Author(s): WILLIAM PARRY and MARK POLLICOTT
Article Title: An analogue of Bauer's theorem for closed orbits of skew products
Year of publication: 2008
Link to published version: http://dx.doi.org/10.1017/S0143385707000557
Publisher statement: None

Ergod. Th. & Dynam. Sys. (2008), **28**, 535–546
doi: 10.1017/S0143385707000557

An analogue of Bauer's theorem for closed orbits of skew products

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(*Received* 26 *April* 2007 *and accepted in revised form* 24 *July* 2007)

Dedicated to the memory of Bill Parry

Abstract. In this article we prove an analogue of Bauer's theorem from algebraic number theory in the context of hyperbolic systems.

0. *Introduction*

Let $\phi : X \to X$ be a hyperbolic dynamical system and, for a finite G, let $\widetilde{\phi} : \widetilde{X} \to \widetilde{X}$ be a G-extension for which $\widetilde{\phi}$ is also hyperbolic. Given ϕ we are interested in describing the possibilities for $\widetilde{\phi}$ in terms of the closed ϕ-orbits.

This is analogous to a classical problem in number theory which asks for a description of all finite Galois extensions of an algebraic number field in terms of its prime ideals. This lies at the heart of class field theory. Let L be a Galois extension of a number field K (i.e. one for which the automorphism group fixing K is transitive). A prime ideal p for K corresponds to an ideal for L, which may be a product of prime ideals from L. We say that p splits if it is a product of distinct prime ideals in L. The following result characterizes a Galois extension in terms of which primes split in it [**6**].

BAUER'S THEOREM. *A Galois extension L of K is uniquely determined by the set of prime ideals that split in it (i.e. if $L_1, L_2 \supset K$ are Galois extensions and an ideal p in K splits in L_1 if and only if it splits in L_2 then L_1 is isomorphic to L_2).*

We can use the analogy of a dynamical system to a number field whereby a G-extension (covering) corresponds to a Galois extension and where closed orbits play the role of primes (cf. [**11, 12, 14**]). In particular, in this note we want to consider analogues of Bauer's theorem for dynamical systems. We are interested in how local information on closed prime orbits and their Frobenius elements classifies skew products.

† Bill Parry died before this paper was completed. I have attempted to write the paper as we had originally planned. All errors are entirely my responsibility. (M.P.)

DYNAMICAL BAUER'S THEOREM. *Let $\phi : X \to X$ be a hyperbolic dynamical system and let $\widetilde{\phi}_i : \widetilde{X}_i \to \widetilde{X}_i$ be a finite G_i covering for which $\widetilde{\phi}$ is also hyperbolic. Assume that a closed ϕ-orbit τ splits with respect to $\widetilde{\phi}_1$ precisely when it splits for $\widetilde{\phi}_2$. Then there exists an isomorphism $\alpha : G_1 \to G_2$.*

This can be viewed as the analogue of the number theoretic result via the dictionary in Table 1, between dynamical systems and algebraic number theory.

TABLE 1. A dictionary.

Dynamical systems	Number theory
$X =$ space	$K =$ number field
$\tilde{X} =$ covering space for X	$L =$ algebraic extension of K
$\tau =$ closed orbit for ϕ (of least period $\lambda(\tau)$)	$p =$ prime ideal for K
$N(\tau) = \exp(\lambda(\tau))$	$N(p) =$ norm of p
$\tilde{\tau} =$ closed orbit for $\tilde{\phi}$	$P =$ prime ideal for K
τ splits	p splits

This can be compared with the well-known dictionary between Riemann surfaces and number theory (cf. [**11, 12, 14**]).

In §1, we present the main result in the context of subshifts of finite type. In §2, we consider some related results for subshifts. In §3, we consider the particular case of abelian extensions, where it is possible to have a more complete understanding of the possible extensions. In §4, we extend these results to hyperbolic diffeomorpisms and flows. Finally, in §5 we compare these results with those known in the particular case of geodesic flows and graphs.

1. *Skew products for subshifts of finite type*

We begin by considering a simple dynamical setting.

Let $\sigma : \Sigma \to \Sigma$ be a mixing subshift of finite type with finite state space $\{0, \ldots, k-1\}$. More precisely, let A be an aperiodic $k \times k$ matrix with 0–1 entries and let

$$\Sigma = \left\{ x = (x_n)_{-\infty}^{\infty} \in \prod_{-\infty}^{\infty} \{0, \ldots, k-1\} : A(x_i, x_{i+1}) = 1, \forall i \in \mathbb{Z} \right\}.$$

With the Tychonoff product topology, Σ is compact and totally disconnected. We define a shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_n = x_{n+1}$.

