MOBILITY AND INEQUALITY OVER CYCLICAL ENDOGENOUS GROWTH

MUN LAI YOKE

(B.Sc., B.Sc.(Hons.), M.Tech.(Knowledge Engineering),
M.Sc.(Financial Engineering) & M.Soc.Sc.(Applied Economics),
National University of Singapore;
M.B.A., Heriot-Watt University;
Postgrad.Dip.Ed.(Credit), Nanyang Technological University)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ECONOMICS

NATIONAL UNIVERSITY OF SINGAPORE

2015

Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Signed: 

Mun Lai Yoke

14 August 2015

Acknowledgements

First, I would like to thank my supervisor, Prof. Zeng Jinli, for helping me bring the thesis to fruition. His systematic approach towards research, coupled with his patience, inspired me to do better. I hope to pay it forward when I teach.

I would also like to thank Prof. Zhang Jie, Dr. Zhu Shenghao, Prof. Liu Haoming, and Prof. Serene Tan for their constructive comments. In the course of my study, I have also learned from other faculty members, both directly, and indirectly at various reading groups. Prof. Aditya Goenka, Prof. Tilak Abeysinghe, Prof. Duan Jin-Chuan, Prof. Heejoon Han, Dr. Tomoo Kikuchi, Dr. Lin Liu, Prof. Luo Xiao, Prof. Chen Yichun, Prof. Davin Chor, Prof. Albert Tsui, Prof. Guillaume Rocheteau, and Prof. Danny Quah, I thank you for your generosity in sharing your research papers, purpose and passion.

Last but not least, I would like to thank all my coursemates at NUS, among them: Zhang Shen, Athakrit Thepmongkol, Jiao Qian, Miao Bin, Vu Thanh Hai, Li Li, Wan Jing, Sreyashi Sen, Jiang Wei, Hong Bei, Zeng Ting, Shao Lei, Liu Xuyuan, Jiang Yushi, Ding Rong, Shen Bo, Sun Yifei, Cao Qian, Cai Xiqian, Wang Jianguang, Yang Guangpu, and Liu Bin for the friendship and intellectual exchanges on economics and beyond, and the Head, Prof. Julian Wright, and staff at the Economics department for the administrative support.

I am the family face;
Flesh perishes, I live on,
Projecting trait and trace
Through time to times anon,
And leaping from place to place
Over oblivion.

The years-heired feature that can
In curve and voice and eye
Despise the human span
Of durance – that is I;
The eternal thing in man,
That heeds no call to die.

– Thomas Hardy, *Heredity*

Contents

Summary

In this thesis, the mobility, inequality, labor and growth trajectories in the presence of idiosyncratic productivity shocks are characterized under a general equilibrium framework, enabling us to analyze the effects of strength of bequest motive, envy, taste for leisure, monopoly power and fiscal policies on intergenerational mobility and growth-inequality trade-offs. (See Figure 1 on page xii.) This thesis provides a mechanism and plausible explanations to (i) the differences in wealth inequality dynamics across countries and within a country across time , (ii) the increase in mobility in developed countries with an increase in innovation subsidies, funded by uniform output tax, and (iii) the increase in inherited wealth inequality in countries with decreasing inheritance tax.

In the first chapter, I extend Matsuyama (1999) endogenous growth model by introducing idiosyncratic labor productivity shocks to households with bequest motives in the spirit of Bossmann et al. (2007). In Matsuyama (1999), due to the presence of fixed cost of innovation and monopoly profits, there are different growth regimes, depending on the capital to variety ratio. Under i.i.d. shocks, the lineage wealth evolution and cross-sectional wealth distributions for Solow and Romer steady states, and period-2 cycles are identical, even though as shown in Matsuyama (1999), the aggregate growth over period-2 cycles are higher than at either of the steady states. (Under i.i.d. labor productivity shocks, I have characterized the global mobility evolution and wealth distribution for the Matsuyama (1999) endogenous growth model, which encompasses growing

through chaotic intervals as discussed in Gardini et al. (2008).) With lineage-correlated shocks, the cross-sectional inequality depends on the nature of the fixed points.

In the second chapter, I introduce "Others regarding behavior" in the form of "Envy" as in Alvarez-Cuadrado and Long (2012) into the benchmark model in Chapter 1. This is because empirical data suggests that in addition to absolute consumption, we are concerned with the relative ranking too, which I shall refer to as "Status Anxiety"[1]. This results in bequest being a luxury good. The key findings in this chapter are: status anxiety reduces the steady state ratio of capital stock to variety of intermediate goods under both Solow and Romer regimes, the potential growth rate of the economy, as well as the lineage mobility. Moreover, as in Chapter 1, the higher the price elasticity of the final goods sector's demand for each intermediate, the lower the intergenerational lineage mobility. However, unlike in Chapter 1, with status anxiety, the inequality of inherited wealth could be higher than the inequality of the wage income under certain conditions.

In the third chapter, I introduce labor-leisure choice. With endogenous labor, when the preference parameter for leisure increases, the intergenerational lineage mobility increases, while cross-sectional wealth inequality increases. In addition, the variation of wage income and leisure over the different regimes and cycles are characterized. The comparative statics on cross-sectional ratio

[1]Inspired by De Botton (2008).

of wages earned to final output are driven by the relative size of the general equilibrium effect of the wages, the contemporaneous substitution effect of consumption to leisure and the inherited wealth effect, adjusted for quality with the contemporaneous labor productivity shock, on demand for leisure.

Finally, I examine with the distributive role of fiscal policies in Chapter 4. Assuming a logarithmic utility function, an increase in inheritance tax decreases cross-sectional inequality, leveling the playing field, under i.i.d. and correlated shocks for both growth regimes and period-2 cycles, strengthening the results from Bossmann et al. (2007), while contrasting with Becker and Tomes (1979), Atkinson (1980) and Davies (1986). Furthermore, an increase in inheritance tax also increases intergenerational lineage mobility under i.i.d. shocks. Besides, the reduction of volatility of growth by fiscal policies such as the innovation subsidy policy rule proposed by Aloi and Lasselle (2007), which stabilizes period-2 cycles, could level the playing field, by decreasing the persistence of inherited wealth during the transition to the Romer steady state, if funded by a uniform value-added tax. However, moving from period-2 cycles to the Romer steady state with the innovation subsidies may change the cross-sectional inequality of wealth if the idiosyncratic labor productivity shocks are correlated.

Figure 1 depicts the structure and coverage of this thesis.

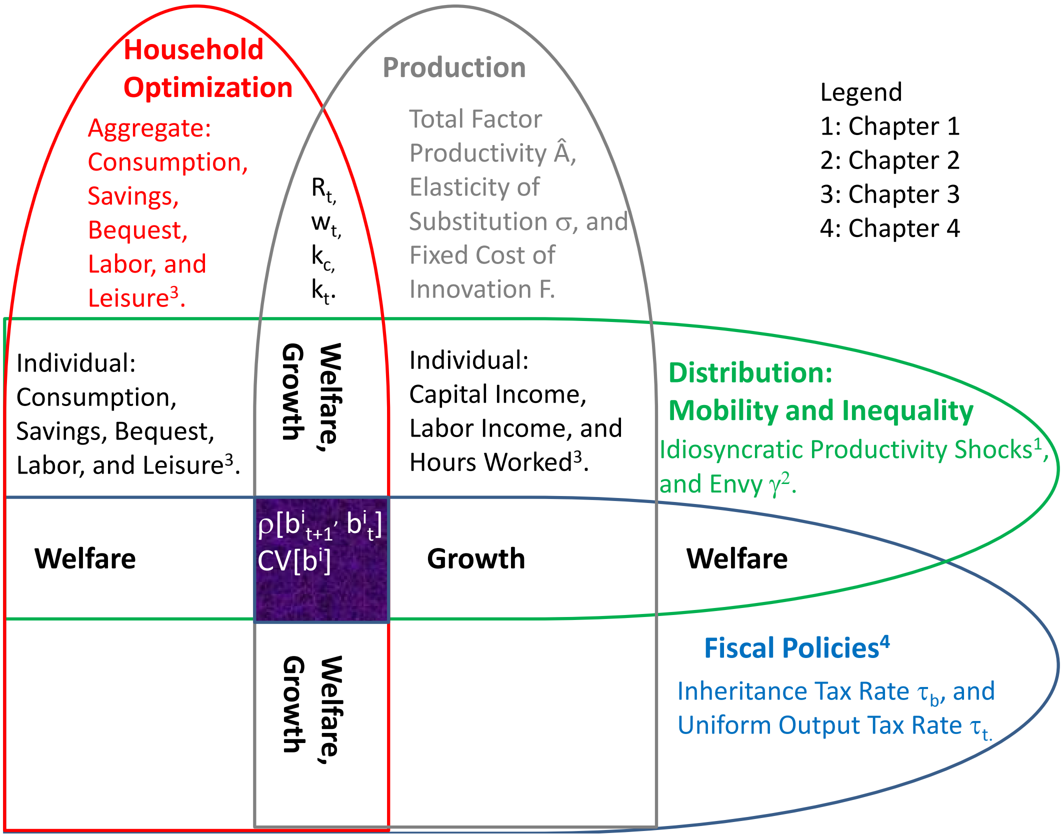

Figure 1: Venn diagram - a general equilibrium analysis of mobility and distribution with fiscal policies.

Legend	
l_t^i	Idiosyncratic labor productivity for an agent from lineage i, born at time, t.
c_t^i	Consumption when young for an agent born at time t.
d_{t+1}^i	Consumption when old for an agent born at time t.
b_t^i	Bequest inherited by an agent from lineage i, born at time, t.
$\beta < 1$	Subjective discount factor
$0 \leq \mu < 1$	Bequest motive.
$w_t^i = w_t l_t^i$	Idiosyncratic wage income with inelastic labor.
w_t	Equilibrium wage rate for each unit of labor productivity.
s_t^i	Amount saved when young of an individual from lineage i, born at time t.
y_t^i	Lifetime resource of an individual from lineage i, born at time t.
R_{t+1}	Equilibrium gross rate of return on capital at time $t+1$.
K_{t-1}	Unconsumed final goods in period $t-1$, available for production in period t.
L_t	Total labor supply at period t.
Y_t	Final goods production at period t.
$\hat{A}$	Total factor productivity.
$[0, N_t]$	Range of intermediate available at period t.
$x_t(z)$	Intermediate input of variety z.
$\sigma > 1$	Direct partial elasticity of substitution between each pair of intermediate goods.
F	Fixed cost of innovation.
$k_t = \frac{K_t}{\theta \sigma F N_t}$	Normalized capital, where $\theta = [1 - \frac{1}{\sigma}]^{1-\sigma}$.
k_c	Critical value of normalized capital, separating the growth regimes.
$\rho[b_{t+1}^i, b_t^i]$	Correlation of bequests of lineage i, between generation t and generation $t+1$.
$CV[b_t^i]$	Cross-sectional coefficient of variation of bequests at time t.
$0 \leq \gamma < 1$	Degree of envy when young.
$0 \leq \eta < 1$	Degree of envy when old,$\eta = \xi \gamma$, where $0 \leq \xi < 1$.
$0 \leq \lambda, \Lambda_t^i < 1$	Taste for leisure and fraction of time consumed as leisure when young by agent of lineage i, born at time t, respectively.
τ_b	Inheritance tax rate levied on the amount bequested.
τ_t	Time dependent uniform tax rate levied on output or value-added tax.
T_t	Tax revenue collected at time t.
κ	Proportion of government subsidy.

Table 1: Legend of key variables and parameters used.

List of Figures

List of Tables

Chapter 1

Mobility and inequality dynamics with idiosyncratic productivity shocks

1.1 Introduction

Growth and equity are central tenets in many modern societies. The pursuit of sustained economic growth has long been enshrined as a high priority mandate in many public offices, and a central theme in the study of macroeconomics. In recent years, this mandate has been tampered with words such as "equitable" and "inclusive". This is evident in the two goals which the World Bank Group[1] has set in 2013, for the world to achieve by 2030: (i) "End extreme poverty by decreasing the percentage of people living on less than $1.25 a day to no more than 3%", and (ii) "Promote shared prosperity by fostering the income growth of the bottom 40% for every country". Citing Easterly (2007) and Berg et al. (2012) respectively, the broadening of the measure of prosperity by World Bank (2013,

[1] World Bank (2013). Retrieved from http://www.worldbank.org/content/dam/Worldbank/document/WB-goals2013.pdf, on March 19, 2015.

pp. 23), from growth of GDP per capita to include distributional dimensions as a proxy for shared prosperity, arises from concerns that structural inequality may hamper the magnitude and duration of growth. Most recently, Piketty (2014) has garnered much attention and provoked discussions from academia[2] to mass media[3], with his book, a bestseller on Amazon[4], documenting the evolution of inequality over the last century. While traditional debates encompass both the normative and positive roles of inequality on growth and political stability, Piketty (2014) draws the spotlight on the role of the difference between returns on capital and growth rate, "r - g", in accentuating the wealth gap. On the other hand, Acemoglu and Robinson (2014) posit that institutional factors and their endogenous evolution are more important than the difference between returns to capital and growth rate, using the economic and political histories of South Africa and Sweden, and Levy and Temin (2007) do the same for the United States, attributing the changes in income distribution over time to a set of economic institutions including the Treaty of Detroit and the Washington Concensus.

In this chapter, by investigating an endogenous growth model with different

[2] AEA 2015 Conference: A Discussion of Thomas Piketty's "Capital in the 21st Century", and Intergenerational Mobility over Time and Across Locations: Establishing the Facts and Explaining the Mechanisms.
Retrieved from: https://www.aeaweb.org/aea/2015conference/program/preliminary.php, on October 27, 2014.

[3] For example, The Guardian (June 17, 2014). Article by Stuart Jeffries, "Piketty mania: how an economics lecture became the hottest gig in town".
Retrieved from http://www.theguardian.com/books/2014/jun/17/thomas-piketty-lse-capitalism-talk, on October 27, 2014.

[4] Amazon Best Sellers of 2014.
Retrieved from http://www.amazon.com/gp/bestsellers/2014/books, on October 27, 2014.

growth regimes, I hope to shed some light on the role of R and G on wealth accumulation and inequality. Specifically, an economy can grow via capital accumulation (neoclassical) or innovation (neo Schumpeterian) or both. The question of interest is: will different growth regimes, determined endogenously, lead to different cross-sectional wealth distributions and mobility for a household lineage under the general equilibrium framework? That is, I explore if the uni-directional impact of growth on inequality could result in different dynamics for inequality and mobility under different growth regimes. I formalize this idea with a model drawn from three existing literature: endogenous growth model with fixed innovation cost to generate equilibrium with growth trajectories arising from accumulation, innovation or both regimes, stochastic idiosyncratic productivity to generate an earnings distribution, and "joy-of-giving" bequest motive to propagate the distribution of wealth.

First, from the literature on growth, innovation induced by temporary monopoly profits mitigates the diminishing returns to scale in neoclassical production, resulting in sustainable growth. Endogenous deterministic cyclical growth, where the magnitude of economic activity arises from agents' optimization, may result from the innovation clustering via quality ladder growth (see Francois and Lloyd-Ellis (2003)), or via lumpy portfolio choice between capital accumulation or innovation due to cost of R&D financing (see Bental and Peled (1996), and Matsuyama (1999, 2001)). In this chapter, I extend Matsuyama (1999) by introducing idiosyncratic labor productivity shocks. This is because, according to

Nolan et al. (2014), earnings constituent the largest share of income, and hence the driver of the income inequality, which leads us to the second strand of the literature upon which I infer.

The evolution of inequality has been attributed to a myriad of factors ranging from technological changes that result in skills premium, globalization that leads to liberalization and enhanced mobility of the factor and product markets, social changes such as assortative marriages and the rise of single-parent households, as well as changes in fiscal policies such as cuts in marginal tax rates[5]. Using data from 30 rich countries over a period of 30 years, Nolan et al. (2014) found that while there is a prevalent upward trend in income inequality in most countries, cross country differences in the dynamics in terms of timing and direction exist.

Second, existing literature using uninsurable idiosyncratic risks to generate heterogeneity among agents includes Quadrini (2000) and Benhabib et al. (2011, 2014a,b), where uninsurable idiosyncratic income arising from labor earnings and/or stochastic investment returns, with or without stochastic death rates, are used to generate wealth distributions exhibiting power laws. A key purpose of this strand of research is to calibrate income and wealth distributions to match observed Pareto distributions in available datasets [6]. For example, with stochastic returns and a minimum investment threshold, Levy (2003) gen-

[5]IMF Staff from Fiscal Affairs Department, supervised by Sanjeev Gupta and Michael Keen (2014) on Fiscal Policy and Income Inequality, January 23.

[6]See Sornette (2006) for examples of stochastic difference equations to generate Pareto distribution.

erates a wealth distribution that converges to the empirically observed Pareto distribution. A mechanism to generate the Pareto distribution is by a continuous mixture of exponential distributions with gamma mixing weights (Bean (2001, pp. 235)), where the exponential distributions arise from the multiplicative changes in idiosyncratic investment returns of every lineage (physical or human) and the gamma weights from the aggregation across lineages where the arrival of death or the arrival rate of "creative destruction" is a Poisson process, as exhibited in Benhabib et al. (2014a) and Jones and Kim (2014) respectively. Alternatively, to generate heterogeneity: Matsuyama (2004, 2007, 2013) attains symmetry breaking with endogenous inequality in multiple equilibrium settings using imperfect credit markets or factor mobility; Lindquist (2004) generates inequality over business cycles with capital skill complementarity; while Zhang (2005) obtains a distribution of investment in human capital using bequest and mean preserving spread.

Third, in this model, the propagation of inequality is explored using "joy-of-giving" bequest, as well as correlated productivity shocks within a lineage. This is similar in spirit to De Nardi (2004), where the "joy of giving" bequest motive together with a stochastic death rate reinforces the wealth concentration, as well as Bossmann et al. (2007), who examine the role of bequests and taxation on bequests on the distribution of wealth under neoclassical growth. In models that use idiosyncratic stochastic capital (sometimes referred to as entrepreneurship or home production) returns, or stochastic bequest motives with jumps, or stochastic death rate to generate the wealth distribution from ex-ante identical agents, the mobility or correlation across generations is pure luck, and/or all is fair (lineage mobility is perfectly random), as with Levy (2003). This is one end of the continuum. However, in this thesis, by using the "joy-of-giving" bequest motive (here in Chapter 1), "status anxiety"(in Chapter 2), and endogenous labor-leisure choice (in Chapter 3), fiscal policies such as inheritance taxes and innovation subsidies (in Chapter 4), I study the direct impact of household optimization on lineage mobility and cross-sectional inequality under different growth regimes, in a variety of contexts. The focus of this work is thus not on the top percentiles, but on the mobility and wealth accumulation of the masses under different growth regimes, to explore the notion of shared prosperity and its plausibility. Additionally, with the stationarity assumption, the dichotomy of the cross-sectional inequality of inherited wealth versus the lineage persistence of inherited wealth under different growth regimes provides some clues to facilitate our understanding on the taxonomy of mobility and inequality under a stationary distribution.

Under plausible conditions, stationary distributions over period-2 growth cycles of inherited wealth trajectory with general equilibrium can be established. To the best of my knowledge, this is novel. The technical difficulty entails finding the conditions for the existence, and stationarity in cross-sectional distributions, along with non-uniqueness of equilibrium all at once. I sidestep this issue by making use of the Matsuyama (1999) model, which has proven the existence and non-uniqueness of equilibrium under suitable conditions, and then applying of time series analysis, via covariance-stationarity (relaxing the strict stationary assumption) to characterize the cross sectional distributions, using the correct normalization so that the distributions under both the Solow (accumulation) and Romer (innovation) regimes can be covariance-stationary and comparable.

Key findings in this chapter, under feasible stationarity conditions, are: in the absence of capital market frictions (for example investment quantum threshold and differentiated accessibility to investment), such that the returns on capital faced by agents are identical, the (i) lineage mobility as well as the (ii) asymptotic cross-sectional inequality under both the Solow and Romer steady states are equivalent; in addition, under i.i.d. idiosyncratic labor productivity shocks, the (i) lineage mobility and the (ii) asymptotic cross-sectional inequality under both the Solow and Romer steady states are also equivalent to that under period-2 cycles. The equivalence result between the two regimes under i.i.d. shocks is highly instructive for future research, and will be discussed in Section 1.4. With mean reverting correlated shocks (iii) lineage mobility is lower than that with i.i.d. shocks under the steady states, and (iv) cross-sectional wealth inequality is higher than that with i.i.d. shocks under both steady states and period-2 cycles. With lineage-correlated shocks, (v) the cross-sectional inequality depends on the nature of the fixed points. Relative to the exogenous parameters in the model, (vi) the higher the bequest motive, the lower the intergenerational mobility and the lower the cross-sectional inequality for both i.i.d. and correlated mean reverting labor productivity shocks at both steady states; (vii) the higher the elasticity of substitution for intermediate goods in the final sector (equivalently, the higher the capital share, the lower the monopoly margin of innovation in this model), the lower the intergenerational mobility and the lower the cross sectional inequality of bequests for i.i.d. shocks.

The rest of the chapter is organized as follows. Section 2 presents basic setup of the model. Section 3 discusses the aggregate, lineage and cross-sectional bequest distributions for the 2 steady states, as well as the period-2 cyclical fixed points. Section 4 examines the distribution equivalence under the 2 regimes. Section 5 explores other comparative statics. In particular, how the distribution of wealth and mobility changes with the bequest motive and price elasticity. Section 6 demonstrates how correlated productivity shocks may be incorporated into the basic theoretical framework, and the role of correlation on wealth distribution and mobility. Section 7 concludes.

1.2 The model

1.2.1 Agents' optimization

Time is discrete. The economy consists of overlapping generations of agents who live for 2 periods: the young period, and the old period. In each generation, there is a continuum of measure 1 of agents. In each family *i*, old agent gives birth to 1 child, keeping the population in the economy constant[7]. When old,

[7]Population changes can easily be incorporated by changing the measure of agents in each generation to n_t, instead of 1.

an agent retires, consumes his savings, and leave a bequest to his child. When young, an agent works inelastically to earn labor income, consumes, and saves.

Young agents within a given generation, t, are heterogenous in two aspects: their idiosyncratic productivity, l_t^i, and the inherited bequest b_t^i. I assume that the labor productivity is drawn from an independent and identical distribution on a positive support, with an expectation, $E[l_t^i] = \bar{l} \equiv 1$, and a finite variance, $Var[l_t^i] = \sigma_l^2$, across agents from the same generation, and for now, independent across time, that is, independent across generations from the same lineage. We shall introduce correlations within the same lineage in the later section of this chapter.

Preference of an agent i born at period t is:

$$U_t\left(c_t^i, d_{t+1,}^i b_{t+1,}^i\right) = \ln c_t^i + \beta\left[\ln d_{t+1}^i + \mu \ln b_{t+1}^i\right], \qquad (1.1)$$

where $l_t^i \sim \text{i.i.d.}(1, \sigma_l^2)$; c_t^i is the consumption when young, and d_{t+1}^i is the consumption when old, respectively, of an agent born at time t; b_{t+1}^i is the bequest left by an agent born at time, t, when old, to his immediate offspring; $\beta < 1$ is the subjective discount factor; and $0 \leq \mu < 1$ is the importance of bequest motive.

First period budget constraint for agent i is:

$$c_t^i + s_t^i = b_t^i + w_t^i \equiv y_t^i, \qquad (1.2)$$

where $w_t^i = w_t l_t^i$; w_t is the equilibrium wage rate for each unit of labor productivity; s_t^i is the amount saved when young; and y_t^i is the lifetime resource of agent i, born at time t.

Second period budget constraint for agent i is:

$$R_{t+1}s_t^i = d_{t+1}^i + b_{t+1}^i, \tag{1.3}$$

where R_{t+1} is the equilibrium gross rate of return on capital at time $t+1$.

The timing of the model is depicted by Figure 1.1.

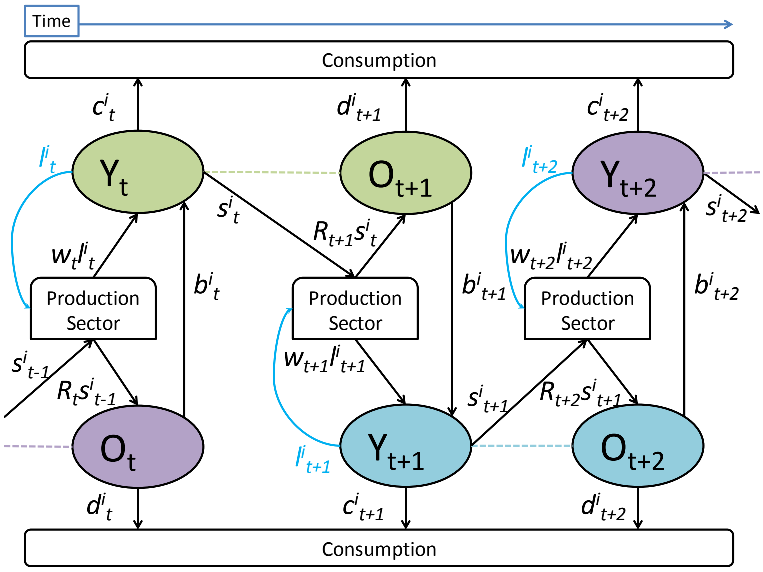

Figure 1.1: Timing of the model.

Optimal choices of agent i are[8,9]:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}y_t^i; \tag{1.4}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)}y_t^i; \tag{1.5}$$

[8] See Appendix A.1 for derivation.

[9] With log utility or other homothetic preferences, the propensity to save is independent of the lifetime income resources. This property allows us to aggregate easily the savings functions over all agents such that the aggregate evolution of the economy does not hinge of the distribution of wealth, while allowing the aggregate evolution of the economy to affect the wealth distribution along the transitional path.

$$d_{t+1}^{i} = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} y_t^i; \tag{1.6}$$

$$b_{t+1}^{i} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} y_t^i. \tag{1.7}$$

Using the macron (overbar accent) to represent the average, the optimal choices of the average agent are[10]:

$$\bar{s}_t = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} \bar{y}_t \equiv \Omega_1 \bar{y}_t; \tag{1.8}$$

$$\bar{c}_t = \frac{1}{1+\beta(1+\mu)} \bar{y}_t; \tag{1.9}$$

$$\bar{d}_{t+1} = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} \bar{y}_t; \tag{1.10}$$

$$\bar{b}_{t+1} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} \bar{y}_t \equiv R_{t+1}\Omega_2 \bar{y}_t. \tag{1.11}$$

1.2.2 Firms' optimization

To incorporate endogenous growth with both accumulation and innovation regimes, the production sector is based on Matsuyama (1999). In this economy, there is a final good (acting as the numeraire), which is produced competitively, using labor and intermediate products that are converted from unconsumed final goods in the previous period, K_{t-1} into a composite via a symmetric CES function. This composite of intermediates are combined with labor via a Cobb-Douglas production into the final goods as follows:

$$Y_t = \hat{A}(L)^{\frac{1}{\sigma}} \left\{ \int_0^{N_t} \left[x_t(z) \right]^{1-\frac{1}{\sigma}} dz \right\}, \tag{1.12}$$

where $\hat{A}$ is the total factor productivity; $x_t(z)$ denotes the intermediate input of variety z in period t; $\sigma \in (1, \infty)$ is the direct partial elasticity of substitution

[10]See Appendix A.1.

between each pair of intermediate goods; and $[0, N_t]$ is the range of intermediate available at period t.

The intermediate sector, at each period t, consist of "old" intermediates in the range $z \in [0, N_{t-1}]$, with $N_0 > 0$ in period 1, that are available competitively, and "new" intermediates in the range $z \in [N_{t-1}, N_t]$, that may be introduced for exclusive sales due to a 1-period patent protection. To produce a unit of "old" intermediate, a units of capital is needed. Hence "old" intermediates are priced at marginal cost: $p_t(z) \equiv p_t^c = aR_t$ for $z \in [0, N_{t-1}]$, where R_t is the gross return on capital at time t. To innovate a "new" variety, a fixed cost, F units of capital per variety, is incurred. Subsequently, a units of capital is required to produce each unit of "new" intermediate. With monopolistic competition, no barriers to entry or exit, 1 period monopolistic rent, and a constant price elasticity σ, "new" intermediates, if introduced, will be sold at price $p_t(z) \equiv p_t^m = \frac{a\sigma R_t}{(\sigma-1)}$ for $z \in [N_{t-1}, N_t]$. Since the final goods sector is perfectly competitive, we can derive the factor prices by their marginal products.

$$p_t^c = \left(1 - \frac{1}{\sigma}\right) \hat{A}(L)^{\frac{1}{\sigma}} (x_t^c)^{-\frac{1}{\sigma}}; \tag{1.13}$$

$$p_t^m = \left(1 - \frac{1}{\sigma}\right) \hat{A}(L)^{\frac{1}{\sigma}} (x_t^m)^{-\frac{1}{\sigma}}; \tag{1.14}$$

$$w_t = \frac{1}{\sigma} \left(\frac{Y_t}{L}\right). \tag{1.15}$$

From equations (1.13) and (1.14), the relative demand for "old" and "new" intermediates can be expressed as:

$$\frac{x_t^c}{x_t^m} = \left(\frac{p_t^c}{p_t^m}\right)^{-\sigma} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma}. \tag{1.16}$$

Innovation is driven by the existence of 1-period monopoly profits, specifically:

$$\pi_t = p_t^m x_t^m - R_t(ax_t^m + F). \tag{1.17}$$

Hence one-period monopoly power and free entry implies:

$$ax_t^m \leq (\sigma - 1)F; \quad N_t \geq N_{t-1}; \quad [ax_t^m - (\sigma - 1)F](N_t - N_{t-1}) = 0. \tag{1.18}$$

As capital is used to produce "old" and/or "new" intermediates, the resource constraint on capital in period t is:

$$K_{t-1} = N_{t-1}ax_t^c + (N_t - N_{t-1})(ax_t^m + F). \tag{1.19}$$

Combining equations (1.16), (1.18), and (1.19), the allocation of the capital resource is to the production of each variety of intermediate inputs, "old" and "new" is:

$$ax_t^c = ax_t^m \left(1 - \frac{1}{\sigma}\right)^{-\sigma} = \min\left\{\frac{K_{t-1}}{N_{t-1}}, \theta\sigma F\right\}; \tag{1.20}$$

and the dynamics for innovation is:

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \max\left\{0, \frac{K_{t-1}/N_{t-1} - \theta\sigma F}{\sigma F}\right\}; \tag{1.21}$$

where $\theta \equiv \left[1 - \frac{1}{\sigma}\right]^{1-\sigma}$, $\theta \in [1, e]$, $e = 2.71828...$, and θ is increasing with σ. From equation (1.21), for innovators to break-even and innovation to occur, the ratio of of available capital to variety, K_{t-1}/N_{t-1} must exceed a threshold level, $\theta\sigma F$.

Equation (1.12) can be written as:

$$Y_t = \hat{A}(L)^{\frac{1}{\sigma}} \left[N_{t-1}(x_t^c)^{1-\frac{1}{\sigma}} + (N_t - N_{t-1})(x_t^m)^{1-\frac{1}{\sigma}}\right]. \tag{1.22}$$

Substituting the complementary slack condition of innovation arising from the free entry and monopoly rent, allocation of capital resource to each variety of intermediate, as well as the dynamics of innovation, i.e. equations (1.18), (1.19), and (1.21) respectively, the total output is:

$$Y_t = \begin{cases} A\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}}\left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1}, \\ AK_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases} \tag{1.23}$$

where $A \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{\frac{1}{\sigma}}$. From equation (1.23), the economy operates in a "Solow regime" when $K_{t-1}/N_{t-1} \leq \theta\sigma F$, where no innovation occurs, and the aggregate production function exhibits diminishing returns to capital as with the standard neoclassical growth model. On the other hand, the economy operates in a "Romer regime" when innovation occurs, where the production of final goods resembles an "AK" growth model as in Rivera-Batiz and Romer (1991).

1.3 Equilibrium steady state and dynamics

1.3.1 Derivation of equilibrium capital stock

By aggregating the inelastic labor from the young, with measure 1 of young agents and $E(l_t^i) = 1$, the labor market clearing condition is:

$$L_t = L = \int_0^1 l_t^i \, di = 1. \tag{1.24}$$

Aggregating the savings from the young, the capital market clearing condition is:

$$K_t = \int_0^1 s_t^i \, di. \tag{1.25}$$

From the competitive Cobb-Douglas final goods sector, the wage share of the economy is:

$$w_t L = \frac{1}{\sigma} Y_t, \tag{1.26}$$

while the capital share of the economy is:

$$R_t K_{t-1} = \left(1 - \frac{1}{\sigma}\right) Y_t. \tag{1.27}$$

Since we have measure 1 of agents in each generation, we can obtain an equivalent expression for the aggregate savings and the aggregate capital stock of the economy, from the optimization choices of the average agent. Specifically:

$$\begin{aligned}
\bar{s}_t &= \frac{\beta(1+\mu)}{1+\beta(1+\mu)} \bar{y}_t \\
&\equiv \Omega_1 \bar{y}_t \\
&\equiv \Omega_1 [\bar{b}_t + w_t] \\
&= \Omega_1 w_t + \Omega_1 \left(\frac{\mu}{1+\mu}\right) R_t \bar{s}_{t-1}.
\end{aligned} \tag{1.28}$$

The unconsumed final goods at time t, will form the available capital stock for production at time $t+1$. Thus by integrating the savings across all young agents, the aggregate savings in the economy is:

$$K_t = S_t = \bar{s}_t = \Omega_1 w_t L + \Omega_1 \left(\frac{\mu}{1+\mu}\right) R_t K_{t-1}, \tag{1.29}$$

and the capital stock as a function of the final output of the economy can be derived by substituting out the factor shares, (1.26) and (1.27), as follow:

$$K_t = \Omega_1 \left(\frac{1}{\sigma}\right) Y_t + \Omega_1 \left(\frac{\mu}{1+\mu}\right)\left(1-\frac{1}{\sigma}\right) Y_t$$
$$= \Omega_1 \Omega_3 Y_t, \tag{1.30}$$

where $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$ and $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$. That is, this economy saves a constant fraction of its output for the next period.

