NETS AND TRANSLATION NETS OF HIGHER DIMENSIONS

GANESH SWAMINATHAN

(MS in CST, University of Mysore)

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Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Ganesh Swaminathan

June 9, 2015

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Contents

Abstract

Finite Geometry has been one of the interesting areas of research in the field of Combinatorics and has seen tremendous advancement in the 20^{th} century after the introduction of *Finite Nets* by R. H. Bruck in 1951. In the early chapters of the thesis, we survey the concept of "*Nets (finite) and Translation Nets*" in two dimension and in three dimension. We also try to fill in the gaps found and formulate equivalent mathematical definitions. Some of the drawbacks in Laskar's definition for *nets of dimension three* led to the redefinition by fixing the parameters involved. Then, the concept of "*partial congruence partition*" in three dimension, denoted by $PCP^{(3)}$, is introduced and the equivalence of $PCP^{(3)}$ and translation nets of dimension three is also proved. The latter chapters cover the concept of Association Scheme and extend the definition of a net and partial congruence partition to n-dimension. Several new parameters for "*Association Scheme of class 3*" are also derived.

In the first chapter, finite geometry and incidence relation are introduced, followed by one of the major topics in finite geometry called *finite affine planes*. The generalization of finite affine planes are given by the concept called ***nets***, introduced by R. H. Bruck. Some of the issues with basic results of Bruck's net are addressed by adding a new axiom. Important results about nets are highlighted and then our focus shifts to a different way of looking at nets, which is, the well-known *latin squares*. A formal proof of equivalence between *nets* and *a set of mutually orthogonal latin squares* is also presented.

Translation nets (in two dimension) are introduced in the second chapter. We present a proof of the equivalence of *partial congruence partition* and *translation nets* in two dimension. The proof is slightly different from the original proof given by A. P. Sprague in 1982. Upper bounds for certain parameters of the nets are then discussed and supporting examples are given.

The discussion about three dimensional case begins in the third chapter with the definition of incidence relation in three dimension and the original definition of *nets of dimension three* (we call it as 3-nets), introduced by R. Laskar in 1971. We address some of the issues found in the Laskar's definition by redefining one of the conditions and modifying the initial condition for the parameters used. Later, it is proved that the new definition recovers the old definition. Some of the results about special cases are also discussed at the end of the chapter.

We introduce the concept of "*partial congruence partition*" in three dimension *($PCP^{(3)}$)* in the fourth chapter. The proof of the equivalence between $PCP^{(3)}$ and *translation nets of dimension three* is presented in this chapter. An example for construction of (4,3,7)-$PCP^{(3)}$ is given, followed by a generalization of such construction. Upper bounds for some of the parameters of 3-nets have also been established along with suitable examples.

Chapter five emphasizes the concept of "*Association scheme*". The the relationship between class three association scheme and a 3-net is revisited. Then, we calculate certain new parameters for "*Association Scheme of class 3*". Chapter six extends the concept of finite nets to n-dimension. In this last chapter we also define *partial congruence partition in n-dimension.* Thus, paving the way to extend the translation nets to any arbitrary finite dimension. We conclude by quoting some of the results which can be extended to n-dimension.

Chapter 1

Introduction

1.1 Finite Affine Planes

Any geometry that studies only a *finite number of points* is said to be a ***finite geometry***. One of the well-studied class of finite plane geometries is the class of *affine planes* (defined in 1.2). The existence of non-intersecting lines in finite affine planes plays a vital role in these studies. We are interested in the generalization of this category of finite geometry where we can define *parallel classes* of lines.

Since we are dealing with finite geometry, any system we define throughout the thesis will have only finite number of points and only finite number of any other objects (*e.g.,* lines, planes, etc).

Definition 1.1 *(Incidence Relation) A plane geometry (2-dimensional) is an incident structure* $(\mathcal{X}, \mathcal{L}, \mathcal{I})$ *where,*

(a) $\mathcal{X}$ *is the set of points and* $\mathcal{L}$ *is the set of lines;*

(b) $\mathcal{X}$ *and* $\mathcal{L}$ *are disjoint sets; and*

(c) $\mathcal{I} \subseteq \mathcal{X} \times \mathcal{L}$ *is the incident relation between points and lines.*

In this thesis, we adopt most of the commonly used terms in the study of geometries without explicitly defining them. For example, the following are equivalent ways to describe the relation $(p, l) \in \mathcal{I}$ for $p \in \mathcal{X}$ and $l \in \mathcal{L}$.

1. A point p is on a line l.
2. A line l passes through a point p.
3. A point p is incident with a line l.
4. A line l contains a point p.

The following are equivalent ways to describe the ***parallel*** relation between lines $l, l' \in \mathcal{L}$, *i.e.,* either $l = l'$ or $\{\ p \in \mathcal{X} : (p, l) \in \mathcal{I}\ \} \bigcap \{\ p \in \mathcal{X} : (p, l') \in \mathcal{I}\ \} = \emptyset$.

1. For any two lines l and l', either they are the same or they have no point in common.
2. Two lines l and l' are parallel (or symbolically $l \parallel l'$).
3. For any two lines l and l', either they are the same or they do not meet.

1.1.1 Definitions and Example

Definition 1.2 *(Finite Affine Plane) A **finite affine plane** is an incidence structure $A = (\mathcal{X}, \mathcal{L}, \mathcal{I})$, satisfying the following conditions:*

(A1) Any two distinct points lie on precisely one line.

i.e., for any $p, p' \in \mathcal{X}$, there exists a unique line $l \in \mathcal{L}$ such that $(p, l) \in \mathcal{I}$ and $(p', l) \in \mathcal{I}$.

(A2) Any line has at least two points on it.

i.e., for any $l \in \mathcal{L}$, $|\{\ p \in \mathcal{X} : (p, l) \in \mathcal{L}\ \}| \geq 2$.

(A3) Whenever a point p is not on a line l, there is precisely one line l' passing through p and having no common point with l.

i.e., for any $p \in \mathcal{X}$ and $l \in \mathcal{L}$ with $(p, l) \notin \mathcal{I}$, there exists a unique line $l' \in \mathcal{L}$ such that $l \parallel l'$ and $(p, l') \in \mathcal{I}$.

(A4) There exists three non-collinear points.

i.e., there exists three distinct points $p_1, p_2, p_3 \in \mathcal{X}$ such that, for all $l \in \mathcal{L}$, $(p_i, l) \notin \mathcal{X}$ for some $i \in \{ 1, 2, 3 \}$.

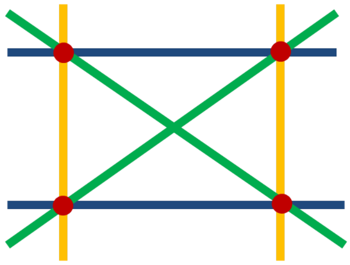
Remark 1.3 *(A2) and (A4) are used to eliminate the trivial cases (e.g., no lines or no points or single line with all the points) whereas the rest of the conditions determines the structure of the geometry. (A1) and (A2) eliminates the scenarios of points not on any line and lines with no points respectively.*

Remark 1.4 *(A3) guarantees the existence of parallel lines and it is often known as **Playfair's axiom**.*

Remark 1.5 *$|\mathcal{X}| \geq 4$ and $|\mathcal{L}| \geq 6$. In words, the minimum number of points and lines in a finite affine plane are 4 and 6 respectively. Example 1.7 shows the simplest form of this geometry.*

Definition 1.6 *(Order of a finite affine plane) We define the **order** of a finite affine plane to be the number of points on a line. We shall later show that for a finite affine plane, every line has the same number of points, proving that it is well-defined.*

Example 1.7 *The most basic example is the rectangle with diagonals, where the set of points are only the vertices of the rectangle (so the intersection of diagonals is not considered as a point and thus they do not meet) and the set of lines are the sides together with the diagonals. Thus, we can see that the order is 2 and there are three parallel classes of lines.*



Points are highlighted in red while the lines that belong to three different parallel classes are highlighted in blue, yellow and green.

1.1.2 Basic results

First we prove that the *parallel property* of finite affine plane is indeed an equivalence relation. Thus, the partition of the lines in the geometry is obtained.

Theorem 1.8 *The lines of a finite affine plane can be partitioned into parallel classes of lines.*

Proof : To prove the existence of partition, it is sufficient to prove that the relation "*parallel*" is an equivalence relation. Reflexive and Symmetric properties are obvious. We only give the details of Transitivity property.

Let l_1, l_2 and l_3 be pairwise distinct lines such that l_1 is parallel to l_2 and l_2 is parallel to l_3. Now, we need to prove that l_1 and l_3 are parallel. Assume on the contrary that l_1 and l_3 are not parallel, and hence they have a common point, say p. Then, observe that the line l_2 does not contain p (because $l_1 \parallel l_2$ and $l_2 \parallel l_3$). By (A3) we must have a unique line parallel to l_2 containing the point p, but we have l_1 and l_3 such that $l_1 \neq l_3$ containing p and parallel to l_2, hence a contradiction. Thus, transitivity holds and therefore "*parallel*" is an equivalence relation.

Remark 1.9 *(A3) and (A4) guarantees the existence of more than one parallel class of lines. In fact, by (A1) and (A4), the minimum number of parallel classes of lines is three.*

Remark 1.10 *Any line of the geometry is in exactly one parallel class and any point of the geometry is in exactly one line from each parallel class.*

Remark 1.11 *In the proof of Theorem 1.8, we have not used the finiteness of points. Thus, the theorem is true for any **affine plane**.*

Theorem 1.12 *Let A be a finite affine plane, and if there exists a line with n points, then:*

(i) *Every line has n points.*

(ii) *Every class of parallel lines has n lines.*

(iii) *Every point is on $n+1$ lines.*

(iv) *There are $n+1$ classes of parallel lines.*

(v) *A has n^2 points and n^2+n lines.*

Proof : Let l be a line with n points.

By Remarks 1.9 and 1.10 and by Theorem 1.8, there exists non-empty parallel classes of lines $\mathcal{C}$ and $\mathcal{C}'$. Let l be a line in $\mathcal{C}$ but not in $\mathcal{C}'$. By the conditions in Definition 1.2 and using Theorem 1.8 we observe the following.

(a) every point of l must lie on a unique line of $\mathcal{C}'$; and

(b) every line in $\mathcal{C}'$ must intersect with l (and every other line $\mathcal{C}$) at a unique point.

By (a), we must have n lines in $\mathcal{C}'$. So, $|l| = |\mathcal{C}'| = n$. Let $\mathcal{C}''$ be another non-empty parallel class of lines. Then, we have $|\mathcal{C}''| = n$ using similar argument.

Let m be a line other than l in $\mathcal{C}$, then m also has n points (since the above argument doesn't depend on the choice of l). Thus, any line on the parallel class $\mathcal{C}$ has n points.

Let l' be a line in $\mathcal{C}'$, then using (b) l' has n points since $\mathcal{C}''$ has n lines. Hence, every line in $\mathcal{C}'$ has n points. Similarly, every line in $\mathcal{C}''$ has n points. Thus, we have shown that every line on each parallel class has n points proving (i).

To complete the proof of (ii), we just reverse the role of l and l' (and thus $\mathcal{C}$ and $\mathcal{C}'$), and see that $\mathcal{C}$ has precisely n lines. Thus, (i) and (ii) are proved.

Let p' be a point on l' distinct from l, then every point of l must join with p' by (A1) giving n (because l has n points on it) parallel classes of lines. By (A2) we can consider the line l' and l to be in the same parallel class and thus there are $n+1$ parallel classes due to p'. So, we have show (iv) from a point not on l. To prove (iv) for a point on l, consider a point p on the line l and reversing the argument for p and p' (and thus for l and l') gives $n+1$ classes of parallel lines [note that this is valid since l' also has n points by (i)]. Thus, we have $n+1$ parallel classes of lines in the geometry. Hence (iv) is proved.

Proof of (iii) is a direct consequence of Remark 1.10 and (iv).

Let p be a point of the geometry and by (iii) we know that there are $(n+1)$ lines passing through p and by (i) we know that each line has n points. Therefore, we have $n(n+1) = n^2 + n$ points but not all of them are distinct. Because we have accounted for the point p, $(n+1)$ times (once for each line passing through p) instead of only one. Thus, subtracting the over counting, we have $n^2 + n - n = n^2$ distinct points in the geometry proving part of (v).

Other part of (v) is quite straightforward. By (iv) we have $n+1$ parallel classes of lines and each parallel class has n lines by (ii). Since, they form a partition they all distinct and hence we have $n(n+1) = n^2 + n$ lines in the geometry.

1.2 Nets

R. H. Bruck in 1951 [1] introduced the concept of (finite) ***nets*** (which Laskar in [3] called them as (finite) nets of dimension two and we call them as 2-nets) which is a generalization of *finite affine planes*.

1.2.1 Definition and Examples

Definition 1.13 *(Net or 2-net) Let s and t be positive integers. An (s,t)-net (or (s,t)-net of dimension two) is an incidence structure $N = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ satisfying the following conditions:*

(N1$^{(2)}$) *These exists a point.*

i.e., $|\mathcal{X}| \geq 1$.

(N2$^{(2)}$) *The lines of the net can be partitioned into t disjoint, non-empty parallel classes such that:*

(a) every point of the net is incident with only one line of each class; and

(b) any two lines from two distinct classes intersect at only one point of the net.

i.e., There exist $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t \subseteq \mathcal{L}$ with

$$\mathcal{L} = \bigcup_{i=1}^{t} \mathcal{C}_i \text{ , } \mathcal{C}_i \neq \emptyset \textit{ for all } i \textit{ and } \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \textit{ for all } i \neq j \textit{ such that:}$$

(a) for all $p \in \mathcal{X}$ and for all i, there exists a unique $l_i \in \mathcal{C}_i$, such that $(p, l_i) \in \mathcal{I}$.

(b) let $l_i \in \mathcal{C}_i$ and $l_j \in \mathcal{C}_j$ such that for all $i \neq j$, then there exists a unique point $p \in \mathcal{X}$ such that $(p, l_i) \in \mathcal{I}$ and $(p, l_j) \in \mathcal{I}$.

(N3$^{(2)}$) *Every line is incident with s points of the net.*

i.e., for any $l \in \mathcal{L}$, $|\{ p \in \mathcal{X} : (p,l) \in \mathcal{L} \}| = s$.

Definition 1.14 *(Order and Degree of a 2-net) We define the **order** of an (s,t)-net to be the number of points on a line (i.e, the parameter s) and **degree** of an (s,t)-net to be the number of parallel classes of lines (i.e., the parameter t). In short, s is called the order of a 2-net and t is called the degree of a 2-net.*

Example 1.15 *A finite affine plane of order s is a $(s, s+1)$-net* (by Theorem 1.12 (iv)).

Remark 1.16 *Bruck's definition does not rule out the existence of a line which contains all the points of the net. Hence most of the basic results discussed in [2] may not be accurate unless $t \geq 2$. To address this, we need to add the following axiom.*

(N4$^{(2)}$) Given a line, there exists a point which is not incident with the line. Equivalently, there does not exist a line which contains all the points of the net.
i.e., there exists $p \in \mathcal{P}$ and $l \in \mathcal{L}$ such that $(p,l) \notin \mathcal{I}$.

1.2.2 Basic results

Using the conditions in the definition of a 2-net and (N4$^{(2)}$) from Remark 1.16, we discuss the basic properties of a 2-net for three different cases of t.

I. If $t \geq 3$, then as pointed out in [2], we can weaken (N3$^{(2)}$) in Definition 1.13 to the following

(N3$'^{(2)}$) There exists a line with s points.
i.e., there exists $l \in \mathcal{L}$ with $|\{\, p \in \mathcal{X} : (p,l) \in \mathcal{L} \,\}| = s$.

Then, we have the following facts as stated in [2].

(i) Every line is incident with s points of the net.

(ii) Every point of the net lies on precisely t distinct lines.

(iii) The net has exactly ts distinct lines each of which fall into t distinct parallel classes of s lines each.

(iv) The net has exactly s^2 points.

The proof of these results are similar to the proof in Theorem 1.12. The following result is an immediate consequence of (N4$^{(2)}$) which is not stated in [2].

(v) Every parallel class of the net contains *at least* two lines.

II. If $t = 2$, we state some of the results below (this special case is not discussed in [2]).

(i) Every point of the net lies on precisely *two* distinct lines.

(ii) Each parallel class of lines has s lines.
(*Proof:* Immediate from the conditions (N3$^{(2)}$) and (N2$^{(2)}$) in 1.13)

(iii) The net has exactly $2s$ distinct lines each of which fall into *two* distinct parallel classes of s lines each.

(iv) The net has exactly s^2 points.
(*Proof:* Immediate from the conditions (N3$^{(2)}$) and (N2$^{(2)}$) and by (b) above)

III. If $t = 1$, we state the corresponding facts below assuming the number of lines in the net is d (this special case is also not discussed in [2]).

(i) Every point of the net lies on precisely *one* line.

(ii) Since $t = 1$, there is only one parallel class of lines having d lines.

(iii) The net has exactly ds points.

1.3 MOLS and Nets

1.3.1 Definitions and Examples

Definition 1.17 *(Latin Square) A **Latin Square** of order n is an $n \times n$ matrix or array consisting of n distinct symbols (or objects) from a symbol set, say S, such that every symbol appear exactly once in each row and exactly once in each column. While the n distinct symbols from the symbol set S can be arbitrary, conventionally it is taken from the set of positive integers $\{ 1, 2, \ldots, n \}$.*

The name ***Latin Square*** was believed to be inspired from the papers of, one of the great mathematicians of all time, *Leonhard Euler*. Euler used *Latin characters* to construct such an array and hence the name.

Example 1.18 *If we take $S = \{ 1, 2, 3 \}$, we get a Latin Square of order 3 as shown below.*

1	2	3
2	3	1
3	1	2

Example 1.19 *If we take $S = \{$ A, B, C, D $\}$, we get a Latin Square of order 4 as given below.*

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

Example 1.20 *Well-known **Sudoku** puzzles are Latin Squares (usually considered to be of order 9 with some additional constraints). Sudoku latin square of order 4 is shown below.*

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

Example 1.21 *Multiplication tables in group theory are also particular type of Latin Squares.*

Remark 1.22 *We can also use a matrix form to represent a Latin Square L, i.e., if L is a latin square of order n associated with a symbol set S, then we write $L = (a_{ij})$, such that $a_{ij} \in S$ and $1 \leq i, j \leq n$, where the entries a_{ij} denote the symbol in i^{th} row and j^{th} column. Hence L can be regarded as a square matrix with entries from its associated symbol set.*

Definition 1.23 *(Mutually orthogonal Latin Square - MOLS) Let $L = (a_{ij})$ and $L' = (b_{ij})$ be two latin squares of order n associated with the symbol set S and S' respectively. We say L and L' are **mutually orthogonal** or simply orthogonal, if the ordered pairs formed by taking an entry a_{ij} of L and its corresponding entry b_{ij} of L' are all different.*

In other words, L and L' are orthogonal if the entries in superimposing of L and L' gives exactly the elements of the set $S \times S'$. By superimposing we mean to form a new matrix M of same order as L and L' whose entries are given by, $M = (a_{ij}b_{ij})$, i.e., for each entry in L, adjoin the corresponding entry in L' to get the respective entry of M.

