PORTFOLIO SELECTION WITH TRANSACTION COSTS AND CAPITAL GAINS TAXES

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To the memory of my mother CHENG Jinmei and

To my father LEI Chongbao

Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety.
I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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3 Aug 2015

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Summary

This thesis is concerned with the continuous-time portfolio selection problem of an investor who faces transaction costs and capital gains taxes. In contrast to the existing literature, we propose a model taking into account an important market feature that taxes are only paid at the end of each calendar year. We focus on the constant absolute risk aversion (CARA) utility, and an extension to the constant relative risk aversion (CRRA) utility is also provided. We find that the investor is inclined to defer realization of capital gains until the beginning of the next calendar year. Moreover, the presence of transaction costs could lead the investor to defer realization of capital losses to the end of each calendar year.

One economically interesting extension of our model is also discussed in the thesis. We introduce labor income with no-borrowing constraint against future labor income. We show that the inability to borrow of a CRRA investor can substantially reduce consumption and investment in the risky asset, and provide an incentive to trade more frequently.

Since closed form solutions of the investor's problem are generally unavailable, we finally conduct asymptotic analysis in terms of small interest rate and tax rate. We focus on the case that transaction costs are absent and taxes are paid immediately after sale. Based on the expansion of Chen and Dai (2013a), we propose a more refined expansion, and obtain an explicit strategy. Our numerical results demonstrate that the explicit strategy is a good approximation of the optimal strategy even for relatively large interest rate and tax rate.

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Chapter 1

Introduction

In this thesis, we study the continuous-time optimal investment and consumption problem with transaction costs and capital gains taxes. This problem is a variant of the classical investment-consumption problem in modern finance.

1.1 Historical Work

The seminal paper "Portfolio Selection" of Markowitz (1952) marks the start of modern finance. In it Markowitz founds modern portfolio theory with the introduction of (single-period) mean-variance optimization and efficient frontiers. Merton (1969, 1971) introduces the time dimension to portfolio theory and lays the theoretical groundwork for intertemporal portfolio selection. In his pioneering work, Merton formulates and solves the continuous-time optimal investment and consumption problem of a constant relative risk aversion (CRRA) investor in a perfectly liquid market. The objective of the investor is to choose how much to consume and how to allocate his wealth between a risk-free asset and a risky asset so as to maximize the expected utility from intertemporal consumption over an infinite time horizon. Provided that the risky asset price follows a geometric Brownian motion, Merton shows that the optimal strategy is to keep a constant fraction of the total wealth in the risky asset, and consume at a constant rate which is proportional to the total wealth.

Since the seminal work of Merton, there has been extensive literature on the optimal investment and consumption problem in financial markets subject to imperfections. Among them, transaction costs have received considerable attention from researchers. Merton's strategy requires the investor to continuously rebalance the portfolio. In the presence of transaction costs, however, it can be infinitely expensive. Therefore, Merton's strategy must be sub-optimal in this case. Magill and Constantinides (1976) introduce transaction costs to Merton's model. They provide a fundamental insight that the optimal trading policy is described by a no-trading region: within it, the investor does not trade the risky asset at all; out of it, he trades a minimal amount so that the fraction of wealth invested in the risky asset reaches the boundary of the no-trading region. From then on, portfolio selection with transaction costs has been extensively studied. We refer the reader to Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Akian, Menaldi, and Sulem (1996), Liu and Loewenstein (2002), Liu (2004), Kabanov and Klüppelberg (2004), Dai and Yi (2009), Chen, Dai, and Zhao (2012), Chen and Dai (2013b), and so forth.

However, very few works have been done on portfolio selection with capital gains taxes, although capital gains taxes represent a much higher percentage than transaction costs in the real market. One can imagine that capital gains taxes must have an appreciable impact on the investor's strategy as well. The first relevant work on capital gains taxes is due to Constantinides (1983). He shows that the optimal trading policy is to realize capital losses immediately and defer realization of capital gains indefinitely until the event of a forced liquidation. However, this policy depends heavily on the assumption of the unrestricted short sales of the risky asset, which is ideal and not realistic.

In a multiple period context, a capital gain or loss for a particular share sold is computed by the difference between the sale price and the original purchase price of this share. Therefore, one needs to keep track of the exact original purchase price of each share, known as the exact tax basis, which incurs strong path dependency. As a consequence, most of the existing literature on capital gains taxes has been restricted in a discrete-time framework with a very limited number of time steps (e.g., Constantinides, 1983, 1984; Dybvig and Koo, 1996; and DeMiguel and Uppal, 2005). To overcome the challenging difficulties caused by path dependency, Dammon, Spatt, and Zhang (2001, 2004) approximate the exact tax basis using the average tax basis which is the weighted average of past purchase prices of the shares held. Their work makes a great breakthrough because the average tax basis is not only a reasonably good approximation of the exact tax basis (cf. DeMiguel and Uppal, 2005; and Dai, Liu, Yang, and Zhong, 2015), but also significantly reduces the path dependency of the problem. It is worth pointing out that the average tax basis is actually used in Canada. Ben Tahar, Soner, and Touzi (2007, 2010) further formulate a continuous-time version of the model proposed by Dammon, Spatt, and Zhang. In the continuous-time framework, the average taxation rule only introduces one more variable, the tax basis. This greatly simplifies the calculation.

In the above mentioned papers on capital gains taxes, capital gains and losses are treated equally, that is, the investor pays taxes immediately for realized capital gains but receives tax rebates immediately for realized capital losses. In this case, the above literature shows that the investor may defer realization of capital gains, but should realize capital losses immediately. However, most tax codes around the world do not pay tax rebates for realized capital losses, but provide the investor with a tax loss carry-forward instead. Taking the current U.S. tax law as an example, up to $3,000 of realized capital losses can apply to offset taxable income each year with the rest carried forward to the future indefinitely. Recently, there are three papers investigating the effect of such provision on the investor's strategy by using the average tax basis. Ehling, Gallmeyer, Srivastava, Tompaidis, and Yang (2013) find it does not significantly affect the investor's strategy if he has large embedded capital gains. Marekwica (2012) shows that the investor should realize capital losses immediately provided that capital losses are fully carried forward. These two papers are both in a discrete-time framework. In a continuous-time framework, Dai, Liu, Yang, and Zhong (2015) take into account asymmetric long-term and short-term tax rates. They find that even if there are large embedded capital losses, the strategy with full carry-forward is significantly different from the strategy with full tax rebate. Besides, it is optimal to realize capital losses immediately in symmetric tax rate case if capital losses are fully carried forward.

So far, transaction costs and capital gains taxes are two distinct issues that have been separately studied in the existing literature on continuous-time portfolio selection except Ben Tahar, Soner, and Touzi (2007).[1] However, this work does not explore the impact of transaction costs on a taxable investor's strategy. The continuous-time portfolio selection problem with transaction costs or capital gains taxes gives rise to a Hamilton-Jacobi-Bellman (HJB) equation with free boundaries. Since its closed form solutions are generally unavailable, it is natural to seek numerical solutions via, for example, the penalty method combined with a finite difference discretization (cf. Forsyth and Vetzal, 2002; and Dai, Kwok, and You, 2007). To simplify the problem, a natural approach is to obtain an asymptotic expansion in terms of small transaction costs or small capital gains taxes, based on that one recovers Merton's problem in the limit case of zero transaction costs or zero capital gains taxes. Regarding the Merton problem with transaction costs, there is a large body of literature on its asymptotic expansions. We refer the reader to Constantinides (1986), Shreve and Soner (1994), Bichuch (2012), Gerhold, Muhle-Karbe, and Schachermayer (2012), Bichuch and Shreve (2013), Soner and Touzi (2013), and references therein.

Regarding the Merton problem with capital gains taxes, there are two extra difficulties concerning asymptotic expansions. Firstly, an additional variable, the tax basis, strongly complicates asymptotic expansions and could make rigorous expansions intractable. Secondly, the tax rate is relatively large, which makes it crucial

[1] The impact of transaction costs on the optimal trading and pricing of taxable securities is studied by Dammon and Spatt (1996) via a discrete-time model. They show that the investor should defer realizing capital losses until tax rebates is much greater than transaction costs.

to find appropriate perturbation parameters. Interestingly, Ben Tahar, Soner, and Touzi (2010) show that one recovers a fictitious Merton problem when the interest rate is zero. In light of this result, Chen and Dai (2013a) conduct formal expansions, and obtain an explicit strategy that effectively approximates the optimal strategy for small interest rate and tax rate. However, their approximation may not perform very well for large interest rate and tax rate.

Most of the papers mentioned above focus on the CRRA utility due to its analytical convenience and support on an empirical basis. Very few assume that the investor is of constant absolute risk aversion (CARA). In a perfectly liquid market, Merton (1969, 1971) shows that the optimal strategy of a CARA investor is to keep a constant dollar amount in the risky asset and consume at a rate that is affine in the total wealth. In the presence of transaction costs, it is shown that the optimal trading policy is to keep the dollar amount in the risky asset within the no-trading region (cf. Liu, 2004; Chen, Dai, and Zhao, 2012; and Chen and Dai, 2013b). To the best of our knowledge, however, the Merton problem of a CARA investor who faces capital gains taxes has not yet been studied by researchers. The reason some researchers are interested in the CARA utility lies in its separability. By virtue of its separability, the multiple risky-asset problem with transaction costs can be reduced to the single risky-asset case provided that the asset returns are uncorrelated (cf. Liu, 2004). In addition, it is feasible to handle problems arising from utility indifference pricing. These utility indifference pricing problems are essentially portfolio selection problems after doing some appropriate transformations. For example, using the approach developed in Dai and Yi (2009), Yi and Yang (2008) solve a sub-problem arising from utility indifference pricing with transaction costs which is studied in Davis, Panas, and Zariphopoulou (1993).

1.2 Contributions of the Thesis

In this thesis, we study the optimal investment and consumption problem of an investor who faces both transaction costs and capital gains taxes. Specifically, the investor under consideration can continuously trade a risk-free asset and a risky asset so as to maximize his expected utility from intertemporal consumption over an infinite time horizon. The price of the risky asset is assumed to follow a geometric Brownian motion. There are constant proportional transaction costs incurred in buying and selling the risky asset. Capital gains and losses on the risky asset are taxed at a constant rate. We approximate the exact tax basis by the average tax basis as in most of the existing literature. We consider both the case where capital losses are fully rebatable and the case where capital losses are fully carried forward. We focus on the CARA utility case, and an extension to the CRRA utility case is also provided.

Our first main contribution in the thesis is to propose a model taking into account the market feature that taxes are only paid at the end of each calendar year (year-end taxes). To the best of our knowledge, this is the first work to investigate the effect of this feature on the investor's strategy. The existing literature always assumes that taxes are paid immediately after sale (instant taxes). Under this assumption, we find that the presence of transaction costs can lead the investor to defer realization of capital losses. In this thesis, we relax this assumption and assume that taxes are only paid at the end of each calendar year as in the real market. We show that the investor tends to avoid realizing capital gains late in this calendar year; moreover, he is inclined to defer realization of capital gains until the beginning of the next calendar year. In addition, the presence of transaction costs could lead the investor to defer realization of capital losses to the end of each calendar year. From the standpoint of the investor's expected utility, we find that the investor can be better off, but not much, by the provision that taxes are paid annually.

In addition, we derive the optimal strategy of a CARA investor facing multiple uncorrelated risky assets. We show that the optimal trading boundaries can be computed separately for each risky asset (up to some constants); and the optimal consumption rate is a linear combination of the dollar amount invested in the risk-free asset and (some transformation of) the investor's value function. A verification theorem is provided for the full rebate and instant tax case. This result enables us to break down the multiple risky-asset problem into the single risky-asset problem, and thus makes it feasible to compute the optimal strategy for a large number of uncorrelated risky assets. It is worth pointing out that even if the asset returns are correlated, the strategy for the uncorrelated return case can be used as a benchmark.

Our second main contribution is to derive the optimal strategy of a CRRA investor who also receives a constant stream of labor income, but is not allowed to borrow against his future labor income. The no-borrowing constraint destroys the homogeneity property of the investor's value function. This increases the dimensionality of the problem and thus strongly complicates the computation. Therefore, we focus on the no-transaction cost, full rebate, and instant tax case. Our results show that the no-borrowing constraint significantly reduces the value of labor income. This is mainly reflected in two aspects. On the one hand, the no-borrowing constraint can greatly decrease consumption and investment in the risky asset, and provide an incentive to trade more frequently. On the other hand, the no-borrowing constraint can significantly reduce the investor's expected utility.

Our third main contribution is to derive a good approximation, in an explicit form, of the optimal strategy even for relatively large interest rate and tax rate. We consider only the no-transaction cost, full rebate, and instant tax case. In addition, we focus on the CARA utility case, and an extension to the CRRA utility case is also provided. Using the approach developed in Chen and Dai (2013a), we firstly provide two asymptotic expansions. After that, we further propose a more refined expansion. Our numerical results show that the explicit strategies implied by these three expansions effectively approximate the optimal strategy for small interest rate and tax rate. Moreover, the strategy implied by the refined expansion can effectively approximate the optimal strategy even for relatively large interest rate and tax rate.

1.3 Organization of the Thesis

The remainder of the thesis is organized as follows. Chapter 2 reviews the basic model of portfolio selection with instant taxes, and solve the optimization problem faced by a CARA investor.

Chapter 3 studies the optimal investment and consumption problem of a CARA investor who faces both transaction costs and year-end taxes. We formulate the model and explore the impact of the provision that taxes are paid annually, as well as the effect of transaction costs, on the investor's strategy. An extension to the CRRA utility case is also provided.

Chapter 4 is devoted to an extension for a CRRA investor who also receives labor income, but is not allowed to borrow against his future labor income. We aim to explore the impact of the no-borrowing constraint on the investor's strategy.

Chapter 5 conducts asymptotic expansions on portfolio selection with instant taxes. We focus on the CARA utility case, and an extension to the CRRA utility case is also provided. Numerical results are presented to demonstrate our theoretical analysis.

Concluding remarks and possible directions for future research are offered in the last chapter.

Appendix A extends our model to include multiple risky assets. We show that the multiple risky-asset problem of a CARA investor can be reduced to the single risky-asset case provided that the asset returns are uncorrelated.

Chapter 2

Merton Problem with Instant Taxes

The continuous-time portfolio selection problem for a constant relative risk aversion (CRRA) investor who faces capital gains taxes is firstly formulated and solved by Ben Tahar, Soner, and Touzi (2007, 2010), and then further developed by Dai, Liu, Yang, and Zhong (2015). These works provide a groundwork for future research studies, including ours. So we devote this chapter to review the tax model, but the utility function is assumed to exhibit constant absolute risk aversion (CARA).

2.1 The Asset Market

Throughout this thesis, unless otherwise mentioned, we consider a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, endowed with a one-dimensional standard $\mathscr{F}_t-$Brownian motion $\{B_t\}_{t\geq 0}$. We consider a financial market consisting of only two investment assets. The first asset is a risk-free money market account growing at a constant, continuously compounded interest rate of $r > 0$.[1] The second asset is a risky stock whose price process P_t evolves according to a geometric Brownian motion:

$$dP_t = \mu P_t dt + \sigma P_t dB_t, \tag{2.1}$$

[1] We restrict $r > 0$ to prevent a CARA investor from unlimited consumption and unlimited investment in the risky asset (cf. Merton's result presented in Theorem 2.1). This restriction can be relaxed for a CRRA investor.

where $\mu > r$ and $\sigma > 0$ are constants representing respectively the expected rate of return and the volatility of the stock. In this thesis, we always assume that short selling of the stock is prohibited, and that wash sales[2] of the stock are allowed.

2.2 Taxation Rules on Capital Gains

We assume that capital gains and losses on the stock are taxed immediately when the investor sells the stock (instant taxes). The amount of tax to be paid for each sale of the stock at time t is determined by the difference between the current stock price P_t and the average tax basis $\bar{P}_t$ which is defined as the weighted average purchase price of the current stock holding. More specifically, if $P_t \geq \bar{P}_t$, the investor would realize a capital gain by selling the stock, and pay a tax $\tau(P_t - \bar{P}_t)$ for each unit of the stock sold, where $\tau \in [0, 1)$ is a constant tax rate. In contrast, if $P_t \leq \bar{P}_t$, the sale of the stock corresponds to the realization of a capital loss. In this thesis, we investigate two different ways to deal with realized capital losses. Following Dai, Liu, Yang, and Zhong (2015), we term the full rebate (FR) case and the full carry-forward (FC) case throughout:

- In the FR case, capital losses are fully rebatable. The investor can use all capital losses to offset taxable ordinary income. Specifically, if $P_t \leq \bar{P}_t$, by selling one unit of the stock, the investor would receive a tax rebate $\tau(\bar{P}_t - P_t)$ which can be immediately reinvested.

- In the FC case, capital losses are fully carried forward. The investor can only carry forward capital losses to offset future gains.

Because a tax loss carry-forward does not pay any interest and bears the risk of never being used, we can imagine that the FC case is a less attractive treatment of capital losses for the investor than the FR case.

[2] A wash sale is selling the stock at a loss and repurchasing the same or substantially identical stock shortly (within 30 days under the U.S. tax law).

Remark 2.1. *The current tax law allows to apply up to $3,000 a year in capital losses to offset taxable ordinary income with the rest carried forward indefinitely to the future. The FR case is more suitable for low income investors whose capital losses are likely less than $3,000 per year. The FC case is more suitable for high income investors whose capital losses are likely much more than $3,000 per year.*

2.3 The Investor's Problem

We denote by x_t the dollar amount invested in the money market account, y_t the dollar amount invested in the stock, and k_t the total cost basis for the stock holding which is the position on the stock evaluated at the average tax basis. The transfers of wealth between the two investment assets are described by two nondecreasing, right-continuous, and $\mathscr{F}_t-$adapted processes L_t and M_t with $L_{0-} = M_{0-} = 0$. On a purchase of the stock, the dollar amount transferred from the money market account to the stock account is given by dL_t. On a sale of the stock, the dollar amount transferred from the stock account to the money market account is given by $y_{t-}dM_t$, where $dM_t \leq 1$ is the proportion of stock shares the investor sells. We assume that the investor derives his utility from intertemporal consumption. The consumption rate c_t is an $\mathscr{F}_t-$adapted process which is integrable on each finite time interval, that is, $\int_0^t |c_s|ds < \infty$ for any $t \geq 0$. Such triple (c, L, M) is called a consumption-investment strategy of the investor. Then we have the following dynamics for x_t, y_t and k_t:

$$dx_t = (rx_t - c_t)dt - dL_t + f(0, y_{t-}, k_{t-}; l)dM_t, \tag{2.2}$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dB_t + dL_t - y_{t-}dM_t, \tag{2.3}$$

$$dk_t = dL_t - k_{t-}dM_t + l(k_{t-} - y_{t-})^+ dM_t, \tag{2.4}$$

where

$$f(x, y, k; l) = x + y - \tau\big[(1-l)(y-k) + l(y-k)^+\big]$$

is the total wealth after liquidation, and $l = 0$ or 1 corresponds to the FR case or the FC case.

2.3.1 The CRRA Utility Case

We firstly assume that the investor preferences are characterized by a power utility function, belonging to the CRRA class, with a constant risk aversion factor p:

$$U(c) = \frac{c^{1-p}}{1-p}, \quad p > 0, \, p \neq 1.$$

In this case, we restrict the set of consumption policies to be such that $c_t \geq 0$ for any $t \geq 0$. In addition, a consumption-investment strategy (c, L, M) is called to be admissible if the unique solution of (2.2)-(2.4) with $(x_0, y_0, k_0) = (x, y, k)$ satisfies the solvency constraint

$$f(x_t, y_t, k_t; l) \geq 0, \, y_t \geq 0, \, k_t \geq 0, \, \forall t \geq 0.$$

Let $\bar{\mathscr{A}}_0(x, y, k)$ denote the set of admissible strategies. The investor's problem is defined by

$$V(x, y, k) = \max_{\bar{\mathscr{A}}_0(x,y,k)} \mathbb{E}_0^{x,y,k} \left[\int_0^\infty e^{-\beta t} U(c_t) dt \right], \; \forall f(x, y, k; l) \geq 0, y \geq 0, k \geq 0,$$

where $\beta > 0$ is a constant discount factor, and $\mathbb{E}_0^{x,y,k}$ is the conditional expectation at time $t = 0$ given that $(x_0, y_0, k_0) = (x, y, k)$. This problem is discussed in Ben Tahar, Soner, and Touzi (2010) and Dai, Liu, Yang, and Zhong (2015). So we will not go into the details of this problem here.

2.3.2 The CARA Utility Case

In the remaining part of this chapter, we assume that the investor preferences are characterized by a CARA (exponential) utility function with a constant risk aversion factor γ:

$$u(c) = -e^{-\gamma c}, \quad \gamma > 0. \tag{2.5}$$

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and

$$\mathscr{S} = \{\, (x, y, k) \in \mathbb{R}^3 \,|\, y > 0,\ k > 0 \,\}. \tag{2.6}$$

Then, the investor's problem is defined by

$$V(x, y, k) = \max_{(c,L,M)\in\mathscr{A}_0(x,y,k)} \mathbb{E}_0^{x,y,k}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right], \quad \forall (x, y, k) \in \bar{\mathscr{S}},$$

where $\bar{\mathscr{S}} = \mathbb{R} \times \mathbb{R}_+^2$ is the closure of $\mathscr{S}$, and $\mathscr{A}_0(x, y, k)$ is the set of admissible strategies defined by the following constraints (2.7)-(2.9). For any consumption-investment strategy (c, L, M), we denote by (x_t, y_t, k_t) the unique solution of (2.2)-(2.4) with $(x_0, y_0, k_0) = (x, y, k)$. The trading in the stock is subject to the no-short-sale constraint:

$$(x_t, y_t, k_t) \in \bar{\mathscr{S}}, \quad \forall t \geq 0. \tag{2.7}$$

Without any constraint other than the no-short-sale constraint on $\mathscr{A}_0(x, y, k)$, the optimal strategy is obviously given by: $c_t = \infty$ for all $t \geq 0$. To prevent the investor from unlimited consumption, we impose two technical conditions on $\mathscr{A}_0(x, y, k)$ as follows:

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\beta t - r\gamma W_t}\right] = 0, \tag{2.8}$$

$$\mathbb{E}\int_0^T \left|y_t e^{-\beta t - r\gamma W_t}\right|^2 dt < \infty, \quad \forall T \in [0, \infty), \tag{2.9}$$

where $W_t = f(x_t, y_t, k_t; l)$. These two restrictions follow from Lo, Mamaysky, and Wang (2001) and Liu (2004). The restriction (2.8) rules out strategies that finance current consumption by running unlimited deficit. The restriction (2.9) is to ensure that $\int_0^T y_t e^{-\beta t - r\gamma W_t} dB_t$ is a martingale. As implied by the proof of the verification theorem Lemma A.1, it is necessary for the Merton solution, which is described below in Theorem 2.1, to be optimal.[3]

Remark 2.2. *In the literature, there is another way to prevent a CARA investor from unlimited consumption in the infinite time horizon case. This way is to weaken the state firstly for the finite time horizon problem, and then let the time horizon $T \to \infty$ to obtain the "infinite time horizon" problem (cf. Chen and Dai, 2013b). By this way, the limit of the finite time horizon problem is actually considered.*

Remark 2.3. *It is natural to impose the no-bankruptcy constraint $f(x_t, y_t, k_t; l) \geq 0$ for all $t \geq 0$. However, this constraint would destroy the separability property of the value function for the CARA utility case as shown in Proposition 2.1.*

Remark 2.4. *The reason we are interested in the CARA utility lies in its separability. By virtue of its separability, the multiple risky-asset problem can be reduced to the single risky-asset case provided that the asset returns are uncorrelated (cf. Proposition A.1). The multiple risky-asset problem is discussed in Appendix A.*

2.4 The Case without Taxes

For the purpose of comparison, we present the main results for the case without capital gains taxes (i.e., $\tau = 0$). In this case, the investor's problem can be rewritten as:

$$V(W) = \max_{(y,c)\in\bar{\mathscr{A}}_0(W)} \mathbb{E}_0^W \left[\int_0^\infty e^{-\beta t} u(c_t) dt \right],$$

subject to

$$dW_t = \left[rW_t - c_t + (\mu - r) y_t \right] dt + \sigma y_t dB_t,$$

where $W_t = x_t + y_t$ is the total wealth, and $\bar{\mathscr{A}}_0(W)$ is the set of admissible strategies defined by the solvency constraint (2.8)-(2.9). This problem permits explicit forms of

[3] The Merton solution allows the investor to incur negative wealth and may require negative consumption. As the initial wealth increases, it is shown by Cox and Huang (1989) that the optimal strategies with the nonnegative wealth and nonnegative consumption constraint converge to the strategies without this constraint. Accordingly, we focus on investors with large initial wealth such as mutual funds and hedge funds, and thus do not impose the nonnegative constraint.

the value function and the optimal strategy, which are presented as follows without proof (cf. Merton, 1969).

Theorem 2.1 (Merton's Result)**.** *In the absence of capital gains taxes, the investor's problem allows an explicit expression of the value function:*

$$V(W) = -\frac{1}{r} e^{-r\gamma W - \frac{\beta - r}{r} - \frac{(\mu - r)^2}{2r\sigma^2}}. \tag{2.10}$$

In addition, the optimal investment and consumption strategy is:

$$y_t^* = \frac{\mu - r}{r\gamma\sigma^2}, \tag{2.11}$$

$$c_t^* = rW_t^* + \frac{\beta - r}{r\gamma} + \frac{(\mu - r)^2}{2r\gamma\sigma^2}, \tag{2.12}$$

where W_t^* is the optimal wealth derived from the above strategy.

Merton's result shows that the optimal investment policy requires the investor to continuously rebalance the portfolio to maintain a constant dollar amount in the stock, and the optimal consumption rate is an affine function of the total wealth.

Remark 2.5. *Merton's single risky-asset problem is extended to allow for multiple risky assets later in Section A.2. To verify the optimality of the corresponding closed form solution, a verification theorem Lemma A.1 is provided there.*

2.5 The Case with Taxes

We now focus on the general case with capital gains taxes (i.e., $\tau > 0$). By the Dynamic Programming Principle, the value function $V(x, y, k)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation[4]

$$\max \left\{ \mathcal{L}_0 V,\, \mathcal{B}_0 V,\, \mathcal{S}_0 V \right\} = 0, \quad (x, y, k) \in \mathscr{S}, \tag{2.13}$$

[4] A heuristic derivation of the HJB equation is provided later in Section 3.1.4.

where

$$\mathcal{L}_0 V = \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + rxV_x - \beta V + u^*(V_x), \tag{2.14}$$

$$\mathcal{B}_0 V = -V_x + V_y + V_k, \tag{2.15}$$

$$\mathcal{S}_0 V = f(0, y, k; l)V_x - yV_y - \left[k - l(k-y)^+\right] V_k, \tag{2.16}$$

and

$$u^*(q) = \sup_c \{u(c) - cq\} = -\frac{q}{\gamma} + \frac{q}{\gamma}\log\frac{q}{\gamma}. \tag{2.17}$$

The optimal consumption rate proves to be

$$c^* = -\frac{1}{\gamma}\log\frac{V_x}{\gamma}, \tag{2.18}$$

and the optimal trading policy is governed by the two free boundaries of the HJB equation (2.13).

We next present an important property of the value function. It provides a groundwork to reduce the dimension of the investor's problem.

Proposition 2.1 (Separability)**.** *The value function is separable:*

$$V(x, y, k) = e^{-r\gamma x} V(0, y, k), \quad (x, y, k) \in \bar{\mathscr{S}}. \tag{2.19}$$

Proof. This proposition follows immediately from Proposition 3.1 as well as Proposition 3.2 (see below). □

The separability property inspires us to make the following transformation to reduce the dimension of the problem:

$$V(x, y, k) = -e^{-r\gamma(x+y) - \phi(z,b)}, \quad z = r\gamma y, \quad b = \frac{k}{y}, \quad (x, y, k) \in \mathscr{S}. \tag{2.20}$$

For expositional convenience, we refer to z as the scaled position on the stock and b as the basis-price ratio. Since V is the solution of the HJB equation (2.13), we can verify that $\phi(z,b)$ satisfies:

$$\max\Big\{\mathcal{L}_1\phi,\,\mathcal{B}_1\phi,\,\mathcal{S}_1\phi\Big\}=0,\quad z>0,\,b>0, \tag{2.21}$$

where

$$\begin{aligned}\mathcal{L}_1\phi &= \frac{1}{2}\sigma^2z^2(\phi_{zz}-\phi_z^2)-\sigma^2zb(\phi_{zb}-\phi_z\phi_b)+\frac{1}{2}\sigma^2b^2(\phi_{bb}-\phi_b^2)+(\mu-\sigma^2z)z\phi_z\\ &\quad +(\sigma^2+\sigma^2z-\mu)b\phi_b-r\phi-\frac{1}{2}\sigma^2z^2+(\mu-r)z+\beta-r\big(1-\log r\big),\\ \mathcal{B}_1\phi &= z\phi_z+(1-b)\phi_b,\\ \mathcal{S}_1\phi &= zf(-1,1,b;l)-z\phi_z+l(b-1)^+\phi_b.\end{aligned}$$

In terms of ϕ, we can rewrite the optimal consumption rate (2.18) as

$$c^*=\frac{1}{\gamma}\big[\phi+r\gamma(x+y)-\log r\big]. \tag{2.22}$$

In contrast to the case without capital gains taxes, the optimal investment s-trategy can be characterized by three regions: the sell region (**SR**), the buy region (**BR**), and the no-trading region (**NTR**). They are defined as follows:

$$\begin{aligned}\mathbf{SR} &= \big\{(z,b)\big|\mathcal{S}_1\phi=0\big\},\\ \mathbf{BR} &= \big\{(z,b)\big|\mathcal{B}_1\phi=0\big\},\\ \mathbf{NTR} &= \big\{(z,b)\big|\mathcal{S}_1\phi<0 \text{ and } \mathcal{B}_1\phi<0\big\}.\end{aligned}$$

In the remainder of this chapter, we aim at locating these regions.

2.5.1 The FR Case

We firstly focus on the FR case (i.e., $l=0$). In this case, the value function has the following two properties, which are analogous to those obtained by Ben Tahar, Soner, and Touzi (2010) in the CRRA utility case.

Proposition 2.2. *If $l = 0$, the value function has lower and upper bounds:*

$$-e^{-r\gamma W - K} \le V(x,y,k) \le -e^{-r\gamma W - \bar{K}}, \quad (x,y,k) \in \bar{\mathscr{S}},$$

where $W = f(x,y,k;0)$,

$$\bar{K} = \frac{\beta - r}{r} + \frac{(\mu - r)^2}{2r\sigma^2} + \log r, \qquad K = \frac{\beta - r}{r} + \frac{((1-\tau)\mu - r)^2}{2r(1-\tau)^2\sigma^2} + \log r. \quad (2.23)$$

Proof. The upper bound can be derived, in the similar way as the proof of Proposition 4.1 of Ben Tahar, Soner, and Touzi (2010), by constructing admissible strategies in the tax-free market. The lower bound can be derived, in the similar way as the proof of Proposition 4.2 of Ben Tahar, Soner, and Touzi (2010), by constructing a sequence of admissible strategies which approximates the value function in a fictitious tax-free market described below. □

The upper bound in Proposition 2.2 is the value function in the tax-free market given by (2.10). It indicates that although the investor may benefit from tax rebates, he cannot perform better than in the tax-free market. The lower bound in Proposition 2.2 is the value function associated with a sub-optimal strategy

$$\begin{aligned} y_t &= \frac{(1-\tau)\mu - r}{r\gamma(1-\tau)^2\sigma^2}, \\ c_t &= rW_t + \frac{\beta - r}{r\gamma} + \frac{((1-\tau)\mu - r)^2}{2r\gamma(1-\tau)^2\sigma^2}, \end{aligned}$$

by which one keeps liquidating the portfolio. It corresponds to the value function in a fictitious tax-free market with a modified expected rate of return $(1-\tau)\mu$ and a modified volatility $(1-\tau)\sigma$ of the stock.

Proposition 2.3 (Optimality of Wash Sales)**.** *Assume $l = 0$. For any $(x,y,k) \in \bar{\mathscr{S}}$, whenever $k \ge y$,*

$$V(x,y,k) = V(W,0,0) = V(W - \tilde{y}, \tilde{y}, \tilde{y}),$$

where $W = f(x,y,k;0)$ and $\tilde{y}$ is any positive constant.

Proof. This property, which holds for any utility function, is a corollary of Proposition 3.5 of Ben Tahar, Soner, and Touzi (2010). □

Proposition 2.3 indicates that whenever the tax basis exceeds the stock price, it is optimal to realize capital losses, or specifically, to rebalance the entire portfolio as follows: $(x_t, y_t, k_t) \to (W_t, 0, 0) \to (W_t - \tilde{y}^*, \tilde{y}^*, \tilde{y}^*)$, where $\tilde{y}^*$ is the optimal position on the stock after realizing capital losses. This policy is observed in practice and is known as a wash sale. Accordingly, we can define the wash sale region (**WSR**) in z-b plane as

$$\mathbf{WSR} = \big\{(z, b)\big|z \geq 0,\, b \geq 1\big\}.$$

This enables us to restrict our attention to $0 \leq b \leq 1$.

Without available analytic solution, we numerically solve the variational equation (2.21) with $l = 0$ by the penalty method with a finite difference discretization (cf. Forsyth and Vetzal, 2002; and Dai, Kwok, and You, 2007). Specifically, the penalty approximation of (2.21) is

$$\mathcal{L}_1\phi + K_P[\mathcal{B}_1\phi]^+ + K_P[\mathcal{S}_1\phi]^+ = 0, \quad z > 0,\, b > 0, \tag{2.24}$$

where $(\cdot)^+ = \max\{\cdot, 0\}$ and K_P is a positive constant.[5] (2.24) is expected to converge to (2.21) as $K_P \to \infty$. Since we are most interested in the **NTR** which is generally much smaller than the state space $(0, \infty) \times (0, \infty)$, we confine ourselves to a truncated domain:

$$[z_{\text{low}}, z_{\text{up}}] \times [b_{\text{low}}, b_{\text{up}}],\;\; b_{\text{low}} = 0,\;\; b_{\text{up}} > 1. \tag{2.25}$$

[5](2.24) can be derived by (3.16) and (2.20).

