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Triangle Trek

Thomas E. Moore
Bridgewater State College, MOORE@bridgew.edu

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by Thomas E. Moore
Department of Mathematics & Computer Science

Viking ships rode the tides in Boston Harbor in September, 1991 in a re-creation of a fanciful trek that Leif Erickson's crew might have made as they explored South along the coast of Vinland. Of course this was just a Scandinavian tease, coming as it did just before the quincentennial celebration of Columbus' discovery of the lands of the Western hemisphere. Native American peoples might well deem both explorations as instances of *rediscovery.*

And that is the point of departure for what follows.

I will tell, more aptly retell, of two examples of discovery/rediscovery in a world of the imagination, a mathematical *terra incognita*. The first tale should be familiar to mathematicians and historians of science but deserves a wider audience. The second tale, which I humbly include, is an anecdote of a person exploration.

Pascal, the Precious Mirror and other Precursors

Blaise Pascal (1623-1662) of Clermont and Paris, France is justly famous inside and outside mathematics. A philosopher and prose-stylist without equal, he made fundamental contributions to geometry and probability. His name is even linked to the history of computer science by his invention at age 19 of a mechanical adding machine (which was, however, a commercial failure). A contemporary of the great jurist-mathematician Pierre de Fermat of Toulouse, it is their correspondence on matters mathematical that prompts us today to call them the co-founders of probability theory. However in this connection we must mention that Girolamo Cardano (1501-1576), in his 1560 book on gambling, *De Ludo Aleae*, gave the formal definition of the probability of an event and so perhaps he deserves a share of the title.

More to the point of this tale is the artifact known today as "Pascal's Triangle." It is a sequence of numbers that occurs in many aspects of calculation including algebra, combinatorics, (the theory of counters), and number theory. Its connection with coin tosses, dice sums and card deals made it inevitable that it appear in the analysis Pascal was urged to do by a friend on games of chance. The actual object we find printed in Pascal's posthumously published *Traite´ du triangle arithme´tique* (1665) in the following form:

1 1 1 1 1 1...
1 2 3 4 5......
1 3 6 10.......
1 4 10.........
1 5..............
1.................
....

The fundamental step of construction, other than placing 1s on the top and left borders is that *any other number is the sum of the number before it and the number above it*. (The reader may pause here and extend it both horizontally and vertically. Beware! Seduced by may patterns in the triangle you may be gone-exploring for quite some time.)

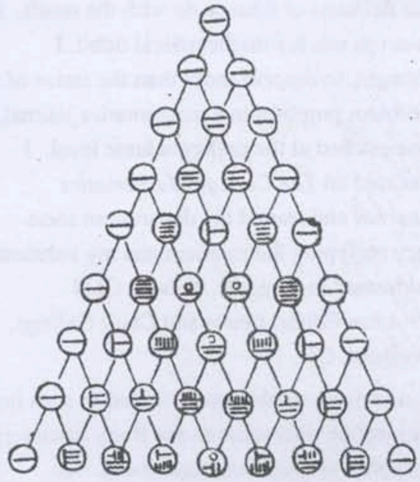
The triangle is usually presented in modern mathematics textbooks in the following form and it is this version that we will refer to as "Pascal's triangle" in the remainder of this article.

1
1 2
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
....

In this presentation a number, other than the obligatory 1s, is obtained by adding each two consecutive numbers already present in a row (such as 3 + 3) and placing the result (observe the location of the 6) between these two summands, in the next row down.

The occurrence of the triangle in Pascal's *Traite´* in 1665 was by no means its first appearance in print.

In China the mathematician Chu-Shih-Chieh printed his book *The precious mirror of the four elements* in 1303 containing the following diagram: XMAS TREE HERE



Chu was concerned with binomial expansions such as we learn in elementary algebra. A few examples will make the connection.