From the savings process (1.30), in conjunction with with the dynamics of innovation (1.21) and the total output (1.23), the unique equilibrium path for any initial condition, K_0 and N_0 is pinned down.

$$\frac{K_t}{\Omega_1\Omega_3} = Y_t = \begin{cases} A\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}}\left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1}, \\ AK_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases} \tag{1.31}$$

where $A \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{\frac{1}{\sigma}}$.

This dynamical system can be normalized by the range of intermediate goods to a 1-dimensional map. Define $\Phi^n(k) \equiv \Phi\left(\Phi^{n-1}(k)\right)$, $\Phi^1(k) \equiv \Phi(k)$, and

$$k_t \equiv \frac{K_t}{\theta\sigma F N_t}.$$

Equilibrium path for initial condition k_0 is given by the sequence $\{\Phi^t(k_0)\}$:

$$k_t = \Phi(k_{t-1}) \equiv \begin{cases} G\left(k_{t-1}\right)^{1-\frac{1}{\sigma}} & \text{if} \quad k_{t-1} \leq k_c = 1, \\ \frac{Gk_{t-1}}{1+\theta(k_{t-1}-1)} & \text{if} \quad k_{t-1} \geq k_c = 1, \end{cases} \tag{1.32}$$

where:

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$[11], $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$, $G \equiv \Omega_1\Omega_3 A$, $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$, $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$, and $A \equiv \frac{\hat{A}}{a}\left(\frac{aL}{\theta\sigma F}\right)^{\frac{1}{\sigma}}$.

[11] $k = 0$ is excluded from the domain in the mapping $k_t = \Phi(k_{t-1})$, as $\Phi'(0) > 1$, making $k = 0$ a repelling fixed point and hence trivial.

1.3.2 Steady state

The mapping $k_t = \Phi(k_{t-1})$ with domain $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ in (1.32) has a unique steady state where $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$ is constant over time. Denote the fixed point as k^* such that $k^* = \Phi(k^*)$ if $k^* < k_c \equiv 1$ and k^{**} such that $k^{**} = \Phi(k^{**})$ if $k^{**} > k_c \equiv 1$.

Lemma 1.1. *G determines the regime under which the steady state lies, hence the potential growth of the economy*[12]*. See Figure 1.2.*

(i) If $G < 1$, the steady state is in the Solow regime, with $k_t = k^* < k_c \equiv 1$.

(ii) If $G > 1$, the steady state is in the Romer regime, with $k_t = k^{**} > k_c \equiv 1$.

Proof. From (1.23) (1.30), $K_t = \Omega_1\Omega_3 Y_t = \Omega_1\Omega_3 A K_{t-1} = G K_{t-1}$. Thus, the growth potential of the economy is determined by the parameter $G = \Omega_1\Omega_3 A$.

At steady state:

(i) If $k_t = k^* < k_c \equiv 1$ (Solow regime), using (1.32) and (1.21), $N_t = N_{t-1}$ and $K_t = K_{t-1}$. At this steady state, there is no innovation, and all goods are supplied competitively. The economy does not grow in the long run due to diminishing returns, as with the neoclassical stationary path. From (1.32), $k^* = (\Omega_1\Omega_3 A)^{\sigma} \equiv G^{\sigma}$. The condition for the existence of this stationary path is $G \equiv \Omega_1\Omega_3 A < (k_c)^{\frac{1}{\sigma}} \equiv (1)^{\frac{1}{\sigma}} = 1$.

[12] See Gardini et al. (2008, pp. 543) for a discussion on border-collision bifurcation, that results in five different regimes depending on the value of the parameter σ, when $G = 1$.

(ii) If $k_t = k^{**} > k_c \equiv 1$, from (1.32), the steady state is $k = k^{**} \equiv 1 + \frac{(G-1)}{\theta} > k_c = 1$ (Romer regime). Thus, the condition for the existence of this stationary path is $G \equiv \Omega_1 \Omega_3 A > 1$. At this steady state, innovation occurs, with K and N growing at the same rate G, along a balanced growth path.

■

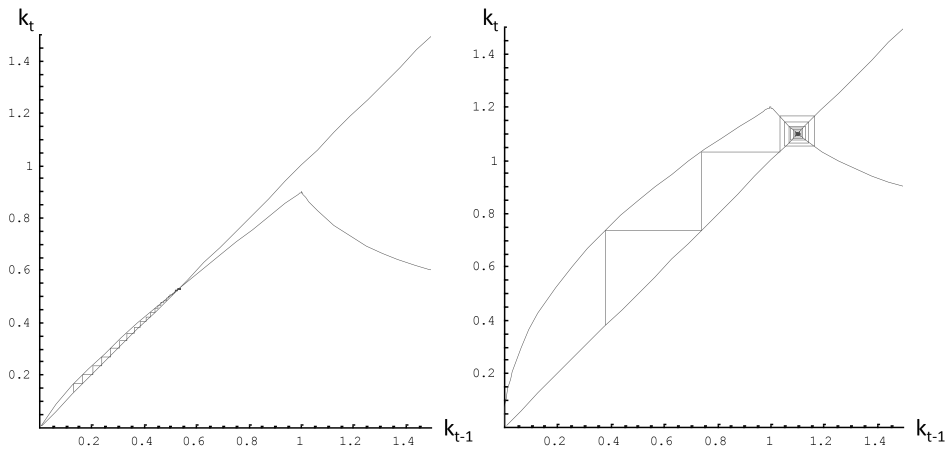

Figure 1.2: The unique steady state (if $G < 1$: Solow, and if $G > 1$: Romer).

1.3.3 Aggregate dynamics

For clarity and completeness, I restate 2 propositions from Matsuyama (1999) on aggregate dynamics in this section. The focus of this thesis remains the characterization of the lineage and cross-sectional wealth distributions that evolve jointly with the aggregate dynamics; and how these distributions may change with the structure of productivity shocks, bequest motives, status anxiety, elastic labor, inheritance tax, or innovation subsidies.

Proposition 1.1. *Matsuyama (1999, pp. 343, Proposition 1)*[13].

(i) *If* $G < 1$*, then for any* $k_0 \in \mathbb{R}_+$*,* $k_t < k_c$ *for all t, and* $\lim_{t\to\infty} k_t = k^*$.

The economy converges to a neoclassical stationary path. (Solow regime)

(ii) *If* $G > \theta - 1$*, then for any* $k_0 \in \mathbb{R}_+$*, there exists a* t' *such that* $\{k_t; t \geq t'\} \subset [k_c, \Phi(k_c)]$ *and* $\lim_{t\to\infty} k_t = k^{**}$.

The economy oscillates around and eventually converges to a balanced growth path. (Romer regime)

(iii) *If* $1 < G < \theta - 1$*, there are period-2 cycles*[14]*;* k_t *fluctuates forever between Solow and Romer regimes for almost all initial conditions, that is for* $k_0 \in \mathbb{R}_+ \backslash D$*, where D is at most countable subset of* $\mathbb{R}_+$.

Proof. See Matsuyama (1999, pp. 344).

I shall provide a sketch of the proof by graphical analysis here.

(i) With $k^* < k_c \equiv 1$ (Solow regime), the slope of the mapping, $k_t = \Phi(k_{t-1})$, is $\Phi^{'}(k_t) = (1 - \frac{1}{\sigma})G(k_{t-1})^{-\frac{1}{\sigma}} > 0$. As $k_{t-1} \to 0$, $\Phi^{'}(k_t) > 1$, and in the steady state, $k^* = G^{\sigma}$, $\Phi^{'}(k_t) = (1 - \frac{1}{\sigma}) < 1$ because $\sigma > 1$. Consequently,

[13] See Matsuyama (1999, pp. 344–345) and Gardini et al. (2008) for a discussion on other possible trajectories. In particular, Gardini et al. (2008) proved that while chaotic regimes may exist, only a fixed point or a cycle of period 2 are possible for stable cycles.

[14] Matsuyama (1999, pp. 346) suggests plausible empirical specification of parameters for cycles to occur.

$k^*$is a globally stable steady state, and for any $k_0 > 0$, the sequence $\{k_t\}_{t=0}^{\infty}$ converges towards k^*.

(ii) $k^{**} > k_c \equiv 1$ (Romer regime), the slope of the mapping, $k_t = \Phi(k_{t-1})$, is $\Phi^{'}(k_t) = \frac{G[1+\theta(k_{t-1}-1)]-Gk_{t-1}\theta}{[1+\theta(k_{t-1}-1)]^2} = \frac{G(1-\theta)}{[1+\theta(k_{t-1}-1)]^2}$. At the steady state, by substituting $k_t = k^{**} \equiv 1 + \frac{(G-1)}{\theta} > k_c = 1$, we have $\Phi^{'}(k^{**}) = \frac{(1-\theta)}{G} < 0$. Since θ varies from 1 to $e = 2.71828...$, as σ varies from 1 to ∞, if $G > \theta - 1$, then $-1 < \Phi^{'}(k^{**}) < 0$. Then, if $G > \theta - 1$ and $k_c < \Phi^2(k_c)$, the steady state is globally stable. For any initial state (even in the Solow regime), the economy will propagate to the Romer regime, remain there, and converge with oscillation towards the steady state (balanced growth path).

(iii) If $1 < G < \theta - 1$, then $\Phi^{'}(k^{**}) < -1$, and k^{**} is locally unstable. It can be shown that $1 < G < \theta - 1$ is equivalent to $\Phi^2(k_c) < k_c < \Phi(k_c)$; thus $[\Phi^2(k_c), \Phi(k_c)]$ represents the trapping region, which includes both the Solow and Romer regimes, and eventually, the economy grows through cycling back and forth between the 2 regimes. See Figure 1.3.

■

Proposition 1.2. *Matsuyama (1999, pp. 346, Proposition 2). Let g_x be the gross growth rate of variable X. Along period-2 cyles:*

(i) $g_N = 1 < G < G(k^L)^{-\frac{1}{\sigma}} = g_K = g_Y$ *in the Solow regime;*

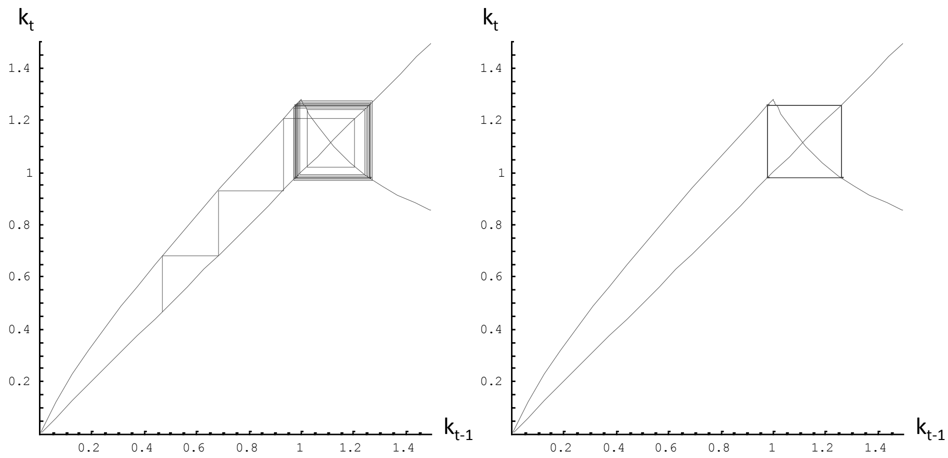

Figure 1.3: The existence of period-2 cycles with $1 < G < \theta - 1$.

(ii) $g_N = 1 + \theta(k^H - 1) > G = g_K = g_Y$ *in the Romer regime;*

(iii) $g_N = g_K = g_Y = (1 + \theta(k^H - 1))^{\frac{1}{2}} = G(k^L)^{-\frac{1}{2\sigma}} > G$ *over the cycles.*

where $k^H = \Phi(k^L) = G(k^L)^{1-\frac{1}{\sigma}}$*, and* $k^L = \Phi(k^H) = \frac{Gk^H}{1+\theta(k^H-1)}$.

Proof. As stated in Matsuyama (1999), by substituting the iterated fixed points of the period-2 cycles: $k_{t-2} = k^H$, $k_{t-1} = k^L$ and $k_t = k^H$ into the variety expansion and total output equations, (1.21) and (1.32), where $k^H = \Phi(k^L) = G(k^L)^{1-\frac{1}{\sigma}}$, and $k^L = \Phi(k^H) = \frac{Gk^H}{1+\theta(k^H-1)}$. $\frac{N_t}{N_{t-1}} = 1$; $\frac{N_t+1}{N_t} = 1 + \theta(k^H - 1)$;
$\frac{Y_t}{Y_{t-1}} = \frac{K_t}{K_{t-1}} = \left(\frac{k^H}{k^L}\right)\left(\frac{N_t}{N_{t-1}}\right) = \frac{k^H}{k^L}$;
$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \left(\frac{k^L}{k^H}\right)\left(\frac{N_{t+1}}{N_t}\right) = \left(\frac{k^L}{k^H}\right)[1 + \theta(k^H - 1)]$. ■

Two features of this endogenous cyclical growth model are worth noting.

First, from the perspective of the production function, the cycle length of this model corresponds to the duration of innovation rent. In practice, depending on the nature of the type of intellectual property (e.g. patents, industrial designs, geographical indications or trademarks), the duration of innovation protection varies. For example, the available term of patent protection for inventions must be available for a minimum of 20 years from the date of filing the patent application under the TRIPS Agreement, administered by the WTO[15]. On the other hand, from the perspective of utility maximizing agents, the length of a period in a 2-period OLG model of life-cycle savings is typically 30 years or longer[16]. Hence, there is a drawback of using a 2-period life cycle preference for analytical tractability as the period length may be incongruent with the patent length or period of monopoly rent in Matsuyama (1999)'s model. Although Matsuyama extended his model, Matsuyama (1999), to an infinite horizon representative agent economy in Matsuyama (2001), the infinite horizon representation is not adopted here as I am keen to study intergenerational mobility, and due to the finiteness of a human life, the rate of convergence matters to each realizable path of a human life. Furthermore, modeling with infinitely-lived households implies that parents value their children's utility and vice-versa, with perfect 2-way altruism and risk-sharing, unless wedges (frictions) are introduced; for example,

[15]WTO - TRIPS Agreement (Article 33).
Retrieved from https://www.wto.org/english/tratop_e/trips_e/intel2_e.htm#patents, on November 8, 2014.

[16]For example, in Jouvet et al. (2010), the length of a full period of their 2-period OLG model is 40 years.

enforcing agents to make their inter-temporal consumption versus savings decision before knowing the realization of their labor productivity shocks in each period. With the warm-glow ("joy-of-giving") bequest motive, parents value bequest itself (as a reduced form representation), and risk-sharing is unilateral and imperfect. In Chapter 2, with the presence of "Status Anxiety", the "conflicts and solidarity across generations"[17], absent in the infinite horizon models, are further highlighted using OLG models. Thus, bringing the model to data, to match the patent length, one might have to fine-tune the number of periods each agent lives [18]. In any case, this model explains mid-term economic fluctuations, not to the short-term business cycle volatility.

Second, growth and innovation are not synchronized[19]. From Proposition 1.2 part (i) and (ii), for period-2 cycles, capital accumulation and total output growth are higher under the Solow regime compared to the Romer regime. This is because the economy reaps the rewards of innovation only after the innovation ends, when the market becomes competitive. Yet, to avoid fate of diminishing returns, we need innovation of new goods, so that the economy can experience

[17]Phrase quoted from David de la Croix's website.
Retrieved from http://perso.uclouvain.be/david.delacroix/olg.html, on October 22, 2015.

[18]Examples include: Bouzahzah et al. (2002, pp. 2096), "Agents are homogenous within generations and live for six periods of life(i.e. from age 18 to age 78), each of them representing 10 years."; and Jouvet et al. (2010, pp. 14), "In our model economy agents live for 9 periods. Therefore we interpret one model period to last 6 years."

[19]In Wälde (2005), both capital accumulation and R&D occur simultaneously, where individuals allocate resources between capital accumulation and R&D as a portfolio. Then, using the amount of resources allocated for R&D to drive the arrival rate of a Poisson process that determines the success rate of innovation in the economy as whole, he generates endogenous stochastic cycles where innovation is procyclical in contrast with Matsuyama (1999) endogenous deterministic cycles where innovation is countercyclical.

indefinite growth. Thus from Proposition 1.2 part (iii), even though cycles induce volatility into the economy, they augment growth over the cycle compared to the balanced growth path.

Proposition 1.3. *An economy with a higher bequest motive, μ, has a higher growth potential.*

Proof. Market clearing (equations (1.26), (1.27) and (1.29)), and the consumer optimization (equation (1.30)) imply $S_t = K_t = \Omega_1\Omega_3 Y_t$. Together with the final output function (1.23),

$$Y_t = \begin{cases} A\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}}\left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1} \\ AK_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases}$$

along the balanced growth path, $K_t = \Omega_1\Omega_3 Y_t = \Omega_1\Omega_3 A K_{t-1} = G K_{t-1}$, where $G \equiv$ gross growth rate.

$$\frac{\partial G}{\partial \mu} = A\left[\Omega_3 \frac{\partial \Omega_1}{\partial \mu} + \Omega_1 \frac{\partial \Omega_3}{\partial \mu}\right] > 0, \tag{1.33}$$

where $\Omega_1 = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$, $\Omega_3 = \frac{1+\sigma\mu}{\sigma(1+\mu)}$, $\frac{\partial \Omega_1}{\partial \mu} = \frac{\beta}{[1+\beta(1+\mu)]^2} > 0$, and $\frac{\partial \Omega_3}{\partial \mu} = \frac{\sigma-1}{\sigma(1+\mu)^2} > 0$. ■

In this model, to perpetuate growth indefinitely, the innovation of new intermediate goods is necessary to circumvent the diminishing returns resulting from growth via factor accumulation à la Neoclassical models. As accumulated capital from previous period is used to produce "old" and/or "new" intermediates in equation (1.19), an increase in the bequest motive would result in a higher aggregate savings rate, hence a higher rate of capital accumulation that relaxes the resource constraint governing the innovation process.

Proposition 1.4. *The normalized steady-state capital stock to variety ratio, k_t, is increasing in the degree of bequest motive μ, in both regimes.*

Proof. From equation (1.32):

when $G < 1$, $k^* = (\Omega_1 \Omega_3 A)^\sigma \equiv G^\sigma$;

when $G > 1$, $k = k^{**} \equiv 1 + \frac{(G-1)}{\theta}$.

$$\frac{\partial k^*}{\partial \mu} = \frac{\partial k^*}{\partial G} \times \frac{\partial G}{\partial \mu} = (\sigma G^{\sigma-1}) A \left[\Omega_3 \frac{\partial \Omega_1}{\partial \mu} + \Omega_1 \frac{\partial \Omega_3}{\partial \mu} \right] > 0. \tag{1.34}$$

$$\frac{\partial k^{**}}{\partial \mu} = \frac{\partial k^{**}}{\partial G} \times \frac{\partial G}{\partial \mu} = \frac{1}{\theta} A \left[\Omega_3 \frac{\partial \Omega_1}{\partial \mu} + \Omega_1 \frac{\partial \Omega_3}{\partial \mu} \right] > 0. \tag{1.35}$$

■

The higher the bequest motive, the higher the marginal utility from savings for each agent. Therefore the higher the aggregate wealth accumulation. Thus, in the presence of endogenous growth, this increase in bequest motive can increase the growth potential of the economy.

1.3.4 Lineage bequest evolution at Solow steady state

Moving on, from the aggregate dynamics of the economy, the lineage bequest evolution of agents from family i, across time can be tracked. This is an advantage of using the time series analysis approach to characterize a stationary distribution if variances of the shocks are finite. We can track or forecast the intergenerational impact of a specific productivity shock through the lineage using the impulse-response function. To do so, the law of motion of bequest for the lineage is derived, then inverted, as shown below.

Let the single asterisk superscript represents the Solow steady state. At the Solow steady state, $k^* = G^\sigma < 1$. Let $Y_{t+1} = Y_t \equiv Y^*$, $R_{t+1} = R_t \equiv R^*$, and $K_{t+1} = K_t \equiv K^*$. From (1.31), in the Solow regime,

$$R_t = \frac{\partial Y_t}{\partial K_{t-1}} = \left(1 - \frac{1}{\sigma}\right) A \left(\theta \sigma F N_{t-1}\right)^{\frac{1}{\sigma}} \left(K_{t-1}\right)^{-\frac{1}{\sigma}} = \left(1 - \frac{1}{\sigma}\right) A \left(k_{t-1}\right)^{-\frac{1}{\sigma}}, \tag{1.36}$$

where $k_t = \frac{K_t}{\theta \sigma F N_t}$. At Solow steady state, $R^* = \left(1 - \frac{1}{\sigma}\right) A \left(k^*\right)^{-\frac{1}{\sigma}}$. From equation (1.32),

$$k^* = G(k^*)^{\frac{\sigma-1}{\sigma}} \Rightarrow \frac{k^{*\sigma}}{k^{*\sigma}} = \frac{G(k^*)^{\sigma-1}}{k^{*\sigma}} \Rightarrow 1 = G^\sigma (k^*)^{-1} \Rightarrow k^* = G^\sigma = (\Omega_1 \Omega_3 A)^\sigma. \tag{1.37}$$

$$R^* = \left(1 - \frac{1}{\sigma}\right) A \left(k^*\right)^{-\frac{1}{\sigma}} = \left(1 - \frac{1}{\sigma}\right) A \left(\Omega_1 \Omega_3 A\right)^{-1} = \frac{\left(1 - \frac{1}{\sigma}\right)}{\Omega_1 \Omega_3}. \tag{1.38}$$

At Solow steady state, the gross rate of return on capital, R^*, is the ratio of capital share of the economy, $\left(1 - \frac{1}{\sigma}\right)$, to the fraction of lifetime resources saved by the young, $\Omega_1 \Omega_3$. Using the bequest of an agent of lineage i, equation (1.7), as well as the capital and labor shares, equations (1.26) and (1.27), we derive the law of motion of the bequests for an agent of lineage i.

$$\begin{aligned} b_{t+1}^i &= \frac{R_{t+1} \beta \mu}{1 + \beta(1 + \mu)} (y_t^i) \\ &= \frac{R_{t+1} \beta \mu}{1 + \beta(1 + \mu)} (b_t^i + l_t^i w_t) \\ &= \frac{R_{t+1} \beta \mu}{1 + \beta(1 + \mu)} (b_t^i + l_t^i \frac{1}{\sigma} Y_t). \end{aligned} \tag{1.39}$$

At Solow steady state, normalizing[20] by the total output, Y_{t+1} and rearranging,

$$\frac{b^i_{t+1}}{Y_{t+1}} - \frac{R^*\beta\mu}{1+\beta(1+\mu)}\frac{b^i_t}{Y_t} - \frac{R^*\beta\mu}{\sigma[1+\beta(1+\mu)]} = \frac{R^*\beta\mu}{1+\beta(1+\mu)}\frac{1}{\sigma}[l^i_t - \bar{l}_t], \quad (1.40)$$

$$\Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{b^i_t}{Y_t} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3\sigma[1+\beta(1+\mu)]} = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma}[l^i_t - \bar{l}_t] \quad (1.41)$$

The ratio of bequest to total output inherited by a young agent from lineage i at any time t, follows a first order non-homogenous difference equation.

Following, to facilitate the discussion on the law of motion of bequests for any lineage, I shall introduce 2 basic concepts that are used in time series analysis. (The standard notations used in time series analysis literature such as ρ for autocorrelations, γ for autocovariances, ϕ for the coeffcients for AR (autoregressive) processes, and θ for the coefficients for MA (moving average) processes are used here[21].)

Definition 1.1. *A time series process, Y_t is said to be covariance-stationary or weakly stationary if neither the mean μ_t, nor the autocovariances γ_{jt} depend on*

[20]While not necessary for the characterization of the law of motion for bequests under the Solow steady state, normalization is needed for the characterization of the Romer steady state, the period-2 cycles, as the total output is growing under the latter 2 trajectories. Normalization applies along the transition to the fixed point(s) as well.

[21]However, to minimize confusion, $E[l^i_t] = \bar{l} = 1$ and σ_l, with the subscript l, are used to represent the mean and standard deviation of the idiosyncratic labor productivity draws respectively, as μ is used to denote the bequest motive of agents' preference, and σ is used to denote the final goods sector's demand for each intermediate in the production function.

the date t[22]*. That is,*

$$\begin{array}{ll} E[Y_t] = E[Y] & \textit{for all } t \\ E[(Y_t - E[Y])\left(Y_{t-j} - E[Y]\right)] = \gamma_j & \textit{for all } t \textit{ and any } j. \end{array} \tag{1.42}$$

If a process is covariance-stationary, the covariance between Y_t and Y_{t-j} is dependent only on the length of time separating the observations, and is independent of t, the date of the observation. Furthermore,

$$\gamma_j = \gamma_{-j} \quad \text{for all integers j.} \tag{1.43}$$

Definition 1.2. *A time series process, Y_t is said to be strictly stationary if, for any values of $j_1, j_2, ...j_n$, the joint density of $(Y_t, Y_{t+j_1}, Y_{t+j_2}, ..., Y_{t+j_n})$ depends only on the intervals separating the dates ($j_1, j_2, ...j_n$) and not the date itself (t)*[23].

If a process is strictly stationary with finite second moments, then it must be covariance-stationary. However, a covariance-stationary process need not be strictly stationary[24], e.g. when higher moments such as $E(Y_t^3)$ are a function of time. In addition, there are strictly stationary processes that are not covariance stationary. For example, a sequence of i.i.d. Cauchy random variables, while strictly stationary, is not covariance stationary since no joint moments exist.

Now define

$$z_{t+1}^{*i} = \frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_1^*}, \tag{1.44}$$

[22]See Hamilton (1994, pp. 45).
[23]See Hamilton (1994, pp. 46).
[24]See Wei (2006, pp. 9) for an instructive example, that resembles the trajectory of a period-2 cycle in this thesis.

where $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma} = \delta_1$, and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. Substituting (1.44) into equation (1.40), we get a first order homogenous difference equation for z_{t+1}^{*i} . That is,

$$z_{t+1}^{*i} - \phi_1^* z_t^{*i} = [l_t^i - \bar{l}_t] \equiv u_{t+1}^i \sim i.i.d.(0, \sigma_l^2). \tag{1.45}$$

Following Conlisk (1974) and Bossmann et al. (2007), I use the correlation of parent-child inherited wealth within a lineage as a measure of the degree of social immobility, and coefficient of variation as a measure of cross-sectional inequality respectively.

Proposition 1.5. *Since* $|\phi_1^*| \equiv \left|\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\right| < 1$, z_{t+1}^{*i} *is a covariance-stationary AR(1) process*[25].

(i) The expected bequest inherited by a young agent of lineage i, *as a ratio of the total output, at time,* $t+1$, *is:*

$$E\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\delta_1}{1-\phi_1^*} = \frac{\frac{1}{\sigma}\phi_1^*}{1-\phi_1^*}. \tag{1.46}$$

(ii) The variance of bequest inherited by a young agent of lineage i, *as a ratio of the total output, at time,* $t+1$, *is:*

$$Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\sigma_l^2\Omega_4^2}{1-\phi_1^{*2}} = \frac{\sigma_l^2}{1-\phi_1^{*2}}\left(\frac{1}{\sigma}\phi_1^*\right)^2. \tag{1.47}$$

(iii) The intergenerational mobility can be measured by the covariance or the correlation coefficient, to gauge the extent of intergenerational transmis-

[25]The distribution of z_{t+1}^{*i} given $z_t^{*i}, z_{t-1}^{*i}, z_{t-2}^{*i}, \ldots$ is identical to the distribution of z_{t+1}^{*i} given z_t^{*i}. That is, the AR(1) process is a Markov process.

sion of inequality. They are respectively:

$$Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right]=\Omega_4^2\frac{\sigma_l^2\phi_1^*}{1-\phi_1^{*2}}=\frac{\sigma_l^2\phi_1^*}{1-\phi_1^{*2}}\left(\frac{1}{\sigma}\phi_1^*\right)^2. \tag{1.48}$$

$$\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right]=\rho_1=\phi_1^*=\frac{R^*\beta\mu}{1+\beta(1+\mu)}=\frac{\frac{\left(1-\frac{1}{\sigma}\right)}{\Omega_1\Omega_3}\beta\mu}{1+\beta(1+\mu)}=\frac{(\sigma-1)\mu}{1+\sigma\mu}. \tag{1.49}$$

In general, for AR(1) processes,

$$Cov\left[\frac{b^i_{t+k}}{Y_{t+k}},\frac{b^i_t}{Y_t}\right]=\phi^*Cov\left[\frac{b^i_{t+k-1}}{Y_{t+k-1}},\frac{b^i_t}{Y_t}\right]=\frac{\sigma_l^2\phi_1^{*k}}{1-\phi_1^{*2}}\left(\frac{1}{\sigma}\phi_1^*\right)^2\forall k\geq 1. \tag{1.50}$$

$$\rho\left[\frac{b^i_{t+k}}{Y_{t+k}},\frac{b^i_t}{Y_t}\right]=\rho_k=\phi_1^*\rho_{k-1}=\left(\frac{(\sigma-1)\mu}{1+\sigma\mu}\right)^k\forall k\geq 1. \tag{1.51}$$

Proof. See Hamilton (1994, Appendix 3.A. pp. 69) for conditions for covariance-stationarity for AR(1)[26]. In essence, equation (1.45) can be expressed as $(1-\phi_1^*L)z^{*i}_{t+1}=u^i_{t+1}$, where L is the lag operator, and for z^{*i}_{t+1} to be stationary, the root of the characteristic equation, $(1-\phi_1^*L)=0$, must lie outside the unit circle, hence $|\phi_1^*|<1$.

$$\begin{aligned}\phi_1^*&=\frac{R^*\beta\mu}{1+\beta(1+\mu)}=\frac{\frac{\left(1-\frac{1}{\sigma}\right)}{\Omega_1\Omega_3}\beta\mu}{1+\beta(1+\mu)}\\&=\frac{(\sigma-1)(1+\mu)\beta\mu}{\Omega_1(1+\sigma\mu)[1+\beta(1+\mu)]}\\&=\frac{[1+\beta(1+\mu)](\sigma-1)(1+\mu)\beta\mu}{\beta(1+\mu)(1+\sigma\mu)[1+\beta(1+\mu)]}\\0\leq\phi_1^*&=\frac{(\sigma-1)\mu}{1+\sigma\mu}<1.\end{aligned} \tag{1.52}$$

[26]Although Hamilton (1994, Appendix 3.A. pp. 69) proves the convergence for the MA(∞) process, an AR(1) process can be viewed as an MA(∞) process (Hamilton (1994, pp. 53)).

The moments for this AR(1) process can be calculated from the difference equation (1.44) directly, assuming covariance stationarity (Hamilton (1994, pp. 53–56)), or by applying the generalized autocovariance generating function for autoregressive moving average, ARMA, processes, together with some algebraic manipulations (Hamilton (1994, pp. 61–63)). That is, for the AR(1) process in equation (1.45), $E\left[z_{t+1}^{*i}\right] = 0$, $Var\left[z_{t+1}^{*i}\right] = \gamma_0 = \sigma_l^2$, $Cov\left[z_{t+j}^{*i}, z_t^{*i}\right] = \gamma_j = \phi_1^{*j}\gamma_0 = \frac{\phi^{*j}\sigma_l^2}{1-\phi_1^{*2}}$, and $\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1^{*j}\gamma_0}{\gamma_0} = \phi_1^{*j}$. ■

The general condition for covariance stationarity for an AR(1) process, $|\phi_1^*| < 1$ implies that $\frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]} < 1$ in this model, where G_{t+1} denotes the gross growth rate of final output. While shown on the preceding proof that this condition is not binding in the current model, it is worth discussing what this covariance-stationarity condition means from the point of view of agents' optimization and the production structure, both under the Solow regime and Romer regime. From equation (1.7) the coefficient of the AR(1) corresponds to the fraction of lifetime resource (the agent gets from wages and from inheritance when young) that is bequested to the next generation. Thus, for a covariance-stationary bequest distribution to exist under the Solow steady state, agents in the economy should not pass on more wealth than the sum of wealth they have inherited and wage income they have earned when they are young, as in the style of Bill Gates[27]. That is, from the perspective of agents' optimization,

$$b^i_{t+1} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} y^i_t,$$
$$|\phi_1^*| = \left|\frac{R^*\beta\mu}{1+\beta(1+\mu)}\right| < 1 \Rightarrow b^i_{t+1} < y^i_t. \tag{1.53}$$

From the economy's perspective, the condition $|\phi_1^*| < 1$ implies that agents' subjective discount rate for the future is large (β is small) relative to the gross return on capital, which is endogenously determined. That is:

$$R^* < \frac{1+\beta(1+\mu)}{\beta\mu},$$
$$\Rightarrow \frac{\overbrace{(1-\frac{1}{\sigma})}^{\text{capital share}}}{\underbrace{\Omega_1\Omega_3}_{\text{savings rate out of lifetime resource of young}}} < \frac{1+\beta(1+\mu)}{\beta\mu}.$$

In Bossmann et al. (2007), the mobility, proxied by the correlation of parent-child inherited wealth, depends on time along the equilibrium path to the long-run steady-state's correlation value. In this chapter, by normalizing the inherited wealth to the total output, the intergenerational mobility ρ_1, determined by the autoregressive coefficient ϕ_1^*, is constant throughout the equilibrium path to the steady-state. Wan and Zhu (2012) do not characterize the intergenerational mobility as they generalize Bossmann et al. (2007) by relaxing the assumption of finite variance of the labor productivity shocks; thereupon they are silent on the lineage bequest evolution.