Let G be a finite group. Consider a skew product $\sigma_f : \Sigma \times G \to \Sigma \times G$ associated to a continuous function $f : \Sigma \to G$, defined by $\sigma_f(x, g) = (\sigma x, f(x)g)$. Since f is locally constant, we can assume after recoding the shift, if necessary, that it only depends on the first two coordinates (i.e. $f(x) = f(x_0, x_1)$). We can associate to a periodic orbit $\tau = \{x, \sigma x, \ldots, \sigma^{n-1}x\}$ for σ its period $|\tau| = n$ and a *Frobenius element*

$$[\tau]_G = f^n(x) := f(\sigma^{n-1}x) \cdots f(\sigma x) f(x).$$

Here $[\tau]_G$ is only defined up to conjugacy, depending on the choice of x within the orbit. However, when $[\tau]_G = 1$ it is well defined.

We say that a closed orbit is prime if it is not the concatenation of the same orbit two or more times. We say that a closed (prime) orbit τ for $\sigma : \Sigma \to \Sigma$ *splits* with respect to f if $[\tau]_G = 1$.

Let G_1 and G_2 be finite groups and let $f_1 : \Sigma \to G_1$ and $f_2 : \Sigma \to G_2$ be two continuous functions and consider the skew products

$$\sigma_{f_1} : \Sigma \times G_1 \to \Sigma \times G_1 \text{ given by } \sigma_{f_1}(x, g) = (\sigma x,\ f_1(x)g) \quad \text{and}$$
$$\sigma_{f_2} : \Sigma \times G_2 \to \Sigma \times G_2 \text{ given by } \sigma_{f_2}(x, g) = (\sigma x,\ f_2(x)g).$$

We can also consider σ_{f_j} as a subshift of finite type with symbol spaces (i, g), where $g \in G_j$ $(j = 1, 2)$. Let us assume that σ_{f_1} and σ_{f_2} are mixing.

The following is a dynamical analogue of Bauer's theorem in the context of subshifts of finite type.

THEOREM 1. *Consider two mixing skew products $\sigma_{f_i} : \Sigma \times G_i \to \Sigma \times G_i$, $i = 1, 2$, over the same subshift of finite type $\sigma : \Sigma \to \Sigma$. Assume that a closed σ-orbit τ splits with respect to f_1 precisely when it splits for f_2. Then:*

(i) *there exists an isomorphism $\alpha : G_1 \to G_2$; and, furthermore,*

(ii) *$\alpha \circ f_1, f_2 : \Sigma \to G_2$ are cohomologous (i.e. there exists a continuous function $u : \Sigma \to G_2$ such that $\alpha \circ f_1 = u \circ \sigma f_2 u^{-1}$).*

In algebraic number theory, the standard proof of Bauer's theorem uses the Chebotarev theorem on equidistribution. Perhaps surprisingly, the proof of Theorem 1 does not depend on the dynamical analogue of the Chebotarev theorem [**9**]. We shall present two different proofs of Theorem 1, the first of which does not use the dynamical Chebotarev theorem and the second of which does.

An immediate corollary of Bauer's theorem is that L_1, L_2 have the same Dedekind zeta function. By analogy with this, Theorem 1 has the following immediate consequence. We associate to σ_f a zeta function defined by

$$\zeta(z, \sigma_f) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Card}(\mathrm{Fix}_n(\sigma_f))\right)$$

where $\mathrm{Fix}_n(\sigma_f)$ denotes the number of points fixed under the n-fold composition of σ_f. This is the Artin–Mazur zeta function [**1**] for σ_f. Since σ_f is also a subshift of finite type, it is well known that $\zeta(z, \sigma_f)$ is the reciprocal of a polynomial.

COROLLARY. *Assume that τ splits with respect to f_1 precisely when it splits with respect to f_2. Then $\sigma_{f_1} = \sigma_{f_2}$ have the same zeta function, i.e. $\zeta(z, \sigma_{f_1}) = \zeta(z, \sigma_{f_2})$.*

Proof of Corollary. Since, by the theorem, $\alpha \circ f_1 = u \circ \sigma f_2 u^{-1}$ we see that a σ_{f_1}-periodic point (x, g) corresponds to a σ_{f_2}-periodic point $(x, \alpha(g))$. □

2. *A proof of Theorem 1*

In this paper we shall give two proofs of Theorem 1. The first, which we present in this section, is more direct in that it does not require the dynamical Chebotarev theorem. It is convenient to introduce a formal group which will prove central to our analysis.