The boundary conditions are as follows:

$$\begin{cases} 1. \text{ at } \{z_{\text{low}}\} \times [b_{\text{low}}, b_{\text{up}}], & \mathcal{B}_1\phi = 0; \\ 2. \text{ at } \{z_{\text{up}}\} \times [b_{\text{low}}, b_{\text{up}}], & \mathcal{S}_1\phi = 0; \\ 3. \text{ at } (z_{\text{low}}, z_{\text{up}}) \times \{b_{\text{up}}\}, & \mathcal{S}_1\phi = 0; \\ 4. \text{ at } (z_{\text{low}}, z_{\text{up}}) \times \{b_{\text{low}}\}, & \text{use equation (2.24) itself.} \end{cases} \tag{2.26}$$

Conditions 1 and 2 are financially intuitive because a CARA investor would buy or sell the stock when the amount invested in the stock is low or high enough. Condition 3 is implied by the optimality of wash sales for $b > 1$. Condition 4 is natural because (2.24) is degenerate at $b = 0$. We apply a finite difference discretization with Newton iteration for nonlinear terms (cf. Dai and Zhong, 2010) to solve (2.24) subject to the boundary conditions (2.26). Numerical results are provided in the next section.

Remark 2.6. *Our numerical results show that increasing the size of the computation domain does not affect the solution in areas of interest, and that z_{low} can be 0. In addition, b can be 0 (it corresponds to $k = 0$). Thus we can extend the state space to $\mathbb{R}^2_+$.*

Remark 2.7. *Standard existence and uniqueness results for the solution cannot be applied due to the nonlinearity of (2.21). When providing numerical estimates of the solution, we are implicitly assuming that a solution exists and abstracting from uniqueness concerns.*

Remark 2.8. *For the model proposed by Ben Tahar, Soner, and Touzi (2010), Bian, Chen, and Dai (2015) show that although the associated HJB equation admits many solutions, the value function corresponds to the minimal viscosity solution of the HJB equation which can be numerically solved by the penalty method.*

2.5.2 The FC Case

We turn to the FC case (i.e., $l = 1$). In this case, the two free boundaries of the reduced HJB equation (2.21) reads:

$$z\phi_z + (1-b)\phi_b = 0, \quad \text{for } b > 1.$$

This indicates that incessant trading is necessary for $b > 1$ and that ϕ can be rewritten as

$$\omega(\eta) = \phi(z,b), \quad \eta = \frac{1}{1-z(1-b)} \in [0,1]. \tag{2.27}$$

Plugging into (2.21), we have

$$\begin{aligned}&\frac{1}{2}\sigma^2(z^*)^2\eta^4(\omega_{\eta\eta}-\omega_\eta^2) + z^*[\mu+\sigma^2 z^*(\eta-1)]\eta^2\omega_\eta - r\omega\\ &\qquad -\frac{1}{2}\sigma^2(z^*)^2 + (\mu-r)z^* + \beta - r\big(1-\log r\big) = 0, \quad \text{in } \eta\in(0,1), \end{aligned} \tag{2.28}$$

with

$$\omega(0) = \bar{K}, \quad \omega(1) = \phi(z,1),$$

where $\bar{K}$ is defined in (2.23), $\phi(z,1)$ is a constant since $\phi_z = 0$ at $b = 1$, and

$$z^* = \frac{r-\mu-\mu\eta^2\omega_\eta}{\sigma^2[\eta^4(\omega_{\eta\eta}-\omega_\eta^2)+2(\eta-1)\eta^2\omega_\eta - 1]}.$$

Following the algorithm given by Dai, Liu, Yang, and Zhong (2015), we can recursively solve (2.21) with $l = 1$ and (2.28) as follows:

1. Give an initial guess K^0. Set $j = 1$ and constant $b_{\text{up}} > 1$.

2. At the j-th iteration, solve $\omega_j(\eta)$ using (2.28) with $\omega_j(0) = \bar{K}$, $\omega_j(1) = K^{j-1}$.

3. Solve $\phi_j(z,b)$ using (2.21) with

$$\phi_j(z,b_{\text{up}}) = \omega_j\left(\frac{1}{1-z(1-b_{\text{up}})}\right).$$

Table 2.1: The Default Values of the Parameters, CARA

This table reports the default values of the parameters.

Variable	Symbol	Default Value
Interest rate	r	0.01
Expected stock return	μ	0.05
Stock return volatility	σ	0.25
Subjective discount rate	β	0.01
Tax rate	τ	0.15

4. Set $K^j = \phi_j(z, 1)$. If $|K^j - K^{j-1}| <$ tolerance, then stop and set $\phi = \phi_j$, $\omega = \omega_j$; otherwise, set $j = j + 1$, and go to Step 2.

In Step 3, we apply the penalty method. The other boundary conditions are similar to (2.26) in the FR case for computation.

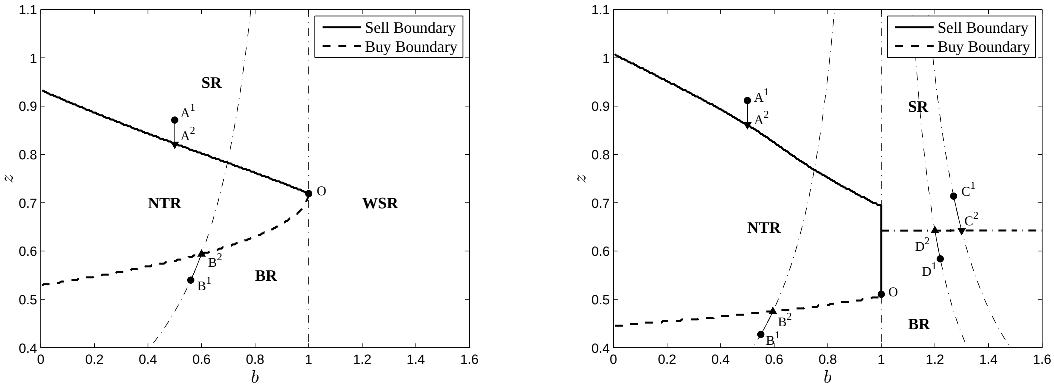
2.6 Numerical Results

In this section, we provide numerical results on the solution of the investor's problem. We use the default values of the parameters summarized in Table 2.1. Risk aversion factor γ is not listed in the table since it does not affect the optimal trading boundaries in z-b plane.

Optimal Trading Policy

Figure 2.1 plots the optimal buy and sell boundaries with the round dot O representing the optimal position at $b = 1$, and produces the expected partition of the state space into three regions in z-b plane. Panel (a) corresponds to the FR case while panel (b) corresponds to the FC case.

In the FR case, as indicated by panel (a) of Figure 2.1, when there are capital gains (i.e., $b < 1$), the investor adopts the following trading policy:



(a) $l = 0$ (b) $l = 1$

Figure 2.1: Trading Boundaries, CARA

This figure shows the optimal trading boundaries. Parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$, $\tau = 0.15$.

- If the current scaled position on the stock z_t is vertically above the sell boundary, the investor would sell vertically downward to reach the sell boundary. For example, sell from A^1 to A^2 in the sub-figure.

- If z_t is vertically below the buy boundary, the investor would buy to reach the buy boundary along the hyperbola

$$z = \frac{z_t(b_t - 1)}{b - 1}. \tag{2.29}$$

  For example, buy from B^1 to B^2 in the sub-figure.

- If z_t is vertically between the buy and sell boundaries, no position adjustment happens.

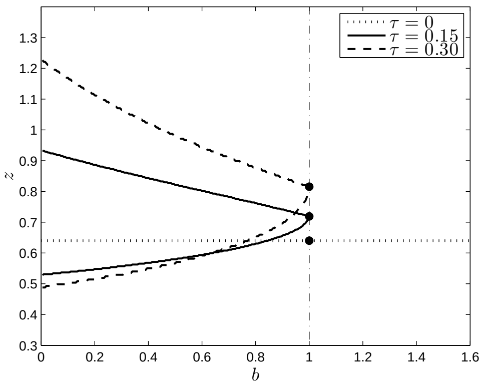
When there are capital losses (i.e., $b > 1$), the investor would sell all of his stock holding and then buy back to $z = \tilde{z}^*$ (the round dot O at $b = 1$ in the sub-figure). This is supported by Proposition 2.3. We point out that $\tilde{z}^*$ can also be regarded as the optimal initial scaled position on the stock.

In the FC case, as indicated by panel (b) of Figure 2.1, when there are capital gains, the optimal trading policy is similar to that in the FR case. When there are capital losses, there is an incessant trading line. The investor would keep trading the stock to stay at the trading line along a hyperbola in the form of (2.29). For example, sell from C^1 to C^2 or buy from D^1 to D^2 in the sub-figure. When the basis-price ratio $b = 1$, the optimal position (the round dot O in the sub-figure) differs from the left limit point of the optimal trading line for $b > 1$. It suggests that the optimal position on the stock is discontinuous at $b = 1$. Comparing panel (b) with panel (a), we find that the FC case has a wider **NTR** and a lower initial position on the stock (the optimal position at $b = 1$) than the FR case.

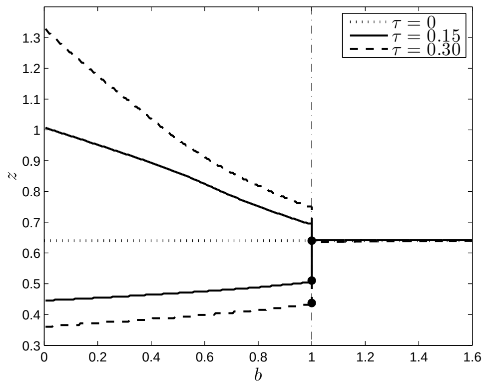
The above findings show that the investor may defer realizing capital gains but would realize losses immediately in both the FR and FC cases. The aim of deferring the realization of capital gains is to save the time value of capital gains taxes, and moreover, in the FC case, to make some of future losses rebatable. In the FR case, the liquidation of the investor's position in case of a capital loss can be explained by the purpose of earning interest on tax rebates earlier and reducing the duration of a sub-optimal position. However, in the FC case, the incessant trading in case of a capital loss can be explained by the purpose of offsetting some potential capital gains. When the basis-price ratio $b = 1$, the investor tends to allocate less money in the stock if he has capital gains so that he could pay less taxes, but tends to allocate more money in the stock if he has capital losses so that he can offset more subsequent capital gains. In addition, since a tax loss carry-forward does not pay any interest and bears the risk of never being used, the investor tends to hold less stock and trade less frequently in the FC case than in the FR case.

Sensitivities

Next, we study the effects of perturbed model parameters on the optimal trading policy.



(a) $l = 0$



(b) $l = 1$

Figure 2.2: Trading Boundaries, Various τ, CARA

This figure shows the optimal trading boundaries for various τ. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

Changes in Tax Rate We firstly investigate the effects of perturbed tax rates. Figure 2.2 plots the optimal trading boundaries with varying tax rates for both the FR and FC cases. We can see that the **NTR** expands at a higher tax rate. It suggests that an investor who pays taxes at a higher rate has less tendency to transact the stock than the one who pays taxes at a lower rate. This can be attributed to the aim of saving tax expenses. Surprisingly, an investor paying taxes at a higher rate seems to allocate more money in the stock after realizing capital losses in the FR case. This can be explained by the investor's expectation of earning more interest on a larger tax rebate in case of a capital loss. However, we find that this expectation may diminish when the money market account becomes more valuable. Figure 2.3 plots the optimal initial scaled position on the stock $\tilde{z}^*$ against tax rate for different levels of interest rate r. It can be seen that when $r = 1\%$, a higher tax rate implies a larger $\tilde{z}^*$. But when $r = 3\%$, a higher tax rate implies a smaller $\tilde{z}^*$.

Changes in Expected Stock Return Figure 2.4 plots the optimal trading boundaries with varying expected rates of stock return. As the expected stock return increases, both of the buy and sell boundaries shift upwards. This is intuitive

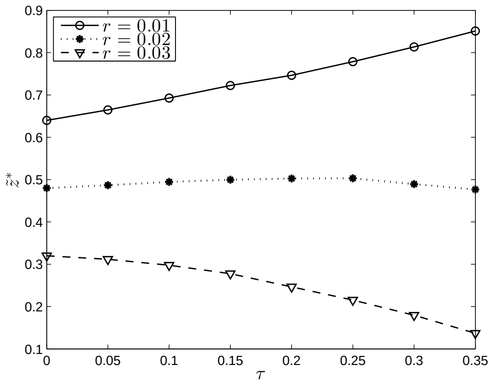
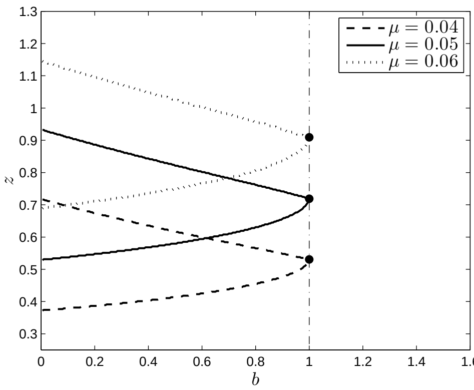
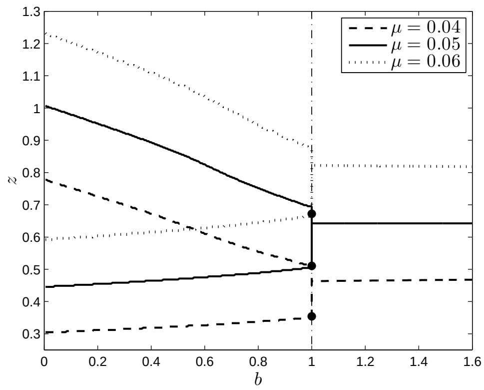


Figure 2.3: Initial Scaled Position on the Stock, FR, CARA

This figure shows the optimal initial scaled position on the stock $\tilde{z}^*$ against τ for different levels of r in the FR case. Other parameter values: $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.



(a) $l = 0$



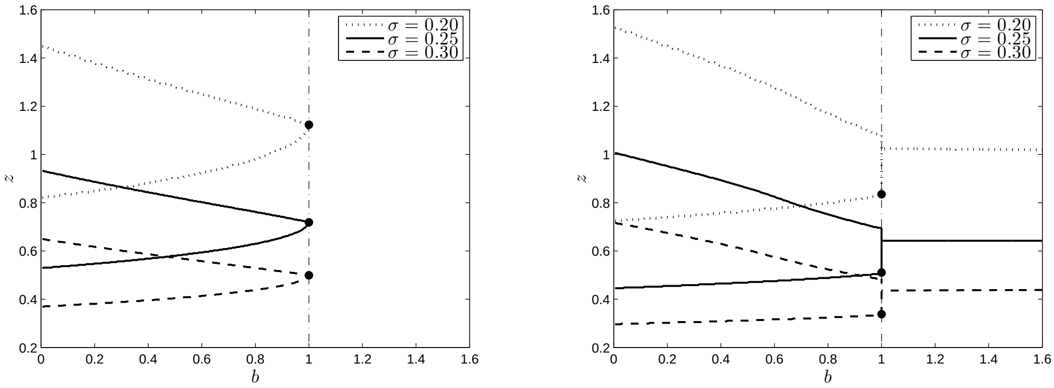
(b) $l = 1$

Figure 2.4: Trading Boundaries, Various μ, CARA

This figure shows the optimal trading boundaries for various μ. Other parameter values: $\tau = 0.15$, $r = 0.01$, $\sigma = 0.25$, $\beta = 0.01$.

since the investor has more incentives to allocate more money into a high-return stock. The figure also shows that the **NTR** widens as the expected stock return rises. This is because the investor tends to hold more on a high-return stock, which may incur a relatively larger amount of taxes.

Changes in Volatility As a measure for variation of stock price, a higher volatility means that the stock price can potentially be spread out over a larger range



(a) $l = 0$ (b) $l = 1$

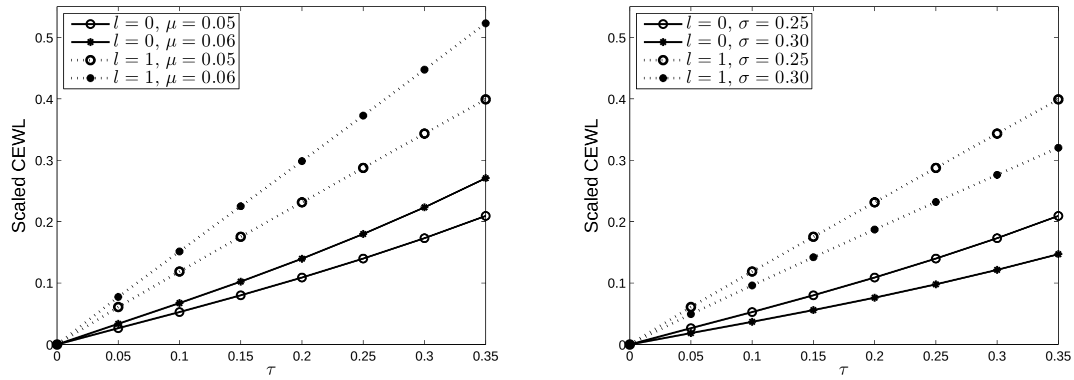
Figure 2.5: Trading Boundaries, Various σ, CARA

This figure shows the optimal trading boundaries for various σ. Other parameter values: $\tau = 0.15$, $r = 0.01$, $\mu = 0.05$, $\beta = 0.01$.

of values. As a result, the investor would feel safer to allocate more money into a low-volatility stock. This is confirmed by Figure 2.5. The figure also shows that as the volatility increases, the **NTR** shrinks. It is inconsistent with the intuition that the investor would have less tendency to trade a high-volatility stock. There can be two reasons for this counterintuitive finding. Firstly, high volatility represents opportunities to buy the stock at a much cheaper price and sell the stock at a much higher price. Secondly, high volatility indicates that the investor would hold less on the stock and thus pay a relatively smaller amount of taxes.

Certainty Equivalent Wealth Loss

In the FR case, although the investor may benefit from tax rebates, he cannot perform better than in the tax-free market. In the FC case, we can imagine that the investor would be even worse off because he no longer qualifies for tax rebates. To examine it, we compute the certainty equivalent wealth loss (CEWL) of Merton from capital gains taxes. To gain the same utility as in the taxable market, the investor in the tax-free market (Merton) can invest less money. The missing money



(a) Various μ (b) Various σ

Figure 2.6: Scaled CEWL, CARA

This figure shows the scaled CEWL of Merton against τ for different levels of μ in panel (a) and for different levels of σ in panel (b). Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

is the CEWL of Merton, Δ_W, which can be computed by

$$V(W, 0, 0; \tau) = V(W - \Delta_W, 0, 0; 0), \tag{2.30}$$

where W and $V(x, y, k; \tau)$ are the initial wealth and the value function of an investor in a market with tax rate τ. To be consistent with the scaled position on the stock $z = r\gamma y$, we report the scaled CEWL of Merton $r\gamma\Delta_W$ instead.

Figure 2.6 plots the scaled CEWL of Merton against tax rate τ for different levels of expected stock return μ in panel (a) and for different levels of stock return volatility σ in panel (b). We can observe that the scaled CEWL significantly increases as the tax rate rises, and the scaled CEWL in the FC case is much lager than that in the FR case. In the default case with $\tau = 0.15$, $\mu = 0.05$, and $\sigma = 0.25$, the scaled CEWL of Merton is 0.08 for the FR case and 0.18 for the FC case. Compared with Merton's optimal scaled position on the stock 0.64, the CEWL-stock ratio is 12.5% for the FR case and 28.1% for the FC case. These observations demonstrate that the investor is substantially worse off by capital gains taxes in the FR case and much worse off in the FC case. Figure 2.6 also shows that a larger expected stock return or a smaller stock return volatility corresponds to a larger value of the scaled CEWL of Merton. This is because in such cases, the investor would allocate more money in the stock, which can lead to a greater effect of capital gains taxes.

Chapter 3

Merton Problem with Transaction Costs and Year-End Taxes

The existing literature on portfolio selection with capital gains taxes always assumes that taxes are paid immediately after sale (instant taxes). However, taxes are actually only paid at the end of each calendar year (year-end taxes) in the real market. Besides the presence of capital gains taxes, the presence of transaction costs is another important feature of the real world. The impact of transaction costs on a taxable investor's strategy has not yet been explored by researchers in a continuous-time framework. In this chapter, we study the continuous-time optimal investment and consumption problem of a CARA investor who faces both transaction costs and year-end taxes. An extension to the CRRA utility case is also provided.

3.1 The Model

Based on the Merton model with instant taxes in the previous chapter, we will propose a Merton model with both transaction costs and year-end taxes in this section. For expositional convenience, we refer to the case where taxes are paid immediately after sale as the instant tax (IT) case, and refer to the case where taxes are only paid at the end of each calendar year as the year-end tax (YT) case throughout this thesis.

3.1.1 The Investor's Problem

We consider a financial market consisting of only two investment assets: a risk-free asset (money market account) and a risky asset (stock). As in the previous chapter, the money market account grows at a constant interest rate of $r > 0$; the price of the stock P_t follows a geometric Brownian motion (2.1) with constant drift $\mu > r$ and constant volatility $\sigma > 0$. We always assume that short selling of the stock is prohibited, and that wash sales of the stock are allowed.

The investor can buy the stock at the ask price of $(1+\theta)P_t$ and sell it at the bid price of $(1-\alpha)P_t$, where $\theta \in [0,\infty)$ and $\alpha \in [0,1)$ are constants representing proportional transaction cost rates for purchasing and selling the stock respectively.

The sales of the stock are subject to taxes on capital gains at a constant tax rate of $\tau \in [0,1)$. The taxation rules in the IT case are presented in Section 2.2. We focus on the YT case. Before the year-end settlement, the investor keeps records of realized capital gains and losses. At the end of the fiscal year, the investor pays a tax on a net realized accumulated capital gain. Our approach to deal with year-end taxes is to discount the amount of the year-end taxes to be paid as cash-flows at times when stock shares are sold. Specifically, before the year-end settlement, the money market account $\hat{x}_t$ and the accumulated taxes to be paid $\hat{\xi}_t$ are given by:

$$\begin{aligned} d\hat{x}_t &= (r\hat{x}_t - c_t)dt - (1+\theta)dL_t + (1-\alpha)y_{t-}dM_t, \\ d\hat{\xi}_t &= -\tau\big[(1-l)((1-\alpha)y_{t-} - k_{t-}) + l((1-\alpha)y_{t-} - k_{t-})^+\big]dM_t, \end{aligned}$$

where the triple (c, L, M) is a consumption-investment strategy which remains the same as described in the first paragraph of Section 2.3, and $l = 0$ or 1 corresponds to the FR case or the FC case. Then the discounted after-tax value of the money market account $x_t = \hat{x}_t + g(t;\lambda)\hat{\xi}_t$, the stock account y_t, and the total cost basis for the stock holding k_t evolve according to the following equations:

$$dx_t = (rx_t - c_t)dt - (1+\theta)dL_t + f(t,0,y_{t-},k_{t-};l,\lambda)dM_t, \tag{3.1}$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dB_t + dL_t - y_{t-}dM_t, \tag{3.2}$$

$$dk_t = (1+\theta)dL_t - k_{t-}dM_t + l(k_{t-} - (1-\alpha)y_{t-})^+ dM_t, \tag{3.3}$$

where

$$f(t,x,y,k;l,\lambda) = x + (1-\alpha)y - g(t;\lambda)\tau\big[(1-l)((1-\alpha)y-k) + l((1-\alpha)y-k)^+\big]$$

is the total wealth after liquidation,

$$g(t;\lambda) = \begin{cases} 1, & \text{for } \lambda = 0, \\ e^{-r(\lceil t \rceil - t)}, & \text{for } \lambda = 1, \end{cases} \tag{3.4}$$

$\lambda = 0$ or 1 corresponds to the IT case or the YT case, and $\lceil t \rceil$ is the ceiling function which is defined as the smallest integer not less than t.[1] For convenience in exposition, we still refer to x_t as the value of the money market account in the YT case.

A consumption-investment strategy (c, L, M) is admissible for $(x,y,k) \in \bar{\mathscr{S}}$ starting from $t \geq 0$ if (x_s, y_s, k_s) given by (3.1)-(3.3) satisfies the solvency constraint

$$(x_s, y_s, k_s) \in \bar{\mathscr{S}}, \ \forall s \geq t, \tag{3.5}$$

$$\lim_{s\to\infty} \mathbb{E}\left[e^{-\beta s - r\gamma W_s}\right] = 0, \tag{3.6}$$

$$\mathbb{E}\int_t^T \left|y_s e^{-\beta s - r\gamma W_s}\right|^2 ds < \infty, \quad \forall T \in [t,\infty), \tag{3.7}$$

where $\bar{\mathscr{S}}$ is the closure of $\mathscr{S}$ given by (2.6) and $W_s = f(s, x_s, y_s, k_s; l, \lambda)$. Let $\mathscr{A}_t(x,y,k)$ denote the set of admissible strategies. We define the value function at

[1] In Ben Tahar, Soner, and Touzi (2007) with $\lambda = l = 0$, $f = x + (1-\alpha)y - \tau(1-\alpha)(y-k)$, where capital gains and losses are also subject to transaction costs. In the real-world tax law, however, transaction costs are one source of capital losses which should be taxed.

time t to be

$$V(t,x,y,k) = \max_{\mathscr{A}_t(x,y,k)} \mathbb{E}_t \left[\int_t^\infty e^{-\beta(s-t)} u(c_s) ds \right], \quad \forall t \geq 0,\ (x,y,k) \in \bar{\mathscr{S}}, \tag{3.8}$$

where $\beta > 0$ is a constant discount factor, $u(\cdot)$ defined in (2.5) is a CARA utility function with a constant risk aversion factor γ.

3.1.2 Separability and Periodicity

This sub-section provides two elementary properties of the value function: separability and periodicity.

Proposition 3.1 (Separability)**.** *The value function is separable:*

$$V(t,x,y,k) = e^{-r\gamma x} V(t,0,y,k), \quad t \geq 0,\ (x,y,k) \in \bar{\mathscr{S}}. \tag{3.9}$$

Proof. The proof is similar to that in Chen and Dai (2013b). Let $(\tilde{c}_s, \tilde{L}_s, \tilde{M}_s) \in \mathscr{A}_t(0,y,k)$ and $(\tilde{x}_s, \tilde{y}_s, \tilde{k}_s)$ be the corresponding solution of (3.1)-(3.3). Consider a consumption-investment strategy $(c_s, L_s, M_s) = (\tilde{c}_s + rx, \tilde{L}_s, \tilde{M}_s)$. We denote by (x_s, y_s, k_s) the corresponding solution of (3.1)-(3.3) with initial position (x,y,k) at time t. Clearly, $y_s = \tilde{y}_s$, $k_s = \tilde{k}_s$, and $\hat{x}_s = x_s - \tilde{x}_s$ is the solution of the following initial value problem:

$$d\hat{x}_s = r(\hat{x}_s - x)ds, \quad \hat{x}_t = x.$$

It has a unique solution $\hat{x}_s \equiv x$. Therefore, $(x_s, y_s, k_s) = (\tilde{x}_s + x, \tilde{y}_s, \tilde{k}_s)$ and it becomes straightforward to verify the solvency constraint (3.5)-(3.7). It then follows that $(c_s, L_s, M_s) \in \mathscr{A}_t(x,y,k)$. Now it is clear that the relation between $(c, L, M) \in \mathscr{A}_t(x,y,k)$ and $(\tilde{c}, \tilde{L}, \tilde{M}) \in \mathscr{A}_t(0,y,k)$ is one-to-one and onto. Due to the separability of the exponential utility function, we have

$$\mathbb{E}_t^{x,y,k} \left[\int_t^\infty e^{-\beta(s-t)} u(c_s) ds; (c, L, M) \right] = e^{-r\gamma x} \mathbb{E}_t^{0,y,k} \left[\int_t^\infty e^{-\beta(s-t)} u(\tilde{c}_s) ds; (\tilde{c}, \tilde{L}, \tilde{M}) \right].$$

Take the supremum over $\mathscr{A}_t(x, y, k)$ on the left-hand side and over $\mathscr{A}_t(0, y, k)$ on the right-hand side. It yields (3.9). □

Proposition 3.2 (Periodicity)**.**
(a) If $\lambda = 0$, the value function V is time-independent.
(b) If $\lambda = 1$, the value function V is one-year periodic:

$$V(t, x, y, k) = V(t+1, x, y, k), \qquad t \geq 0,\ (x, y, k) \in \bar{\mathscr{S}}. \tag{3.10}$$

Proof. The proof is similar to the proof of Proposition 3.6. □

3.1.3 The HJB Equation

It turns out that the value function is governed by the following HJB equation

$$\max\left\{V_t + \mathcal{L}_0 V,\ \mathcal{B}_0 V,\ \mathcal{S}_0 V\right\} = 0, \quad t \geq 0,\ (x, y, k) \in \mathscr{S}, \tag{3.11}$$

where $\mathcal{L}_0$ is the same as (2.14), and

$$\begin{aligned} \mathcal{B}_0 V &= -(1+\theta)V_x + V_y + (1+\theta)V_k, \\ \mathcal{S}_0 V &= f(t, 0, y, k; l, \lambda)V_x - yV_y - \left[k - l(k - (1-\alpha)y)^+\right]V_k. \end{aligned}$$

A heuristic derivation of (3.11) is provided in the next sub-section.

Due to the separability property, we can make the following transformation to reduce the dimension:

$$V(t, x, y, k) = -e^{-r\gamma(x+(1-\alpha)y)-\phi(t,z,b)}, \quad z = r\gamma y, \quad b = \frac{k}{y}, \quad t \geq 0,\ (x, y, k) \in \mathscr{S}.$$

Then the HJB equation (3.11) is reduced to

$$\max\left\{\phi_t + \mathcal{L}_1\phi,\ \mathcal{B}_1\phi,\ \mathcal{S}_1\phi\right\} = 0, \quad t \geq 0,\ z > 0,\ b > 0, \tag{3.12}$$

where

$$
\begin{aligned}
\mathcal{L}_1\phi &= \frac{1}{2}\sigma^2 z^2(\phi_{zz}-\phi_z^2) - \sigma^2 zb(\phi_{zb}-\phi_z\phi_b) + \frac{1}{2}\sigma^2 b^2(\phi_{bb}-\phi_b^2) \\
&\quad +(\mu-(1-\alpha)\sigma^2 z)z\phi_z + (\sigma^2+(1-\alpha)\sigma^2 z-\mu)b\phi_b - r\phi \\
&\quad -\frac{1}{2}(1-\alpha)^2\sigma^2 z^2 + (1-\alpha)(\mu-r)z + \beta - r\big(1-\log r\big), \qquad (3.13) \\
\mathcal{B}_1\phi &= -(\theta+\alpha)z + z\phi_z + (1+\theta-b)\phi_b, \\
\mathcal{S}_1\phi &= zf(t,-1+\alpha,1,b;l,\lambda) - z\phi_z + l(b-1+\alpha)^+\phi_b.
\end{aligned}
$$

Proposition 3.2 follows that: if $\lambda = 0$, ϕ is time-independent; and if $\lambda = 1$, ϕ is one-year periodic:

$$\phi(t+1,z,b) = \phi(t,z,b), \quad t \geq 0,\ z > 0,\ b > 0.$$

Motivated by this, if $\lambda = 1$, we can use the following iterative algorithm to solve (3.12) in one period $t \in [0,1]$:

1. Give an initial guess of $\phi_0(0,z,b)$, and set $j = 1$.

2. At the j-th iteration, solve for $\phi_j(t,z,b)$ using (3.12), $t \in [0,1)$, with terminal condition
$$\phi_j(1,z,b) = \phi_{j-1}(0,z,b).$$

3. If $|\phi_j(1,z,b) - \phi_j(0,z,b)| <$ tolerance, then stop and set $\phi = \phi_j$; otherwise, set $j = j+1$, and go to Step 2.

In Step 2 or for the case $\lambda = 0$, we apply the penalty method.

Remark 3.1. *In the FR case, although transaction costs could inhibit trading, potential tax rebates can be larger relative to even substantial transaction costs as long as b is large enough, say $b = b_{up}$. This motivates us to impose similar boundary conditions as (2.26) for computation.*

Remark 3.2. *In the FC case with $\theta + \alpha = 0$,*

$$z\phi_z + (1-b)\phi_b = 0, \quad \text{for } b > 1.$$

In this case, a similar special treatment as in Section 2.5.2 is conducted for $\lambda = 1$. In the FC case with $\theta + \alpha > 0$, we further make a change of variable

$$\xi = \frac{b}{1+b}. \tag{3.14}$$

Then (3.12) reads

$$\max\left\{\phi_t + \mathcal{L}_1^1\phi,\, \mathcal{B}_1^1\phi,\, \mathcal{S}_1^1\phi\right\} = 0, \quad t \geq 0,\, z > 0,\, 0 < \xi < 1,$$

where

$$\begin{aligned}
\mathcal{L}_1^1\phi &= \frac{1}{2}\sigma^2 z^2(\phi_{zz} - \phi_z^2) - \sigma^2 z\xi(1-\xi)(\phi_{z\xi} - \phi_z\phi_\xi) + \frac{1}{2}\sigma^2\xi^2(1-\xi)^2(\phi_{\xi\xi} - \phi_\xi^2) \\
&\quad + (\mu - (1-\alpha)\sigma^2 z)z\phi_z + (\sigma^2(1-\xi) + (1-\alpha)\sigma^2 z - \mu)\xi(1-\xi)\phi_\xi - r\phi \\
&\quad - \frac{1}{2}(1-\alpha)^2\sigma^2 z^2 + (1-\alpha)(\mu - r)z + \beta - r\big(1 - \log r\big), \\
\mathcal{B}_1^1\phi &= -(\theta+\alpha)z + z\phi_z + (1 + \theta - (2+\theta)\xi)(1-\xi)\phi_\xi, \\
\mathcal{S}_1^1\phi &= -g(t;\lambda)\tau z\frac{(1-\alpha-(2-\alpha)\xi)^+}{1-\xi} - z\phi_z + ((2-\alpha)\xi - 1 + \alpha)^+(1-\xi)\phi_\xi.
\end{aligned}$$

At $\xi = 1$, the above equation degenerates into the one with only transaction costs. The other boundary conditions are similar to (2.26) for computation.