[27]See Daily Mail (June 9, 2011). Interview with Bill Gates by Caroline Graham, "This is not the way I'd imagined Bill Gates... A rare and remarkable interview with the world's second richest man."
Retrieved from http://www.dailymail.co.uk/home/moslive/article-2001697/Microsofts-Bill-Gates-A-rare-remarkable-interview-worlds-second-richest-man.html, on November 14, 2014.

1.3.5 Cross-sectional distributions at Solow steady state

Proposition 1.6. *Since* $|\phi_1^*| < 1$*, the cross-sectional inequality of bequests, measured by the coefficient of variation,* $\sigma_l\sqrt{\frac{1-\phi_1^*}{1+\phi_1^*}}$*, which is less than the cross sectional inequality of wages,* σ_l*. The higher the standard deviation of the idiosyncratic productivity shocks or cross sectional inequality of wages,* σ_l*, the higher the cross sectional inequality.*

Proof. From the asymptotic distribution for a covariance-stationary process[28], the inequality of labor income and bequests in the asymptotic cross-sectional distribution using coefficient of variation[29] are:

$$CV\left[w^i\right] = CV\left[w_t^i\right] = \frac{\sqrt{Var\left[w_t^i\right]}}{E\left[w_t^i\right]} = \frac{\sigma_l \overline{w}_t}{\overline{w}_t} = \sigma_l. \tag{1.54}$$

$$CV\left[b^i\right] = CV\left[\frac{b^i}{Y^*}\right] = CV\left[\frac{b_{t+1}^i}{Y^*}\right] = \frac{\sqrt{Var\left[\frac{b_{t+1}^i}{Y^*}\right]}}{E\left[\frac{b_{t+1}^i}{Y^*}\right]} = \frac{\sigma_l\sqrt{\frac{1}{1-\phi_1^{*2}}}}{\frac{1}{(1-\phi_1^*)}} = \sigma_l\sqrt{\frac{1-\phi_1^*}{1+\phi_1^*}}. \tag{1.55}$$

∎

The implications of Proposition 1.6 is that, if the wealth inheritance is a covariance-stationary process, and in the absence of heterogenous capital returns, and stochastic mortality (hence stochastic bequests), the intergenerational transfers serve the purpose of private insurance or risk pooling across generations within the same lineage for idiosyncratic labor productivity shocks, to

[28] See Hamilton (1994, pp. 186–195: Section 7.2. Limit Theorems for Serially Dependent Observations).

[29] The coefficient of variation, as a measure of inequality, is scale invariant.

equalise the lifetime resource, $y_t^i = b_t^i + w_t^i$. This result concurs with Bossmann et al. (2007) and Wan and Zhu (2012) as expected, where only the neoclassical growth model is analysed. To establish this stationary asymptotic cross-sectional distribution of wealth inherited to total output, I use the covariance-stationarity assumption and associated properties of the time series; whereas Bossmann et al. (2007) use 2-series theorem, and Wan and Zhu (2012) use Theorem 1 of Brandt (1986), Lorenz dominance and convex order. In subsequent sections, we shall extend the results of Bossmann et al. (2007) to cover the case of endogenous growth with innovation in the style of Romer.

Furthermore, from Proposition 1.6, an exogenous increase in σ_l will lead to an increase in cross-sectional inequality of bequest to output ratio. Thus, any change in labor force demographics, distribution of human capital, labor-augmenting technology, or degree of openness of an economy that may increase the spread of idiosyncratic productivity of the labor force, will henceforth result in a higher cross-sectional inequality of bequest to output. Examples of changes in spread of productivity or human capital brought about by education, technology or globalized production chains include Davies et al. (2005), Acemoglu and Autor (2012) and Costinot et al. (2012) respectively. A time trend in the wage dispersion is also reported in a study by Krueger et al. (2010). They find a keen and sustained sharp increase in cross-sectional wage dispersion during the last thirty years for Canada, UK and the US, with the variance of male log wages increasing by about 40%. However, according to them, observable characteristics such as experience and education, only partially account for this increase, and they attribute this increase in wage dispersion as largely residual in its nature. In any case, an exogenous increase in labor productivity dispersion will correspondingly lead to an increase in wage dispersion and an increase cross-sectional inequality of wealth in this chapter, hence a plausible driver for the increase in income inequality in the United States over the last thirty years, as depicted by Piketty (2014, pp. 24, Figure I.1).

1.3.6 Lineage bequest evolution at Romer steady state

Let the double-asterisk superscript represents the Romer steady state. At the Romer steady state, $k^{**} \equiv 1 + \frac{(G-1)}{\theta} > 1$. Let $Y_{t+1} = GY_t$, $K_{t+1} = GK_t$, and $R_{t+1} = R_t \equiv R^{**}$. At the Romer regime, from the capital share at equilibrium, (1.27), and final output production, (1.23), gross return on capital at the Romer regime is:

$$R_t = \frac{(1-\frac{1}{\sigma})Y_t}{K_{t-1}} = \frac{(1-\frac{1}{\sigma})AK_{t-1}}{K_{t-1}} = \left(1-\frac{1}{\sigma}\right)A. \tag{1.56}$$

At Romer steady state, substituting $R^{**} = (1-\frac{1}{\sigma})A$ into the law of motion of the bequests for an agent of lineage i (1.39), normalizing by the total output, Y_{t+1}, where the output growth G is $\Omega_1\Omega_3 A$, and rearranging,

$$\frac{b^i_{t+1}}{Y_{t+1}} - \frac{R^{**}\beta\mu}{1+\beta(1+\mu)}\frac{b^i_t}{GY_t} - \frac{R^{**}\beta\mu}{\sigma G[1+\beta(1+\mu)]} = \frac{R^{**}\beta\mu}{1+\beta(1+\mu)}\frac{1}{\sigma G}[l^i_t - \bar{l}_t], \tag{1.57}$$

$$\Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{b^i_t}{Y_t} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3\sigma[1+\beta(1+\mu)]}$$
$$= \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma}[l^i_t - \bar{l}_t].$$

At the Romer steady state, the ratio of bequest to total output inherited by a young agent from lineage i at any time t, is a first order non-homogenous equation. Note that the law of motion of lineage bequest normalized by output is identical to that in the Solow regime, as stated in equation (1.40).

Proposition 1.7. *The dynamics of lineage bequest normalized by the final output, and hence the asymptotic cross-sectional distributions are identical under both the Solow and Romer steady states.*

Proof. As shown in equations (1.40) and (1.57). ■

While the dynamics are identical, a subtle difference exists in terms of interpretation for the agents' optimization for stationarity. Arising from the condition for the existence of covariance stationarity, unlike in the Solow steady state, agents under the Romer regime can pass on more wealth than the sum of wealth they have inherited and wage income they have earned when young to their next generation, provided that is less than the gross growth of the economy. That is, from the agent's perspective,

$$b^i_{t+1} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}y^i_t, \quad \text{and} \quad |\phi^*_1| = \left|\frac{R^{**}\beta\mu}{G[1+\beta(1+\mu)]}\right| < 1,$$
$$\Rightarrow b^i_{t+1} < Gy^i_t.$$

From the economy's perspective, $\phi_1^* = \frac{R^{**}\beta\mu}{G[1+\beta(1+\mu)]} = \frac{(1-\frac{1}{\sigma})A\beta\mu}{A\Omega_1\Omega_2[1+\beta(1+\mu)]} < 1$, implies that agents subjective discount rate for the future is large relative to the ratio of gross return on capital to gross output growth, $\frac{R^{**}}{G}$ or equivalently relative to the ratio of capital share to savings rate. That is:

$$\frac{R^{**}}{G} < \frac{[1+\beta(1+\mu)]}{\beta\mu} \Rightarrow \frac{\overbrace{1-\frac{1}{\sigma}}^{\text{capital share}}}{\underbrace{\Omega_1\Omega_3}_{\text{savings rate out of lifetime resource of young}}} < \frac{1+\beta(1+\mu)}{\beta\mu}$$

That is, for models with constant capital share and homothetic preference such that agents save a constant fraction of income, it is the interplay between the capital share to savings rate relative to the discount rate and bequest motive that determines the stationarity of the bequest distribution, and not the moot point on capital returns to growth ratio, R/G alone. The right hand side of the inequality represents the preference primitives, while the left hand side embeds the institutional structure as elucidated by Acemoglu and Robinson (2014).

In the current model, this covariance stationary condition is slack and not binding, as the AR(1) coefficient which dependent only on the elasticity of substitution (or monopoly power) and the strength of the bequest motive, σ and μ respectively, is always less than 1.

1.3.7 Cross-sectional distributions at Romer steady state

Proposition 1.8.

(i) The gross growth rate of cross-sectional ***expected value*** *of bequests in the*

Romer steady state is $(\Omega_1\Omega_3 A) > 1$*, while that equals 1 in the Solow steady state.*

(ii) The gross growth rate of cross-sectional ***variance*** *of bequests in the Romer steady state is* $(\Omega_1\Omega_3 A)^2 > 1$*, while that equals 1 in the Solow steady state.*

(iii) However, using CV as a measure of inequality, the cross sectional inequality of bequests to total income under the Romer and Solow steady states are equivalent, if the stationarity assumption for the law of motion holds in both steady states.

Proof. Unlike the Solow steady state where, $Y_{t+1} = Y_t = Y^*$, in the Romer steady state, Y^{**} is growing, that is, $Y_{t+1} = \Omega_1\Omega_3 AY_t$. $E\left[b^i_{t+1}\right] = (\Omega_1\Omega_3 A)E\left[b^i_t\right]$, and $Var\left[b^i_{t+1}\right] = (\Omega_1\Omega_3 A)^2 Var\left[b^i_t\right]$. But, CV as an inequality measure is invariant to scale changes. Thus with covariance-stationarity, $|\phi^*_1| < 1$, $CV\left[\frac{b^i}{Y^{**}}\right] = CV\left[\frac{b^i}{Y^*}\right] = CV\left[b^i\right]$. ∎

While Wolff (1992) finds that the Great Depression has a considerable effect on the inequality of wealth, Krueger et al. (2010) do not find such a link between wealth inequality and recessions, citing the recessions in Sweden and Italy in the 1990s as examples, as those recessions were not associated with significant changes in asset prices. Thus, changes in growth rate alone, although an easy suspect, is not a sufficient factor driving changes in wealth inequality.

1.3.8 Lineage bequest evolution and cross-sectional distributions for period-2 cycles

Having done the groundwork to characterize the dynamics of the bequest distributions for both the Solow and Romer steady states, the derivation of lineage bequest evolution for period-2 cycle follows. Note that for period-2 cycles, output growth occurs at both the Solow and Romer regimes. Let the superscripts L and H denote the fixed points with the lower and higher normalized capital respectively. Let R^L be the gross return on capital for the fixed point, k^L; R^H be the gross return on capital for the fixed point, k^H; and G^L be the gross growth rate of output for the period-2 fixed point in the Solow regime, and G, as stated earlier, is the gross growth rate along the balanced growth path in the Romer regime.

From the capital share at equilibrium, equation (1.27), the savings rate, equation(1.30), and the gross growth rate of variables along period-2 cycles in Proposition 1.2, we get:

$$R_t = \frac{(1-\frac{1}{\sigma})Y_t}{K_{t-1}} = \frac{(1-\frac{1}{\sigma})}{K_{t-1}}\frac{K_t}{\Omega_1\Omega_3},$$

$$\frac{R^L}{G^L} = \frac{\frac{(1-\frac{1}{\sigma})}{\Omega_1\Omega_3}\frac{K^L}{K^H}}{\frac{Y^L}{Y^H}} = \frac{(1-\frac{1}{\sigma})}{\Omega_1\Omega_3} = \frac{R^H}{G}. \tag{1.58}$$

Using the iterated fixed points of the period-2 cycles: $k_{t+1} = k^H$, $k_t = k^L$, $k_{t-1} = k^H$, and $k_{t-2} = k^L$, by recursive substitution into the law of motion of bequest, equation (1.39), and normalizing using the final output, we get, at Romer fixed point:

$$\frac{b^i_{t+1}}{Y_{t+1}} - \phi^R \underbrace{\left[\phi^S \frac{b^i_{t-1}}{Y_{t-1}} + \frac{\phi^S}{\sigma} + \frac{\phi^S}{\sigma} u^i_t\right]}_{=\frac{b^i_t}{Y_t}} - \frac{\phi^R}{\sigma} = \frac{\phi^R}{\sigma} u^i_{t+1},$$

$$\Rightarrow \ \frac{b^i_{t+1}}{Y_{t+1}} \ - \ \phi^R \phi^S \frac{b^i_{t-1}}{Y_{t-1}} \ - \ \frac{\phi^R \phi^S}{\sigma}\left[1 + \frac{1}{\phi^S}\right] \ = \ \frac{\phi^R}{\sigma} u^i_{t+1} \ + \ \frac{\phi^R \phi^S}{\sigma} u^i_t, \quad (1.59)$$

at Solow fixed point:

$$\frac{b^i_t}{Y_t} - \phi^S \underbrace{\left[\phi^R \frac{b^i_{t-2}}{Y_{t-2}} + \frac{\phi^R}{\sigma} + \frac{\phi^R}{\sigma} u^i_{t-1}\right]}_{=\frac{b^i_{t-1}}{Y_{t-1}}} - \frac{\phi^S}{\sigma} = \frac{\phi^S}{\sigma} u^i_t,$$

$$\Rightarrow \ \frac{b^i_t}{Y_t} \ - \ \phi^R \phi^S \frac{b^i_{t-2}}{Y_{t-2}} \ - \ \frac{\phi^R \phi^S}{\sigma}\left[1 + \frac{1}{\phi^R}\right] \ = \ \frac{\phi^S}{\sigma} u^i_t \ + \ \frac{\phi^R \phi^S}{\sigma} u^i_{t-1}, \quad (1.60)$$

where $\phi^R \equiv \frac{R^H \beta\mu}{G[1+\beta(1+\mu)]} = \phi^S \equiv \frac{R^L \beta\mu}{G^L[1+\beta(1+\mu)]} = \phi^*_1 \equiv \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. (The extra notations are introduced to facilitate the exposition, even though the parameter values are the same.)

To characterize the Romer fixed point, let $z^{Hi}_{t+1} = \frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\frac{1}{\Omega_4}\delta_2}{1-\phi^R\phi^S}$, where $\Omega_4 = \frac{\phi^R}{\sigma}$, $\delta_2 = \frac{\phi^R\phi^S}{\sigma}[1+\frac{1}{\phi^S}]$, and $\phi^H_2 = \phi^R\phi^S$, such that:

$$z^{Hi}_{t+1} - \phi^H_2 z^{Hi}_{t-1} = [l^i_t - \bar{l}_t] + \phi^S[l^i_{t-1} - \bar{l}_{t-1}] \equiv u^i_{t+1} + \phi^S u^i_t, \quad (1.61)$$

where u^i_{t+1} and $u^i_t \sim$ i.i.d.$(0, \sigma^2_l)$.

To characterize the Solow fixed point, let $z^{Li}_t = \frac{1}{\Omega_4}\frac{b^i_t}{Y_t} - \frac{\frac{1}{\Omega_4}\delta_2}{1-\phi^R\phi^S}$, where $\Omega_4 = \frac{\phi^S}{\sigma}$, $\delta_2 = \frac{\phi^R\phi^S}{\sigma}[1+\frac{1}{\phi^R}]$, and $\phi^L_2 = \phi^R\phi^S$, such that:

$$z^{Li}_t - \phi^L_2 z^{Li}_{t-2} = [l^i_{t-1} - \bar{l}_{t-1}] + \phi^R[l^i_{t-2} - \bar{l}_{t-2}] \equiv u^i_t + \phi^R u^i_{t-1}, \quad (1.62)$$

where u^i_t and $u^i_{t-1} \sim$ i.i.d.$(0, \sigma^2_l)$.

Proposition 1.9. *Since* $\left|\phi_2^H\right| = \left|\phi_2^L\right| = \left|(\phi_1^*)^2\right| < 1$*, then* z_{t+1}^{Hi} *and* z_t^{Li} *are covariance stationary ARMA(2,1) processes, with*

$$E\left[\frac{b_{t+1}^{Hi}}{Y_{t+1}}\right] = E\left[\frac{b_t^{Li}}{Y_t}\right] = \frac{\delta_2}{1-\phi^R\phi^S} = \frac{\frac{\phi^R\phi^S}{\sigma}[1+\frac{1}{\phi^S}]}{1-\phi^R\phi^S} = \frac{\frac{1}{\sigma}\phi_1^*}{1-\phi_1^*}; \tag{1.63}$$

$$Var\left[\frac{b_{t+1}^{Hi}}{Y_{t+1}}\right] = Var\left[\frac{b_t^{Li}}{Y_t}\right] = \frac{\sigma_l^2\Omega_4^2}{(1-\phi_1^{*2})} = \frac{\sigma_l^2}{(1-\phi_1^{*2})}\left(\frac{1}{\sigma}\phi_1^*\right)^2; \tag{1.64}$$

$$Cov\left[\frac{b_{t+1}}{Y_{t+1}}, \frac{b_t}{Y_t}\right] = \frac{\sigma_l^2\phi_1^*}{1-\phi_1^{*2}}\left(\frac{1}{\sigma}\phi_1^*\right)^2; \tag{1.65}$$

$$Cov\left[\frac{b_{t+2}}{Y_{t+2}}, \frac{b_t}{Y_t}\right] = \frac{\sigma_l^2\phi^{*2}}{(1-\phi_1^{*2})}\left(\frac{1}{\sigma}\phi_1^*\right)^2; \tag{1.66}$$

$$Cov\left[\frac{b_{t+k}}{Y_{t+k}}, \frac{b_t}{Y_t}\right] = \phi^R\phi^S Cov\left[\frac{b_{t+k-2}}{Y_{t+k-2}}, \frac{b_t}{Y_t}\right] = \frac{\sigma_l^2\phi^{*k}}{(1-\phi_1^{*2})}\left(\frac{1}{\sigma}\phi_1^*\right)^2 \forall k \geq 2; \tag{1.67}$$

$$\rho\left[\frac{b_{t+1}^i}{Y_{t+1}}, \frac{b_t^i}{Y_t}\right] = \rho_1 = \frac{Cov\left[\frac{b_{t+1}}{Y_{t+1}}, \frac{b_t}{Y_t}\right]}{\sqrt{Var\left[\frac{b_{t+1}^{Hi}}{Y_{t+1}}\right] Var\left[\frac{b_t^{Li}}{Y_t}\right]}} = \phi_1^*; \tag{1.68}$$

$$\rho\left[\frac{b_{t+k}^i}{Y_{t+k}}, \frac{b_t^i}{Y_t}\right] = \rho_k = \phi^R\phi^S\rho_{k-2} = \left(\frac{(\sigma-1)\mu}{1+\sigma\mu}\right)^k \forall k \geq 2. \tag{1.69}$$

Proof. Equations (1.61) and (1.62) can be expressed as $(1-\phi_2 L^2)z_{t+1}^{H\,or\,L,i} = u_{t+1}^i + \phi_1^* u_t^i$, where L is the lag operator, and for $z_{t+1}^{H\,or\,L,i}$ to be stationary, the roots of the characteristic equations, $(1-\phi_2 L^2) = 0$, must lie outside the unit circle, hence $|\phi_2| < 1$, or equivalently, $|\phi_1^{*2}| < 1$. From (1.52), $0 \leq \phi_1^* < 1$, thus $\left|\phi_2^H\right| = \left|\phi_2^L\right| = \left|(\phi_1^*)^2\right| < 1$, and the conditions for covariance-stationarity for the ARMA(2,1) process are met[30]. To compute the moments, the notation is simplified by dropping the superscripts, resulting in the following ARMA(2,1)

[30] See Hamilton (1994).

process:

$$z_t = \phi_2 z_{t-2} + u_t + \phi_1 u_{t-1}$$
$$= \phi_1^{*2} z_{t-2} + u_t + \phi_1^* u_{t-1} \tag{1.70}$$

where u_t and $u_{t-1} \sim$ i.i.d.$(0, \sigma_l^2)$.

$$\gamma(0) = E[z_t z_t] = \phi_1^{*2} E[z_t z_{t-2}] + E[z_t u_t] + \phi_1^* E[z_t u_{t-1}]$$
$$= \phi_1^{*2}\gamma(2) + \sigma_l^2 + \phi_1^* E[(\phi_1^{*2} z_{t-2} + u_t^i + \phi_1^* u_{t-1}^i)u_{t-1}]$$
$$= \phi_1^{*2}\gamma(2) + \sigma_l^2 + \phi_1^{*2}\sigma_l^2. \tag{1.71}$$

$$\gamma(1) = E[z_{t-1} z_t] = \phi_1^{*2} E[z_{t-1} z_{t-2}] + E[z_{t-1} u_t] + \phi_1^* E[z_{t-1} u_{t-1}]$$
$$= \phi_1^{*2}\gamma(1) + E[(\phi_1^{*2} z_{t-3} + u_{t-1}^i + \phi_1^* u_{t-2}^i)u_t] + \phi_1^* E[(\phi_1^{*2} z_{t-3} + u_{t-1}^i + \phi_1^* u_{t-2}^i)u_{t-1}]$$

$$\gamma(1) = \phi_1^{*2}\gamma(1) + \phi_1^* \sigma_l^2$$
$$= \frac{\phi_1^* \sigma_l^2}{(1 - \phi_1^{*2})}. \tag{1.72}$$

$$\gamma(2) = E[z_{t-2} z_t] = \phi_1^{*2} E[z_{t-2} z_{t-2}] + E[z_{t-2} u_t] + \phi_1^* E[z_{t-2} u_{t-1}]$$
$$= \phi_1^{*2}\gamma(0) + E[(\phi_1^{*2} z_{t-4} + u_{t-2}^i + \phi_1^* u_{t-3}^i)u_t] + \phi_1^* E[(\phi_1^{*2} z_{t-4} + u_{t-2}^i + \phi_1^* u_{t-3}^i)u_{t-1}]$$

$$\gamma(2) = \phi_1^{*2}\gamma(0). \tag{1.73}$$

Substituting $\gamma(2)$ from (1.73) into $\gamma(0)$ from (1.71), we get:

$$\gamma(0) = \phi_1^{*4}\gamma(0) + \sigma_l^2 + \phi_1^{*2}\sigma_l^2$$
$$= \sigma_l^2 \frac{(1 + \phi_1^{*2})}{(1 - \phi_1^{*4})} = \frac{\sigma_l^2}{(1 - \phi_1^{*2})}. \tag{1.74}$$

$$\gamma(k) = E[z_{t-k}z_t] = \phi_1^{*2}E[z_{t-k}z_{t-2}] + E[z_{t-k}u_t] + \phi_1^{*}E[z_{t-k}u_{t-1}] \quad \forall k \geq 2$$

$$= \phi_1^{*2}E[z_{t-k}z_{t-2}] = \phi_1^{*2}\gamma(k-1) \quad \forall k \geq 2. \tag{1.75}$$

■

Proposition 1.10. *With i.i.d. labor productivity shocks, for period-2 cycles, the mean and variances of the bequest to output ratio as well as the intergenerational mobilities are the same for both the Solow and Romer fixed points, and they are also identical to those under the Solow or Romer steady states.*

Proof. By inspection of the moments in Proposition 1.9. ■

1.4 Comments on the distribution equivalence between Solow and Romer regimes

1.4.1 Implications for generating evolving inequality of wealth over time

For linear models, in the absence of financial frictions, there is no distinction on whether the lifetime resource at young is obtained through earned wages or through bequest as long as they are indistinguishable when invested. Whether the real wages are stagnant or growing is immaterial; as long as the labor share to output ratio is a constant, both the Solow and Romer regimes yield equivalent dynamics and distributions after normalising for growth.

Thus, a way to generate an evolving cross-sectional inequality of inherited wealth over time is via an exogenous change in the primitive parameters in the model such as the spread of the idiosyncratic productivity shocks, as discussed in Proposition 1.6. Other factors include exogenous fiscal shocks, such as changes in inheritance tax or innovation subsidies. This approach is adopted by Piketty (2011), where to match the long-run evolution of inheritance from 1820, as well as for projection into 2050, he simulated the data with exogenous changes in the differentiated savings rates, bequest ratios, and capital taxes, as well as exogenous capital losses due to destruction shocks, for different time segments.

Yet another alternative is the introduction of financial frictions (e.g. credit constraints or indivisible lumpy investment), or differentiated capital embedded in endogenous growth such that the rich can have favorable access to profitable investment projects during regimes when returns on capital are higher; or finally, a production mechanism, where capital share is not a constant fraction of final output. Particularly, to obtain cyclical inequality dynamics over different growth regimes, the key is to generate heterogenous returns on wealth (via heterogenous capital returns or human capital returns) *both* across agents within the same cohort and across the different regimes. In this model, while the returns on capital are different across the 2 regimes, the capital share are unchanged over the 2 regimes, and all agents have equal access to the capital markets with their wealth, without any distinction between wealth earned through labor income or inherited, resulting in equivalence across the 2 different regimes.

For future research, extensions with models of innovation satisfying both conditions will be looked into using this covariance-stationary technique to study cyclical evolution of inequality. For this thesis, in Chapter 4, as with existing literature, I shall fall back on using exogenous changes to explain the evolving inequality across time. These exogenous fiscal policy changes are tax rate on bequest and innovation subsidies, and they are empirically motivated. While the impact of bequest has been explored under neoclassical growth by Bossmann et al. (2007) and Wan and Zhu (2012), and the impact of investment subsidy by García-Peñalosa and Turnovsky (2007), I hope to complement the existing research by providing a tractable analytical characterization of impact of these fiscal policies on the transition and lineage bequest evolution of inequality (i.e. intergenerational mobility), beyond the cross-sectional distribution consequences at steady states, for growth under both regimes and cycles, without using numerical simulations.

1.4.2 Implications for policy makers

Further, for pedagogical purposes, the distributional equivalence under different growth regimes is informative in defining desideratum in the economy's fiscal policies. Unlike Piketty (2011, 2014), the difference between the gross return on capital and the gross growth rate of an economy, consequently $r - g$, does not drive the cross-sectional inequality of inherited wealth in this model. Serving as the pivotal points to this result here are (i) the constant labor to output share in the long run (allowing for exogenous occasional changes in the labor shares via changes in the price elasticity of final sector's demand during transition to the stationary distribution), and (ii) the absence of financial frictions in the economy. The degree of validity of these pivots are empirically falsifiable, by bringing closer together macro-data such as national accounts and micro-data on household labor and finances. Schneider (2011) provides a survey on the labor share literature, both theoretical and empirical.

The second pivotal point: access to the capital markets by agents, rich or poor, merits discussion. The lynchpin in Piketty (2011, 2014) is the "class saving" model, where the savings rate out of labor and capital income are exogenously differentiated. In the benchmark case (Piketty (2011, pp.1107)), agents consume 100% of their labor income by assumption, thus all savings in the economy come only from the returns to inherited wealth. In addition, Piketty (2011, pp. 1105) assumes growth to be exogenous, driven by changes in productivity, within a Cobb-Douglas production function, with 2 factors of production: non-human capital K_t, and human capital $H_t = L_t e^{gt}$, where L_t is the labor supply and g is the exogenous rate of productivity growth. With this set-up, Piketty (2011, pp. 1112–1113) posits that even if workers save out of their labor income, the steady-state inheritance converges quickly to the benchmark "class savings" model when the growth rate approaches 0 (e.g. in the region of 1 to 2%). Piketty (2011, pp. 1115) also explores a finite-horizon wealth-in-utility saving model, with the same production function, arriving at the same steady-state convergence with the "class-savings" model, for agents with identical preference, but ex-ante heterogenous endowments of wealth.

Thus, implicit in Piketty (2011, pp.1074) "$r > g$ logic": if rate of returns to private wealth exceeds rate of growth of the economy, inequality is exacerbated, is the limited ability for the workers without inheritance to participate in the capital market, hence its dividends from economic growth. Based on his "$r > g$ logic", Piketty (2011, 2014) postulates an increase in taxes on capital to regulate the size of the steady-state inheritance flows to curb the rising inequality of wealth, without impinging upon the exogenously driven growth rate of the economy. Note that the mobility in his "class saving" model is zero, with permanent stratification of the society, except when taxes and redistribution are taken to the extreme of ex-post equality for all.

In contrast, the "$r > g$ logic" does not apply in this model of endogenous growth with innovation, since the wealth distributions under Solow and Romer regimes are equivalent, if all agents, regardless of their wealth, are able to optimise their savings and consumption decisions, and participate in capital markets with homogenous returns. While the preference in this model with bequest motive is similar to the reduced form of the finite-horizon wealth-in-utility savings model used by Piketty (2011, pp. 1115), with idiosyncratic labor productivity draws, I manage to avoid the social stratification that results in his model with labor income mobility via productivity draws, while achieving a non-degenerate distribution of inherited wealth, regardless whether agents were ex-ante homogenous or heterogenous in terms of their wealth. Moreover, unlike Piketty (2011), with the production function used in this model, both the savings rate and growth rate are endogenously determined. Thus, I posit that improving access to capital markets (both physical and human) and the reduction of frictions (such as investment quantum through vehicles such as investment trusts, and quality public schools), and informational disadvantage (through improved reporting standards and investment literacy programmes) may be helpful to promote mobility, and to cap inequality arising from capital market inaccessibility (and hence the inability to partake in the growth dividends of the economy).

1.4.3 Technical implications

With the distributional equivalence of the inherited wealth to output ratio under i.i.d. productivity shocks, I have effectively characterized the global lineage evolution of inherited wealth to output ratio, as well as the long-run cross-sectional distribution of inherited wealth to output ratio. This has novel implications when combined with the results from Gardini et al. (2008) and Mitra (2001). Mitra (2001) presents a sufficient condition for topological chaos for unimodal maps that do not satisfy the Li-Yorke condition. Gardini et al. (2008) have characterized the global aggregate dynamics of the Matsuyama (1999) model, and shown that the economy can grow via an attracting chaotic interval, 2-cyclical chaotic intervals, 4-cyclical chaotic intervals, attracting period-2 cycles, attracting fixed point in Romer regime, or converge to an attracting fixed point in the Solow regime, depending on the parameter values of σ and G in the Matsuyama (1999) model, as it is topologically equivalent to a skewed tent map.

Furthermore, I can subsequently relax the finite variance assumption of the idiosyncratic shocks. If the idiosyncratic shocks, l_t is stationary and ergodic, with a finite mean, I can proceed to characterize the cross-sectional wealth distribution and compare the inequality using convex order and Lorenz dominance as in Zhu (2013a) and Wan and Zhu (2012). By relaxing the finite variance assumption, I can then bring the model to the wealth data that displays fat-tails (e.g. in Piketty (2011) and Piketty (2014)), where the right-tail cannot be captured with the finite variance assumption.

As noted by Stachurski (2009), for stable invariant distribution to exist, there is positive probability of moving between any 2 points of space within a finite time, to ensure distribution is ergodic. Thus, by incorporating bequest motive and i.i.d. idiosyncratic productivity shocks into the Matsuyama (1999), the economy is never stratified globally, since the global wealth distribution is stable and invariant, even though the aggregate growth dynamics can be chaotic.

1.5 Comparative statics

1.5.1 Impact of an increase in bequest motive, μ, on lineage and cross-sectional distribution

Proposition 1.11. *Without bequests, correlation in wealth is* 0. *That is, wealth is determined by own i.i.d. labor income, and there is perfect mobility. In the presence of a bequest motive, the higher the bequest motive,* μ*, the higher the correlation of the ratio of bequests to total output across generations from the* *same dynasty, the lower the intergenerational mobility.*

Proof. From the correlation equation (1.49),

$$\rho_1 = \phi_1^* = \frac{(\sigma-1)\mu}{1+\sigma\mu},$$

if $\mu > 0$,

$$\frac{\partial \rho_1^*}{\partial \mu} = \frac{\sigma-1}{(1+\sigma\mu)^2} > 0. \tag{1.76}$$

■

Proposition 1.12. *The higher the bequest motive, μ, the lower the asymptotic cross-sectional spread of bequests, the lower the cross-sectional bequest inequality in both regimes.*

Proof. From (1.55),

$$sign\frac{\partial CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial \mu} = sign\frac{\partial\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2}{\partial \mu}$$
$$= sign\frac{\partial\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2}{\partial \phi_1^*}\frac{\partial \phi_1^*}{\partial \mu}$$
$$= sign\left\{\sigma_l^2 \underbrace{\left[(1+\phi_1^*)^{-1}(-1)+(1-\phi_1^*)(-1)(1+\phi_1^*)^{-2}\right]}_{\leq 0}\right\} \underbrace{\frac{\partial \phi_1^*}{\partial \mu}}_{\geq 0 \text{ from (1.76)}} \leq 0. \tag{1.77}$$

■

With the "joy-of-giving" bequest motive, an increase in bequest motive reduces both intergenerational mobility and cross-sectional inequality of inherited wealth under i.i.d. labor productivity shocks. That is, bequest functions as an intergenerational risk pooling device for idiosyncratic labor productivity shocks. This extends the robustness of role of bequest in intergenerational risk sharing by Becker and Tomes (1979) and Bossmann et al. (2007) to a model of endogenous growth (with innovation and cycles). This result contrasts with Zhu (2014), where investment risk is introduced into Becker and Tomes (1979). The difference in the resulting wealth inequality in relations to the strength of the bequest motive highlight the need to read beyond the headlines (R versus G), to identify the fundamental driver(s) of inequality (e.g. heterogeneous returns on capital or investment versus heterogeneous returns on labor), with bequest acting merely as a vehicle to propagate the heterogeneity. In this model, the returns on capital are homogeneous, so too the strength of the bequest motive (see Charles and Hurst (2002) for models with varying bequest motives in agents' preferences), thus bequest itself is rank preserving, mitigating the productivity risks of subsequent generations within a lineage.