Remark 1.24 *The set of ordered pairs formed using the entries of L and L' is exactly the set $S \times S'$.*

Definition 1.25 *(A set of MOLS) Let $\mathscr{L}$ be a set consisting of latin squares $L_i, i = 1, 2, \ldots, t$; such that all of them are of same order, say s. We call the set $\mathscr{L} = \{ L_1, L_2, \ldots, L_t \}$ a set of t mutually orthogonal latin squares, if any two distinct latin squares in $\mathscr{L}$ are orthogonal. ($i.e.$, for any $i \neq j$, $L_i L_j$ is orthogonal). We call the set $\mathscr{L}$ in short as **a set of MOLS**.*

Example 1.26 *Let L and L' be two latin squares of order 3 associated with the same symbol set $S = \{ 1, 2, 3 \}$ as given below.*

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \qquad\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$$

Latin Square L Latin Square L'

Superimposing these two latin squares L and L', we get the following matrix M and observe that the entries in M is the set $S \times S$.

$$\begin{array}{ccc} 1\,1 & 2\,2 & 3\,3 \\ 2\,3 & 3\,1 & 1\,2 \\ 3\,2 & 1\,3 & 2\,1 \end{array}$$

Superimposed matrix M

Note that L' is just an permutation of entries in second and third row of L (keeping first row fixed).

1.3.2 Basic results

We list some of the interesting results about Latin Squares and MOLS without proof.

Results on Latin Squares:

1. Latin square of order n exists for all n.
2. A permutation of rows or columns of a latin square gives back a latin square.
3. A permutation on the n-symbols of a latin square also results in a latin square.

Definition 1.27 *(Complete set of MOLS) Let $\mathscr{L} = \{ L_1, L_2, \ldots, L_t \}$ a set of MOLS, where each L_i's are of order s. We call the set $\mathscr{L}$* ***complete*** *if $|\mathscr{L}| = s-1$, i.e., $t = s - 1$.*

Results on MOLS:

1. Let $L_1, L_2, \ldots, L_t$ be a set of MOLS of order $s > 1$. Then $\boxed{t \leq s - 1}$.
2. There exists a complete set of MOLS of order s if and only if there exists a finite affine plane of order s.
3. There exist pairs of orthogonal Latin Squares of every odd order.
4. Euler conjectured that there are no pairs of mutually orthogonal latin squares for order $s > 2$ of the form $s \equiv 2\ (mod\ 4)$. It was proved to be false by the joint efforts of Bose, Shrikhande and Parker except for $s = 6$; the only case where Euler was true, as proved by G. Tarry.

1.3.3 A set of MOLS $\Leftrightarrow$ Nets

Theorem 1.28 *(Equivalence of MOLS and 2-nets) A set of $t-2$ $(t \geq 3)$ mutually orthogonal latin squares of order s (as defined in 1.25) is equivalent to an (s, t)-net (as defined in 1.13).*

Proof : Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ be a (s,t)-net such that $t \geq 3$. Take two parallel classes of lines as follows:

$$\mathcal{C} = \{ l_1, l_2, \ldots, l_s \} \text{ and}$$
$$\mathcal{C}' = \{ l'_1, l'_2, \ldots, l'_s \}$$

Then we can create a coordinate system A such that for every point $p \in \mathcal{X}$, we identify p with the ordered pair (i,j) where $p \in l_i$ and $p \in l'_j$, *i.e.,*

$$A := \{ (i,j) : p \in l_i, p \in l'_j, l_i \in \mathcal{C} \text{ and } l'_j \in \mathcal{C}' \}$$

It is easy to see that $|A| = s^2$ (by (N2$^{(2)}$) in Definition 1.13).Given a parallel class of lines $\mathcal{C}''$ as below,

$$\mathcal{C}'' = \{ l''_1, l''_2, \ldots, l''_s \} \text{ , where } \mathcal{C}'' \neq \mathcal{C}, \mathcal{C}'$$

define a $s \times s$ array by

$$L := (a_{ij}) \text{ such that } (a_{ij}) = k \text{ if } (i,j) \in l''_k$$

Since any two distinct line of the net from different classes of lines intersect at exactly one point, every entry in L appear exactly once in each row and each column. This proves that L is a latin square (obtained from 3 parallel classes of planes).

Let $\mathcal{C}''' = \{ l'''_1, l'''_2, \ldots, l'''_s \}$, where $\mathcal{C}''' \neq \mathcal{C}, \mathcal{C}', \mathcal{C}''$. Then we can obtain another latin square L' by defining another $s \times s$ array as follows.

$$L' := (b_{ij}) \text{ such that } (b_{ij}) = k \text{ if } (i,j) \in l'''_k$$

Since there are t parallel classes of lines, we can proceed in the same manner to obtain $t-2$ latin squares of order s, say $\mathscr{L}$. Let L_1 and L_2 be any two latin squares of the set $\mathscr{L}$ obtained by parallel classes of lines $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Then, since the intersection of a line from $\mathcal{C}$ and a line from $\mathcal{C}'$ is contained in exactly one line from $\mathcal{C}_1$ and exactly one line from $\mathcal{C}_2$, we have L_1 and L_2 are orthogonal latin square.

Now just reversing the argument would yield an (s,t)-net from a set of $t-2$ MOLS.

Chapter 2

Translation Nets in 2-dimension

2.1 Definitions

Definition 2.1 *(Regular or Sharply transitive group action) Let $\mathcal{X}$ be a finite set of points. Let G be a group of permutations on $\mathcal{X}$. Then we call that G acts on $\mathcal{X}$. Furthermore, G is said to be* ***regular*** *or* ***acting regularly*** *on $\mathcal{X}$ (sometimes it is also referred as sharply transitive) if for all $x, y \in \mathcal{X}$, there exists a unique $g \in G$ such that $x^g = y$.*

Definition 2.2 *(Homomorphism of a geometry) Let B and B' be two geometries with incidence structures $B = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ and $B' = (\mathcal{X}', \mathcal{L}', \mathcal{I}')$. A* ***homomorphism*** *from B to B' is a mapping*

$$f : \mathcal{X} \cup \mathcal{L} \longrightarrow \mathcal{X}' \cup \mathcal{L}'$$

such that, for all $p \in \mathcal{X}$ and $l \in \mathcal{L}$,

(i) $p^f \in \mathcal{X}', l^f \in \mathcal{L}'$ *and*

(ii) *if $(p, l) \in \mathcal{I}$ then $(p^f, l^f) \in \mathcal{I}'$.*

Definition 2.3 *(Isomorphism of a geometry) We call a homomorphism f from $B = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ to $B' = (\mathcal{X}', \mathcal{L}', \mathcal{I}')$ as an* ***isomorphism*** *if f is bijective.*

Definition 2.4 *(Automorphism of a geometry) We call an isomorphism f from $B = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ to $B' = (\mathcal{X}', \mathcal{L}', \mathcal{I}')$ as an* ***automorphism*** *if $B = B'$, i.e, f is an isomorphism from B to itself.*

Definition 2.5 *(Automorphism group of a geometry) Let B be a geometry with an incidence structure $B = (\mathcal{X}, \mathcal{L}, \mathcal{I})$. Define Aut$(B)$ as follows:*

$$Aut(B) := \{ \, f : f \text{ is an automorphism on } B \, \}$$

A subset $\mathcal{G}$ of Aut(B) is called an ***automorphism group of*** *B if the set $\mathcal{G}$ forms a group using the composition of mappings as the binary operator.*

Definition 2.6 *(Translation Net and Translation Group) Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ be an (s, t)-net. The net N with an automorphism group $\mathcal{G}$ such that*

(a) $\mathcal{G}$ acts regularly on $\mathcal{X}$ and

(b) $\mathcal{G}$ fixes each parallel class of lines

is called an ***(s,t)-translation net*** *or simply a* ***translation net*** *and the group $\mathcal{G}$ is called the* ***translation group****.*

Lemma 2.7 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ be a 2-net and $\mathcal{G}$ be the corresponding automorphism group of N such that $\mathcal{G}$ acts regularly on the points of N, then there exists a bijection*

$$\sigma : \mathcal{G} \longrightarrow \mathcal{X}$$

i.e., there exists a bijection between the points of the net and the automorphism group of the net. Furthermore, if we define a binary operation on $\mathcal{X}$ by

$$xy = \sigma(\sigma^{-1}(x)\sigma^{-1}(y)) \,, \quad \text{for } x, y \in \mathcal{X}$$

then $\mathcal{X}$ is a group isomorphic to $\mathcal{G}$.

Proof : Let $x \in \mathcal{X}$ be fixed. Since $\mathcal{G}$ acts regularly on $\mathcal{X}$, by Definition 2.1, for any $y \in \mathcal{X}$ there exists a unique $g \in \mathcal{G}$ such that $x^g = y$. Hence the map σ defined by

$$\sigma : \mathcal{G} \longrightarrow \mathcal{X}$$

$$g \mapsto x^g$$

gives a 1-1 correspondence between $\mathcal{X}$ and $\mathcal{G}$. The definition of group structure in $\mathcal{X}$ implies

$$\sigma(g)\sigma(h) = \sigma(gh)\ , \quad \text{for all } \ g, h \in \mathcal{G}$$

and thus $\mathcal{X}$ is a group isomorphic to $\mathcal{G}$.

Remark 2.8 *By Definition 2.6 (a) and by the above Lemma 2.7, it is important to note that every point of a net N can be identified with an element of $\mathcal{G}$ and the lines of N can be regarded as subsets of $\mathcal{G}$ (because the lines of a net can be considered as a subset of the points on the net).*

Remark 2.9 *Let l be line of a net N and $\mathcal{C}$ be its parallel class. By (b) in Definition 2.6 implies that l^g also belongs to $\mathcal{C}$.*

Definition 2.10 *(Partial Congruence Partition - PCP) Let G be a group of order $s^2 \geq 1$. Define $\mathcal{H}$ to be the set containing some subgroups of G, such that the following conditions hold:*

(a) $|\mathcal{H}| = t$

(b) $|H| = s$*, for all* H *in* $\mathcal{H}$

(c) $H \cap H' = \{\ e\ \}$*, for any* H, H' *in* $\mathcal{H}$*; where* e *is the identity element of the group* G*.*

In other words, $\mathcal{H} \subseteq \{\ H\ :\ H \leq G$ *and* $|H| = s\ \}$ *, such that* $|\mathcal{H}| = t$ *and* $H \cap H' = \{e\}$*, for any* H, H' *in* $\mathcal{H}$*; where* e *is the identity element of the group* G*.*

Then $\mathcal{H}$ is called a ***partial congruence partition (PCP) in G*** *with parameters s and t, in short, (s,t)-PCP or just PCP in G, where the elements of $\mathcal{H}$ are often referred as components. Because of (b), axiom (c) is equivalent to*

(c′) $HH' = G$, for any two distinct components H, H' of $\mathcal{H}$.

2.2 PCP ⇔ Translation Nets

First, we list some basic results from group theory in the following Lemmas.

Lemma 2.11 *Let G be a group and H, K be any two subgroups of G. If $aH \cap bK \neq \emptyset$ for distinct $a, b \in G$, then $\exists\ g \in G$ such that $aH \cap bK = g\,(H \cap K)$. Moreover, g is unique if $H \cap K = \{\, e \,\}$.*

Proof : Since $aH \cap bK \neq \emptyset, \exists\ x \in aH \cap bK$, which means $x \in aH$ and $x \in bK$. Thus, we have

$$x \in aH \ \Rightarrow\ xH = aH$$
$$x \in bK \ \Rightarrow\ xK = bK$$

Thus, $aH \cap bK = xH \cap xK = x(H \cap K)$.
If $H \cap K = \{\, e \,\}$, then observe that $aH \cap bK = x(H \cap K) = x\{\, e \,\} = \{\, x \,\}$.

Lemma 2.12 *Let G be a group and H, K be any two subgroups of G such that $HK = G$. Then, for distinct $a, b \in G$, we have $aH \cap bK \neq \emptyset$.*

Proof : First we observe that,

$$aH \cap bK \neq \emptyset \Longleftrightarrow a^{-1}\,(aH \cap bK) \neq \emptyset \Longleftrightarrow H \cap cK \neq \emptyset, \text{ for } c = a^{-1}b \in G.$$

Since $G = HK$ and $c \in G$, we write $c = hk$, for some $h \in H, k \in K$. So, $H \cap cK = H \cap hK$. Now observe that $h \in hK$ and $h \in H$ giving $h \in H \cap hK$ proving $H \cap cK \neq \emptyset$ and thus the result follows.

□

Next we show that a *PCP* with the appropriate geometric structure (defined in Proposition 2.13) yields an (s, t)-net.

Proposition 2.13 *(PCP ⇒ 2-net) Let G be a group of order $s^2 \geq 1$ and $\mathcal{H}$ be the corresponding PCP as defined in 2.10. Define a plane geometry $(\mathcal{X}, \mathcal{L}, \mathcal{I})$ where*

(i) $\mathcal{X} = G$*;*

(ii) $\mathcal{L} = \{ \, gH : g \in G$ *and* $H \in \mathcal{H} \, \}$ *and*

(iii) $\mathcal{I} = \{ \, (g, hH) : g, h \in G, H \in \mathcal{H}$ *and* $g \in hH \, \}$.

Then $(\mathcal{X}, \mathcal{L}, \mathcal{I})$ is an (s, t)-net.

Proof : To prove that it forms an (s, t)-net, we prove the conditions in Definition 1.13 are satisfied.

(N1$^{(2)}$) Since $s^2 \geq 1$, this is true.

(N2$^{(2)}$) For each $H \in \mathcal{H}$, define

$$\mathcal{C}_H := \{ \, gH : g \in G \, \}.$$

Then $\{ \, \mathcal{C}_H \, \}_{H \in \mathcal{H}}$ gives us a partition of $\mathcal{L}$. The set $\mathcal{C}_H$ is often referred as the set of all *left cosets* of H in G. There are t non-empty, distinct subgroups in $\mathcal{H}$, giving t disjoint, non-empty parallel classes. To see (a) and (b), observe the following:

(a) Let $g \in G$. Then for all $H \in \mathcal{H}$, we have $g \in gH$ and $g \notin hH$ for any $h \in G, h \neq g$, because $gH \cap hH$ is either empty or they are equal.

(b) Let aH and bK be the lines of distinct classes for distinct $H, K \in \mathcal{H}$. Need to show $aH \cap bK = \{ \, x \, \}$, for one $x \in G$. Note that $|H| = |K| = s$ and $H \cap K = \{ e \}$. Thus, $G = HK$ satisfying the conditions of Lemmas 2.11 and 2.12 above, giving the desired result.

(N3$^{(2)}$) By the construction of $H \in \mathcal{H}$, $|H| = s \Rightarrow |gH| = s$, for all $g \in G$. Thus, all lines have s points.

Theorem 2.14 *(A. P. Sprague, 1982) Partial congruence partition-PCP with the setting in Proposition 2.13 gives rise to translation nets and conversely, every translation net produce a PCP.*

Proof :

($\Rightarrow$) This direction is trivial from Proposition 2.13.

($\Leftarrow$) Let $\mathcal{G}$ be a translation group of a (s,t)-net N such that points of N are identified with elements of $\mathcal{G}$ and lines are subsets of $\mathcal{G}$. So, we have $|\mathcal{G}| = s^2 \geq 1$. Let $\mathcal{H}$ be the set of all lines containing the identity element. It is obvious that $|H| = s$, for all $H \in \mathcal{H}$ and $H \cap K = \{ e \}$, for all distinct $H, K \in \mathcal{H}$. Now it remains to show that every $H \in \mathcal{H}$ is indeed a subgroup.

Take any $H \in \mathcal{H}$, then by Remark 2.9, for any line H in N and for any $g \in \mathcal{G}$,

$$H^g := \{ gh : h \in H \}$$

is also a line in the same parallel class of H. Hence, either $H^g = H$ or $H^g \cap H = \emptyset$. Since $e \in H$, then for any $g \in H$, we have $g \in H^g \Rightarrow H^g = H$. Thus, H is a subgroup. That is, the lines of N containing the identity are the subgroups of $\mathcal{G}$.

2.3 Upper bounds for t and T_G

We have established 1-1 correspondence between *PCP* and *Translation Nets* in Theorem 2.14 and hence the existence of translation nets is a purely group-theoretic problem. In order to find the maximum number of parallel class for which an (s,t)-translation nets can exist, we formulate the problem in group-theoretic language as follows:

Let G be a group of order $s^2 \geq 1$. Find the number T_G (defined below) precisely or at least find a bound for it.

$$T_G := max \; \{ \; t \leq s+1 : \text{ there exists an } (s,t)\text{-}PCP \text{ in } G \; \} \qquad (2.1)$$

Lemma 2.15 *Let G be a group of order $s^2 > 1$ and $\mathcal{H}$ be the corresponding PCP in G. Then, the number of components in PCP is at most $s + 1$.*

Proof : Let t be the number of components in the *PCP*. By (b) and (c) in Definition 2.10, we note that there are $(s - 1)$ non-identity elements in H, for each $H \in \mathcal{H}$. Thus, we have $t \cdot (s - 1)$ non-identity elements of G in $\mathcal{H}$.

Hence, the total number of elements in $\mathcal{H}$ is $[t \cdot (s - 1) + 1]$. But $|G| = s^2$. So, we must have

$$\begin{aligned} & t \cdot (s-1) + 1 \leq s^2 \\ \Rightarrow \qquad & t \cdot (s-1) \leq s^2 - 1 \\ \Rightarrow \qquad & t \leq (s+1) \end{aligned}$$

Remark 2.16 *By Proposition 2.13 we can talk about the degree of a net given a PCP. Hence from Lemma 2.15 we see that the degree of a net is at most $s + 1$. It is a well know fact that t in Lemma 2.15 attains its maximum if and only if any two points of the net are joined by exactly one line and if this happens then the net is just an* ***finite affine plane of order s****, as pointed out in Example 1.15.*

Remark 2.16 gives an example where the maximum upper bound for a 2-net is attained. In the next example we construct a *PCP* (equivalently translation net) for which the maximum upper bound T_G is attained.