3.1.4 Heuristic Derivation of the HJB Equation

This sub-section presents a heuristic derivation of the HJB equation (3.11). Here, we restrict the original problem (3.8) to a restricted class of admissible strategies in which L and M are absolutely continuous with bounded derivatives, i.e.,

$$L_t = \int_0^t \tilde{l}_s ds, \quad M_t = \int_0^t \tilde{m}_s ds, \quad 0 \le \tilde{l}_s, \tilde{m}_s \le K.$$

By the Dynamic Programming Principle, we can rewrite the restricted problem, denoted by $\tilde{V}(t,x,y,k)$, in an iterative form:

$$\tilde{V}(t,x,y,k) = \max_{c,\tilde{l},\tilde{m}} \mathbb{E}_t \left[\int_t^{t'} e^{-\beta(s-t)} u(c_s) ds + e^{-\beta(t'-t)} \tilde{V}(t', x_{t'}, y_{t'}, k_{t'}) \right]. \tag{3.15}$$

Assuming that $\tilde{V}$ is smooth enough, we may apply Itô's formula between t and t':

$$\begin{aligned} & e^{-\beta t'} \tilde{V}(t', x_{t'}, y_{t'}, k_{t'}) - e^{-\beta t} \tilde{V}(t,x,y,k) \\ & = \int_t^{t'} e^{-\beta s} \left(\tilde{V}_t + \mathcal{L}_0^c \tilde{V} + \tilde{l} \mathcal{B}_0 \tilde{V} + \tilde{m} \mathcal{S}_0 \tilde{V} \right) (s, x_s, y_s, k_s) ds \\ & \quad + \int_t^{t'} e^{-\beta s} \sigma y_s \tilde{V}_y(s, x_s, y_s, k_s) dB_s, \end{aligned}$$

where

$$\mathcal{L}_0^c \tilde{V} = \frac{1}{2} \sigma^2 y^2 \tilde{V}_{yy} + \mu y \tilde{V}_y + (rx - c) \tilde{V}_x - \beta \tilde{V}.$$

Taking the limit $t' \to t$ in (3.15), it is easy to verify that $\tilde{V}$ satisfies the following Bellman equation:

$$\max_{c,\tilde{l},\tilde{m}} \left\{ \tilde{V}_t + \mathcal{L}_0^c \tilde{V} + u(c) + \tilde{l} \mathcal{B}_0 \tilde{V} + \tilde{m} \mathcal{S}_0 \tilde{V} \right\} = 0.$$

The optimal strategy is

$$c^* = -\frac{1}{\gamma} \log \frac{\tilde{V}_x}{\gamma}, \qquad \tilde{l} = \begin{cases} K, & \text{if } \mathcal{B}_0 \tilde{V} \ge 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \tilde{m} = \begin{cases} K, & \text{if } \mathcal{S}_0 \tilde{V} \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

This suggests that transactions either take place at maximum rate or not at all. The optimal strategy yields that

$$\tilde{V}_t + \mathcal{L}_0\tilde{V} + K\left[\mathcal{B}_0\tilde{V}\right]^+ + K\left[\mathcal{S}_0\tilde{V}\right]^+ = 0. \tag{3.16}$$

This is the penalty approximation of the HJB equation (3.11). Since transactions take place at maximum rate, i.e., infinite speed, then we have the following inequalities:

$$V_t + \mathcal{L}_0 V \leq 0, \quad \mathcal{B}_0 V \leq 0, \quad \mathcal{S}_0 V \leq 0,$$
$$\text{at least one of the above is zero,}$$

by taking the limit $K \to \infty$. We can rewrite the above inequalities in a variational form:

$$\max\left\{V_t + \mathcal{L}_0 V,\, \mathcal{B}_0 V,\, \mathcal{S}_0 V\right\} = 0, \quad t \geq 0,\, (x, y, k) \in \mathscr{S},$$

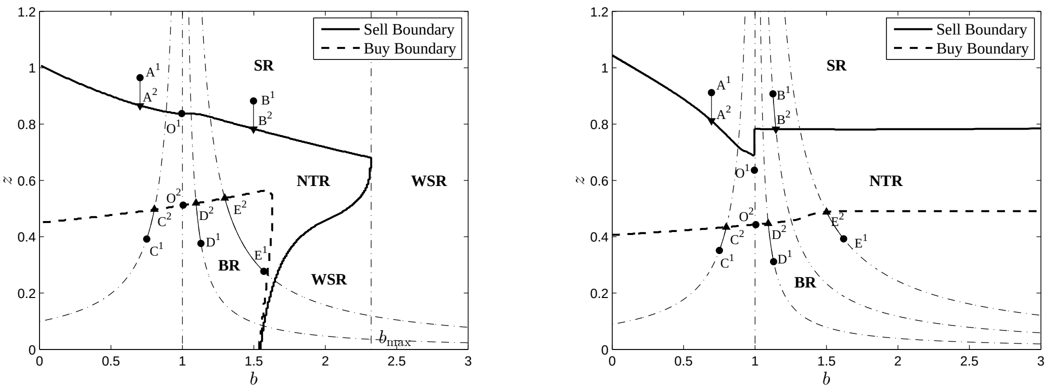
which is the same as (3.11).

Remark 3.3. *In the heuristic derivation of the HJB equation (3.11), we are implicitly assuming that the restricted problem $\tilde{V}$ goes to the original problem V as $K \to \infty$. We point out that the above derivation is firstly proposed by Davis and Norman (1990) for portfolio selection with transaction costs.*

3.2 Optimal Strategy with Transaction Costs

In this section, we provide numerical results to explore the impact of transaction costs on the investor's strategy in the IT case. We use the same default parameters listed in Table 2.1, and moreover, set the default transaction cost rates $\theta = \alpha = 0.5\%$.

Figure 3.1 plots the optimal buy and sell boundaries, and produces the expected partition of the state space into three regions in z-b plane. The transaction direction is marked in the figure. Panel (a) corresponds to the FR case while panel (b)



(a) $l = 0$ (b) $l = 1$

Figure 3.1: Trading Boundaries with Transaction Costs, CARA

This figure shows the optimal trading boundaries with transaction costs. Parameter values: $\tau = 0.15$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$, $\theta = \alpha = 0.005$.

corresponds to the FC case. In each panel, the round dot O^1 represents the optimal position at $b = 1-\alpha$ after sale, and the round dot O^2 represents the optimal position at $b = 1+\theta$ after purchase. We can see that transaction costs can dramatically affect the optimal trading boundaries.

Specifically, in the FR case, the **NTR** exists when the basis-price ratio $b \leq b_{\max}$, where $b_{\max} > 1$ is a constant (e.g., $b_{\max} = 2.32$ in panel (a) of the figure). In the FC case, however, the **NTR** exists for all $b \geq 0$. In both the FR and FC cases, if the current state (z_t, b_t) lies in the **BR**, the investor would buy to reach the buy boundary along the hyperbola

$$z = \frac{z_t(b_t - 1 - \theta)}{b - 1 - \theta}.$$

For example, buy from C^1 to C^2 for $b < 1 + \theta$; buy from D^1 to D^2 or from E^1 to E^2 for $b > 1 + \theta$ in each panel of the figure. If the current state lies in the **SR** and there is a capital gain ($b < 1 - \alpha$), the investor would sell vertically downward to reach the sell boundary. However, if the current state lies in the **SR** and there is a capital loss ($b > 1 - \alpha$), there are differences between the FR and FC cases:

- In the FR case, if z_t is vertically above the upper part of the sell boundary, the investor would sell vertically downward to reach the sell boundary. For example, sell from B^1 to B^2 in panel (a) of the figure. The remaining part of the **SR** consists of two regions: the region below the lower part of the sell boundary and the region with $b > b_{\max}$. This part is the **WSR** where the investor would sell all of his stock holding and then buy back to O^2 in panel (a) of the figure.

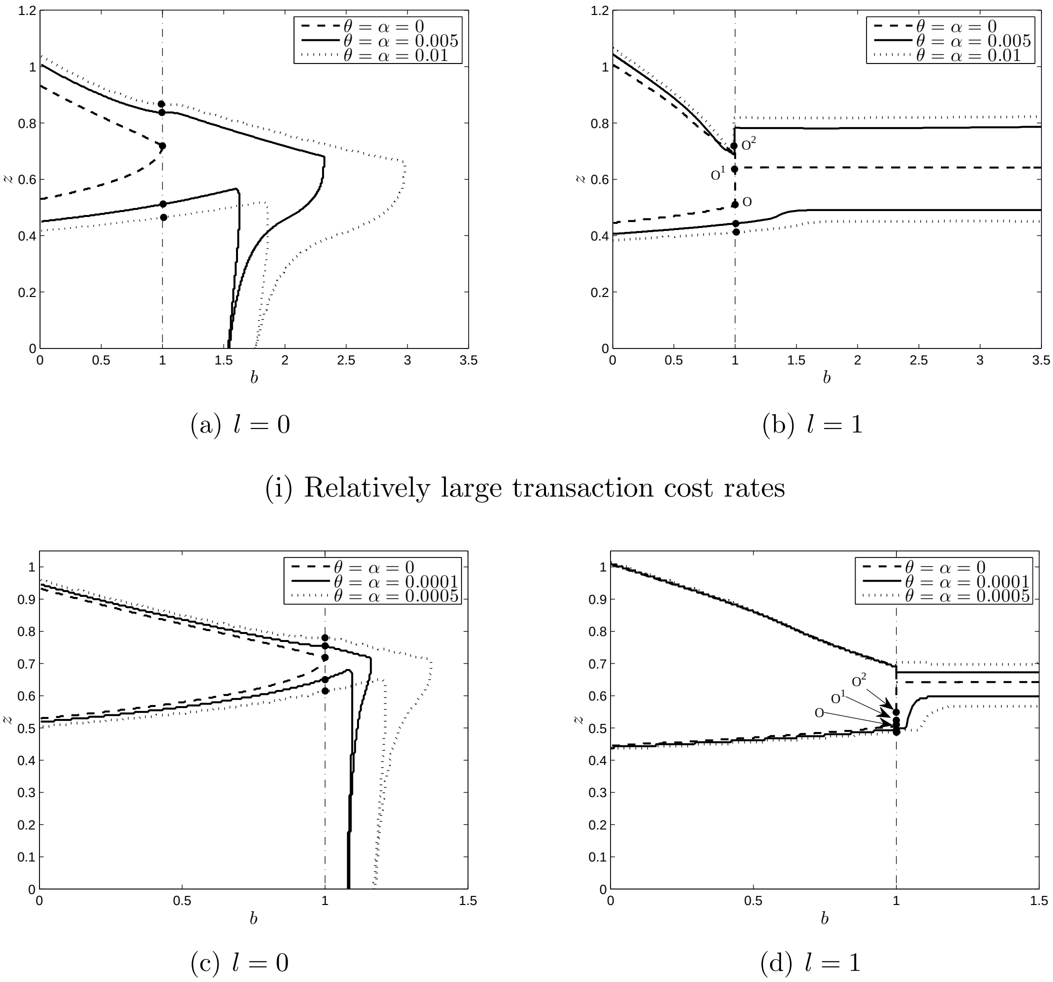
- In the FC case, the investor would sell to the sell boundary along the hyperbola

$$z = \frac{z_t(b_t - 1 + \alpha)}{b - 1 + \alpha}.$$

  For example, sell from B^1 to B^2 in panel (b) of the figure. In this subfigure, moreover, the round dots O^1 and O^2 show that the sell boundary is discontinuous at $b = 1 - \alpha$, but the buy boundary is continuous at $b = 1 + \theta$.

The above findings show that transaction costs can lead the investor to defer realization of capital losses in both the FR and FC cases. This results from the trade-off between saving transaction costs and the desire to receive tax rebates in the FR case or offset future capital gains in the FC case. In the FR case, when available tax rebates are large enough to compensate for transaction costs, the investor may sell some portion of the stock. When available tax rebates are much larger, he may do a wash sale to earn interest earlier on tax rebates (after subtracting transaction costs). In the FC case, although the investor could offset some future capital gains, he may still prefer deferring realization of capital losses since tax loss carry-forward does not pay any interest and bears the risk of never being used.

We next investigate how perturbed transaction cost rates affect the investor's trading policy. Figure 3.2 plots the optimal trading boundaries for various transaction cost rates for both the FR and FC cases. The upper panels show that the shape of the **NTR** with positive transaction costs is significantly different from that with zero transaction costs. The lower panels present the optimal trading boundaries for



(a) $l = 0$ (b) $l = 1$

(i) Relatively large transaction cost rates

(c) $l = 0$ (d) $l = 1$

(ii) Relatively small transaction cost rates

Figure 3.2: Trading Boundaries, Various Transaction Cost Rates, CARA

This figure shows the optimal trading boundaries with various transaction cost rates. Other parameter values: $\tau = 0.15$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

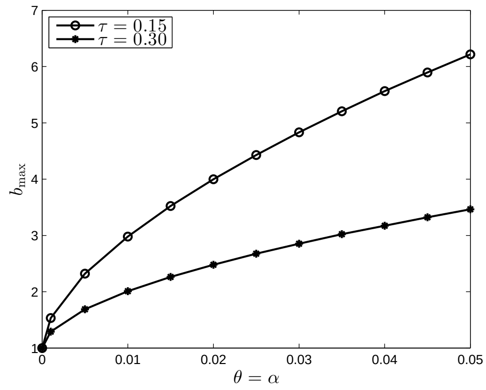


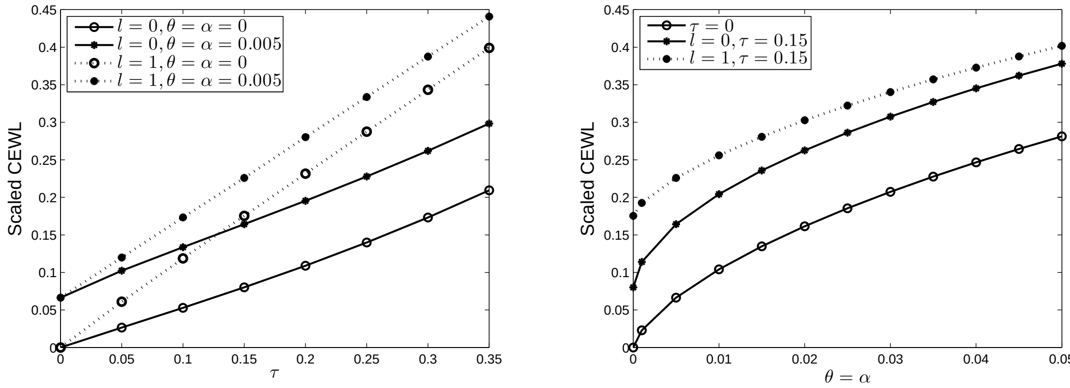
Figure 3.3: $b_{\max}$ of the **NTR**, FR, CARA

This figure shows the greatest value of b, $b_{\max}$, within the **NTR** in the FR case. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

quite small transaction cost rates. From it, we can have a clear look at how the **NTR** with positive transaction costs converges to that with zero transaction costs. Note that in the right panels, O is the optimal position at $b = 1$ for zero transaction costs, O^1 and O^2 are the optimal positions at $b = 1 - \alpha$ after sale for low and high transaction cost rates respectively.

It is intuitive that the **NTR** tends to expand as transaction cost rate increases. However, Figure 3.2 shows that one part of the **NTR** with 0.5% transaction costs is not completely contained in the **NTR** with 1% in the FR case. In this part, an investor paying 1% transaction costs would buy the stock but the one paying 0.5% would not transact. This can happen because the investor paying 1% transaction costs may pay less transaction costs because he tends to buy a smaller amount of stock.

Our results show that there is a greatest value of the basis-price ratio, $b_{\max}$, within the **NTR** in the FR case. Figure 3.3 plots $b_{\max}$ as a function of transaction cost rate for different levels of tax rate. We can observe that $b_{\max}$ is significantly increased by a higher transaction cost rate or a lower tax rate. This is because a larger basis-price ratio is needed to generate large enough tax rebates so that the investor would be willing to do a wash sale.



(a) Against τ (b) Against $\theta = \alpha$

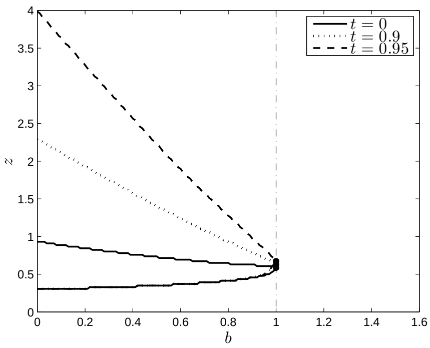
Figure 3.4: Scaled CEWL, with Transaction Costs, CARA

This figure shows the scaled CEWL of Merton in the presence of transaction costs. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

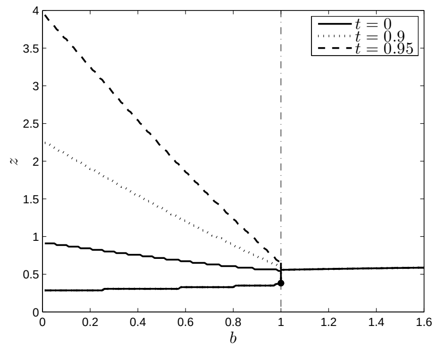
Our previous results as well as the existing literature show that transaction costs and capital gains taxes separately have a significantly negative effect on the investor's expected utility. How will they jointly affect the investor's expected utility? To address this, we compute the scaled CEWL of Merton from transaction costs and capital gains taxes. Figure 3.4 plots it against tax rate in panel (a) and against transaction cost rate in panel (b). In the presence of transaction costs and capital gains taxes, we find that the investor would be even worse off by either a higher tax rate or a higher transaction cost rate. Moreover, this effect can be relatively weakened a bit in the FC case.

3.3 Optimal Strategy with Year-End Taxes

In this section, we provide numerical results in the YT case to explore the impact of the provision that taxes are only paid at the end of each calendar year on the investor's strategy. The default values of the parameters are as follows: interest rate $r = 0.05$, expected stock return $\mu = 0.09$, stock return volatility $\sigma = 0.25$, subjective discount rate $\beta = 0.05$, tax rate $\tau = 0.15$, and transaction cost rates $\theta = \alpha = 0.005$.

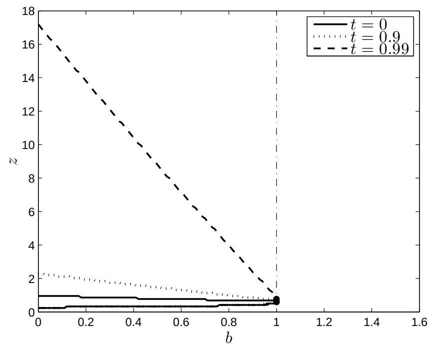


(a) $l = 0$

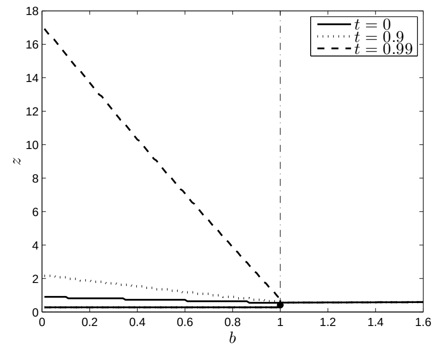


(b) $l = 1$

(i) $t = 0, 0.9, 0.95$

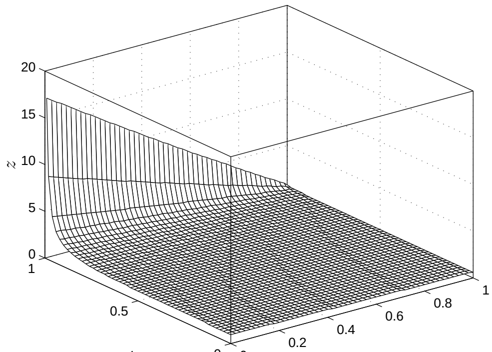


(c) $l = 0$

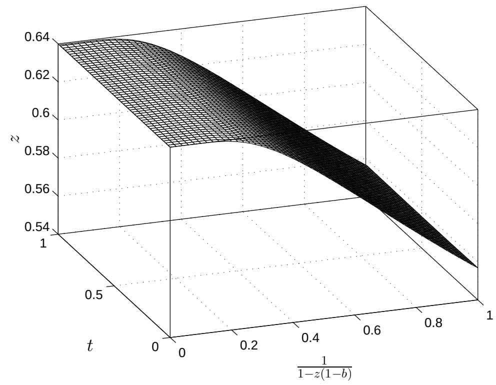


(d) $l = 1$

(ii) $t = 0, 0.9, 0.99$



(e) $l = 0$; sell boundary

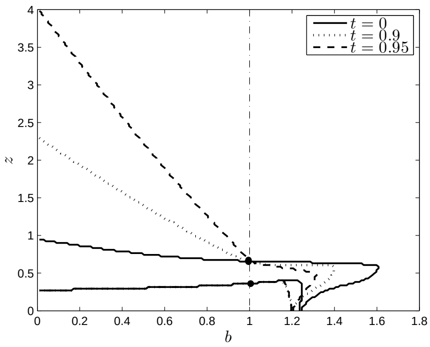


(f) $l = 1$; trading line for $b > 1$

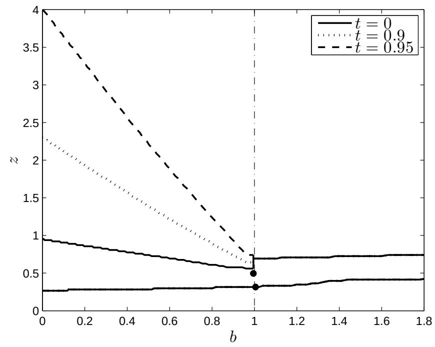
(iii) $t \in [0, 0.99]$

Figure 3.5: Trading Boundaries without Transaction Costs, YT, CARA

This figure shows the optimal trading boundaries without transaction costs in the YT case. Parameter values: $r = 0.05$, $\mu = 0.09$, $\sigma = 0.25$, $\beta = 0.05$, $\tau = 0.15$, $\theta = \alpha = 0$.

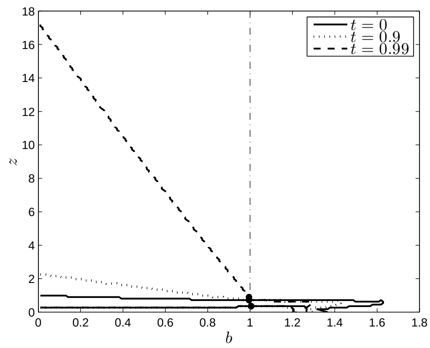


(a) $l = 0$

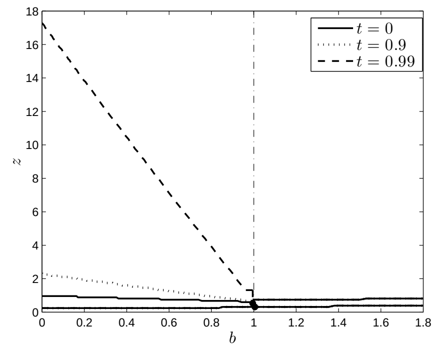


(b) $l = 1$

(i) $t = 0, 0.9, 0.95$

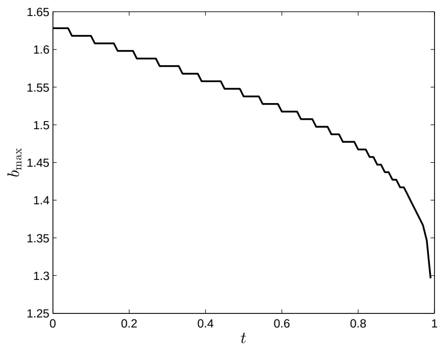


(c) $l = 0$

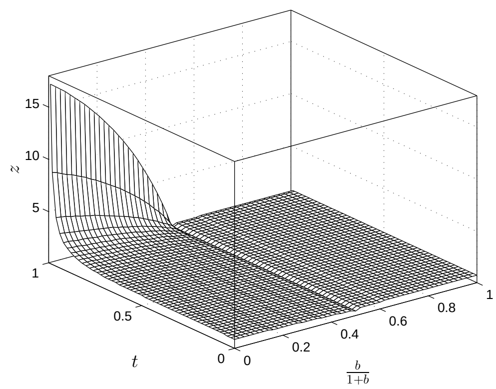


(d) $l = 1$

(ii) $t = 0, 0.9, 0.99$



(e) $l = 0$; $b_{\max}$ of the **NTR**



(f) $l = 1$; sell boundary

(iii) $t \in [0, 0.99]$

Figure 3.6: Trading Boundaries with Transaction Costs, YT, CARA

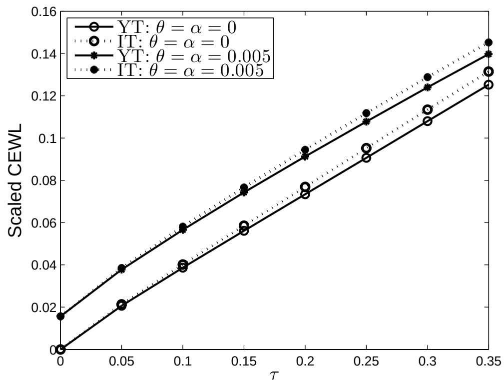
This figure shows the optimal trading boundaries with transaction costs in the YT case. Parameter values: $r = 0.05$, $\mu = 0.09$, $\sigma = 0.25$, $\beta = 0.05$, $\tau = 0.15$, $\theta = \alpha = 0.005$.

We remark that a relatively large r is set to distinguish between the YT case and the IT case, regarding the investor's expected utility.

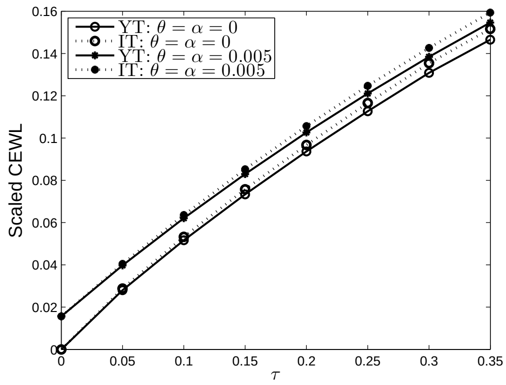
Figure 3.5 plots the optimal trading boundaries without transaction costs in the YT case. The left panels correspond to the FR case while the right panels correspond to the FC case. In each case, the two upper panels show the optimal buy and sell boundaries against the basis-price ratio b at time $t = 0, 0.9, 0.95, 0.99$. We can observe that given t, the trading boundaries have similar shapes as those in the IT case. Besides, the sell boundary for $b < 1$ rises dramatically as time goes by, especially near the end of the fiscal year. In particular, the sell boundary for $b = 0$ can go to infinity as $t \to 1$. This is easy to verify by using the free boundary condition on the sell boundary of (3.12) as well as the periodicity of the value function. To have a clear look at changes of the sell boundary for $b < 1$ over time, in panel (e), we plot the optimal sell boundary against both b and t in the FR case. However, other parts of the trading boundaries vary extremely slightly or even do not vary against t, for example, the optimal trading line for $b > 1$ in the FC case as shown in panel (f). These observations indicate that when possible, the investor tends to avoid realizing capital gains late in this calendar year; moreover, he is inclined to defer realization of capital gains until the beginning of the next calendar year. In this way, the investor can put off the tax liability.

In the presence of transaction costs, the optimal trading boundaries in the YT case are plotted in Figure 3.6. It can be seen that in the FR case, the **NTR** for $b > 1$ shrinks and the greatest value of b within the **NTR** declines as time goes on. This observation suggests that the investor has an incentive to realize capital losses at the end of each calendar year. This is because only at this moment he can receive tax rebates. However, the figure also shows that this incentive disappears in the FC case. The reason is that the investor no longer qualifies for tax rebates.

To explore the effect of the provision that taxes are paid annually on the investor's expected utility, we finally compute the scaled CEWL of Merton from transaction costs and capital gains taxes. Figure 3.7 plots it against tax rate for different levels



(a) $l = 0$



(b) $l = 1$

Figure 3.7: Scaled CEWL, YT, CARA

This figure shows the scaled CEWL of Merton in the YT Case. Other parameter values: $r = 0.05$, $\mu = 0.09$, $\sigma = 0.25$, $\beta = 0.05$.

of transaction cost rate. For the purpose of comparison, results for the IT case are also presented. We can see that the scaled CEWL in the YT case is lower than that in the IT case and the difference between them is widening as tax rate increases. But the difference is not large. With the parameters we use in the figure, the scaled CEWL is decreased by at most 4.8% for the YT case relative to the IT case. This shows that an investor in the YT case can perform better, but not much, than the one in the IT case.

3.4 The CRRA Utility Case

The most widely used utility function in the financial literature exhibits CRRA due to its analytical convenience and support on an empirical basis. In this section, we provide an extension to the CRRA utility.

3.4.1 Problem Formulation

In this section, we assume that the investor preferences are characterized by a CRRA utility function with a constant risk aversion factor γ:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,\, \gamma \neq 1,$$

In this case, we restrict the set of consumption policies to be such that $c_t \geq 0$ for any $t \geq 0$.

Different from the CARA utility case, we assume that the money market account is also subject to taxes on capital gains at a constant rate of $\tau_i \in [0, 1)$. These taxes are also only paid at the end of each calendar year in the YT case.[2] Then, the money market account x_t, the stock account y_t, and the total cost basis for the stock holding k_t are governed by the following equations:

$$dx_t = \big[(1 - \tau_i g(t;\lambda))rx_t - c_t\big]dt - (1+\theta)dL_t + f(t, 0, y_{t-}, k_{t-}; l, \lambda)dM_t, \quad (3.17)$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dB_t + dL_t - y_{t-}dM_t, \quad (3.18)$$

$$dk_t = (1+\theta)dL_t - k_{t-}dM_t + l(k_{t-} - (1-\alpha)y_{t-})^+ dM_t, \quad (3.19)$$

where

$$f(t, x, y, k; l, \lambda) = x + (1-\alpha)y - g(t;\lambda)\tau\big[(1-l)((1-\alpha)y - k) + l((1-\alpha)y - k)^+\big]$$

is the total wealth after liquidation,

$$g(t;\lambda) = \begin{cases} 1, & \text{for } \lambda = 0, \\ \frac{1}{\tau_i + (1-\tau_i)e^{r(\lceil t \rceil - t)}}, & \text{for } \lambda = 1. \end{cases}$$

[2] This can affect the separability of the value function in the CARA utility case.

We define the solvency region to be

$$\mathscr{S}_t = \left\{(x,y,k) \in \mathbb{R}^3 \middle| y > 0,\ k > 0,\ f(t,x,y,k;l,\lambda) > 0\right\},$$

and the value function at time t to be

$$V(t,x,y,k) = \max_{\mathscr{A}_t(x,y,k)} \mathbb{E}_t^{x,y,k}\left[\int_t^\infty e^{-\beta(s-t)}U(c_s)ds\right], \quad \forall (x,y,k) \in \bar{\mathscr{S}_t}, t \geq 0, \quad (3.20)$$

where $\bar{\mathscr{S}_t}$ is the closure of $\mathscr{S}_t$, $\beta > 0$ is a constant discount factor, and $\mathscr{A}_t(x,y,k)$ is the set of admissible strategies (c, L, M) such that the unique solution of (3.17)-(3.19) with initial endowment (x,y,k) satisfies $(x_s, y_s, k_s) \in \bar{\mathscr{S}_s}$ for all $s \geq t$. Then the value function is governed by the following HJB equation

$$\max\left\{V_t + \bar{\mathcal{L}}_0 V,\ \bar{\mathcal{B}}_0 V,\ \bar{\mathcal{S}}_0 V\right\} = 0, \quad t \geq 0,\ (x,y,k) \in \mathscr{S}_t, \qquad (3.21)$$

where

$$\begin{aligned}
\bar{\mathcal{L}}_0 V &= \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + (1-\tau_i g(t;\lambda)) r x V_x - \beta V + U^*(V_x), &(3.22)\\
\bar{\mathcal{B}}_0 V &= -(1+\theta)V_x + V_y + (1+\theta)V_k, &(3.23)\\
\bar{\mathcal{S}}_0 V &= f(t,0,y,k;l,\lambda)V_x - yV_y - \left[k - l(k-(1-\alpha)y)^+\right]V_k, &(3.24)
\end{aligned}$$

and

$$U^*(q) = \sup_{c>0}\left\{U(c) - cq\right\} = \frac{\gamma}{1-\gamma} q^{1-1/\gamma}.$$

The optimal consumption rate proves to be

$$c^* = (V_x)^{-1/\gamma}. \qquad (3.25)$$

By the homogeneity property of the value function Proposition 3.3, we can reduce the dimensionality of the problem:

$$V(t,x,y,k) = y^{1-\gamma}\Phi(t,Z,b), \quad Z = \frac{x}{y}, \quad b = \frac{k}{y}, \quad \forall t \geq 0,\ (x,y,k) \in \mathscr{S}_t, \tag{3.26}$$

It can be verified that $\Phi(t,Z,b)$ satisfies

$$\max\left\{\Phi_t + \bar{\mathcal{L}}_1\Phi,\ \bar{\mathcal{B}}_1\Phi,\ \bar{\mathcal{S}}_1\Phi\right\} = 0, \quad t \geq 0,\ f(t,Z,1,b;l,\lambda) > 0,\ b > 0,$$

where

$$\begin{aligned}
\bar{\mathcal{L}}_1\Phi &= \frac{1}{2}\sigma^2 Z^2\Phi_{ZZ} + \frac{1}{2}\sigma^2 b^2\Phi_{bb} + \sigma^2 Zb\Phi_{Zb} - \left[\mu - (1-\tau_i g(t;\lambda))r - \gamma\sigma^2\right] Z\Phi_Z \\
&\quad -(\mu-\gamma\sigma^2)b\Phi_b + \left[(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right) - \beta\right]\Phi + U^*(\Phi_Z), \\
\bar{\mathcal{B}}_1\Phi &= (1-\gamma)\Phi - (1+\theta+Z)\Phi_Z + (1+\theta-b)\Phi_b, \\
\bar{\mathcal{S}}_1\Phi &= -(1-\gamma)\Phi + f(t,Z,1,b;l,\lambda)\Phi_Z + l(b-1+\alpha)^+\Phi_b.
\end{aligned}$$

We point out that a further transformation like (5.34) should be made to solve the above equation. Numerical methods are similar to those in the CARA utility case.