1.5.2 Impact of an increase in price elasticity, σ, on lineage and cross-sectional distribution

Proposition 1.13. *The higher the price elasticity of the final sector's demand for each intermediate, σ (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin for innovation), the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility.*

Proof.

$$\frac{\partial \rho_1}{\partial \sigma} = \frac{1}{(1+\sigma\mu)^2}\left[\underbrace{\sigma(1-\mu)}_{\geq 0} + \underbrace{\mu(\sigma^2+1)}_{\geq 0}\right] \geq 0. \tag{1.78}$$

■

Proposition 1.14. *The higher the price elasticity of the final sector's demand for each intermediate, σ (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin for innovation), the lower the asymptotic cross-sectional spread of bequests, the lower the cross-sectional bequest inequality in both regimes.*

Proof.

$$\begin{aligned} sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]}{\partial \sigma} &= sign\frac{\partial \left(CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]\right)^2}{\partial \sigma} \\ &= sign\frac{\partial \left(CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]\right)^2}{\partial \phi_1^*}\frac{\partial \phi_1^*}{\partial \sigma} \\ = sign\left\{\sigma_l^2 \underbrace{\left[(1+\phi_1^*)^{-1}(-1) + (1-\phi_1^*)(-1)(1+\phi_1^*)^{-2}\right]}_{\leq 0}\right\} & \underbrace{\frac{\partial \phi_1^*}{\partial \sigma}}_{\geq 0 \text{ from (1.78)}} \leq 0. \end{aligned} \tag{1.79}$$

■

The impact of price elasticity of the final sector's demand for each intermediate, hence the monopoly margin for innovation is absent in Becker and Tomes (1979); Charles and Hurst (2002); Bossmann et al. (2007); García-Peñalosa and Turnovsky (2007) and García-Peñalosa and Turnovsky (2011). In Becker and

Tomes (1979) and Charles and Hurst (2002), production is not explicitly modelled, and changes in the endowment generating process is used to study the distribution of income or wealth and the intergenerational mobility. The papers by Bossmann et al. (2007) and García-Peñalosa and Turnovsky (2011) use the Neoclassical growth model, where the former has inelastic labor, and the latter, elastic labor with agents holding different initial capital endowments. García-Peñalosa and Turnovsky (2007) use an endogenous growth model with elastic labor, but no innovation.

Piketty and Zucman (2014) discuss the role of capital share on the wealth to income ratio arising from changes in technology, with a 2-factor CES production function. If the capital-labor elasticity of substitution is larger than 1, capital share in the national income increases as wealth to income ratio increases (due to the changes in the relative prices of the factors of production), and an exogenous decrease in growth rate (due to a decrease in population or productivity growth) results in a higher wealth-income ratio in the long run.

In this model, the returns on capital across all agents who saved are homogenous and thus rank preserving, although it can vary across time. Thus mobility is driven by the idiosyncratic labor productivity shocks. A decrease in wage share will decrease intergenerational mobility and decrease cross-sectional wealth inequality. Thus, the higher the price elasticity of the final sector's demand for each intermediate, the lower the monopoly margin for innovation, the lower the wage share (since labor is used only in the production of the final good via a Cobb-Douglas production), the lower the intergenerational mobility, and the lower the cross-sectional bequest inequality. As a result of the Cobb-Douglas production function in the final good sector, with labor and intermediates as factor inputs, unlike the CES used in Piketty and Zucman (2014), the capital share in this thesis is constant regardless of the growth regimes, and the endogenous growth rate is a function of the price elasticity of the final sector's demand for each intermediate good.

Financial frictions such as credit constraints, indivisible investment quantum, and differential access to the capital markets (e.g. closed-end funds and exclusive contracts by venture capitalists in start-ups), or stochastic idiosyncratic capital returns with lineage correlation, if present, will further stratify the mobility structure by introducing heterogeneity in the capital returns faced by agents.

1.6 Correlated productivity shocks

1.6.1 Characterization of lineage bequest evolution with correlated productivity shocks at the Solow and Romer steady states

Empirical estimates of the elasticity of son's earnings with respect to father's earnings from different data sources and countries range from 0.11 to 0.57, suggesting that presence of intergenerational correlation of income. (See Mazumder (2005); Solon (2004, 2002, 1992); Zimmerman (1992)). By using 2 intergenerational links via parent's bequest motives and productivity inheritance, De Nardi

(2004) finds that voluntary bequest motive (also known as "warm-glow" bequest by Andreoni (1989)) explains the existence of large inheritance, while earnings persistence further concentrates the cross-sectional distribution of inheritance, when calibrated against both the US and Swedish data. Thus, in this section, we study the impact of correlated productivity (alternative reduced-form interpretations include correlated genes, health, human or social capital) that affects idiosyncratic labor efficiency. As with Davies and Kuhn (1991), Bossmann et al. (2007) and Wan and Zhu (2012), adopting a mean-reverting correlated idiosyncratic labor productivity process, let:

$$l_t^i = \bar{l}_t + \nu(l_{t-1}^i - \bar{l}_{t-1}) + \varepsilon_t^i, \tag{1.80}$$

where $\bar{l}_t = \bar{l}_{t-1} = \bar{l} = 1$, $0 < \nu < 1$, and $\varepsilon_t^i \sim$ i.i.d. $\left(0, \sigma_l^2\right)$, such that σ_l^2 is finite, and that the i.i.d. shocks represented by ε_t^i has a lower bound sufficient to keep $l_t^i > 0$. The process is initiated by $l_0^i = \bar{l} + \varepsilon_0^i$. For illustration, the following exposition on correlated shocks is based on either one of the steady states, e.g. Solow. From equation (1.41),

$$\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b_t^i}{Y_t} - \frac{\delta_1}{\Omega_4} = [l_t^i - \bar{l}_t] \tag{1.81}$$

where $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma} = \delta_1$, and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. Thus, the mean reverting correlated idiosyncratic labor productivity shocks implies:

$$\left(l_t^i - \bar{l}_t\right) - \nu(l_{t-1}^i - \bar{l}_{t-1}) = \varepsilon_t^i \sim i.i.d.(0, \sigma_l^2), \tag{1.82}$$

$$\Rightarrow \left(\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b^i_t}{Y_t} - \frac{\delta_1}{\Omega_4}\right) - \nu\left(\frac{1}{\Omega_4}\frac{b^i_t}{Y_t} - \frac{1}{\Omega_4}\phi_1^*\frac{b^i_{t-1}}{Y_{t-1}} - \frac{\delta_1}{\Omega_4}\right)$$
$$= \varepsilon^i_t \sim i.i.d.(0, \sigma_l^2)$$

$$\Rightarrow \frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{1}{\Omega_4}(\phi_1^* + \nu)\frac{b^i_t}{Y_t} - \frac{1}{\Omega_4}(-\phi_1^*\nu)\frac{b^i_{t-1}}{Y_{t-1}} - \frac{\delta_1}{\Omega_4}(1-\nu)$$
$$= \varepsilon^i_t \sim i.i.d.(0, \sigma_l^2). \tag{1.83}$$

With mean reverting correlated shocks as specified in (1.80), the ratio of bequest to total output inherited by a young agent from lineage i at any time t, is a second order non-homogenous difference equation.

Let the superscript m denotes the mean-reverting correlated labor productivity draw. Define $z^{mi}_{t+1} = \frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - (1-\nu)\left(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_1^m-\phi_2^m}\right)$. Lineage dynamics from (1.83) can be expressed as follows:

$$z^{mi}_{t+1} - \phi_1^m z^{mi}_t - \phi_2^m z^{mi}_{t-1} = \left[l^i_t - \bar{l}_t\right] - \nu\left[l^i_{t-1} - \bar{l}_{t-1}\right] \equiv \varepsilon^i_t \sim \text{ i.i.d. } (0, \sigma_l^2), \tag{1.84}$$

where $\phi_1^m = (\phi_1^* + \nu)$, $\phi_2^m = (-\phi_1^*\nu)$, $\delta_z^m = (1-\nu)(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_1^m-\phi_2^m})$, and
$\phi_1^* = \frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]} = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$, and $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\sigma\Omega_1\Omega_3[1+\beta(1+\mu)]} = \delta_1$.

Expanding, we get an AR(2) process for the dynamics of z^{mi}_{t+1} and correspondingly the dynamics of $\frac{b^i_{t+1}}{Y_{t+1}}$ from equations (1.84) and (1.83). This is equivalent to the mixing of 2 ARMA processes, that is, for equation (1.82), the sum of 2 AR(1) processes is an AR(2) process.

Proposition 1.15. *Since $|\phi_2^m| < 1$ and $|\phi_1^m| < 2$, z^{mi}_{t+1} is a covariance stationary AR(2) process.*

$$E\left[\frac{b^i_{t+1}}{Y_{t+1}}\right] = \frac{\delta_1}{1-\phi_1^*}. \tag{1.85}$$

$$Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right] = \frac{\sigma_l^2\Omega_4^2}{(1-\phi_2^m-\phi_1^m)(1-\phi_2^m+\phi_1^m)}\left(\frac{1-\phi_2^m}{1+\phi_2^m}\right)$$
$$= \frac{\sigma_l^2\phi_1^{*2}}{\sigma^2(1-\phi_1^{*2})(1-\nu^2)}\left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right). \tag{1.86}$$

$$Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right] = \frac{\sigma_l^2\Omega_4^2}{(1-\phi_2^m)(1-\phi_2^m-\phi_1^m)(1-\phi_2^m+\phi_1^m)}\left(\frac{1-\phi_2^m}{1+\phi_2^m}\right)$$
$$= \frac{\sigma_l^2\phi_1^{*2}}{\sigma^2(1-\phi_1^{*2})(1-\nu^2)(1-\phi_1^*\nu)}. \tag{1.87}$$

$$Cov\left[\frac{b^i_{t+2}}{Y_{t+2}},\frac{b^i_t}{Y_t}\right] = \phi_1^m Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right] + \phi_2^m Cov\left[\frac{b^i_t}{Y_t},\frac{b^i_t}{Y_t}\right]$$
$$= (\phi_1^*+\nu)Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right] + (-\phi_1^*\nu)Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]. \tag{1.88}$$

$$\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right] = \rho_1 = \frac{\phi_1^m}{1-\phi_2^m} = \frac{\phi_1^*+\nu}{1+\phi_1^*\nu}. \tag{1.89}$$

$$\rho\left[\frac{b^i_{t+2}}{Y_{t+2}},\frac{b^i_t}{Y_t}\right] = \rho_2 = \frac{\phi_1^{m2}}{1-\phi_2^m} + \phi_2^m = \frac{\phi_1^{*2}+2\phi_1^*\nu+\nu^2}{1+\phi_1^*\nu} - \phi_1^*\nu. \tag{1.90}$$

$$Cov\left[\frac{b^i_{t+k}}{Y_{t+k}},\frac{b^i_t}{Y_t}\right] = \phi_1^m Cov\left[\frac{b^i_{t+k-1}}{Y_{t+k-1}},\frac{b^i_t}{Y_t}\right] + \phi_2^m Cov\left[\frac{b^i_{t+k-2}}{Y_{t+k-2}},\frac{b^i_t}{Y_t}\right] \ \forall k \geq 1$$
$$= (\phi_1^*+\nu)Cov\left[\frac{b^i_{t+k-1}}{Y_{t+k-1}},\frac{b^i_t}{Y_t}\right] + (-\phi_1^*\nu)Cov\left[\frac{b^i_{t+k-2}}{Y_{t+k-2}},\frac{b^i_t}{Y_t}\right] \ \forall k \geq 1. \tag{1.91}$$

$$\rho_k = \phi_1^m\rho_{k-1} + \phi_2^m\rho_{k-2} \quad \forall k \geq 1$$
$$= (\phi_1^*+\nu)\rho_{k-1} + (-\phi_1^*\nu)\rho_{k-2} \quad \forall k \geq 1. \tag{1.92}$$

Proof. See Hamilton (1994) for conditions for covariance-stationarity for AR(2). Essentially, the equation (1.84) can be expressed as $(1-\phi_1^m L-\phi_2^m L^2)z^{*i}_{t+1} = \varepsilon^i_t$, where L is the lag operator, and for z_{t+1}^{mi} to be stationary, the roots of the characteristic equation, $(1 - \phi_1^m L - \phi_2^m L^2) = 0$, must lie outside the unit circle. Let the roots be L_i where $i = 1$ & 2. For $|L_i| > 1$, $\left|\frac{1}{L_i}\right| < 1$, which implies $\left|\frac{1}{L_1} \cdot \frac{1}{L_2}\right| = |\phi_2^m| < 1$, and $\left|\frac{1}{L_1} + \frac{1}{L_2}\right| = |\phi_1^m| < 2$.

$$|\phi_1^m| = |\phi_1^* + \nu| = \left| \underbrace{\frac{(\sigma-1)\mu}{1+\sigma\mu}}_{<1 \text{ from (1.52)}} + \underbrace{\nu}_{<1 \text{ by assumption}} \right| < 2 \tag{1.93}$$

$$|\phi_2^m| = |-\phi_1^* \nu| = \left| \underbrace{\frac{(\sigma-1)\mu}{1+\sigma\mu}}_{<1 \text{ from (1.52)}} \times \underbrace{\nu}_{<1 \text{ by assumption}} \right| < 1 \tag{1.94}$$

The moments for this AR(2) process can be calculated from the difference equation (1.84) directly, assuming covariance stationarity. To simplify notation, superscripts are dropped, and the time index on the moving average innovations are rescaled, by defining $u_t^i \equiv \varepsilon_{t-1}^i$.

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + u_t, \tag{1.95}$$

$$\begin{aligned} \gamma(0) = E[z_t z_t] &= \phi_1 E[z_t z_{t-1}] + \phi_2 E[z_t z_{t-2}] + E[z_t u_t] \\ &= \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_l^2, \end{aligned} \tag{1.96}$$

$$\begin{aligned} \gamma(k) = E[z_{t-k} z_t] &= \phi_1 E[z_{t-k} z_{t-1}] + \phi_2 E[z_{t-k} z_{t-2}] + E[z_{t-k} u_t] \\ &= \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + 0 \quad \forall k \geq 1. \end{aligned} \tag{1.97}$$

From the variance(1.96) and covariances (1.97) respectively,

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad \forall k \geq 1. \tag{1.98}$$

$\blacksquare$

The introduction of mean reverting idiosyncratic productivity shocks that are correlated across generations of the same lineage, but i.i.d. across agents from the cohort does not change the expected ratio of bequest to final output. However, the variance and covariances are modified, hence affecting the lineage mobility and cross-sectional inequality.

Proposition 1.16. *In the presence of autocorrelated productivity shocks, the correlation of the ratio of bequests to total output across generations from the same dynasty is higher; hence the intergenerational inherited wealth mobility is lower. The higher the autocorrelation of intergenerational productivity shocks, ν, the higher the correlation of bequest to output ratio across generations, the lower the intergenerational mobility at the steady states.*

Proof.

$$\frac{\rho_1 = \rho\left[\frac{b^i_{t+1}}{Y_{t+1}}, \frac{b^i_t}{Y_t}\right] \text{ with } 0 < \nu < 1}{\rho_1 = \rho\left[\frac{b^i_{t+1}}{Y_{t+1}}, \frac{b^i_t}{Y_t}\right] \text{ with } \nu = 0} = \frac{\frac{\phi_1^* + \nu}{1 + \phi_1^* \nu}}{\phi_1^*} = \frac{1 + \frac{\nu}{\phi_1^*}}{1 + \phi_1^* \nu} > 1 \text{ if } 0 < \mu < 1. \tag{1.99}$$

$$\rho_2 = \rho\left[\frac{b^i_{t+2}}{Y_{t+2}}, \frac{b^i_t}{Y_t}\right] \text{ with } 0 < \nu < 1 = \frac{\phi_1^{*2} + \nu^2 + \phi_1^* \nu (1 - \phi_1^* \nu)}{1 + \phi_1^* \nu} > 0 = \rho_2 \text{ with } \nu = 0. \tag{1.100}$$

$$\begin{aligned} \frac{\partial \rho_1}{\partial \nu} &= \frac{1}{(1 + \phi_1^* \nu)^2} \cdot [1 + \phi_1^* \nu - (\phi_1^* + \nu)\phi_1^*] \\ &= \frac{(1 - \phi_1^{*2})}{(1 + \phi_1^* \nu)^2} \geq 0. \end{aligned} \tag{1.101}$$

$$\frac{\partial \rho_2}{\partial \nu} = \frac{1}{(1+\phi_1^*\nu)^2} \cdot \left[(2\nu + \phi_1^* - 2\phi_1^{*2}\nu)(1+\phi_1^*\nu) - \left[\phi_1^{*2} + \nu^2 + \phi_1^*\nu(1-\phi_1^*\nu)\right]\phi_1^*\right]$$
$$= \frac{(2\nu + \phi_1^* - 2\phi_1^{*2}\nu)(1-\phi_1^{*2})}{(1+\phi_1^*\nu)^2} \geq 0. \tag{1.102}$$

■

While intuitive, the impact of correlated productivity shocks (or "correlation of abilities" by Bossmann et al. (2007)) on the lineage mobility is not discussed in both Wan and Zhu (2012) and Bossmann et al. (2007). This is because to characterize the lineage mobility, I invoke the covariance-stationarity property of the time series not used by Bossmann et al. (2007). This property is not available in Wan and Zhu (2012) as they seek to generalize Bossmann et al. (2007) by relaxing the finite variance assumption of the labor productivity shocks.

1.6.2 Impact on cross-sectional distribution at steady states with correlated productivity shocks

Proposition 1.17. *In the presence of correlated productivity shocks, the cross sectional inequality of the ratio of bequests to total output is more than the economy with i.i.d. productivity shocks. The higher the autocorrelation of intergenerational productivity shocks, ν, the higher the asymptotic cross-sectional spread of bequests, the higher the cross-sectional bequest inequality at the steady states.*

Proof. Since the expected value of the bequest to output ratio are identical under both i.i.d. shocks and mean reverting correlated shocks as specified, the ratio of the asymptotic coefficients of variation of bequest are:

$$\frac{\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2 \; with\, 0<\nu<1}{\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2 \; with\, \nu=0} = \frac{Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] \; with\, 0<\nu<1}{Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] \; with\, \nu=0}$$
$$= \frac{\frac{1}{(1-\phi_2^m-\phi_1^m)(1-\phi_2^m+\phi_1^m)}\left(\frac{1-\phi_2^m}{1+\phi_2^m}\right)}{\frac{1}{1-\phi_1^*}}$$
$$= \left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right)\left(\frac{1}{1-\nu^2}\right) > 1. \qquad (1.103)$$

$$sign\frac{\partial CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial \nu} = sign\frac{\partial\left\{CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right\}^2}{\partial \nu}$$
$$= sign\,\frac{\partial}{\partial \nu}\left[\sigma_l^2\left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right)\frac{(1-\phi_1^*)^2}{(1-\phi_1^{*2})(1-\nu^2)}\right]$$
$$= sign\,\sigma_l^2\left(\frac{1-\phi_1^*}{1+\phi_1^*}\right)\left[\frac{1}{(1-\nu^2)}\cdot\frac{2\phi_1^*}{(1+\phi_1^*\nu)^2}+\left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right)\cdot\frac{2\nu}{(1-\nu^2)^2}\right] \geq 0.$$

■

This result is consistent with the results from Bossmann et al. (2007) and Wan and Zhu (2012), derived using different assumptions and methods, as stated in Section 1.3.5.

From Propositions 1.16 and 1.17, persistence in the labor productivity shocks within a lineage is thus undesirable from the Rawlsian's "veil of ignorance" perspective (Rawls (2009)) and instructive for policy makers.

1.6.3 Role of bequest motive on lineage bequest evolution and cross-sectional distribution at steady states with correlated shocks

Proposition 1.18. *In the presence of correlated productivity shocks, the higher the bequest motive, μ, the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility at the steady states.*

Proof. If $0 < \nu < 1$,

$$\frac{\partial \rho_1}{\partial \phi_1^*} = \frac{1}{(1+\phi_1^*\nu)^2} \cdot [1+\phi_1^*\nu - (\phi_1^*+\nu)\nu] = \frac{(1-\nu^2)}{(1+\phi_1^*\nu)^2} \geq 0. \tag{1.104}$$

$$\frac{\partial \rho_1}{\partial \mu} = \frac{\partial \rho_1}{\partial \phi_1^*} \cdot \frac{\partial \phi_1^*}{\partial \mu} = \frac{(1-\nu^2)}{(1+\phi_1^*\nu)^2} \frac{\sigma-1}{(1+\sigma\mu)^2} > 0. \tag{1.105}$$

$$\frac{\partial \rho_2}{\partial \phi_1^*} = \frac{(2\phi_1^*+\nu-2\nu^2\phi_1^*)(1-\nu^2)}{(1+\phi_1^*\nu)^2} \geq 0.$$

$$\frac{\partial \rho_2}{\partial \mu} = \frac{\partial \rho_2}{\partial \phi_1^*} \cdot \frac{\partial \phi_1^*}{\partial \mu} = \frac{(2\phi_1^*+\nu-2\nu^2\phi_1^*)(1-\nu^2)}{(1+\phi_1^*\nu)^2} \frac{\sigma-1}{(1+\sigma\mu)^2} > 0. \tag{1.106}$$

■

Proposition 1.19. *In the presence of correlated productivity shocks, an increase in bequest motive decreases cross sectional inequality of the ratio of bequest to total output at the steady states.*

Proof. If $0 < \nu < 1$,

$$
\begin{aligned}
sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]}{\partial \mu} = sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]\right)^2}{\partial \mu} &= sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]\right)^2}{\partial \phi_1^*}\frac{\partial \phi_1^*}{\partial \mu} \\
&= sign\,\frac{\partial}{\partial \phi_1^*}\left[\frac{\sigma_l^2}{(1-\nu^2)}\left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right)\left(\frac{1-\phi_1^*}{1+\phi_1^*}\right)\right]\frac{\partial \phi_1^*}{\partial \mu} \\
&= sign\,\frac{\sigma_l^2}{(1-\nu^2)}\left[\left(\frac{1-\phi_1^*}{1+\phi_1^*}\right)\frac{2\nu}{(1-\phi_1^*\nu)^2}+\left(\frac{1+\phi_1^*\nu}{1-\phi_1^*\nu}\right)\frac{-2}{(1+\phi_1^*)^2}\right]\frac{\partial \phi_1^*}{\partial \mu} \\
&= sign\,\frac{\sigma_l^2}{(1-\nu^2)}\underbrace{\left[\frac{2\nu(1-\phi_1^{*2})-2(1-\phi_1^{*2}\nu^2)}{(1+\phi_1^*)^2(1-\phi_1^*\nu)^2}\right]}_{<0,\text{ if } 0<\nu<1.}\underbrace{\frac{\sigma-1}{(1+\sigma\mu)^2}}_{\geq 0}\leq 0. \quad (1.107)
\end{aligned}
$$

∎

That is, at steady states, bequest remains an intergenerational risk pooling device, even with intergenerational mean reverting correlated shocks, reducing the cross-sectional inequality of wealth at the steady states. However, with correlated shocks, we have introduced persistence to the outcome of wealth within a lineage, by slowing the exponential decay property of a covariance-stationary AR process, and increasing the "memory" of both good and bad draws of labor productivity shocks, thus lowering the intergenerational mobility.

1.6.4 Characterization of lineage bequest evolution and cross-sectional distributions for period-2 cycle with correlated shocks

Similarly via the mixing of ARMA processes, with correlated productivity shocks, the lineage bequest evolution for the period-2 cycle follows an ARMA(4,2) process.

From equation (1.41),

$$\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b^i_t}{Y_t} - \frac{\delta_1}{\Omega_4} = [l^i_t - \bar{l}_t] \tag{1.108}$$

where $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma} = \delta_1$, and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. From the mean reverting shocks,

$$\left(l^i_t - \bar{l}_t\right) - \nu(l^i_{t-1} - \bar{l}_{t-1}) = \varepsilon^i_t, \text{ and} \tag{1.109}$$

$$\nu\left(l^i_{t-1} - \bar{l}_{t-1}\right) - \nu^2(l^i_{t-2} - \bar{l}_{t-2}) = \nu\varepsilon^i_{t-1}, \text{ we get:} \tag{1.110}$$

$$\left(l^i_t - \bar{l}_t\right) - \nu(l^i_{t-1} - \bar{l}_{t-1}) + \nu\left(l^i_{t-1} - \bar{l}_{t-1}\right) - \nu^2(l^i_{t-2} - \bar{l}_{t-2}) = \varepsilon^i_t + \nu\varepsilon^i_{t-1}. \tag{1.111}$$

Substituting (1.108) into (1.111),

$$\left(\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b^i_t}{Y_t} - \frac{\delta_1}{\Omega_4}\right) - \nu^2\left(\frac{1}{\Omega_4}\frac{b^i_{t-1}}{Y_{t-1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b^i_{t-2}}{Y_{t-2}} - \frac{\delta_1}{\Omega_4}\right) = \varepsilon^i_t + \nu\varepsilon^i_{t-1}$$

$$\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\nu^2}{\Omega_4}\frac{b^i_{t-1}}{Y_{t-1}} = \phi_1^*\left(\frac{1}{\Omega_4}\frac{b^i_t}{Y_t} - \frac{\nu^2}{\Omega_4}\frac{b^i_{t-2}}{Y_{t-2}}\right) + \frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4} + \varepsilon^i_t + \nu\varepsilon^i_{t-1} \tag{1.112}$$

By iterative substitution, we get an ARMA(4,2) process for each of the fixed point in period 2 cycle:

$$\begin{aligned}\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\nu^2}{\Omega_4}\frac{b^i_{t-1}}{Y_{t-1}} \\ - \phi_1^*\left[\phi_1^*\left(\frac{1}{\Omega_4}\frac{b^i_{t-1}}{Y_{t-1}} - \frac{\nu^2}{\Omega_4}\frac{b^i_{t-3}}{Y_{t-3}}\right) + \frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4} + \varepsilon^i_{t-1} + \nu\varepsilon^i_{t-2}\right] \\ - \frac{\delta_1}{\Omega_4} + \frac{\nu^2\delta_1}{\Omega_4} = \varepsilon^i_t + \nu\varepsilon^i_{t-1}\end{aligned} \tag{1.113}$$

$$\begin{aligned}\frac{1}{\Omega_4}\frac{b^i_{t+1}}{Y_{t+1}} - \left(\phi_1^{*2} + \nu^2\right)\frac{1}{\Omega_4}\frac{b^i_{t-1}}{Y_{t-1}} - \left(-\phi_1^{*2}\nu^2\right)\frac{1}{\Omega_4}\frac{b^i_{t-3}}{Y_{t-3}} \\ - (1+\phi_1^*)\left(\frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4}\right) = \varepsilon^i_t + (\phi_1^* + \nu)\,\varepsilon^i_{t-1} + \phi_1^*\nu\varepsilon^i_{t-2}\end{aligned} \tag{1.114}$$

Let the superscript cm represents the period-2 cycle with mean reverting shocks. Define $z_{t+1}^{cmi} = \frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - (1+\phi_1^*)(1-\nu^2)\left(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_2^{cm}-\phi_4^{cm}}\right)$. Lineage dynamics of period-2 cycles with mean reverting productivity shocks from (1.114) can be expressed as follows:

$$z_{t+1}^{cmi} - \phi_2^{cm} z_{t-1}^{cmi} - \phi_4^{cm} z_{t-3}^{cmi} = \left[l_t^i - \bar{l}_t\right] - \nu^2\left[l_{t-2}^i - \bar{l}_{t-2}\right] = \varepsilon_t^i + \theta_1\varepsilon_{t-1}^i + \theta_2\varepsilon_{t-2}^i \tag{1.115}$$

where $\phi_2^{cm} = \left(\phi_1^{*2} + \nu^2\right)$, $\phi_4^{cm} = \left(-\phi_1^{*2}\nu^2\right)$, $\theta_1 = (\phi_1^* + \nu)$, $\theta_2 = (\phi_1^*\nu)$, $\delta_z^{cm} = (1+\phi_1^*)(1-\nu^2)\left(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_2^{cm}-\phi_4^{cm}}\right)$, $\phi_1^* = \frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]} = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$, $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\sigma\Omega_1\Omega_3[1+\beta(1+\mu)]} = \delta_1$, and $\varepsilon_t^i, \varepsilon_{t-1}^i, \varepsilon_{t-2}^i \sim$ i.i.d.$(0, \sigma_l^2)$.

For each of the fixed points, the asymptotic cross-sectional distribution can be derived from the lineage bequest evolution,

$$z_{t+1}^{cmi} - \phi_2^{cm} z_{t-1}^{cmi} - \phi_4^{cm} z_{t-3}^{cmi} = \varepsilon_t^i + \theta_1\varepsilon_{t-1}^i + \theta_2\varepsilon_{t-2}^i. \tag{1.116}$$

Proposition 1.20. *Since $\left|\phi_4^{cm}\right| < 1$, and $\left|\phi_2^{cm}\right| < 2$, z_{t+1}^{cmi} is a covariance stationary ARMA(4,2) process.*

$$E\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\delta_1}{1-\phi_1^*}. \tag{1.117}$$

$$Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \sigma_l^2\Omega_4^2\frac{\phi_2\theta_2(1+\phi_4) + (1-\phi_4)\left(1+\theta_1^2+\theta_2^2\right)}{(1+\phi_4)(1-\phi_4-\phi_2)(1-\phi_4+\phi_2)}$$

$$\begin{aligned} Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] &= \sigma_l^2\Omega_4^2\left[\frac{1}{(1-\nu^4)(1-\nu^2\phi_1^{*2})(1-\phi_1^{*4})}\right] \\ &\quad \times [1+\phi_1^{*2} + \nu(2\phi_1^* + \phi_1^{*3}) + \nu^2(1+2\phi_1^{*2}+\phi_1^{*4}) \\ &\quad + \nu^3(\phi_1^* + 2\phi_1^{*3} - \phi_1^{*5}) + \nu^4(\phi_1^{*2}+\phi_1^{*4}) + \nu^5(-\phi_1^{*3})]. \end{aligned} \tag{1.118}$$

$$Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right]=\sigma_l^2\frac{\theta_1+\theta_1\theta_2}{(1-\phi_2-\phi_4)(1+\phi_4)}=\sigma_l^2\frac{(\phi_1^*+\nu)}{(1-\nu^2)(1-\phi_1^{*2})(1-\nu\phi_1^*)}. \tag{1.119}$$

$$Cov\left[\frac{b^i_{t+2}}{Y_{t+2}},\frac{b^i_t}{Y_t}\right]=\frac{\phi_2 Var\left[\frac{b^i_t}{Y_t}\right]+\theta_2\sigma_l^2}{1-\phi_4}. \tag{1.120}$$

$$Cov\left[\frac{b^i_{t+3}}{Y_{t+3}},\frac{b^i_t}{Y_t}\right]=(\phi_2+\phi_4)Cov\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right]. \tag{1.121}$$

$$Cov\left[\frac{b^i_{t+4}}{Y_{t+4}},\frac{b^i_t}{Y_t}\right]=\frac{\phi_2\theta_2\sigma_l^2}{1-\phi_4}+\left(\frac{\phi_2}{1-\phi_4}+\phi_4\right)Var\left[\frac{b^i_t}{Y_t}\right]. \tag{1.122}$$

$$\begin{aligned}\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_t}{Y_t}\right]=\rho_1=&[(1+\phi_1^{*2})(1+\nu^2)(\nu+\phi_1^*)(1+\nu\phi_1^*)]\div\dots\\ &\dots[1+\phi_1^{*2}+\nu(2\phi_1^*+\phi_1^{*3})+\nu^2(1+2\phi_1^{*2}+\phi_1^{*4})+\dots\\ &\dots\nu^3(\phi_1^*+2\phi_1^{*3}-\phi_1^{*5})+\nu^4(\phi_1^{*2}+\phi_1^{*4})+\nu^5(-\phi_1^{*3})].\end{aligned} \tag{1.123}$$

$$\begin{aligned}\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_{t-1}}{Y_{t-1}}\right]&=\rho_2=\frac{\phi_2}{1-\phi_4}+\frac{\theta_2\sigma_l^2}{(1-\phi_4)\gamma(0)}\\ &=\frac{(\phi_1^{*2}+\nu^2)}{(1+\phi_1^{*2}\nu^2)}+[(\phi_1^*+\nu)(1-\nu^4)(1-\nu^2\phi_1^{*2})(1-\phi_1^{*4})]\div\dots\\ &\dots\{(1+\phi_1^{*2}\nu^2)[1+\phi_1^{*2}+\nu(2\phi_1^*+\phi_1^{*3})+\nu^2(1+2\phi_1^{*2}+\phi^{*4})+\dots\\ &\dots\nu^3(\phi_1^*+2\phi_1^{*3}-\phi^{*5})+\nu^4(\phi_1^{*2}+\phi_1^{*4})+\nu^5(-\phi_1^{*3})]\}.\end{aligned} \tag{1.124}$$

$$\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_{t-2}}{Y_{t-2}}\right]=\rho_3=(\phi_2+\phi_4)\rho_1=(\phi_1^{*2}+\nu^2-\phi_1^{*2}\nu^2)\rho_1. \tag{1.125}$$

$$\rho\left[\frac{b^i_{t+1}}{Y_{t+1}},\frac{b^i_{t-3}}{Y_{t-3}}\right]=\rho_4=\phi_2\rho_2+\phi_4=(\phi_1^{*2}+\nu^2)\rho_2-\phi_1^{*2}\nu^2. \quad (1.126)$$

$$\begin{aligned}Cov\left[\frac{b^i_{t+k}}{Y_{t+k}},\frac{b^i_{t}}{Y_{t}}\right]&=\phi_2^{cm}Cov\left[\frac{b^i_{t+k-2}}{Y_{t+k-2}},\frac{b^i_{t}}{Y_{t}}\right]+\phi_4^{cm}Cov\left[\frac{b^i_{t+k-4}}{Y_{t+k-4}},\frac{b^i_{t}}{Y_{t}}\right]\ \forall k\geq 3\\&=(\phi_1^{*2}+\nu^2)Cov\left[\frac{b^i_{t+k-1}}{Y_{t+k-1}},\frac{b^i_{t}}{Y_{t}}\right]-\phi_1^{*2}\nu^2Cov\left[\frac{b^i_{t+k-2}}{Y_{t+k-2}},\frac{b^i_{t}}{Y_{t}}\right]\ \forall k\geq 3.\end{aligned} \quad (1.127)$$

$$\rho\left[\frac{b^i_{t+k}}{Y_{t+k}},\frac{b^i_{t}}{Y_{t}}\right]=\rho_k=\phi_2\rho_{k-2}+\phi_4\rho_{k-4}=(\phi_1^{*2}+\nu^2)\rho_{k-2}-\phi_1^{*2}\nu^2\rho_{k-4}\forall k\geq 3. \quad (1.128)$$

Proof. Essentially, the equation (1.116) can be expressed as:

$$(1-\phi_2^{cm}L^2-\phi_4^{cm}L^4)z^{cmi}_{t+1}=\varepsilon^i_t+\theta_1\varepsilon^i_{t-1}+\theta_2\varepsilon^i_{t-2}, \quad (1.129)$$

where L is the lag operator, and for z^{cmi}_{t+1} to be stationary, the roots of the characteristic equation, $(1-\phi_2^{cm}L^2-\phi_4^{cm}L^4)=0$, must lie outside the unit circle. Let the roots be L_i where $i=1,2,3$ and 4. $(1-\phi_2^{cm}L^2-\phi_4^{cm}L^4)$ is a biquadratic equation. Let $B=L^2$, and B_1 and B_2 be the roots of $(1-\phi_2^{cm}B-\phi_4^{cm}B^2)=0$. Let L_1 and L_2 be the positive and negative square root of B_1, and L_3 and L_4 be the positive and negative square root of B_2. For $|L_i|>1$ or $\left|\frac{1}{L_i}\right|<1$, the following 2 conditions have to be met: $\left|\frac{1}{B_1}\cdot\frac{1}{B_2}\right|=\left|\phi_4^{cm}\right|<1$, and $\left|\frac{1}{B_1}+\frac{1}{B_2}\right|=\left|\phi_2^{cm}\right|<2$.

$$\begin{aligned}|\phi_2^{cm}|&=|\phi_1^{*2}+\nu^2|\\&=\left|\left[\underbrace{\frac{(\sigma-1)\mu}{1+\sigma\mu}}_{\text{between 0 and 1from (1.52)}}\right]^2+\left[\underbrace{\nu}_{\text{between 0 and 1 by assumption}}\right]^2\right|<2,\end{aligned} \quad (1.130)$$

$$|\phi_4^{cm}| = \left|-\phi_1^{*2}v^2\right| = \left| - \left[\frac{(\sigma-1)\mu}{1+\sigma\mu}\right]^2 [v]^2 \right| < 1. \tag{1.131}$$

With covariance-stationarity, we can compute the covariances and correlations of the lineage bequest evolution. The algebraic manipulations are shown in Appendix A.3. ■

By comparing Propositions 1.15 and 1.20, it can be seen that the equivalance of asymptotic cross-sectional distribution between the steady states and period-2 cycles in Proposition 1.10 no longer holds if the idiosyncratic productivity shocks are correlated. Unlike the steady states where the growth rates are identical for every generation, under period-2 cycles, the growth rates of faced by alternating generations are different. If productivity shocks are i.i.d. across each generation, the difference in the growth rates between the Solow and Romer regime will be immaterial once the inherited wealth is normalized by the total output, to keep the scale constant for comparison. However, if the productivity shocks are correlated across generations within a lineage, the good or bad luck is propagated via *both* the bequests as well as the next generation's productivity draw, hence differences in the growth rates between the Solow and Romer regimes will be accumulated over the cycle, resulting in a difference in the variance of the asymptotic cross-sectional distributions between the steady states and the period-2 cycles for correlated labor productivity shocks as stated in the following proposition.