Definition. Fix a symbol, say zero, in the symbol space $\{0, \ldots, k-1\}$. We can consider the set of allowed words $w = 0x_1x_2 \ldots x_n0$ (i.e. $x_1, \ldots, x_n \in \Sigma$ satisfying $A(0, x_1) = A(x_1, x_2) = \cdots = A(x_{n-1}, x_n) = A(x_n, 0) = 1$) beginning and ending with the symbol zero. This becomes a semi-group under the operation of concatenation, i.e.

$$0x_1x_2 \ldots x_n0 \star 0x_1'x_2' \ldots x_m'0 \mapsto 0x_1x_2 \ldots x_n0x_1' \ldots x_m'0.$$

Let us introduce relations $w^N = I$, for any $w \in W$, where I is the identity element and N is the least common multiple of the cardinalities $|G_1|$ and $|G_2|$ of the groups. We then have the presentation

$$W = \langle w = 0x_1x_2 \ldots x_n0 : w^N = I\rangle.$$

LEMMA 2.1. *W is a group.*

Proof. By construction we already have that W is a semi-group. The identity is the trivial element and for each w we have an inverse $w^{-1} = w^{N-1}$. □

As before, we can assume without loss of generality that $f_1(x) = f_1(x_0, x_1)$ and $f_2(x) = f_2(x_0, x_1)$. We can define homomorphisms $\alpha_i : W \to G_i$ by

$$\alpha_i(0x_1x_2 \ldots x_n0) = f_i(0, x_1) f_i(x_1, x_2) \cdots f_i(x_{n-1}, x_n) f_i(x_n, 0),$$

for $i = 1, 2$. If $w = 0x_1x_2 \ldots x_n0$ then we can abbreviate the right-hand side of the above identity to $f_i(w)$, for $i = 1, 2$. By definition, this is well defined on the original semi-group. Moreover, since $f_i(w^N) = f_i(w)^N = 1$, for $i = 1, 2$, we have that the relations on the groups are respected.

LEMMA 2.2.
(1) *The homomorphisms $\alpha_i : W \to G_i$ are surjective, $i = 1, 2$.*
(2) *Under the hypothesis of Theorem 1,* $\ker(\alpha_1) = \ker(\alpha_2)$.

Proof. Part (1) comes from the assumption that σ_{f_i} is mixing. In particular, for any $g \in G_i$ we can find $w = 0x_1 \ldots x_n0$ such that $f_i(0, x_1) \cdots f_i(x_n, 0) = g$.

We next turn to the proof of part (2). We need to know that $\alpha_1(w) = 1$ if and only if $\alpha_2(w) = 1$, for all $w \in W$. By hypothesis, this is true for prime orbits and so we need only extend this to non-prime orbits. If $w \in W$ is not a prime orbit then we can consider the concatenation ww' with a second word w', chosen so that $\alpha_i(w') = 1$, for $i = 1, 2$, and ww' is a prime orbit. It is easy to see that this can always be done. By construction, $\alpha_i(ww') = \alpha_i(w)\alpha_i(w') = \alpha_i(w)1 = \alpha_i(w)$, for $i = 1, 2$. Thus, $\alpha_1(w) = 1$ if and only if $\alpha_1(ww') = 1$ if and only if $\alpha_2(ww') = 1$ if and only if $\alpha_2(w) = 1$, as required. □

To prove part (i) of Theorem 1, we show that G_1 and G_2 are isomorphic by observing that

$$G_1 \equiv_{\alpha_1^{-1}} W/\ker(\alpha_1) = W/\ker(\alpha_2) \equiv_{\alpha_2} G_2.$$

To prove part (ii) of Theorem 1, we construct a conjugacy relating the functions f_1 and f_2 using a modified form of the non-abelian Livsic theorem.

LEMMA 2.3. (Non-abelian Livsic theorem) [**8**] *If σ is transitive and $\alpha(f_1)$, f_2 give the same weights to closed orbits (at least those containing the symbol zero) then $\alpha(f_1)$, f_2 are cohomologous.*

Proof. Recall that we have assumed (without loss of generality) that each $f_i(x)$ depends only on the two terms x_0, x_1 in the sequence $x = (x_n)_{n=-\infty}^{\infty}$. Again we fix a symbol zero and since σ is mixing we can choose for each symbol j an admissible word $w^+(j) = 0 j_1 \ldots j_n j$ (i.e. $A(0, j_1) = A(j_1, j_2) = \cdots = A(j_{n-1}, j_m) = A(j_n, j) = 1$) and denote

$$f_i(w^+(j)) = f_i(0, j_1) f_i(j_1, j_2) \cdots f_i(j_n, j) \in G_i.$$

Similarly, we can choose an admissible word $w^-(j) := j l_1 \ldots l_m 0$ and denote

$$f_i(w(j)) = f_i(j, l_1) f_i(l_1, l_2) \cdots f_i(l_m, 0) \in G_i.$$

Whenever $A(j, j') = 1$ we see that the concatenation of $w^-(j)$, (j, j') and $w^+(j)$ corresponds to a closed orbit and we have by hypothesis that

$$f_2(w^+(j)) f_2(j, j') f_2(w^-(j')) = \alpha(f_1(w^+(j)) f_1(j, j') f_1(w^-(j'))) \quad \text{and}$$
$$f_2(w^+(j')) f_2(w^-(j')) = \alpha(f_1(w^+(j')) f_1(w^-(j')))$$

which gives that

$$f_2(j, j') = h(j) \alpha(f_1(j, j')) h(j')^{-1}$$

where $h(j) = f_1(w^+(j))^{-1} \alpha(f_1(w^+(j))$, as required. □

For $w \in W$ we have by construction $\alpha(\alpha_1(w)) = (\alpha_2 \alpha_1^{-1}) \alpha_1(w) = \alpha_2(w)$, where $\alpha : G_1 \to G_2$ is the isomorphism defined in part (1). In particular, since w corresponds to a periodic point, Lemma 2.3 applies and we deduce that $\alpha(f_1)$, f_2 are cohomologous. This completes the proof of the theorem. □

For later use, we denote by $W(1)$ the common kernel.