3.4.2 Theoretical Results

This sub-section provides some theoretical results that facilitates our subsequent analysis.

Proposition 3.3 (Homogeneity)**.** *The value function V has the homogeneity property:*

$$V(t,px,py,pk) = p^{1-\gamma}V(t,x,y,k), \quad t \geq 0,\ (x,y,k) \in \bar{\mathscr{S}}_t,$$

for any positive constant p.

Proof. This property follows immediately from the fact that $(C,L,M) \in \mathscr{A}_t(x,y,k)$ if and only if $(pC,pL,M) \in \mathscr{A}_t(px,py,pk)$. □

Proposition 3.4. *Assume $l = \lambda = 0$ and $\theta = \alpha = 0$. Denote $W = x + y - \tau(y-k)$.*

(a) The value function has lower and upper bounds:

$$\frac{K_0^{-\gamma}}{1-\gamma}W^{1-\gamma} \leq V(x,y,k) \leq \frac{\bar{K}_1^{-\gamma}}{1-\gamma}W^{1-\gamma},$$

where

$$\bar{K}_1 = \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\left[(1-\tau_i)r + \frac{(\mu-(1-\tau_i)r)^2}{2\gamma\sigma^2}\right], \tag{3.27}$$

$$K_0 = \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\left[(1-\tau_i)r + \frac{((1-\tau)\mu-(1-\tau_i)r)^2}{2\gamma(1-\tau)^2\sigma^2}\right]. \tag{3.28}$$

(b) If $r = 0$, then $\bar{K}_1 = K_0$ and

$$V(x,y,k) = \frac{\bar{K}_1^{-\gamma}}{1-\gamma}W^{1-\gamma}.$$

(c) Whenever $k \geq y$, it is optimal to do a wash sale, and

$$V(x,y,k) = V(W,0,0) = V((1-\tilde{\pi})W, \tilde{\pi}W, \tilde{\pi}W),$$

where $\tilde{\pi}$ is any positive constant.

Proof. The proof is provided in Ben Tahar, Soner, and Touzi (2010). □

In part (a) of Proposition 3.4, the upper bound is the value function in a tax-free market with a modified interest rate $(1-\tau_i)r$; the lower bound is the value function in a tax-free market with a modified interest rate $(1-\tau_i)r$, a modified expected stock return $(1-\tau)\mu$, and a modified stock return volatility $(1-\tau)\sigma$. Proposition 3.4 facilitates asymptotic analysis in Chapter 5.

Proposition 3.5. *If $r = 0$ and $l = 0$, the investor's problem reduces to a tax-free problem with the same transaction cost parameter (θ, α). Moreover, if $(\hat{c}^*, \hat{L}^*, \hat{M}^*)$ is the optimal strategy of the tax-free problem, then the strategy $(c^*, L^*, M^*) = (\hat{c}^*, \hat{L}^*/(1-\tau), \hat{M}^*)$ is optimal for the original taxable problem.*

Proof. If $r = 0$ and $l = 0$, set $\hat{x}_t = x_t + \tau k_t$ and $\hat{y}_t = (1-\tau)y_t$. Then it follows from (3.17)-(3.19) that

$$\begin{aligned} d\hat{x}_t &= -c_t dt - (1+\theta)d\hat{L}_t + (1-\alpha)\hat{y}_{t-}dM_t, \\ d\hat{y}_t &= \mu\hat{y}_{t-}dt + \sigma\hat{y}_{t-}dB_t + d\hat{L}_t - \hat{y}_{t-}dM_t, \end{aligned}$$

where $\hat{L} = (1-\tau)L$. Besides, the no-bankruptcy constraint translates into $\hat{x}_t + (1-\alpha)\hat{y}_t \geq 0$. Therefore, the investor's problem, in terms of new variables, reduces to a tax-free problem with transaction costs. □

Proposition 3.6 (Periodicity)**.**
(a) If $\lambda = 0$, the value function V is time-independent.
(b) If $\lambda = 1$, the value function V is one-year periodic:

$$V(t,x,y,k) = V(t+1,x,y,k), \quad \forall\, t \geq 0,\, (x,y,k) \in \bar{\mathscr{S}}_t.$$

Proof. (a) Assume $\lambda = 0$. It is sufficient to prove

$$V(t,x,y,k) = V(0,x,y,k), \quad \forall\, t \geq 0,\, (x,y,k) \in \bar{\mathscr{S}}_t.$$

Its proof is similar to the proof of the case $\lambda = 1$ in part (b).
(b) Assume $\lambda = 1$. Clearly, $\bar{\mathscr{S}}_t = \bar{\mathscr{S}}_{t+1}$. Starting from $s = t+1$, given $(c_s, L_s, M_s) \in \mathscr{A}_{t+1}(x,y,k)$, we denote by (x_s, y_s, k_s) the solution of (3.17)-(3.19). Then we have

$$\begin{aligned} V(t+1,x,y,k) &= \max_{(c_s,L_s,M_s)\in\mathscr{A}_{t+1}(x,y,k)} \mathbb{E}_{t+1}^{x,y,k}\left[\int_{t+1}^{\infty} e^{-\beta(s-t-1)}U(c_s)ds\right] \\ &= \max_{(\tilde{c}_v,\tilde{L}_v,\tilde{M}_v)\in\mathscr{A}_t(x,y,k)} \mathbb{E}_{t}^{x,y,k}\left[\int_{t}^{\infty} e^{-\beta(v-t)}U(\tilde{c}_v)dv\right] \\ &= V(t,x,y,k). \end{aligned}$$

The second equality follows from a change of variable $v = s-1$, and the time translation $(\tilde{c}_v, \tilde{L}_v, \tilde{M}_v) = (c_{v+1}, L_{v+1}, M_{v+1})$, $(\tilde{x}_v, \tilde{y}_v, \tilde{k}_v) = (x_{v+1}, y_{v+1}, k_{v+1})$. □

Table 3.1: The Default Values of the Parameters, CRRA

This table reports the default values of the parameters.

Variable	Symbol	Default Value
Interest rate	r	0.01 (IT), 0.05 (YT)
Expected stock return	μ	0.07 (IT), 0.11 (YT)
Stock return volatility	σ	0.25
Subjective discount rate	β	0.01 (IT), 0.05 (YT)
Risk aversion factor	γ	3
Tax rate for interest	τ_i	0.35
Tax rate for capital gains from stock	τ	0.15
Proportional transaction cost for purchase	θ	0.005
Proportional transaction cost for sale	α	0.005

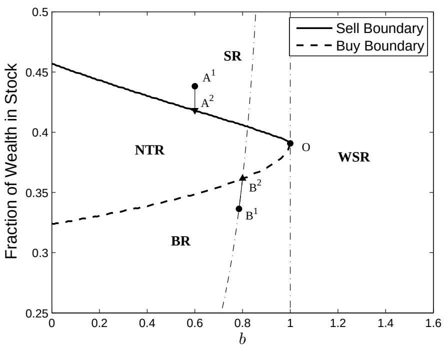
3.4.3 Numerical Results

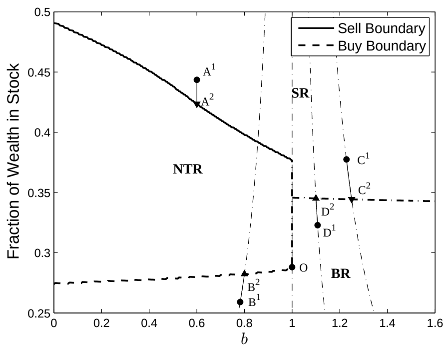
In this sub-section, we provide numerical results to characterize the optimal trading strategy. The default values of the parameters are reported in Table 3.1.

We firstly focus on the IT case. Figure 3.8 plots the optimal trading boundaries against the basis-price ratio b. The vertical axis denotes the amount invested in the stock as a fraction of the total wealth after liquidation, i.e., $\pi = \frac{y}{f(t,x,y,k;l,\lambda)}$. The upper panels correspond to zero transaction costs, the middle panels correspond to positive transaction costs, and the lower panels correspond to different levels of transaction cost rate. The left panels correspond to the FR case while the right panels correspond to the FC case. Comparing with Figure 2.1, Figure 3.1, and Figure 3.2, we find that the corresponding figures between the CRRA and CARA utility cases have similar shapes (the vertical axis represents different variables). Thus, a CRRA investor adopts a similar trading policy. The differences are:

- If the current state (π_t, b_t) lies in the **BR**, the investor would buy to reach the buy boundary along the hyperbola

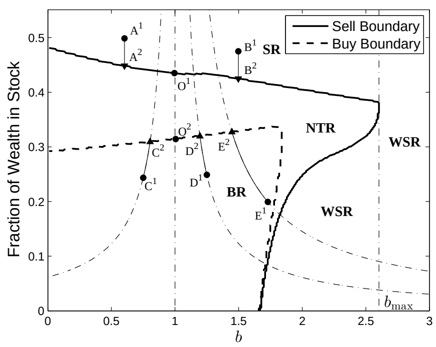
$$\pi = \frac{m}{1 + \theta - b - m(\theta - f(t, -1, 1, b; l, \lambda))},$$

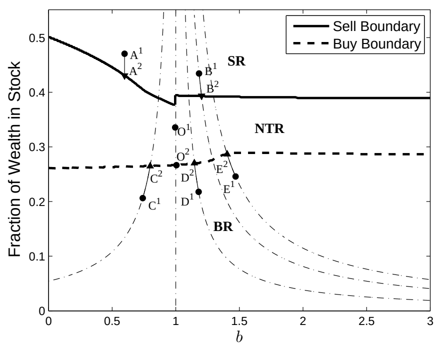




(a) $l = 0$ (b) $l = 1$

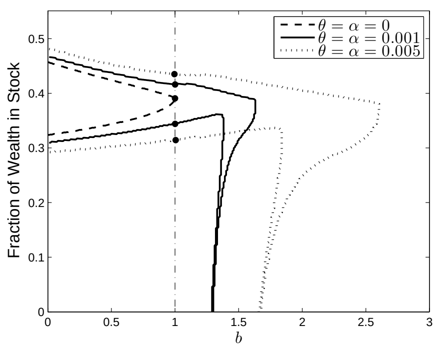
(i) $\theta = \alpha = 0$

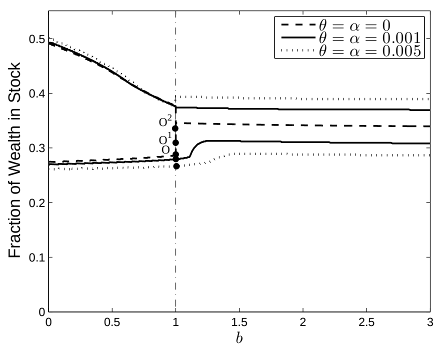




(c) $l = 0$ (d) $l = 1$

(ii) $\theta = \alpha = 0.005$



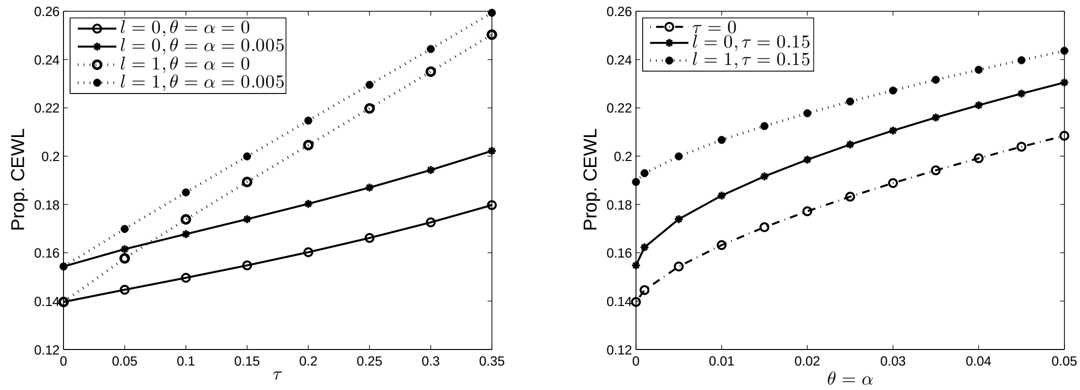


(e) $l = 0$ (f) $l = 1$

(iii) Various $\theta = \alpha$

Figure 3.8: Trading Boundaries, IT, CRRA

This figure shows the optimal trading boundaries in the IT case. The vertical axis denotes the amount invested in the stock as a fraction of the liquidated wealth. Other parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = 0.35$, $\tau = 0.15$.



(a) Against τ (b) Against $\theta = \alpha$

Figure 3.9: Prop. CEWL, IT, CRRA

This figure shows the proportional (prop.) CEWL of Merton in the IT case. Other parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $\tau_i = 0.35$.

where

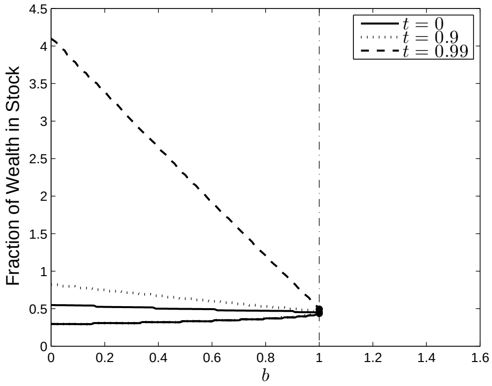
$$m = \frac{\pi_t(1 + \theta - b_t)}{1 + \pi_t(\theta - f(t, -1, 1, b_t; l, \lambda))}.$$

- In the FC case, if (π_t, b_t) lies in the **SR** and there is a capital loss ($b > 1 - \alpha$), the investor would sell to reach the sell boundary along the hyperbola

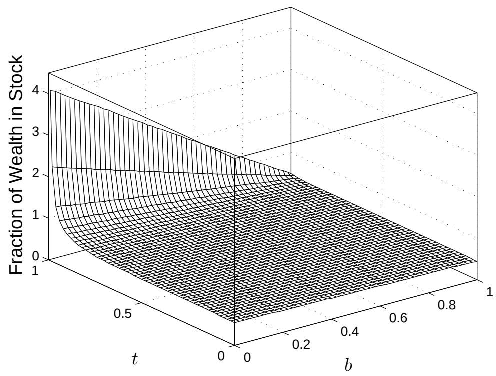
$$\pi = \frac{\pi_t(1 - \alpha - b_t)}{1 - \alpha - b}.$$

To investigate the joint effect of transaction costs and capital gains taxes on the investor's expected utility, we plot the proportional CEWL, as a fraction of the initial wealth, of Merton from transaction costs and capital gains taxes against tax rate on the stock in panel (a) and against transaction cost rate in panel (b) of Figure 3.9. Comparing with Figure 3.4, we have similar findings as in the CARA utility case except that: when the transaction cost rate and the tax rate on the stock are both zero, the proportional CEWL is still positive. This missing amount of Merton is due to taxes on the money market account.

We now turn to the YT case. We plot the optimal trading boundaries for the case with zero transaction costs in Figure 3.10 and for the case with positive transaction

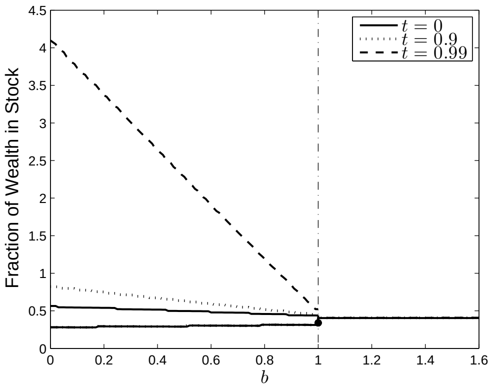


(a) $t = 0, 0.9, 0.99$

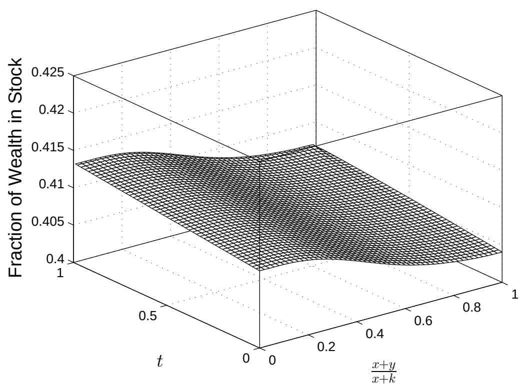


(b) Sell boundary, $t \in [0, 1)$

(i) $l = 0$



(c) $t = 0, 0.9, 0.99$

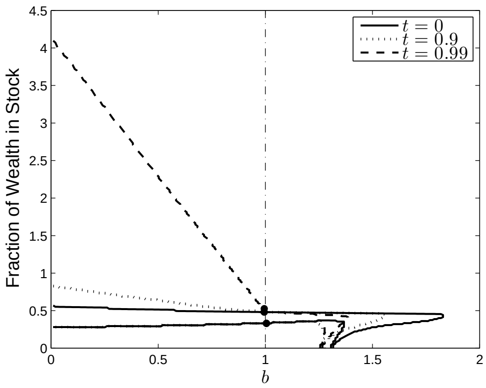


(d) Trading line for $b > 1$, $t \in [0, 1)$

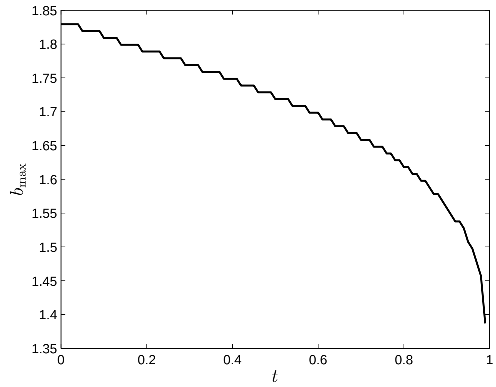
(ii) $l = 1$

Figure 3.10: Trading Boundaries without Transaction Costs, YT, CRRA

This figure shows the optimal trading boundaries without transaction costs in the YT case. The vertical axis denotes the amount invested in the stock as a fraction of the liquidated wealth. Parameter values: $r = 0.05$, $\mu = 0.11$, $\sigma = 0.25$, $\beta = 0.05$, $\gamma = 3$, $\tau_i = 0.35$, $\tau = 0.15$, $\theta = \alpha = 0$.

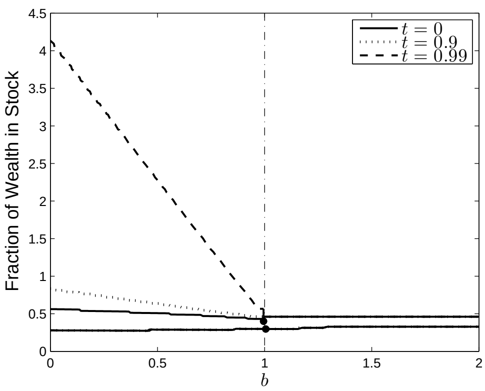


(a) $t = 0, 0.9, 0.99$

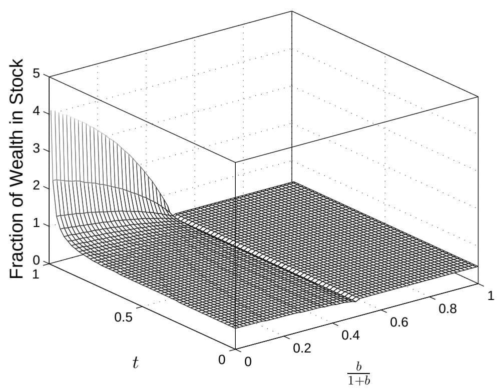


(b) $b_{\max}$ of the **NTR**, $t \in [0, 1)$

(i) $l = 0$



(c) $t = 0, 0.9, 0.99$

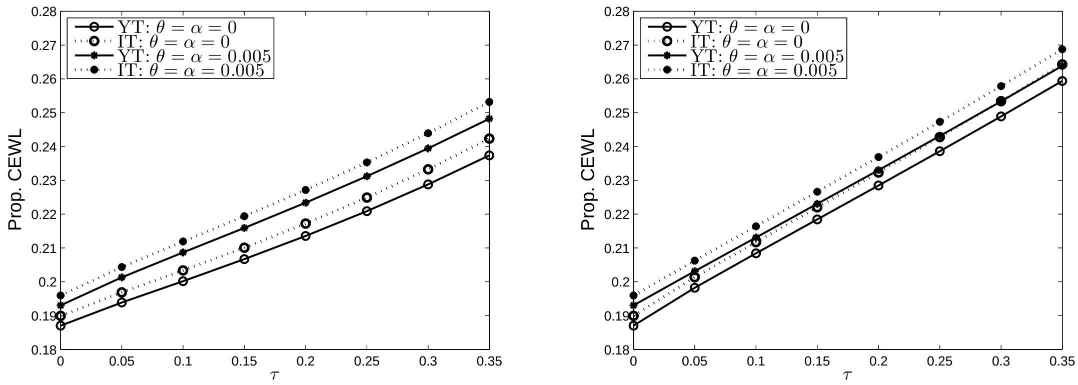


(d) Sell boundary, $t \in [0, 1)$

(ii) $l = 1$

Figure 3.11: Trading Boundaries with Transaction Costs, YT, CRRA

This figure shows the optimal trading boundaries with transaction costs in the YT case. The vertical axis denotes the amount invested in the stock as a fraction of the liquidated wealth. Parameter values: $r = 0.05$, $\mu = 0.11$, $\sigma = 0.25$, $\beta = 0.05$, $\gamma = 3$, $\tau_i = 0.35$, $\tau = 0.15$, $\theta = \alpha = 0.005$.



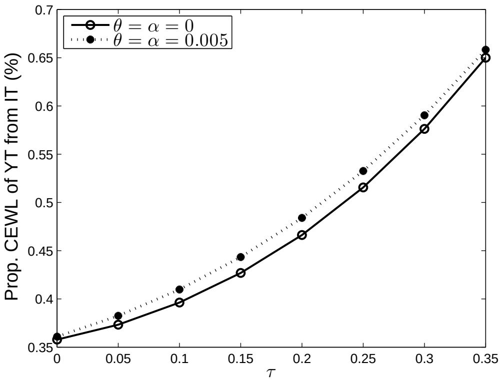
(a) $l = 0$ (b) $l = 1$

Figure 3.12: Prop. CEWL, YT, CRRA

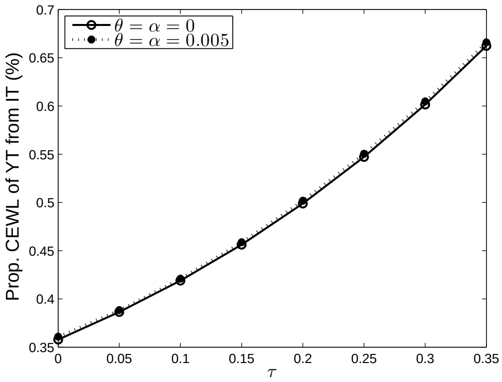
This figure shows the proportional (prop.) CEWL of Merton in the YT Case. Other parameter values: $r = 0.05$, $\mu = 0.11$, $\sigma = 0.25$, $\beta = 0.05$, $\gamma = 3$, $\tau_i = 0.35$.

costs in Figure 3.11 respectively. These figures have similar shapes as Figure 3.5 and Figure 3.6. Therefore, the effect of the provision that taxes are paid annually on the optimal trading policy is similar to that in the CARA utility case.

Figure 3.12 plots the proportional CEWL of Merton from transaction costs and capital gains taxes against tax rate on the stock for different levels of transaction cost rate. The figure shows that an investor in the YT case is better off than the one in the IT case. In addition, the difference of the proportional CEWL between the YT and IT cases mainly comes from taxes on the money market account. To eliminate the effect of taxes on the money market account, we plot the proportional CEWL of the YT case from the IT case in Figure 3.13. We can observe that the proportional CEWL is less than 1% with the parameters we use in the figure. It suggests that an investor in the YT case is better off, but not much, than the one in the IT case. This finding is similar to that in the CARA utility case.



(a) $l = 0$



(b) $l = 1$

Figure 3.13: Prop. CEWL of the YT Case from the IT Case, CRRA

This figure shows the proportional (prop.) CEWL of the YT case from the IT case. Other parameter values: $r = 0.05$, $\mu = 0.11$, $\sigma = 0.25$, $\beta = 0.05$, $\gamma = 3$, $\tau_i = 0.35$.

Chapter 4

Labor Income and Borrowing Constraints

In addition to financial income received from the portfolio, the investor can also receive labor income in the real market. In a perfectly liquid market, Merton (1971) provides an extension in which the investor has a deterministic stream of labor income. He shows that the investor adopts the optimal strategy as if he has no labor income but instead capitalizes the lifetime labor income flow at the risk-free interest rate to his wealth. In this case, the wealth may go below zero (cf. He and Pages, 1993). In the real world, however, it may be not possible for the investor to borrow against future labor income. As a consequence, the investor can only choose an investment and consumption strategy such that the wealth is nonnegative. In this chapter, we extend our model to incorporate a constant stream of labor income. Our aim is to explore the impact of the no-borrowing constraint on the investor's strategy. Since negative wealth is permitted with CARA utility, we focus on the CRRA utility, under which the total wealth is restricted by the no-bankruptcy constraint.

4.1 Problem Formulation

The portfolio selection problem without labor income is presented in Section 3.4. In this chapter, we assume that the investor receives labor income at a constant rate of $I > 0$. Other settings and symbols, unless otherwise mentioned, remain the same as in Section 3.4. Then the money market account x_t, the stock account y_t, and the total cost basis for the stock holding k_t are given by:

$$dx_t = \big[(1-\tau_i g(t;\lambda))rx_t - c_t + I\big]dt - (1+\theta)dL_t + f(t,0,y_{t-},k_{t-};l,\lambda)dM_t, \quad (4.1)$$

$$dy_t = \mu y_{t-}dt + \sigma y_{t-}dB_t + dL_t - y_{t-}dM_t, \quad (4.2)$$

$$dk_t = (1+\theta)dL_t - k_{t-}dM_t + l(k_{t-} - (1-\alpha)y_{t-})^+ dM_t. \quad (4.3)$$

We assume that the investor is not allowed to borrow against his future labor income, that is, the total wealth after liquidation is restricted by the no-bankruptcy constraint, i.e.,

$$W_t = f(t,x_t,y_t,k_t;l,\lambda) \geq 0, \quad \text{for all } t \geq 0.$$

Then we define the solvency region as

$$\mathscr{S}_t = \big\{(x,y,k) \in \mathbb{R}^3 \big| y > 0,\ k > 0,\ f(t,x,y,k;l,\lambda) > 0\big\}.$$

For any $(x,y,k) \in \bar{\mathscr{S}_t}$, the investor's value function at time t is defined as

$$V(t,x,y,k) = \max_{\mathscr{A}_t(x,y,k)} \mathbb{E}_t^{x,y,k}\left[\int_t^\infty e^{-\beta(s-t)}U(c_s)ds\right],$$

where $\mathscr{A}_t(x,y,k)$ is the set of admissible strategies (c,L,M) such that the unique solution of (4.1)-(4.3) $(x_s,y_s,k_s) \in \bar{\mathscr{S}_s}$ for all $s \geq t$. It turns out that V is characterized by

$$\max\Big\{V_t + \bar{\mathcal{L}}_0 V + IV_x,\ \bar{\mathcal{B}}_0 V,\ \bar{\mathcal{S}}_0 V\Big\} = 0, \quad t \geq 0,\ (x,y,k) \in \mathscr{S}_t, \quad (4.4)$$

where the operators $\bar{\mathcal{L}}_0$, $\bar{\mathcal{B}}_0$, and $\bar{\mathcal{S}}_0$ are defined in (3.22)-(3.24).

4.2 The Case with Only Labor Income and Taxes

The no-borrowing constraint destroys the homogeneity property of the value function (cf. Proposition 3.3). As a consequence, the 3-dimensional (time-dependent) HJB equation (4.4) cannot be reduced, and thus the computational demand increases substantially. To get some idea about the impact of the no-borrowing constraint on the investor's strategy, we focus on the no-transaction cost, FR and IT case (i.e., $l = \lambda = 0$, $\theta = \alpha = 0$). In this case, it is easy to verify that the value function V is time-independent (cf. Proposition 3.6). So we abbreviate $V(t, x, y, k)$ and write $V(x, y, k)$.

To eliminate the non-homogeneous term IV_x in (4.4), we make a change of variable

$$\bar{x} = x + \frac{I}{(1-\tau_i)r}.$$

Then the value function $V(\bar{x}, y, k)$ satisfies

$$\max\left\{\bar{\mathcal{L}}_0 V(\bar{x}, y, k),\, \bar{\mathcal{B}}_0 V(\bar{x}, y, k),\, \bar{\mathcal{S}}_0 V(\bar{x}, y, k)\right\} = 0, \tag{4.5}$$

in the domain

$$y > 0,\ k > 0,\ \bar{W} = \bar{x} + y - \tau(y - k) > \frac{I}{(1-\tau_i)r}.$$

This HJB equation has the same form as the one without labor income but with a different solution domain. We remark that if the investor is free to borrow against future labor income, i.e., $\bar{W}_t \geq 0$, the HJB equation (4.5) is the same as the one without labor income but in terms of $(\bar{x}, y, k)$. It suggests that the investor adopts the optimal strategy as if he has no labor income but instead capitalizes the labor income flow at the after-tax interest rate to his wealth.

We futher make a change of variables

$$V(\bar{x}, y, k) = \frac{\bar{K}_1^{-\gamma}}{1-\gamma}\bar{W}^{1-\gamma}e^{(1-\gamma)\Psi(\bar{z}, b, \xi)}, \tag{4.6}$$

$$\bar{z} = \frac{(1-\tau)y}{\bar{W}}, \quad b = \frac{k}{y}, \quad \xi = \frac{\bar{W}}{1+\bar{W}}. \tag{4.7}$$

where $\bar{K}_1$ is a constant defined in (3.27). It can be verified that $\Psi(\bar{z}, b, \xi)$ satisfies

$$\max\left\{\mathcal{L}_I\Psi,\, \mathcal{B}_I\Psi,\, \mathcal{S}_I\Psi\right\} = 0, \quad \bar{z} > 0,\, b > 0,\, \xi \in \left(\frac{I}{I + (1-\tau_i)r}, 1\right), \tag{4.8}$$

where

$$\begin{aligned}
\mathcal{L}_I\Psi &= \frac{1}{2}\sigma^2\bar{z}^2(1-\bar{z})^2\left[\Psi_{\bar{z}\bar{z}} + (1-\gamma)\Psi_{\bar{z}}^2\right] - \sigma^2\bar{z}(1-\bar{z})b\left[\Psi_{\bar{z}b} + (1-\gamma)\Psi_{\bar{z}}\Psi_b\right] \\
&\quad + \frac{1}{2}\sigma^2 b^2\left[\Psi_{bb} + (1-\gamma)\Psi_b^2\right] + \sigma^2\bar{z}^2(1-\bar{z})\xi(1-\xi)\left[\Psi_{\bar{z}\xi} + (1-\gamma)\Psi_{\bar{z}}\Psi_\xi\right] \\
&\quad + \frac{1}{2}\sigma^2\bar{z}^2\xi^2(1-\xi)^2\left[\Psi_{\xi\xi} + (1-\gamma)\Psi_\xi^2\right] - \sigma^2 b\bar{z}\xi(1-\xi)\left[\Psi_{b\xi} + (1-\gamma)\Psi_b\Psi_\xi\right] \\
&\quad + \left[\left(\mu - \bar{r} - \gamma\sigma^2\bar{z}\right)(1-\bar{z}) + \frac{\bar{r}\tau b\bar{z}}{(1-\tau)}\right]\bar{z}\Psi_{\bar{z}} + \left[\sigma^2 - \mu - (1-\gamma)\sigma^2\bar{z}\right]b\Psi_b \\
&\quad + \left[\bar{r} + (\mu - \bar{r})\bar{z} + \sigma^2\bar{z}^2(1-\gamma-\xi) - \frac{\bar{r}\tau b\bar{z}}{(1-\tau)}\right]\xi(1-\xi)\Psi_\xi \\
&\quad - \frac{1}{2}\gamma\sigma^2\bar{z}^2 + (\mu - \bar{r})\bar{z} + \bar{r} - \frac{\beta}{1-\gamma} - \frac{\bar{r}\tau b\bar{z}}{(1-\tau)} \\
&\quad + \bar{K}_1 U^*\left(e^{\Psi}(1 - \bar{z}\Psi_{\bar{z}} + \xi(1-\xi)\Psi_\xi)\right), \qquad \bar{r} = (1-\tau_i)r, \\
\mathcal{B}_I\Psi &= \bar{z}\Psi_{\bar{z}} + (1-b)\Psi_b, \\
\mathcal{S}_I\Psi &= -\Psi_{\bar{z}}.
\end{aligned}$$

The transformation (4.6)-(4.7) is inspired by the transformation (5.33)-(5.34) for the case without labor income. We apply the penalty method to solve the above equation. Two of the boundary conditions are: at $\xi = 1$, the above reduced HJB equation degenerates to the one in the free borrowing case; at $\xi = \frac{I}{I+(1-\tau_i)r}$, use $\mathcal{S}_I\Psi = 0$. The latter boundary condition, corresponding to zero wealth, is natural since the investor cannot borrow against future labor income. The other boundary conditions are similar to (2.26) for computation.