Proposition 1.21. *In the presence of correlated productivity shocks, the cross-*

sectional inequality of the ratio of bequests to total output under period-2 cycles is greater (less) than that under steady states if the correlation of mean reverting productivity shocks, $\nu \geq (<) [\phi_1^{*3} + \nu^2(\phi_1^* + \phi_1^{*5}) + \nu^4(\phi_1^{*3})]$.

Proof. By taking a simple difference of the variances of the cross-sectional distributions between the steady states and the period-2 cycles with correlated shocks. ■

The threshold compares ν on the left against a polynomial of ν with coefficients that are functions of ϕ_1^*, and implicitly the bequest motive μ, on the right. This is because inheritance via bequest narrows the cross-sectional wealth inequality through risk sharing across generations of the same lineage as the rate of returns on capital, while different across generations of the same lineage in a period-2 cycle, is the same for each cohort. However, inheritance via ability through the correlated productivity shocks increases the cross-sectional wealth as the good or bad productivity draws across generations of the same lineage persist in a period-2 cycle, and are different across agents of the same cohort.

1.6.5 Impact on cross-sectional distributions under period-2 cycles with correlated productivity shocks

Proposition 1.22. *The higher the autocorrelation of intergenerational productivity shocks,* ν*, the higher the asymptotic cross-sectional spread of bequests, the higher the cross-sectional bequest inequality at the fixed points of the period-2 cycles.*

Proof.

$$sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]}{\partial \nu} = sign\frac{\partial\left\{CV\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]\right\}^2}{\partial \nu}$$
$$= sign\,(1-\phi_1^*)^2\frac{\partial\gamma(0)}{\partial\nu} \geq 0. \tag{1.132}$$

■

Just like the steady states, for period-2 cycle, cross-sectional inequality increases with the persistence of inherited ability. The impact of correlation of labor productivity on lineage mobility less clear.

$$\frac{\partial\rho_1}{\partial\nu} = \frac{1}{[\gamma(0)]^2}\left[\underbrace{\gamma(0)\frac{\partial\gamma(1)}{\partial\nu}}_{\geq 0} - \underbrace{\gamma(1)\frac{\partial\gamma(0)}{\partial\nu}}_{\geq 0}\right] \gtreqless 0. \tag{1.133}$$

1.6.6 Role of bequest motive on lineage bequest evolution and cross-sectional distributions under period-2 cycles with correlated shocks

Under period-2 cycles, with correlated productivity shocks, the role of bequests on lineage and cross-sectional inequality depends on the interaction of the bequest motive and the magnitude of the correlation of the mean reverting shocks. This is due to the interaction between the magnitude of the "ability inheritance" with the autoregressive coefficient in the absence of correlated shocks, $\phi_1^* = \frac{(\sigma-1)\mu}{1+\sigma\mu}$, arising from the bequest motive, resulting in the comparative statics for lineage mobility and the cross-sectional inequality with respect to bequest motive being ambiguous, unlike the steady state cases in Propositions 1.18 and 1.19.

If $0 < \nu < 1$,

$$\frac{\partial \rho_1}{\partial \phi_1^*} = \frac{1}{[\gamma(0)]^2}\left[\underbrace{\gamma(0)\frac{\partial \gamma(1)}{\partial \phi_1^*}}_{\geq 0} - \underbrace{\gamma(1)\frac{\partial \gamma(0)}{\partial \phi_1^*}}_{\geq 0}\right] \gtreqless 0, \tag{1.134}$$

$$\Rightarrow \frac{\partial \rho_1}{\partial \mu} = \frac{\partial \rho_1}{\partial \phi_1^*}\frac{\partial \phi_1^*}{\partial \mu} \gtreqless 0. \tag{1.135}$$

$$sign\frac{\partial CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial \mu} = sign\frac{\partial\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2}{\partial \mu} = sign\frac{\partial\left(CV\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]\right)^2}{\partial \phi_1^*}\frac{\partial \phi_1^*}{\partial \mu} \tag{1.136}$$

$$= sign\,\frac{\partial}{\partial \phi_1^*}\left[(1-\phi_1^*)^2 \cdot \gamma(0)\right]\frac{\partial \phi_1^*}{\partial \mu} \tag{1.137}$$

$$= sign\left[\gamma(0)2(1-\phi_1^*)(-1) + (1-\phi_1^*)^2\frac{\partial \gamma(0)}{\partial \phi_1^*}\right]\frac{\partial \phi_1^*}{\partial \mu} \tag{1.138}$$

$$= sign\left[\underbrace{\gamma(0)2(1-\phi_1^*)(-1)}_{\leq 0} + \underbrace{(1-\phi_1^*)^2\frac{\partial \gamma(0)}{\partial \phi_1^*}}_{\geq 0}\right]\underbrace{\frac{\sigma - 1}{(1+\sigma\mu)^2}}_{\geq 0} \gtreqless 0. \tag{1.139}$$

1.7 Conclusion

Using covariance-stationarity, and normalizing the inherited wealth with total output, it can be seen that intergenerational mobility and cross-sectional inequality at either Solow or Romer steady states are identical for both idiosyncratic and mean reverting correlated shocks. The existence of a covariance-stationary distributions over period-2 cycles with fixed points on Solow and Romer growth regimes is proven and characterized. While Matsuyama (1999) has shown that the aggregate growth over period-2 cycles are higher than at either of the steady states, we extend that result by showing that wealth inequality over period-2 cycles, measured in terms of coefficient of variation, are identical to the steady states under i.i.d. idiosyncratic productivity shocks.

More explicitly, under i.i.d. idiosyncratic productivity shocks, for all fixed points, at steady states or period-2 cycles, the cross-sectional inequality of wealth are identical, even though the growth rates differ. Thus, from this chapter, it can be seen that differences in the rate of returns on capital versus the output growth could just be a red herring to the evolution of wealth inequality in the last century. The crux may lie with the confluence of the exogenous and endogenous institutional structures related to fiscal policies (Benabou (2000); García-Peñalosa and Turnovsky (2007, 2011), education and technology (Davies et al. (2005); Acemoglu and Autor (2012), and trade and industrial relations (Garicano and Rossi-Hansberg (2006); Costinot et al. (2012, 2013)). In the following chapters, I shall explore the role of consumption externality, elastic labor, and the role of inheritance tax and innovation subsidies under specific context.

In addition, under i.i.d. idiosyncratic productivity shocks, the higher price elasticity of the final sector's demand for each intermediate brought about by exogenous changes in production technology or contracting, (and correspondingly, higher capital share as well as lower monopoly margin for innovation), the lower the intergenerational mobility, and the lower the asymptotic cross-sectional spread of bequests. This is because mobility is driven by the idiosyncratic labor productivity shocks, and a decrease in wage share will lower intergenerational mobility as capital returns are homogeneous and thus wealth inherited are rank preserving.

With correlated productivity shocks, intergenerational mobility is lower than that with i.i.d. shocks for the steady states. The higher the correlation in the mean reverting shocks, the higher the cross-sectional inequality for both the steady states and period-2 cycles. This means that any structural or institutional changes in educational policy or credit policy that can exogenously increase the correlation of labor productivity within a lineage will increase cross-sectional inequality.

Finally, with the "joy-of-giving" bequest motive, an increase in bequest motive reduces intergenerational mobility and cross-sectional inequality of inherited wealth under both Solow and Romer steady states for both i.i.d. productivity shocks and mean reverting correlated shocks, and for period-2 cycles under i.i.d. shocks. That is, bequest can act as intergenerational risk pooling mechanism within a lineage, but across lineages, bequest is a stratifying mechanism.

Chapter 2

Status anxiety and its impact on mobility, inequality and growth trajectories

2.1 Introduction

"Social status is a ranking of individuals (or groups of individuals) in a given society, based on their traits, assets and actions. [. . .] the interest in social status as a factor in explaining behavior stems from empirical observation that there is a substantial agreement among different members of society about the relative position of a given individual (or social position). It is this concordance of ranking which gives social status its force as an incentive mechanism. [. . .] Because of these social rewards, each individual seeks to increase his social status through group affiliation, investments in assets (including human and social capital) and the appropriate choice of actions," Weiss and Fershtman (1998, pp 802).

Like a good that is fixed in supply, even for a growing economy, it is interesting to examine how social comparison may affect the individual and aggregate consumption and saving patterns, and the growth of an economy. Veblen (1899) argues that wealthy individuals consume conspicuous goods and services to signal their wealth levels in order to gain self esteem. Consequently, if everyone does that, all may end up with similar relative consumption levels, but lower savings, hence lowering aggregate investment and growth. However, the relative consumption preferences may not negatively affect savings and growth if individuals are willing to trade current consumption for their future status. The hypothesis in this chapter is that status anxiety, as reflected by a relative consumption preference, could affect (i) the normalized steady state capital stock in both Solow and Romer growth regimes, (ii) the growth potential of an economy, and hence (iii) the evolution of mobility and inequality of an economy. This is because, as observed by De Nardi (2015), the inequality of cross-sectional wealth distribution is higher than the inequality of cross-sectional labor earnings and income. Thus, in this chapter, I explore the impact of envy, through the introduction of consumption externality, to raise the savings rate of the rich, hence wealth concentration, as observed in the data.

The incorporation of status anxiety, as envy, into the preference as relative consumption is empirically motivated. Di Tella and MacCulloch (2006) observe that since World War II in the United States, happiness responses are flat in the face of considerable increases in average income. They propose two explanations for the paradox that have a stronger empirical basis: that happiness is based on relative rather than absolute income and that happiness adapts to changes in the level of income. Neumark and Postlewaite (1998) find that married women are more likely to work outside the home if their sisters' husband earn more than their own husbands, supporting the importance of preference interdependence. Relative utility is supported from the panel data for the Netherlands in a model estimated by Van de Stadt et al. (1985), where both one's past consumption and consumption of others influence utility. Using CEX data, Charles et al. (2007) show that accounting for differences in income characteristics of a reference group explains most of the racial differences in visible consumption. In addition, they find that the relative importance of interpersonal comparisons decreases with age. Experimental research also supports the importance of relative consumption. Alpizar et al. (2005) find that most individuals are concerned with both relative income and relative consumption of particular goods. The degree of concern varies in the expected direction depending on the properties of the good.

Related literature can be classified along three dimensions: (i) the use of status in utility, (ii) the existence of heterogeneous consumption and comparison utility, and (iii) the generation of earnings and wealth distribution from idiosyncratic risks and symmetry breaking as discussed in Chapter 1. There are existing theoretical models that incorporate status directly into the utility function. Becker et al. (2005) introduce status as a complement to other consumption goods that can be bought directly in a market. Moldovanu et al. (2007) use the relative positions in utility to study the optimal allocation of prizes in contests. Most recently, Rayo (2013) incorporates public perceptions of one's type into the utility and apply that to monopolistic design and pricing of positional goods that consumers use to signal their types.

Similar in spirit, models with heterogeneous consumption and comparison utility are well-entrenched and driven by empirics. Carroll and Weil (1994) and Carroll et al. (2000) suggest that habit formation may be needed to explain time series features of consumption data. Campbell and Cochrane (1995) use a model where individual compare their own consumption to a habit stock based on past aggregate consumption to explain aggregate stock prices. In Constantinides (1990)'s asset pricing model, individuals care about how their current consumption compares to their own consumption in the recent past, while Abel (1990) uses of both past aggregate comparison and past individual comparison. Bakshi and Chen (1996) find that by incorporating "catching up with the Joneses" type of status concerns into investors' preference, the standard deviation of the implied intertemporal marginal rate of substitution (stochastic discount factor) can better match the Hansen-Jagannathan volatility bounds using data from NYSE value-weighted index and long-term government bonds. Parallel to this study, Turnovsky and Monteiro (2007) examine the impact of consumption externalities on efficient capital accumulation with a Cobb-Douglas production technology. This chapter differs from theirs in two aspects: the functional form for the consumption externality, and more importantly, the production function with innovation and cyclical endogenous growth. As a result, consumption externality alone results in long-run distortionary effects in Turnovsky and Monteiro (2007) if and only if labor is elastically supplied; whilst in this chapter, even with inelastic labor, consumption externality will distort long-run growth potential.

The closest paper that studies the impact of envy on both the growth and inequality evolution dynamics is Gershman (2014). I shall briefly discuss the modelling similarities and differences here, and consequently some differences in the results arise. There are two types of externalities in Gershman (2014), namely: negative consumption externality (envy); and positive intergenerational production externality through learning-by-doing knowledge spillover (where all agents output are perfect substitutes). In this chapter, like Gershman (2014), there is negative consumption externality, but a threshold technological externality, generated via threshold capital to variety ratio for innovative growth through fixed cost and monopoly rent, hence driving the results that inequality may affect growth through the threshold quantum of accumulated capital (not present in another paper on envy by Alvarez-Cuadrado and Long (2012) with neoclassical growth). There are several key modelling differences. First, in Gershman (2014), to generate asymmetry, there are two groups of agents with initially different endowments of capital to start with, whilst I utilise a continuum of agents with ex-ante identical draw of labor productivities. (Agents can have ex-ante identical capital resources.) Second, in Gershman (2014), agents can engage in destructive behavior with their labor hours. In my model, with property rights well-enforced, it is equivalent to having the destruction side eliminated, resulting in only the "Keeping Up with Jones" behavior and equilibrium in Gershman (2014). Third, the endogenous growth is achieved through individuals' learning-by-doing that generates externality via the future productivity (based on a linear (perfect substitutes) combination of both agents output). Using Matsuyama (1999) framework for production, in this chapter, endogenous growth is attained through innovation by variety expansion with fixed cost, where agents affect the growth through the accumulation of capital through savings and bequest versus consumption when young and old. Among the findings from Gershman (2014), for the long run "Keeping up with the Jones" equilibrium, which occurs if initial inequality is low, and/or tolerance for inequality is high, agents become identical, with no more envy driving future generations, otherwise agents are stratified in the other possible equilibria, with zero mobility. Either way, there is no chance for any agent to leapfrog the initial rich. In this model, distribution is sustained by both the idiosyncratic labor productivity shocks and bequest to the next generation. That is, there is an element of luck that may help push an agent from a poor lineage out of poverty. That is, there is mobility in my model, and one advantage of using covariance-stationarity for the distributional equilibrium concept, is that the intergenerational mobility can be tracked explicitly through the stationary covariances, and forecast with mean square error specifications can be made for future generations for any lineage of interest.

Key questions to be addressed are: (i) how envy may influence the growth potential of an economy, (ii) how envy may affect aggregate saving and steady state aggregate capital under the different growth regimes, (iii) how envy may impact individual savings and bequest rates, such that they may vary across the income distribution, and (iv) how envy affects the cross sectional wealth inequality and inter-generational social mobility under the different growth regimes. This complements the work by Alvarez-Cuadrado and Long (2012), where they address whether bequests will result in a concentration of wealth in the presence of envy, and how a central planner can implement a progressive consumption tax so that agents can internalise the consumption externality introduced through envy.

This chapter is organized as follows. In Section 2, the model is set-up. In Section 3, the aggregate equilibrium is characterized. In Section 4, the impact of envy on lineage and cross-sectional dynamics are explored using the Solow steady state for exposition purposes, and Section 5 concludes.

2.2 The model

The basic model combines the household problem in Alvarez-Cuadrado and Long (2012) with the firms' problem in Matsuyama (1999). The household problem is analogous to Bossmann et al. (2007) where the latter abstracts from social comparison.

2.2.1 Agents' optimization

The set-up of the agent's problem is similar to that in Chapter 1, Section 1.2.1. However, unlike the agents in Chapter 1, agents in this chapter are concerned about both their absolute as well as the relative consumption among agents in the same generation. This difference is reflected in the agents' preference in the form of an additive specification[1] of relative consumption as in Ljungqvist and Uhlig (2000) and Alvarez-Cuadrado and Long (2012). The choice and implication of this additive utility form will be discussed later in this chapter.

The preference of an agent i born in period t is:

$$U_t\left(\widehat{c}_t^i, \widehat{d}_{t+1}^i, b_{t+1,}^i\right) = \ln(c_t^i - \gamma\bar{c}_t) + \beta\left[\ln(d_{t+1}^i - \eta\bar{d}_{t+1}) + \mu\ln(b_{t+1}^i)\right], \quad (2.1)$$

where γ, η and ξ are parameters of envy, with $0 \leq \gamma, \eta, \xi < 1, \eta = \xi\gamma$.

First period budget constraint for an agent i is:

$$c_t^i + s_t^i = b_t^i + w_t^i \equiv y_t^i. \quad (2.2)$$

Second period budget constraint for an agent i is:

$$R_{t+1}s_t^i = d_{t+1}^i + b_{t+1}^i. \quad (2.3)$$

Optimal choices of the average agent are[2]:

$$\bar{s}_t = \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t \equiv \Omega_1\bar{y}_t; \quad (2.4)$$

[1] Carroll et al. (2000) explore the use of multiplicative consumption externalities cast within a CRRA utility, while Arrow and Dasgupta (2009) investigate the existence of economic distortion due to consumption externality using various structural forms of felicity functions such as multiplicative consumption externality with elastic labor for infinitely lived agents.

[2] See Appendix B.1.

$$\bar{c}_t = \frac{(1-\xi\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t; \tag{2.5}$$

$$\bar{d}_{t+1} = \frac{R_{t+1}\beta(1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t; \tag{2.6}$$

$$\bar{b}_{t+1} = \frac{R_{t+1}\beta\mu(1-\gamma)(1-\xi\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t \equiv R_{t+1}\Omega_2\bar{y}_t. \tag{2.7}$$

Optimal choices of an agent i are[3]:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}\left[y_t^i - \phi_s\bar{y}_t\right]; \tag{2.8}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)}\left[y_t^i + \phi_c\bar{y}_t\right]; \tag{2.9}$$

$$d_{t+1}^i = \frac{R_{t+1}\beta}{1+\beta(1+\mu)}\left[y_t^i + \phi_d\bar{y}_t\right]; \tag{2.10}$$

$$b_{t+1}^i = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\left[y_t^i - \phi_b\bar{y}_t\right]; \tag{2.11}$$

where

$$\phi_s = \frac{(1+\mu)(1-\xi\gamma)\gamma - \xi\gamma(1-\gamma)}{(1+\mu)\{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}}; \tag{2.12}$$

$$\phi_c = \frac{\beta[(1+\mu)(1-\xi\gamma)\gamma - \xi\gamma(1-\gamma)]}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}; \tag{2.13}$$

$$\phi_d = \frac{\xi\gamma\mu(1-\gamma)\beta + \xi\gamma(1-\gamma) - (1-\xi\gamma)\gamma}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}; \tag{2.14}$$

$$\phi_b = \frac{(1-\xi\gamma)\gamma + \xi\gamma(1-\gamma)\beta}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}. \tag{2.15}$$

As stated in Alvarez-Caudrado and Long (2012) Proposition 1, the income elasticities for consumption when young, consumption when old and bequests

[3] See Appendix B.1.

for each individual are correspondingly:

$$\varepsilon_{c^i}^{y^i} \equiv \left[1+\phi_c(\frac{\overline{y}_t}{y_t^i})\right]^{-1} < \varepsilon_{d^i}^{y^i} \equiv \left[1+\phi_d(\frac{\overline{y}_t}{y_t^i})\right]^{-1} < 1 < \varepsilon_{b^i}^{y^i} \equiv \left[1-\phi_d(\frac{\overline{y}_t}{y_t^i})\right]^{-1}. \quad (2.16)$$

Thus, this specification of envy in the utility generates different elasticities for consumption when young, consumption when old and bequests for each individual such that bequest is a luxury good. In addition, as stated in Alvarez-Caudrado and Long (2012) Proposition 2, the change of income elasticities of savings and bequests with respect to a change in income are respectively:

$$\frac{\partial \frac{s_t^i}{y_t^i}}{\partial y_t^i} = \frac{\phi_c \overline{y}_t}{[1+\beta(1+\mu)](y_t^i)^2} > 0; \quad (2.17)$$

$$\frac{\partial \frac{b_{t+1}^i}{y_t^i}}{\partial y_t^i} = \frac{R_{t+1}\beta\mu\phi_b\overline{y}_t}{[1+\beta(1+\mu)](y_t^i)^2} > 0. \quad (2.18)$$

Thus, this specification of envy in the utility generates income elasticity of savings and bequests that are dependent on one's income. This implies that a non-degenerate distribution of bequest can be obtained even with inelastic labor, and ex-ante identical agents. Hence, the 2 reasons for adopting this additive consumption externality structure rather than the multiplicative form. Empirically, using data from PSID, SCF and CES, Dynan et al. (2004) find that savings rate, as well as the marginal propensity to save are positively related to lifetime income.

2.2.2 Firms' optimization

The production sector is identical to that described in Chapter 1, Section 1.2.2.

2.3 Equilibrium steady state and dynamics

2.3.1 Derivation of equilibrium capital stock

The market clearing conditions are:

$$L_t = L = \int_0^1 l_t^i \, di = 1; \tag{2.19}$$

$$K_t = \int_0^1 s_t^i \, di = \bar{s}_t \equiv S_t. \tag{2.20}$$

Given the competitive Cobb-Douglas final goods sector, factor shares of the economy are:

$$w_t L = \frac{1}{\sigma} Y_t; \tag{2.21}$$

$$R_t K_{t-1} = \left(1 - \frac{1}{\sigma}\right) Y_t. \tag{2.22}$$

From the optimization decision of the average agent, and by substituting the lifetime resource (B.11), bequest (B.13), and wage (2.21) of the average agent into his savings, we can obtain an equivalent expression for the aggregate savings, and the aggregate capital stock of the economy, since we have measure 1 of agents in each generation. Specifically,

$$\begin{aligned}\bar{s}_t &= \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]} \bar{y}_t \\ &\equiv \widetilde{\Omega}_1[\bar{b}_t + w_t] \\ &= \widetilde{\Omega}_1 w_t + \widetilde{\Omega}_1 \left[\frac{\mu(1-\xi\gamma)}{1+\mu(1-\xi\gamma)}\right] R_t \bar{s}_{t-1}.\end{aligned} \tag{2.23}$$

The unconsumed final goods at time t, will form the available capital stock for production at time $t+1$. Thus by integrating the savings across all young agents, the aggregate savings in the economy is:

$$K_t = S_t = \bar{s}_t = \widetilde{\Omega}_1 w_t L + \widetilde{\Omega}_1 \left[\frac{\mu(1-\xi\gamma)}{1+\mu(1-\xi\gamma)}\right] R_t K_{t-1}, \tag{2.24}$$

and the capital stock as a function of the final output of the economy can be derived by substituting out the factor shares, (2.21) and (2.22), as follow:

$$\begin{aligned} K_t &= \widetilde{\Omega}_1 \left(\frac{1}{\sigma}\right) Y_t + \widetilde{\Omega}_1 \left[\frac{\mu(1-\xi\gamma)}{1+\mu(1-\xi\gamma)}\right] \left(1-\frac{1}{\sigma}\right) Y_t \\ &= \widetilde{\Omega}_1 \widetilde{\Omega}_3 Y_t, \end{aligned} \tag{2.25}$$

where $\widetilde{\Omega}_1 \equiv \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$ and $\widetilde{\Omega}_3 \equiv \frac{1+\sigma\mu(1-\xi\gamma)}{\sigma[1+\mu(1-\xi\gamma)]}$. That is, this economy saves a constant fraction of its output for the next period.

From the savings accumulation process (2.25), in conjunction with the dynamics of innovation (1.21) and the total output (1.23), the unique equilibrium path for any initial condition, K_0 and N_0 is pinned down.

$$\frac{K_t}{\widetilde{\Omega}_1 \widetilde{\Omega}_3} = Y_t = \begin{cases} A\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}} \left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1}, \\ A K_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases} \tag{2.26}$$

where $A \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{\frac{1}{\sigma}}$.

This dynamical system can be normalized by the range of intermediate goods to a 1-dimensional map. Define $\Phi^n(k) \equiv \Phi\left(\Phi^{n-1}(k)\right)$, $\Phi^1(k) \equiv \Phi(k)$, and

$$k_t \equiv \frac{K_t}{\theta\sigma F N_t}.$$

Equilibrium path for initial condition k_0 is given by the sequence $\{\Phi^t(k_0)\}$:

$$k_t = \Phi(k_{t-1}) \equiv \begin{cases} \widetilde{G}(k_{t-1})^{1-\frac{1}{\sigma}} & \text{if} \quad k_{t-1} \le k_c = 1, \\ \frac{\widetilde{G}k_{t-1}}{1+\theta(k_{t-1}-1)} & \text{if} \quad k_{t-1} \ge k_c = 1, \end{cases} \tag{2.27}$$

where

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$[4], $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$, $\widetilde{G} \equiv \widetilde{\Omega}_1\widetilde{\Omega}_3 A$, $\widetilde{\Omega}_1 \equiv \frac{\beta(1+\mu(1-\xi\gamma))(1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$,

$\widetilde{\Omega}_3 \equiv \frac{1+\sigma\mu(1-\xi\gamma)}{\sigma[1+\mu(1-\xi\gamma)]}$, and $A \equiv \frac{\hat{A}}{a}\left(\frac{aL}{\theta\sigma F}\right)^{\frac{1}{\sigma}}$.

The mapping $k_t = \Phi(k_{t-1})$ with domain $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ in (2.27) is similar to equation (1.32) in Chapter 1, with changes in the parameter value of $\widetilde{G}$. That is, status anxiety in the form suggested by Alvarez-Cuadrado and Long (2012), when incorporated into Matsuyama (1999) with idiosyncratic productivity shocks and bequest motives, results in corresponding topology and dynamics as in Chapter 1. From here, I shall highlight the differences that status anxiety may bring to the economy compared to the results in Chapter 1, as a consequence of the change in $\widetilde{G}$. Furthermore, compared to Alvarez-Cuadrado and Long (2012) where the production function is homogenous of degree one to capital and labor, the results in this chapter include their results for the case when $\widetilde{G} < 1$, where the economy converges to the Solow steady state. When $\widetilde{G} > 1$, new insights arising from status anxiety will be discussed.

[4] $k = 0$ is excluded from the domain in the mapping $k_t = \Phi(k_{t-1})$, as $\Phi'(0) > 1$, making $k = 0$ a repelling fixed point and hence trivial.

2.3.2 Aggregate steady state and dynamics

Proposition 2.1. *The growth potential of the economy is decreasing in the degree of envy γ. However, in the absence of a bequest motive, but the presence of envy, the growth potential of the economy is increasing in the degree of relative envy when old, ξ.*

Proof. Market clearing (equations (2.21), (2.22) and (2.24)), and the consumer optimization (equation (2.25)) imply $S_t = K_t = \widetilde{\Omega}_1\widetilde{\Omega}_3 Y_t$. Together with the final output function (1.23),

$$Y_t = \begin{cases} A\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}}\left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1} \\ AK_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases}$$

along the balanced growth path, $K_t = \widetilde{\Omega}_1\widetilde{\Omega}_3 Y_t = \widetilde{\Omega}_1\widetilde{\Omega}_3 A K_{t-1} = \widetilde{G}K_{t-1}$, where $\widetilde{G} \equiv$ gross growth rate.

$$\frac{\partial \widetilde{G}}{\partial \gamma} = A\left[\widetilde{\Omega}_3\frac{\partial \widetilde{\Omega}_1}{\partial \gamma} + \widetilde{\Omega}_1\frac{\partial \widetilde{\Omega}_3}{\partial \gamma}\right] \leq 0. \tag{2.28}$$

$$\frac{\partial \widetilde{G}}{\partial \xi} = A\left[\widetilde{\Omega}_3\frac{\partial \widetilde{\Omega}_1}{\partial \xi} + \widetilde{\Omega}_1\frac{\partial \widetilde{\Omega}_3}{\partial \xi}\right] > \text{or} \leq 0. \quad \text{If}\, \mu = 0, \frac{\partial \widetilde{G}}{\partial \xi} \geq 0. \tag{2.29}$$

The algebraic manipulations to sign $\frac{\partial \widetilde{\Omega}_1}{\partial \gamma}$, $\frac{\partial \widetilde{\Omega}_1}{\partial \xi}$, $\frac{\partial \widetilde{\Omega}_3}{\partial \gamma}$, and $\frac{\partial \widetilde{\Omega}_3}{\partial \xi}$ are shown in Appendix B.2. ■

Proposition 2.2. *The normalized steady-state capital stock to variety ratio, k_t, is decreasing in the degree of envy γ, in both the regimes. However, in the absence of a bequest motive, but the presence of envy, the normalized steady-state capital stock to variety ratio, k_t, is increasing in the degree of relative envy when old, ξ.*

Proof. From equation (2.27):

when $\widetilde{G} < 1$, $k^* = (\widetilde{\Omega}_1\widetilde{\Omega}_3 A)^\sigma \equiv \widetilde{G}^\sigma$;

when $\widetilde{G} > 1$, $k = k^{**} \equiv 1 + \frac{(\widetilde{G}-1)}{\theta}$.

$$\frac{\partial k^*}{\partial \gamma} = \frac{\partial k^*}{\partial \widetilde{G}} \times \frac{\partial \widetilde{G}}{\partial \gamma} = (\sigma \widetilde{G}^{\sigma-1})A\left[\widetilde{\Omega}_3 \frac{\partial \widetilde{\Omega}_1}{\partial \gamma} + \widetilde{\Omega}_1 \frac{\partial \widetilde{\Omega}_3}{\partial \gamma}\right] \leq 0. \tag{2.30}$$

$$\frac{\partial k^{**}}{\partial \gamma} = \frac{\partial k^{**}}{\partial \widetilde{G}} \times \frac{\partial \widetilde{G}}{\partial \gamma} = \frac{1}{\theta}A\left[\widetilde{\Omega}_3 \frac{\partial \widetilde{\Omega}_1}{\partial \gamma} + \widetilde{\Omega}_1 \frac{\partial \widetilde{\Omega}_3}{\partial \gamma}\right] \leq 0. \tag{2.31}$$

$$\begin{aligned}\frac{\partial k^*}{\partial \xi} &= \frac{\partial k^*}{\partial \widetilde{G}} \times \frac{\partial \widetilde{G}}{\partial \xi} = (\sigma \widetilde{G}^{\sigma-1})A\left[\widetilde{\Omega}_3 \frac{\partial \widetilde{\Omega}_1}{\partial \xi} + \widetilde{\Omega}_1 \frac{\partial \widetilde{\Omega}_3}{\partial \xi}\right] > \text{or} \leq 0, \\ &\geq 0 \quad \text{if } \mu = 0. \end{aligned} \tag{2.32}$$

$$\begin{aligned}\frac{\partial k^{**}}{\partial \xi} &= \frac{\partial k^{**}}{\partial \widetilde{G}} \times \frac{\partial \widetilde{G}}{\partial \xi} = \frac{1}{\theta}A\left[\widetilde{\Omega}_3 \frac{\partial \widetilde{\Omega}_1}{\partial \xi} + \widetilde{\Omega}_1 \frac{\partial \widetilde{\Omega}_3}{\partial \xi}\right] > \text{or} \leq 0, \\ &\geq 0 \quad \text{if } \mu = 0. \end{aligned} \tag{2.33}$$

The algebraic manipulations to sign $\frac{\partial \widetilde{\Omega}_1}{\partial \gamma}$, $\frac{\partial \widetilde{\Omega}_1}{\partial \xi}$, $\frac{\partial \widetilde{\Omega}_3}{\partial \gamma}$, and $\frac{\partial \widetilde{\Omega}_3}{\partial \xi}$ are shown in Appendix B.2. ■

Comparing with Gershman (2014, Proposition 2), where an increase in envy increases the aggregate final output in the "Keeping up with the Jones" equilibrium, (where $\tau = 0$, that is no destruction of private capital), the results from Proposition 2.1 and 2.2 differ in that with bequest, growth potential is decreasing with envy, γ; without bequest, growth potential increasing with envy in the old age, ξ.