Example 2.17 *Consider a 2-dimensional vector space whose entries are from $\mathbb{F}_q$, where q is a prime power, that is, let $V = \{ \; (x, y) : x, y \in \mathbb{F}_q \; \}$. Then, we have $|V| = q^2$ and note that V is a group with operation being addition. Next,* *we consider the following collection of subgroups of V.*

$$\mathcal{H} := \{ \langle (0,1) \rangle \} \bigcup \{ \langle (1,y) \rangle : y \in \mathbb{F}_q \}$$

Observe that $\mathcal{H}$ is a PCP in V (refer Definition 2.10) and $|\mathcal{H}| = q+1$. Hence the value of the parameters are $s = q$ and $t = q+1$, giving a $(q, q+1)$-PCP and hence an example of $(s, s+1)$-translation net for $s = q$. Since q is a prime power, say $q = p^n$ for some positive integer n and a prime p, we have $|V| = q^2 = p^{2n}$ and thus $T_G = q + 1 = p^n + 1$.

Remark 2.18 *Translation nets can be regarded as a generalizations of* ***translation planes****, studied by André (one of the pioneers to study translation planes) [11] (another reference is Lüneburg [12]); and it may be worthy to note that the translation group of a translation plane is* ***elementary abelian****.*

Remark 2.19 *If G is an elementary abelian group of order $p^{2n} > 1$ (where p is a prime number), then $T_G = p^n + 1$.*

Since the upper bound is achievable for elementary abelian group (Remark 2.19), the next natural question dealt by D. Jungnickel, 1981 [13] is to ask for the upper bounds of T_G if G is not elementary abelian. We present some useful results below to illustrate the upper bound for T_G.

Lemma 2.20 *If H is a subgroup of a group G then for any $g \in G, gHg^{-1}$ is a group isomorphic to H.*

Lemma 2.21 *If H and K are subgroups of a group G such that $HK = G$ and then for any $a, b \in G$, we have $(aHa^{-1}) \cap (bKb^{-1}) = c(H \cap K)c^{-1}$, for some $c \in G$.*

Proof : Let $a \in G = HK = KH$, then we can write $a = kh$ for some $k \in K$ and $h \in H$. Now we see that, $aHa^{-1} = khHh^{-1}k^{-1} = kHk^{-1}$. Hence,

$$(aHa^{-1}) \cap K = k(H \cap K)k^{-1} \tag{2.2}$$

Let $H' = aHa^{-1}$, then by Lemma 2.20,

$$|H| = |aHa^{-1}| = |H'| \tag{2.3}$$

and from equation (2.2), we have

$$|H' \cap K| = |k(H \cap K)k^{-1}| = |H \cap K| \text{ [by equation (2.3)]} \tag{2.4}$$

Since $HK = G$, we have

$$|G| = |HK| = \frac{|H| \cdot |K|}{|H \cap K|} \tag{2.5}$$

Now we observe the following

$$\begin{aligned} |H'K| &= \frac{|H'| \cdot |K|}{|H' \cap K|} \\ &= \frac{|H| \cdot |K|}{|H \cap K|} \text{ , by equations (2.3) and (2.4)} \\ &= |G| \text{ , by equation (2.5)} \end{aligned}$$

Thus, $\boxed{H'K = G}$ and hence we apply the same arguments above (in the beginning of the proof) to H' instead of H and set $b = h'k'$, for some $h' \in H$ and $k' \in K$. So, we get

$$\begin{aligned} (bKb^{-1}) \cap H' &= h'(K \cap H')h'^{-1} \\ &= h'(H' \cap K)h'^{-1} \\ &= h'(aHa^{-1} \cap K)h'^{-1} \\ &= h'k(H \cap K)k^{-1}h'^{-1} \text{ , by equation (2.2)} \\ &= c(H \cap K)c^{-1} \text{ , where } c = h'k \in G \end{aligned}$$

Hence, $(aHa^{-1}) \cap (bKb^{-1}) = H' \cap (bKb^{-1}) = c(H \cap K)c^{-1}$, for some $c \in G$.

Remark 2.22 *In Lemma 2.21 above, if we also have $H \cap K = \{ e \}$, then*

$$(aHa^{-1}) \cap (bKb^{-1}) = \{ cec^{-1} \} = \{ e \}.$$

Lemma 2.23 *Let G be a finite group. Suppose p is a prime divisor of $|G|$. Let P be a Sylow p-subgroup of G. For any subgroup H of G, there exists $g \in G$ such that $gHg^{-1} \cap P$ is a Sylow p-subgroup of gHg^{-1}.*

Proof : Let Q be a Sylow p-subgroup of H. By Sylow theorems, we know that Q must be contained in a Sylow p-subgroup of G, say, P'. Also there exists $g \in G$ such that $gP'g^{-1} = P$. Hence $gHg^{-1} \cap P = gQg^{-1}$ is a Sylow p-subgroup of gHg^{-1}.

Theorem 2.24 *Let G be a group of order s^2. Suppose p is a prime divisor of s. Let P be a Sylow p-subgroup of G. If there exists a PCP of G with t subgroups, then there exists a PCP of P with t subgroups.*

Proof : Let $\mathcal{H}$ be a *PCP* of G. For each $H \in \mathcal{H}$, we can choose $g \in G$ such that with $gHg^{-1} \cap P$ is a Sylow p-subgroup of gHg^{-1} (by Lemma 2.23). Now if we replace H by gHg^{-1}, then the resultant is still a *PCP* of G (because of Lemmas 2.20, 2.21 and Remark 2.22). Let us define the set $\mathcal{H}'$ as follows:

$$\mathcal{H}' = \{ (g_i H_i g_i^{-1}) \cap P : H_i \in \mathcal{H}, 1 \leq i \leq t \}$$

Then we prove that $\mathcal{H}'$ is a *PCP* of P, *i.e.,* the elements in $\mathcal{H}'$ satisfies the conditions (a) to (c) in Definition 2.10.

(a) Since $|\mathcal{H}| = t$, we have $|\mathcal{H}'| = t$.

(b) Since p divides s, we have $|H| = s = p^r \cdot n$, for some positive integer n. Now by Lemma 2.23, we have $(gHg^{-1}) \cap P$ is a sylow p-subgroup of gHg^{-1} and by Lemma 2.20, $|gHg^{-1}| = |H| = s = p^r \cdot n$. Hence $|gHg^{-1} \cap P| = p^r$.

(c) $(g_1H_1g_1^{-1} \cap P) \bigcap (g_2H_2g_2^{-1} \cap P) = (g_1H_1g_1^{-1} \cap g_2H_2g_2^{-1}) \cap P = \{e\} \cap P = \{\, e \,\}$, where the second last equal to is because of Lemma 2.21.

Theorem 2.25 *Let G be a group of order $s^2 > 1$ such that, $|G| = s^2 = p^a \cdot n$, and let R be the Sylow-p-subgroup of G. Then any (s,t)-PCP in G induces a PCP in R, $i.e., T_G \leq T_R$.*

The proof of Theorem 2.25 was due to Frohardt [14] (but the result appeared in [7]). It serves as a motivation to study the existence of partial congruence partition in p-groups. Hachenberger in [15,16] carried out the study for groups of order p^{2n} for $n \geq 2$. We shall state the result for groups of order p^{2n} that are not elementary abelian and $n \geq 4$ from [16], while the groups of order p^{2n} for the case $n = 2$ and $n = 3$ are handled in [15] and [16] respectively.

Theorem 2.26 *(Hachenberger) Let p be any prime and let G be a group of order p^{2n} which is not elementary abelian. If $n \geq 4$, then*

$$T_G \leq (p^{n-1} - 1)(p-1)^{-1} = p^{n-2} + \ldots + p + 1\,.$$

2.4 Normal components in *PCP*

Definition 2.27 *(Normal component) Let G be a group of order $s^2 > 1$ and $\mathcal{H}$ be the corresponding PCP. Recall from the definition of PCP in 2.10, elements of $\mathcal{H}$ are called components. Also notice that the elements are nothing but subgroups of G. We call the elements of $\mathcal{H}$ as **normal components** if they are **normal subgroups** of G.*

The corresponding definition with respect to translation nets can be found in [6], which introduces central collineation to nets.

A. P. Sprague in [6] opened the gates for the study of normal components in *PCP* (or in translation net depending on the context) by discussing Sylow-subgroup properties of G with specific order. Later Hachenberger and Jungnickel [17,18] studied the normal components for certain specific cases. It may be an useful observation based on the results stated below from [6], which points out that, it is advantageous to classify *PCP* into the following:

(a) there exists no normal component;

(b) there exists exactly one normal component;

(c) there exists exactly two normal components and

(d) there exists at least three normal components.

Theorem 2.28 *Let G be a group of order s^2 and $\mathcal{H}$ be the corresponding PCP of G such that it has at least two subgroups which are normal in G, i.e., $G \cong H \times H'$ for any $H, H' \in \mathcal{H}$. Then all the components of $\mathcal{H}$ are isomorphic and hence are all normal.*

Theorem 2.29 *Let N be a translation net and G be a translation group associated with N such that at least three parallel classes of N are normal. Then G is abelian.*

More details about the discussion on p-groups and cases including two normal components can be seen in [7] and [8].

Chapter 3

Nets of Dimension Three

Inspired by the definition of 2-net given by Bruck, in 1971 R. Laskar took the next step to extend the definition to a 3-dimensional finite geometry. Laskar considered ***planes*** along with points and lines in the definition of Bruck to define a ***finite net of dimension three*** [3] with various parameters. We simply call it a 3-net instead of a 3-dimensional net.

3.1 Definitions and Examples

Definition 3.1 *(Incidence Relation for 3-dimensional geometry) A 3-dimensional plane geometry is an incident structure* $(\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ *where,*

(a) $\mathcal{X}$ *is the set of points,* $\mathcal{L}$ *is the set of lines and* $\mathcal{P}$ *is the set of planes;*

(b) $\mathcal{X}, \mathcal{L}$ *and* $\mathcal{P}$ *are pairwise disjoint sets; and*

(c) $\mathcal{I} \subseteq (\mathcal{X} \times \mathcal{L}) \bigcup (\mathcal{X} \times \mathcal{P}) \bigcup (\mathcal{L} \times \mathcal{P})$ *is the incident relation between points and lines, points and planes and lines and planes.*

Since there are three objects under consideration (*points, lines* and *planes*), we can talk about the following types of relationships. Again, we adopt most of the commonly used terms in the study of geometries without explicitly defining them. We list below some of these terminologies for the different types of relationships.

1. ***point-plane relationship:***

   (i) The following are equivalent ways to describe the relation: $(p, \Pi) \in \mathcal{I}$ for $p \in \mathcal{X}$ and $\Pi \in \mathcal{P}$.

      (a) A point p is on a plane Π.

      (b) A plane Π passes through a point p.

      (c) A point p is incident with a plane Π.

      (d) A plane Π contains a point p.

   (ii) A point $p \in \mathcal{X}$ is not on a plane $\Pi \in \mathcal{P}$ is equivalent to $(p, \Pi) \notin \mathcal{I}$.

2. ***line-plane relationship:*** As it will be pointed out in Remark 3.2, there are three types of line-plane interactions; but we consider only two types (type (i) and (iii) in Remark 3.2) and describe the relationship below.

   (i) The following are equivalent ways to describe the relation: $(l, \Pi) \in \mathcal{I}$ for $l \in \mathcal{L}$ and $\Pi \in \mathcal{P}$.

      (a) A line l is on a plane Π.

      (b) A line l is incident with a plane Π.

      (c) A plane Π contains a line l.

   (ii) A line $l \in \mathcal{L}$ is not on a plane $\Pi \in \mathcal{P}$ is equivalent to $(l, \Pi) \notin \mathcal{I}$.

3. ***point-line relationship:***

   (i) (Coplanar) A point $p \in \mathcal{X}$ and a line $l \in \mathcal{L}$ lie on a plane is equivalent to saying that there exists a plane $\Pi \in \mathcal{P}$ such that $(p, \Pi) \in \mathcal{I}$ and $(l, \Pi) \in \mathcal{I}$. We call p and l are ***coplanar***. Here, we discuss the incidence relation between a point and a line which are coplanar. In a plane $\Pi \in \mathcal{P}$, we refer to the paragraph after Definition 1.1 for the equivalent ways to describe the relation: a point $p \in \mathcal{X}$ and a line $l \in \mathcal{L}$ such that $(p, \Pi) \in \mathcal{I}$, $(l, \Pi) \in \mathcal{I}$ and $(p, l) \in \mathcal{I}$.

(ii) (Non-Coplanar) A point $p \in \mathcal{X}$ and a line $l \in \mathcal{L}$ do not lie on a plane is equivalent to saying that for any plane $\Pi \in \mathcal{P}$ we have $(p, \Pi) \notin \mathcal{I}$ or $(l, \Pi) \notin \mathcal{I}$. In such a situation the point and the line is called ***non-coplanar*** and there is no incidence relation between a point and a line.

4. ***line-line relationship:***

(i) (Coplanar lines) Two lines $l, l' \in \mathcal{L}$ lie on a plane is equivalent to saying that there exists a plane $\Pi \in \mathcal{P}$ such that $(l, \Pi) \in \mathcal{I}$ and $(l', \Pi) \in \mathcal{I}$. The lines l and l' are called *coplanar lines*. Now we can discuss the ***parallel*** relation between two coplanar lines. Inside a plane $\Pi \in \mathcal{P}$, the equivalent ways to describe the following relation is defined in 1.1 and in (ii) below (non coplanar lines are parallel): for $l, l' \in \mathcal{L}$ and a plane $\Pi \in \mathcal{P}$ such that $(l, \Pi) \in \mathcal{I}$ and $(l', \Pi) \in \mathcal{I}$, either $(l, \Pi) = (l', \Pi)$ or $\{\, p \in \mathcal{X} : (p, \Pi) \in \mathcal{I}$ and $(p, l) \in \mathcal{I} \,\} \bigcap \{\, p \in \mathcal{X} : (p, \Pi) \in \mathcal{I}$ and $(p, l') \in \mathcal{I} \,\} = \emptyset$

(ii) (Non-Coplanar lines) Two lines $l, l' \in \mathcal{L}$ do not lie on a plane is equivalent to saying that for any plane $\Pi \in \mathcal{P}$ we have either $(l, \Pi) \in \mathcal{I}$ but $(l', \Pi) \notin \mathcal{I}$ or $(l', \Pi) \in \mathcal{I}$ but $(l, \Pi) \notin \mathcal{I}$. Then the lines l and l' are called *non coplanar lines*.

5. ***plane-plane relationship:*** The following are equivalent ways to describe the ***parallel*** relation between two planes $\Pi, \Pi' \in \mathcal{P}$: either $\Pi = \Pi'$ or $\{\, p \in \mathcal{X} : (p, \Pi) \in \mathcal{I} \,\} \bigcap \{\, p \in \mathcal{X} : (p, \Pi') \in \mathcal{I} \,\} = \emptyset$.

(a) For any two planes Π and Π', either they are the same or they have no point in common.

(b) Two planes Π and Π' are parallel (or symbolically $\Pi \parallel \Pi'$).

(c) For any two planes Π and Π', either they are the same or they do not intersect.

Remark 3.2 *There are three scenarios possible for the interaction between a line and a plane:*

(i) a line and a plane do not meet at all;

(ii) a line and a plane meet at a point;

(iii) a line and a plane meet at a line, i.e., the line is completely contained in the plane.

The line-plane relationship defined in 3.1 refers only to the scenarios (i) and (iii) above (i.e., we do not consider the scenario (ii)).

Remark 3.3 *In this thesis, we also adopt the following equivalent ways of writing the incidence relation between points, lines and planes.*

(i) $(p,l) \in \mathcal{I} \iff p \in l \iff$ *a point* $p \in \mathcal{X}$ *lies on a line* $l \in \mathcal{L}$.

(ii) $(p,\Pi) \in \mathcal{I} \iff p \in \Pi \iff$ *a point* $p \in \mathcal{X}$ *lies on a plane* $\Pi \in \mathcal{P}$.

(iii) $(l,\Pi) \in \mathcal{I} \iff l \in \Pi \iff$ *a line* $l \in \mathcal{L}$ *lies on a plane* $\Pi \in \mathcal{P}$.

(iv) $(p,l) \notin \mathcal{I} \iff p \notin l \iff$ *a point* $p \in \mathcal{X}$ *does not lie on a line* $l \in \mathcal{L}$.

(v) $(p,\Pi) \notin \mathcal{I} \iff p \notin \Pi \iff$ *a point* $p \in \mathcal{X}$ *does not lie on a plane* $l \in \mathcal{L}$.

(vi) $(l,\Pi) \notin \mathcal{I} \iff l \notin \Pi \iff$ *a line* $l \in \mathcal{L}$ *does not lie on a plane* $\Pi \in \mathcal{P}$.

First, we present below the original definition for "finite nets of dimension three" by Laskar [3].

Definition 3.4 *(3-dimensional net or 3-net) Let* s, t, β *and* r *be* ***positive integers****. An* (s,t,β,r)*-net (or a net of dimension three) is an incidence structure* $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ *satisfying the following conditions:*

(N1$^{(3)}$) *If a point p lies on a line l, and the line l lies on a plane Π, then p lies on Π.*

(N2$^{(3)}$) *Any two non-parallel lines intersecting at a point, lie on the same plane.*

(N3$^{(3)}$) *Points and lines incident with a plane form a (s,t)-net.*

(N4$^{(3)}$) *The planes of the net can be partitioned into β disjoint, non-empty parallel classes such that:*

- *(a) every point of the net is incident with only one plane of each class;*
- *(b) any two planes from two distinct classes intersect at only one line of the net.*

(N5$^{(3)}$) *Two lines are identified as non-intersecting if they do not have a common point. The lines of the net are partitioned into a finite number of maximal non-intersecting sets of lines such that:*

- *(a) every point of the net is incident with only one line of each set;*
- *(b) any two lines from distinct sets intersect at **at most** one point of the net.*

(N6$^{(3)}$) *There exists a line incident with r planes.*

Remark 3.5 *In (N5$^{(3)}$), by the maximality of non-intersecting sets of lines, we mean that for any two distinct classes of lines $\mathcal{C}_i$ and $\mathcal{C}_j$ (for some finite index i, j) and a line l_i in $\mathcal{C}_i$, there exists at least one line l_j in $\mathcal{C}_j$ such that l_i and l_j are not parallel.*

Definition 3.6 *(Order and Degree of a 3-net) We define the **order** of a 3-net to be the number of points in a line (i.e, the parameter s) and **degree** of a 3-net to be the number of parallel classes of planes (i.e., the parameter β). In short, s is called the order of a 3-net and β is called the degree of a 3-net.*

Remark 3.7 *Laskar's definition does not rule out the existence of lines which has no point and do not lie on any plane. Hence the basic results in [3] (or in section 3.2 below) does not hold if the parameters s, t, β and r are not greater than 1. To address this, we need to assume that the parameters s, t, β and r are greater than 1.*

Example 3.8 *Consider a 3-dimensional vectors (as triplets) whose entries are from $\mathbb{Z}_n$, that is, let $V = \{\ (abc) : a, b, c \in \mathbb{Z}_n\ \}$. Here, just for the sake of convenience we write the triplet as (abc) instead of (a, b, c). Then, $|V| = n^3$ and V is a group with operation being addition. If we set the collection of planes to be $\mathcal{K} = \{\ K_x, K_y, K_z\ \}$, where the planes are given by*

$$K_x = \{\ (x\ b\ c) : b, c \in \mathbb{Z}_n\ \}$$
$$K_y = \{\ (a\ y\ c) : a, c \in \mathbb{Z}_n\ \}$$
$$K_z = \{\ (a\ b\ z) : a, b \in \mathbb{Z}_n\ \}$$

and the collection of lines $\mathcal{H} = \{\ H_{xy}, H_{yz}, H_{xz}\ \}$, where the lines formed by the intersection of planes are given by

$$H_{xy} = \{\ (x\ y\ c) : c \in \mathbb{Z}_n\ \} = K_x \cap K_y$$
$$H_{yz} = \{\ (a\ y\ z) : a \in \mathbb{Z}_n\ \} = K_y \cap K_z$$
$$H_{xz} = \{\ (x\ b\ z) : b \in \mathbb{Z}_n\ \} = K_x \cap K_z\ .$$

Then they form a 3-net.