4.3 The Case with Only Labor Income

For the purpose of comparison, we also investigate the simple case with only labor income. Then the evolution of the wealth process $W_t = x_t + y_t$ is described by

$$dW_t = \big[rW_t + (\mu - r)y_t - c_t + I\big]dt + \sigma y_t dB_t.$$

The value function

$$V(W) = \max_{y_t, c_t} \mathbb{E}_0^W \left[\int_0^\infty e^{-\beta t} U(c_t) dt\right]$$

satisfies the following HJB equation

$$\max_{y,c} \left\{ \frac{1}{2}\sigma^2 y^2 V_{WW} + [rW + (\mu - r)y - c + I]V_W - \beta V + U(c) \right\} = 0, \quad W > 0.$$

After letting $\bar{W} = W + \frac{I}{r}$ and using similar transformation as (4.6)-(4.7), we can use a numerical method to solve the resulting equation. Here we present a different method, a dual approach, to derive a closed form of the optimal strategy.[1]

The dual of the value function is

$$\phi(\zeta) = \sup_{W>0} [V(W) - W\zeta] := I^{1-\gamma}\psi(\eta), \quad \eta = I^\gamma \zeta.$$

We can verify that

$$\zeta = V_W, \quad \phi_\zeta = -W, \quad \phi_{\zeta\zeta} = -\frac{1}{V_{WW}},$$
$$W = -I\psi_\eta, \quad V = I^{1-\gamma}(\psi - \eta\psi_\eta).$$

Then $\psi(\eta)$ satisfies

$$\frac{1}{2}\beta_3 \eta^2 \psi_{\eta\eta} - (\beta_1 - \beta_2)\eta\psi_\eta - \beta_2\psi + U^*(\eta) + \eta = 0, \quad \eta \in (0, \bar{\eta}),$$

[1] The investor's problem here is a special case in Dybvig and Liu (2011). Refer to it for more details.

with boundary condition

$$\psi_\eta(\bar{\eta}) = 0, \quad \psi_{\eta\eta}(\bar{\eta}) = 0,$$

where

$$\beta_1 = r, \quad \beta_2 = \beta, \quad \beta_3 = \frac{(\mu - r)^2}{\sigma^2}.$$

Note that $\bar{\eta}$ corresponds to $W = 0$. The above equation has a closed form of solution

$$\psi(\eta) = A_+ \eta^{\alpha_-} + A_- \eta^{\alpha_+} + \frac{1}{\bar{K}_2} U^*(\eta) + \frac{1}{\beta_1}\eta$$

where

$$\begin{aligned}
\bar{K}_2 &= \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\left[r + \frac{(\mu - r)^2}{2\gamma\sigma^2}\right], \\
\alpha_\pm &= \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 \pm \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}, \\
A_- &= \frac{1}{\bar{K}_2} \frac{\frac{\gamma-1}{\gamma} - \alpha_-}{\alpha_+(\alpha_+ - \alpha_-)} \bar{\eta}^{\frac{\gamma-1}{\gamma} - \alpha_+} - \frac{1}{\beta_1} \frac{1 - \alpha_-}{\alpha_+(\alpha_+ - \alpha_-)} \bar{\eta}^{1-\alpha_+}, \\
A_+ &= \frac{1}{\bar{K}_2} \frac{\alpha_+ - \frac{\gamma-1}{\gamma}}{\alpha_-(\alpha_+ - \alpha_-)} \bar{\eta}^{\frac{\gamma-1}{\gamma} - \alpha_-} - \frac{1}{\beta_1} \frac{\alpha_+ - 1}{\alpha_-(\alpha_+ - \alpha_-)} \bar{\eta}^{1-\alpha_-} = 0, \\
\bar{\eta} &= \left(\frac{1}{\bar{K}_2}\beta_1 \frac{\alpha_+ - \frac{\gamma-1}{\gamma}}{\alpha_+ - 1}\right)^\gamma.
\end{aligned}$$

Besides, the optimal strategy is

$$y^* = \frac{\mu - r}{\sigma^2} I\eta\psi_{\eta\eta}, \quad c^* = I\eta^{-\frac{1}{\gamma}}.$$

4.4 The Case with Only Labor Income and Transaction Costs

We also consider the case with only labor income and transaction costs. In this case, the value function $V(x, y)$ is associated with the following HJB equation

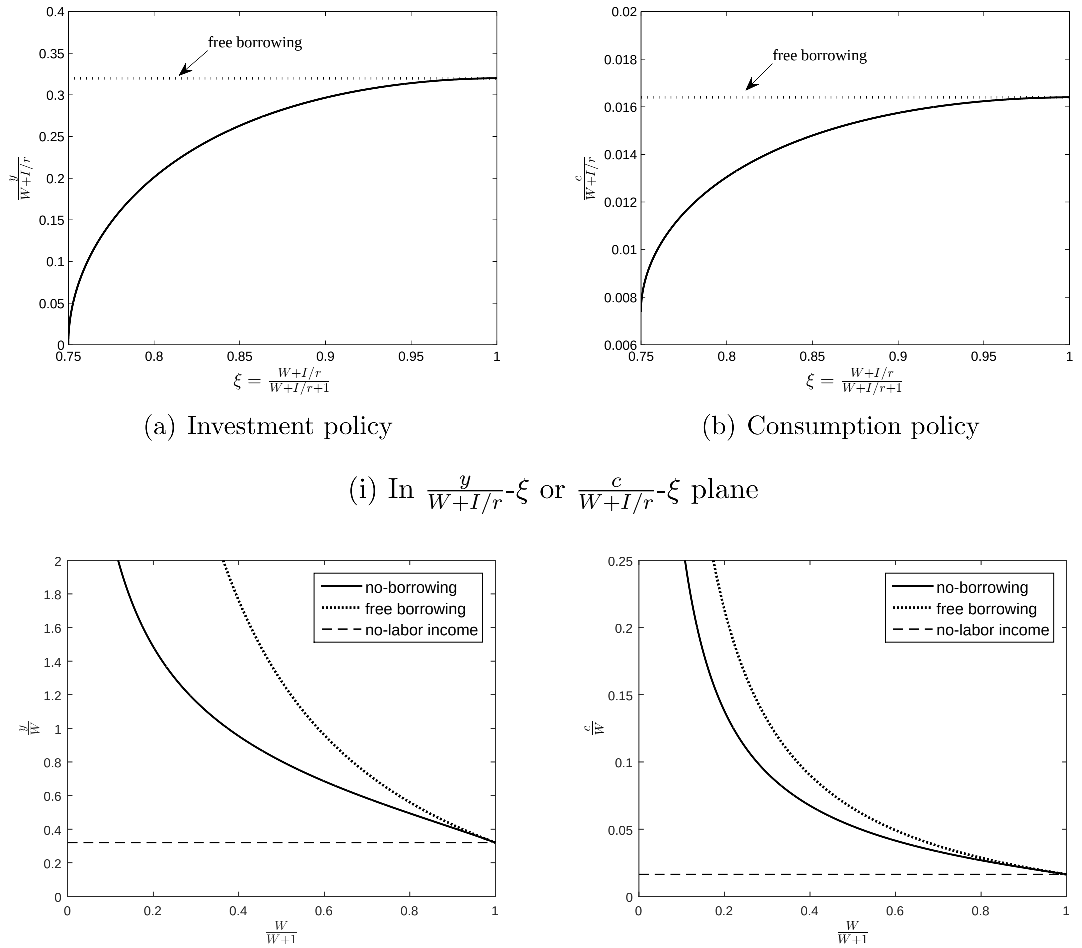
$$\max\left\{\bar{\mathcal{L}}_0 V + IV_x,\ -(1+\theta)V_x + V_y,\ (1-\alpha)V_x - V_y\right\} = 0, \quad x + (1-\alpha)y > 0, \quad (4.9)$$

where $\bar{\mathcal{L}}_0$ is the same as (3.22), and θ and α are proportional transaction cost rates for buying and selling the stock. After letting $\bar{x} = x + \frac{I}{r}$ and using similar transformation as (4.6)-(4.7), we can use the penalty method to solve the resulting equation.

4.5 Numerical Results

In this section, we present numerical results to explore the impact of the no-borrowing constraint on the investor's optimal strategy. The values of the parameters are copied from Table 3.1. Since tax rate for interest τ_i just reduces interest rate by factor $(1 - \tau_i)$, we set $\tau_i = 0$. In addition, we set the labor income rate $I = 0.03$. Hence, ξ varies from 0.75 to 1.

Figure 4.1 shows the optimal trading line and consumption rate in the case with only labor income. Panel (a) and panel (b) respectively plot the amount invested in the stock and the consumption rate as a fraction of the liquidated wealth plus the capitalized value of labor income ($\bar{W}$) against ξ. We can observe that they are both increasing functions of ξ, and go to those of the free borrowing case as $\xi \to 1$. It suggests that the inability to borrow can substantially make the investor consume at a lower rate and invest less in the stock, especially when the ratio of labor income to the total wealth is high. In the extreme case of $\xi = 0.75$ or the liquidated wealth $W = 0$, the investor would sell all of his stock holding. In addition, we find that the



(a) Investment policy

(b) Consumption policy

(i) In $\frac{y}{W+I/r}$-ξ or $\frac{c}{W+I/r}$-ξ plane

(c) Investment policy

(d) Consumption policy

(ii) In $\frac{y}{W}$-$\frac{W}{W+1}$ or $\frac{c}{W}$-$\frac{W}{W+1}$ plane

Figure 4.1: Optimal Strategy with Only Labor Income

This figure shows the optimal trading line and consumption rate for the case with only labor income. Parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $I = 0.03$, $\tau = \tau_i = 0$, $\theta = \alpha = 0$.

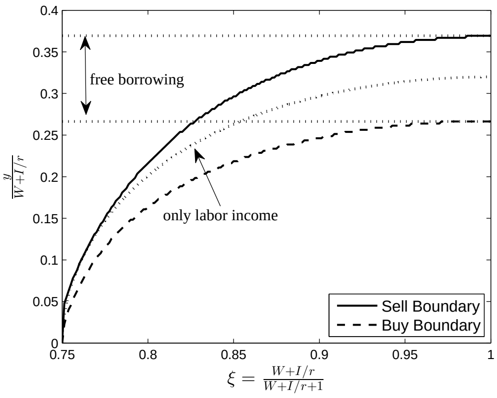


Figure 4.2: Trading Boundaries with Labor Income and Transaction Costs

This figure shows the optimal trading boundaries for the case with only labor income and transaction costs. Parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $I = 0.03$, $\tau = \tau_i = 0$, $\theta = \alpha = 0.005$.

optimal consumption rate is smaller than the labor income rate. It suggests that the investor would consume a part of his current income if the liquidated wealth is zero. To compare with the zero labor income case, we plot the amount invested in the stock or the consumption rate as a fraction of the liquidated wealth (W) in panel (c) or (d) of the figure. We find that labor income can dramatically increase the investor's holding of the stock and consumption rate even for the no-borrowing case. Moreover, after relaxing the no-borrowing constraint, the investor would consume much more and allocate much more wealth into the stock.

In a perfectly liquid market, the investor would keep trading to stay at the trading line. In the presence of transaction costs, however, there is a **NTR** as shown by Figure 4.2. Within the **NTR**, the investor allows the fraction of the stock to fluctuate. In addition, as ξ rises, the buy and sell boundaries both increase, and the **NTR** widens.

In the presence of capital gains taxes, we plot the optimal trading boundaries in Figure 4.3. Panel (a) or (b) shows the buy boundary or the sell boundary against b and ξ. Panels (c)-(e) show the trading boundaries against b for $\xi = 0.75, 0.85, 1$ respectively. Panel (f) corresponds to the free borrowing case. It can be seen that given ξ, the trading boundaries have similar shapes as those of the case without

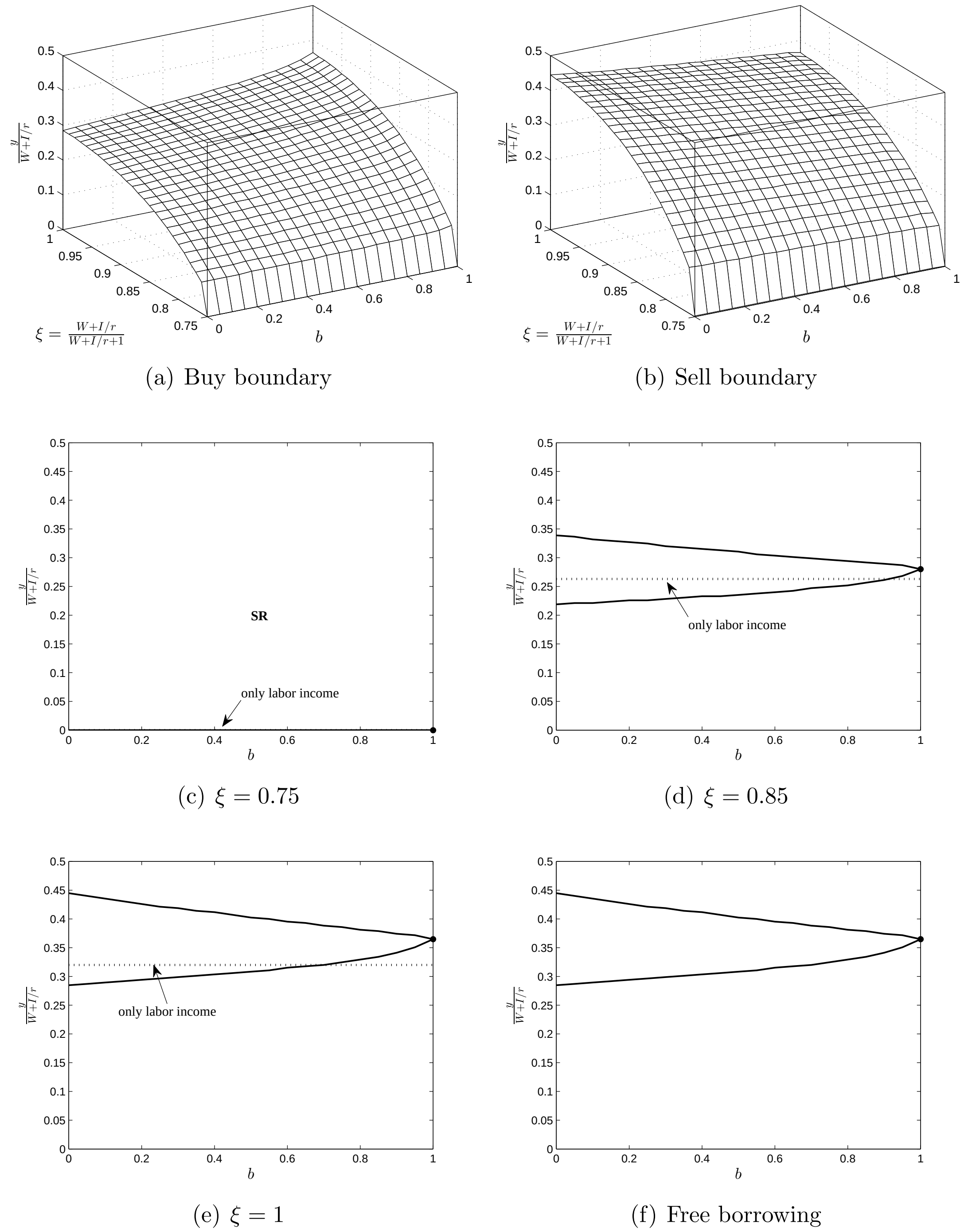
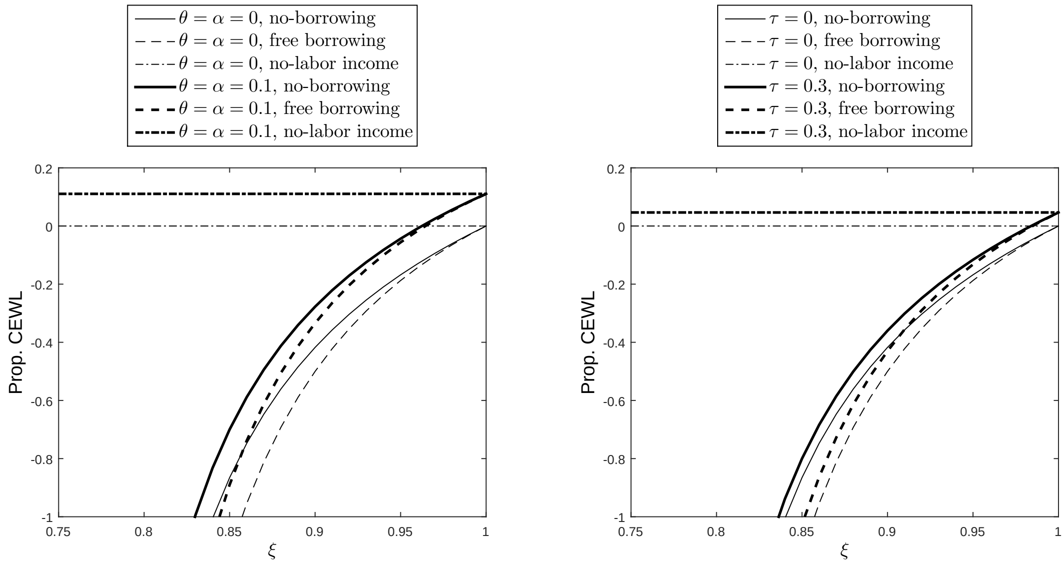


Figure 4.3: Trading Boundaries with Labor Income and Taxes

This figure shows the optimal trading boundaries for the case with only labor income and capital gains taxes. Parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $I = 0.03$, $\tau = 0.15$, $\tau_i = 0$, $\theta = \alpha = 0$.



(a) Various $\theta = \alpha$, $\tau = 0$

(b) Various τ, $\theta = \alpha = 0$

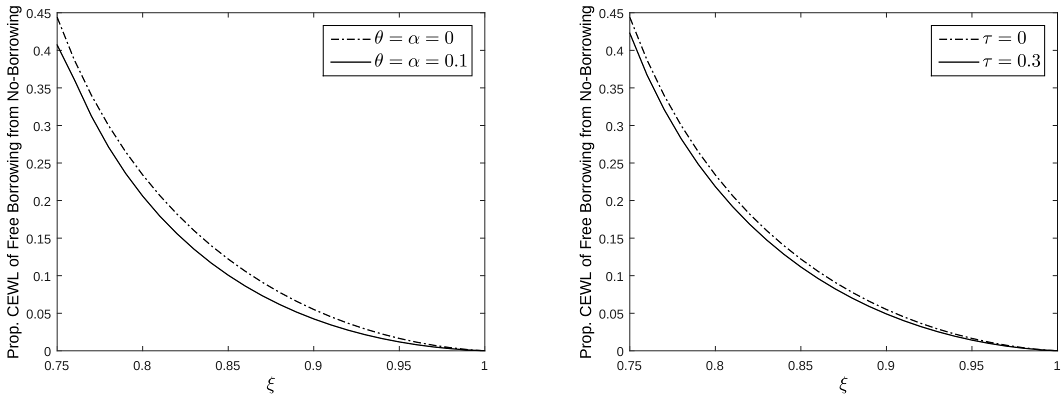
Figure 4.4: Prop. CEWL, with Labor Income

This figure shows the proportional (prop.) CEWL of Merton in the presence of labor income. Other parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $I = 0.03$, $\tau_i = 0$.

labor income (the vertical axis represents different variables). The region for $b > 1$ is still the **WSR**. Hence, the investor would adopt a similar trading policy as if he has no labor income. In addition, both of the buy and sell boundaries increase and the **NTR** widens as ξ rises.

The above findings show that if the investor is not allowed to borrow against his future labor income, the value of labor income would be significantly reduced. This is mainly reflected in two aspects. On the one side, the investor tends to consume much less and allocate much less into the stock. On the other side, he has an incentive to trade more frequently, which can incur a relatively large amount of transaction costs and taxes.

Lastly, we examine the effect of the no-borrowing constraint on the investor's expected utility. Figure 4.4 shows the proportional CEWL, as a fraction of the initial wealth (W), of Merton from labor income, capital gains taxes and transaction costs as a function of ξ. Panel (a) corresponds to the case with transaction costs but



(a) Various $\theta = \alpha$, $\tau = 0$ (b) Various τ, $\theta = \alpha = 0$

Figure 4.5: Prop. CEWL of Free Borrowing from No-Borrowing

This figure shows the proportional (prop.) CEWL, as a fraction of $\bar{W}$, of the free borrowing case from the no-borrowing case in the presence of labor income. Other parameter values: $r = 0.01$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 3$, $I = 0.03$, $\tau_i = 0$.

without capitals gains taxes while panel (b) corresponds to the opposite case. We can see that the proportional CEWL of Merton reduces dramatically and even becomes negative as ξ decreases. It suggests that labor income can have a significant positive effect on the investor's expected utility. Moreover, the investor can be better off than Merton as long as labor income can compensate for expenses from transactions costs and capital gains taxes. The figure also shows that the no-borrowing constraint could lessen the effect of labor income.

To have a better look at the effect of the no-borrowing constraint on the investor's expected utility, the proportional CEWL, as a fraction of the initial wealth plus capitalized value of labor income ($\bar{W}$), of the free borrowing case from the no-borrowing case is plotted in Figure 4.5. We find that the proportional CEWL is a decreasing function of ξ. This suggests that the larger the ratio of labor income to the total wealth is, the stronger is the negative effect of the no-borrowing constraint on the investor's expected utility. In the extreme case of zero wealth, the proportional CEWL can be more than 40% with default parameters.

Chapter 5

Asymptotic Analysis for Merton Problem with Instant Taxes

Since closed form solutions are generally unavailable, Chen and Dai (2013a) provide asymptotic expansions for portfolio selction with (instant) capital gains taxes and CRRA utility. They obtain an explicit strategy that effectively approximates the optimal strategy for small interest rate and tax rate. Based on their work, in this chapter, we propose a more refined expansion for the Merton problem with instant taxes and CARA utility. The aim is to find an explicit strategy that can effectively approximate the optimal strategy even for relatively large interest rate and tax rate. An extension to the CRRA utility case is also provided.

5.1 Model Restatement

The Merton problem with instant taxes is presented in Chapter 2. We consider only the FR case. Instead of (2.20), we make the following transformation:

$$V(x, y, k) = -e^{-r\gamma[x+y-\tau(y-k)]-K-\psi(z,b)}, \quad z = r\gamma y, \quad b = \frac{k}{y},$$

where K is defined in (2.23). Then the HJB equation (2.13) with $l = 0$ becomes

$$\max\left\{\tilde{\mathcal{L}}\psi + \tilde{f},\ z\psi_z + (1-b)\psi_b,\ -\psi_z\right\} = 0, \quad z \geq 0,\, b \geq 0, \tag{5.1}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}}\psi &= z^2\psi_{zz} - 2zb\psi_{zb} + b^2\psi_{bb} + az\psi_z + (2-a)b\psi_b - \frac{2r}{\sigma^2}\psi, \\
a &= -z\psi_z + b\psi_b - 2(1-\tau)z + \frac{2\mu}{\sigma^2}, \\
\tilde{f} &= -(1-\tau)^2(z-z_*)^2 + \frac{2r\tau}{\sigma^2}(1-b)z, \quad z_* = \frac{(1-\tau)\mu - r}{(1-\tau)^2\sigma^2}.
\end{aligned}$$

The two free boundaries of the above equation follows that $\psi_z = 0$ at $b = 1$. By the optimality of wash sales as shown in Proposition 2.3, we have

$$\psi(z,b) = \psi(z,1) \equiv \text{constant}, \quad \text{for all } z \geq 0, b \geq 1.$$

This enables us to restrict our attention to $0 \leq b \leq 1$. In addition, by Proposition 2.2, we have

$$0 \leq \psi(z,b) \leq \bar{K} - K = \frac{(\mu-r)^2 - \left(\mu - \frac{r}{1-\tau}\right)^2}{2r\sigma^2} \leq \frac{r\tau}{1-\tau}\frac{\mu-r}{r\sigma^2}. \tag{5.2}$$

5.2 The Free Boundary Conditions

We assume that $\psi \in C^1(\mathbb{R}_+^2)$. We also assume that there exist two functions $z^{\pm}(\cdot) \in C^1([0,1)) \cap C([0,1])$ such that:

$$z^-(b) < z^+(b), \quad \text{for all } 0 \leq b < 1,$$

$$z^-(1) = z^+(1),$$

and the trading or no-trading regions in $b < 1$ can be written as

$$\begin{aligned} \mathbf{SR} &= \Big\{(z,b)\Big|z \geq z^+(b),\, 0 \leq b < 1\Big\}, \\ \mathbf{BR} &= \Big\{(z,b)\Big|z \leq z^-(b),\, 0 \leq b < 1\Big\}, \\ \mathbf{NTR} &= \Big\{(z,b)\Big|z^-(b) < z < z^+(b),\, 0 \leq b < 1\Big\}. \end{aligned}$$

In other words, $z^{\pm}(b)$ are the optimal buy and sell boundaries in $b \leq 1$. The main purpose of this chapter is to find a good approximation of $z^{\pm}(b)$. Before that, we derive free boundary conditions at $z^{\pm}(b)$. In the remainder of this chapter, the operator $\tilde{\mathcal{L}}$ in (5.1) is treated as linear.

5.2.1 Free Boundary Conditions on the Sell Boundary

In **SR**, $\psi_z = 0$. It follows that there exists a function g of one variable such that $\psi(z,b) = g(b)$ and

$$0 \geq \tilde{\mathcal{L}}\psi + \tilde{f} = \tilde{\mathcal{L}}g + \tilde{f}, \quad \forall\, z > z^+(b),$$

where $\tilde{\mathcal{L}}g$ is a linear function of z:

$$\tilde{\mathcal{L}}g = b^2 g''(b) + \Big[2 - bg'(b) + 2(1-\tau)z - \frac{2\mu}{\sigma^2}\Big]bg'(b) - \frac{2r}{\sigma^2}g(b).$$

Set $\psi^1(z,b) = \psi(z,b) - g(b)$. Then $\psi^1 \equiv 0$ in **SR**, and $\psi^1 \leq 0$ in the whole space since $\psi^1_z = \psi_z \geq 0$. Hence, at $(z^+(b), b)$, ψ^1 attains its global maximum; and moreover, $\psi^1 = 0$, $\psi^1_z = \psi^1_b = 0$, and $\tilde{\mathcal{L}}\psi^1 \leq 0$ (in certain weak sense). So, at $(z^+(b) - 0, b)$,

$$0 = \tilde{\mathcal{L}}\psi + \tilde{f} = \tilde{\mathcal{L}}\psi^1 + \tilde{\mathcal{L}}g + \tilde{f} \leq \tilde{\mathcal{L}}g + \tilde{f}.$$

Thus, $\tilde{\mathcal{L}}g + \tilde{f} = 0$ and $\tilde{\mathcal{L}}\psi^1 = 0$ at $(z^+(b), b)$. Therefore,

$$\begin{aligned} \psi(z,b) = g(b), \quad \tilde{\mathcal{L}}g + \tilde{f} \leq 0, \qquad & \text{if} \quad z \geq z^+(b), \\ \psi_z = 0, \quad \tilde{\mathcal{L}}\psi = \tilde{\mathcal{L}}g, \quad \tilde{\mathcal{L}}g + \tilde{f} = 0, \qquad & \text{if} \quad z = z^+(b). \end{aligned} \tag{5.3}$$

5.2.2 Free Boundary Conditions on the Buy Boundary

In **BR**, $z\psi_z + (1-b)\psi_b = 0$. Assume $(b-1)z^-(b)$ is monotone in $b \in [0,1]$. Then there exists a function h of one variable such that $\psi(z,b) = h((b-1)z)$ and

$$0 \geq \tilde{\mathcal{L}}\psi + \tilde{f} = \tilde{\mathcal{L}}h + \tilde{f}, \quad \forall\ 0 \leq z < z^-(b),$$

where

$$\tilde{\mathcal{L}}h = z^2 h''(\eta) - \left[zh'(\eta) - 2(1-\tau)z + \frac{2\mu}{\sigma^2}\right] zh'(\eta) - \frac{2r}{\sigma^2}h(\eta)\bigg|_{\eta=(b-1)z}.$$

Set $\psi^2(z,b) = \psi(z,b) - h((b-1)z)$. Then $\psi^2 \equiv 0$ in **BR**, and $\psi^2 \leq 0$ in the whole space since $z\psi^2_z + (1-b)\psi^2_b \leq 0$. Hence, at $(z^-(b), b)$, ψ^2 attains its local maximum; and moreover, $\psi^2 = 0$, $\psi^2_z = \psi^2_b = 0$, and $\tilde{\mathcal{L}}\psi^2 \leq 0$ (in certain weak sense). So, at $(z^-(b)+0, b)$,

$$0 = \tilde{\mathcal{L}}\psi + \tilde{f} = \tilde{\mathcal{L}}\psi^2 + \tilde{\mathcal{L}}h + \tilde{f} \leq \tilde{\mathcal{L}}h + \tilde{f}.$$

Thus, $\tilde{\mathcal{L}}h + \tilde{f} = 0$ and $\tilde{\mathcal{L}}\psi^2 = 0$ at $(z^-(b), b)$. Therefore,

$$\begin{aligned} \psi(z,b) = h((b-1)z), \quad \tilde{\mathcal{L}}h + \tilde{f} \leq 0, \qquad & \text{if} \quad z \in [0, z^-(b)], \\ z\psi_z + (1-b)\psi_b = 0, \quad \tilde{\mathcal{L}}\psi = \tilde{\mathcal{L}}h, \quad \tilde{\mathcal{L}}h + \tilde{f} = 0, \qquad & \text{if} \quad z = z^-(b). \end{aligned} \tag{5.4}$$

5.3 The Main Theoretical Results

In this section, we present the main theoretical results derived from asymptotic analysis on (5.1) in terms of small interest rate and tax rate. We restrict attention to $\tau < 1 - \frac{r}{\mu}$ such that $z_* > 0$.

Proposition 5.1. *We denote*

$$\epsilon = \sqrt{\frac{2r\tau z_*}{(1-\tau)^2\sigma^2}}, \quad A = \frac{4}{3z_*}, \quad z_1(b) = z_* + \frac{\epsilon^2(1-b)}{2z_*}. \tag{5.5}$$

We have the following approximations to the reduced value function $\psi(z,b)$ and the optimal trading boundaries $z^{\pm}(b)$ in $b \in [0,1]$:

(a) leading order expansion

$$\begin{cases} \psi(z,b) = & (1-\tau)^2\epsilon^{8/3}A^{2/3}\frac{\sigma^2}{2r}m_0 + o(\epsilon^{8/3}) \\ z^{\pm}(b) \approx & z_1(b) \pm \epsilon\sqrt{1-b} \end{cases}, \tag{5.6}$$

(b) first order tip expansion

$$\begin{cases} \psi(z,b) = & (1-\tau)^2\epsilon^{8/3}A^{2/3}\frac{\sigma^2}{2r}m_0 + o(\epsilon^{8/3}) \\ z^{\pm}(b) \approx & z_1(b) \pm \epsilon\sqrt{1-b}\left(\frac{g_0'}{\sqrt{p}}\right)^{1/3}\Big|_{p=\frac{1-b}{\delta}} \end{cases}, \tag{5.7}$$

(c) second order tip expansion

$$\begin{cases} \psi(z,b) = & (1-\tau)^2\epsilon^{8/3}A^{2/3}\frac{\sigma^2}{2r}(m_0 + m_1\delta) + o(\epsilon^{10/3}) \\ z^{\pm}(b) \approx & z_1(b) + \epsilon\sqrt{1-b}\left(\frac{g_0'}{\sqrt{p}}\right)^{1/3}\chi^{\pm}(p)\Big|_{p=\frac{1-b}{\delta}} \end{cases}, \tag{5.8}$$

where $\delta = (A\epsilon)^{2/3}$,

$$\chi^{\pm}(p) = \pm 1 \pm \frac{\delta}{3}\frac{g_1'}{g_0'} - \delta\left(\frac{1}{2} - \frac{1}{4}\delta p\right)\frac{pg_0'' + g_0'}{(pg_0')^{2/3}},$$

m_0 *and* g_0 *are determined by*

$$\begin{cases} (1-\delta p)^2 g_0''(p) = m_0 - p + (pg_0'(p))^{2/3}, \quad p \in [0, \frac{1}{\delta}], \\ g_0(0) = 0, \quad g_0'(0) = 0, \end{cases} \tag{5.9}$$

m_1 *and* g_1 *are determined by*

$$\begin{cases} (1-\delta p)^2 g_1''(p) = \frac{2}{3}\left(\frac{p^2}{g_0'}\right)^{1/3} g_1'(p) - (1-\frac{1}{2}\delta p)(pg_0'' + g_0') \\ \qquad\qquad +\hat{c}(1-\delta p)g_0' + m_1, \quad p \in [0, \frac{1}{\delta}], \\ g_1(0) = 0, \quad g_1'(0) = 0, \end{cases} \tag{5.10}$$

and $\hat{c} = 2 + 2(1-\tau)z_* - 2\mu/\sigma^2$.