In this model, agents save to finance consumption when old (positional), and for bequest to their immediate offspring (non-positional). An increase in envy, denoted by γ, will cause the agents to allocate more of their lifetime resources to positional goods (consumption when young and old), hence lowering savings and available aggregate capital stock for innovation.

The savings impact of an increase in the *relative* importance of envy when old, ξ, in the presence of bequests, is ambiguous, as positional and non-positional motives for savings compete. Without bequests, the ambiguity is removed.

The results extend the Proposition 3 from Alvarez-Cuadrado and Long (2012, pp. 960) to endogenous growth model. Envy not only affects the steady state normalized capital stock in *both* Solow and Romer regime, it also determines on the *growth potential* of an economy with endogenous growth. Contrastingly, in Turnovsky and Monteiro (2007) where labor is also inelastic, and in Alonso-Carrera et al. (2008), with pure altruism, where consumption externality extends beyond agents from the same generation, the steady state capital stock is not affected by consumption externality. Furthermore, in Alvarez-Cuadrado and Long (2012), there is no long run growth as a consequence of the neoclassical growth framework.

2.4 Impact of envy on lineage bequest evolution and distribution

In the following section, I shall highlight the differences that envy might bring to the evolution of mobility and inequality, as well as a result that is robust to envy. While I develop the exposition using the case of Solow steady state, the following Propositions (2.4, 2.5, 2.6 and 2.7) hold correspondingly when applied to the Romer steady state or the period-2 cycles, given the distributional equivalence under i.i.d. shocks, as infered from Proposition 1.10, compared to their respective benchmarks in Chapter 1, without envy, by setting $\gamma = 0$.

2.4.1 Lineage bequest evolution

At Solow steady state

At the Solow steady state, $\widetilde{k}^* = \widetilde{G}^{\sigma} < 1$. Let $Y_{t+1} = Y_t \equiv \widetilde{Y}^*$, $R_{t+1} = R_t \equiv \widetilde{R}^*$, and $K_{t+1} = K_t \equiv \widetilde{K}^*$. From (2.26), in the Solow regime,

$$R_t = \frac{\partial Y_t}{\partial K_t} = \left(1 - \frac{1}{\sigma}\right) A \left(\theta \sigma F N_{t-1}\right)^{\frac{1}{\sigma}} \left(K_{t-1}\right)^{-\frac{1}{\sigma}} = \left(1 - \frac{1}{\sigma}\right) A \left(k_{t-1}\right)^{-\frac{1}{\sigma}}, \tag{2.34}$$

where $k_t = \frac{K_t}{\theta \sigma F N_t}$. At Solow steady state, $\widetilde{R}^* = \left(1 - \frac{1}{\sigma}\right) A \left(\widetilde{k}^*\right)^{-\frac{1}{\sigma}}$. From equation (2.27),

$$\widetilde{k}^* = \widetilde{G}(\widetilde{k}^*)^{\frac{\sigma-1}{\sigma}} \Rightarrow \frac{\widetilde{k}^{*\sigma}}{\widetilde{k}^{*\sigma}} = \frac{\widetilde{G}(\widetilde{k}^*)^{\sigma-1}}{\widetilde{k}^{*\sigma}} \Rightarrow 1 = \widetilde{G}^{\sigma}(\widetilde{k}^*)^{-1} \Rightarrow \widetilde{k}^* = \widetilde{G}^{\sigma} = (\widetilde{\Omega}_1 \widetilde{\Omega}_3 A)^{\sigma}. \tag{2.35}$$

$$\widetilde{R}^* = \left(1 - \frac{1}{\sigma}\right) A \left(\widetilde{k}^*\right)^{-\frac{1}{\sigma}} = \left(1 - \frac{1}{\sigma}\right) A \left(\widetilde{\Omega}_1 \widetilde{\Omega}_3 A\right)^{-1} = \frac{\left(1 - \frac{1}{\sigma}\right)}{\widetilde{\Omega}_1 \widetilde{\Omega}_3}. \tag{2.36}$$

Using the bequest of an agent of lineage i, equation (2.11), the savings of the average agent, equation (2.4), as a function of the average lifetime resources, as well as the capital and labor shares, equations (2.21) and (2.22), we derive the law of motion of the bequests for an agent of lineage i.

$$\begin{aligned} b^i_{t+1} &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(y^i_t - \phi_b \bar{y}_t) \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(y^i_t - \phi_b \frac{\bar{s}_t}{\widetilde{\Omega}_1}) \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(y^i_t - \phi_b \frac{\widetilde{\Omega}_1\widetilde{\Omega}_3 Y_t}{\widetilde{\Omega}_1}) \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(b^i_t + l^i_t w_t - \phi_b \widetilde{\Omega}_3 Y_t) \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(b^i_t + l^i_t \frac{1}{\sigma} Y_t - \phi_b \widetilde{\Omega}_3 Y_t), \end{aligned} \tag{2.37}$$

where $\phi_b \equiv \frac{(1-\xi\gamma)\gamma+\xi\gamma(1-\gamma)\beta}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$.

At the Solow steady state, normalizing[5] by the total output, Y_{t+1} and rearranging,

$$\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\widetilde{R}^*\beta\mu}{1+\beta(1+\mu)}\frac{b^i_t}{Y_t} - \frac{\widetilde{R}^*\beta\mu}{\sigma[1+\beta(1+\mu)]}\left[1-\sigma\phi_b\widetilde{\Omega}_3\right] = \frac{\widetilde{R}^*\beta\mu}{1+\beta(1+\mu)}\frac{1}{\sigma}[l^i_t - \bar{l}_t]. \tag{2.38}$$

The ratio of bequest to total output inherited by a young agent from lineage i at any time t, is an autoregressive AR(1) process with a deterministic drift (time trend). In the presence of envy, value of the AR(1) coefficient as well as the drift changes.

Define

$$\widetilde{z}^{*i}_{t+1} = \frac{1}{\widetilde{\Omega}_4}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\frac{1}{\widetilde{\Omega}_4}\widetilde{\delta}_1}{1-\widetilde{\phi}^*_1}, \tag{2.39}$$

where $\widetilde{\Omega}_4 = \frac{\widetilde{R}^*\beta\mu}{1+\beta(1+\mu)}\frac{1}{\sigma}$, $\widetilde{\delta}_1 = \frac{\widetilde{R}^*\beta\mu}{\sigma(1+\beta(1+\mu))}[1-\sigma\phi_b\widetilde{\Omega}_3]$, and $\widetilde{\phi}^*_1 = \frac{\widetilde{R}^*\beta\mu}{1+\beta(1+\mu)}$.

[5]While not necessary for the characterization of the law of motion for bequests under the Solow steady state, normalization is needed for the characterization of the Romer steady state and the period-2 cycles as the total output is growing under the latter 2 trajectories.

Substituting (2.39) into equation (2.38), we get a detrended autoregressive process of order 1, AR(1), for z_t^{*i}. That is,

$$\widetilde{z}_{t+1}^{*i} - \widetilde{\phi}_1^{*}\widetilde{z}_t^{*i} = [l_t^i - \bar{l}_t] \equiv u_{t+1}^i \sim i.i.d.(0, \sigma_l^2). \tag{2.40}$$

Proposition 2.3. *Since* $|\widetilde{\phi}_1^{*}| \equiv |\frac{\widetilde{R}^{*}\beta\mu}{1+\beta(1+\mu)}| < 1$, $\widetilde{z}_{t+1}^{*i}$ *is a covariance-stationary*[6] *AR(1) process.*

(i) The expected bequest inherited by a young agent of lineage i, as a ratio of the total output, at time, t + 1, is:

$$E\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\widetilde{\delta}_1}{1-\widetilde{\phi}_1^{*}}. \tag{2.41}$$

(ii) The variance of bequest inherited by a young agent of lineage i, as a ratio of the total output, at time, t + 1, is:

$$Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\sigma_l^2\widetilde{\Omega}_4^2}{1-\widetilde{\phi}_1^{*2}}. \tag{2.42}$$

(iii) The intergenerational mobility can be measured by the covariance or the correlation coefficient, to gauge the extent of intergenerational transmission of inequality. They are respectively:

$$Cov\left[\frac{b_{t+1}^i}{Y_{t+1}}, \frac{b_t^i}{Y_t}\right] = \widetilde{\Omega}_4^2 \frac{\widetilde{\phi}_1^{*}\sigma_l^2}{1-\widetilde{\phi}_1^{*2}}. \tag{2.43}$$

$$\rho\left[\frac{b_{t+1}^i}{Y_{t+1}}, \frac{b_t^i}{Y_t}\right] = \widetilde{\rho}_1 = \widetilde{\phi}_1^{*} = \frac{\widetilde{R}^{*}\beta\mu}{1+\beta(1+\mu)} = \frac{\frac{(1-\frac{1}{\sigma})}{\widetilde{\Omega}_1\widetilde{\Omega}_3}\beta\mu}{1+\beta(1+\mu)}. \tag{2.44}$$

[6] See Chapter 1 Definitions 1.1 and 1.2, or Hamilton (1994, pp.45–46) for the definition of stationarity, strict stationarity and covariance (weak) stationarity.

Proof. As in Chapter 1, Proposition 1.5. ■

Proposition 2.4. *The higher the envy, γ, the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility.*

Proof. From the correlation equation of Proposition 2.3,

$$\begin{aligned}
\frac{\partial \widetilde{\rho}_1}{\partial \gamma} &= \frac{\left(1-\frac{1}{\sigma}\right)\beta\mu}{1+\beta(1+\mu)}\left(\frac{\partial \frac{1}{\widetilde{\Omega}_1\widetilde{\Omega}_3}}{\partial \gamma}\right) \\
&= \frac{\left(1-\frac{1}{\sigma}\right)\beta\mu}{1+\beta(1+\mu)}\left[\frac{-1}{(\widetilde{\Omega}_1\widetilde{\Omega}_3)^2}\right]\left(\frac{\partial \widetilde{\Omega}_1\widetilde{\Omega}_3}{\partial \gamma}\right) \\
&= \underbrace{\frac{\left(1-\frac{1}{\sigma}\right)\beta\mu}{1+\beta(1+\mu)}}_{\geq 0}\underbrace{\left[\frac{-1}{(\widetilde{\Omega}_1\widetilde{\Omega}_3)^2}\right]}_{\leq 0}\underbrace{\left[\widetilde{\Omega}_3\frac{\partial \widetilde{\Omega}_1}{\partial \gamma}+\widetilde{\Omega}_1\frac{\partial \widetilde{\Omega}_3}{\partial \gamma}\right]}_{\leq 0} \\
&\geq 0.
\end{aligned} \tag{2.45}$$

The algebraic manipulations to sign $\frac{\partial \widetilde{\Omega}_1}{\partial \gamma}$, and $\frac{\partial \widetilde{\Omega}_3}{\partial \gamma}$ are shown in Appendix B.2.

■

Although Alvarez-Cuadrado and Long (2012) derived the the intergenerational correlation in terms of the returns of capital and bequest motives, this comparative statics, absent in Alvarez-Cuadrado and Long (2012), adds a new dimension to the intergenerational mobility results discussed in Chapter 1. Specifically, the higher the envy, the lower the intergenerational mobility. The introduction of envy creates a distribution in agents' propensity to save, with the poorer agents spending a greater fraction of their lifetime resources on positional goods, hence reducing the mobility of their lineage.

Proposition 2.5. *The higher the price elasticity of the final goods sector's demand for each intermediate, σ (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin for innovation), the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility.*

Proof. From the correlation equation of Proposition 2.3,

$$\begin{aligned}\widetilde{\rho}_1 = \widetilde{\phi}_1^* &= \frac{\widetilde{R}^*\beta\mu}{1+\beta(1+\mu)} = \frac{\frac{\left(1-\frac{1}{\sigma}\right)}{\widetilde{\Omega}_1\widetilde{\Omega}_3}\beta\mu}{1+\beta(1+\mu)} \\ &= \frac{(\sigma-1)\beta\mu[1+\mu(1-\xi\gamma)]}{\widetilde{\Omega}_1[1+\beta(1+\mu)][1+\sigma\mu(1-\xi\gamma)]} \\ &= \underbrace{\frac{\{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}\mu}{(1-\gamma)\}[1+\beta(1+\mu)]}}_{\equiv\mathbf{D}\geq 0}\frac{(\sigma-1)}{[1+\sigma\mu(1-\xi\gamma)]},\end{aligned} \tag{2.46}$$

as $\widetilde{\Omega}_1 \equiv \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$, and $\widetilde{\Omega}_3 \equiv \frac{1+\sigma\mu(1-\xi\gamma)}{\sigma[1+\mu(1-\xi\gamma)]}$.

$$\begin{aligned}\frac{\partial\widetilde{\rho}_1}{\partial\sigma} &= \mathbf{D}\frac{\partial\frac{(\sigma-1)}{[1+\sigma\mu(1-\xi\gamma)]}}{\partial\sigma} \\ &= \mathbf{D}\frac{1}{[1+\sigma\mu(1-\xi\gamma)]^2}\left\{\underbrace{\sigma[1-\mu(1-\xi\gamma)]}_{\geq 0}+\underbrace{\mu(1-\xi\gamma)(\sigma^2+1)]}_{\geq 0}\right\} \\ &\geq 0.\end{aligned} \tag{2.47}$$

■

Thus, the increase intergenerational mobility as a result of an increase in wage share, as in Chapter 1, is robust to the envy specification here. Again, this result is new relative to Alvarez-Cuadrado and Long (2012).

2.4.2 Cross-sectional distributions

At Solow steady state

Proposition 2.6. *Assuming* $|\widetilde{\phi}_1^*| < 1$ *and* $\sigma\phi_b\widetilde{\Omega}_3 < 1$*, the cross sectional inequality of the ratio of bequests to total output is more than (less than) the cross sectional inequality of wages if* $\frac{1-\widetilde{\phi}_1^*}{1+\widetilde{\phi}_1^*} > (<)(1-\sigma\phi_b\widetilde{\Omega}_3)^2$*. In the absence of envy, that is,* $\gamma = 0$*, the cross sectional inequality the ratio of bequests to total output is less than the cross sectional inequality of wages.*

Proof. From the asymptotic distribution for a covariance-stationary process, the inequality of bequests and labor income in the asymptotic cross sectional distribution using coefficient of variation[7] are:

$$CV\left[w^i\right] = CV\left[w_t^i\right] = \frac{\sqrt{Var\left[w_t^i\right]}}{E\left[w_t^i\right]} = \frac{\sigma_l \overline{w}_t}{\overline{w}_t} = \sigma_l. \tag{2.48}$$

$$\begin{aligned} CV\left[b^i\right] = CV\left[\frac{b^i}{Y^*}\right] = CV\left[\frac{b_{t+1}^i}{Y^*}\right] &= \frac{\sqrt{Var\left[\frac{b_{t+1}^i}{Y^*}\right]}}{E\left[\frac{b_{t+1}^i}{Y^*}\right]} = \frac{\sigma_l\sqrt{\frac{1}{1-\widetilde{\phi}_1^{*2}}}}{\frac{(1-\sigma\phi_b\widetilde{\Omega}_3)}{(1-\widetilde{\phi}_1^*)}} \\ &= \frac{\sigma_l}{(1-\sigma\phi_b\widetilde{\Omega}_3)}\sqrt{\frac{1-\widetilde{\phi}_1^*}{1+\widetilde{\phi}_1^*}}. \end{aligned} \tag{2.49}$$

■

Unlike in Chapter 1, in the presence of envy, bequests can become a source of wealth concentration. The risk pooling effect of intergenerational transfers is diminished with positional concerns, as the rich has a higher propensity to save.

[7] The coefficient of variation, as a measure of inequality, is scale invariant.

Thus, in contrast to Chapter 1, the inequality of inherited wealth could be higher than the inequality of the wage income with status anxiety.

Proposition 2.7.

(i) If envy $\gamma > 0$, the relative inequality of bequest, capital holdings, consumption when old and young are:

$$CV[b_{t+1}^i] > CV[s_t^i] > CV[d_{t+1}^i] > CV[c_t^i]. \tag{2.50}$$

(ii) The differences in relative inequality of bequest, capital holdings, consumption when old and young are identical under both the Solow regime and the Romer regime.

(iii) In the absence of envy, that is, if envy $\gamma = 0$, the relative inequality of bequest, capital holdings, consumption when old and young are:

$$CV[b_{t+1}^i] = CV[s_t^i] = CV[d_{t+1}^i] = CV[c_t^i], \tag{2.51}$$

where at steady state, $CV[w^i] \geq CV[b^i] = CV[s^i] = CV[d^i] = CV[c^i]$ (from Proposition 1.6).

Proof. Using the optimization equations of agent i, (2.8), (2.9), (2.10), (2.11), the first period budget constraint (B.7), and the assumption of $Cov[w_t^i, b_t^i] = 0$ by construction, we have:

$$c_t^i + s_t^i = b_t^i + w_t^i \equiv y_t^i.$$

Taking variance and expectations on the first period budget constraint,

$$Var\left[y_t^i\right] = Var\left[b_t^i\right] + Var\left[w_t^i\right] + \underbrace{2Cov\left[b_t^i, w_t^i\right]}_{=0 \text{ (by construction)}} = Var\left[b_t^i\right] + Var\left[w_t^i\right]; \tag{2.52}$$

$$E\left[y_t^i\right] = E\left[b_t^i\right] + E\left[w_t^i\right] = \overline{b}_t^i + \overline{w}_t^i; \tag{2.53}$$

$$E\left[s_t^i\right] = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}(1-\phi_s)\left[\overline{b}_t^i + \overline{w}_t^i\right]; \tag{2.54}$$

$$Var\left[s_t^i\right] = \left(\frac{\beta(1+\mu)}{1+\beta(1+\mu)}\right)^2 Var\left[y_t^i\right]; \tag{2.55}$$

$$E\left[c_t^i\right] = \frac{1}{1+\beta(1+\mu)}(1+\phi_c)\left[\overline{b}_t^i + \overline{w}_t^i\right]; \tag{2.56}$$

$$Var\left[c_t^i\right] = \left(\frac{1}{1+\beta(1+\mu)}\right)^2 Var\left[y_t^i\right]; \tag{2.57}$$

$$E\left[d_{t+1}^i\right] = \frac{R_{t+1}\beta}{1+\beta(1+\mu)}(1+\phi_d)\left[\overline{b}_t^i + \overline{w}_t^i\right]; \tag{2.58}$$

$$Var\left[d_{t+1}^i\right] = \left(\frac{R_{t+1}\beta}{1+\beta(1+\mu)}\right)^2 Var\left[y_t^i\right]; \tag{2.59}$$

$$E\left[b_{t+1}^i\right] = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(1-\phi_b)\left[\overline{b}_t^i + \overline{w}_t^i\right]; \tag{2.60}$$

$$Var\left[b_{t+1}^i\right] = \left(\frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\right)^2 Var\left[y_t^i\right]; \tag{2.61}$$

$$CV[b_{t+1}^i] = \frac{\sqrt{Var\left[b_{t+1}^i\right]}}{E\left[b_{t+1}^i\right]} = \frac{\sqrt{Var\left[y_{t+1}^i\right]}}{(1-\phi_b)\left[\overline{b}_t^i + \overline{w}_t^i\right]}; \tag{2.62}$$

$$CV[s_t^i] = \frac{\sqrt{Var\left[s_t^i\right]}}{E\left[s_t^i\right]} = \frac{\sqrt{Var\left[y_{t+1}^i\right]}}{(1-\phi_s)\left[\overline{b}_t^i + \overline{w}_t^i\right]}; \tag{2.63}$$

$$CV[d_{t+1}^i] = \frac{\sqrt{Var\left[d_{t+1}^i\right]}}{E\left[d_{t+1}^i\right]} = \frac{\sqrt{Var\left[y_{t+1}^i\right]}}{(1+\phi_d)\left[\overline{b}_t^i + \overline{w}_t^i\right]}; \tag{2.64}$$

$$CV[c_t^i] = \frac{\sqrt{Var\left[c_t^i\right]}}{E\left[c_t^i\right]} = \frac{\sqrt{Var\left[y_{t+1}^i\right]}}{(1+\phi_c)\left[\overline{b}_t^i + \overline{w}_t^i\right]}. \tag{2.65}$$

Since

$$\phi_s = \frac{(1+\mu)(1-\xi\gamma)\gamma - \xi\gamma(1-\gamma)}{(1+\mu)\{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}}$$
$$\leq \phi_b = \frac{(1-\xi\gamma)\gamma + \xi\gamma(1-\gamma)\beta}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]} \times \frac{1+\mu}{1+\mu}$$
$$\Rightarrow (1-\phi_b) \geq (1-\phi_s). \quad (2.66)$$

$$\phi_c = \frac{\beta[(1+\mu)(1-\xi\gamma)\gamma - \xi\gamma(1-\gamma)]}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$$
$$\geq \ \phi_d = \frac{\xi\gamma\mu(1-\gamma)\beta + \xi\gamma(1-\gamma) - (1-\xi\gamma)\gamma}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}$$
$$\geq 0 \Rightarrow (1+\phi_c) \geq (1+\phi_d) \geq (1-\phi_s) \geq (1-\phi_b). \quad (2.67)$$

Part (ii) follows from Proposition 1.8. ■

This strengthens the applicability of the "stable steady state" results from Alvarez-Cuadrado and Long (2012, pp. 962–964) on the relative inequality of bequest, capital holdings, consumption when old and young, in which there is no long-run growth under the neoclassical framework, to include Matsuyama (1999)'s production function with endogenous growth through innovation and cycles, where the fixed points are not unique. Specifically, even though as shown in Matsuyama (1999), the aggregate growth over period-2 cycles are higher than at either of the steady states, with i.i.d. shocks, the lineage wealth dynamics and cross-sectional wealth distributions for Solow and Romer steady states, and period-2 cycles are identical as in Chapter 1, despite the presence of "Status Anxiety". As discussed in Section 1.4.3, under i.i.d. productivity shocks, the global lineage evolution of inherited wealth to output ratio, as well as the long-run cross-sectional distribution of inherited wealth to output ratio in the presence of "Status Anxiety" have been characterized in this thesis, including chaotic trajectories, not just at the steady state(s). Compared to Alvarez-Cuadrado and Long (2012), besides introducing long-run growth through innovation, incorporating the production function of Matsuyama (1999) enables one to study the effect of price elasticity of the final goods sector's demand for each intermediate on intergenerational lineage mobility. As in Chapter 1, the higher the price elasticity of the final goods sector's demand for each intermediate, the lower the intergenerational lineage mobility as shown in Proposition 2.5.

2.5 Conclusion

This chapter shows that status anxiety can affect both the aggregate bequest and lineage bequest evolution of an economy. Status anxiety can decrease the steady state ratio of capital stock to variety of intermediate goods in an endogenous growth model with innovation and fixed cost of innovation. Furthermore, status anxiety can affect the nature of the steady state (Solow or Romer) of the economy, and hence its growth potential.

On the distributional front, an increase in status anxiety decreases intergenerational mobility as agents shift resources to positional goods, reducing the allocation to non-positional goods such as bequest. On a cross-sectional front, the presence of status anxiety can result in the inequality of wealth being more than or less than that of wage. This contrasts with Chapter 1, where in the absence of status anxiety, the cross-sectional inequality of wealth is always less than that of wages.

Thus, envy is not only a factor that reduces the intergenerational risk pooling through bequest, resulting in a higher concentration of inherited wealth; it is also a factor that drives the relative distributions of household consumption when young and old, savings and bequests through the nature of the expenditure: positional (conspicuous) or non-positional, leading to the immiseration of the poor.

Chapter 3

Endogenous labor supply and its impact on mobility, inequality and growth trajectories

3.1 Introduction

Theoretically, inheritance creates a disincentive to labor due to the marginal utility of wealth, with leisure being a normal good. The magnitude of this disincentive affects the relative importance of inheritance and life-cycle savings on total wealth of a country, as well as the evolution of relative inequality between inheritance and lifetime income (from earnings and inheritance). Using EITM (Estate-Income Tax Match) and PSID (Panel Study of Income Dynamics) data respectively, Holtz-Eakin et al. (1993) and Joulfaian and Wilhelm (1994) find a negative relationship between labor force participation and size of inheritance for the US. In addition, Joulfaian and Wilhelm (1994) also find that the impact of inheritance on hours worked from prime-age worker is statistically significant, but small. This concurs with the findings of Halvorsen et al. (2012), using a panel data that covers all registered inheritances in Norway, with women being more responsive than men. Moreover, Halvorsen et al. (2012) also note that young inheritors respond via the intensive margin more than the extensive margin, while older inheritors do the reverse.

In this chapter, I explore the aggregate and distributional implications of endogenous labor supply, in the presence of bequests and idiosyncratic labor productivity shocks, within an endogenous growth framework. This is related to the literature of incomplete markets by Aiyagari (1994) and Huggett (1997), where infinitely-lived agents, with inelastic labor, are subjected to idiosyncratic labor productivity shocks, with limited ability to insure against the income risk. This financial friction results in an increase demand for savings for precautionary motive, and consequently, capital and output are higher. By introducing endogenous labor-leisure choice into an incomplete market model with infinitely lived agents, Marcet et al. (2007) study the capital accumulation with 2-state Markov chain in labor productivity, and show that ex post wealth effect reduces labor supply, offsetting the effects of precautionary savings. Zhu (2013b) generalize that result to cover multiple-state Markov chain as well as more general utility functions. Using a simple OLG model with "warm glow" bequest motive and a log-linear preference for leisure, I study the wealth effect on aggregate labor supply in the presence of uninsurable labor-income risk, and the general equilibrium effect on wage per unit efficient labor analytically. One can interpret the idiosyncratic productivity risk as any inputs that complements with capital, such as heterogenous technology adoption, entrepreneurship or human capital.

The analytical simplification is done by abstracting away the intertemporal effect of consumption-savings choice, retaining only the intratemporal effect, by using the log-linear assumption for the preference of leisure in conjunction with 2-period lived agents in an OLG structure: to link labor effort to contemporaneous average wage, contemporaneous idiosyncratic productivity, and contemporaneous idiosyncratic inherited wealth.

By using a production function with innovation and fixed cost (Matsuyama (1999)) instead of the neoclassical production used in the previous literature, there are a few new findings at the aggregate level. In addition, by using covariance-stationarity, I characterize the changes in lineage mobility and cross-sectional inequality of wealth that result from elastic labor over different growth regimes including period-2 cycles. For the aggregate supply of labor, the higher the price elasticity of the final goods sector's demand for each intermediate (and correspondingly, the higher the capital share, the lower the wage share, as well as lower monopoly margin for innovator), the lower the aggregate labor supply at equilibrium. Furthermore, if the labor share is relatively small, an increase in bequest motive will reduce the aggregate labor (wealth effect dominates), which is consistent with Marcet et al. (2007) and Zhu (2013b). However, if the labor share is relatively large, an increase in the bequest motive will result in a higher aggregate labor supply (precautionary savings for uninsurable labor risk dominates), as with Aiyagari (1994). Still on the aggregate front, the growth potential of the economy, and steady state normalized capital stock is decreasing with the preference for leisure. On the lineage bequest evolution, the higher the preference for leisure, the higher the intergenerational mobility as the elastic labor provides a channel to level the playing field of the asset-poor via the disincentive to work among the asset-rich. On the cross-sectional distribution, the higher the preference for leisure, the higher the cross-sectional inequality of bequest. Under period-2 growth cycles, the inequality of leisure consumed at both period-2 fixed points are identical, but are higher compared to the steady state inequality of leisure consumption, as the alternating growth regimes induce correlations with endogenous labor-leisure choice. Finally, the higher the price elasticity of the final goods sector's demand for each intermediate (and correspondingly, the higher the capital share, the lower the wage share, as well as lower monopoly margin for innovator), the higher the cross-sectional inequality of the ratio of wage income earned to total output.

This work complements a related thread of literature, where unlike the agents in this chapter, in García-Peñalosa and Turnovsky (2007) and García-Peñalosa and Turnovsky (2011), the infinitely-lived agents have identical labor productivity, but ex-ante heterogeneous endowments of physical capital. In García-Peñalosa and Turnovsky (2007), economy-wide capital stock exerts an externality such that the aggregate output is essentially AK in form, and each agent's labor supply is linear in his capital endowment. Hence income inequality is determined by the initial distribution of capital among the agents and the aggregate labor supply at equilibrium. There is no income dynamics, and faster growth results in more unequal income distribution. In García-Peñalosa and Turnovsky (2011), with a neoclassical growth model, the wealth distribution dynamics are similar under both elastic and inelastic labor, while the income distribution varies as poorer agents supply more labor.

This chapter is organized as follows. Section 2 formalizes the model. Section 3 characterizes the aggregate equilibrium, and studies the impact of model's primitive on aggregate labor supply. Section 4 analyzes the impact of elastic labor on lineage bequest evolution and cross-sectional distributions, and Section 5 concludes.

3.2 The model

3.2.1 Agents' optimization

Time is discrete. The economy consists of overlapping generations of agents who live for 2 periods: the young period, and the old period. In each generation, there is a continuum of measure 1 of agents. In each family *i*, old agent gives birth to 1 child, keeping the population in the economy constant[1]. When old, agents retire, consume their savings, and leave a bequest to their child. When young, agents choose the proportion of time endowment spent between work and leisure; consume, and save.

Young agents within a given generation, t, are heterogenous in 2 aspects: their idiosyncratic productivity, l_t^i, and the inherited bequest b_t^i. I assume that

[1]Population changes can easily be incorporated by changing the measure of agents in each generation to n_t, instead of 1.

the labor productivity is drawn from an independent and identical distribution on a positive support, with an expectation, $E[l_t^i] = \bar{l} \equiv 1$, and a finite variance, $Var[l_t^i] = \sigma_l^2$, across agents from the same generation, and independent across time, that is, independent across generations from the same lineage. The problem of an agent i born at time t is:

$$\underset{c_t^i,\, d_{t+1}^i,\, \Lambda_t^i}{Max} \; U_t\left(c_t^i, d_{t+1,}^i b_{t+1,}^i\right) \equiv \ln c_t^i + \lambda \ln \Lambda_t^i + \beta \left[\ln d_{t+1}^i + \mu \ln b_{t+1}^i\right] \quad (3.1)$$

$$\text{subject to:} \qquad c_t^i + s_t^i = b_t^i + w_t l_t^i [1 - \Lambda_t^i] \equiv y_t^i, \quad (3.2)$$

$$\text{and} \qquad R_{t+1} s_t^i = d_{t+1}^i + b_{t+1}^i. \quad (3.3)$$

where c_t^i is the consumption when young, and d_{t+1}^i is the consumption when old, respectively, of an agent born at time t; b_{t+1}^i is the bequest left by an agent born at time, t, when old, to his immediate offspring born at time t; $\beta < 1$ is the subjective discount factor; $0 \leq \mu, \lambda < 1$ are the importance of bequest and leisure motives respectively; Λ_t^i is the fraction of time spend on leisure by young; $l_t^i \sim$ i.i.d.$(1, \sigma_l^2)$; $w_t^i = w_t l_t^i$; w_t is the equilibrium wage rate for each unit of labor productivity; s_t^i is the amount saved when young; y_t^i is the lifetime resource of an individual i for agent i, born at time t; and R_{t+1} is the equilibrium gross rate of return on capital at time $t+1$.