$(a + b)^0 = 1$

$(a + b)^1 = 1a + b$

$(a + b)^2 = 1a^2 + ab + 1b^2$

$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

This list begins a systematic calculation of binomial expansions. The patterns exhibited here suggest the general result known as the binomial theorem, which is useful not only in algebra but in probability as well. For our present purposes we just point out one of these patterns. If the reader will suppress all symbols on the right hand side of the equalities above, except the constant coefficients, then we are left (on the right) with:

1

1 1

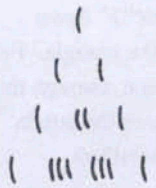
1 2 1

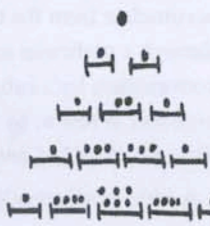
1 3 3 1

....

Even Chu wrote that the triangle was ancient knowledge in his own time and refers to a Chinese work of 1050 A.D., not extant.

The triangle also appears in the work of Persian mathematicians such as Al-Tusi (1200 - 1275) in his charmingly entitled Collection on Arithmetic by means of board and dust [1]. On Jordanus de Nemore (circa 1225) in his *De arithmetica* constructed the triangle and noted some patterns. Manuscript copies of the work offer various manifestations of the triangle such as the two that follow.





Its first appearance in a European book was in an arithmetic of 1557 by Peter Apian, an astronomer at the University of Ingoldstadt. The triangle also occurs in the *General trattato* (1556) of Niccolo Fontana (Tartaglia) as well as in the *Opus novum de proportionibus* (1570) of Cardano.

In these works it is the triangle's connection with dice throws that is important. In order to put this connection in front of the reader, we make a table which shows in how many ways up to 6 dice can make a certain total (by adding the spots that show). Thus one die can make a sum of 1,2,3,4,5, or 6 and can do each in only one way. But two dice can sum from 2 through 12. They can deliver the sum of 2 only by throwing "snake eyes" (1&1) while a 3 can be had in two ways (1&2 as well as 2&1). The sum of 7 is the most frequent total obtainable with two dice, in six ways. Three dice allow a sum of 3 in one way but a sum of 4 in three ways (1&1&2 and 1&2&1 and 2&1&1). And so on.

This leads to the following table.

Sum	**1**	**2**	**3**	**4**	**5**	**6**
1	**1**	**1**	**1**	**1**	**1**	**1**
2	**0**	**1**	**2**	**3**	**4**	**5**
3	**0**	**0**	**1**	**3**	**6**	**10**
4	**0**	**0**	**0**	**1**	**4**	**10**
5	**0**	**0**	**0**	**0**	**1**	**5**
6	**0**	**0**	**0**	**0**	**0**	**1**

For example, the next to last row shows that five dice give a sum of 5 in only one way but give a sum of 6 in five ways. If the reader suppresses the 0s in the table, the Pascal's triangle emerges, now appearing as the columns of this table.

So, with the many appearances of the arithmetical triangle in print long before Pascal's 1665 treatise, why do modern authors insist on calling it "Pascal's triangle?"

A survey of the works alluded to above that contain the triangle and appeared prior to Pascal's treatise shows that the use of the triangle in those works is narrower in scope than that of Pascal. In fact Pascal's treatise includes all the classic applications: the construction of figurate numbers, the theory of counting (combinatorics), the expansion of binomials to powers, as well as its use in dividing the stakes in unfinished games of chance. Moreover, as a work of pure mathematics it is penetrating in its discoveries of the relationships among the numbers in the triangle. These relationships are not merely set out as claims to the truth but proved to be the truth. Indeed on such proof employs the technique, attributed to Pascal and now called mathematical induction, which is perhaps the key proof method in computer science.

Therefore, despite the earlier manifestations of the triangle, we do indeed proclaim it as "Pascal's triangle."

A personal reminiscence of the triangle

In teaching Topics in Mathematics to liberal arts majors I usually include a brief foray into probability and this necessarily

• *Triangle Trek - Continued from previous page.*

introduces my students to "Pascal's triangle."

Now there are many relationships among the numbers in the triangle and observing patterns among them and generalizing to a conjecture is a good exercise in inductive thinking. For example, the next figure indicates the sum of the numbers in each row. The results are suggestive.

Row 0 (so-called)............1...............sums to 1.

Row 1 1   1...........sums to 2.