In addition, for each of the above expansions, we denote by $\bar{\psi}(z, b)$ *the approximated reduced value function. Then we have the following approximation for the optimal consumption:*

$$c^* \approx \frac{\bar{\psi} + r\gamma W + K - \log r}{\gamma}, \tag{5.11}$$

where $W = x + y - \tau(y - k)$ *is the liquidated wealth and* K *is defined in (2.23).*

We remark that the unknown m_0 is determined by δ only. Via a shooting method, we can obtain m_0 as a function of δ. The result is shown in Figure 5.1. As implied by our numerical results, we will approximate $g_0'(p)$ by a quadratic function:[1]

$$g_0'(p) \approx \eta_{01}p + \eta_{02}p^2. \tag{5.12}$$

Panel (a) of Figure 5.2 shows a numerical example of m_0 and $g_0'(p)$, which is a numerical evidence for the uniqueness of m_0 (on the condition that the limit $\lim_{p \to \frac{1}{\delta}} g'(p)$ exists). Therefore, we can obtain an explicit approximation of the optimal trading boundaries for the first order tip expansion:

$$z^{\pm}(b) \approx z_1(b) \pm \epsilon(1-b)^{2/3}\delta^{-1/6}(\eta_{01} + \eta_{02}p)^{1/3}\Big|_{p=\frac{1-b}{\delta}}. \tag{5.13}$$

We also remark that the unknown m_1 is determined by δ and $\hat{c}$ as well as $g_0'(p)$. As a first order linear ordinary differential equation of the function $g_1'(p)$, (5.10) has a unique solution

$$g_1'(p) = e^{\bar{A}(p)} \int_0^p e^{-\bar{A}(\tilde{p})} \bar{H}(\tilde{p}) d\tilde{p},$$

[1] A higher order approximation would complicate the computation enormously for the second order tip expansion.

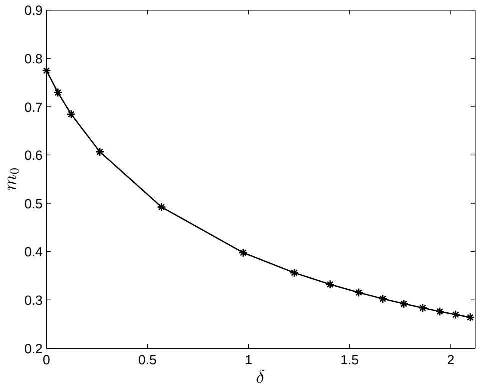


Figure 5.1: Constant m_0 in (5.9)

This figure shows the constant m_0 in (5.9) as a function of δ.

where

$$\begin{aligned}\bar{A}(p) &= \int_0^p \frac{2}{3}\Big(\frac{\tilde{p}^2}{g_0'(\tilde{p})}\Big)^{1/3}\frac{1}{(1-\delta\tilde{p})^2}d\tilde{p},\\ \bar{H}(p) &= \frac{-(1-\frac{1}{2}\delta p)(pg_0''(p)+g_0'(p))+\hat{c}(1-\delta p)g_0'(p)+m_1}{(1-\delta p)^2}.\end{aligned}$$

Set

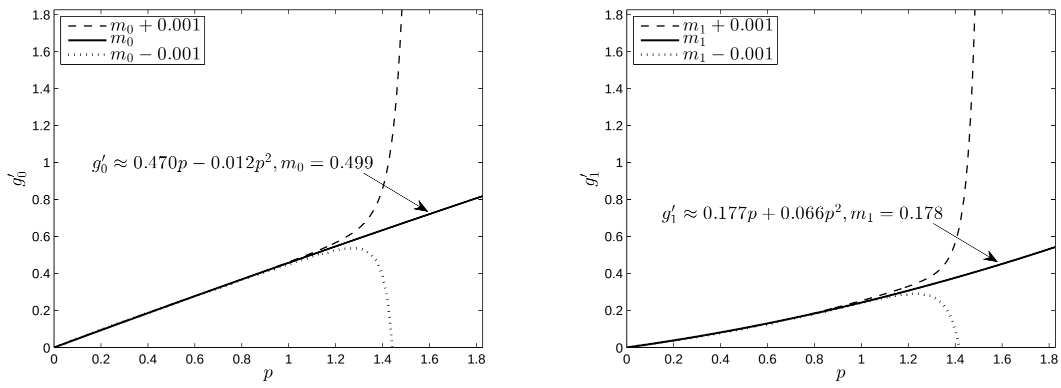
$$\int_0^{1/\delta} e^{-\bar{A}(p)}\bar{H}(p)dp = 0.$$

Then we have

$$m_1 = -\frac{\int_0^{1/\delta} e^{-\bar{A}(p)}\frac{-(1-\frac{1}{2}\delta p)(pg_0''(p)+g_0'(p))+\hat{c}(1-\delta p)g_0'(p)}{(1-\delta p)^2}dp}{\int_0^{1/\delta} e^{-\bar{A}(p)}\frac{1}{(1-\delta p)^2}dp}. \tag{5.14}$$

If we approximate $g_0'(p)$ by (5.12), by directly computation, we have

$$\bar{A}(p) = \begin{cases} \frac{1}{2}\eta_{01}^{-1/3}p^{4/3}, & \text{if } \delta = \eta_0 = 0,\\ 2\eta_{01}^{-1/3}(\eta_0+\delta)^{-4/3}\left[F\left(\left(\frac{(\eta_0+\delta)p}{1+\eta_0 p}\right)^{1/3}\right) - F(0)\right] & \text{otherwise,}\end{cases} \tag{5.15}$$



(a) m_0 and $g_0'(p)$

(b) m_1 and $g_1'(p)$

Figure 5.2: Numerical Example for m_0 and $g_0'(p)$, m_1 and $g_1'(p)$

This figure shows a numerical example for m_0 and $g_0'(p)$, m_1 and $g_1'(p)$. Parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$, $\tau = 0.15$.

where $\eta_0 = \frac{\eta_{02}}{\eta_{01}}$ and

$$F(p) = -\frac{\log(p^2+p+1)}{18} - 3^{-3/2}\arctan\left(\frac{2p+1}{\sqrt{3}}\right) + \frac{\log|p-1|}{9} - \frac{1}{3}\frac{p}{p^3-1}.$$

In this case, we can numerically compute m_1 by evaluating the relative integrals, and then derive the solution of $g_1'(p)$. As implied by our numerical results, we will approximate $g_1'(p)$ by a quadratic function:

$$g_1'(p) \approx \eta_{11}p + \eta_{12}p^2. \tag{5.16}$$

A numerical example of m_1 and $g_1'(p)$ is shown in panel (b) of Figure 5.2. As a consequence, we can obtain an explicit approximation of the optimal trading boundaries for the second order tip expansion:

$$\begin{aligned} z^{\pm}(b) \quad &\approx \quad z_1(b) + \epsilon(1-b)^{2/3}\delta^{-1/6}(\eta_{01}+\eta_{02}p)^{1/3}\left[\pm 1 \pm \frac{\delta}{3}\frac{\eta_{11}+\eta_{12}p}{\eta_{01}+\eta_{02}p}\right] \\ &\quad -\epsilon(1-b)^{1/3}\delta^{7/6}\left(\frac{1}{2}-\frac{1}{4}\delta p\right)\frac{2\eta_{01}+3\eta_{02}p}{(\eta_{01}+\eta_{02}p)^{1/3}}\Bigg|_{p=\frac{1-b}{\delta}}. \end{aligned} \tag{5.17}$$

Proof of Proposition 5.1. By (5.5), $\tilde{f}$ in (5.1) can be rewritten as

$$\tilde{f}(z,b) = (1-\tau)^2\Big[-(z-z_1(b))^2 + \epsilon^2(1-b) + \frac{\epsilon^4(1-b)^2}{4z_*^2}\Big].$$

The result for the optimal consumption is clear. We next prove three expansions.

(a) Proof for leading order expansion. Let

$$\zeta = \frac{z - z_1(b)}{\epsilon},$$

then

$$\tilde{f} = (1-\tau)^2\epsilon^2\Big[-\zeta^2 + 1 - b + \frac{\epsilon^2}{4z_*^2}(1-b)^2\Big].$$

Away from the tip $(z^{\pm}(1), 1)$, we can express the solution as:[2]

$$\psi(z,b) = (1-\tau)^2\Big[\epsilon^{8/3}m_\epsilon + \epsilon^3 g_\epsilon(b) + \epsilon^4 v^\epsilon(\zeta,b)\Big], \quad \text{for} \quad b\in[0,1], \qquad (5.18)$$

where m_ϵ is a constant to be determined, and

$$\begin{aligned} v^\epsilon(\zeta,b) &= 0, \quad \text{if} \quad \zeta \geq \frac{z^+(b)-z_1(b)}{\epsilon}, \\ g_\epsilon(1) &= 0. \end{aligned}$$

Using (5.18), by direct computation, we have

$$\begin{aligned} \frac{\tilde{\mathcal{L}}\psi + \tilde{f}}{(1-\tau)^2\epsilon^2} &= z_*^2 v^\epsilon_{\zeta\zeta} - \zeta^2 + 1 - b - \frac{2r}{\sigma^2}\epsilon^{2/3}m_\epsilon + O(\epsilon), \\ \frac{z\psi_z + (1-b)\psi_b}{(1-\tau)^2\epsilon^3} &= z_* v^\epsilon_\zeta + (1-b)g'_\epsilon(b) + O(\epsilon), \\ \frac{-\psi_z}{(1-\tau)^2\epsilon^3} &= -v^\epsilon_\zeta. \end{aligned}$$

[2]The first term follows from (5.20) in part (b). The second term is implied by (5.3). The order of ψ is at least $O(\epsilon^2)$ by (5.2). The specified orders of the second and the third terms are determined in the following computation.

Thus, the reduced HJB equation (5.1) becomes

$$\max\left\{z_*^2 v^\epsilon_{\zeta\zeta} - \zeta^2 + 1 - b + O(\epsilon^{2/3}),\ z_* v^\epsilon_\zeta + (1-b)g'_\epsilon(b) + O(\epsilon),\ -v^\epsilon_\zeta\right\} = 0.$$

Sending $\epsilon \downarrow 0$, then $\bar{V} = \lim_{\epsilon\downarrow 0} v^\epsilon$, $\bar{G} = \lim_{\epsilon\downarrow 0} g_\epsilon$ solves

$$\max\left\{z_*^2 \bar{V}_{\zeta\zeta} - \zeta^2 + 1 - b,\ z_*\bar{V}_\zeta + (1-b)\bar{G}'(b),\ -\bar{V}_\zeta\right\} = 0, \quad (\zeta, b) \in \mathbb{R} \times [0,1],$$
$$\bar{V}(\infty, b) = 0, \quad \bar{G}(1) = 0.$$

This equation has the same form as the corresponding one in Chen and Dai (2013a). It has a unique solution

$$\bar{V}(\zeta, b) = \begin{cases} 0, & \text{if } \zeta \geq \sqrt{1-b}, \\ \frac{1}{12z_*^2}\left[\zeta - \sqrt{1-b}\right]^4 + \frac{\sqrt{1-b}}{3z_*^2}\left[\zeta - \sqrt{1-b}\right]^3, & \text{if } |\zeta| \leq \sqrt{1-b}, \\ \frac{4}{3z_*^2}(1-b)^{3/2}\zeta, & \text{if } \zeta \leq -\sqrt{1-b}, \end{cases}$$
$$\bar{G}(b) = \frac{2A}{3}(1-b)^{3/2},$$

with the buy and sell boundaries

$$\zeta^+(b) = \inf\{\zeta | \bar{V}_\zeta = 0\} = \sqrt{1-b},$$
$$\zeta^-(b) = \sup\{\zeta | z_*\bar{V}_\zeta + (1-b)\bar{G}'(b) = 0\} = -\sqrt{1-b}.$$

Therefore, we have the expansion (5.6) with m_0 to be determined later by the first order tip expansion.

(b) Proof for first order tip expansion. In part (a), $\bar{G}(b) = \frac{2A}{3}(1-b)^{3/2}$ is not in C^2. It indicates that a tip expansion is needed near $b = 1$. Near the tip $(z^{\pm}(1), 1)$, we use the stretched variable

$$q = \frac{z - z_1(b)}{\delta^2}, \quad p = \frac{1-b}{\delta}.$$

Then

$$\tilde{f} = (1-\tau)^2\epsilon^2\delta\Big[-q^2A^2 + p + \frac{p^2}{4z_*^2A^2}\delta^4\Big]. \tag{5.19}$$

Near the tip $(z^{\pm}(1), 1)$, similar to (5.18), we can express the solution as:[3]

$$\psi(z,b) = (1-\tau)^2\epsilon^2\left[\frac{\delta\sigma^2}{2r}m_\delta + \delta^3 g_\delta(p) + \delta^5 v^\delta(p,q)\right], \tag{5.20}$$

where m_δ is a positive constant to be determined, and

$$\begin{aligned} v^\delta(p,q) &= 0, \quad \text{if} \quad q \geq \frac{z^+(b) - z_1(b)}{\delta^2}, \\ g_\delta(0) &= 0, \quad g_\delta'(0) = 0. \end{aligned}$$

Using (5.20), by direct computation, we have

$$\begin{aligned} \frac{\tilde{\mathcal{L}}\psi + \tilde{f}}{(1-\tau)^2\epsilon^2\delta} &= z_*^2 v_{qq}^\delta + 2z_*(1-\delta p)\delta v_{pq}^\delta + (1-\delta p)^2 g_\delta'' \\ &\quad -\hat{c}(1-\delta p)\delta g_\delta' - m_\delta - q^2A^2 + p + O(\delta^2), \qquad (5.21) \\ \frac{z\psi_z + (1-b)\psi_b}{(1-\tau)^2\epsilon^2\delta^3} &= z_* v_q^\delta - p g_\delta' - p\delta^2 v_p^\delta + O(\delta^2), \qquad (5.22) \\ \frac{-\psi_z}{(1-\tau)^2\epsilon^2\delta^3} &= -v_q^\delta. \qquad (5.23) \end{aligned}$$

Expand the solution and the optimal trading boundaries in the first order as

$$\begin{aligned} m_\delta = m_0 + O(\delta), \quad g_\delta = g_0 + O(\delta), \quad v^\delta = v^0 + O(\delta), \\ q^{\pm}(p) := \frac{z^{\pm}(b) - z_1(b)}{\delta^2}\bigg|_{b=1-p\delta} = q_0^{\pm}(p) + O(\delta). \end{aligned}$$

Sending $\delta \downarrow 0$, then (m_0, g_0, v^0) solves

$$\max\left\{z_*^2 v_{qq}^0 + (1-\delta p)^2 g_0'' - m_0 - q^2A^2 + p,\ z_* v_q^0 - p g_0',\ -v_q^0\right\} = 0,$$

$$v^0(p,\infty) = 0, \quad g_0(0) = 0, \quad g_0'(0) = 0,$$

[3] The first term of (5.20) is implied by (5.19).

in $(p,q) \in [0, \frac{1}{\delta}] \times \mathbb{R}$. This equation has the same form as the corresponding one in Chen and Dai (2013a). It has a unique solution

$$v^0 = \begin{cases} 0, & \text{if} \quad q \geq q_0^+(p), \\ \frac{A^2}{z_*^2}\left[\frac{q^4 - q_0^+(p)^4}{12} - \frac{q_0^+(p)^3}{3}(q - q_0^+(p)) - \frac{q_0^+(p)^2}{2}(q - q_0^+(p))^2\right], & \text{if} \quad q \in [q_0^-(p), q_0^+(p)], \\ \frac{1}{z_*} p g_0'(p)(q + q_0^+(p)) - \frac{4A^2}{3z_*^2} q_0^+(p)^4, & \text{if} \quad q \leq q_0^-(p), \end{cases}$$

m_0 and g_0 are determined by (5.9), and the expansion is given by (5.7).

(c) Proof for second order tip expansion. Following the proof of (b), expand the solution and the optimal trading boundaries in the second order as

$$m_\delta = m_0 + m_1\delta + O(\delta^2), \quad g_\delta = g_0 + g_1\delta + O(\delta^2), \quad v^\delta = v^0 + v^1\delta + O(\delta^2),$$
$$q^\pm(p) := \frac{z^\pm(b) - z_1(b)}{\delta^2}\bigg|_{b=1-p\delta} = q_0^\pm(p) + q_1^\pm(p)\delta + O(\delta^2).$$

Hence, function (5.21) can be written as

$$\begin{aligned} &z_*^2 v_{qq}^0 + (1-\delta p)^2 g_0'' - m_0 - q^2A^2 + p \\ &\quad + \delta\left[z_*^2 v_{qq}^1 + (1-\delta p)^2 g_1'' - m_1 + 2z_*(1-\delta p)v_{pq}^0 - \hat{c}(1-\delta p)g_0'\right] + O(\delta^2), \end{aligned} \tag{5.24}$$

where

$$v_{pq}^0(p,q) = \begin{cases} 0, & \text{if} \quad q \geq q_0^+(p), \\ -2\frac{A^2}{z_*^2} q_0^+(p) q_0^{+'}(p)(q - q_0^+(p)), & \text{if} \quad q \in [q_0^-(p), q_0^+(p)], \\ \frac{1}{z_*}(g_0'(p) + p g_0''(p)), & \text{if} \quad q \leq q_0^-(p), \end{cases}$$

and

$$q_0^\pm(p) = \pm\frac{1}{A}\Big(p g_0'(p)\Big)^{1/3}. \tag{5.25}$$

Since

$$z_*^2 v_{qq}^0 + (1-\delta p)^2 g_0'' - m_0 - q^2A^2 + p = 0, \quad q \in [q_0^-(p), q_0^+(p)],$$

we have

$$z_*^2 v_{qq}^1 + (1-\delta p)^2 g_1'' - m_1 + 2z_*(1-\delta p)v_{pq}^0 - \hat{c}(1-\delta p)g_0' = 0, \quad q \in [q_0^-(p), q_0^+(p)]. \tag{5.26}$$

At the sell boundary $q = q_0^+ + \delta q_1^+ + O(\delta^2)$, on the one side, function (5.23) is equal to 0. Expand it in the second order

$$v_q^0(p, q_0^+) + v_{qq}^0(p, q_0^+)q_1^+\delta + \delta v_q^1(p, q_0^+) + O(\delta^2) = 0,$$

which yields

$$v_q^1(p, q_0^+) = -v_{qq}^0(p, q_0^+)q_1^+ = 0.$$

On the other side, function (5.21) or (5.24) is equal to 0. Substituting $q = q_0^+ + \delta q_1^+ + O(\delta^2)$ into (5.24), one can get

$$-2A^2 q_0^+ q_1^+ + (1-\delta p)^2 g_1'' - m_1 - \hat{c}(1-\delta p)g_0' = 0. \tag{5.27}$$

So (5.26) can be rewritten as

$$z_*^2 v_{qq}^1 + 2z_*(1-\delta p)v_{pq}^0 + 2A^2 q_0^+ q_1^+ = 0, \quad q \in [q_0^-(p), q_0^+(p)]. \tag{5.28}$$

Integrating it with respect to q,

$$v_q^1 = 2\frac{A^2}{z_*^3}(1-\delta p)q_0^+ q_0^{+'}(q-q_0^+)^2 - 2\frac{A^2}{z_*^2}q_0^+ q_1^+(q-q_0^+), \quad q \in [q_0^-(p), q_0^+(p)]. \tag{5.29}$$

Similarly, at the buy boundary $q = q_0^- + \delta q_1^- + O(\delta^2)$, function (5.22) is equal to 0. Expanding it in the second order, one gets

$$\begin{aligned} pg_1'(p) &= z_* v_{qq}^0(p, q_0^-)q_1^- + z_* v_q^1(p, q_0^-) - p\delta v_p^0(p, q_0^-) \\ &= 4\frac{A^2}{z_*^2}(2-\delta p)q_0^+(p)^3 q_0^{+'}(p) + 4\frac{A^2}{z_*}q_0^+(p)^2 q_1^+(p). \end{aligned}$$

Thus,

$$q_1^+(p) = \frac{pg_1'(p) - 4\frac{A^2}{z_*^2}(2-\delta p)q_0^+(p)^3 q_0^{+'}(p)}{4\frac{A^2}{z_*}q_0^+(p)^2}. \tag{5.30}$$

Meanwhile, at $q = q_0^- + \delta q_1^- + O(\delta^2)$, (5.21) or (5.24) is equal to 0. One can get

$$-2A^2 q_0^- q_1^- + (1-\delta p)^2 g_1'' - m_1 + (2-\delta p)(g_0' + pg_0'') - \hat{c}(1-\delta p)g_0' = 0. \tag{5.31}$$

Equations (5.27) and (5.31) with $q_0^- = -q_0^+$ imply

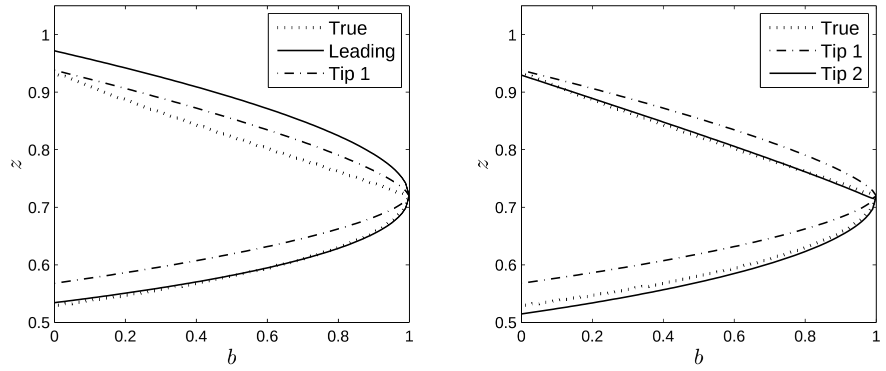
$$q_1^-(p) = -q_1^+(p) - (2-\delta p)\frac{pg_0''(p) + g_0'(p)}{2A^2 q_0^+(p)}. \tag{5.32}$$

Substituting (5.30) and (5.25) into (5.27), we have $g_1(p)$ satisfies (5.10). In addition, (5.20), (5.30) and (5.32) imply that the expansion is given by (5.8). □

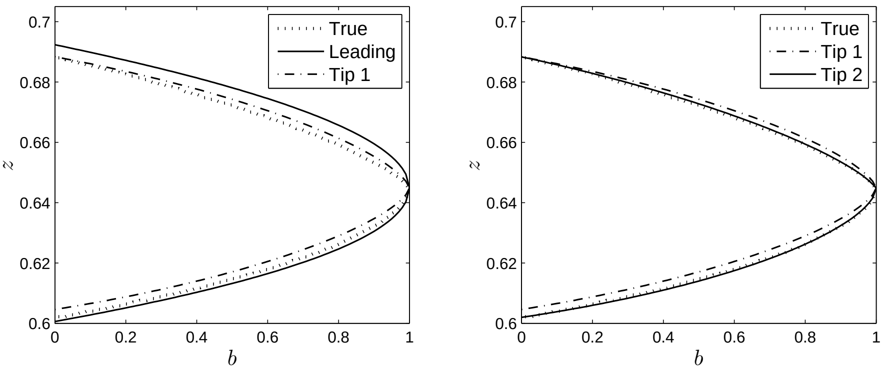
5.4 Numerical Results

In this section, we present numerical results to justify our asymptotic analysis. For comparison, we always employ the penalty method combined with a finite difference scheme to solve (5.1), so as to generate benchmark values (marked with "True" in the tables and figures of this section).

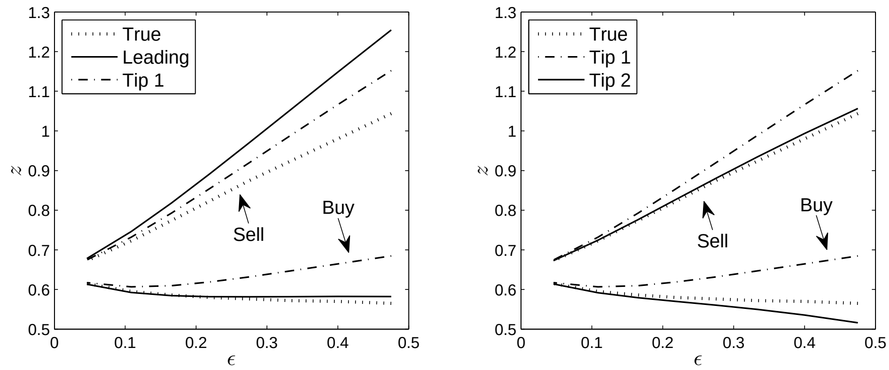
In Figure 5.3, we examine the accuracy of the approximated trading boundaries implied by three expansions (5.6), (5.13) and (5.17). Panel (a) shows the approximated trading boundaries for a relative large ϵ. Panel (b) corresponds to a relative small ϵ. Panel (c) shows the approximated trading boundaries at $b = 0.5$ against ϵ. We can see that all of the three approximations perform very well for small ϵ. But the one implied by the leading order expansion and the one implied by the first order tip expansion do not perform very well for large ϵ. We can also observe that for the second order tip expansion, its sell boundary closely matches the optimal sell boundary, and its buy boundary is reasonably good even for large ϵ.



(a) $\tau = 0.15$, $\epsilon = 0.219$



(b) $\tau = 0.01$, $\epsilon = 0.046$



(c) At $b = 0.5$, τ varies from 0.01 to 0.35

Figure 5.3: Approximations to Trading Boundaries, CARA

This figure shows approximations to the optimal trading boundaries in the CARA utility case. The "True", "Leading", "Tip 1", and "Tip 2" correspond to the benchmark, the leading order expansion, the first order tip expansion, and the second order tip expansion respectively. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

Table 5.1: Scaled CEWL for Approximated Strategies, CARA

This table reports the scaled CEWL of Merton for approximated strategies with varying τ in the CARA utility case. The "Upper", "True", "Leading", "Tip 1", and "Tip 2" correspond to the upper bound, the benchmark, the leading order expansion, the first order tip expansion, and the second order tip expansion respectively. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

τ	ϵ	Upper	True	Leading	Tip 1	Tip 2
35%	0.475	0.3214	0.2093	0.2311	0.2190	0.2105
30%	0.400	0.2596	0.1731	0.1891	0.1799	0.1738
25%	0.334	0.2044	0.1398	0.1511	0.1444	0.1402
20%	0.274	0.1550	0.1090	0.1164	0.1119	0.1092
15%	0.219	0.1104	0.0801	0.0845	0.0817	0.0802
10%	0.165	0.0701	0.0528	0.0550	0.0536	0.0529
5%	0.109	0.0335	0.0266	0.0273	0.0268	0.0266
1%	0.046	0.0065	0.0056	0.0057	0.0056	0.0056

In Table 5.1, we examine the effects of the approximated investment and consumption strategies on the investor's expected utility. Given an explicit strategy, we can compute the corresponding value function,[4] and the corresponding scaled CEWL of Merton from capital gains taxes. In Table 5.1, we report the scaled CEWL with different levels of tax rate. The upper bound of the scaled CEWL (i.e., $\bar{K} - K$, as suggested by (5.2)) is also reported. The table shows that the approximated strategies do not significantly affect the investor's expected utility especially when ϵ is small. In addition, the approximated strategy implied by the second order tip

[4] Substitute the approximated consumption (5.11) into the original HJB equation (2.13). The operator $\tilde{\mathcal{L}}\psi + \tilde{f}$ in (5.1) can be approximated by

$$\bar{\mathcal{L}}\psi + \tilde{f} = z^2\psi_{zz} - 2zb\psi_{zb} + b^2\psi_{bb} + az\psi_z + (2-a)b\psi_b - \frac{2r}{\sigma^2}\bar{\psi} + \tilde{f} + \frac{2r}{\sigma^2} - \frac{2r}{\sigma^2}e^{\psi-\bar{\psi}}.$$

The approximated buy region and sell region, denoted by $\overline{\mathbf{BR}}$ and $\overline{\mathbf{SR}}$, are implied by the explicit strategy (5.6), (5.13) or (5.17). Then, we can compute the corresponding value function of an investor who adopts the above explicit strategy by solving the following penalty approximation

$$-\bar{\mathcal{L}}\psi - \tilde{f} = K_P[z\psi_z + (1-b)\psi_b]\mathbf{1}_{\overline{\mathbf{BR}}} + K_P[-\psi_z]\mathbf{1}_{\overline{\mathbf{SR}}},$$

where K_P is the penalty parameter, and $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

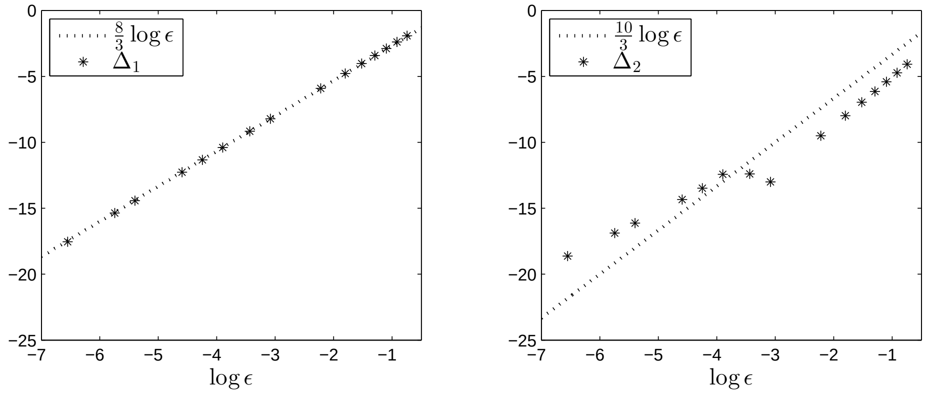


Figure 5.4: Order of Approximations to the Value Function, CARA

This figure shows the order of approximations to the reduced value function with varying τ from 0.001% to 35%. Other parameter values: $r = 0.01$, $\mu = 0.05$, $\sigma = 0.25$, $\beta = 0.01$.

expansion is superior to the one implied by the first order tip expansion. In particular, when the tax rate is 35%, compared with the benchmark, the scaled CEWL is increased by 4.6% for the first order tip expansion but is only increased by 0.57% for the second order tip expansion.

Finally, we examine the order of approximations to the reduced value function $\psi(z, 1)$.[5] The left panel of Figure 5.4 plots

$$\log(\epsilon) \mapsto \Delta_1 = \log\left(\frac{\psi(z,1)}{(1-\tau)^2 A^{2/3}\frac{\sigma^2}{2r}m_0}\right).$$

The first order tip expansion (5.7) implies that it should be close to the straight line $\log(\epsilon) \mapsto \frac{8}{3}\log(\epsilon)$. The sub-figure shows a good fit for that the first order tip expansion is of order $O(\epsilon^{8/3})$. The right panel of Figure 5.4 plots

$$\log(\epsilon) \mapsto \Delta_2 = \log\left(\frac{\psi(z,1) - (1-\tau)^2\epsilon^{8/3}A^{2/3}\frac{\sigma^2}{2r}m_0}{(1-\tau)^2 A^{4/3}\frac{\sigma^2}{2r}m_1}\right).$$

By the second order tip expansion (5.8), it should be close to the straight line $\log(\epsilon) \mapsto \frac{10}{3}\log(\epsilon)$. However, the sub-figure does not show a good fit for that the

[5] $\psi(z, 1)$ can be regarded as the scaled deferral value, which is the scaled certainly equivalent wealth gain of the case where the investor cannot defer realizing any capital gain or loss.

second order tip expansion is of order $O(\epsilon^{10/3})$. The reason is that when we compute m_1 via (5.14), we use a quadratic function to approximate g_0'. It may not be a good enough approximation to g_0'. We emphasize that conducting the second order tip expansion is mainly for the theoretical interest and the purpose of finding a more refined approximation of the optimal strategy.

5.5 The CRRA Utility Case

In this section, we present an extension to the CRRA utility. The mathematical formulation of the investor's problem is presented in Section 3.4.1. We consider only the zero transaction cost, FR and IT case (i.e., $l = \lambda = 0$, $\theta = \alpha = 0$). As in Chen and Dai (2013a), we assume that the tax rate for interest is the same as the tax rate on the stock (i.e., $\tau_i = \tau$). Then the investor's value function $V(x, y, k)$ satisfies the HJB equation (3.21) with $l = \lambda = 0$, $\theta = \alpha = 0$, and $\tau_i = \tau$. Following Chen and Dai (2013a), we make the following transformation to reduce the dimension:

$$K_0 = \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\left[(1-\tau)r + \frac{(\mu - r)^2}{2\sigma^2\gamma}\right], \quad W = x + y - \tau(y - k), \tag{5.33}$$

$$V(x, y, k) = \frac{K_0^{-\gamma}}{1-\gamma} W^{1-\gamma} e^{(1-\gamma)w(\xi,b)}, \quad \xi = \frac{(1-\tau)y}{W}, \quad b = \frac{k}{y}. \tag{5.34}$$

It follows that[6]

$$\max\left\{\mathcal{L}^1 w + f_1,\, \xi w_\xi + (1-b)w_b,\, -w_\xi\right\} = 0, \quad \xi \geq 0,\, b \geq 0, \tag{5.35}$$

[6] This reduced HJB equation remains valid for the logarithmic utility function $U(c) = \log(c)$ with $\gamma = 1$. In this case, in the HJB equation (3.21), $U^*(q) = -1 - \log(q)$. In addition, the dimensional reduction transformation is

$$V(x, y, k) = \frac{1}{\beta}\left[w(\xi, b) + \log(W) + \log(\beta) + \frac{(\mu - r)^2}{2\sigma^2\beta} - \frac{\beta - (1-\tau)r}{\beta}\right].$$

where

$$\begin{aligned}
\mathcal{L}^1 w &= \xi^2(1-\xi)^2 w_{\xi\xi} - 2\xi(1-\xi)bw_{\xi b} + b^2 w_{bb} + K_1\xi w_\xi + K_2 b w_b - K_3 w, \\
f_1 &= -\gamma(\xi-\xi_*)^2 + \frac{2r\tau}{\sigma^2}\xi(1-b), \quad \xi_* = \frac{\mu-r}{\gamma\sigma^2}, \\
K_1 &= \left[-2\gamma(\xi-\xi_*) + (1-\gamma)\big(\xi(1-\xi)w_\xi - bw_b\big)\right](1-\xi) + \frac{2r\tau[1+(b-1)\xi]}{\sigma^2} + K_4, \\
K_2 &= 2\left(1-\frac{\mu}{\sigma^2}\right) + (1-\gamma)\left[-2\xi - \xi(1-\xi)w_\xi + bw_b\right], \\
K_3 &= -\hat{K}\frac{U^*(q)-U^*(1)}{q-1}\bigg|_{q=e^w(1-\xi w_\xi)} \frac{e^w-1}{w}, \quad \hat{K} = \frac{2K_0}{\sigma^2}, \\
K_4 &= -\hat{K}\frac{U^*(q)-U^*(1)}{q-1}\bigg|_{q=e^w(1-\xi w_\xi)} e^w.
\end{aligned}$$

5.5.1 The Main Theoretical Results

In this sub-section, we present the main theoretical results derived from asymptotic analysis on (5.35). The operator $\mathcal{L}^1$ in (5.35) is degenerate at $\xi = 1$. It would lead to different expansions for $\xi_* = 1$ and $\xi_* \neq 1$. Here, we only study the case $\xi_* \neq 1$ that is of primary interest. Refer to Chen and Dai (2013a) for the case $\xi_* = 1$.