At time $t+1$, an old agent maximizes his second period utility, defined as:

$$V \equiv \ln d_{t+1}^i + \mu \ln b_{t+1}^i, \quad (3.4)$$

by choosing his second period consumption and bequest for his immediate offspring, given his second period budget constraint. Substituting (3.3) into (3.4), and differentiating the latter with respect to d^i_{t+1} for the first order condition, the optimal choices for an old agent are:

$$d^i_{t+1} = \frac{1}{1+\mu}(R_{t+1}s^i_t); \tag{3.5}$$

$$b^i_{t+1} = \frac{\mu}{1+\mu}(R_{t+1}s^i_t). \tag{3.6}$$

By substituting these choices, rewrite the second period value function as:

$$V\left(R_{t+1}s^i_t\right) \equiv (1+\mu)\ln(R_{t+1}) + (1+\mu)\ln\left(s^i_t\right) + \ln\left(\frac{1}{1+\mu}\right) + \mu\ln\left(\frac{\mu}{1+\mu}\right). \tag{3.7}$$

At time t, subject to the first period budget constraint, the young agent chooses consumption and savings to maximize his utility:

$$\ln(c^i_t) + \lambda\ln\Lambda^i_t + \beta V\left(R_{t+1}s^i_t\right). \tag{3.8}$$

Substituting (3.2) and (3.7) into (3.8), and differentiating the latter with respect to s^i_t for the first order condition:

$$\frac{1}{b^i_t + w_t l^i_t[1-\Lambda^i_t] - s^i_t} = \frac{\beta(1+\mu)}{s^i_t}, \tag{3.9}$$

which implies that the optimal savings for agent i as a function of his lifetime resource is a constant fraction, as with Bossmann et al. (2007).

$$s^i_t = \frac{\beta(1+\mu)\{b^i_t + w_t l^i_t[1-\Lambda^i_t]\}}{1+\beta(1+\mu)} = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}y^i_t. \tag{3.10}$$

Substituting (3.2) and (3.7) into (3.8), and differentiating the latter with respect to Λ^i_t for the first order condition:

$$\frac{w_t l_t^i}{b_t^i + w_t l_t^i [1 - \Lambda_t^i] - s_t^i} = \frac{\lambda}{\Lambda_t^i}, \tag{3.11}$$

The right hand side of (3.11) is the marginal benefit of leisure, while the left hand side is the quality(wealth) adjusted marginal cost of leisure. Substituting the optimal savings equation (3.10) into (3.11), we get the optimal amount of leisure for an agent *i*:

$$\Lambda_t^i = \frac{\lambda}{1 + \beta(1+\mu) + \lambda} \left(\frac{b_t^i}{w_t l_t^i} + 1 \right). \tag{3.12}$$

That is, the optimal amount of leisure to consume for an individual of lineage *i* is directly proportional to the size of the inherited wealth, but inversely proportional to the idiosyncratic labor productivity draw and the wage rate per unit productivity (i.e. the opportunity costs of leisure). This helps to explain why even for the very rich, such as Bill Gates and Warren Buffet[2], they may continue to work despite their wealth and age, as their labor productivities are uttermost.

From equation (3.12), each agent's sensitivity of individual labor supply to relative wealth differs, by virtue of the idiosyncratic labor productivity draws, unlike in García-Peñalosa and Turnovsky (2011), where the sensitivity of individual labor supply to relative capital, which depends on the aggregate leisure, is thus identical for all agents. This is because in García-Peñalosa and Turnovsky (2011), agents are endowed with only heterogenous capital ex-ante, but the op-

[2]Ranked by Forbes (2015, March 2) as the World's richest person for the past 21 years and World's third richest respectively.
Retrieved from http://www.forbes.com/sites/chasewithorn/2015/03/02/forbes-billionaires-full-list-of-the-500-richest-people-in-the-world-2015/, on May 1, 2015.

portunity cost per unit of leisure are identical with no idiosyncratic labor productivity shocks. As a result, each infinitely-lived agent operating in the neoclassical growth model in García-Peñalosa and Turnovsky (2011) will have a steady-state relative wealth, and the higher the steady state relative wealth, the more leisure he consumes, equalizing the impact of wealth inequality on the distribution of income. Contrastingly, in this chapter, there is no steady-state relative wealth for the OLG agents of each lineage i, as a result of the idiosyncratic labor productivity draws, but there exists a law of motion of inherited wealth to total output of the economy to characterize the wealth dynamics of the lineage. Additionally, the evolution of the distribution of inherited wealth and the distribution of maximum idiosyncratic labor income that could be earned by an agent of lineage i, drive the distribution of leisure together. Besides, with the possibility of different growth regimes, as well as period-2 cycles, as a result of the production function with innovation and fixed cost introduced by Matsuyama (1999), different aggregate wage rate per efficient labor can be generated, which introduces differences in the asymptotic cross-sectional distribution of wealth between the steady-states and the fixed points of period-2 cycles, as will be shown in Section 3.4.2.

Note that with the choice of log preference for leisure for analytical tractability, this model can only be used to analyze the intensive margin of labor supply, and is silent on the extensive margin and related issues such as unemployment and retirement decisions.

Substituting (3.10) into (3.5), (3.6), and (3.2), the optimal choices of an agent i are:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} y_t^i; \tag{3.13}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)} y_t^i; \tag{3.14}$$

$$d_{t+1}^i = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} y_t^i; \tag{3.15}$$

$$b_{t+1}^i = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} y_t^i; \tag{3.16}$$

and the optimal choices of the average agent are:

$$\overline{\Lambda}_t = \frac{\lambda}{1+\beta(1+\mu)+\lambda} \left(\frac{\overline{b}_t}{w_t \overline{l}_t} + 1 \right); \tag{3.17}$$

$$\overline{s}_t = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} \overline{y}_t \equiv \Omega_1 \overline{y}_t; \tag{3.18}$$

$$\overline{c}_t = \frac{1}{1+\beta(1+\mu)} \overline{y}_t; \tag{3.19}$$

$$\overline{d}_{t+1} = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} \overline{y}_t; \tag{3.20}$$

$$\overline{b}_{t+1} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} \overline{y}_t \equiv R_{t+1} \Omega_2 \overline{y}_t. \tag{3.21}$$

With the log-linear preference for leisure and the 2-period OLG structure, it does not change the intertemporal optimization, thus the optimal choices of an agent i correspond to those in Chapter 1 with inelastic labor, as agents continue to save the same fixed proportion of their lifetime resources for consumption at old age and bequest. By eliminating the intertemporal substitution effect, the labor effort is driven by the wealth effect relative to the idiosyncratic productivity shocks. This chapter thus focuses on the idiosyncratic change in labor effort arising from the contemporaneous heterogeneous productivity shocks and the heterogenous wealth endowment, that accumulates from the transmission of past idiosyncratic productivity shocks and labor choices from agents of the same lineage. Overall, the endogenous labor-leisure choice has a scaling effect by changing the quantum of lifetime resources available for the young. Furthermore, it changes the coefficient and drift in the first order difference equation governing the accumulation of intergenerational transfer of wealth through bequest as will be discussed in the subsequent section.

3.2.2 Firms' optimization

The production sector is similar to that described in Chapter 1, Section 1.2.2, except that aggregate labor, L, is replaced by an elastic aggregate labor supply $\overline{L}_t$, where $\overline{L}_t = \int_0^1 [1-\Lambda_t^i] l_t^i \, di$. The final goods production function is thus:

$$Y_t = \hat{A}(\overline{L}_t)^{\frac{1}{\sigma}} \left\{ \int_0^{N_t} [x_t(z)]^{1-\frac{1}{\sigma}} dz \right\}. \tag{3.22}$$

3.3 Equilibrium steady state and dynamics

3.3.1 Derivation of equilibrium capital stock

The market clearing conditions are:

$$\overline{L}_t = \int_0^1 [1-\Lambda_t^i] l_t^i \, di; \tag{3.23}$$

$$K_t = \int_0^1 s_t^i \, di = \overline{s}_t \equiv S_t. \tag{3.24}$$

Given the competitive Cobb-Douglas final goods sector, capital share of the economy is:

$$R_t K_{t-1} = \left(1 - \frac{1}{\sigma}\right) Y_t, \tag{3.25}$$

and the wage share of the economy is:

$$
\begin{aligned}
\frac{1}{\sigma}Y_t &= w_t\overline{L}_t && (3.26)\\
&= w_t\int_0^1 l_t^i\,di - w_t\int_0^1 \Lambda_t^i l_t^i\,di\\
&= w_t\left[1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\int_0^1\left(\frac{b_t^i}{w_t}+l_t^i\right)di\right]\\
&= w_t\left[1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\left(\frac{\overline{b}_t^i}{w_t}\right)\right];
\end{aligned}
$$

$$
\Rightarrow \overline{L}_t = \left[1-\underbrace{\frac{\lambda}{1+\beta(1+\mu)+\lambda}}_{\text{disutility from working}} - \underbrace{\frac{\lambda}{[1+\beta(1+\mu)+\lambda]}\left(\frac{\overline{b}_t^i}{w_t}\right)}_{\text{disutility from working with increase in bequest}}\right]. \tag{3.27}
$$

Combining the average inheritance and savings equations, (3.21) and (3.18), with the capital market share (3.25), we get:

$$
\overline{b}_t = \frac{\mu}{1+\mu}R_t\overline{s}_{t-1} = \frac{\mu}{1+\mu}\left(1-\frac{1}{\sigma}\right)Y_t. \tag{3.28}
$$

Substituting the wage share (3.26) and the average inheritance (3.21), in the aggregate effective labor supply (3.27), we get:

$$
\begin{aligned}
\overline{L}_t &= 1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\left(1+\frac{\overline{b}_t^i}{w_t}\right)\\
&= 1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\left(1+\frac{\frac{\mu}{1+\mu}\left(1-\frac{1}{\sigma}\right)Y_t}{\frac{1}{\sigma\overline{L}_t}Y_t}\right)\\
\Rightarrow \overline{L}_t &= \frac{[1+\beta(1+\mu)](1+\mu)}{[1+\beta(1+\mu)+\lambda](1+\mu)+\lambda\mu(\sigma-1)}\\
\overline{L}\equiv\overline{L}_t &= \frac{1+\mu+\beta(1+\mu)^2}{1+\mu+\beta(1+\mu)^2+\lambda+\lambda\mu\sigma}\leq 1. && (3.29)
\end{aligned}
$$

Due to the log-linear formulation of the preference for leisure and OLG framework, the aggregate labor and leisure are constant. Since we have measure

1 of agents in each generation, we can obtain an equivalent expression for the aggregate savings, and the aggregate capital stock, of the economy from the optimization choices of the average agent. Specifically:

$$\begin{aligned}\bar{s}_t &= \frac{\beta(1+\mu)}{1+\beta(1+\mu)}\bar{y}_t \\ &\equiv \Omega_1 \bar{y}_t \\ &\equiv \Omega_1 [\bar{b}_t + w_t \bar{L}] \\ &= \Omega_1 w_t \bar{L} + \Omega_1 \left(\frac{\mu}{1+\mu}\right) R_t \bar{s}_{t-1}. \end{aligned} \tag{3.30}$$

The unconsumed final goods at time t, will form the available capital stock for production at time $t+1$. Thus by integrating the savings across all young agents, the aggregate savings in the economy is:

$$K_t = S_t = \bar{s}_t = \Omega_1 w_t \bar{L} + \Omega_1 \left(\frac{\mu}{1+\mu}\right) R_t K_{t-1}, \tag{3.31}$$

and the capital stock as a function of the final output of the economy can be derived by substituting out the factor shares, (3.25) and (3.26), as follow:

$$\begin{aligned}K_t &= \Omega_1 \left(\frac{1}{\sigma}\right) Y_t + \Omega_1 \left(\frac{\mu}{1+\mu}\right)\left(1 - \frac{1}{\sigma}\right) Y_t \\ &= \Omega_1 \Omega_3 Y_t, \end{aligned} \tag{3.32}$$

where $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$ and $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$. That is, this economy saves a constant fraction of its output for the next period.

From the savings accumulation process (3.32), in conjunction with the dynamics of innovation (1.21) and the total output (1.23) (in Chapter 1, replacing the L with the elastic labor supply $\overline{L}$, the unique equilibrium path for any initial condition, K_0 and N_0 is pinned down.

$$\frac{K_t}{\Omega_1\Omega_3} = Y_t = \begin{cases} \overline{A}\left[\theta\sigma F N_{t-1}\right]^{\frac{1}{\sigma}}\left[K_{t-1}\right]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1}, \\ \overline{A}K_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases} \tag{3.33}$$

where $\overline{A} \equiv \frac{\hat{A}}{a}\left[\frac{a\overline{L}}{\theta\sigma F}\right]^{\frac{1}{\sigma}}$.

This dynamical system can be normalized by the range of intermediate goods to a 1-dimensional map. Define $\Phi^n(k) \equiv \Phi\left(\Phi^{n-1}(k)\right)$, $\Phi^1(k) \equiv \Phi(k)$, and

$$k_t \equiv \frac{K_t}{\theta\sigma F N_t}.$$

Equilibrium path for initial condition k_0 is given by the sequence $\{\Phi^t(k_0)\}$:

$$k_t = \Phi(k_{t-1}) \equiv \begin{cases} \overline{G}\left(k_{t-1}\right)^{1-\frac{1}{\sigma}} & \text{if} \quad k_{t-1} \leq k_c = 1, \\ \\ \frac{\overline{G}k_{t-1}}{1+\theta(k_{t-1}-1)} & \text{if} \quad k_{t-1} \geq k_c = 1, \end{cases} \tag{3.34}$$

where:

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$[3], $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$, $\overline{G} \equiv \Omega_1\Omega_3\overline{A}$, $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$, $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$, and $\overline{A} \equiv \frac{\hat{A}}{a}(\frac{a\overline{L}}{\theta\sigma F})^{\frac{1}{\sigma}}$. The aggregate dynamics is thus similar to Chapter 1, except for the scaling of the output due to the introduction of a log-linear leisure-labor choice, where $\overline{L} \leq 1$.

The mapping $k_t = \Phi(k_{t-1})$ with domain $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ in (3.34) has a unique steady state where $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$ is constant over time. Denote the fixed point as k^* such that $k^* = \Phi(k^*)$ if $k^* < k_c \equiv 1$ and k^{**} such that $k^{**} = \Phi(k^{**})$ if $k^{**} > k_c \equiv 1$.

[3] $k = 0$ is excluded from the domain in the mapping $k_t = \Phi(k_{t-1})$, as $\Phi'(0) > 1$, making $k = 0$ a repelling fixed point and hence trivial.

3.3.2 Comparative statics of equilibrium aggregate labor supply

As labor enters into the final goods production in the form of a Cobb-Douglas technology in equation (3.22), the higher the aggregate labor supply, the lower the equilibrium wage per unit of effective labor, keeping the factor share constant. Furthermore, from equation (3.32), it can be seen that the higher the aggregate labor supply, the higher the labor augmented total productivity factor, $\overline{A} \equiv \frac{\hat{A}}{a}\left[\frac{a\overline{L}}{\theta\sigma F}\right]^{\frac{1}{\sigma}}$.

Proposition 3.1. *The equilibrium aggregate labor supply, $\overline{L}$, is constant, and it is determined by the primitives of the model as follows:*

(i) The higher the taste for leisure, λ, the lower the aggregate labor supply at equilibrium, $\overline{L}$.

(ii) The higher the price elasticity of the final goods sector's demand for each intermediate, σ (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin of innovator), lower the aggregate labor supply at equilibrium, $\overline{L}$.

(iii) The higher subjective discount factor, β, the higher the aggregate labor supply at equilibrium, $\overline{L}$.

(iv) If $\sigma \leq (>) \frac{1+2\beta(1+\mu)}{1+\beta(1-\mu)}$, the higher the bequest motive, μ, the higher (lower) the aggregate labor supply at equilibrium, $\overline{L}$.

Proof. From (3.29):

$$\overline{L} \equiv \overline{L}_t = \frac{1+\mu+\beta(1+\mu)^2}{1+\mu+\beta(1+\mu)^2+\lambda+\lambda\mu\sigma} \leq 1.$$

(i)

$$\frac{\partial \overline{L}}{\partial \lambda} = -\frac{(1+\mu\sigma)\left[1+\mu+\beta(1+\mu)^2\right]}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \leq 0. \tag{3.35}$$

(ii)

$$\frac{\partial \overline{L}}{\partial \sigma} = -\frac{\lambda\sigma\left[1+\mu+\beta(1+\mu)^2\right]}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \leq 0. \tag{3.36}$$

(iii)

$$\frac{\partial \overline{L}}{\partial \beta} = \frac{(1+\mu)^2}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \times \left\{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right] - \left[1+\mu+\beta(1+\mu)^2\right]\right\},$$

$$\Rightarrow \frac{\partial \overline{L}}{\partial \beta} = \frac{(1+\mu)^2(\lambda+\lambda\sigma\mu)}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \geq 0. \tag{3.37}$$

(iv)

$$\frac{\partial \overline{L}}{\partial \mu} = \frac{1}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \times \left\{\left[1+2\beta(1+\mu)\right]\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right] - \left[1+\mu+\beta(1+\mu)^2\left[1+2\beta(1+\mu)+\lambda\sigma\right]\right\}$$

$$= \frac{1}{\left[1+\lambda+\mu+\lambda\mu\sigma+\beta(1+\mu)^2\right]^2} \times \left\{\lambda\left[1-\sigma+\beta(1+\mu)\left[2+(\mu-1)\sigma\right]\right]\right\} \lesseqgtr 0,$$

$$\Rightarrow \text{If } \sigma \leq (>) \frac{1+2\beta(1+\mu)}{1+\beta(1-\mu)}, \frac{\partial \overline{L}}{\partial \mu} \geq (<) 0. \tag{3.38}$$

■

The results from Proposition 3.1(i) and (iii) follow from the log-linear preference for leisure and OLG structure assumed.

From Proposition 3.1(ii), the higher the price elasticity of the final goods sector's demand for each intermediate (and correspondingly, the higher the capital share, the lower the wage share, as well as lower monopoly margin for innovator), the lower the aggregate labor supply at equilibrium. The driver of this result is the Cobb-Douglas assumption of the final good sector, where labor and intermediate goods are the factor inputs.

From Proposition 3.1(iv), it can be seen that the wealth effect of bequest on aggregate labor supply is related to the price elasticity, σ (and correspondingly, capital share, as well as the monopoly margin of innovator). If the capital share is small enough relative to the labor share, an increase in bequest motive will encourage the agent with average wealth to work more, since he gets a relatively bigger slice from the output gains due to his diligence, and a higher aggregate labor supply results (precautionary savings for uninsurable labor risk dominates), as with Aiyagari (1994). Conversely, if the labor share is relatively small, an increase in bequest motive will reduce the aggregate labor (wealth effect dominates), which is consistent with Marcet et al. (2007) and Zhu (2013b).

3.3.3 Aggregate steady state and dynamics

Proposition 3.2. *The growth potential of the economy is decreasing in the degree of leisure, λ.*

Proof. Market clearing (equations (3.26), (3.25), and the consumer optimization (equation (3.32)) imply $S_t = K_t = \widetilde{\Omega}_1 \widetilde{\Omega}_3 Y_t$. Together with the final output function (3.33),

$$Y_t = \begin{cases} \overline{A}\,[\theta\sigma F N_{t-1}]^{\frac{1}{\sigma}}\,[K_{t-1}]^{1-\frac{1}{\sigma}} & \text{if} \quad K_{t-1} \leq \theta\sigma F N_{t-1} \\ \overline{A} K_{t-1} & \text{if} \quad K_{t-1} \geq \theta\sigma F N_{t-1}, \end{cases}$$

along the balanced growth path, $K_t = \Omega_1\Omega_3 Y_t = \Omega_1\Omega_3 \overline{A} K_{t-1} = \overline{G} K_{t-1}$, where $\overline{G} \equiv$ gross growth rate.

$$\begin{aligned} \frac{\partial \overline{G}}{\partial \lambda} &= \Omega_1\Omega_3 \frac{\partial \overline{A}}{\partial \lambda} \\ &= \underbrace{\Omega_1\Omega_3 \frac{\hat{A}}{a}\left(\frac{a}{\theta\sigma F}\right)^{\frac{1}{\sigma}} \cdot \frac{1}{\sigma}\overline{L}^{\left(\frac{1}{\sigma}-1\right)}}_{\geq 0} \cdot \underbrace{\frac{\partial \overline{L}}{\partial \lambda}}_{\leq 0} \leq 0. \end{aligned} \tag{3.39}$$

■

Proposition 3.3. *The normalized steady-state capital stock to variety ratio, k_t, is decreasing in the preference for leisure, λ, in both the regimes.*

Proof. From equation (3.34):

when $\overline{G} < 1$, $k^* = (\Omega_1\Omega_2\overline{A})^{\sigma} \equiv \overline{G}^{\sigma}$;

when $\overline{G} > 1$, $k = k^{**} \equiv 1 + \frac{(\overline{G}-1)}{\theta}$.

$$\frac{\partial k^*}{\partial \lambda} = \frac{\partial k^*}{\partial \overline{G}} \times \frac{\partial \overline{G}}{\partial \lambda} = (\sigma \overline{G}^{\sigma-1}) \left(\Omega_1\Omega_3 \frac{\partial \overline{A}}{\partial \lambda}\right) \leq 0. \tag{3.40}$$

$$\frac{\partial k^{**}}{\partial \lambda} = \frac{\partial k^{**}}{\partial \overline{G}} \times \frac{\partial \overline{G}}{\partial \lambda} = \frac{1}{\theta}\left(\Omega_1\Omega_3\frac{\partial \overline{A}}{\partial \lambda}\right) \leq 0. \tag{3.41}$$

■

In this model, agents work to finance consumption when young and old, and for bequest to their immediate offspring. An increase in the preference for leisure, denoted by λ, will cause the agents to allocate less of their time on work, hence lowering savings and available aggregate capital stock for innovation.

Thus, unlike neoclassical growth models, both the normalized steady state capital stock *and* growth potential of the economy are affected by the preference for leisure, creating greater leeway for policy engagement through labor income tax or subsidies. The results extend the findings from Bossmann et al. (2007, Appendix B.2. pp. 1268-1269) to endogenous growth models.

3.4 Impact of leisure on lineage bequest evolution and distribution

In the following section, I shall highlight the differences that the endogenous labor-leisure choice might bring to the evolution of mobility and inequality, compared to their respective benchmarks in Chapter 1, with inelastic labor, by setting $\lambda = 0$.

While I develop the following exposition using the steady states, the results are applicable to the fixed points of period-2 cycles, given the distributional equivalence under i.i.d. shocks, as infered from Proposition 1.10.

3.4.1 Lineage bequest evolution

From the capital share at equilibrium, equation (3.25), the savings rate, equation (3.32), and from total output equation (3.33) , we get:

$$\frac{R_t}{G_t} = \frac{(1-\frac{1}{\sigma})Y_t/K_{t-1}}{Y_t/Y_{t-1}} = \frac{(1-\frac{1}{\sigma})\left(\frac{1}{\Omega_1\Omega_3}\right)K_t/K_{t-1}}{Y_t/Y_{t-1}} = \frac{1-\frac{1}{\sigma}}{\Omega_1\Omega_3}, \tag{3.42}$$

where G_t denotes the gross output growth rate at time t. Using the bequest of an agent of lineage i, equation (3.16), as well as the capital and labor shares, equations (3.25) and (3.26), and aggregate labor, equation (3.29), we derive the law of motion of the bequests for an agent of lineage i.

$$\begin{aligned} b^i_{t+1} &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}(y^i_t) \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\left[b^i_t + w_t l^i_t(1-\Lambda^i_t)\right] \\ &= \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\left\{b^i_t + w_t l^i_t\left[1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\left(\frac{b^i_t}{w_t l^i_t}+1\right)\right]\right\} \end{aligned} \tag{3.43}$$

Normalizing[4] by the total output and rearranging,

$$\begin{aligned} \frac{b^i_{t+1}}{Y_{t+1}} = \frac{R_{t+1}\beta\mu}{Y_{t+1}[1+\beta(1+\mu)]}&\left[1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]b^i_t \\ &+\frac{Y_t}{Y_{t+1}\sigma\bar{L}}\frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\left[1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]l^i_t, \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]}&\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]\frac{b^i_t}{Y_t} \\ -\frac{1}{\sigma\bar{L}}\frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]}&\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right] \\ = \frac{1}{\sigma\bar{L}}\frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]}&\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right][l^i_t-\bar{l}_t], \end{aligned}$$

[4] While not necessary for the characterization of the law of motion for bequests under the Solow steady state, normalization is needed for the characterization of the Romer steady state and the period-2 cycles as the total output is growing under the latter 2 trajectories. Normalization applies along the transition to the fixed point(s) as well.

$$\Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]\frac{b^i_t}{Y_t}$$
$$- \frac{1}{\sigma\bar{L}}\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]$$
$$= \frac{1}{\sigma\bar{L}}\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right][l^i_t - \bar{l}_t]. \quad (3.44)$$

The ratio of bequest to total output inherited by a young agent from lineage i at any time t, with elastic labor, is a first order non-homogenous difference equation.

Let the supercript E denotes elastic labor. Define

$$z^{Ei}_{t+1} = \frac{1}{\Omega_5}\frac{b^i_{t+1}}{Y_{t+1}} - \frac{\frac{1}{\Omega_5}\delta_3}{1-\phi^E_1}, \quad (3.45)$$

where $\Omega_5 = \frac{1}{\sigma\bar{L}}\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]$, $\delta_3 = \frac{1}{\sigma\bar{L}}\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]$, and $\phi^E_1 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]$. Substituting (3.45) into equation (3.44), we get a first order homogenous difference equation for z^{Ei}_{t+1} . That is,

$$z^{Ei}_{t+1} - \phi^E_1 z^{Ei}_t = [l^i_t - \bar{l}_t] \equiv u^i_{t+1} \sim i.i.d.(0,\sigma^2_l). \quad (3.46)$$

Proposition 3.4. *Since* $\left|\phi^E_1\right| \equiv \left|\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]\right| < 1$, z^{Ei}_{t+1} *is a covariance-stationary AR(1) process.*

(i) The expected bequest inherited by a young agent of lineage i*, as a ratio of the total output, at time,* $t+1$*, is:*

$$E\left[\frac{b^i_{t+1}}{Y_{t+1}}\right] = \frac{\delta_3}{1-\phi^E_1} = \frac{\frac{1}{\sigma\bar{L}}\phi^E_1}{1-\phi^E_1}. \quad (3.47)$$

(ii) The variance of bequest inherited by a young agent of lineage i*, as a ratio*

of the total output, at time, $t+1$, is:

$$Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right] = \frac{\sigma_l^2 \Omega_5^2}{1-\phi_1^{E2}} = \frac{\sigma_l^2}{1-\phi_1^{E2}}\left(\frac{1}{\sigma \overline{L}}\phi_1^E\right)^2. \tag{3.48}$$

(iii) The intergenerational mobility can be measured by the covariance or the correlation coefficient, to gauge the extent of intergenerational transmission of inequality. They are respectively:

$$Cov\left[\frac{b^i_{t+1}}{Y_{t+1}}, \frac{b^i_t}{Y_t}\right] = \Omega_5^2 \frac{\sigma_l^2 \phi_1^E}{1-\phi_1^{E2}} = \frac{\sigma_l^2 \phi_1^E}{1-\phi_1^{E2}}\left(\frac{1}{\sigma \overline{L}}\phi_1^E\right)^2. \tag{3.49}$$

$$\rho\left[\frac{b^i_{t+1}}{Y_{t+1}}, \frac{b^i_t}{Y_t}\right] = \rho_1 = \phi_1^E = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3}\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]. \tag{3.50}$$

Proof. As in Chapter 1, Proposition 1.5.

$$\begin{aligned}\phi_1^E &\equiv \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right] \\ &= \frac{(\sigma-1)\mu}{1+\sigma\mu}\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right] < 1. \end{aligned} \tag{3.51}$$

For the AR(1) process in equation (3.46), $E\left[z^{Ei}_{t+1}\right] = 0$, $Var\left[z^{Ei}_{t+1}\right] = \gamma_0 = \sigma_l^2$, $Cov\left[z^{Ei}_{t+j}, z^{Ei}_t\right] = \gamma_j = \phi_1^{Ej}\gamma_0 = \frac{\phi^{Ej}\sigma_l^2}{1-\phi_1^{E2}}$, and $\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\phi_1^{Ej}\gamma_0}{\gamma_0} = \phi_1^{Ej}$. ■

Proposition 3.5. *The higher the preference for leisure, λ, the lower the correlation of the ratio of bequests to total output across generations from the same dynasty, the higher the intergenerational mobility.*

Proof. From the correlation equation of Proposition 3.50,

$$\begin{aligned}\frac{\partial \phi_1^E}{\partial \lambda} &= \frac{(\sigma-1)\mu[1+\beta(1+\mu)]}{1+\sigma\mu}\frac{\partial\left[\frac{1}{1+\beta(1+\mu)+\lambda}\right]}{\partial\lambda} \\ &= \underbrace{\frac{(\sigma-1)\mu[1+\beta(1+\mu)]}{1+\sigma\mu}}_{\geq 0}\underbrace{\frac{-1}{[1+\beta(1+\mu)+\lambda]^2}}_{\leq 0} \leq 0.\end{aligned} \tag{3.52}$$

The higher the preference for leisure, the higher the intergenerational mobility as the elastic labor provides a channel to level the playing field of the asset-poor via the disincentive to work among the asset-rich. That is, with endogenous labor, wealth effect from inheritance discourages labor. Hence bequest has an indirect effect on mobility via the marginal value of hours worked, besides the direct effect via intergenerational risk pooling. This impact of leisure on lineage mobility heterogenous wealth has not been discussed in the existing heterogeneous agent literature because the models do not cover elastic labor (as with Wan and Zhu (2012)), and/or because the models do not explicitly characterize the lineage bequest evolution with elastic labor, (as with Bossmann et al. (2007), Zhu (2013b), and Marcet et al. (2007)).

Proposition 3.6. *With endogenous labor, the higher the price elasticity of the final sector's demand for each intermediate, σ (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin for innovation), the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility.*

Proof.

$$\begin{aligned}\frac{\partial \phi_1^E}{\partial \sigma} &= \frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda} \cdot \frac{\partial \frac{(\sigma-1)\mu}{1+\sigma\mu}}{\partial \sigma} \\ &= \left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right] \cdot \frac{1}{(1+\sigma\mu)^2}\left[\sigma(1-\mu)+\mu(\sigma^2+1)\right] \geq 0, \quad (3.53)\end{aligned}$$

■

Proposition 3.7. *With endogenous labor, the higher the bequest motive, μ, the higher the correlation of the ratio of bequests to total output across generations from the same dynasty, the lower the intergenerational mobility.*

Proof.

$$\frac{\partial \phi_1^E}{\partial \mu} = \left[\frac{(\sigma-1)\mu}{1+\sigma\mu}\right]\left[\frac{\beta\lambda}{[1+\beta(1+\mu)+\lambda]^2}\right] + \left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]\left[\frac{\sigma-1}{[1+\sigma\mu]^2}\right]$$
$$\geq 0, \tag{3.54}$$

■

It can be seen from Propositions 3.6 and 3.7 that the results relating to comparative statics of the lineage bequest evolution in Chapter 1 (Propositions 1.78 and 1.76) are robust to the introduction of endogenous labor-leisure choice. In this model, the returns on capital across all agents who saved are homogenous and thus rank preserving, although it can vary across time. Thus mobility is driven by the idiosyncratic labor productivity shocks. A decrease in wage share will decrease intergenerational mobility. With the "joy-of-giving" bequest motive, an increase in bequest motive reduces intergenerational mobility under i.i.d. labor productivity shocks. That is, bequest functions as an intergenerational risk pooling device for idiosyncratic labor productivity shocks within a lineage, offsetting the intragenerational mobility arising from the idiosyncratic labor productivity draws.

3.4.2 Cross-sectional distributions

Cross-sectional distribution of bequest

Proposition 3.8. *The cross -sectional inequality of bequest with elastic labor is higher than that with inelastic labor. The higher the preference for leisure, the higher the cross-sectional inequality of bequest.*

Proof. Note that $\phi_1^E = \frac{(\sigma-1)\mu}{1+\sigma\mu}\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right] \leq \frac{(\sigma-1)\mu}{1+\sigma\mu} = \phi_1^* < 1$. From the asymptotic distribution for a covariance-stationary process[5], the inequality of bequests and labor income in the asymptotic cross-sectional distribution using coefficient of variation[6] are:

$$\begin{aligned} CV\left[b^i\right] \text{ with elastic labor} &= CV\left[\frac{b^i}{Y^E}\right] = CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right] \\ &= \frac{\sqrt{Var\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]}}{E\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]} = \frac{\sigma_l\sqrt{\frac{1}{1-\phi_1^{E2}}}}{\frac{1}{(1-\phi_1^E)}} = \sigma_l\sqrt{\frac{1-\phi_1^E}{1+\phi_1^E}}. \end{aligned} \tag{3.55}$$

$$\left[\frac{CV[\frac{b^i_{t+1}}{Y^E_{t+1}}]\ with\ 0<\lambda<1}{CV[\frac{b^i_{t+1}}{Y_{t+1}}]\ with\ \lambda=0}\right]^2 = \frac{1-\phi_1^E}{1-\phi_1^*}\cdot\frac{1+\phi_1^*}{1+\phi_1^E} > 1. \tag{3.56}$$

[5]See Hamilton (1994, pp. 186–195: Section 7.2. Limit Theorems for Serially Dependent Observations).

[6]The coefficient of variation, as a measure of inequality, is scale invariant.

If $0 < \lambda < 1$,

$$sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]}{\partial\lambda} = sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\lambda}$$

$$= sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\phi^E_1}\frac{\partial\phi^E_1}{\partial\lambda}$$

$$= sign\,\sigma_l^2\underbrace{\left\{\frac{-2}{\left(1+\phi^E_1\right)^2}\right\}}_{<0}\underbrace{\frac{\partial\rho_1}{\partial\lambda}}_{\leq 0 \text{ from (3.52)}} \geq 0. \quad (3.57)$$

■

With endogenous labor, cross-sectional variation of inherited wealth generates cross-sectional variation of labor-leisure choice, in addition to the baseline idiosyncratic variation of labor productivity. Hence bequest has an indirect effect on cross-sectional inequality via the marginal value of work arising from wealth effect.