3.2 Basic Results

We list some of the basic but important results obtained from the conditions in Definition 3.4, assuming s, t, β and r are **greater than one**. We only prove Lemmas 3.13, 3.14 and Theorem 3.15, while we refer to [3] and [10] for the proof of the remaining results.

Theorem 3.9 *(Basic properties of 3-net)*

(i) Every line is incident with at least one plane.

(ii) Every plane contains exactly s^2 points and ts lines.

(iii) Every line is incident with s points.

(iv) Every point is incident with β planes.

Theorem 3.10 *Every line is incident with r planes. Equivalently every line of the net is incident with the same number of planes.*

Theorem 3.11 *Every point of the net is incident with*

$$u = \frac{t \cdot \beta}{r} \text{ lines.}$$

Theorem 3.12 *The number of planes in a parallel class is* **s**. *In other words, the number of planes in a parallel class is equal to the number of lines of a parallel class in a plane.*

□

Let r be the number of planes passing through any line of a 3-net N and u be the number of lines passing through any point. We first count u in two ways as given in Lemma 3.13 and 3.14 below.

Lemma 3.13

$$u = \frac{t + (r-1)t^2}{r} .$$

Proof : Let $p \in \mathcal{X}$ be a point of the net. For any plane $\Pi \in \mathcal{P}$ such that p is incident with Π, we have t lines passing through p on Π by (N3$^{(3)}$).

For each of these t lines, there are $(r-1)$ planes $\Pi' \neq \Pi$ passing through the line (because by assumption, there are r planes passing through any line of the net). On each of these planes, again there are t lines passing through p. Hence, we have $(r-1)t$ lines passing through p in the planes other than Π.

If we fix a line in Π passing through p, then there are $1 + (r-1)t$ lines passing through p, and since there are t such lines in Π passing through p, we have $t[1 + (r-1)t] = t + (r-1)t^2$ lines in total passing through p. Note that each line passing through p is counted r times and hence the total number of lines passing through any point in N is given by

$$u = \frac{t + (r-1)t^2}{r} \ .$$

$\square$

Lemma 3.14

$$u = 1 + (t-1)r \ .$$

Proof : Let $p \in \mathcal{X}$ be a point of the net and $l \in \mathcal{L}$ be a line containing p [this is possible because there is no stand alone point by (N3$^{(3)}$) and (N4$^{(3)}$)]. There are r planes passing through l, and in each of these planes there are $(t-1)$ lines (other than l) passing through p. Thus, the total number of lines passing through p is $1 + (t-1)r$.

Theorem 3.15 *Let N be a 3-net with the parameters s, t, β and r. Then the following holds:*

(i) $r = t$.

(ii) $\beta = t^2 - t + 1$.

Proof :

(i) Lemmas 3.13 and 3.14 give two ways of counting the number of lines passing through a given point of the net. Hence, they are one and the same. So, we have

$$\frac{t + (r-1)t^2}{r} = 1 + (t-1)r$$

Simplifying, we get $(t-r)[rt-(t+r)+1] = 0$. Hence, either $(t-r) = 0$ or $rt - (t+r) + 1 = 0$, *i.e.,* $t = r$ or $rt + 1 = t + r$. We shall show that latter case is not possible.

Since r and t are integers greater than two, we have $rt + 1 > r + t$ for all r, t and hence $rt + 1 > r + t$. Thus $\boxed{r = t}$.

(ii) From Theorem 3.11,

$$u = \frac{t \cdot \beta}{r}.$$

Since $r = t$ by (i) above and $t > 1$, $\boxed{u = \beta}$.

Now using $r = t$ and $u = \beta$ in Lemma 3.14, we get $\beta = 1 + (t-1)t = t^2 - t + 1$.

Lemma 3.16 *The total number of points, lines and planes in N are s^3, $s^2(t^2 - t + 1)$ and $s(t^2 - t + 1)$ respectively.*

3.3 Redefining old definition

Sometimes the meaning of "maximal non-intersecting sets" in the definition of 3-net may be misleading or unclear. We address this issue by giving a simpler axiom to replace (N5$^{(3)}$):

$$\text{(}N5'^{(3)}\text{) Every line is incident with at least one plane.} \tag{3.1}$$

Sometimes it is convenient to define the 3-nets (nets of dimension three) in terms of Mathematical language, as used in the incidence relation. We perform the following to give a new definition of a 3-net and define it Mathematically in Definition 3.17.

(i) Assume the parameters s, t, β and r to be **greater than one**.

(ii) Use the conditions (N1$^{(3)}$) to (N4$^{(3)}$) in Definition 3.4.

(iii) Replace (N5$^{(3)}$) with (N5$'^{(3)}$).

(iv) Remove (N6$^{(3)}$) (because it can be proved from other conditions and hence we consider it as redundant).

Definition 3.17 *(Mathematical reformulation of a 3-net) Let s, t, β and r be integers **greater than one**. An (s, t, β, r)-net (or net of dimension three) is an incidence structure $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ satisfying the following conditions:*

(N1$^{(3)}$) For $p \in \mathcal{X}, l \in \mathcal{L}$ and $\mathit{\Pi} \in \mathcal{P}$, if $(p, l) \in \mathcal{I}$ and $(l, \mathit{\Pi}) \in \mathcal{I}$, then $(p, \mathit{\Pi}) \in \mathcal{I}$.

(N2$^{(3)}$) For $p \in \mathcal{X}, l_1, l_2 \in \mathcal{L}$, if $(p, l_1) \in \mathcal{I}$ and $(p, l_2) \in \mathcal{I}$, there is a $\mathit{\Pi} \in \mathcal{P}$ such that $(l_1, \mathit{\Pi}) \in \mathcal{I}$ and $(l_2, \mathit{\Pi}) \in \mathcal{I}$.

(N3$^{(3)}$) For each $\mathit{\Pi} \in \mathcal{P}$, let us define

$$\mathcal{X}' = \{\, p \in \mathcal{X} : (p, \mathit{\Pi}) \in \mathcal{I} \,\};$$
$$\mathcal{L}' = \{\, l \in \mathcal{L} : (l, \mathit{\Pi}) \in \mathcal{I} \,\}; \text{ and}$$
$$\mathcal{I}' = \{\, (p, l) : (p, l) \in \mathcal{I}, p \in \mathcal{X}' \text{ and } l \in \mathcal{L}' \,\}.$$

Then $(\mathcal{X}', \mathcal{L}', \mathcal{I}')$ forms a 2-net.

(N4$^{(3)}$) There exist $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\beta \subseteq \mathcal{P}$ such that

(a) $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\beta$ is a partition of $\mathcal{P}$;

(b) for each $\mathcal{B}_i$, and for any $\mathit{\Pi}, \mathit{\Pi}' \in \mathcal{B}_i$ with $\mathit{\Pi} \neq \mathit{\Pi}'$, $\mathit{\Pi}$ and $\mathit{\Pi}'$ are parallel;

(c) let $\mathcal{B}_i$ and $\mathcal{B}_j$ be any two distinct classes of planes. For each $\Pi \in \mathcal{B}_i$ and $\Pi' \in \mathcal{B}_j$ there is a unique $l \in \mathcal{L}$ such that $(l, \Pi) \in \mathcal{I}$ and $(l, \Pi') \in \mathcal{I}$; and

(d) for each $p \in \mathcal{X}$ and for each $i \in \{ 1, 2, \ldots, \beta \}$, there is a unique $\Pi \in B_i$ such that $(p, \Pi) \in \mathcal{I}$.

The sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\beta$ are called the parallel classes of planes.

(N5$'^{(3)}$) For all $l \in \mathcal{L}$, there is a $\Pi \in \mathcal{P}$ such that $(l, \Pi) \in \mathcal{I}$.

It is natural to ask whether the new axiom in equation (3.1) is equivalent to the old one, which we answer in Theorem 3.22. Prior to that, it is important to redefine the maximal non-intersecting classes with a precise relation between any two given lines.

Definition 3.18 *(Maximal non-intersecting $\sim$ classes) The **maximal non intersecting classes** of lines is defined by the relation given below:*

"two lines are related if there exist two parallel classes $\mathcal{B}_1$ and $\mathcal{B}_2$ of planes such that each of the two lines are the intersection of a plane from $\mathcal{B}_1$ with a plane from $\mathcal{B}_2$".

Before proving that this relation is an equivalence relation, we first observe a useful result in the form of a lemma below.

Lemma 3.19 *Let $l \sim l'$. If $\mathcal{B}$ is a class of plane such that there exists $\Pi \in \mathcal{B}$ and l is incident with Π, then there exists $\Pi' \in \mathcal{B}$ such that l' is incident with Π'.*

Proof : Since $l \sim l'$, there exist two parallel classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$ such that

$$l = \Pi_1 \cap \Pi_2 \text{ for some } \Pi_1 \in \mathcal{B}_1 \text{ and } \Pi_2 \in \mathcal{B}_2 \text{ and}$$
$$l' = \Pi_1' \cap \Pi_2' \text{ for some } \Pi_1' \in \mathcal{B}_1 \text{ and } \Pi_2' \in \mathcal{B}_2.$$

Since Π_1' and Π_2 belong to different classes of planes, they intersect at a line, say $l'' = \Pi_2 \cap \Pi_1'$. Let p be a point incident with $l'' \in \Pi_2$ and $\mathcal{B}$ be a class of plane such that $l \in \Pi$ for some $\Pi \in \mathcal{B}$. Since $\Pi_2 \in \mathcal{B}_2$, and $\mathcal{B}_2, \mathcal{B}$ are different class of planes, by (N4)$^{(3)}$)(a) there exists a plane $\Pi'' \in \mathcal{B}$ incident with p. Let m be the line of intersection of Π'' and Π_2 incident with p. Now, observe the following:

$$l \text{ is parallel to } l'' \text{ because } \Pi_1 \text{ is parallel to } \Pi_1' \text{ and}$$
$$l \text{ is parallel to } m \text{ because } \Pi \text{ is parallel to } \Pi''.$$

l, l'' and m lie on the plane Π_2, so by the transitivity of *parallel class for lines* in Π_2, l'' is parallel to m. $p \in l$ and $p \in m$ implies that $m = l''$. Thus, for any $\Pi \in \mathcal{B}$ incident with l there is a plane $\Pi'' \in \mathcal{B}$ incident with l''.

Now, we repeat the above argument replacing l by l'' and Π by Π'', and consider the lines l'' and l' to get the required $\Pi' \in \mathcal{B}$.

Proposition 3.20 *The relation defined in 3.18 is indeed an equivalence relation.*

Proof : It is easy to see that the relation satisfies the *Reflexive* and *Symmetric* properties. So, it remains to show the *Transitivity*.

$l_1 \sim l_2 \Rightarrow l_1$ and l_2 are defined by two classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$.

$l_2 \sim l_3 \Rightarrow$ by Lemma 3.19 l_2 and l_3 are defined by two classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$.

Hence, l_1 and l_3 are defined by two classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$, proving $l_1 \sim l_3$.

Remark 3.21 *From Lemma 3.19, we observe that each class of lines can be defined by two distinct classes of planes, say $\mathcal{B}_i$ and $\mathcal{B}_j$, for some $i, j \in \{1, 2, \ldots, \beta\}$.*

Theorem 3.22 *(New axiom ⇒ old axiom) The (old) axiom (N5$^{(3)}$) can be proved by our new axiom (N5$'^{(3)}$), i.e., the following holds:*

(i) property of maximal non-intersecting sets as stated in Remark 3.5.

(ii) (N5$^{(3)}$) (a) and (b).

Proof :

(i) Let $\mathcal{C}_i$ and $\mathcal{C}_j$ be two distinct classes of lines. Let l_i be a line from $\mathcal{C}_i$. We need to find a line in $\mathcal{C}_j$ intersecting l_i at a point. Let p be a point on l_i and let $\mathcal{C}_j$ be defined by two classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$ (using Remark 3.21). Then, there exists $\Pi_1 \in \mathcal{B}_1$ and $\Pi_2 \in \mathcal{B}_2$ such that $p \in \Pi_1 \cap \Pi_2$. Let $\Pi_1 \cap \Pi_2 = l_j$, then $p \in l_j$. Hence, l_j is the required line.

(ii) (a) Let p be a point on the net and $\mathcal{C}$ be a class of lines. Again by the Remark 3.21, $\mathcal{C}$ is defined by two classes of planes $\mathcal{B}_1$ and $\mathcal{B}_2$. We need to show that p lie on exactly one line in $\mathcal{C}$. By (N4$^{(3)}$), p lie on exactly one plane from each class of $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\Pi_1 \in \mathcal{B}_1$ and $\Pi_2 \in \mathcal{B}_2$ be such planes containing p. Since Π_1 and Π_2 belongs to different class of planes, they intersect at exactly one line l and since p lie on Π_1 and Π_2, p lie on l. This l is the unique line in $\mathcal{C}$ containing p, as p cannot lie on any other planes in $\mathcal{B}_1$ and $\mathcal{B}_2$ by (N4$^{(3)}$).

(b) Any two lines belong to either same plane or different planes. If they belong to different plane, they do not intersect, and if they belong to same plane they intersect at *at most* one point by (N3$^{(3)}$).

3.3.1 Another reformulation (using point-line incidence relation)

We reformulate (N5$'^{(3)}$) in equation (3.1) as follows:

$$\text{(}N5''^{(3)}\text{) } \textit{Every line is incident with at least one point.} \tag{3.2}$$

We may consider (N5$'^{(3)}$) in equation (3.1) as reformulation with respect to planes, while (N5$''^{(3)}$) in equation (3.2) may be regarded as reformulation with respect to points. Again, one can ask the equivalence of (N5$'^{(3)}$) and (N5$''^{(3)}$) and we discuss this below in Theorem 3.23.

Theorem 3.23 *If we fix the conditions N1$^{(3)}$ to N4$^{(3)}$ in Definition of a 3-net, then we have (N5$'^{(3)}$) $\Longleftrightarrow$ (N5$''^{(3)}$).*

Proof :

($\Rightarrow$) Since every line is incident with at least one plane, we may treat the lines and its corresponding incident plane just as a 2-net. In 2-net there is no stand alone lines (by (N3$^{(2)}$) in Definition 1.13), *i.e.,* every line is incident with at least one point.

($\Leftarrow$) Let l be a line and p be a point on l. Since $\beta > 1$ and by (N4$^{(3)}$)(b), there exists at least 2 planes containing p, say Π and Π' (which belong to different classes of planes). If l is incident with Π or Π', we are done.

If not, let $l' \neq l$ be the line of intersection of Π and Π'. Since $p \in \Pi, \Pi'$ $\Rightarrow p \in l' \Rightarrow l$ and l' intersect at p. Hence, by (N2$^{(3)}$), l and l' lie on the same plane, say Π''.

3.4 Special cases

The integers s, t, β and r are assumed to be greater than one for geometrical purposes. But, the trivial cases where either one of them is equal to 1 cannot be completely ignored. So, we discuss some results for some of the special cases such as $\beta = 1$ and $t = 1$. Theorem 3.24 discusses about *maximal non-intersecting classes* for the special case $\beta = 1$. Theorem 3.25 discusses about the special case $t = 1$. In both these theorems, we use the conditions in Definition 3.17 instead of the conditions in Definition 3.4. In particular, we will use (N5$'^{(3)}$) rather than (N5$^{(3)}$) (a) and (b).

Theorem 3.24 *Let N be a 3-net as defined in 3.17 with the parameters $\beta = 1$ and s, t and r are greater than 1. Then the maximal non-intersecting classes exists and (N5$^{(3)}$) (a) and (b) is valid, i.e., Theorem 3.22 holds for the special case $\beta = 1$.*

Proof : Let $\mathcal{P}$ be the collection of planes. For each plane $\Pi \in \mathcal{P}$, there are t classes of parallel lines on the plane. We label them as $\mathcal{C}_1(\Pi)$, $\mathcal{C}_2(\Pi)$, $\ldots$, $\mathcal{C}_t(\Pi)$. Let us define

$$\mathcal{C}_i = \bigcup_{\Pi \in \mathcal{P}} \mathcal{C}_i(\Pi), \quad i = 1, 2, \ldots, t.$$

Then $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$ are the required *maximal non-intersecting classes* of lines.

Let l_i be incident with a plane Π and let $\mathcal{C}_i$ be the class of lines containing l_i. Then, l_i belongs to $\mathcal{C}_i(\Pi)$, for some $i = 1, 2, \ldots, t$. Let $\mathcal{C}_j$ be a different class of lines ($j = 1, 2, \ldots, t$ and $i \neq j$) so that $l_i \notin \mathcal{C}_j$. Then $\mathcal{C}_j(\Pi)$ is a different class of lines in Π such that there exists a line l_j which intersect with l_i (in fact, every line in $\mathcal{C}_j(\Pi)$ will intersect with l_i). Thus, property of *maximal non-intersecting sets* as stated in Remark 3.5 holds.

When $\beta = 1$, it is nothing but a collection of 2-dimension nets and hence (N5$^{(3)}$) (a) and (b) is obvious.

□

Since the proof of Theorem 1.2 in [3] does not hold when $t = 1$, we give below a modified proof for the special case.