Proposition 5.2. *Assume $\xi_* \neq 1$. Let*

$$\varepsilon = \sqrt{\frac{2r\tau\xi_*}{\gamma\sigma^2}}, \; A = \frac{4}{3\xi_*(1-\xi_*)^2}, \; \xi_1(b) = \xi_* + \frac{\varepsilon^2(1-b)}{2\xi_*}. \tag{5.36}$$

We have the following approximations to the reduced value function $w(\xi, b)$ and the optimal trading boundaries $\xi^{\pm}(b)$ in $b \in [0,1]$:

(a) leading order expansion

$$\begin{cases} w(\xi,b) = & \gamma A^{2/3}\varepsilon^{8/3}\frac{m_0}{\hat{K}} + o(\varepsilon^{8/3}) \\ \xi^{\pm}(b) \approx & \xi_1(b) \pm \varepsilon\sqrt{1-b} \end{cases}, \tag{5.37}$$

(b) first order tip expansion

$$\begin{cases} w(\xi,b) = & \gamma A^{2/3}\varepsilon^{8/3}\frac{m_0}{\hat{K}} + o(\varepsilon^{8/3}) \\ \xi^{\pm}(b) \approx & \xi_1(b) \pm \varepsilon\sqrt{1-b}\left(\frac{g_0'}{\sqrt{p}}\right)^{1/3}\Big|_{p=\frac{1-b}{\delta}} \end{cases}, \tag{5.38}$$

(c) second order tip expansion

$$\begin{cases} w(\xi,b) = & \gamma A^{2/3}\varepsilon^{8/3}\frac{m_0+m_1\delta}{\hat{K}} + o(\varepsilon^{10/3}) \\ \xi^{\pm}(b) \approx & \xi_1(b) + \varepsilon\sqrt{1-b}\left(\frac{g_0'}{\sqrt{p}}\right)^{1/3}\chi^{\pm}(p)\Big|_{p=\frac{1-b}{\delta}} \end{cases}, \tag{5.39}$$

where $\delta = (A\varepsilon)^{2/3}$,

$$\chi^{\pm}(p) = \pm 1 \pm \frac{\delta}{3}\frac{g_1'}{g_0'} - \delta\left(\frac{(1-\xi_*)(1-\delta p)}{2} + \frac{\delta p}{3\xi_* A}\right)\frac{pg_0''+g_0'}{(pg_0')^{2/3}},$$

m_0 *and* g_0 *are determined by*

$$\begin{cases} (1-\delta p)^2 g_0''(p) = m_0 - p + (pg_0'(p))^{2/3}, \quad p\in[0,\frac{1}{\delta}], \\ g_0(0)=0, \quad g_0'(0)=0, \end{cases} \tag{5.40}$$

m_1 *and* g_1 *are determined by*

$$\begin{cases} (1-\delta p)^2 g_1''(p) = \frac{2}{3}\left(\frac{p^2}{g_0'}\right)^{1/3} g_1'(p) - \left[1-\xi_* - \frac{1}{2}(1-\xi_*^2)\delta p\right](pg_0''+g_0') \\ \qquad\qquad +\hat{c}(1-\delta p)g_0' + m_1, \quad p\in[0,\frac{1}{\delta}], \\ g_1(0)=0, \quad g_1'(0)=0, \end{cases} \tag{5.41}$$

and $\hat{c} = 2 - 2\mu/\sigma^2 - 2(1-\gamma)\xi_*$.

In addition, for each of the above expansions, we denote by $\bar{w}(\xi,b)$ *the approximated reduced value function. Then we have the following approximation for the optimal consumption:*

$$\left(\frac{c}{W}\right)^* \approx K_0 e^{\frac{\gamma-1}{\gamma}\bar{w}}. \tag{5.42}$$

We remark that (5.40) is the same as (5.9). We still approximate $g_0'(p)$ by a quadratic function in the form of (5.12). Then, we can obtain an explicit approximation of the optimal trading boundaries for the first order tip expansion:

$$\xi^{\pm}(b) \approx \xi_1(b) \pm \varepsilon(1-b)^{2/3}\delta^{-1/6}(\eta_{01}+\eta_{02}p)^{1/3}\Big|_{p=\frac{1-b}{\delta}}. \tag{5.43}$$

We also remark that (5.41) has the same form as (5.10). Similarly, we have

$$m_1 = -\frac{\int_0^{1/\delta} e^{-\bar{A}(p)} \frac{-\left[1-\xi_*-\frac{1}{2}(1-\xi_*^2)\delta p\right](pg_0''(p)+g_0'(p))+\hat{c}(1-\delta p)g_0'(p)}{(1-\delta p)^2} dp}{\int_0^{1/\delta} e^{-\bar{A}(p)} \frac{1}{(1-\delta p)^2} dp},$$

where $\bar{A}(p)$ is the same as the one in (5.14). If we approximate $g_0'(p)$ by (5.12), we can numerically compute m_1 via evaluating the relative integrals, and then derive the solution of $g_1'(p)$. We still approximate $g_1'(p)$ by a quadratic function in the form of (5.16). Then, we can obtain an explicit approximation of the optimal trading boundaries for the second order tip expansion:

$$\begin{aligned}\xi^{\pm}(b) \;\approx\;& \xi_1(b) + \varepsilon(1-b)^{2/3}\delta^{-1/6}(\eta_{01}+\eta_{02}p)^{1/3}\left[\pm 1 \pm \frac{\delta}{3}\frac{\eta_{11}+\eta_{12}p}{\eta_{01}+\eta_{02}p}\right] \\ & -\varepsilon(1-b)^{1/3}\delta^{7/6}\left(\frac{(1-\xi_*)(1-\delta p)}{2} + \frac{\delta p}{3\xi_* A}\right)\frac{2\eta_{01}+3\eta_{02}p}{(\eta_{01}+\eta_{02}p)^{1/3}}\Bigg|_{p=\frac{1-b}{\delta}}\end{aligned} \tag{5.44}$$

Proof of Proposition 5.2. The proofs for (a) leading order expansion and (b) first order tip expansion are provided in Chen and Dai (2013a). We only prove (c) second order tip expansion. Near the tip $(\xi^{\pm}(1), 1)$, we use the stretched variable:

$$q = \frac{\xi - \xi_1(b)}{\delta^2}, \quad p = \frac{1-b}{\delta}.$$

Then, we can rewrite the optimal trading boundaries as

$$q^{\pm}(p) := \frac{\xi^{\pm}(b) - \xi_1(b)}{\delta^2}\Big|_{b=1-p\delta}.$$

We can also rewrite the reduced value function as

$$w=\gamma\varepsilon^2\left[\frac{\delta}{\hat{K}}m_\delta+\delta^3 g_\delta(p)+\delta^5 v^\delta(p,q)\right], \tag{5.45}$$

where m_δ is a positive constant to be determined, and

$$\begin{aligned}
v^\delta(p,q) &= 0, \quad \text{if} \quad q\geq q^+(p),\\
g_\delta(0) &= 0, \quad g_\delta'(0)=0.
\end{aligned}$$

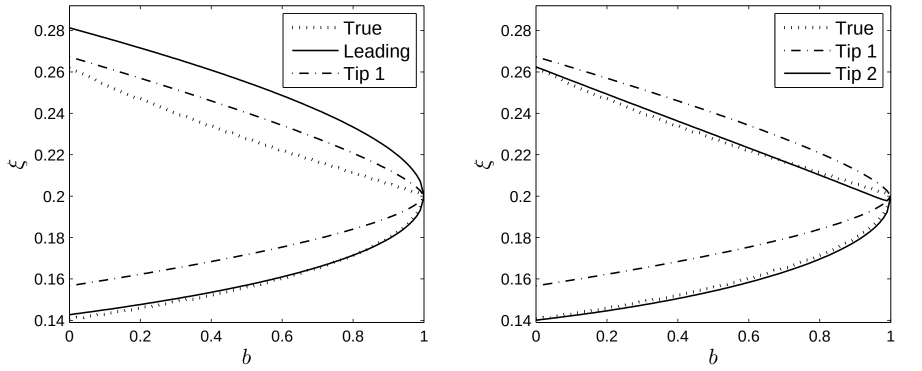
It follows that

$$\begin{aligned}
\frac{\mathcal{L}^1 w+f_1}{\gamma\varepsilon^2\delta} &= av^\delta_{qq}+2\delta(1-\delta p)\xi_*(1-\xi_*)v^\delta_{pq}+(1-\delta p)^2 g_\delta''\\
&\quad -\hat{c}\delta(1-\delta p)g_\delta'-m_\delta-q^2A^2+p+O(\delta^2),\\
\frac{\xi w_\xi+(1-b)w_b}{\gamma\varepsilon^2\delta^3} &= \xi_* v^\delta_q-pg_\delta'-p\delta^2 v^\delta_p+O(\delta^2),\\
\frac{-w_\xi}{\gamma\varepsilon^2\delta^3} &= -v^\delta_q,
\end{aligned}$$

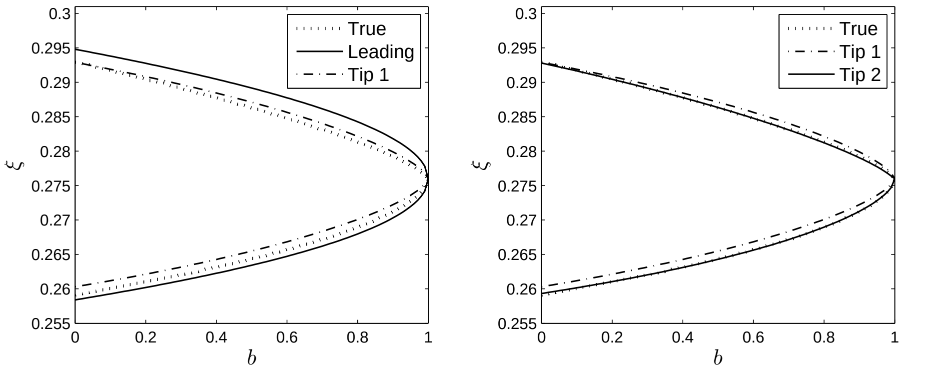
where $a=[\xi_*(1-\xi_*)]^2$. These approximated operators have similar forms of (5.21)-(5.23) in the CARA utility case. So the subsequent proof for the second order tip expansion is similar. The result for the optimal consumption is straightforward. □

5.5.2 Numerical Results

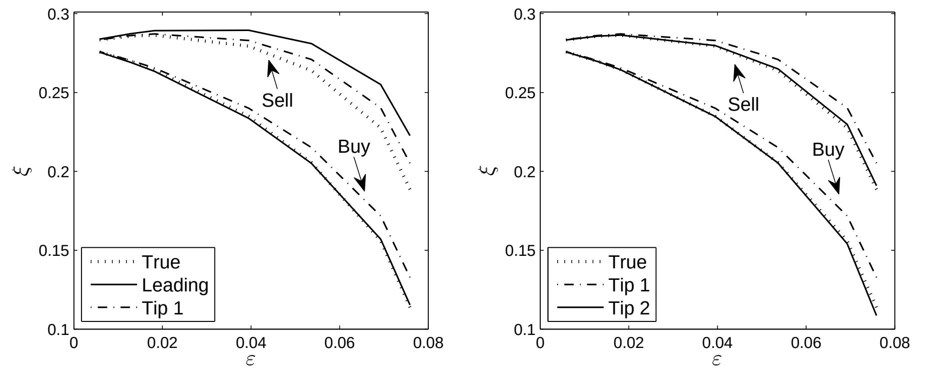
We finally present numerical results to justify the efficiency of our asymptotic analysis. Figure 5.5 shows approximations to the optimal trading boundaries. Table 5.2 reports the proportional CEWL of Merton from capital gains taxes for problems adopting the approximated strategies. Figure 5.6 examines the order of approximations to the reduced value function $w(\xi,1)$. The left and right panels respectively



(a) $r = 2\%$, $\varepsilon = 0.069$



(b) $r = 0.1\%$, $\varepsilon = 0.018$



(c) At $b = 0.5$, r varies from 0.01% to 3%

Figure 5.5: Approximations to Trading Boundaries, CRRA

This figure shows approximations to the optimal trading boundaries in the CRRA utility case. The "True", "Leading", "Tip 1", and "Tip 2" correspond to the benchmark, the leading order expansion, the first order tip expansion, and the second order tip expansion respectively. Other parameter values: $\tau = 0.15$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 4$.

Table 5.2: Prop. CEWL for Approximated Strategies, CRRA

This table reports the proportional (prop.) CEWL of Merton for approximated strategies with varying r in the CRRA utility case. The "Upper", "True", "Leading", "Tip 1", and "Tip 2" correspond to the upper bound, the benchmark, the leading order expansion, the first order tip expansion, and the second order tip expansion respectively. Other parameter values: $\tau = 0.15$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 4$.

r	ε	Upper	True	Leading	Tip 1	Tip 2
3.00%	0.076	17.5264%	16.2350%	16.5206%	16.3393%	16.2409%
2.00%	0.069	14.9224%	13.7262%	13.9591%	13.8068%	13.7292%
1.00%	0.054	10.1144%	9.3239%	9.4462%	9.3611%	9.3247%
0.50%	0.039	6.0955%	5.6683%	5.7210%	5.6820%	5.6686%
0.10%	0.018	1.4514%	1.3805%	1.3860%	1.3814%	1.3805%
0.05%	0.013	0.7431%	0.7128%	0.7147%	0.7131%	0.7128%
0.01%	0.006	0.1515%	0.1476%	0.1478%	0.1476%	0.1476%

plot

$$\log(\varepsilon) \mapsto \Delta_1 = \log\left(\frac{w(\xi,1)}{\gamma A^{2/3}\frac{m_0}{\hat{K}}}\right) \approx \frac{8}{3}\log(\varepsilon),$$

$$\log(\varepsilon) \mapsto \Delta_2 = \log\left(\frac{w(\xi,1) - \gamma\varepsilon^{8/3}A^{2/3}\frac{m_0}{\hat{K}}}{\gamma A^{4/3}\frac{m_1}{\hat{K}}}\right) \approx \frac{10}{3}\log(\varepsilon).$$

According to these figures, we have similar findings as in the CARA utility case.

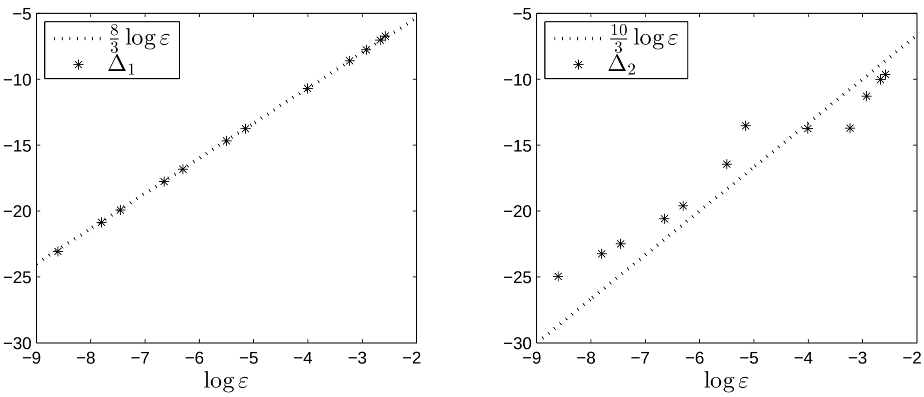


Figure 5.6: Order of Approximations to the Value Function, CRRA

This figure shows the order of approximations to the reduced value function with varying r from 0.00001% to 3%. Other parameter values: $\tau = 0.15$, $\mu = 0.07$, $\sigma = 0.25$, $\beta = 0.01$, $\gamma = 4$.

Chapter 6

Conclusion and Future Work

This thesis contributes to the literature on continuous-time portfolio selection with transaction costs and capital gains taxes. The existing literature always assumes that taxes are paid immediately after sale (instant taxes). In contrast to the existing literature, we propose a model which considers the market feature that taxes are only paid at the end of each calendar year (year-end taxes). We consider both the case where capital losses are fully rebatable and the case where capital losses are fully carried forward. We focus on the constant absolute risk aversion (CARA) utility, and an extension to the constant relative risk aversion (CRRA) utility is also provided. It turns out that the investor tends to avoid realizing capital gains late in this calendar year. Moreover, he is inclined to defer realization of capital gains until the beginning of the next calendar year. In addition, the presence of transaction costs could lead the investor to defer realization of capital losses to the end of each calendar year.

After that, we extend our model to incorporate a constant stream of labor income with no-borrowing constraint against future labor income. We show that the inability to borrow of a CRRA investor can significantly decrease consumption and investment in the risky asset, and provide an incentive to trade more frequently. Since the no-borrowing constraint destroys the homogeneity property of the investor's value function, here we focus on the no-transaction cost, full rebate, and instant tax case. More general cases are left as a future research topic.

Since closed form solutions are generally unavailable, we finally conduct asymptotic analysis in terms of small interest rate and tax rate. Based on the expansion of Chen and Dai (2013a), we propose a more refined expansion as well as an explicit strategy. The explicit strategy can effectively approximate the optimal strategy even for relatively large interest rate and tax rate by our numerical results. Here we consider only the no-transaction cost, full rebate, and instant tax case. More general cases are left for future work.

Finally, our study can be further extended to incorporate more realistic market features such as multiple correlated risky assets, stochastic labor income, portfolio constraints, and the provision that up to $3,000 of realized capital losses can apply to offset taxable income each year. Such extensions are economically interesting but mathematically challenging.

Appendix A

The Multiple Risky-Asset Problem

The existing literature on continuous-time portfolio selection with capital gains taxes is restricted to single risky asset. In this appendix, we study the continuous-time optimal investment and consumption problem of an investor who has access to multiple risky assets subject to transaction costs and capital gains taxes. We consider only the CARA utility. By virtue of its separability, we show that the multiple risky-asset problem can be reduced to the single risky-asset case provided that the asset returns are uncorrelated.

A.1 Problem Formulation

We consider a financial market consisting of one risk-free asset and n risky assets. The risk-free asset is a money market account growing at a constant interest rate of $r > 0$. The i^{th} risky asset, $i = 1, 2, \ldots, n$, is a stock whose price process P_{it} satisfies a geometric Brownian motion

$$dP_{it} = \mu_i P_{it} dt + \sigma_i P_{it} dB_{it},$$

where $\mu_i > r$ and $\sigma_i > 0$ are constants representing respectively the expected rate of return and the volatility of the i^{th} stock. The processes $\{B_{it}\}_{t\geq 0}$ are standard

Brownian motions on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with constant coefficients of correlation ρ_{ij}, namely, $\mathbb{E}(dB_i dB_j) = \rho_{ij}dt$. We assume that short selling of any stock is prohibited and that wash sales of any stock are allowed.

The investor can buy the i^{th} stock at the ask price of $(1+\theta_i)P_{it}$ and sell it at the bid price of $(1-\alpha_i)P_{it}$. The constants $\theta_i \in [0,\infty)$ and $\alpha_i \in [0,1)$ account for proportional transaction costs incurred in buying and selling the i^{th} stock.

The sales of the i^{th} stock are subject to taxes on capital gains and losses at a constant rate of $\tau_i \in [0,1)$. The average tax basis is used for each stock. We consider only the FR and IT case.

We denote by x_t the dollar amount invested in the money market account, y_{it} the dollar amount invested in the i^{th} stock, $y = (y_1, y_2, \ldots, y_n)$, k_{it} the total cost basis for the holding of the i^{th} stock, and $k = (k_1, k_2, \ldots, k_n)$. An investment policy (L, M) with $L = (L_1, L_2, \ldots, L_n)$ and $M = (M_1, M_2, \ldots, M_n)$ is defined as follows: L_{it} and M_{it} are nondecreasing, right-continuous, and $\mathscr{F}_t-$adapted processes with $L_{i,0-} = M_{i,0-} = 0$; the dollar amount transferred from the money market account to the i^{th} stock account when buying the i^{th} stock is given by dL_{it}; the dollar amount transferred from the i^{th} stock account to the money market account when selling the i^{th} stock is given by $y_{i,t-}dM_{it}$, where $dM_{it} \leq 1$ is the proportion of the i^{th} stock shares the investor sells. A consumption policy c is an $\mathscr{F}_t-$adapted process which is integrable on each finite time interval, that is, $\int_0^t |c_s| ds < \infty$ for any $t \geq 0$. Then we have the following dynamics for x_t, y_{it} and k_{it}:

$$dx_t = (rx_t - c_t)dt - \sum_{i=1}^{n}(1+\theta_i)dL_{it} + \sum_{i=1}^{n} f_i(0, y_{i,t-}, k_{i,t-})dM_{it}, \tag{A.1}$$

$$dy_{it} = \mu_i y_{i,t-}dt + \sigma_i y_{i,t-}dB_{it} + dL_{it} - y_{i,t-}dM_{it}, \tag{A.2}$$

$$dk_{it} = (1+\theta_i)dL_{it} - k_{i,t-}dM_{it}, \tag{A.3}$$

where

$$f_i(x, y_i, k_i) = x + (1-\alpha_i)y_i - \tau_i((1-\alpha_i)y_i - k_i).$$

We denote by $\mathscr{A}_0(x, y, k)$ the set of admissible strategies (c, L, M) such that the implied (x_t, y_t, k_t) from (A.1)-(A.3) with $(x_0, y_0, k_0) = (x, y, k)$ satisfies the solvency constraint

$$y_{it} \geq 0,\ k_{it} \geq 0,\ \forall t \geq 0,\ i = 1, 2, \dots, n, \tag{A.4}$$

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\beta t - r\gamma W_t}\right] = 0, \tag{A.5}$$

$$\mathbb{E}\int_0^T \left|y_t e^{-\beta t - r\gamma W_t}\right|^2 dt < \infty, \quad \forall T \in [0, \infty), \tag{A.6}$$

where $W_t = x_t + \sum_{i=1}^n f_i(0, y_{it}, k_{it})$ is the liquidated wealth at time t, and $|\cdot|$ is the Euclidean norm. Then, the investor's problem is to choose an admissible strategy so as to maximize the expected utility of intertemporal consumption:

$$V(x, y, k) = \max_{\mathscr{A}_0(x,y,k)} \mathbb{E}_0\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right], \quad \forall (x, y, k) \in \mathbb{R} \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \tag{A.7}$$

where $\beta > 0$ is a constant discount factor, and $u(\cdot)$ defined in (2.5) is a CARA utility function with a constant risk aversion factor $\gamma > 0$.

A.2 The Case without Transaction Costs and Taxes

In the absence of both transaction costs and capital gains taxes, the investor's problem can be rewritten as

$$V(W) = \max_{(y,c)\in\bar{\mathscr{A}}_0(W)} \mathbb{E}_0\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right], \tag{A.8}$$

subject to

$$dW_t = \left[rW_t - c_t + \sum_{i=1}^n (\mu_i - r)y_{it}\right] dt + \sum_{i=1}^n \sigma_i y_{it} dB_{it},$$

where $W_t = x_t + \sum_{i=1}^{n} y_{it}$ is the total wealth, and $\bar{\mathscr{A}_0}(W)$ is the set of admissible strategies defined by the solvency constraint (A.5)-(A.6). This problem permits explicit forms of the value function and the optimal strategy, which are presented in the following theorem.

Theorem A.1 (Merton's Result with Multiple Risky Assets). *In the absence of both transaction costs and capital gains taxes (i.e., $\theta_i = \alpha_i = \tau_i = 0$, $i = 1, 2, \ldots, n$), the investor's problem allows an explicit expression of the value function:*

$$V(W) = -\frac{1}{r} e^{-r\gamma W - a_n}, \tag{A.9}$$

where the scalar

$$a_n = \frac{\beta - r}{r} + \frac{(\mu - r)\varsigma^{-1}(\mu - r)'}{2r},$$

the vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, and the matrix $\varsigma = (\rho_{ij}\sigma_i\sigma_j)$.[1] *In addition, the optimal investment and consumption strategy is:*

$$y_t^* = \frac{\mu - r}{r\gamma}\varsigma^{-1}, \tag{A.10}$$

$$c_t^* = rW_t^* + \frac{a_n}{\gamma}, \tag{A.11}$$

where W_t^* is the optimal wealth derived from the above strategy.

Proof. It can be verified that the value function satisfies

$$\max_{y_i, c}\left\{\left[\frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j y_i y_j\right]V_{WW} + \left[rW - c + \sum_{i=1}^{n}(\mu_i - r)y_i\right]V_W - \beta V + u(c)\right\} = 0.$$

In matrix form, it can be rewritten as

$$\max_{y, c}\left\{\frac{1}{2}(y\varsigma y')\,V_{WW} + [rW - c + (\mu - r)y']\,V_W - \beta V + u(c)\right\} = 0. \tag{A.12}$$

[1] ς is known as the variance-covariance matrix of the stocks. It is symmetric and positive definite. ς^{-1} denotes the inverse of ς. For any vector or matrix Q, Q' denotes the transpose of Q.

The first order condition follows that the optimal (y^*, c^*) is given by

$$y^* = -(\mu - r)\varsigma^{-1}\frac{V_W}{V_{WW}}, \quad c^* = -\frac{1}{\gamma}\log\frac{V_W}{\gamma}.$$

We conjecture that the value function is of the form (A.9). The above equations imply that the optimal strategy is given by (A.10)-(A.11). Substituting (A.9)-(A.11) into (A.12), we can derive the value of a_n. The proof is completed based on the following lemma. □

The solution (A.9)-(A.11) of (A.12) specified in Theorem A.1 is a candidate for the optimal strategy and the value function of the optimal control problem (A.8). In the following lemma, we verify that this solution is indeed optimal.

Lemma A.1 (Verification Theorem for Merton's Result)**.** *The strategy (A.10)-(A.11) and the function (A.9) specified in Theorem A.1 are the optimal strategy and the value function of the optimal control problem (A.8).*

Proof. (a) We firstly show that the strategy (A.10)-(A.11) is admissible. By this strategy, we have

$$r\gamma W_t^* = r\gamma W - (\beta - r)t + \frac{1}{2}\sum_{i=1}^{n}(\mu_i - r)z_i t + \sum_{i=1}^{n}\sigma_i z_i B_{it}, \quad \forall t \geq 0,$$

and

$$\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j z_i z_j = \sum_{i=1}^{n}(\mu_i - r)z_i, \tag{A.13}$$

where $z_i = r\gamma y_i^*$ are constants, $i = 1, 2, \ldots, n$. It follows that

$$e^{-\beta t - r\gamma W_t^*} = e^{-rt - r\gamma W} N_0(t),$$

where

$$N_0(t) = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\mu_i - r)z_i t - \sum_{i=1}^{n}\sigma_i z_i B_{it}\right\}$$

is a martingale by (A.13). Then it is straightforward to check that (y_t^*, W_t^*) is subject to the solvency constraint (A.5)-(A.6).

(b) For any $(y_t, c_t) \in \bar{\mathscr{A}_0}(W)$, we denote by W_t the resulting total wealth and

$$V(W; y, c) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right]$$

the resulting value function. We next prove that

$$V(W) \geq V(W; y, c), \quad \forall (y, c) \in \bar{\mathscr{A}_0}(W), \tag{A.14}$$

where $V(W)$ is the claimed optimal value function (A.9). Applying Itô's formula on $V(W)$, we have

$$\begin{aligned} e^{-\beta t} V(W_t) &= V(W) + \int_0^t e^{-\beta s} \mathcal{L}_0^c V(W_s) ds + \sum_{i=1}^n \int_0^t e^{-\beta s} \sigma_i y_{is} V_W(W_s) dB_{is} \\ &\leq V(W) - \int_0^t e^{-\beta s} u(c_s) ds + \sum_{i=1}^n \int_0^t e^{-\beta s} \sigma_i y_{is} V_W(W_s) dB_{is}, \end{aligned}$$

where the inequality follows from that V is the solution of (A.12), and

$$\mathcal{L}_0^c V = \frac{1}{2}(y \varsigma y') V_{WW} + [rW - c + (\mu - r) y'] V_W - \beta V.$$

Taking the expectation,

$$V(W) \geq \mathbb{E}\left[e^{-\beta t} V(W_t)\right] + \mathbb{E}\left[\int_0^t e^{-\beta s} u(c_s) ds\right] - \mathbb{E}\left[\sum_{i=1}^n \int_0^t e^{-\beta s} \sigma_i y_{is} V_W(W_s) dB_{is}\right].$$

In the right-hand side of the above inequality, by (A.5), the first term goes to 0 as $t \to \infty$; and by (A.6), the third term is 0 for any fix t. Taking the limit $t \to \infty$ and using the monotone convergence theorem, we have

$$V(W) \geq \mathbb{E}\left[\int_0^\infty e^{-\beta s} u(c_s) ds\right].$$

This proves (A.14) by the arbitrariness of $(y, c) \in \bar{\mathscr{A}}_0(W)$.

(c) Choosing the strategy (y^*, c^*) given by (A.10)-(A.11), we have equality in all inequalities of part (b). Therefore,

$$V(W) \geq \max_{(y,c)\in\bar{\mathscr{A}}_0(W)} V(W; y, c) \geq V(W; y^*, c^*) = V(W).$$

This completes the proof. □

A.3 The Case with Transaction Costs and Taxes

We turn to the general case with transaction costs and capital gains taxes. It turns out that the value function satisfies the following HJB equation:

$$\max\left\{\bar{\mathcal{L}}V,\ \max_{1\leq i\leq n} \bar{\mathcal{B}}_i V,\ \max_{1\leq i\leq n} \bar{\mathcal{S}}_i V\right\} = 0, \quad y_i > 0,\ k_i > 0, \tag{A.15}$$

where

$$\begin{aligned} \bar{\mathcal{L}}V &= \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j y_i y_j V_{y_i y_j} + \sum_{i=1}^{n} \mu_i y_i V_{y_i} + rxV_x - \beta V + u^*(V_x), \quad \text{(A.16)} \\ \bar{\mathcal{B}}_i V &= -(1+\theta_i)V_x + V_{y_i} + (1+\theta_i)V_{k_i}, \\ \bar{\mathcal{S}}_i V &= f_i(0, y_i, k_i)V_x - y_i V_{y_i} - k_i V_{k_i}, \end{aligned}$$

and u^* is the same as (2.17). The optimal consumption proves to be

$$c^* = -\frac{1}{\gamma}\log\frac{V_x}{\gamma}.$$

The value function has the separability property:

$$V(x, y, k) = e^{-r\gamma x}V(0, y, k), \quad (x, y, k) \in \mathbb{R} \times \mathbb{R}_+^n \times \mathbb{R}_+^n. \tag{A.17}$$

The proof is only a slight variation of the proof of Proposition 3.1, and is thus omitted here. Due to the separability property, we can make the transformation:

$$V(x,y,k) = -e^{-r\gamma x - \varphi(z,b) - \log r}, \quad y_i > 0,\ k_i > 0,$$
$$z_i = r\gamma y_i, \quad b_i = \frac{k_i}{y_i}, \quad z = (z_1, z_2, \ldots, z_n), \quad b = (b_1, b_2, \ldots, b_n).$$

Then the HJB equation (A.15) can be reduced to

$$\max\left\{\mathcal{L}\varphi,\ \max_{1\le i\le n} \mathcal{B}_i\varphi,\ \max_{1\le i\le n} \mathcal{S}_i\varphi\right\} = 0, \quad z_i > 0,\ b_i > 0, \tag{A.18}$$

where

$$\begin{aligned}
\mathcal{L}\varphi &= \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j z_i z_j\big(\varphi_{z_i z_j} - \varphi_{z_i}\varphi_{z_j}\big) - \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j z_i b_j\big(\varphi_{z_i b_j} - \varphi_{z_i}\varphi_{b_j}\big) \\
&\quad -\frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j z_j b_i\big(\varphi_{z_j b_i} - \varphi_{z_j}\varphi_{b_i}\big) + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j b_i b_j\big(\varphi_{b_i b_j} - \varphi_{b_i}\varphi_{b_j}\big) \\
&\quad +\sum_{i=1}^{n} \mu_i z_i \varphi_{z_i} + \sum_{i=1}^{n} \big(\sigma_i^2 - \mu_i\big) b_i \varphi_{b_i} - r\varphi + \beta - r, \qquad \text{(A.19)} \\
\mathcal{B}_i\varphi &= -(1+\theta_i)z_i + z_i\varphi_{z_i} + (1+\theta_i - b_i)\varphi_{b_i}, \\
\mathcal{S}_i\varphi &= f_i(0,1,b_i) - \varphi_{z_i}.
\end{aligned}$$

The above equation is a $2n$-dimensional HJB equation with $2n$ free boundaries. In the simplest case $n = 2$, it is a 4-dimensional variational inequality, which is difficult to solve even numerically. Interestingly, if the asset returns are uncorrelated, we can compute the optimal strategy separately for each stock. This uncorrelated return case is discussed in the next section. For the correlated return case, we leave it to future work. It is worth pointing out that even if the asset returns are correlated, the strategy of the uncorrelated return case can be used as a benchmark.