Remark. The definition of the group W was, at least in part, motivated by the group Γ_P, introduced in [**10**]. More precisely, Γ_P is defined by the products

$$P(0, x_1) P(x_1, x_2) \cdots P(x_{n-1}, x_n) P(x_n, 0),$$

where P is a Markov matrix, and it is shown to be a finitary isomorphism invariant for the associated Markov shift.

The next theorem describes the situation when the skew products are not assumed to be mixing.

THEOREM 2. *Consider two skew products $\sigma_{f_i} : \Sigma \times G_i \to \Sigma \times G_i$, $i = 1, 2$, over the same shift $\sigma : \Sigma \to \Sigma$. Assume that a closed σ-orbit τ splits with respect to f_1 precisely when it splits respect to f_2. Then there exist*

(i) *an isomorphism $\alpha : G_1' \to G_2'$ between subgroups $G_1' < G_1$ and $G_2' < G_2$, and*

(ii) *continuous functions $f_1' : \Sigma \to G_1'$ and $f_2' : \Sigma \to G_2'$ with $f_1' \sim f_1$, $f_2' \sim f_2$,*

such that $f_2' \sim \alpha(f')$.

The proof of this theorem makes use of the following result.

LEMMA 2.4. *Consider a mixing component for σ_f. By adding coboundaries, if necessary, this component takes the form $\Sigma \times G$ where G is a subgroup of G, i.e. there exists $f' : \Sigma \to G'$ such that $f \sim f'$ and $\sigma_{f'} : \Sigma \times G' \to \Sigma \times G'$ is ergodic.*

This lemma is proved in [**8**].

Proof of Theorem 2. By Lemma 2.4 we can choose a cohomologous function $f_1' \sim f_1$ so that an ergodic component of $\sigma_{f_1'} : \Sigma \times G_1 \to \Sigma \times G_1$ takes the form $\Sigma \times G_1'$. Similarly, for some $f_2' \sim f_2$ an ergodic component of $\sigma_{f_2'} : \Sigma \times G_2 \to \Sigma \times G_2$ takes the form $\Sigma \times G_2'$. Moreover, the skew products $\sigma_{f_1'} : \Sigma \times G_1' \to \Sigma \times G_1'$ and $\sigma_{f_2'} : \Sigma \times G_2' \to \Sigma \times G_2'$ retain the property that a closed σ-orbit τ splits with respect to f_1' precisely when it splits with respect to f_2' (since the property of the weight of a closed orbit being one is unchanged under adding coboundaries). Thus, we can apply Theorem 1 to $\sigma_{f_1'} : \Sigma \times G_1' \to \Sigma \times G_1'$ and $\sigma_{f_2'} : \Sigma \times G_2' \to \Sigma \times G_2'$ to deduce the result. □

3. *The Chebotarev-type theorems*

In this section we shall present a proof of Theorem 1 which is closer in spirit to that of Bauer's theorem in algebraic number theory. This shows a close connection with the dynamical version of the Chebotarev theorem, which we recall below [**6**]. Moreover, this has the additional advantage that it yields results for homogeneous extensions. We recall the following result, which describes the proportion of prime ideals that split.

CHEBOTAREV'S THEOREM. *Given a Galois extension L of K, with Galois group G, the prime ideals p which split have density $1/|G|$, i.e.*

$$\lim_{n\to+\infty} \frac{\mathrm{Card}\{p : p \text{ splits and } N(p) \le n\}}{\mathrm{Card}\{p : N(p) \le n\}} = \frac{1}{|G|}.$$

To prove Theorem 2, we shall use the following version of a dynamical analogue of the Chebotarev theorem, which is a particular case of a result due to the present authors [**9**].