Theorem 3.25 *Let N be a 3-net as defined in 3.17 with the parameters $t = 1$ and s, β and r are greater than 1. Then every line is incident with exactly β planes, i.e., $r = \beta$.*

Proof : Let β be the number of classes of planes. Let l be a line and p be a point on l. Suppose l lies on a plane Π and let $\mathcal{B}$ be the class of planes which contains Π. For each $i \in 1, 2, \ldots, \beta - 1$ and for each parallel class of planes $\mathcal{B}_i$ other than the one containing Π, there exists a plane Π_i in each class that contains the point p. For each i, let l_i be the line of intersection of Π and Π_i. Since $t = 1$, Π and $\Pi_i's$ are $(s, 1)$-nets and there is only one parallel class of lines in Π and Π_i. Since l and l_i are lines on Π containing a common point p, they are not parallel and hence they must be the same, that is $l = l_i$, for all i. Hence, l is incident with β planes (Π and $\Pi_i, i = 1, 2, \ldots, \beta - 1$).

Chapter 4

Translation Nets in 3-dimension

We attempt to introduce the definition of Partial Congruence Partition (*PCP*) in 3-dimension for the first time. Several examples and results (similar to the two dimensional cases) that we attempted to prove are successful. I acknowledge that one of the most useful references is Mai Thi Thanh Hien's Honours project. Suggestions from my supervisor are immense and it helped me in proving the important results (sections 4.2 and 4.3) of this chapter. Definitions, examples, main results and properties of a *PCP* in 3-dimension are new research materials.

4.1 Definitions and Examples

Definition 4.1 *(Translation Net and Translation Group in 3-dimension)*
Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be an (s, t, β, r)-net. N with an automorphism group $\mathcal{G}$ (regard automorphism as permutation) such that

(a) $\mathcal{G}$ acts regularly on $\mathcal{X}$; and

(b) $\mathcal{G}$ fixes each parallel class of planes,

*is called a (s, t, β, r)-translation net or simply a **translation net of dimension three** and the group $\mathcal{G}$ is called the **translation group of dimension three**. We call the former as 3d-translation net while the latter as 3d-translation group.*

The immediate question one can ask is, whether there is a PCP setting similar to the one of dimension two. We answer this question in the following definition.

Definition 4.2 *(Partial Congruence Partition - PCP in 3-dimension) Let G be a group of order s^3, $s \geq 2$. Let $\mathcal{K} \subseteq \{ K : K \leq G$ and $|K| = s^2 \}$ (i.e., set of some subgroups of G of order s^2). Let*

$\mathcal{H} = \{ H = K \cap K' : K, K' \in \mathcal{K}, K \neq K' \}$, such that the following conditions hold:

(a) for all $H, H' \in \mathcal{H}$, either $H = H'$ or $H \cap H' = \{ e \}$, where e is the identity element of G;

(b) for all $H \in \mathcal{H}, |H| = s$ and by the choice of $K \in \mathcal{K}$ we have, for all $K \in \mathcal{K}, |K| = s^2$;

(c) for all $K \in \mathcal{K}$, there are exactly t distinct subgroups $H \in \mathcal{H}$ such that $H \subset K, t \geq 2$; and

(d) for all $H, H' \in \mathcal{H}$, there exists a subgroup $K \in \mathcal{K}$ such that $H \cup H' \subset K$.

Then $(\mathcal{K}, \mathcal{H})$ is called a ***partial congruence partition in G of dimension three***[1]*, denoted by PCP$^{(3)}$*[1]*, where the elements of $\mathcal{K}$ are called plane components and the elements of $\mathcal{H}$ are called line components. If we assume $|\mathcal{K}| = \beta$, we call the partial congruence partition as (s, t, β)-PCP*[1] *in G.*

Example 4.3 *Let $\mathbb{F}_n$ be the finite field with n elements, and consider the following*

(i) $G = \mathbb{F}_n^3$, i.e., G is the set of 3-dimensional vectors of $\mathbb{F}_n$;

(ii) $\mathcal{K}$ be the collection of all the 2-dimensional subspaces of G; and

(iii) $\mathcal{H}$ be the collection of all the 1-dimensional subspaces of G.

[1] *Going forward, to distinguish the PCP in Definition 2.10 and the one in Definition 4.2, we denote the former by PCP$^{(2)}$ and the latter by PCP$^{(3)}$.*

Then $(\mathcal{K}, \mathcal{H})$ *forms a* $PCP^{(3)}$ *of* G*.*

Example 4.4 *(Construction of a PCP$^{(3)}$ for s=4) Consider the field $\mathbb{F}_4$ with 4 elements, $\mathbb{F}_4 = \{\, 0, 1, a, a^2 \,\}$, where $a + 1 = a^2$, $a + a^2 = 1$, $a^2 + a^2 = 0$ and $a^2 + 1 = a$.*

Define G to be the set of 3-dimensional vectors in the vector space $\mathbb{F}_4^{(3)}$, i.e., $G = \{\, xyz : x, y, z \in \mathbb{F}_4 \,\}$; where we have abused the notation (x, y, z) of a 3-dimensional vectors by simply writing it as xyz. Note that $|G| = 4^3$ and G has a group structure with respect to addition based on the structure in $\mathbb{F}_4$ as shown below.

+	0	1	a	a^2
0	0	1	a	a^2
1	1	0	a^2	a
a	a	a^2	0	1
a^2	a^2	a	1	0

Now, we need to construct the collection of subgroups $\mathcal{K}$ and $\mathcal{H}$. Let us first form the elements of $\mathcal{K}$ as shown below.

$$\begin{aligned}
K_1 &= \langle\, 010, 100 \,\rangle = \{\, x(010) + y(100) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 010, 0a0, 0a^20, 100, a00, a^200, 110, a10, a^210, 1a0, aa0, a^2a0, \\
&\qquad 1a^20, aa^20, a^2a^20 \,\} \\
K_2 &= \langle\, 001, 100 \,\rangle = \{\, x(001) + y(100) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 001, 00a, 00a^2, 100, a00, a^200, 101, a01, a^201, 10a, a0a, a^20a, \\
&\qquad 10a^2, a0a^2, a^20a^2 \,\} \\
K_3 &= \langle\, 001, 010 \,\rangle = \{\, x(001) + y(010) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 001, 00a, 00a^2, 010, 0a0, 0a^20, 011, 0a1, 0a^21, 01a, 0aa, 0a^2a, \\
&\qquad 01a^2, 0aa^2, 0a^2a^2 \,\}
\end{aligned}$$

$$
\begin{aligned}
K_4 &= \langle\, 100, 011 \,\rangle = \{\, x(100) + y(011) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 100, a00, a^200, 011, 0aa, 0a^2a^2, 111, 1aa, 1a^2a^2, a11, aaa, \\
&\qquad aa^2a^2, a^211, a^2aa, a^2a^2a^2 \,\} \\
K_5 &= \langle\, 010, 101 \,\rangle = \{\, x(010) + y(101) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 010, 0a0, 0a^20, 101, a0a, a^20a^2, 111, a1a, a^21a^2, 1a1, aaa, \\
&\qquad a^2aa^2, 1a^21, aa^2a, a^2a^2a^2 \,\} \\
K_6 &= \langle\, 001, 110 \,\rangle = \{\, x(001) + y(110) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 001, 00a, 00a^2, 110, aa0, a^2a^20, 111, aa1, a^2a^21, 11a, aaa, \\
&\qquad a^2a^2a, 11a^2, aaa^2, a^2a^2a^2 \,\} \\
K_7 &= \langle\, 110, 011 \,\rangle = \{\, x(110) + y(011) : x, y \in \mathbb{F}_4 \,\} \\
&= \{\, 000, 110, aa0, a^2a^20, 011, 0aa, 0a^2a^2, 101, 1a^2a, 1aa^2, aa^21, \\
&\qquad a0a, a1a^2, a^2a1, a^21a, a^20a^2 \,\}
\end{aligned}
$$

Define the set $\mathcal{K} := \{\, K_1, K_2, \ldots, K_7 \,\}$. Then, $|K_i| = 4^2$ for all $i = 1, 2, \ldots, 7$. Next, we construct the collection of subgroups $\mathcal{H}$ by the intersection of the elements in $\mathcal{K}$.

$$
\begin{aligned}
H_1 &= \langle\, 100 \,\rangle = K_1 \cap K_2 \cap K_4 = \{\, 000, 100, a00, a^200 \,\} \\
H_2 &= \langle\, 010 \,\rangle = K_1 \cap K_3 \cap K_5 = \{\, 000, 010, 0a0, 0a^20 \,\} \\
H_3 &= \langle\, 001 \,\rangle = K_2 \cap K_3 \cap K_6 = \{\, 000, 001, 00a, 00a^2 \,\} \\
H_4 &= \langle\, 110 \,\rangle = K_1 \cap K_6 \cap K_7 = \{\, 000, 110, aa0, a^2a^20 \,\} \\
H_5 &= \langle\, 101 \,\rangle = K_2 \cap K_5 \cap K_7 = \{\, 000, 101, a0a, a^20a^2 \,\} \\
H_6 &= \langle\, 011 \,\rangle = K_3 \cap K_4 \cap K_7 = \{\, 000, 011, 0aa, 0a^2a^2 \,\} \\
H_7 &= \langle\, 111 \,\rangle = K_4 \cap K_5 \cap K_6 = \{\, 000, 111, aaa, a^2a^2a^2 \,\}
\end{aligned}
$$

Define the set $\mathcal{H} := \{H_1, H_2, \ldots, H_7\}$. Then, $|H_i| = 4$ for all $i = 1, 2, \ldots, 7$. It is straightforward to see the parameters value are *(i)* $s = 4$*;*

(ii) since for each $K \in \mathcal{K}$ *there are three* $H_i's$ *contained in it (e.g.,* K_1 *contains* H_1, H_2 *and* H_4*), then* $t = 3$ *and*

(iii) since $|\mathcal{K}| = 7$*, then* $\beta = 7$*.*

Hence, the group G *together with the collection* $\mathcal{K}$ *and* $\mathcal{H}$ *as defined above forms a* $(4, 3, 7)$*-PCP*$^{(3)}$.

Example 4.5 *(Generalized construction of Example 4.4) Let* $\mathbb{F}_n$ *be a finite field with* n *elements,* $\mathcal{K}$ *be a collection of 2-dimensional subspaces of* $\mathbb{F}_n^3$ *and* $\mathcal{H}$ *be a collection of 1-dimensional subspaces of* $\mathbb{F}_n^3$ *such that it generate a 3-dimension net. Let* $\mathbb{F}_k$ *be a finite extension of* $\mathbb{F}_n$*, where the the order is* $k = n^m$*, for some* m*. For each* K *in* $\mathcal{K}$ *take a basis* $\{ x, y \}$ *for* K*. Then,* $K = \{ ax + by : a, b \in \mathbb{F}_n \}$*. Define the following:*

$$K' := \{ ax + by : a, b \in \mathbb{F}_k \} \text{ and}$$
$$\mathcal{K}' := \{ K' : K \in \mathcal{K} \}.$$

We check whether this $\mathcal{K}'$ *generate a PCP*$^{(3)}$ *of* $\mathbb{F}_k^3$ *(and hence it generates a 3-dimensional net).*

To check this, we need to prove the following properties [note that (ii) and (iii) corresponds to the properties in the Definition of $PCP^{(3)}$].

(i) Definition of K' is independent of the choice of basis for K.

(ii) For each $K' \in \mathcal{K}'$, we have $|K'| = k^2$.

(iii) Construct a collection $\mathcal{H}' = \{ H' = K_1' \cap K_2' : K_1', K_2' \in \mathcal{K}' \}$ satisfying the conditions (a) to (d) in the definition of $PCP^{(3)}$.

Proof :

(i) If there exists a different basis $\{ x', y' \}$ of K, we show that the definition for K' still holds. Since $x', y' \in K$ and $\{ x, y \}$ is a basis of K, we can write and x' and y' as follows:

$$x' = a_1 x + b_1 y \text{ , for some } a_1, b_1 \in \mathbb{F}_n$$
$$y' = a_2 x + b_2 y \text{ , for some } a_2, b_2 \in \mathbb{F}_n$$

Then for any $u \in K'$, we have

$$\begin{aligned} u &= ax' + by' \text{ , for some } a_1, b_1 \in \mathbb{F}_k \\ &= a(a_1 x + b_1 y) + b(a_2 x + b_2 y) \\ &= (aa_1 + a_2 b)x + (ab_1 + bb_2)y \\ &= a'x + b'y \text{ , where } a' = aa_1 + a_2 b \in \mathbb{F}_k \text{ and } b' = ab_1 + bb_2 \in \mathbb{F}_k \end{aligned}$$

Hence, the definition of K' holds for different bases of K.

(ii) It is clear from the definition of $K' \in \mathcal{K}'$ that $|K'| = |\mathbb{F}_k|^2 = k^2$.

(iii) We only show the construction of the set $\mathcal{H}'$; while the conditions (a) to (d) in the definition of $PCP^{(3)}$ are easy to check.

For each $H \in \mathcal{H}$, since H is a non-trivial intersection of two 2-dimensional subspaces, H is a 1-dimensional subspace of $\mathbb{F}_n^3$, say $H = \{ az : a \in F \}$ for some non-zero vector $z \in H$. Now, define $H' = \{ az : a \in \mathbb{F}_k \}$ and $\mathcal{H}' = \{ H' : H \in \mathcal{H} \}$. Suppose for $H \in \mathcal{H}$, $H = K_1 \cap K_2$ where $K_1, K_2 \in \mathcal{K}$. We need to show that H' is exactly the set $K_1' \cap K_2'$.

Let $K_1 = \{ax_1 + by_1 : a, b \in \mathbb{F}_n\}$ and $K_2 = \{ax_2 + by_2 : a, b \in \mathbb{F}_n\}$ where x_1, y_1 and x_2, y_2 are the vectors chosen to define K_1' and K_2' respectively.

Since $H \subset K_1,\ z = a_1x_1 + b_1y_1$, for some $a_1, b_1 \in \mathbb{F}_n$.

Since $H \subset K_2,\ z = a_2x_2 + b_2y_2$, for some $a_2, b_2 \in \mathbb{F}_n$.

But then $z \in K_1' \cap K_2'$, by the definition of K_1' and K_2'. Hence $H' \subset K_1' \cap K_2'$.

The intersection of two different 2-dimensional subspaces is either the zero vector or a 1-dimensional subspace. As $H' \subset K_1' \cap K_2'$, $K_1' \cap K_2' \neq \{ 0 \}$ and hence is a 1-dimensional subspace. So, $K_1' \cap K_2' = H'$.

4.2 PCP$^{(3)}$ $\Leftrightarrow$ 3d-Translation Nets

Before we prove the equivalence of $PCP^{(3)}$ and 3d-translation nets, we first prove a useful lemma from the conditions in Definition 3.17.

Lemma 4.6 *If $l \in \mathcal{L}$ then there exists $\Pi_1, \Pi_2 \in \mathcal{P}$ such that $(l, \Pi_1) \in \mathcal{I}$ and $(l, \Pi_2) \in \mathcal{I}$.*

Proof : Let $p \in \mathcal{X}$ such that $(p, l) \in \mathcal{I}$. Since $\beta > 1$, there exist Π and Π' ($\Pi \neq \Pi'$) in $\mathcal{P}$ from different classes of planes such that $(p, \Pi) \in \mathcal{I}$ and $(p, \Pi') \in \mathcal{I}$ (by (N4$^{(3)}$(a) in Definition 3.17). If $(l, \Pi) \in \mathcal{I}$ and $(l, \Pi') \in \mathcal{I}$, we are done.

If not, we might have either $(l, \Pi) \notin \mathcal{I}$ or $(l, \Pi') \notin \mathcal{I}$, or both. WLOG, we assume $(l, \Pi) \notin \mathcal{I}$. Since $t > 1$, we must have l_1 and l_2 in Π ($i.e.$, $(l_1, \Pi) \in \mathcal{I}$ and $(l_2, \Pi) \in \mathcal{I}$) such that $(p, l_1) \in \mathcal{I}$ and $(p, l_2) \in \mathcal{I}$.

Consider l and l_1, since $(p, l) \in \mathcal{I}$ and $(p, l_1) \in \mathcal{I}$, by (N2$^{(3)}$) there exists a $\Pi_1 \in \mathcal{P}$ such that $(l, \Pi_1) \in \mathcal{I}$ and $(l_1, \Pi_1) \in \mathcal{I}$. Similarly for l and l_2, there is a $\Pi_2 \in \mathcal{P}$ such that $(l, \Pi_2) \in \mathcal{I}$ and $(l_2, \Pi_2) \in \mathcal{I}$. Now we have $(l, \Pi_1) \in \mathcal{I}$ and

$(l, \Pi_2) \in \mathcal{I}$. So, it remains to show that $\Pi_1 \neq \Pi_2$. Observe that $(l_i, \Pi) \in \mathcal{I}$ and $(l_i, \Pi_i) \in \mathcal{I}$ for $i = 1, 2$ implies $\Pi_1 \neq \Pi_2$ (as $l_1 \neq l_2$).

Proposition 4.7 *(PCP$^{(3)}$ $\Rightarrow$ 3-net) Let G be a group of order s^3 ($s \geq 2$) and $(\mathcal{K}, \mathcal{H})$ be a PCP$^{(3)}$ as defined in 4.2. Define a plane geometry $(\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ where*

(i) $\mathcal{X} = G$;

(ii) $\mathcal{L} = \{ gH : g \in G \text{ and } H \in \mathcal{H} \}$;

(iii) $\mathcal{P} = \{ gK : g \in G \text{ and } K \in \mathcal{K} \}$; *and*

(iv) $\mathcal{I} = \{ (g, hH) : g, h \in G, H \in \mathcal{H} \text{ and } g \in hH \} \bigcup \{ (g, kK) : g, k \in G, K \in \mathcal{K} \text{ and } g \in kK \} \bigcup \{ (hH, kK) : h, k \in G, H \in \mathcal{H}, K \in \mathcal{K} \text{ and } hH \subset kK \}$.

Then $(\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ is a 3-net.

Proof : To prove that the plane geometry forms a 3-net, we prove the conditions in Definition 3.17 (note that $s \geq 2$ and axiom (c) in Definition 4.2 guarantees that s, t, r and β are greater than one).

(*N1*$^{(3)}$) If $(g, hH) \in \mathcal{I}$ for some $g, h \in G$ and $H \in \mathcal{H}$ and $hH \subset kK$ for some $k \in G$ and $kK \in \mathcal{K}$, then clearly $(g, kK) \in \mathcal{I}$.

(*N2*$^{(3)}$) Let $gH, hH' \in \mathcal{L}$ be two lines of the geometry such that $gH \cap hH' \neq \emptyset$, for some $g, h \in G$ and $H, H' \in \mathcal{H}$. Since $gH \cap hH^{'} \neq \emptyset$, and $H \cap H^{'} = \{ e \}$, by Lemma 2.11, we have $gH \cap hH' = \{ x \}$, for some $x \in G$.