A.4 The Uncorrelated Return Case

In this section, we assume that the asset returns are uncorrelated, i.e., (ρ_{ij}) is an $n \times n$ identity matrix. Then the operator (A.19) becomes

$$\begin{aligned}\mathcal{L}\varphi &= \sum_{i=1}^{n}\Big[\frac{1}{2}\sigma_i^2 z_i^2\big(\varphi_{z_i z_i} - \varphi_{z_i}^2\big) - \sigma_i^2 z_i b_i\big(\varphi_{z_i b_i} - \varphi_{z_i}\varphi_{b_i}\big) + \frac{1}{2}\sigma_i^2 b_i^2\big(\varphi_{b_i b_i} - \varphi_{b_i}^2\big)\\ &\quad + \mu_i z_i \varphi_{z_i} + \big(\sigma_i^2 - \mu_i\big) b_i \varphi_{b_i} + \frac{\beta - r}{n}\Big] - r\varphi.\end{aligned}$$

We conjecture that the reduced value function can be decomposed to

$$\varphi(z, b) = \sum_{i=1}^{n} \varphi^i(z_i, b_i).$$

Then the operator above can be rewritten as

$$\mathcal{L}\varphi(z, b) = \sum_{i=1}^{n} \mathcal{L}^i \varphi^i(z_i, b_i),$$

where

$$\begin{aligned}\mathcal{L}^i\varphi^i &= \frac{1}{2}\sigma_i^2 z_i^2\big(\varphi^i_{z_i z_i} - (\varphi^i_{z_i})^2\big) - \sigma_i^2 z_i b_i\big(\varphi^i_{z_i b_i} - \varphi^i_{z_i}\varphi^i_{b_i}\big) + \frac{1}{2}\sigma_i^2 b_i^2\big(\varphi^i_{b_i b_i} - (\varphi^i_{b_i})^2\big)\\ &\quad + \mu_i z_i \varphi^i_{z_i} + \big(\sigma_i^2 - \mu_i\big) b_i \varphi^i_{b_i} - r\varphi^i + \frac{\beta - r}{n}. \qquad \text{(A.20)}\end{aligned}$$

Within the **NTR** where none of the stocks is traded, $\mathcal{L}\varphi = 0$. Then the n partial differential equations (PDEs)

$$\mathcal{L}^i\varphi^i - \varepsilon_i = 0, \quad i = 1, 2, \ldots, n, \qquad \text{(A.21)}$$

should be satisfied for some constants ε_i such that $\sum_{i=1}^{n} \varepsilon_i = 0$. In the trading region $\mathbf{BR}^i$ where the i^{th} stock is purchased, $\mathcal{B}_i\varphi = 0$. Then

$$\mathcal{B}_i\varphi^i = -(1 + \theta_i) z_i + z_i \varphi^i_{z_i} + (1 + \theta_i - b_i)\varphi^i_{b_i} = 0 \qquad \text{(A.22)}$$

should be satisfied. In the trading region $\mathbf{SR}^i$ where the i^{th} stock is sold, $\mathcal{S}_i\varphi = 0$. Then

$$\mathcal{S}_i\varphi^i = f_i(0,1,b_i) - \varphi^i_{z_i} = 0 \tag{A.23}$$

should be satisfied.

It can be observed that the PDE system (A.21) and boundary conditions (A.22)-(A.23) are completely separable in (z_i, b_i). This suggests that the optimal investment policy in the i^{th} stock depends only on (z_i, b_i). Therefore, we can compute the optimal trading boundaries separately for each stock provided that $\varepsilon_i = 0$ for all $i = 1, 2, \dots, n$. Next, we show that $\varepsilon_i = 0$ for all $i = 1, 2, \dots, n$.

Suppose $\varphi^i(z_i, b_i)$ is the solution of

$$\max\left\{\mathcal{L}^i\varphi^i - \varepsilon_i,\ \mathcal{B}_i\varphi^i,\ \mathcal{S}_i\varphi^i\right\} = 0.$$

Consider a variation of it:

$$\max\left\{\mathcal{L}^i\varphi^i - \varepsilon_i - \bar{\eta}_i,\ \mathcal{B}_i\varphi^i,\ \mathcal{S}_i\varphi^i\right\} = 0,$$

where $\bar{\eta}_i$ is a constant. Then $\bar{\varphi}^i(z_i, b_i) = \varphi^i(z_i, b_i) - \bar{\eta}_i/r$ is the solution of the variation. We can see that the undetermined ε_i does not affect the optimal trading boundaries. In addition, since $\sum_{i=1}^n \varepsilon_i = 0$, $\varphi(z, b) = \sum_{i=1}^n \varphi^i(z_i, b_i)$ is also independent of ε_i. Without loss of generality, we can set $\varepsilon_i = 0$ for all $i = 1, 2, \dots, n$.

Lastly, we provide a verification theorem which shows the validity of the above conjectured separate policy and the form of the value function.

Proposition A.1. *Assume that the asset returns are uncorrelated. For all* $i = 1, 2, \dots, n$*, let* $\varphi^i(z_i, b_i)$ *be the solution of*

$$\max\left\{\mathcal{L}^i\varphi^i,\ \mathcal{B}_i\varphi^i,\ \mathcal{S}_i\varphi^i\right\} = 0, \quad z_i > 0,\ b_i > 0,$$

satisfying certain regularity conditions,[2] *where the operators* $\mathcal{L}^i$, $\mathcal{B}_i$ *and* $\mathcal{S}_i$ *are defined in (A.20), (A.22) and (A.23). Define the* $\mathbf{BR}^i$, $\mathbf{SR}^i$ *and* $\mathbf{NTR}^i$ *as follows:*

$$\begin{aligned}\mathbf{BR}^i &= \{(z_i, b_i) \big| \mathcal{B}_i\varphi^i = 0\},\\ \mathbf{SR}^i &= \{(z_i, b_i) \big| \mathcal{S}_i\varphi^i = 0\},\\ \mathbf{NTR}^i &= \{(z_i, b_i) \big| \mathcal{S}_i\varphi^i < 0 \text{ and } \mathcal{B}_i\varphi^i < 0\}.\end{aligned}$$

Then the investor's optimal trading policy for the i^{th} *stock follows from the* $\mathbf{BR}^i$, $\mathbf{SR}^i$ *and* $\mathbf{NTR}^i$*. In addition, the optimal consumption policy is*

$$c_t^* = rx_t^* + \frac{1}{\gamma}\sum_{i=1}^{n}\varphi^i\left(r\gamma y_{it}^*, \frac{k_{it}^*}{y_{it}^*}\right), \tag{A.24}$$

where x_t^*, y_{it}^* *and* k_{it}^* *are the implied processes by the above strategy. Moreover, the value function is*

$$V(x, y, k) = -\exp\left\{-r\gamma x - \sum_{i=1}^{n}\varphi^i\left(r\gamma y_i, \frac{k_i}{y_i}\right) - \log r\right\}.$$

Proposition A.1 shows that when there are multiple uncorrelated risky assets subject to transaction costs and capital gains taxes, we can compute the optimal trading boundaries separately for each risky asset (up to some constants). In addition, the optimal consumption rate is a linear combination of the dollar amount invested in the risk-free asset and the reduced value function. This result greatly reduces the dimensionality of the computation and makes it feasible to compute the optimal strategy for a large number of uncorrelated risky assets.

The rest of this appendix is devoted to the proof of Proposition A.1. The investor's problem belongs to the class of combined stochastic control as studied by Brekke and Øksendal (1998), Korn (1997, 1998), and Liu (2004). In particular, Liu

[2] By Lemma A.2 and the proof of Lemma A.3, (1) φ^i is required to satisfy Itô's formula in a generalised sense, (2) the $\mathbf{NTR}^i$ is required to be bounded, and (3) in its closure, φ^i, $\varphi^i_{z_i}$ and $\varphi^i_{b_i}$ are required to be bounded. The condition (2) would be satisfied by our numerical results for the single risky-asset case.

(2004) provides a proof for the case with only transaction costs. Our proof is a variation of the proofs in the above papers. We begin with some terminology and notation, then provide the verification theorem, and finally show that the conditions of Proposition A.1 satisfy all of the conditions of the verification theorem.

An impulse control $\chi = (T_j, \zeta^j)_{j=0,1,2,\cdots}$ is a sequence of trading times T_j, and trading amounts or proportions $\zeta^j = (dL_{T_j}, -dM_{T_j}) \in \mathcal{Q}$ such that

$$0 \leq T_j \leq T_{j+1} \text{ a.s., } \forall j, \tag{A.25}$$

$$T_j \text{ is a stopping time and } \zeta^j \text{ is } \mathscr{F}_{T_j}-\text{measurable, } \forall j, \tag{A.26}$$

$$\mathbb{P}\left(\lim_{j\to\infty} T_j \leq A\right) = 0, \; \forall A \geq 0, \tag{A.27}$$

where $\mathcal{Q} = \mathbb{R}_+^n \times [-1, 0]^n$ and (L, M) is an investment policy. Given an impulse control χ and a consumption policy c, the pair $\eta = (\chi, c)$ is called a combined stochastic control. It is admissible if (c, L, M) is an admissible strategy as defined in Section A.1; that is, the implied (x_t, y_t, k_t) from (A.1)-(A.3) satisfies (A.4)-(A.6). We denote by $\mathcal{W}$ the set of admissible combined stochastic controls.

Let $\mathcal{H}$ be the space of all measurable functions $h : \mathbb{R}^{2n+1} \to \mathbb{R}$. Let $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ be the maximum operator

$$\mathcal{M}h(x, y, k) = \sup_{\zeta \in \mathcal{Q}\setminus\{0\}} h\left(\tilde{x}(\zeta), \tilde{y}(\zeta), \tilde{k}(\zeta)\right), \tag{A.28}$$

and $\hat{\zeta}^h(x, y, k)$ be such that

$$\mathcal{M}h(x, y, k) = h\left(\tilde{x}(\hat{\zeta}^h(x, y, k)), \tilde{y}(\hat{\zeta}^h(x, y, k)), \tilde{k}(\hat{\zeta}^h(x, y, k))\right), \tag{A.29}$$

where

$$\begin{aligned}\tilde{x}(\zeta) &= x - \sum_{i=1}^{n}\left[(1+\theta_i)\zeta_i^+ - f_i(0, y_i, k_i)\zeta_{i+n}^-\right],\\ \tilde{y}_i(\zeta) &= y_i + \zeta_i^+ - y_i\zeta_{i+n}^-,\\ \tilde{k}_i(\zeta) &= k_i + (1+\theta_i)\zeta_i^+ - k_i\zeta_{i+n}^-,\end{aligned}$$

ζ_i and ζ_{i+n} are the i^{th} and $(i+n)^{\text{th}}$ elements of ζ.

For a given consumption policy c, define the operator $\mathcal{L}^c$ as follows:

$$\mathcal{L}^c J(x, y, k) = \sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2 y_i^2 J_{y_i y_i} + \mu_i y_i J_{y_i}\right] + rxJ_x - \beta J - cJ_x,$$

for any function J whose derivatives involved exist.

We now provide the verification theorem in the following lemma. It gives sufficient conditions under which an admissible combined stochastic control solves the investor's problem.

Lemma A.2 (Verification Theorem)**.**

(a) Suppose there exists a sufficiently regular function $v(x, y, k)$ satisfying the following conditions:[3]

1. *$\mathcal{L}^c v(x, y, k) + u(c) \leq 0$, for every consumption policy c,*
2. *$v(x, y, k) \geq \mathcal{M}v(x, y, k)$,*
3. *the following two equations*

$$\forall T \in [0, \infty), \quad \mathbb{E}\int_0^T \left|e^{-\beta t} y_t v_y(x_t, y_t, k_t)\right|^2 dt < \infty, \tag{A.30}$$

$$\lim_{T\to\infty} \mathbb{E}[e^{-\beta T} v(x_T, y_T, k_T)] = 0, \tag{A.31}$$

hold for any (x_t, y_t, k_t) following from an admissible combined stochastic

[3] In the proof, v is only required to satisfy Itô's formula in a generalised sense.

control with $(x_0, y_0, k_0) = (x, y, k)$*, where* $v_y = diag\{v_{y_1}, v_{y_2}, \ldots, v_{y_n}\}$ *is a diagonal matrix,*

4. $\{e^{-\beta t} v(x_t, y_t, k_t)\}_{t \geq 0}$ *is uniformly integrable.*

Then

$$v(x, y, k) \geq V^{\eta}(x, y, k),\ \forall \eta \in \mathcal{W},$$

where $V^{\eta}(x, y, k)$ *is the total expected utility obtained when applying the combined stochastic control* η*.*

(b) Define

$$NT = \{(x, y, k) : v(x, y, k) > \mathcal{M}v(x, y, k)\}.$$

Suppose the conditions in part (a) hold and there exists a function $\hat{c}(x, y, k) : NT \to \mathbb{R}$ *such that*

$$\mathcal{L}^{\hat{c}(x,y,k)} v(x, y, k) + u(\hat{c}(x, y, k)) = 0 \tag{A.32}$$

for all $(x, y, k) \in NT$*. Define the impulse control* $\hat{\chi} = (\hat{T}_j, \hat{\zeta}^j)_{j=1,2,\cdots}$ *inductively as follows:* $\hat{T}_0 = 0$ *and for* $j = 0, 1, 2, \ldots,$

$$\begin{aligned}
\hat{T}_{j+1} &= \inf\{t > \hat{T}_j : (x_t^{(j)}, y_t^{(j)}, k_t^{(j)}) \notin NT\},\\
\hat{\zeta}^{j+1} &= \hat{\zeta}^{v}(x_t^{(j)}, y_t^{(j)}, k_t^{(j)}),
\end{aligned}$$

where $(x_t^{(j)}, y_t^{(j)}, k_t^{(j)})$ *is the result of applying the combined stochastic control* $\hat{\eta}_j = ((\hat{T}_m, \hat{\zeta}^m)_{m=1,2,\cdots,j}, \hat{c})$*, and* $\hat{\zeta}^v$ *is as defined in (A.29). If the combined stochastic control* $\hat{\eta} = (\hat{\chi}, \hat{c})$ *is admissible, then*

$$v(x, y, k) = V(x, y, k),$$

and $\eta^* = \hat{\eta}$ *is optimal, where* $V(x, y, k)$ *is the value function defined in (A.7).*

Proof. (a) Let $T \in [0, \infty)$ be fixed, and $\eta = (\chi, c) \in \mathcal{W}$ with $\chi = (T_j, \zeta^j)_{j=1,2,\cdots}$ be any admissible combined stochastic control. For all $m = 0, 1, 2, \cdots$, define $\delta_m = T_m \wedge T$ with $T_0 = 0$. Let (x_t, y_t, k_t) follow from η with $(x_0, y_0, k_0) = (x, y, k)$. For every m, we can write

$$
\begin{aligned}
&e^{-\beta\delta_m} v(x_{\delta_m}, y_{\delta_m}, k_{\delta_m}) - v(x, y, k) \\
&\quad = \sum_{i=1}^{m} \left[e^{-\beta\delta_i} v(x_{\delta_i^-}, y_{\delta_i^-}, k_{\delta_i^-}) - e^{-\beta\delta_{i-1}} v(x_{\delta_{i-1}}, y_{\delta_{i-1}}, k_{\delta_{i-1}}) \right] \\
&\qquad + \sum_{i=1}^{m} \mathbf{1}_{\{T_i < T\}} e^{-\beta\delta_i} \left[v(x_{\delta_i}, y_{\delta_i}, k_{\delta_i}) - v(x_{\delta_i^-}, y_{\delta_i^-}, k_{\delta_i^-}) \right].
\end{aligned}
$$

By Itô's formula,

$$
\begin{aligned}
&e^{-\beta\delta_i} v(x_{\delta_i^-}, y_{\delta_i^-}, k_{\delta_i^-}) - e^{-\beta\delta_{i-1}} v(x_{\delta_{i-1}}, y_{\delta_{i-1}}, k_{\delta_{i-1}}) \\
&\quad = \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} \mathcal{L}^c v(x_s, y_s, k_s) ds + \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} y_s v_y(x_s, y_s, k_s) \sigma dB_s \\
&\quad \le - \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} u(c_s) ds + \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} y_s v_y(x_s, y_s, k_s) \sigma dB_s
\end{aligned}
\tag{A.33}
$$

where the inequality follows from condition 1, $\sigma = diag\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ and $B = (B_1, B_2, \cdots, B_n)'$. Condition 2 implies that

$$
v(x_{\delta_i}, y_{\delta_i}, k_{\delta_i}) - v(x_{\delta_i^-}, y_{\delta_i^-}, k_{\delta_i^-}) \le 0. \tag{A.34}
$$

After combining the above equations, we have

$$
\begin{aligned}
v(x, y, k) \ \ge \ \mathbb{E}\Bigg[& e^{-\beta\delta_m} v(x_{\delta_m}, y_{\delta_m}, k_{\delta_m}) + \sum_{i=1}^{m} \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} u(c_s) ds \\
& - \sum_{i=1}^{m} \int_{\delta_{i-1}}^{\delta_i} e^{-\beta s} y_s v_y(x_s, y_s, k_s) \sigma dB_s \Bigg].
\end{aligned}
\tag{A.35}
$$

By (A.30), for any fixed m,

$$
\mathbb{E}\left[\int_0^{\delta_m} e^{-\beta s} y_s v_y(x_s, y_s, k_s) \sigma dB_s \right] = 0.
$$

By (A.27) and condition 4,

$$\lim_{m\to\infty} \mathbb{E}\left[e^{-\beta\delta_m} v(x_{\delta_m}, y_{\delta_m}, k_{\delta_m})\right] = \mathbb{E}\left[e^{-\beta T} v(x_T, y_T, k_T)\right].$$

Let $m \to \infty$ in (A.35) and use the monotone convergence theorem. We have

$$v(x, y, k) \geq \mathbb{E}\left[e^{-\beta T} v(x_T, y_T, k_T)\right] + \mathbb{E}\left[\int_0^T e^{-\beta s} u(c_s) ds\right].$$

Letting $T \to \infty$, by (A.31) and the monotone convergence theorem, we have

$$v(x, y, k) \geq \mathbb{E}\left[\int_0^\infty e^{-\beta s} u(c_s) ds\right].$$

Therefore, $v(x, y, k) \geq V^\eta(x, y, k)$ for all $\eta \in \mathcal{W}$.

(b) Given $\hat{\eta} = (\hat{\chi}, \hat{c})$ satisfying the conditions of part (b), we have equality in (A.33) and (A.34). Combining this with the result in part (a), we have

$$v(x, y, k) \geq \sup_{\eta \in \mathcal{W}} V^\eta(x, y, k) \geq V^{\hat{\eta}}(x, y, k) = v(x, y, k).$$

This completes the proof. □

In the above verification theorem, it is required that the combined stochastic control $\hat{\eta}$ is admissible. Next, we show that the strategy specified in Proposition A.1 forms an admissible combined stochastic control.

Lemma A.3. *The strategy specified in Proposition A.1 forms an admissible combined stochastic control.*

Proof. Let $\hat{\eta} = (\hat{\chi}, \hat{c})$ with $\hat{\chi} = (T_j, \hat{\zeta}^j)_{j=0,1,2,\cdots}$ be the strategy specified in Proposition A.1, where T_j is the time when the investor trades and $\hat{\zeta}^j$ is the trading amount or proportion. Let (x_t, y_t, k_t) follow from $\hat{\eta}$ with $(x_0, y_0, k_0) = (x, y, k)$, $z_{it} = r\gamma y_{it}$ and $b_{it} = \frac{k_{it}}{y_{it}}$. Since the prescribed trading policy is to trade the i^{th} stock whenever (z_{it}, b_{it}) is outside the $\mathbf{NTR}^i$, clearly T_j is a stopping time with $0 \leq T_j \leq T_{j+1}$ a.s.

and $\hat{\zeta}^j$ is $\mathscr{F}_{T_j}-$measurable. For all $t \in (0, \infty)$, we have $\mathbb{P}\{(z_{it}, b_{it}) \in \mathbf{NTR}^i\} = 1$, and the boundary of the $\mathbf{NTR}^i$ can be reached in finite expected time as implied by the propositions in Section 5.5 of Karatzas and Shreve (1988). This implies (A.27). Thus, $\hat{\chi}$ is an impulse control. Clearly (A.4) holds. To complete the proof, we only need to prove (A.5)-(A.6).

For all $t \in (0, \infty)$ and $m = 0, 1, 2, \cdots$, by (A.1) and (A.24),

$$\begin{aligned} r\gamma x_{t\wedge T_m} &= r\gamma x - \sum_{i=1}^{n} \int_0^{t\wedge T_m} r\varphi^i(z_{is}, b_{is})ds \\ &\quad + \sum_{i=1}^{n}\sum_{j=0}^{m} \mathbf{1}_{\{T_j<t\}} \left[-(1+\theta_i)r\gamma\hat{\zeta}_i^{j+} + f_i(0,1,b_{i,T_j-})z_{i,T_j-}\hat{\zeta}_{i+n}^{j-}\right]. \end{aligned}$$

In $\mathbf{BR}^i$, $-(1+\theta_i)z_i + z_i\varphi^i_{z_i} + (1+\theta_i - b_i)\varphi^i_{b_i} = 0$. It implies there exists a function h^i such that

$$\varphi^i(z_{is}, b_{is}) = h^i(z_{is}(b_{is} - 1 - \theta_i)) + (1+\theta_i)z_{is}.$$

In $\mathbf{SR}^i$, $f_i(0,1,b_i) - \varphi^i_{z_i} = 0$. It implies there exists a function g^i such that

$$\varphi^i(z_{is}, b_{is}) = g^i(b_{is}) + f_i(0,1,b_{is})z_{is}. \tag{A.36}$$

Since $z_i(b_i - 1 - \theta_i)$ or b_i does not change when the investor buys or sells the i^{th} stock, we have

$$-(1+\theta_i)r\gamma\hat{\zeta}_i^{j+} + f_i(0,1,b_{i,T_j-})z_{i,T_j-}\hat{\zeta}_{i+n}^{j-} = \varphi^i(z_{iT_j-}, b_{iT_j-}) - \varphi^i(z_{iT_j}, b_{iT_j}).$$

We then write

$$\begin{aligned} &\sum_{j=0}^{m} \mathbf{1}_{\{T_j<t\}} \left[\varphi^i(z_{iT_j-}, b_{iT_j-}) - \varphi^i(z_{iT_j}, b_{iT_j})\right] \\ &\quad = \varphi^i(z_i, b_i) - \varphi^i(z_{i,t\wedge T_m}, b_{i,t\wedge T_m}) \\ &\qquad + \sum_{j=1}^{m} \left[\varphi^i(z_{i,t\wedge T_j-}, b_{i,t\wedge T_j-}) - \varphi^i(z_{i,t\wedge T_{j-1}}, b_{i,t\wedge T_{j-1}})\right]. \end{aligned}$$

By Itô's formula, we further write

$$\begin{aligned}
&\varphi^i(z_{i,t\wedge T_j-}, b_{i,t\wedge T_j-}) - \varphi^i(z_{i,t\wedge T_{j-1}}, b_{i,t\wedge T_{j-1}}) \\
&= \int_{t\wedge T_{j-1}}^{t\wedge T_j-} \Big[\Big(\frac{1}{2}\sigma_i^2 z_{is}^2 (\varphi^i_{z_i z_i} - (\varphi^i_{z_i})^2) - \sigma_i^2 z_{is} b_{is} (\varphi^i_{z_i b_i} - \varphi^i_{z_i}\varphi^i_{b_i}) \\
&\quad + \frac{1}{2}\sigma_i^2 b_{is}^2 (\varphi^i_{b_i b_i} - (\varphi^i_{b_i})^2) + \mu_i z_{is}\varphi^i_{z_i} + (\sigma_i^2 - \mu_i) b_{is}\varphi^i_{b_i}\Big) ds \\
&\quad + \frac{1}{2}\sigma_i^2 \Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big)^2 ds + \sigma_i\Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big) dB_{is}\Big] \\
&= \int_{t\wedge T_{j-1}}^{t\wedge T_j-} \Big[\Big(r\varphi^i(z_{is}, b_{is}) - \frac{\beta - r}{n}\Big) ds \\
&\quad + \frac{1}{2}\sigma_i^2 \Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big)^2 ds + \sigma_i\Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big) dB_{is}\Big],
\end{aligned}$$

where the second equality follows from that $\mathcal{L}^i\varphi^i = 0$ in the $\mathbf{NTR}^i$. Therefore,

$$\begin{aligned}
r\gamma x_{t\wedge T_m} &= r\gamma x - (\beta - r)(t\wedge T_m) + \sum_{i=1}^n \left[\varphi^i(z_i, b_i) - \varphi^i(z_{i,t\wedge T_m}, b_{i,t\wedge T_m})\right] \\
&\quad + \sum_{i=1}^n \int_0^{t\wedge T_m} \left[\frac{1}{2}\sigma_i^2 \Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big)^2 ds + \sigma_i\Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big) dB_{is}\right].
\end{aligned}$$

By (A.27), as $m \to \infty$,

$$\begin{aligned}
r\gamma x_t &= r\gamma x - (\beta - r)t + \sum_{i=1}^n \left[\varphi^i(z_i, b_i) - \varphi^i(z_{it}, b_{it})\right] \\
&\quad + \sum_{i=1}^n \int_0^t \left[\frac{1}{2}\sigma_i^2 \Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big)^2 ds + \sigma_i\Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big) dB_{is}\right].
\end{aligned}$$

Since $W_t = x_t + \sum_{i=1}^n f_i(0, y_{it}, k_{it})$, we have

$$e^{-\beta t - r\gamma W_t} = e^{-rt - r\gamma x - \sum_{i=1}^n \left[f_i(0,1,b_{it})z_{it} + \varphi^i(z_i,b_i) - \varphi^i(z_{it},b_{it})\right]} N(t),$$

where

$$N(t) = \exp\left\{-\sum_{i=1}^n \int_0^t \left[\frac{1}{2}\sigma_i^2 \Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big)^2 ds + \sigma_i\Big(z_{is}\varphi^i_{z_i} - b_{is}\varphi^i_{b_i}\Big) dB_{is}\right]\right\}.$$

Since for all $t \in [0, \infty)$, z_{it}, b_{it}, $\varphi^i(z_{it}, b_{it})$, $\varphi^i_{z_i}(z_{it}, b_{it})$ and $\varphi^i_{b_i}(z_{it}, b_{it})$ are all bounded and $\mathbb{E}[N(t)] = 1$, we have

$$0 \leq \lim_{t\to\infty} \mathbb{E}\left[e^{-\beta t - r\gamma W_t}\right] \leq \lim_{t\to\infty}\left[K_3 e^{-rt}\mathbb{E}(N(t))\right] = 0,$$

where K_3 is some finite constant. This proves (A.5). In addition,

$$\begin{aligned} e^{-2\beta t - 2r\gamma W_t} &\leq K_4 e^{-2rt} N(t)^2, \\ \mathbb{E}[N(t)^2] &\leq e^{K_5 t}, \end{aligned}$$

for some finite constants K_4 and K_5. Thus, for all $T \in [0, \infty)$ and $i = 1, 2, \cdots, n$,

$$\mathbb{E}\int_0^T |y_{it} e^{-\beta t - r\gamma W_t}|^2 dt = \frac{1}{r^2\gamma^2}\mathbb{E}\int_0^T |z_{it} e^{-\beta t - r\gamma W_t}|^2 dt < \infty.$$

This proves (A.6). □

Proof of Proposition A.1. We are now ready to prove Proposition A.1. We only need to prove that all conditions of the verification theorem Lemma A.2 are satisfied. Lemma A.3 shows that the strategy specified in Proposition A.1 forms an admissible combined stochastic control. We now show that conditions 3 and 4 of part (a) are satisfied. Let

$$v(x, y, k) = -\frac{1}{r} e^{-r\gamma x - \sum_{i=1}^n \varphi^i(z_i, b_i)}$$

be the proposed value function. We have

$$\lim_{T\to\infty} \mathbb{E}[e^{-\beta T} v(x_T, y_T, k_T)] = -\frac{1}{r}\lim_{T\to\infty} \mathbb{E}[e^{-\beta T - r\gamma x_T - \sum_{i=1}^n \varphi^i(z_{iT}, b_{iT})}] = 0, \qquad \text{(A.37)}$$

where the last equality can be proved in a similar way as the proof of (A.5) in Lemma A.3. The above equation implies that for any fixed $t \geq 0$,

$$\mathbb{E}[|e^{-\beta t} v(x_t, y_t, k_t)|] < \infty.$$

Then, $e^{-\beta t}v(x_t, y_t, k_t)$ is in L^1 (Lebesgue space) and converges to 0 in L^1. Thus, condition 4 holds by Theorem 13.7 in Williams (1991). It is straightforward to check that

$$z_i v_{y_i}(x, y, k) = \gamma\left(z_i\varphi^i_{z_i} - b_i\varphi^i_{b_i}\right)e^{-r\gamma x - \sum_{i=1}^n \varphi^i(z_i, b_i)}.$$

So for all $T \in [0, \infty)$,

$$\begin{aligned}
\mathbb{E}\int_0^T \left|e^{-\beta t}y_t v_y(x_t, y_t, k_t)\right|^2 dt &= \mathbb{E}\int_0^T \left[e^{-2\beta t}\sum_{i=1}^n \left(z_{it}\varphi^i_{z_i} - b_{it}\varphi^i_{b_i}\right)^2 v^2(x_t, y_t, k_t)\right] dt \\
&\leq \mathbb{E}\int_0^T K_6\left[e^{-\beta t}v(x_t, y_t, k_t)\right]^2 dt \\
&= \mathbb{E}\int_0^T K_6\left[\frac{1}{r}e^{-\beta t - r\gamma x_t - \sum_{i=1}^n \varphi^i(z_{it}, b_{it})}\right]^2 dt \\
&< \infty, \qquad\qquad (A.38)
\end{aligned}$$

where K_6 is some finite constant. The first inequality holds because for all $t \in [0, \infty)$, z_{it}, b_{it}, $\varphi^i_{z_i}(z_{it}, b_{it})$ and $\varphi^i_{b_i}(z_{it}, b_{it})$ are all bounded. The last inequality can be proved in a similar way as the proof of (A.6) in Lemma A.3. By (A.37) and (A.38), condition 3 also holds.

Next, we show condition 2 of part (a) holds. By the definition (A.28), we have

$$\mathcal{M}v(x, y, k) = -\frac{1}{r}e^{-r\gamma x - \sum_{i=1}^n \psi^i(z_i, b_i)},$$

where

$$\psi^i(z_i, b_i) = \sup_{(\zeta_i, \zeta_{i+n})}\left\{\varphi^i\left(\tilde{z}_i(\zeta_i, \zeta_{i+n}), \tilde{b}_i(\zeta_i, \zeta_{i+n})\right) - (1 + \theta_i)r\gamma\zeta_i^+ + f_i(0, 1, b_i)z_i\zeta_{i+n}^-\right\},$$

$(\zeta_i, \zeta_{i+n}) \in \mathbb{R}_+ \times [-1, 0]$, $(\zeta_i, \zeta_{i+n}) \neq 0$ for at least one i,

$$\tilde{z}_i(\zeta_i, \zeta_{i+n}) = z_i + r\gamma\zeta_i^+ - z_i\zeta_{i+n}^-,$$

and $\tilde{b}_i(\zeta_i, \zeta_{i+n})$ satisfies

$$\begin{aligned} &\tilde{z}_i(\zeta_i, \zeta_{i+n})(\tilde{b}_i(\zeta_i, \zeta_{i+n}) - 1 - \theta_i) = z_i(b_i - 1 - \theta_i), && \text{if } \zeta_i > 0, \\ &\tilde{b}_i(\zeta_i, \zeta_{i+n}) = b_i, && \text{if } \zeta_{i+n} < 0. \end{aligned}$$

We denote by

$$\omega^i(\zeta_i, \zeta_{i+n}) = \varphi^i\left(\tilde{z}_i(\zeta_i, \zeta_{i+n}), \tilde{b}_i(\zeta_i, \zeta_{i+n})\right) - (1+\theta_i) r\gamma\zeta_i^+ + f_i(0, 1, b_i) z_i \zeta_{i+n}^-.$$

In $\mathbf{NTR}^i$, since $\mathcal{S}_i\varphi^i < 0$, it is straightforward to check that $\omega^i_{\zeta_{i+n}} > 0$ in $\zeta_{i+n} < 0$. So $\omega^i(\zeta_i, \zeta_{i+n}) < \omega^i(\zeta_i, 0)$ for any $\zeta_{i+n} < 0$. Similarly, since $\mathcal{B}_i\varphi^i < 0$ in $\mathbf{NTR}^i$, we have $\omega^i_{\zeta_i} < 0$ in $\zeta_i > 0$. So $\omega^i(\zeta_i, \zeta_{i+n}) < \omega^i(0, \zeta_{i+n})$ for any $\zeta_i > 0$. Thus, $\omega^i(\zeta_i, \zeta_{i+n}) < \omega^i(0, 0)$ for any $(\zeta_i, \zeta_{i+n}) \neq 0$. It follows that

$$\mathcal{M}v(x, y, k) < v(x, y, k), \quad \text{in } \mathbf{NTR}^i.$$

In a similar way, we have

$$\mathcal{M}v(x, y, k) = v(x, y, k), \quad \text{in } \mathbf{SR}^i \text{ and } \mathbf{BR}^i.$$

Therefore, condition 2 holds. In addition, NT is the **NTR** where none of the stocks is traded.

Finally, we write

$$\begin{aligned} \bar{\mathcal{L}}v(x, y, k) &= \sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2 y_i^2 v_{y_i y_i} + \mu_i y_i v_{y_i}\right] + r x v_x - \beta v + u^*(v_x) \\ &= |v(x, y, k)| \sum_{i=1}^{n} \mathcal{L}^i \varphi^i(z_i, b_i), \end{aligned}$$

where $\bar{\mathcal{L}}$ is defined in (A.16). Then it is easy to verify that for any consumption policy c,

$$\mathcal{L}^c v(x,y,k) + u(c) \leq \max_c \left\{ \mathcal{L}^c v(x,y,k) + u(c) \right\} = \mathcal{L}^{c^*} v(x,y,k) + u(c^*) = \bar{\mathcal{L}} v(x,y,k),$$

where c^* is defined in (A.24). Since $\bar{\mathcal{L}} v(x,y,k) = 0$ in NT, (A.32) is satisfied in NT with $\hat{c} = c^*$. Since $\bar{\mathcal{L}} v(x,y,k) \leq 0$ in the whole region, condition 1 also holds. Then the proof is completed. □

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PORTFOLIO SELECTION WITH TRANSACTION COSTS AND CAPITAL GAINS TAXES

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