PROPOSITION 3.1. (Dynamical Chebotarev theorem) *Let σ_f be a mixing skew product. The closed σ-orbits τ which split have density $1/|G|$, i.e.*

$$\lim_{n\to+\infty} \frac{\mathrm{Card}\{\tau : |\tau| \le n,\ [\tau]_G = 1\}}{\mathrm{Card}\{\tau : |\tau| \le n\}} = \frac{1}{|G|}.$$

We shall now show how to deduce Theorem 1 from Proposition 3.1. Consider the skew product $\sigma^{\overline{f}}$ associated to $\overline{f} = (f_1, f_2) : \Sigma \to G_1 \times G_2$. Typically, this will not be mixing but applying Lemma 2.4 with $G = G_1 \times G_2$ there exists a cohomologous function $\overline{f}' \sim \overline{f}$, with $\overline{f}' : \Sigma \to H$ valued in $H \subset G_1 \times G_2$ such that $\sigma_{\overline{f}'} : \Sigma \times H \to \Sigma \times H$ is mixing. The canonical projections $\pi_i : H \to G_i$ $(i = 1, 2)$ are homomorphisms and, since σ_{f_1} and σ_{f_1} are assumed to be mixing, they are surjective.

Each closed σ-orbit $\tau = \{x, \sigma x, \ldots, \sigma^{n-1}x\}$ has weights

$$f_i(\tau) := f_i(\sigma^{n-1}x) \cdots f_i(\sigma x) f_i(x), \quad i = 1, 2$$

and, by hypothesis, the set of closed σ-orbits τ with weights $f_1(\tau) = 1 \in G_1$ is the same as those for which $f_2(\tau) = 1 \in G_2$, and thus the same for those for which $\overline{f}'(\tau) = 1 \in H$.

By the Chebotarev theorem the densities are $1/|G_1| = 1/|G_2| = 1/|H|$. In particular, $\pi_1 : H \to G_1$ and $\pi_2 : H \to G_2$ are isomorphisms, simply because the groups have the same cardinality. By symmetry, G_2 is isomorphic to H and thus $\alpha := \pi_2\pi_1^{-1} : G_1 \to G_2$ is an isomorphism. Finally, Lemma 2.3 applies again to show that $\alpha(f_1)$, f_2 are cohomologous. □

One advantage of this proof is that it extends to give a modest result for homogeneous spaces. Assume that we consider a subgroup $H < G$ and the space of right cosets G/H. The associated homogeneous extension $\sigma_f : \Sigma \times G/H \to \Sigma \times G/H$ is then defined by $\sigma_f(x, gH) = (\sigma x, f(x)gH)$. In this case, we say that a closed σ-orbit τ splits when lifted to $\Sigma \times G/H$ if $f(\tau) \in H$. Clearly, this gives less information than knowing that τ splits in G, and so $f(\tau) = 1$. However, modifying the above proof we see that we have the following.

PROPOSITION 3.2. *Consider two mixing homogeneous extensions $\sigma_{f_i} : \Sigma \times G_i/H_i \to \Sigma \times G_i/H_i$, $i = 1, 2$, over the same subshift of finite type $\sigma : \Sigma \to \Sigma$. Assume that τ splits with respect to f_1 precisely when it splits for f_2. Then $|G_1/H_1| = |G_2/H_2|$.*

More precisely, the Chebotarev theorem for homogeneous extensions corresponding to Proposition 3.1 gives that the proportion of orbits τ which split is $|H|/|G|$ [**7**]. Consider the homogeneous extension $\sigma_{\overline{f}}$ associated to $\overline{f} = (f_1, f_2) : \Sigma \to G_1/H_1 \times G_2/H_2$. Typically, this will not be mixing but applying Lemma 2.4 there exists a cohomologous function $\overline{f}' \sim \overline{f}$, with $\overline{f}' : \Sigma \to H_0$ valued in $H_0 \subset G_1 \times G_2$ such that $\sigma_{\overline{f}'} : \Sigma \times H_0/H_1 \times H_2 \to \Sigma \times H_0/H_1 \times H_2$ is mixing. The canonical projections $\pi_i : H_0/H_i \to G_i/H_i$ $(i = 1, 2)$ are homomorphisms since σ_{f_1} and σ_{f_2} are assumed to be mixing.

By hypothesis, the set of closed σ-orbits τ with weights $f_1(\tau) \in H_1 \subset G_1$ is the same as those for which $f_2(\tau) \in H_2 \subset G_2$, and thus the same for those for which $\overline{f}'(\tau) \in (H_1 \times H_2) \cap H$. By the Chebotarev theorem the densities are $|H_1|/|G_1| = |(H_1 \times H_2) \cap H|/|H| = |H_2|/|G_2|$, as required. □

4. *Abelian extensions*

In the case of abelian Galois groups the classification of number fields is well understood (although for non-abelian extensions very little is known). Correspondingly, in the particular case of extensions by finite abelian groups G we have stronger results than in the previous section. We begin by considering the analogous problem of which finite abelian groups can occur for G in the dynamical context.

Given a continuous function $f : \Sigma \to G$ we can associate the formal group W as defined in §2.