By axiom (d) of *PCP*$^{(3)}$ there exists some $K \in \mathcal{K}$ such that $H \cup H' \subset K$ and hence $H, H' \subset K$. Now we have

$$x \in gH \text{ implies } xH = gH, \text{ and } H \subset K \text{ implies } xH \subset xK,$$

together we have $xH = gH \subset xK$. Similarly for $x \in hH'$ we have

$$x \in hH' \text{ implies } xH' = hH', \text{ and } H' \subset K \text{ implies } xH' \subset xK,$$

again combining the two statements we have $xH' = hH' \subset xK$. This implies that gH and $hH^{'}$ lie on xK for some $K \in \mathcal{K}$.

($N3^{(3)}$) For any $K \in \mathcal{K}$, the conditions (a), (b) and (c) in Definition 4.2 gives the *PCP*$^{(2)}$ for a 2-net. Thus, by Proposition 2.13, we get a (s,t)-net in K.

($N4^{(3)}$) Since we are dealing with a finite group, counting the number of elements in $\mathcal{K}$ will give us the desired partition number, let us say β. For each $K \in \mathcal{K}$, define

$$\mathcal{B}_K := \{\, gK : g \in G \,\}.$$

Then $\{\, \mathcal{B}_K \,\}_{K \in \mathcal{K}}$ gives us a partition of $\mathcal{P}$. The set $\mathcal{B}_K$ is often referred as the set of all *left cosets* of K in G. There are β such non-empty, distinct subgroups in $\mathcal{K}$, giving β disjoint, non-empty parallel classes. To see (a) and (b), observe the following:

(a) Let $g \in G$. Then, for all $K \in \mathcal{K}$, we have $g \in gK$ and $g \notin hK$ for any $h \in G, h \neq g$, because $gK \cap hK$ is either empty or they are equal.

(b) Let aK_1, bK_2 be planes from distinct classes ($K_1 \neq K_2$). By Lemma 2.11 and by the definition of $\mathcal{H}$, we get $aK_1 \cap bK_2 = g(K_1 \cap K_2) = gH$, for some $g \in G, H \in \mathcal{H}$ and the uniqueness is by the choice of g, K_1 and K_2.

($N5'^{(3)}$) Let $H \in \mathcal{H}$, then $H = K \cap K'$ for some $K, K' \in \mathcal{K}$ (by Definition of $PCP^{(3)}$ in 4.2). Thus $H \subset K$ and hence for any $g \in G$ we have $gH \subset gK$, proving that every line is incident with at least one plane.

Remark 4.8 *We can say a stronger fact about ($N5'^{(3)}$) in the proof of Proposition 4.7. Since $H = K \cap K'$, we can conclude that $r \geq 2$, i.e., every line is incident with at least two planes.*

Theorem 4.9 *Partial congruence partition-PCP$^{(3)}$ with the setting in Proposition 4.7 gives rise to a 3d-net and conversely, every translation net of dimension three produces a PCP$^{(3)}$.*

Proof :

($\Rightarrow$) This direction is trivial from Proposition 4.7.

($\Leftarrow$) To prove this direction we need to construct a group of order s^3 with proper subgroups of orders s^2 and s, satisfying the Definition 4.2. First we construct the group and it's subgroups.

Let $\mathcal{G}$ be the translation group of a 3-net, $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$, such that the points of N are identified with elements of $\mathcal{G}$, while the lines and the planes are subsets of $\mathcal{G}$. Let $\mathcal{K}$ be the set of planes containing identity element e. We need to prove that the elements of $\mathcal{K}$ are indeed subgroups of $\mathcal{G}$. Let $K \in \mathcal{K}$, then to prove that K is a subgroup, it is enough to prove the closure of K, since we are dealing with a finite set, *i.e.,* if $g, h \in K$ then we need to show $gh \in K$. By Definition 4.1(b), if K is a plane of the net, then for any $g \in \mathcal{G}$,

$$K^g := \{ \, gh : h \in K \, \}$$

is also a plane in the same parallel class of K. Then, either $K^g = K$ or $K^g \cap K = \emptyset$. Since $e \in K$, then for any $g \in K$, we have $g \in K^g \Rightarrow K^g = K$. Hence for any $g, h \in K$, we have $gh \in K^g = K$. Thus, K is a subgroup of $\mathcal{G}$, that is, the planes of N containing the identity are the subgroups of $\mathcal{G}$. Now, observe the following properties:

(i) By Theorem 3.9 (ii), we get the order of the subgroups $|K| = s^2$ and

(ii) Since $|\mathcal{G}| = |\mathcal{X}|$, by Lemma 3.16, we get the order of the group $|\mathcal{G}| = s^3$.

Next, we prove the conditions (a) to (d) in the Definition 4.2.

(a) Lemma 4.6 shows that the lines of N are intersection of planes and we have shown above that the planes are subgroups of $\mathcal{G}$. Hence, the lines of N are the subgroups H of K. Now form the collection $\mathcal{H}$ of subgroups of intersection of K's such that the pairwise intersection of every element in this collection is trivial.

(b) Every line of the net inside a plane contains s points. Hence, the subgroups in $\mathcal{H}$ has precisely s points each.

(c) This is true by axiom (N3$^{(3)}$) in Definition 3.17.

(d) This is true by axiom (N2$^{(3)}$) in Definition 3.17.

Thus $(\mathcal{K}, \mathcal{H})$ is a *PCP*$^{(3)}$.

4.3 Upper bounds for β and $T_G^{(3)}$

Similar to 2-dimensional case, we have established 1-1 correspondence between *PCP*$^{(3)}$ and *3d-Translation Nets* in Theorem 4.9. Hence, the existence of translation nets in three dimension is again a purely group-theoretic problem, as in the case of two dimension. In order to find the maximum number of parallel class of planes for which a (s, t, β)-translation nets can exist, it is therefore we formulate this in group-theoretic language as follows:

Let G be a group of order $s^3 \geq 1$. Find the number $T_G^{(3)}$ (defined below) precisely or at least find a bound for it.

$$T_G^{(3)} := max \ \{ \ \beta \leq s^2 + s + 1 : \text{ there exists an } (s, t, \beta)\text{-}PCP \text{ (or } PCP^{(3)}) \text{ in } G \ \} \tag{4.1}$$

Lemma 4.10 *Let G be a group of order $s^3 > 1$ and $\mathcal{K}$ be the corresponding PCP$^{(3)}$ in G. Then, the number of plane components in PCP$^{(3)}$ is at most s^2+s+1.*

Proof : Let β be the number of plane components in the *PCP*$^{(3)}$ of G. By Theorem 3.15(ii), we have

$$\beta = t^2 - t + 1,$$

and by Lemma 2.15 we have

$$t \leq s + 1.$$

Thus,

$$\begin{aligned}\beta &= t^2 - t + 1 \\ &\leq (s+1)^2 - (s+1) + 1 \\ &= s^2 + s + 1\end{aligned}$$

Remark 4.11 *Lemma 4.10 above tells that the number of parallel classes of planes (β) in a 3-net is at most $s^2 + s + 1$.*

Remark 4.12 *Following the proof of Lemma 4.10, we see that the maximum value of β can be attained if and only if the maximum value of t is attained.*

Example 4.13 *(Construction of a PCP$^{(3)}$ with maximum value of β) We take $s = 2$ and construct a PCP$^{(3)}$ with maximum value of β, i.e., $\beta = 7$. Consider the field $\mathbb{F}_2$ with 2 elements. Let us define G to be the set of 3-dimensional vectors in the vector space $\mathbb{F}_2^{(3)}$, i.e., $G = \{\, xyz : x, y, z \in \mathbb{F}_2 \,\}$; where we have abused the notation (x, y, z) of a 3-dimensional vectors by simply writing it as xyz. Note that $|G| = 2^3$ and G has a group structure with respect to addition based on the structure in $\mathbb{F}_2$. Now we construct the collection of subgroups $\mathcal{K}$ and $\mathcal{H}$. First, let us form the elements of $\mathcal{K}$ as shown below.*

$$\begin{aligned}K_1 &= \langle\, 010, 100 \,\rangle = \{\, 000, 010, 100, 110 \,\} \\ K_2 &= \langle\, 001, 100 \,\rangle = \{\, 000, 001, 100, 101 \,\} \\ K_3 &= \langle\, 001, 010 \,\rangle = \{\, 000, 001, 010, 011 \,\}\end{aligned}$$

$$K_4 = \langle\, 100, 011 \,\rangle = \{\, 000, 100, 011, 111 \,\}$$

$$K_5 = \langle\, 010, 101 \,\rangle = \{\, 000, 010, 101, 111 \,\}$$

$$K_6 = \langle\, 001, 110 \,\rangle = \{\, 000, 001, 110, 111 \,\}$$

$$K_7 = \langle\, 110, 011 \,\rangle = \{\, 000, 110, 011, 101 \,\}$$

Define the set $\mathcal{K} := \{\, K_1, K_2, \ldots, K_7 \,\}$. *Then,* $|K_i| = 2^2$ *for all* $i = 1, 2, \ldots, 7$. *Next, we construct the collection of subgroups* $\mathcal{H}$ *by the intersection of the elements in* $\mathcal{K}$.

$$H_1 = \langle\, 100 \,\rangle = K_1 \cap K_2 \cap K_4 = \{\, 000, 100 \,\}$$

$$H_2 = \langle\, 010 \,\rangle = K_1 \cap K_3 \cap K_5 = \{\, 000, 010 \,\}$$

$$H_3 = \langle\, 001 \,\rangle = K_2 \cap K_3 \cap K_6 = \{\, 000, 001 \,\}$$

$$H_4 = \langle\, 110 \,\rangle = K_1 \cap K_6 \cap K_7 = \{\, 000, 110 \,\}$$

$$H_5 = \langle\, 101 \,\rangle = K_2 \cap K_5 \cap K_7 = \{\, 000, 101 \,\}$$

$$H_6 = \langle\, 011 \,\rangle = K_3 \cap K_4 \cap K_7 = \{\, 000, 011 \,\}$$

$$H_7 = \langle\, 111 \,\rangle = K_4 \cap K_5 \cap K_6 = \{\, 000, 111 \,\}$$

Define the set $\mathcal{H} := \{H_1, H_2, \ldots, H_7\}$. *Then,* $|H_i| = 2$ *for all* $i = 1, 2, \ldots, 7$. *It is straightforward to see the parameters value are*

(i) $s = 2$;

(ii) since for each $K \in \mathcal{K}$ *there are three* $H_i's$ *contained in it (e.g.,* K_1 *contains* H_1, H_2 *and* H_4*), then* $t = 3$ *and*

(iii) since $|\mathcal{K}| = 7$*, then* $\beta = 7$.

Hence, the group G *together with the collection* $\mathcal{K}$ *and* $\mathcal{H}$ *as defined above forms a* $(2, 3, 7)$*-PCP*$^{(3)}$.

Example 4.14 *(Generalized construction of PCP*$^{(3)}$ *with maximum value of* β*)* *Let* $\mathbb{F}_n$ *be a finite field and* $G = \mathbb{F}_n^3$*,* $\mathcal{K}$ *be the collection of all the 2-dimensional* *subspaces of G and $\mathcal{H}$ be the collection of all the 1-dimensional subspaces of G, then $(\mathcal{K}, \mathcal{H})$ is a $PCP^{(3)}$ of G, as seen in Example 4.3. If we set $s = n$ and then we calculate the value of β using the following result.*

> *If V is an m-dimensional vector space over $\mathbb{F}_n$, then the number of k-dimensional subspaces of V is*
>
> $$\frac{(n^m-1)(n^m-n)\cdots(n^m-n^{k-1})}{(n^k-1)(n^k-n)\cdots(n^k-n^{k-1})}$$

Here, to find out the value of β means, to find the number of elements in the collection $\mathcal{K}$. Hence, use $m = 3$ and $k = 2$ in the above result and we get

$$\begin{aligned}\beta &= \frac{(n^3-1)(n^3-n)}{(n^2-1)(n^2-n)}\\ &= \frac{(n-1)(n^2+n+1)(n)(n^2-1)}{(n^2-1)(n)(n-1)}\\ &= (n^2+n+1)\end{aligned}$$

Since $n = s$, we have $\boxed{\beta = s^2+s+1}$.

4.4 Some results from group theory

Theorem 4.15 *Let G be a nilpotent group of order s^3 $(s \geq 2)$ and p be a prime divisor of s. Let P be a Sylow p-subgroup of G. If there exists a collection of subgroups $\mathcal{K}$ and $\mathcal{H}$ as defined in Definition 4.2, i.e., if there exists a (s, t, β)-PCP$^{(3)}$ of G, then there exists a PCP$^{(3)}$ of P.*

Proof : Let $(\mathcal{K}, \mathcal{H})$ be a *PCP*$^{(3)}$ of G such that $|G| = s^3 = p^{3r} \cdot n$, for some positive integers r and n. Since G is nilpotent, P is a normal subgroup (unique Sylow p-subgroup) of G. For any $K \in \mathcal{K}$ (note that $|K| = s^2 = p^{2r} \cdot m$, for some positive integer m), we have $K \cap P$ is a Sylow p-subgroup of K and hence $|K \cap P| = p^{2r}$. Also for any $H \in \mathcal{H}$, by the same reasoning we have $|H \cap P| = p^r$.

Now replace K by $K \cap P$ and H by $H \cap P$ and form the new collection of subgroups $\mathcal{K}'$ and $\mathcal{H}'$ respectively. By above details, the conditions (a) and (b) in the definition of *PCP*$^{(3)}$ for ($\mathcal{K}'$,$\mathcal{H}'$) of the group P are verified. Axiom (c) is true because the new collection $\mathcal{H}'$ is induced by the old collection $\mathcal{H}$. To prove the axiom (d), let $H_1', H_2' \in \mathcal{H}'$ such that

(i) $H_1' = H_1 \cap P$ and $H_2' = H_2 \cap P$, where $H_1, H_2 \in \mathcal{H}$ and

(ii) $(H_1 \cup H_2) \subset K$ for some $K \in \mathcal{K}$.

Then,

$$\begin{aligned} H_1' \cup H_2' &= (H_1 \cap P) \cup (H_2 \cap P) \\ &= (H_1 \cup H_2) \cap P \\ &\subset K \cap P \\ &= K', \text{ for some } K' \in \mathcal{K}' \text{ such that } K' = K \cap P \end{aligned}$$

□

In the above theorem we discussed *PCP*$^{(3)}$ from nilpotent groups. But, it may not be possible to generalize the result to any groups. However, we can still obtain a weaker result (for any group), which we prove in the Theorem 4.18. First, we observe some useful lemmas.

Lemma 4.16 *Suppose* $(\mathcal{K}, \mathcal{H})$ *is a PCP*$^{(3)}$ *on a group* G *of order* s^3*. If* $H \in \mathcal{H}$ *and* $K \in \mathcal{K}$ *such that* $H \not\subset K$*, then* $H \cap K = \{ e \}$ *and* $HK = G$*.*

Proof : Every subgroup $H \in \mathcal{H}$ is formed by the intersection of $K's$ in $\mathcal{K}$, but $H \not\subset K$ implies H contains no other element of K except the identity element e. Thus H intersects trivially with K. The second part is trivial from first part by counting the number of elements in HK.

Lemma 4.17 *Suppose* $(\mathcal{K}, \mathcal{H})$ *is a PCP*$^{(3)}$ *on a group* G *of order* s^3*. Then for any two distinct* $K_1, K_2 \in \mathcal{K}$*, we have* $K_1K_2 = G$*.*

Proof : Let $K_1 \neq K_2$ be two elements of $\mathcal{K}$ in the *PCP*$^{(3)}$ of G. Then $K_1 \cap K_2 = H$ for some $H \in \mathcal{H}$ such that $|H| = s$, by axiom (b) of *PCP*$^{(3)}$. Hence,

$$|K_1K_2| = \frac{|K_1| \cdot |K_2|}{|K_1 \cap K_2|} = \frac{s^2 \cdot s^2}{s} = s^3.$$

Since $K_1K_2 \subseteq G$, we get $K_1K_2 = G$.

Theorem 4.18 *Let G be a group of order s^3 $(s \geq 2)$ and p be a prime divisor of s. Let P be a Sylow p-subgroup of G. If there exists a collection of subgroups $\mathcal{K}$ and $\mathcal{H}$ as defined in Definition 4.2, i.e., if there exists a (s, t, β)-PCP$^{(3)}$ of G, then there exists collections $\mathcal{K}'$ and $\mathcal{H}'$ such that $(\mathcal{K}', \mathcal{H}')$ satisfies the conditions (a) to (c) in Definition of PCP$^{(3)}$ for P.*

Proof : Let $(\mathcal{K}, \mathcal{H})$ be a *PCP*$^{(3)}$ of G. For each $K \in \mathcal{K}$, there is a $g \in G$ such that with $gKg^{-1} \cap P$ is a Sylow p-subgroup of gKg^{-1} (by Lemma 2.23). Now, replace K by gKg^{-1} and define the new collection of subgroups $\mathcal{K}'$ as follows:

$$\mathcal{K}' = \{\ (g_iK_ig_i^{-1}) \cap P : K_i \in \mathcal{K}, 1 \leq i \leq \beta\ \}$$

Then, for each i, $\left|(g_iK_ig_i^{-1}) \cap P\right| = p^{2r}$ and moreover, $|\mathcal{K}'| = \beta$ since $|\mathcal{K}| = \beta$.

