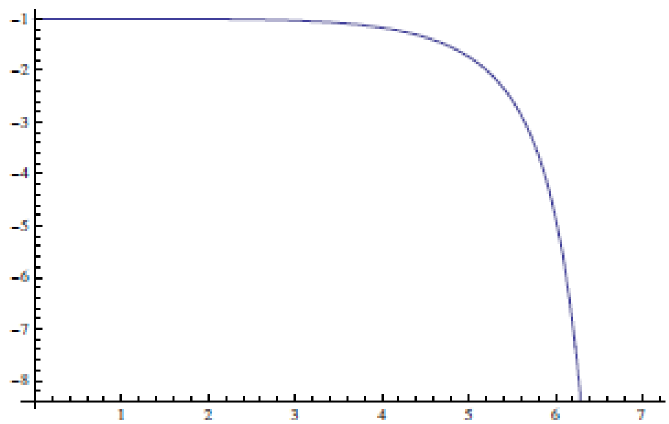
```
(* Tom Version of Psi0 *)
C0101 = 2 * m / rsol ^ 2;
C0202 = m * (2 m - rsol) / rsol ^ 2;
C0303 = m * (2 m - rsol) * (Sin[thetasol]) ^ 2 / rsol ^ 2;
C1212 = -m / (2 m - rsol);
C1313 = -m * (Sin[thetasol]) ^ 2 / (2 m - rsol);
C2323 = -2 m * rsol * (Sin[thetasol]) ^ 2;
(* The 1/4 comes from 1/sqrt(2) in each tetrad vector not included here *)

Psi0 =
  (1 / 4) * (C0101 * (tdotsol ^ 2 * mr ^ 2 - 2 * tdotsol * mr * vrsol * mt + vrsol ^ 2 * mt ^ 2) + C0202
      (tdotsol ^ 2 * mtheta ^ 2 - 2 * tdotsol * mtheta * vthetasol * mt + vthetasol ^ 2 * mt ^ 2) +
     C0303 * (tdotsol ^ 2 * mphi ^ 2 - 2 * tdotsol * mphi * phidotsol * mt + phidotsol ^ 2 * mt ^ 2)
     C1212 * (vrsol ^ 2 * mtheta ^ 2 - 2 * vrsol * mtheta * vthetasol * mr + vthetasol ^ 2 * mr ^ 2)
     C1313 * (vrsol ^ 2 * mphi ^ 2 - 2 * vrsol * mphi * phidotsol * mr + phidotsol ^ 2 * mr ^ 2) +
     C2323 * (vthetasol ^ 2 * mphi ^ 2 -
        2 * vthetasol * mphi * phidotsol * mtheta + phidotsol ^ 2 * mtheta ^ 2));

Psi0re = Re[Psi0];
Psi0im = Im[Psi0];
Psi0replot = Plot[Psi0re, {s, 0, sfin}, PlotRange -> All]
Psi0implot = Plot[Psi0im, {s, 0, sfin}, PlotRange -> All]
```



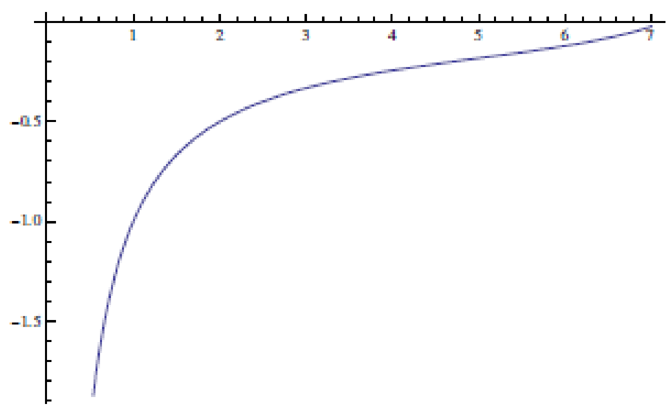



```
udot = -u[s] ^ 2 (sigre[s] ^ 2 + sigim[s] ^ 2) - 1;
sigdotre = 2 sigre[s] / u[s] + Psi0re;
sigdotim = 2 sigim[s] / u[s] + Psi0im;
```