PROPOSITION 4.1. *Consider any cofinite subgroup $W' < W$ for which $H := W/W'$ is abelian. Then we can find $f : \Sigma \to H$ such that $W' = W(1)$.*

Proof. Since W is abelian, a typical element of this group can be represented by

$$\frac{0x_1 \ldots x_n 0}{0y_1 \ldots y_n 0}.$$

For each symbol i we can choose a word $l(i)$ beginning at zero and ending at i and a word $r(i)$ beginning at i and ending at zero. Consider the element of W defined by

$$F(i, j) = \frac{l(i)(i, j)r(j)}{l(i)r(i)} \in W.$$

We can then define a function of two variables $f : \Sigma \to W/W'$ by $f(i, j) = F(i, j)W'$ with values in W/W'. Note that $\alpha(0i_1 \dots i_k 0) = (0i_1 \dots i_k 0)$, so that $\alpha(0i_1 \dots i_k 0) = (0i_1 \dots i_k 0)W'$. Hence the kernel $W(1) = \ker(\alpha)$ is equal to W'. □

5. *Hyperbolic systems*

The results on skew products over subshifts of finite type can be extended to skew products over hyperbolic systems using subshifts of finite type.

Let $\phi : M \to M$ be a diffeomorphism on a compact manifold. Let $X \subset M$ be a closed ϕ-invariant set. We say that $\phi : X \to X$ is hyperbolic if there exists a $D\phi$-invariant splitting $T_X M = E^s \oplus E^u$ and $C > 0$, $0 < \lambda < 1$ such that:

(i) $\|D\phi^n v\| \leq C\lambda^n \|v\|$ for $v \in E^s$, $n \geq 0$;

(ii) $\|D\phi^{-n} v\| \leq C\lambda^n \|v\|$ for $v \in E^u$, $n \geq 0$.

Moreover, we want to assume that:

(iii) $X = \bigcap_{n=-\infty}^{\infty} \phi^{-n} U$ for some open neighbourhood $U \supset X$;

(iv) the periodic orbits are dense in X.

Finally, we shall require that $\phi : X \to X$ is mixing.

Under these hypotheses, we can model $\phi : X \to X$ by a subshift of finite type using a standard result due to Bowen [**2**].

PROPOSITION 5.1. *There exists a subshift of finite type $\sigma : \Sigma \to \Sigma$ and a continuous surjective map $\pi : \Sigma \to X$ such that:*

(1) *$\pi\sigma = \phi\pi$; and*

(2) *π is bounded-to-one.*

The proof involves the use of Markov partitions T_j, $j = 1, \dots, k$ (which correspond to the images of 1-cylinders, of the form $T_j = \pi(\{x \in \Sigma : x_0 = j\})$). The elements of these partitions can be chosen arbitrarily small. The interiors of these sets are arranged to be disjoint and a periodic ϕ-point which does not lie in the boundary corresponds to a unique periodic σ-orbit.

Let $\widetilde{\phi} : \widetilde{X} \to \widetilde{X}$ be a hyperbolic transformation and let G be a freely acting finite group on $\widetilde{X}$ such that $\widetilde{\phi} g = g\widetilde{\phi}$ for all $g \in G$. We say that a closed ϕ-orbit τ splits with respect to $\widetilde{\phi}$ if it lifts to a closed $\widetilde{\phi}$-orbit of the same period. The analogue of Theorem 1 is the following result.

THEOREM 3. *Let $\widetilde{\phi}_1 : \widetilde{X}_1 \to \widetilde{X}_1$ and $\widetilde{\phi}_2 : \widetilde{X}_2 \to \widetilde{X}_2$ be mixing hyperbolic transformations. Assume that there are free actions of finite groups G_i on $\widetilde{X}_i$ $(i = 1, 2)$ with a common quotient $\phi : X \to X$. If a closed ϕ-orbit τ splits with respect to $\widetilde{\phi}_1$ precisely when it splits for $\widetilde{\phi}_1$ then G_1 and G_2 are isomorphic.*

Proof. Using Proposition 5.1, we can associate to $\phi : X \to X$ a shift space $\sigma : \Sigma \to \Sigma$. We can consider lifts of the Markov sections for X to $\widetilde{X}_i$ $(i = 1, 2)$. These can be identified with $T_j \times \{g\}$, where $j = 1, \ldots, k$ and $g \in G_i$. Moreover, assuming the elements of the Markov partition are chosen sufficiently small, we can define locally constant functions $f_i : \Sigma \to G_i$ $(i = 1, 2)$ such that with this identification we can write $\widetilde{\phi}_i(x, g) = (\phi x, f_i(x))$. In particular, we can model the maps $\widetilde{\phi}_i : \widetilde{X}_i \to \widetilde{X}_i$ by skew products $\sigma_{f_i} : \Sigma \times G_i \to \Sigma \times G_i$.