Let us define the new collection of subgroups $\mathcal{H}'$.

$$\mathcal{H}' = \{\ H' = K_1' \cap K_2' : K_1', K_2' \in \mathcal{K}'\ \}$$

(a) Let $H_1', H_2' \in \mathcal{H}'$, and $K_i' \in \mathcal{K}'$ such that

(i) $H_1' \neq H_2'$, $H_1' = K_1' \cap K_2'$, $H_2' = K_3' \cap K_4'$;

(ii) $K_i' = g_iK_ig_i^{-1}$ for $K_i \in \mathcal{K}$ and $i = 1, 2, 3, 4$;

(iii) $K_1 \cap K_2 = H_1, K_3 \cap K_4 = H_1$, for $H_1, H_2 \in \mathcal{H}$ and

(iv) either $H_1 \not\subseteq K_3$ or $H_1 \not\subseteq K_4$

Then,

$$
\begin{aligned}
H_1' \cap H_2' &= (K_1' \cap K_2') \bigcap (K_3' \cap K_4') \\
&= [(g_1K_1g_1^{-1}) \cap (g_2K_2g_2^{-1})] \bigcap [(g_3K_3g_3^{-1}) \cap (g_4K_4g_4^{-1})] \bigcap P \\
&= [g(K_1 \cap K_2)g^{-1}] \bigcap [(g_3K_3g_3^{-1}) \cap (g_4K_4g_4^{-1})] \bigcap P\ , \\
&\qquad\qquad \text{for some } g \in G \text{ [by Lemmas 4.17 and 2.21]} \\
&= [(gH_1g^{-1}) \ \cap \ (g_3K_3g_3^{-1})] \cap (g_4K_4g_4^{-1}) \cap P \\
&= \{\, e \,\} \ \cap \ (g_4K_4g_4^{-1}) \cap P, \ \text{if } H_1 \not\subseteq K_3 \text{ then use Lemma 4.16} \\
&\quad \text{and Remark 2.22 to get trivial intersection for the first two} \\
&\quad \text{terms (else we must have } H_1 \not\subseteq K_4\text{) and apply same argument} \\
&\quad \text{to } 1^{st} \text{ and } 3^{rd} \text{ term} \\
&= \{\, e \,\}
\end{aligned}
$$

(b) For $H' \in \mathcal{H}'$, we have

$$
\begin{aligned}
H' &= K_1' \cap K_2', \text{ for some } K_1', K_2' \in \mathcal{K}' \\
&= (g_1K_1g_1^{-1}) \cap (g_2K_2g_2^{-1}) \cap P, \text{ for some } g_1, g_2 \in G \text{ and } K_1, K_2 \in \mathcal{K} \\
&= g(K_1 \cap K_2)g^{-1} \cap P, \text{ for some } g \in G \text{ [by Lemmas 4.17 and 2.21]} \\
&= gHg^{-1} \cap P, \text{ for some } H \in \mathcal{H}
\end{aligned}
$$

By Lemma 2.23, $gHg^{-1} \cap P$ is a Sylow p-subgroup of gHg^{-1} and hence $|H'| = |gHg^{-1} \cap P| = p^r$.

(c) True by the definition of $\mathcal{K}'$ and $\mathcal{H}'$.

Theorem 4.19 *Let G be an abelian group of order s^3 $(s \geq 2)$. Suppose $(\mathcal{K}, \mathcal{H})$ is a PCP$^{(3)}$ of G. Then $G \cong H_1 \times H_2 \times H_3$, for three some $H_1, H_2, H_3 \in \mathcal{H}$.*

Proof : Let $K_1, K_2 \in \mathcal{K}$ such that $K_1 \cap K_2 = H_3$. Then, by Definition 4.2 axiom (c), there exists $H_1 \subset K_1$ and $H_2 \subset K_2$, where $H_1, H_2 \in \mathcal{H}$ and

$H_1 \neq H_3$, $H_2 \neq H_3$. Then, we have $H_3 \not\subset H_1 \times H_2$. By Definition 4.2 conditions (a) and (b), we see that each H_i $(i = 1, 2, 3)$ intersect trivially and $|G| = |H_1 H_2 H_3| = s^3$. Hence, using these facts and that $H_3 \not\subset H_1 \times H_2$, we conclude $G \cong H_1 \times H_2 \times H_3$.

Theorem 4.20 *Let G be group of order s^3 $(s \geq 2)$. Suppose $(\mathcal{K}, \mathcal{H})$ is a PCP$^{(3)}$ of G. Let $H \in \mathcal{H}$ such that H is a characteristic subgroup of some $K \in \mathcal{K}$ and $H \not\subset K'$ for some $K' \neq K$ in $\mathcal{K}$. If K, K' are normal subgroups of G, then $G \cong H \times K'$.*

Proof : Since $H \in \mathcal{H}$ and $K' \in \mathcal{K}$ such that $H \not\subset K'$, by Lemma 4.16 we have $H \cap K' = \{ e \}$ and $G = HK'$.

Since K' and K are normal subgroups of G, and H is characteristic in K, we have H is a normal subgroup of G. Hence, G is a direct product of H and K'.

Chapter 5

Association Scheme

In 1952, Bose and Shimamoto [19] introduced the concept of an association scheme of class n or simply n-class association scheme. In this chapter we establish a relationship between 3-net and association scheme of class three [3] and calculate various parameters that are not listed in [3].

5.1 Definition and Example

Definition 5.1 *Let A be a finite set and R_i's be non-empty subsets of the cartesian product of A, where $i \in \{0, 1, \cdots, n\}$, satisfying the following conditions:*

(AS1) $R_0 = \{ (x, x) : x \in A \}$*;*

(AS2) $A \times A = \bigcup_{i=0}^{i=n} R_i$ *and* $R_i \cap R_j = \emptyset$ *for* $i \neq j$*;*

(AS3) for each $i \in \{0, 1, \cdots, n\}$*, define* R_i^T *as follows:*

$$R_i^T := \{ (y, x) \in A \times A : (x, y) \in R_i \} ,$$

then $R_i^T = R_j$ *for some* $j \in \{0, 1, \cdots, n\}$*; and*

(AS4) for each $i, j, k \in \{0, 1, \cdots, n\}$ *and for each* $(x, y) \in R_i$ *, the number of* $z \in A$ *such that* $(x, z) \in R_j$ *and* $(z, y) \in R_k$ *, denoted by* p^i_{jk} *, is a constant.*

$S = (A; R_0, R_1, \cdots, R_n)$ *is an* ***association scheme of class n*** *on the finite set* A.

Furthermore, we say S *is* ***symmetric*** *if*

(AS5) for all $i = 1, 2, \cdots, n$, $R_i^T = R_i$ *; and*

we say S *is* ***commutative*** *if*

(AS6) for all $i, j, k = 1, 2, \cdots, n$, $p^i_{jk} = p^i_{kj}$.

Remark 5.2 *It is a well know fact that a symmetric association scheme is always a commutative association scheme.*

Example 5.3 *(Strongly Regular Graphs) Let* $G = (V, E)$ *be a strongly regular graph with* V *being the vertex set and* E *being the edge set of the graph (please refer to [4] for definition of strongly regular graph with parameters of an association scheme). Let us define the classes (*R_i*'s, for* $i = 0, 1, 2$*) in the following way:*

$R_0 = \{ (v, v) : v \in V \}$;

$R_1 = \{ (u, v) :$ *for* $u, v \in V$, *there exists an* $e \in E$ *such that* $e = uv \}$; *and*

$R_2 = \{ (u, v) :$ *for* $u, v \in V$, *there does not exists any* $e \in E$ *such that* $e = uv \}$.

Then $(G; R_0, R_1, R_2)$ *is a* ***symmetric association scheme of class 2*** *on* G.

5.2 2-net and Association Scheme of Class 2

Using the incidence relationship between the points and lines we prove that a 2-net gives rise to an association scheme of class 2. First, we define some of the notations below.

Notations : In this chapter, we will use the following notations for all $i, j, k = 1, 2, 3$.

(i) $n_i(p)$ denotes the number of i^{th} associates of $p \in \mathcal{X}$;

(ii) n_i denotes number of i^{th} associates of any point on the net; and

(iii) if $p, p' \in \mathcal{X}$ are i^{th} associates, the number of j^{th} associates of p which are k^{th} associates of p' is denoted by $p^i_{jk}(p, p')$ and the same is simply denoted by p^i_{jk} for any $p, p' \in \mathcal{X}$.

Theorem 5.4 *(2-net $\Rightarrow$ Association Scheme of Class 2) Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{I})$ be a 2-net. Define the following*

(a) $R_0 = \{ (x, x) : x \in \mathcal{X} \}$;

(b) $R_1 = \{ (x, y) : x, y \in l$ *for some* $l \in \mathcal{L}$ *and* $x \neq y \}$; *and*

(c) $R_2 = \{ (x, y) : x, y \notin l$ *for any* $l \in \mathcal{L}$ *and* $x \neq y \}$.

Then $S = (\mathcal{X}; R_0, R_1, R_2)$ is an association scheme of class 2 on $\mathcal{X}$.

Proof :

(AS1) R_0 satisfies (AS1) trivially.

(AS2) For any two points of the net p_1 and p_2, only one of the following scenarios can happen:

(i) there exists $l \in \mathcal{L}$ such that $(p_1, l) \in \mathcal{I}$ and $(p_2, l) \in \mathcal{I}$ (*i.e.,* they are joined by a line) or

(ii) for any $l \in \mathcal{L}$ one has $(p_1, l) \notin \mathcal{I}$ and $(p_2, l) \notin \mathcal{I}$ (*i.e.,* they are not joined by a line)

Hence, the definition of R_1 and R_2 coincide exactly with these possible scenarios, where R_1 corresponds to (i) while R_2 corresponds to (ii). They are also disjoint, *i.e.,* both the scenarios cannot happen at the same time. Thus, R_1 and R_2 gives a partition of $\mathcal{X} \times \mathcal{X}$ and thus satisfying (AS2).

(AS3) It is trivial to see that $R_i^T = R_i$ for all i and so it satisfies (AS6). Thus (AS3) is also satisfied (with $j = i$).

(AS4) We calculate the parameters below.

(i) $n_1 = t(s-1)$,

(ii) $n_2 = s^2 - t(s-1)$,

(iii) $p_{11}^1 = (s-2) + (t-1)(t-2)$,

(iv) $p_{11}^2 = t(t-1)$,

(v) $p_{22}^1 = n_2(p) - p_{21}^1(p, p')$,

(vi) $p_{21}^1 = p_{12}^1 = n_1(p) - p_{11}^1(p, p')$.

Now, it is clear that all the parameters are dependent on s and t, which are constants. Hence, p_{jk}^i is constant for all i, j, k.

5.3 3-net and Association Scheme of Class 3

In this section we use the incidence relationship between the points, lines and planes to define the association between them as in [3].

Definition 5.5 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net and let us define the following.*

(a) $R_0 = \{ (x, x) : x \in \mathcal{X} \}$*;*

(b) $R_1 = \{ (x, y) : x, y \in l$ *for some* $l \in \mathcal{L}$ *and* $x \neq y \}$*;*

(c) $R_2 = \{ (x, y) : x, y \notin R_0 \cup R_1$ *and* $x, y \in \Pi$ *for some* $\Pi \in \mathcal{P} \}$*; and*

(d) $R_3 = \mathcal{X}^2 \setminus (R_0 \cup R_1 \cup R_2)$.

We call the set of points x, y with $(x, y) \in R_1$ as ***first associates****, the set of points x, y with $(x, y) \in R_2$ as* ***second associates*** *and the set of points x, y with $(x, y) \in R_3$ as* ***third associates****.*

In words, we say that two distinct points of N are first associates if they lie on the same line, second associates if they do not lie on the same line but lie on the same plane and third associates if they do no lie on the same plane.

The following theorem is proved by R. Laskar in sections 2, 3 and 4 of [3].

Theorem 5.6 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net and R_0, R_1, R_2 and R_3 be defined as in Definition 5.5. Then $S = (\mathcal{X}; R_0, R_1, R_2, R_3)$ is an association scheme of class three on $\mathcal{X}$.*

Before we begin the calculation of parameters that are not listed in [3], we state some of the important results from [3], which are very much useful in our calculation.

Theorem 5.7 *Let p be a point of a 3-net and l be a line of the net such that p and l are non-coplanar. Then, there are $(\beta - r)$ second associates of p on l.*

Theorem 5.8 *Let p be a point of a 3-net and Π be a plane of the net such that p and Π are non-coplanar. Then*

(i) there are $(\beta - 1)$ lines in Π which are coplanar with p;

(ii) there are v points on Π such that they are first associates of p, where

$$v = \frac{(\beta - 1)((t - 1)}{t} \text{ ; and}$$

(iii) there are w points on Π such that they are second associates of p, where

$$w = \left(\frac{1}{t}\right)[(\beta - 1)(s - t + 1) + (st - \beta + 1)(\beta - r)].$$

Notations : In the following results, below notations are used, in addition to the ones defined in Section 5.2. For all $i, j, k = 1, 2, 3$;

(i) $n_i^{\Pi}(p)$ denotes number of i^{th} associates of a point p on the plane Π; and

(ii) if p, p' lie on a same plane Π such that they are i^{th} associates in Π, the number of j^{th} associates of p which are k^{th} associates of p' in Π is denoted by ${}^{\Pi}p^i_{jk}$.

Lemma 5.9 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net as in Definition 3.4 and for all $i, j, k = 1, 2, 3$, let n_i and p^i_{jk} be the notations as defined above, and u be the number of lines passing through any given point of the net. Then, for any two points p and p' of N, we have*

(i) $p^1_{13} = p^1_{31} = 0$,

(ii) $p^2_{13} = p^2_{31} = (u - t)(s - \beta + r)$,

(iii) $p^3_{13} = p^3_{31} = u(s - \beta + r - 1)$.

Proof :

(i) Let p and p' be first associates. Then p^1_{13} means that the number of first associates of p which are third associates of p'. That is same as saying, the number of first associates of p other than p' which are not the first or second associates of p', which is represented by the following equation.

$$p^1_{13} = (n_1(p) - 1) - p^1_{11}(p, p') - p^1_{12}(p, p') \tag{5.1}$$

Here, we use the following results from [3], in the above equation (5.1).

$$n_1(p) = u(s - 1) \ ;$$

$$p^1_{11}(p, p') = (s - 2) + (u - 1)(t - 2) \ ; \text{and}$$

$$p^1_{12}(p, p') = (u - 1)(s - t + 1) \ .$$

Thus,

$$\begin{aligned}
p_{13}^1 &= [u(s-1)-1] - p_{11}^1(p,p') - p_{12}^1(p,p') \\
&= u(s-1) - 1 - (s-2) - (u-1)(t-2) - (u-1)(s-t+1) \\
&= u(s-1) - 1 - s + 2 - (u-1)(t-2+s-t+1) \\
&= u(s-1) - s + 1 - (u-1)(s-1) \\
&= (s-1)(u-u+1) - (s-1) \\
&= (s-1) - (s-1) \\
&= 0
\end{aligned}$$

(ii) Let p and p' be second associates and use the same logic in the proof of (i) to see that

$$p_{13}^2 = n_1(p) - p_{11}^2(p,p') - p_{12}^2(p,p').$$

From [3], $p_{11}^2(p,p') = t(t-1)$ and use the value of $p_{12}^2(p,p')$ from Lemma 5.11(iii). Thus,

$$\begin{aligned}
p_{13}^2 &= u(s-1) - t(t-1) - t(s-t) - (u-t)(\beta-r-1) \\
&= u(s-1) - t(t-1+s-t) - (u-t)(\beta-r-1) \\
&= u(s-1) - t(s-1) - (u-t)(\beta-r-1) \\
&= (s-1)(u-t) - (u-t)(\beta-r-1) \\
&= (u-t)(s-1-\beta+r+1) \\
&= (u-t)(s-\beta+r)
\end{aligned}$$

(iii) Let p and p' be third associates and similar to the proof of (i) and (ii) above, we see that

$$p_{13}^3 = n_1(p) - p_{11}^3(p,p') - p_{12}^3(p,p').$$

From [3], $p_{11}^3(p,p') = 0$ and use the value of p_{12}^3 from Lemma 5.14(i). Thus,

$$\begin{aligned} p_{13}^3 &= u(s-1) - u(\beta - r) \\ &= u(s - \beta + r - 1) \end{aligned}$$

Remark 5.10 *Part (i) above can be proved easily by observing that the first associates of p are coplanar with p'. Because, if we let l passes through p and p', then any other line passing through p is coplanar with l and so its points are coplanar with the points on l. So, the first associates of p can only be a first or a second associate but not a third associate of p'. Method of proof given above is just to follow a general principle for (i), (ii) and (iii).*

Lemma 5.11 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net as in Definition 3.4 and for all $i, j, k = 1, 2, 3$, let $n_i^{\Pi}(p)$, ${}^{\Pi}p_{jk}^i$ and p_{jk}^i be the notations as defined above. Let u be the number of lines passing through any given point of the net and v, w be the notation in Theorem 5.8. Then, for any second associates p and p' of N, we have*

(i) *there is exactly one plane containing p and p',*

(ii) ${}^{\Pi}p_{11}^2 = p_{11}^2$,

(iii) $p_{21}^2 = p_{12}^2 = t(s-t) + (u-t)(\beta - r - 1)$,

(iv) $p_{32}^2 = p_{23}^2 = \beta(s-1)(s-t+1) - [s^2 - t(s-1)] - [(\beta - 1)(v+w)]$,

(v) $p_{33}^2 = n_3(p) - p_{31}^2(p,p') - p_{32}^2(p,p')$.

Proof :

(i) Let $\Pi, \Pi' \in \mathcal{P}$ be distinct planes containing the points p and p'. Since p, p' are second associates, let l be the line containing one of them, say p, such that l does not contain p'. For a point to lie on two planes, it must lie on the line of intersection of the two planes. Let l' be the intersection of Π and Π'. Since two planes intersect at exactly one line, p and p' must lie on l'. But this makes p and p' to be first associates contradicting that they are second associates. Hence, there is only one plane incident with any two points which are second associates.

(ii) It is an immediate consequence of part (i) above.

For the proof of (iii) to (vi) we let Π be the only plane containing both p and p'.

(iii) We calculate p^2_{21} in two parts :

(a) the number of second associates of p which are first associates of p' on Π, denoted by ${}^{\Pi}p^2_{21}$; and

(b) the number of second associates of p which are first associates of p' on planes other than Π, denoted by ${}^{-\Pi}p^2_{21}$.

Then,

$$p^2_{21} = ({}^{\Pi}p^2_{21}) + ({}^{-\Pi}p^2_{21}).$$

(a) The number of second associates of p which are first associates of p' inside Π, is nothing but the number of first associates of p' in Π, take away the number of first associates of p which are first associates of p' in Π, *i.e.,*

$${}^{\Pi}p^2_{21} = n^{\Pi}_1(p') - \ {}^{\Pi}p^2_{11} = n^{\Pi}_1(p') - p^2_{11}.$$

Next we calculate $n^{\Pi}_1(p')$, the number of first associates of p' in Π. By the axiom (N3$^{(3)}$) of a 3-net, there are t lines passing through p' in Π and for each of these lines, there are $(s-1)$ points other than p'.