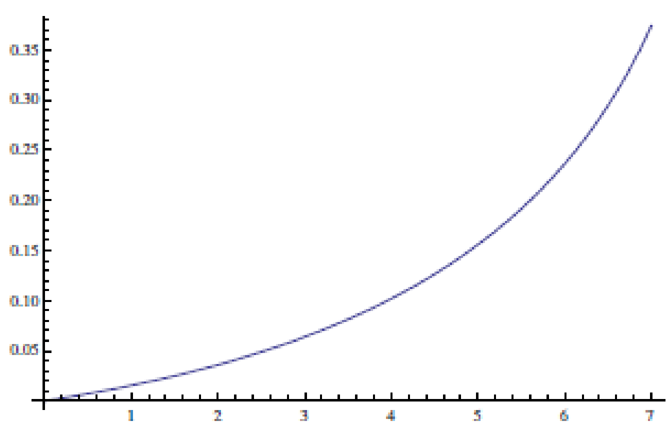
```
Plot[udotval, {s, h, 7.1}]
```



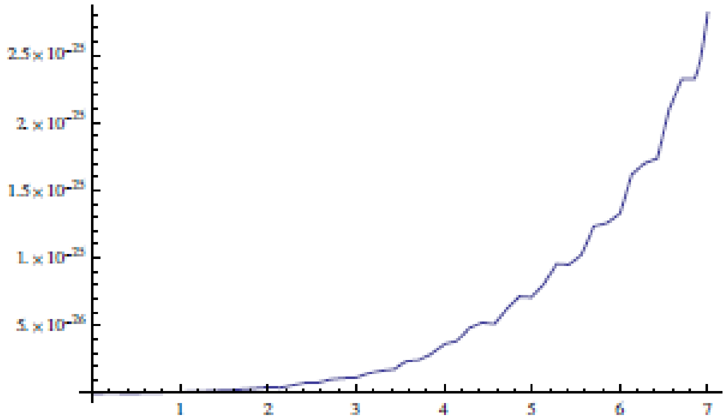

```
Plot[1 / u1, {s, h, 7}, AxesOrigin → {0, 0}]
```




```
Plot[sigma1re, {s, h, 7}]
```



```
Plot[sigma1im, {s, h, 7}]
```




```
rhostart1 = First[(1 / u[s]) /. scalars1 /. s → 7]
sigma1restart = First[sigre[s] /. scalars1 /. s → 7]
sigma1imstart = First[sigim[s] /. scalars1 /. s → 7]
```

```
-0.0233056
```

```
0.374375
```

2.80993×10^{-25}

```
rhodot = rho[s] * rho[s] + sigre1[s] * sigre1[s] + sigim1[s] * sigim1[s];
sigdotim1 = 2 * rho[s] * sigim1[s] + Psi0im;
sigdotre1 = 2 * rho[s] * sigre1[s] + Psi0re;
```

```
scalars3 = NDSolve[{rho'[s] == rhodot, sigre1'[s] == sigdotre1, sigim1'[s] == sigdotim1,
     rho[7] == rhostart1, sigre1[7] == sigma1restart, sigim1[7] == sigma1imstart},
    {rho[s], sigre1[s], sigim1[s]}, {s, 7, sfin}];
```

NDSolve::ndsz : At s == 8.926952622995424`, step size is effectively zero; singularity or stiff system suspected. >>

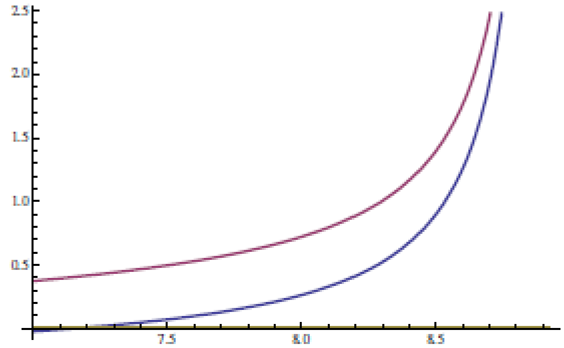
```
x /. s → 8.9
y /. s → 8.9
```

```
11.3312
```

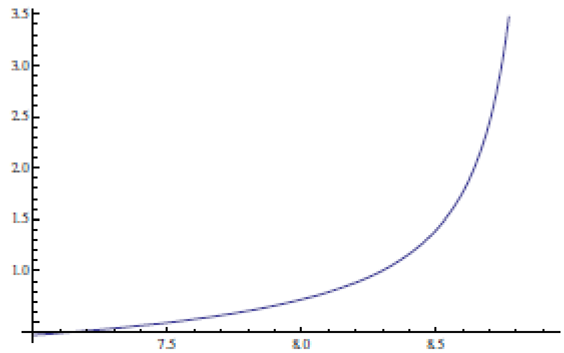
```
2.30946
```

```
rho2 = First[rho[s] /. scalars3];
sigma2re = First[sigre1[s] /. scalars3];
sigma2im = First[sigim1[s] /. scalars3];
```

```
Plot[{rho2, sigma2re, sigma2im}, {s, 7, 8.92}, PlotStyle → Thick]
```