We need to check that the splitting hypothesis for ϕ translates into a corresponding hypothesis for σ, and thus that Theorem 1 applies to give the result. Assume that $\sigma^n x = x$ corresponds to the string $0x_1 \ldots x_{n-1}0$ and that $\alpha_i(0x_1 \ldots x_{n-1}0) = e$. If the closed orbit τ containing $\pi(x)$ does not lie on the boundary of an element of the Markov partition then it has ϕ-period n and $f_i^n(x) = [\tau]_{G_i}]$ $(i = 1, 2)$. However, if $\xi = \pi(x)$ lies in the boundary of an element of the Markov partition, then n may no longer be the *prime* period of τ. However, in this case we can choose a second periodic point $\sigma^m x' = x'$ corresponding to the string $0x'_1 \ldots x'_{m-1}0$, say, such that neither its image $\xi' = \pi(x')$ nor the image $\xi = \pi(xx'^N)$ of the concatenation $x * x'^N$ lie on the boundaries of the Markov partition.

If $f_i^n(x) = e$ then $f_i^{n+mN}(x * x'^N) = f_i^n(x)[f_i^n(x')]^N = f_i^n(x) = e$. By construction, the periodic orbit for ξ' splits with respect to $\widetilde{\phi}$. Thus, by hypothesis, we deduce that $f_1^n(x) = e$ if and only if $f_2^n(x) = e$. We can now apply Theorem 1 to deduce that G_1 and G_2 are isomorphic. □

There is an analogous theory for hyperbolic flows. Let $\phi_t : M \to M$ be a flow on a compact manifold. Let $X \subset M$ be a closed ϕ-invariant set. We say that $\phi_t : X \to X$ is hyperbolic if there exists a $D\phi_t$-invariant splitting $T_X M = E^0 \oplus E^s \oplus E^u$ where E^0 is a one-dimensional bundle tangent to the flow direction and there exist $C > 0$, $\lambda > 0$ such that:

(i) $\|D\phi_t v\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^s$;

(ii) $\|D\phi_{-t} v\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^u$.

Moreover, we want to assume that:

(iii) $X = \bigcap_{t=-\infty}^{\infty} \phi_{-t} U$ for some open neighbourhood $U \supset X$;

(iv) the periodic orbits are dense in X.

Finally, we shall require that $\phi_t : X \to X$ is mixing.

We can model a hyperbolic flow $\phi_t : X \to X$ by a suspended flow for a subshift of finite type. Let $\sigma : \Sigma \to \Sigma$ be a mixing subshift of finite type and let $r : \Sigma \to \mathbb{R}^+$ be a Hölder continuous function. We write

$$\Sigma^r = \{(x, u) \in \Sigma \times \mathbb{R} : 0 \le u \le r(x)\}$$

where $(x, r(x))$ and $(\sigma x, 0)$ are identified. We define a suspended flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ locally by $\sigma_t^r(x, u) = (x, u + t)$.

We can now use the following standard result due to Bowen [**3**].

PROPOSITION 5.2. *There exists a shift $\sigma : \Sigma \to \Sigma$, a Hölder function $r : \Sigma \to \mathbb{R}$ and a continuous map $\pi : \Sigma^r \to X$ such that:*

(1) $\pi\sigma_t^r = \phi_t\pi$; *and*

(2) π *is bounded-to-one.*

The proof involves the use of Markov sections T_i, $i = 1, \ldots, k$, transverse to the flow, which correspond to images $\pi(\{x \in \Sigma : x_0 = i\} \times \{0\})$. As for diffeomorphisms, a periodic ϕ-point which does not intersect the boundary corresponds to a unique periodic σ^r-orbit.

As for diffeomorphisms, we say that a closed ϕ-orbit τ splits with respect to $\widetilde{\phi}$ if it lifts to a closed $\widetilde{\phi}$-orbit of the same period.

THEOREM 4. *Let $\widetilde{\phi}_1 : \widetilde{X}_1 \to \widetilde{X}_1$ and $\widetilde{\phi}_2 : \widetilde{X}_2 \to \widetilde{X}_2$ be two mixing skew products. Assume that there are free actions of finite groups G_i on $\widetilde{X}_i$ $(i = 1, 2)$ with a common quotient $\phi : X \to X$. Assume that a closed ϕ-orbit τ splits with respect to $\widetilde{\phi}_1$ precisely when it splits for $\widetilde{\phi}_2$. Then G_1 and G_2 are isomorphic.*