Hence, the number of first associates of p' is

$$n_1^{\mathit{\Pi}}(p') = t(s-1).$$

Thus,

$$\begin{aligned} {}^{\mathit{\Pi}}p_{21}^2 &= t(s-1) - t(t-1) \\ &= t(s-1-t+1) \\ &= t(s-t) \end{aligned}$$

(b) There are $(u-t)$ lines passing through p' such that the lines are not incident with $\mathit{\Pi}$. By Theorem 5.7 each of these lines intersect with $(\beta - r - 1)$ planes (other than $\mathit{\Pi}$) containing p (but not containing the lines through p, otherwise that will make these planes contain both p and p', giving a contradiction). Hence,

$$^{-\mathit{\Pi}}p_{21}^2 = (u-t)(\beta - r - 1).$$

Finally, we get

$$p_{21}^2 = t(s-t) + (u-t)(\beta - r - 1).$$

(iv) No point of $\mathit{\Pi}$ will contribute to our count, so we consider only the points on planes other than $\mathit{\Pi}$. To find the third associates of p which are second associates of p', it is the same as saying that count the second associates of p' outside $\mathit{\Pi}$ and take away the first and second associates of p on the planes passing through p' from that count. For the latter, we just use Theorem 5.8 (ii) and (iii) by considering the planes through p' and the point p (note that none of these planes will contain p, and hence we can apply Theorem 5.8). Thus,

$$\begin{aligned} p_{32}^2 &= \underbrace{\text{second associates of } p' \text{ outside } \Pi} - \underbrace{\text{first and second associates of } p \text{ on the planes passing through } p'} \\ &= [n_2(p') - {}^{\Pi}n_2(p')] - [(\beta-1)(v+w)] \\ &= \{[\beta(s-1)(s-t+1)] - [s^2 - t(s-1)]\} - \{[(\beta-1)(v+w)]\} \end{aligned}$$

where we have used ${}^{\Pi}n_2(p') = s^2 - t(s-1)$.

(v) Similar to the proof of Lemma 5.9(i).

Lemma 5.12 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net as in Definition 3.4 and for all $i, j, k = 1, 2, 3$, let $n_i^{\Pi}(p)$, ${}^{\Pi}p_{jk}^i$ and p_{jk}^i be the notations as defined above. Let u be the number of lines passing through any given point of the net and v, w be the notation in Theorem 5.8. Then, for any first associates p and p' of N and l be the line passing through p and p', we have*

(i) $p_{21}^1 = p_{12}^1 = (u-1)(s-t+1)$,

(ii) $p_{32}^1 = p_{23}^1 = (\beta - r)[s^2 - t(s-1) - w]$,

(iii) $p_{33}^1 = n_3(p) - p_{31}^1(p,p') - p_{32}^1(p,p')$.

Proof :

(i) Since l contains p and p', any point on l is a first associate of p' but not a second associate of p and hence it cannot contribute to our required count. Also, the third associates of p cannot be first associate of p'. So, it remains to remove the first associates of p which are first associates of p' other than the points on the line l from the overall first associates of p' other than the points on l, *i.e.*,

$$p_{21}^1 = \underbrace{\text{first associates of } p' \text{ not on } l}_{(a)} - \underbrace{\text{first associates of } p \text{ which are first associates of } p' \text{ not on } l}_{(b)}$$

(a) There are $(u-1)$ lines through p' other than l and for each of these lines, there are $(s-1)$ points other than p'. Hence,

$$(a) = (u-1)(s-1).$$

(b) Since each point of the net passes through u lines, p' passes through $(u-1)$ lines (ignoring the line l which passes through p) that lie on the same plane as p'. Every such line intersect with $(t-2)$ lines passing through p'. Hence, for each of these $(u-1)$ lines, there are $(t-2)$ points other than p. Thus,

$$(b) = (u-1)(t-2).$$

Hence, we have

$$\begin{aligned} p_{21}^1 &= (u-1)(s-1) - (u-1)(t-2) \\ &= (u-1)(s-1-t+2) \\ &= (u-1)(s-t+1) \end{aligned}$$

(ii) The number of second associates of p' on a plane containing p' is $s^2 - t(s-1)$. There are exactly r planes containing l. Hence, there are exactly r planes containing both p and p'. So, $(\beta - r)$ planes contain only p (but not l and p). Thus, we have $(\beta - r)[s^2 - t(s-1)]$ second associates on the planes only incident with p'. Of these, there are points which lie on the same plane as p which needs to be removed. By Theorem 5.8 (iii), this count is w for each plane containing p and there are exactly $(\beta - r)$ such planes. Hence, the points that needs to be removed from the previous count is $(\beta - r)w$. Thus, the total count is

$$p_{32}^1 = (\beta - r)[s^2 - t(s-1)] - (\beta - r)w = (\beta - r)[s^2 - t(s-1) - w]\,.$$

(iii) Similar to the proof of Lemma 5.9(i).

Remark 5.13 *Part (i) in Lemma 5.12 above can also be proved using the same logic in Lemma 5.9 (i), i.e., using the equation*

$$p^1_{21} = n_1(p) - 1 - p^1_{11}(p, p') - p^1_{31}(p, p').$$

Lemma 5.14 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net as in Definition 3.4 and for all $i, j, k = 1, 2, 3$, let $n^{\Pi}_i(p)$, ${}^{\Pi}p^i_{jk}$ and p^i_{jk} be the notations as defined above. Let u be the number of lines passing through any given point of the net and v, w be the notation in Theorem 5.8. Then, for any third associates p and p' of N, we have*

(i) $p^3_{21} = p^3_{12} = u(\beta - r)$,

(ii) $p^3_{32} = p^3_{23} = \beta[s^2 - t(s - 1) - w]$,

(iii) $p^3_{33} = n_3(p) - p^3_{31}(p, p') - p^3_{32}(p, p')$.

Proof :

(i) There are u lines through p' and since p and p' are non-coplanar, each of these lines are non-coplanar with p. By Theorem 5.7, for each of these u lines, there are $(\beta - r)$ second associates. Thus, $p^3_{21} = u(\beta - r)$.

(ii) The second associates of p' on a plane containing p' is $s^2 - t(s - 1)$. Since there are β planes containing p', we have $\beta[s^2 - t(s - 1)]$ second associates on the planes incident with p'. Of these, there are points which lie on the same plane as p which needs to be removed. By Theorem 5.8 (iii), this count is w for each plane containing p and there are β such planes. Hence, the points that needs to be removed from the previous count is βw. Thus, the total count is

$$p^3_{32} = \beta[s^2 - t(s - 1)] - \beta w = \beta[s^2 - t(s - 1) - w].$$

(iii) Similar to the proof of Lemma 5.9(i).

Lemma 5.15 *Let $N = (\mathcal{X}, \mathcal{L}, \mathcal{P}, \mathcal{I})$ be a 3-net as in Definition 3.4 and for all $i, j, k = 0, 1, 2, 3$, let n_i and p^i_{jk} be the notations as defined in Section 5.2. Then, we have the following results (where part (i), (ii) and (iii) takes only non-zero values of i, j, and k).*

$$(i) \qquad p^0_{jk} = \begin{cases} 0 & j \neq k \\ n_j & j = k \end{cases} \qquad\qquad (iv) \qquad p^i_{00} = \begin{cases} 0 & i \neq 0 \\ 1 & i = 0 \end{cases}$$

$$(ii) \qquad p^i_{0k} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases} \qquad\qquad (v) \qquad p^0_{j0} = \begin{cases} 0 & j \neq 0 \\ 1 & j = 0 \end{cases}$$

$$(iii) \qquad p^i_{j0} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad\qquad (vi) \qquad p^0_{0k} = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

Proof : Let p and p' be distinct points of the net N. We prove part (i) and (ii) only, as the proof of the remaining results are similar and easy.

(i) Let p be the zero associate (*i.e.,* p itself) and p' be a j^{th} associate of p. Then, if we want p' to be a k^{th} associate of p, we must have $j = k$; else p' does not exists. For example, if $j = 1$ and $k = 2$, then p' cannot be first associate of p and second associate of p at the same time. Thus, $p^0_{jk} = n_j$ when $j = k$ and zero otherwise.

(ii) Let p and p' be i^{th} associates. It is clear that the only possible values of p^i_{0k} are 0 and 1. If we want p to be a k^{th} associate of p', we must have $i = k$. For example, if p and p' are first associates ($i = 1$), since the zero associate of p is just p, p can only be first associate of p' ($k = 1$). Thus, $p^i_{0k} = 1$ when $i = k$ and zero otherwise.

Chapter 6

Higher dimensional Nets

6.1 Finite nets of dimension n

J. Dunbar and R. Laskar extended the notion of Bruck's net to an arbitrary dimension n in 1978 [10]. We do not give the original definition of *finite nets of dimension* n; but we present below the equivalent definition using incidence relation and call it as n-net.

Definition 6.1 *(n-dimensional net or n-net) Let s, t, r, n and β_i, $1 \le i \le n$, be integers greater than one. Then a system $N^{(n)}$ consisting of $(n+1)$-tuple $(\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_{n-1}, \mathcal{I})$, where each $\mathcal{N}_i$, $0 \le i \le n-1$, is a non-empty finite collection of undefined objects called i-sets or i-flats, together with incidence structure $\mathcal{I}$ defined by*

$$\mathcal{I} \subseteq \bigcup_{i,j:\, i \neq j, i<j} \mathcal{N}_i \times \mathcal{N}_j$$

satisfying the following conditions is called n-dimensional net. For convenience we shall simply call it as n-net instead of n-dimensional net.

(N1$^{(n)}$) *For each $i = 1, 2, \cdots, n-2$ and for $N \in \mathcal{N}_{i-1}$, $N' \in \mathcal{N}_i$ and $N'' \in \mathcal{N}_{i+1}$, such that $(N, N') \in \mathcal{I}$ and $(N', N'') \in \mathcal{I}$, then $(N, N'') \in \mathcal{I}$.*

(N2$^{(n)}$) *For each $i = 1, 2, \cdots, n-2$ and for $M \in \mathcal{N}_{i-1}$ and $N, N' \in \mathcal{N}_i$ such that $(M, N) \in \mathcal{I}$ and $(M, N') \in \mathcal{I}$, there exists $M' \in \mathcal{N}_{i+1}$ such that*

$(N, M') \in \mathcal{I}$ and $(N', M') \in \mathcal{I}$.

(N3$^{(n)}$) *For each $i = 1, 2, \cdots, n-1$ and for $N \in \mathcal{N}_i$, there exists $N' \in \mathcal{N}_{i+1}$ such that $(N, N'') \in \mathcal{I}$.*

(N4$^{(n)}$) *The 0-sets and $(n-1)$-sets on $N^{(n)}$ are called points and hyperplanes respectively. For each $l = 1, 2, \cdots, n-1$, there exists $\mathcal{B}_1^l, \mathcal{B}_2^l, \cdots, \mathcal{B}_{\beta_l}^l$ such that*

(a) $\mathcal{B}_1^l, \mathcal{B}_2^l, \cdots, \mathcal{B}_{\beta_l}^l$ is a partition of $\mathcal{N}_l$;

(b) For each $\mathcal{B}_k^l(k = 1, 2, \cdots, \beta_l)$ and for any $N_i, N_j \in \mathcal{B}_k^l$, either they are the same or they are parallel;
i.e., $N_i = N_j$ or $\{ N_0 \in \mathcal{N}_0 : (N_0, N_i) \in \mathcal{I} \} \bigcap \{ N_0 \in \mathcal{N}_0 : (N_0, N_j) \in \mathcal{I} \} = \emptyset$.

(c) For each $N_i \in \mathcal{B}_k^l$ and $N_j \in \mathcal{B}_m^l$ such that $m \neq k$, there exists a unique $N \in \mathcal{N}_{l-1}$ such that $(N_i, N) \in \mathcal{I}$ and $(N_j, N) \in \mathcal{I}$. Further, for any $N \in \mathcal{N}_l$ $(l \leq n-3)$ such that $(N, N_i) \in \mathcal{I})$ and $(N, N_j) \in \mathcal{I})$, then $N \in \mathcal{N}_l$;

(d) For each $N_0 \in \mathcal{N}_0$ and for each $k = 1, 2, \cdots, \beta_l$, there is a unique $N \in \mathcal{B}_k^l$ such that $(N_0, N) \in \mathcal{I}$.

The sets $\mathcal{B}_1^l, \mathcal{B}_2^l, \ldots, \mathcal{B}_{\beta_l}^l$ are called the parallel classes of hyperplanes.

(N5$^{(n)}$) *(a) For each $N_2 \in \mathcal{N}_2$, if we define*

$$\mathcal{N}_0' = \{ N_0 \in \mathcal{N}_0 : (N_0, N_2) \in \mathcal{I} \}$$
$$\mathcal{N}_1' = \{ N_1 \in \mathcal{N}_1 : (N_1, N_2) \in \mathcal{I} \}$$
$$\mathcal{I}' = \{ (N_0, N_1) : (N_0, N_1) \in \mathcal{I}, N_0 \in \mathcal{N}_0' \text{ and } N_1 \in \mathcal{N}_1' \}$$

then $(\mathcal{N}_0', \mathcal{N}_1', \mathcal{I}')$ forms a 2-net.

(b) For any i such that, $3 \leq i \leq n-1$, let $\mathcal{N}_i$ be a i-set. A finite net of

dimension i, say $N^{(i)}$, i.e.,

$$N^{(i)} = (\mathcal{N}_0, \mathcal{N}_1, \cdots, \mathcal{N}_{i-1}) \textit{ is a finite net of dimension } i,$$

is formed by the k-sets defined by

$$\mathcal{N}'_k := \{N \in \mathcal{N}_k : (N, N_i) \in \mathcal{I}, N_i \in \mathcal{N}_i\}, \textit{ for all } k \textit{ such that } 0 \leq k < i.$$

We also have β_i, the number of parallel classes of each hyperplane $\mathcal{N}_k$.

Example 6.2 *A finite affine n-space with s points on each line [10].*

Definition 6.3 *(Partial Congruence Partition - PCP in n-dimension) Let G be a group of order s^n, $s \geq 2$. Let us define the following collections of subgroups of G*

$$\mathcal{K}^{n-1} \subseteq \{\, H^{n-1} : H^{n-1} \leq G \text{ and } \left|H^{n-1}\right| = s^{n-1} \,\}$$

(i.e., $\mathcal{K}^{n-1}$ is a collection of some subgroups of G of order s^{n-1})

$$\mathcal{K}^{n-2} = \{\, H^{n-2} = H_1^{n-1} \cap H_2^{n-1} : H_1^{n-1}, H_2^{n-1} \in \mathcal{K}^{n-1} \text{ and } \left|H^{n-2}\right| = s^{n-2} \,\}$$

$$\mathcal{K}^{n-3} = \{\, H^{n-3} = H_1^{n-2} \cap H_2^{n-2} : H_1^{n-2}, H_2^{n-2} \in \mathcal{K}^{n-2} \text{ and } \left|H^{n-3}\right| = s^{n-3} \,\}$$

$$\vdots$$

$$\mathcal{K}^{1} = \{\, H^{1} = H_1^{2} \cap H_2^{2} : H_1^{2}, H_2^{2} \in \mathcal{K}^{2} \text{ and } \left|H^{1}\right| = s \,\}$$

such that the following conditions hold:

(a) for all $H_1^1, H_2^1 \in \mathcal{K}^1$, either $H_1^1 = H_2^1$ or $H_1^1 \cap H_2^1 = \{\, e \,\}$, where e is the identity element of G;

(b) for all $i = 1, 2, \cdots, n-1$ and for all $H^i \in \mathcal{K}^i$, there are exactly β_i distinct subgroups $H^{i-1} \in \mathcal{K}^{i-1}$ such that $H^{i-1} \subset H^i, t \geq 2$;

(c) for all $i = 1, 2, \cdots, n-1$ and for all $H_1^i, H_2^i \in \mathcal{K}^i$, there exists a subgroup $H^{i+1} \in \mathcal{K}^{i+1}$ such that $H_1^i \cup H_2^i \subset H^{i+1}$;

(d) For any i such that, $3 \leq i \leq n-1$, and for k such that $0 \leq k < i$,

$$(\mathcal{K}^k, \mathcal{K}^{k-1}, \cdots, \mathcal{K}^1) \text{ is a PCP}^{(i)}\text{, i.e., PCP of dimension } i.$$

Then $(\mathcal{K}^{n-1}, \mathcal{K}^{n-2}, \cdots, \mathcal{K}^1)$ is called a ***partial congruence partition of dimension n in G****, denoted by PCP$^{(n)}$. If we assume $|\mathcal{K}^{n-1}| = \beta^{(n-1)}$ (i.e., the number of elements in $\mathcal{K}^{n-1}$; not the $(n-1)^{th}$ power of β), then we call $(s, t, \beta^{(n-1)})$-PCP in G, where the elements of $\mathcal{K}^{n-1}$ are called hyperplane components and the elements of $\mathcal{K}^1$ are called line components.*

Remark 6.4 *$\mathcal{K}^{n-1}, \mathcal{K}^{n-2}, \cdots, \mathcal{K}^1$ can be defined recursively as follows. For $i = 1, 2, \cdots, n-1$,*

$$\mathcal{K}^i = \{\ H^i = H_1^{i+1} \cap H_2^{i+1} : H_1^{i+1}, H_2^{i+1} \in \mathcal{K}^{i+1} \text{ and } \left|H^i\right| = s^i\ \}.$$

Proposition 6.5 *(PCP$^{(n)}$ $\Rightarrow$ n-net) Let G be a group and $(\mathcal{K}^{n-1}, \mathcal{K}^{n-2}, \cdots, \mathcal{K}^1)$ be a PCP$^{(n)}$ as defined in 6.3. Define a plane geometry $N^{(n)} = (\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_{n-1}, \mathcal{I})$ where*

(i) $\mathcal{N}_0 = G$;

(ii) for each $i = 1, 2, \cdots, n-1$ define the sets $\mathcal{N}_i$ recursively as follows:

$$\mathcal{N}_i = \{\ gH^i : g \in G \text{ and } H^i \in \mathcal{K}^i\ \}\ ;\text{ and}$$

(iii) $\mathcal{I} = \{\ (g, hH^i) : g, h \in G, H^i \in \mathcal{K}^i$ and $g \in hH^i$ for each $i = 1, 2, \cdots, n-1\ \} \bigcup \{\ (hH^i, kH^{i+1}) : h, k \in G, H^i \in \mathcal{K}^i$ such that $hH^i \subset kH^{i+1}$ for each $i = 1, 2, \cdots, n-2\ \}$.

Then $(\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_{n-1}, \mathcal{I})$ is an n-net.

6.2 Conclusion

Since we have defined $PCP^{(n)}$, then we can follow the same group theoretic approach as discussed in two dimension and three dimension cases and get the similar results for n-dimensional case. We state some of them below. The proofs should be very similar but are really tedious and long to write down. Hence, we do not go into those details.

First major result we can immediately infer is that, one can recover a n-net from $PCP^{(n)}$ using the incidence relation and by doing a proper setting of points and hyperplanes from the definition of $PCP^{(n)}$. We can also define translation nets of dimension n and prove the equivalency between translation nets of dimension n and $PCP^{(n)}$.

The upper bounds for each β_i can be calculated using the similar approach in Chapters two and four. Also, it is not difficult to use the examples in these chapters to construct examples with maximum value of β_i, for each i. Later, one can use the sylow theorems in group theory and can attempt to express the groups in translation nets of dimension n as direct product of it's subgroups.

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