Row 2 1   2   1........sums to 4.

Row 3............1   3   3   1......sums to 8.

Row 4........1   4   6   4   1..sums to 10.

Looking more closely at the totals, we notice $16=(2)(2)(2)(2)=2^4$. So we can say that row 4 sums to 2^4. Likewise the total in row 3 is 8 and $8 = (2)(2)(2)=2^3$ so that row 3 sums to 2^3. Each total above has a similar description. We therefore conjecture that what we have observed in these first rows of the triangle will persist throughout each and every one of the infinite number of rows of the triangle. That is, each row sums to a power of 2. More precisely, row n sums to 2^n.

It is this observation of patterns that is at the heart of inductive mathematical thinking and the triangle is highly valued as a rich source of such patterns.

In attempting to lead the students to similar observations in a recent version of this course, I had them focus on the distribution of odd numbers in the triangle. They easily spotted rows in which all the numbers are odd, for instance rows 0,1,3,7, and 15. Row 7 of the triangle, for example, consists of the odd numbers 1 7 21 35 35 21 71. The also observed a pattern in the corresponding row names. For example, just as $7=8-1=2^3-1$ and $15=16-1=2^4-1$, all the rows found to have only odd numbers had row names of the form 2^k-1.

Again I mention the importance of the triangle as a resource for problems of pattern observation and inductive thinking, quite apart from its applications to gambling or probability.

A more subtle observation results from counting the number of odd numbers as the accumulate from the topmost "1" down through a particular row in the triangle. For convenience let's call this total, through the numbers in row n, by the name Oddtotaln. Thus Oddtotal2=5 and Oddtotal3=9.

A little work revealed the data in Table A below.

Table A

Row Name n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Oddtotal n	**1**	**3**	**5**	**9**	**11**	**15**	**19**	**27**	**29**	**33**	**37**	**45**	**49**	**57**	**65**	**81**	**83**

My eye was attracted to the subtotals 1,3,9,27,81 which are successive powers of 3, without an omission. The is, $1=3^0$, $3=3^1$, $9=3^2$, $27=3^3$ and so on. This was new to me! Excitement ran through me, perhaps not unlike that shared by the men aboard the vessels of Columbus at their landfall, as I stepped onto *terra mathematica*. My conjecture sprang to life a s the statement, "Let k be any nonnegative integer. Then Oddtotaln = 3^k is solvable for $n=2^k-1$." I should point out that the importance of all of this for me, more generally for any mathematician, is not that a discovery like this will have any practical application. That would be an unlooked for bonus. Rather it is the discovery process itself. It is simply the search for truth, albeit mathematical truth. And the excitement is the rush we used to refer to as the thrill of the chase!

In due course I had a proof of my conjecture (another discovery process) and faced the decision of what to do with the result. It was too much a mathematical tidbit, I thought, to deserve more than the status of a problem proposal in a mathematics journal, one pitched at the undergraduate level. I decided on *The College Mathematics Journal* and started the department secretary on typing the problem and my solution, addressed to Roger B. Nelson, CMJ Problem Editor, Lewis and Clark College, Portland, OR.

As this was going on I decided to scan my immediate references to see if my discovery had after all been someone else's. An article in a 1988 issue of *The Mathematical Intelligencer* by Marta Sved, University of Adelaide, South Australia, did mention my result, unmotivated and without proof but, and this is the kicker, referred to its occurrence *as a problem proposal* submitted by her to the journal *Mathematics Magazine*! Sure enough, as I quickly checked, it was my problem, posed in a 1985 issue.

What could I salvage here?

Perhaps the solution to her proposal would differ from mine sufficiently to warrant publication after all. Checking a subsequent issue of the same journal revealed no fewer than three different solutions sent in by the readership. Moreover one of them was exactly like mine and submitted by none other than ... Roger B. Nelson of Lewis and Clark College! How embarrassing my submission would have been - but I was in time to pluck it from the tray of outgoing mail!

This then was my brush with the triangle and priorities in mathematics. At least one lesson should be drawn. *Do not, dear reader, boldly go where others probably have gone before!* ❧