Proof. Using Proposition 5.2, we can associate to $\phi : X \to X$ a suspension flow $\sigma_t^r : \Sigma^r \to \Sigma^r$. We can consider the lifts $T_i \times G_i$ of the Markov sections to $\widetilde{X}_1$ and $\widetilde{X}_2$, respectively. Moreover, providing the elements of the Markov partition are sufficiently small, we can model the flows $\widetilde{\phi}_{i,t} : \widetilde{X}_i \to \widetilde{X}_i$ by a suspension flow, using the function $r : \Sigma \times G_i \to \mathbb{R}$ defined by $r(x, g) = r(x)$, over the skew product $\sigma_{f_i} : \Sigma \times G_i \to \Sigma \times G_i$. Assume that $\sigma^n x = x$ corresponds to the string $0x_1 \ldots x_{n-1}0$ and that $\alpha_i(0x_1 \ldots x_{n-1}0) = e$. Since $\pi\sigma_t^r = \phi_t\pi$ the image $\xi = \pi(x, 0)$ lies on a closed ϕ-orbit τ of period $r^n(x)$. However, if $\phi_t(\xi)$ lies in the boundary of an element of the Markov section, for some $t \in \mathbb{R}$, then this may not be the *prime* period. As in the proof of Theorem 3, we can choose a second periodic point $\sigma^m x' = x'$ corresponding to the string $0x_1' \ldots x_{m-1}'0$ such that neither the image $\xi' = \pi(x', 0)$ nor the image $\xi = \pi(xx'^N, 0)$, using the concatenation $x * x'^N$, lie on the boundaries of the Markov sections. As before, if $f_i^n(x) = e$ then $f_i^{n+mN}(x * x'^N) = e$ and the periodic orbit τ' for ξ' splits with respect to $\widetilde{\phi}_i$. By construction, and the original hypothesis, we can deduce that $f_1^n(x) = e$ if and only if $f_2^n(x) = e$. We can now apply Theorem 1 to deduce that G_1 and G_2 are isomorphic. □

6. *Application to geodesic flows and graphs*

In this final section we want to place our results into the context of surfaces and graphs. There is an interesting precedent for this in the work of Sunada and Buser on the construction of isospectral examples, based on Exercise 6.4 from [**5**, pp. 362–363] for non-Galois field extensions. Interestingly, this exercise is used to illustrate the need for the Galois hypothesis in Bauer's theorem, which appears as Exercise 6.1 in the same section.

Let V_1 and V_2 be two compact surfaces of negative curvature which are covers for a common third surface V_0, say. Assume that the covering groups are G_1 and G_2, respectively. We say that a closed geodesic γ on V_0 splits with respect to G_i if and only if it lifts to a closed geodesic on V_i of the same length $(i = 1, 2)$.

PROPOSITION 6.1. *Assume that V_1, V_2 are regular covers for V_0. If we assume that a closed geodesic on V_0 splits with respect to V_1 if and only if it splits with respect to V_2 then G_1 and G_2 are isomorphic.*

Proof. We can consider the associated geodesic flows, which are hyperbolic. Since closed geodesics correspond to closed orbits for the geodesic flows, the result follows easily from Theorem 4. □

We compare this with the well-known construction of Sunada of pairs of surfaces with a common cover. Let H be a finite group [**13**]. Two subgroups $H_1, H_2 \subset H$ are called *almost conjugate* if for any $g \in H$ with conjugacy class $[g] \subset H$ we have that $|[g] \cap H_1| = |[g] \cap H_2|$. Recall that two negatively curved surfaces are isospectral if they have the same number of closed geodesics of any given length.

PROPOSITION 6.2. (Sunada's theorem) *Let V be compact negatively curved surfaces upon which a finite group H acts by isometries and let H_i, $i = 1, 2$, be subgroups which act freely. Assume that subgroups $H_1, H_2 \subset H$ are almost conjugate; then the factors $V_i = V/H_i$ of V are isospectral.*

For an excellent account of this theorem we refer the reader to [**4**].

There are analogous results for graphs. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two finite graphs which are covers for a third graph $\mathcal{G}_0$, with covering groups G_1 and G_2, respectively. We say that a closed path γ on $\mathcal{G}_0$ splits with respect to G_i if it lifts to a closed path on $\mathcal{G}_i$ of the same length ($i = 1, 2$).

PROPOSITION 6.3. *Assume that $\mathcal{G}_1$, $\mathcal{G}_2$ are regular covers for $\mathcal{G}_0$. If we assume that a closed path on $\mathcal{G}_0$ splits with respect to $\mathcal{G}_1$ if and only if it splits with respect to $\mathcal{G}_2$ then we can deduce that G_1 and G_2 are isomorphic.*

Proof. We can associate to each of the graphs a subshift of finite type. Since closed paths in the graph correspond to closed orbits for the shift map, the result follows immediately from Theorem 1. □

We can compare this with analogues of Sunada's result for graphs, as studied by Buser, cf. [**4**] . Recall that two graphs are isospectral if they have the same number of closed loops of any given length.

PROPOSITION 6.4. (Buser's theorem) *Let $\mathcal{G}$ be a finite connected graph upon which a finite group G acts by isometries and let H_i, $i = 1, 2$, be subgroups which act freely. Assume that subgroups $H_1, H_2 \subset H$ are almost conjugate; then the factors $\mathcal{G}_i = \mathcal{G}/H_i$ of $\mathcal{G}$ are isospectral.*

Acknowledgements. M.P. is very grateful to Richard Sharp for his helpful comments on the exposition. He also thanks the referee for his suggestions.

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