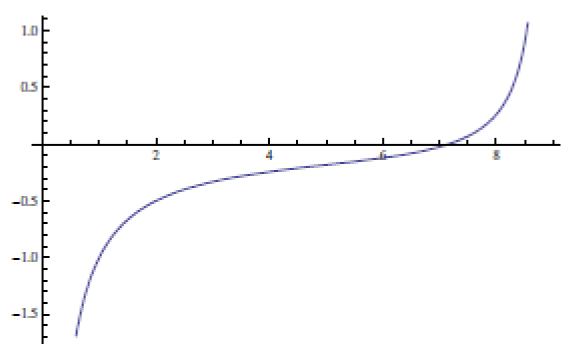

```
Plot[sigma2re, {s, 7, 8.92}]
```



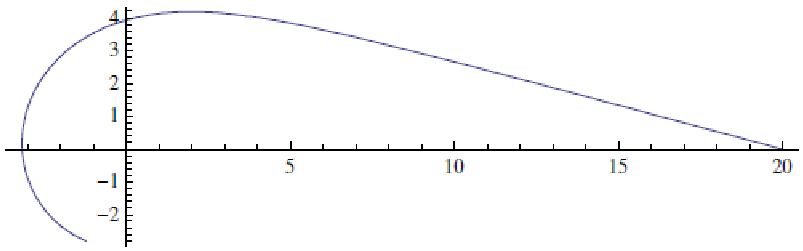

```
pwrho = Piecewise[{{1 / u1, s < 7}, {rho2, s > 7}}]
```

$$\begin{cases} \frac{1}{\text{InterpolatingFunction}[\{\{0.00001, 7.15319\}\}, <>][s]} & s < 7 \\ \text{InterpolatingFunction}[\{\{7., 8.92695\}\}, <>][s] & s > 7 \\ 0 & \text{True} \end{cases}$$

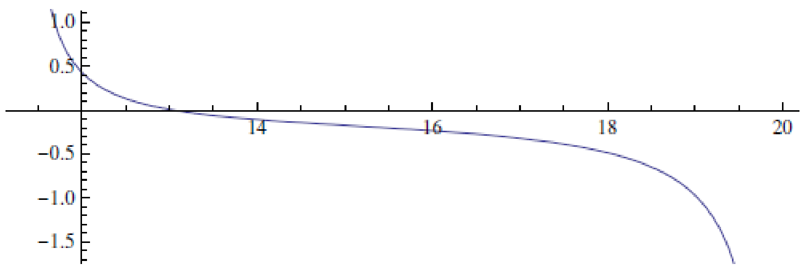
```
Plot[pwrho, {s, h, 8.92}, AxesOrigin → {0, 0}]
```



```
ParametricPlot[{x, y}, {s, 0, 25}]
```




```
ParametricPlot[{x, pwrho}, {s, h, 8.91}]
```



References

[1] "Lecture Notes on General Relativity - S. Carroll." Lecture Notes on General Relativity - S. Carroll. N.p., n.d. Web. 22 Apr. 2015.

[2] Atwater, H. A. Introduction to General Relativity. Oxford: Pergamon, 1974. Print.

[3] Foster, J., and J. D. Nightingale. A Short Course in General Relativity. London: Longman, 1979. Print.

[4] Grocutt, Emma. "What Is Gravitational Lensing?" CHFTLenS. N.p., n.d. Web. 09 Sept. 2014. http://www.cfhtlens.org/public/what-gravitational-lensing

[5] Penrose, Roger, and Wolfgang Rindler. Spinors and Space-time. Vol. 2. Cambridge: Cambridge UP, 1986. Print.

[6] Redd, Nola T. "Einstein's Theory of General Relativity." Space.com. N.p., 18 Sept. 2012. Web. 09 Sept. 2014.

[7] Thornton, Stephen T., and Jerry B. Marion. Classical Dynamics of Particles and Systems. Belmont, CA: Brooks/Cole, 2004. Print.

[8] Wald, Robert M. General Relativity. Chicago: U of Chicago, 1984. Print.