Proposition 3.9. *With endogenous labor, as with inelastic labor, the higher price elasticity of the final goods sector's demand for each intermediate, σ, (and correspondingly, an increase in capital share, a decrease in wage share, as well as a decrease in monopoly margin for innovation), the higher the cross-sectional inequality of the ratio of inherited wealth to total output.*

Proof.

$$
\begin{aligned}
sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]}{\partial\sigma} = sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\sigma}
&= sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\phi^E_1}\frac{\partial\phi^E_1}{\partial\sigma} \\
&= sign\,\sigma_l^2\underbrace{\left\{\frac{-2}{\left(1+\phi^E_1\right)^2}\right\}}_{<0}\underbrace{\frac{\partial\rho_1}{\partial\sigma}}_{\geq 0 \text{ from (3.53)}} \leq 0. \quad (3.58)
\end{aligned}
$$

■

Proposition 3.10. *With endogenous labor, as with inelastic labor, the higher the bequest motive, the lower the cross-sectional inequality of bequest.*

Proof.

$$
\begin{aligned}
sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]}{\partial\mu} = sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\mu}
&= sign\frac{\partial\left(CV\left[\frac{b^i_{t+1}}{Y^E_{t+1}}\right]\right)^2}{\partial\phi^E_1}\frac{\partial\phi^E_1}{\partial\mu} \\
&= sign\,\sigma_l^2\underbrace{\left\{\frac{-2}{\left(1+\phi^E_1\right)^2}\right\}}_{<0}\underbrace{\frac{\partial\rho_1}{\partial\mu}}_{\geq 0 \text{ from (3.54)}} \leq 0. \quad (3.59)
\end{aligned}
$$

■

It can be seen from Propositions 3.9 and 3.10 that the results relating to comparative statics of the cross-sectional wealth inequality in Chapter 1 (Propositions 1.78 and 1.76) are robust to the introduction of endogenous labor-leisure choice. In this model, the returns on capital across all agents who saved are homogenous and thus rank preserving, although it can vary across time. As a result of the Cobb-Douglas production function in the final good sector, a decrease in wage share will decrease cross-sectional wealth inequality. Thus the higher the price elasticity of the final sector's demand for each intermediate, the lower the monopoly margin for innovation, the lower the wage share, the lower the cross-sectional bequest inequality. With the "joy-of-giving" bequest motive, an increase in bequest motive reduces the cross-sectional inequality of inherited wealth under i.i.d. labor productivity shocks. That is, bequest functions as a intergenerational risk pooling device for idiosyncratic labor productivity shocks. This extends the robustness of role of bequest in intergenerational risk sharing by Becker and Tomes (1979) and Bossmann et al. (2007) to a model of endogenous growth (with innovation and cycles) with inelastic and elastic labor, in Chapter 1 and 3 respectively.

Cross-sectional distribution of leisure and wage income earned

From equation (3.12), the expectation and variance of leisure consumed for an agent i are:

$$E\left[\Lambda_t^i\right] = \frac{\lambda}{1+\beta(1+\mu)+\lambda}\left(1+\frac{\overline{b}_t^i}{w_t}\right) = \frac{\lambda+\lambda\mu\sigma}{1+\lambda+\mu+\beta(1+\mu)^2+\lambda\mu\sigma} \tag{3.60}$$

$$
\begin{aligned}
Var\left[\Lambda_t^i\right] &= \left[\frac{\lambda}{w_t[1+\beta(1+\mu)+\lambda]}\right]^2 Var\left[\frac{b_t^i}{l_t^i}\right] \\
&\approx \left[\frac{\lambda}{w_t[1+\beta(1+\mu)+\lambda]}\right]^2 \left\{\frac{\mu_{b_t^i}^2}{\mu_{l_t^i}^2}\left[\frac{\sigma_{b_t^i}^2}{\mu_{b_t^i}^2} - \frac{2Cov\left[b_t^i, l_t^i\right]}{\mu_{b_t^i}\mu_{l_t^i}} + \frac{\sigma_{l_t^i}^2}{\mu_{l_t^i}^2}\right]\right\} \qquad (3.61) \\
&= \left[\frac{\lambda}{w_t[1+\beta(1+\mu)+\lambda]}\right]^2 \left\{\frac{\mu_{b_t^i}^2}{\mu_{l_t^i}^2}\left[\frac{\sigma_{b_t^i}^2}{\mu_{b_t^i}^2} + \frac{\sigma_{l_t^i}^2}{\mu_{l_t^i}^2}\right]\right\} \qquad (3.62)
\end{aligned}
$$

Leisure consumed by an agent i, being a ratio of 2 random variables: contemporaneous idiosyncratic inherited wealth, b_t^i, and idiosyncratic labor productivity shock, l_t^i, which can be approximated by a Taylor expansion since both b_t^i and l_t^i have distributions with support $[0,\infty)$ (see Stuart and Ord (1994, pp. 351)).

Proposition 3.11.

(i) If $\overline{G} = \Omega_1\Omega_3\overline{A} < 1$ or $\overline{G} = \Omega_1\Omega_3\overline{A} > \theta - 1$, for i.i.d. labor productivity shocks, the inequality of leisure consumed, measured by the coefficient of variation, are identical under either the Solow or Romer steady state.

(ii) If $1 < \overline{G} = \Omega_1\Omega_3\overline{A} < \theta - 1$, under period-2 cycles exists, and the inequality of leisure consumed at both period-2 fixed points are identical, but are higher compared to the steady states inequality of leisure consumption in part (i).

Proof. By applying Propositions 1.1 and 1.10 to the moments of Λ_t^i in equations (3.60) and (3.61). ■

From equation (3.12), each agent's sensitivity of individual labor supply to relative wealth differs, due to the idiosyncratic labor productivity draws. The distribution of hours worked is determined by the joint evolution of the distribution of inherited wealth and the distribution of maximum idiosyncratic labor income that could be earned by an agent of lineage i. Different aggregate wage rate per efficient labor over the different steady states and the fixed points of the period-2 cycles are generated. Therefore, under period-2 growth cycles, the inequality of leisure consumed at both period-2 fixed points are identical, but are higher compared to the steady state inequality of leisure consumption, as the alternating aggregate wage rate per efficient labor over growth regimes increases the variability of the cross-sectional distribution of leisure consumed. Thus, extending the prior studies on the distributional implications of endogenous labor-leisure choice, this chapter covers both the AK growth model (as in García-Peñalosa and Turnovsky (2007)) and the neoclassical growth model (as in García-Peñalosa and Turnovsky (2011)), as well as period-2 cycle, which is novel. However, diverging from García-Peñalosa and Turnovsky (2007) and García-Peñalosa and Turnovsky (2011) where agents are endowed with ex-ante heterogenous amount of capital, agents in this chapter can be ex-ante identical in wealth. The lineage bequest evolution and mobility, as well as the cross-sectional inequality of the distributions of wealth, leisure consumed and wage-income are engendered by the idiosyncratic labor productivity shocks (absent in the other two papers) and the magnitude of the motives for bequest and leisure.

From equation (3.12), the ratio of wage income earned to the total output is:

$$\begin{aligned}\frac{w_t l_t^i (1-\Lambda_t^i)}{Y_t} &= \frac{w_t l_t^i}{Y_t} - \frac{w_t l_t^i}{Y_t}\left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\left(1+\frac{b_t^i}{w_t l_t^i}\right) \\ &= \frac{w_t l_t^i}{Y_t}\left(1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right) - \left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{b_t^i}{Y_t}\right)\end{aligned} \tag{3.63}$$

The cross-sectional expectation and variance of wages earned of an agent i from generation t, normalized by the final output at time t, are respectively:

$$\begin{aligned}&E\left[\frac{w_t l_t^i (1-\Lambda_t^i)}{Y_t}\right] \\ &\quad = \left(1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right) E\left[\frac{w_t l_t^i}{Y_t}\right] - \left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right) E\left[\frac{b_t^i}{Y_t}\right] \\ &= \left(1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\frac{1}{\sigma \bar{L}} - \left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{1}{\sigma \bar{L}}\right)\left(\frac{\phi_1^E}{1-\phi_1^E}\right) \\ &\qquad = \frac{1}{\sigma \bar{L}}\left(1-\frac{\lambda}{(1+\beta(1+\mu)+\lambda)\left(1-\phi_1^E\right)}\right) = \frac{1}{\sigma},\end{aligned} \tag{3.64}$$

$$\begin{aligned}&Var\left[\frac{w_t l_t^i (1-\Lambda_t^i)}{Y_t}\right] \\ &\quad = \left(1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)^2 Var\left[\frac{w_t l_t^i}{Y_t}\right] + \left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)^2 Var\left[\frac{b_t^i}{Y_t}\right] \\ &\qquad\qquad \text{since } Cov\left[\frac{w_t l_t^i}{Y_t}, \frac{b_t^i}{Y_t}\right] = 0 \text{ (by construction).} \\ &= \left(\frac{1}{\sigma \bar{L}}\right)^2 \sigma_l^2 \left[\left(1-\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)^2 + \left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)^2 \left(\frac{\phi_1^{E2}}{1-\phi_1^{E2}}\right)\right] \\ &\qquad = \left(\frac{1}{\sigma \bar{L}}\right)^2 \sigma_l^2 \left\{\frac{[1+\beta(1+\mu)]^2\left(1-\phi_1^{E2}\right)+\lambda^2 \phi_1^{E2}}{[1+\beta(1+\mu)+\lambda]^2\left(1-\phi_1^{E2}\right)}\right\}.\end{aligned} \tag{3.65}$$

Thus, the inequality of the cross-sectional wages earned to final output is:

$$CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right] = \frac{\sqrt{Var[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}]}}{E\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}$$

$$= \sigma_l\left(\frac{1}{\bar{\bar{L}}}\right)\sqrt{\frac{[1+\beta(1+\mu)]^2\left(1-\phi_1^{E2}\right)+\lambda^2\phi_1^{E2}}{[1+\beta(1+\mu)+\lambda]^2\left(1-\phi_1^{E2}\right)}}$$

$$= \sigma_l \underbrace{\left(\frac{1}{\bar{\bar{L}}}\right)}_{\text{GE wage share effect}} \sqrt{\underbrace{\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]^2}_{\text{substitution effect}} + \underbrace{\left[\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]^2 \frac{\phi_1^{E2}}{\left(1-\phi_1^{E2}\right)}}_{\text{wealth effect}}}. \tag{3.66}$$

Proposition 3.12. *With endogenous labor, the cross-sectional inequality of the ratio of wage income earned to total output may be increasing or decreasing with an increase in the preference for leisure, λ.*

Proof.

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial\lambda} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial\lambda}$$

$$= sign\,\sigma_l^2\frac{\partial}{\partial\lambda}\left\{\underbrace{\left(\frac{1}{\bar{\bar{L}}}\right)^2}_{\text{H}}\underbrace{\left\{\underbrace{\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]^2}_{\text{E}}+\underbrace{\left[\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]^2}_{\text{F}}\underbrace{\frac{\phi_1^{E2}}{\left(1-\phi_1^{E2}\right)}}_{\text{G}}\right\}}_{\text{I}}\right\}, \tag{3.67}$$

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial\lambda} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial\lambda} = sign\,\sigma_l^2\left[\text{I}\underbrace{\frac{\partial\text{H}}{\partial\lambda}}_{\geq 0}+\text{H}\underbrace{\frac{\partial\text{I}}{\partial\lambda}}_{\leq 0}\right] \lesseqgtr 0. \tag{3.68}$$

$$\frac{\partial\text{H}}{\partial\lambda} = \underbrace{2\left(\frac{1}{\bar{\bar{L}}}\right)\cdot(-1)\cdot\left(\frac{1}{\bar{\bar{L}}}\right)^2\underbrace{\frac{\partial\bar{L}}{\partial\lambda}}_{\leq 0\text{ from (3.35)}}}_{\geq 0}, \tag{3.69}$$

$$\frac{\partial \mathrm{I}}{\partial \lambda} = \underbrace{\frac{\partial \mathrm{E}}{\partial \lambda}}_{\leq 0} + \left[\mathrm{F} \underbrace{\frac{\partial \mathrm{G}}{\partial \lambda}}_{\leq 0} + \mathrm{G} \underbrace{\frac{\partial \mathrm{F}}{\partial \lambda}}_{\leq 0} \right], \tag{3.70}$$

$$\frac{\partial \mathrm{E}}{\partial \lambda} = \underbrace{2\left(\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{-[1+\beta(1+\mu)]}{[1+\beta(1+\mu)+\lambda]^2}\right)}_{\leq 0}, \tag{3.71}$$

$$\frac{\partial \mathrm{F}}{\partial \lambda} = \underbrace{2\left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{-\lambda}{[1+\beta(1+\mu)+\lambda]^2}\right)}_{\leq 0}, \tag{3.72}$$

$$\frac{\partial \mathrm{G}}{\partial \lambda} = \underbrace{\frac{2\phi_1^E}{1-\phi_1^{E2}} \cdot \frac{\partial \phi_1^E}{\partial \lambda} - \frac{\phi_1^E \cdot (-1)}{\left(1-\phi_1^{E2}\right)^2} \cdot \underbrace{\frac{\partial \phi_1^E}{\partial \lambda}}_{\leq 0 \text{ from (3.52)}}}_{\leq 0}. \tag{3.73}$$

■

With an increase in the preference for leisure, the general equilibrium wage share effect (price effect from equilibrium wages) on the cross-sectional inequality of labor earnings is positive, but the substitution and wealth effects (quantity effect from hours worked) on the cross-sectional inequality of labor earnings are negative.

Proposition 3.13. *With endogenous labor, the higher price elasticity of the final goods sector's demand for each intermediate, σ, (and correspondingly, an increase in capital share, a decrease in wage share, as well as a decrease in monopoly margin for innovation), the higher the the cross-sectional inequality of the ratio of wage income earned to total output.*

Proof.

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial\sigma} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial\sigma}$$

$$= sign\,\sigma_l^2\frac{\partial}{\partial\sigma}\left\{\underbrace{\left(\frac{1}{\overline{\overline{L}}}\right)^2}_{\text{H}}\underbrace{\left\{\underbrace{\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]^2}_{\text{E}}+\underbrace{\left[\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]^2}_{\text{F}}\underbrace{\frac{\phi_1^{E2}}{(1-\phi_1^{E2})}}_{\text{G}}\right\}}_{\text{I}}\right\}, \tag{3.74}$$

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial\sigma} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial\sigma} = sign\,\sigma_l^2\left[\text{I}\underbrace{\frac{\partial\text{H}}{\partial\sigma}}_{\geq 0}+\text{H}\underbrace{\frac{\partial\text{I}}{\partial\sigma}}_{\geq 0}\right]\geq 0. \tag{3.75}$$

$$\frac{\partial\text{H}}{\partial\sigma} = \underbrace{2\left(\frac{1}{\overline{\overline{L}}}\right)\cdot(-1)\cdot\left(\frac{1}{\overline{\overline{L}}}\right)^2\underbrace{\frac{\partial\overline{L}}{\partial\sigma}}_{\leq 0\text{ from (3.36)}}}_{\geq 0}, \tag{3.76}$$

$$\frac{\partial\text{I}}{\partial\sigma} = \underbrace{\frac{\partial\text{E}}{\partial\sigma}}_{=0}+\left[\text{F}\underbrace{\frac{\partial\text{G}}{\partial\sigma}}_{\geq 0}+\text{G}\underbrace{\frac{\partial\text{F}}{\partial\sigma}}_{=0}\right], \tag{3.77}$$

$$\frac{\partial\text{E}}{\partial\sigma} = \frac{\partial\text{F}}{\partial\sigma} = 0, \tag{3.78}$$

$$\frac{\partial\text{G}}{\partial\sigma} = \underbrace{\frac{2\phi_1^E}{1-\phi_1^{E2}}\cdot\frac{\partial\phi_1^E}{\partial\sigma}-\frac{\phi_1^E\cdot(-1)}{\left(1-\phi_1^{E2}\right)^2}\cdot\underbrace{\frac{\partial\phi_1^E}{\partial\sigma}}_{\geq 0\text{ from (3.53)}}}_{\geq 0}. \tag{3.79}$$

■

From equation (3.74) and (3.77), when the price elasticity increases, the wealth effect will lead to an increase in the inequality of wage income earned due to the increase in cross-sectional inequality from inherited wealth; furthermore, from equation (3.74) and (3.76), there is a general equilibrium effect that increases the wage share, through the increase in the value of per unit effective labor intensifying the inequality of labor productivity endowment. Thus, the inequality of wage income to output ratio increases with an increase in price elasticity, σ (and correspondingly, an increase in capital share, a decrease in wage share, as well as a decrease in monopoly margin for innovation), in the presence of elastic labor.

Proposition 3.14. *With endogenous labor, the cross-sectional inequality of the ratio of wage income earned to total output may be increasing or decreasing with an increase in the bequest motive,* μ.

Proof.

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial \mu} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial \mu}$$

$$= sign\,\sigma_l^2\frac{\partial}{\partial\mu}\left\{\underbrace{\left(\frac{1}{\bar{\bar{L}}}\right)^2}_{\mathrm{H}}\underbrace{\left\{\underbrace{\left[\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right]^2}_{\mathrm{E}}+\underbrace{\left[\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right]^2}_{\mathrm{F}}\underbrace{\frac{\phi_1^{E2}}{(1-\phi_1^{E2})}}_{\mathrm{G}}\right\}}_{\mathrm{I}}\right\}, \tag{3.80}$$

$$sign\frac{\partial CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]}{\partial \mu} = sign\frac{\partial\left\{CV\left[\frac{w_t l_t^i(1-\Lambda_t^i)}{Y_t}\right]\right\}^2}{\partial \mu} = sign\,\sigma_l^2\left[\mathrm{I}\underbrace{\frac{\partial \mathrm{H}}{\partial\mu}}_{\lesseqqgtr 0}+\mathrm{H}\underbrace{\frac{\partial \mathrm{I}}{\partial\mu}}_{\lesseqqgtr 0}\right]\lesseqqgtr 0. \tag{3.81}$$

$$\frac{\partial \mathrm{H}}{\partial \mu} = \underbrace{2\left(\frac{1}{\overline{\overline{L}}}\right)\cdot(-1)\cdot\left(\frac{1}{\overline{\overline{L}}}\right)^2 \underbrace{\frac{\partial \overline{L}}{\partial \mu}}_{\lesseqgtr 0 \text{ from (3.38)}}}_{\lesseqgtr 0}, \tag{3.82}$$

$$\text{if } \sigma \leq (>)\frac{1+2\beta(1+\mu)}{1+\beta(1-\mu)}, \frac{\partial \overline{L}}{\partial \mu} \geq (<)0.$$

$$\frac{\partial \mathrm{I}}{\partial \mu} = \underbrace{\frac{\partial \mathrm{E}}{\partial \mu}}_{\geq 0} + \left[\mathrm{F}\underbrace{\frac{\partial \mathrm{G}}{\partial \mu}}_{\geq 0} + \mathrm{G}\underbrace{\frac{\partial \mathrm{F}}{\partial \mu}}_{\leq 0}\right], \tag{3.83}$$

$$\frac{\partial \mathrm{E}}{\partial \mu} = \underbrace{2\left(\frac{1+\beta(1+\mu)}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{\beta\lambda}{[1+\beta(1+\mu)+\lambda]^2}\right)}_{\geq 0}, \tag{3.84}$$

$$\frac{\partial \mathrm{F}}{\partial \mu} = \underbrace{2\left(\frac{\lambda}{1+\beta(1+\mu)+\lambda}\right)\left(\frac{-\beta\lambda}{[1+\beta(1+\mu)+\lambda]^2}\right)}_{\leq 0}, \tag{3.85}$$

$$\frac{\partial \mathrm{G}}{\partial \mu} = \underbrace{\frac{2\phi_1^E}{1-\phi_1^{E2}}\cdot\frac{\partial \phi_1^E}{\partial \mu} - \frac{\phi_1^E\cdot(-1)}{\left(1-\phi_1^{E2}\right)^2}\cdot \underbrace{\frac{\partial \phi_1^E}{\partial \mu}}_{\geq 0 \text{ from (3.54)}}}_{\geq 0}. \tag{3.86}$$

■

As stated in Proposition 3.1(iv), the wealth effect of bequest on aggregate labor supply is related to the price elasticity, σ (and correspondingly, capital share, as well as the monopoly margin of innovator). If the capital share is small enough relative to the labor share, an increase in bequest motive will lead to a higher aggregate labor supply results (precautionary savings for uninsurable labor risk dominates), as with Aiyagari (1994), therefore a negative general equilibrium wage effect (price effect) on the cross-sectional inequality of wage income. Conversely, if the labor share is relatively small, an increase in bequest motive will reduce the aggregate labor (wealth effect dominates), which is consistent with Marcet et al. (2007) and Zhu (2013b). The substitution effect of an increase in bequest motive on the cross-sectional inequality of wage income is positive due to an increase in the hours worked (quantity effect) for all agents (converging to the inelastic labor case). However, the wealth effect of an increase in bequest motive on the cross-sectional inequality of wage income is ambiguous as impact on the hours worked (quantity effect) is unclear, due to the heterogeneity of the wealth inherited and the heterogeneity of the indiosyncratic labor productivity draw. If the preference for leisure, λ, is sufficiently low, and the wage share, $\frac{1}{\sigma}$, is sufficiently high, an increase in the bequest motive could increase the cross-sectional inequality of wage income to output ratio, in the presence of elastic labor.

3.5 Conclusion

In the presence of bequest motive, the wealth effect of inheritance brings about greater heterogeneity in the endogenous consumption of leisure, especially over endogenous growth cycles, when measured in terms of coefficient of variation. The comparative statics of the lineage mobility and cross-sectional inequality of the ratio of inherited wealth to output for inelastic labor under i.i.d. idiosyncratic shocks to changes in the bequest motive and price elasticity of the final sector's demand for each intermediate (and correspondingly, higher capital share, lower wage share, as well as lower monopoly margin for innovation), remain robust even when labor-leisure choice is made endogenously. This extends the robustness of role of bequest in intergenerational risk sharing by Becker and Tomes (1979) and Bossmann et al. (2007) to a model of endogenous growth (with innovation and cycles) with both inelastic and elastic labor.

However, with elastic labor, in the presence of a bequest motive, the cross-sectional inequality of the ratio of wages earned to final output is no longer identical to the inequality of the idiosyncratic labor productivity shocks as in Chapters 1 and 2. The comparative statics on cross-sectional ratio of wages earned to final output are driven by the relative size of the general equilibrium effect of the wages, the contemporaneous substitution effect of consumption to leisure and the inherited wealth effect, adjusted for quality with the contemporaneous labor productivity shock, on demand for leisure. Further theoretical research could be informed and directed by empirical evidence, through the merging of micro-datasets individual consumption, labor, savings and inheritances registered.

Chapter 4

Fiscal policies and their impact on mobility, inequality and growth trajectories

4.1 Introduction

In this chapter, I consider the impact of fiscal policies on inequality and mobility. With logarithmic utility, a decrease in inheritance tax increases cross-sectional inequality, tilting the playing field, under i.i.d. and correlated labor productivity shocks, for both Solow and Romer regimes as well as for period-2 cycles, thus strengthening the results from Bossmann et al. (2007), while contrasting with Becker and Tomes (1979), Atkinson (1980) and Davies (1986). Furthermore, an increase in inheritance tax also increases intergenerational lineage mobility under i.i.d. shocks. On the other hand, the reduction of volatility of growth by fiscal policies such as the innovation subsidy policy rule proposed by Aloi and Lasselle (2007), which stabilizes period 2 cycles, could level the playing field, by decreasing the persistence of inherited wealth during the transition to the Romer steady state, if funded by a uniform value-added tax. However, moving from period-2 cycles to the Romer steady state with the innovation subsidies may change the cross-sectional inequality of wealth if the idiosyncratic labor productivity shocks are correlated.

Section 2 motivates and investigates the impact of changes in inheritance tax rate in the case of logarithmic preference, while Section 3 motivates and examines the impact of innovation subsidies, funded by uniform tax, on the growth and distribution dynamics. The results in this chapter can be extended to include the case with status anxiety (consumption externality) in the logarithmic preference, discussed in Chapter 2. Section 4 concludes.

4.2 The impact of inheritance tax

4.2.1 Motivation for exploring changes in inheritance tax rate

As highlighted by Piketty (2014, pp. 503, Figure 14.2), during the last century, there are significant changes in the top marginal tax rate of the inheritance tax; particularly for the United States and the United Kingdom, from the onset of the 70's, these top marginal tax rates have fallen significantly, from more than 70% to 40% or less. In this section, while I am cognizant of the possible endogeneity of inheritance tax rate and income distribution, I shall abstract the political economy aspect from the following analysis. I refer interested readers to the papers by Benabou (2000) and Acemoglu and Robinson (2001) for related theoretical treatment, and Banerjee and Duflo (2003) for empirical literature. I adopt the same approach as Bossmann et al. (2007), Wan and Zhu (2012) and Zhu (2013a) by looking at the comparative statics of an exogenous change in the inheritance tax rate, but applied to a different production function, hence extending the applicability of their results to endogenous growth models with innovation, and different growth regimes, including periodic cycles.

4.2.2 Impact of exogenous changes in inheritance tax rate

The government's budget constraint

Suppose the government now levies an inheritance tax at a flat rate τ_b on all inheritance, to fund lumpsum transfers to the young, as a new redistributive policy. All young agents receives the same lumpsum subsidy, g_t. The government runs a balanced budget at all times.

$$g_t = \tau_b \int_0^1 b_t^i \; di = \tau_b \bar{b}_t. \tag{4.1}$$

Agent's optimization

The first period budget constraint for individual from lineage i at time t is:

$$c_t^i + s_t^i = b_t^i + w_t^i + g_t \equiv y_t^i, \tag{4.2}$$

where b_t^i is now interpreted as the after-tax bequest. Parents, mindful of the inheritance tax, will face a new second period budget constraint:

$$R_{t+1} s_t^i = d_{t+1}^i + (1+\tau_b) b_{t+1}^i. \tag{4.3}$$

From the second period budget constraint, we get:

$$b_{t+1}^i = \frac{R_{t+1} s_t^i - d_{t+1}^i}{1+\tau_b}. \tag{4.4}$$

Substituting the new budget constraints into the household optimization as shown in Appendix C.1, we get the optimal choice of bequest left by an old agent to his immediate offspring to be:

$$b_{t+1}^{i}=\frac{\mu}{(1+\mu)(1+\tau_b)}R_{t+1}s_t^i. \tag{4.5}$$

Note that in this model with logarithmic utility and "joy-of-giving", taxing bequests with transfers does not affect the aggregate capital stock, as individuals save a constant fraction out of their own disposable income when young, and consume a constant fraction out of their own disposable income when old, independent of the tax rate, as implied by (4.2), (C.12) and (C.14):

$$s_t^i=\frac{\beta(1+\mu)}{1+\beta(1+\mu)}y_t^i;$$

$$d_{t+1}^i=\frac{R_{t+1}\beta}{1+\beta(1+\mu)}y_t^i.$$

General equilibrium

Applying the market clearing conditions and working through as in previous section, we get the difference equation governing the law of motion of lineage bequest:

$$\begin{aligned}b_{t+1}^{i}&=\frac{R_{t+1}\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)}\left[y_t^i\right]\\&=\frac{R_{t+1}\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)}\left[b_t^i+l_t^iw_t+g_t\right]\\&=\frac{R_{t+1}\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)}\left[b_t^i+l_t^i\frac{1}{\sigma}Y_t+g_t\right].\end{aligned} \tag{4.6}$$

From the balanced budget of the government (4.1), capital share (1.27) at equilibrium, capital market clearing (1.30), and the bequest and savings of the average agent, (4.5) and(1.8) respectively, we get:

$$g_t = \tau_b \bar{b}_t = \tau_b \frac{R_t \beta\mu}{[1+\beta(1+\mu)](1+\tau_b)} \bar{y}_t \tag{4.7}$$

$$= \tau_b \frac{R_t s_{t-1} \beta\mu}{\Omega_1 [1+\beta(1+\mu)](1+\tau_b)} = \tau_b \frac{\Omega_1 \Omega_3 Y_t \beta\mu}{\Omega_1 [1+\beta(1+\mu)](1+\tau_b)}. \tag{4.8}$$

Normalizing by the final output,

$$\frac{b^i_{t+1}}{Y_{t+1}} - \frac{R_{t+1}\beta\mu}{G_{t+1}(1+\tau_b)[1+\beta(1+\mu)]} \frac{b^i_t}{Y_t} - \frac{R_{t+1}\beta\mu \left[1 + \frac{\tau_b}{(1+\tau_b)} \left[\frac{\beta\mu}{1+\beta(1+\mu)}\right] \sigma\Omega_3\right]}{G_{t+1}\sigma(1+\tau_b)[1+\beta(1+\mu)]}$$

$$= \frac{R_{t+1}\beta\mu}{G_{t+1}\sigma(1+\tau_b)[1+\beta(1+\mu)]} [l^i_t - \bar{l}_t], \tag{4.9}$$

$$\Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3(1+\tau_b)[1+\beta(1+\mu)]} \frac{b^i_t}{Y_t} - \frac{(1-\frac{1}{\sigma})\beta\mu \left[1 + \frac{\tau_b}{(1+\tau_b)} \left[\frac{\beta\mu}{1+\beta(1+\mu)}\right] \sigma\Omega_3\right]}{\Omega_1\Omega_3\sigma(1+\tau_b)[1+\beta(1+\mu)]}$$

$$= \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3(1+\tau_b)[1+\beta(1+\mu)]} \frac{1}{\sigma} [l^i_t - \bar{l}_t]. \tag{4.10}$$

The ratio of bequest to total output inherited by a young agent from lineage i at any time t, is a first order non-homogenous difference equation.

Let the superscript τ_b denotes the case with inheritance tax rate. Define

$$z^{\tau_b i}_{t+1} = \frac{1}{\Omega_5} \frac{b^i_{t+1}}{Y_{t+1}} - \frac{\frac{1}{\Omega_5}\delta_4}{1-\phi_1^{\tau_b}}, \tag{4.11}$$

where $\Omega_5 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma}$, $\delta_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma} \left[1 + \frac{\tau_b}{(1+\tau_b)} \frac{\beta\mu}{1+\beta(1+\mu)} \sigma\Omega_3\right]$, $\phi_1^{\tau_b} = \frac{\phi_1^*}{1+\tau_b}$ and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. Substituting (4.11) into equation (4.10), we get a first order homogenous difference equation for $z^{\tau_b i}_{t+1}$. That is,

$$z^{\tau_b i}_{t+1} - \phi_1^{\tau b} z^{\tau_b i}_t = [l^i_t - \bar{l}_t] \equiv u^i_{t+1} \sim i.i.d.(0, \sigma_l^2). \tag{4.12}$$

Proposition 4.1. *Since* $\left|\phi_1^{\tau b}\right| \equiv \left|\frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)}\right| < 1$, $z_{t+1}^{\tau_b i}$ *is a covariance-stationary AR(1) process.*

(i) The expected after-tax bequest inherited by a young agent of lineage i, *as a ratio of the total output, at time,* $t+1$, *is:*

$$E\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\delta_4}{1-\phi_1^{\tau_b}} = \frac{\frac{1}{\sigma}\phi_1^*\left[1+\frac{\tau_b\sigma\Omega_3\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)}\right]}{1+\tau_b-\phi_1^*}. \tag{4.13}$$

(ii) The variance of after-tax bequest inherited by a young agent of lineage i, *as a ratio of the total output, at time,* $t+1$, *is:*

$$Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\sigma_l^2\Omega_5^2}{1-(\phi_1^{\tau_b})^2} = \frac{\sigma_l^2\left(\frac{\phi_1^*}{\sigma}\right)^2}{(1+\tau_b)^2-\phi_1^{*2}}. \tag{4.14}$$

(iii) The intergenerational mobility can be measured by the covariance or the correlation coefficient, to gauge the extent of intergenerational transmission of inequality. They are respectively:

$$Cov\left[\frac{b_{t+1}^i}{Y_{t+1}},\frac{b_t^i}{Y_t}\right] = \sigma_l^2\frac{\Omega_5^2\phi_1^{\tau_b}}{1-(\phi_1^{\tau_b})^2} = \sigma_l^2\frac{\frac{\phi_1^*}{1+\tau_b}}{(1+\tau_b)^2-\phi_1^{*2}}\left(\frac{1}{\sigma}\phi_1^*\right)^2; \tag{4.15}$$

$$\rho\left[\frac{b_{t+1}^i}{Y_{t+1}},\frac{b_t^i}{Y_t}\right] = \rho_1^{\tau_b} = \phi_1^{\tau_b} = \frac{\phi_1^*}{1+\tau_b} = \frac{(\sigma-1)\mu}{(1+\sigma\mu)(1+\tau_b)}. \tag{4.16}$$

Proof. As with proof for Proposition 1.5.

$$\phi_1^{\tau_b} = \frac{\phi_1^*}{1+\tau_b} = \frac{(\sigma-1)\mu}{(1+\sigma\mu)(1+\tau_b)} < 1. \tag{4.17}$$

Proposition 4.2. *For an economy with i.i.d. productivity shocks, the higher the inheritance tax, the lower the correlation of intergenerational bequest in the* *same dynasty, the higher the intergenerational mobility;*

furthermore, the cross sectional inequality of after-tax inherited wealth plus government transfer is lower the higher the inheritance tax.

Proof. Since $\rho_1^{\tau_b} = \phi_1^{\tau_b} = \frac{(\sigma-1)\mu}{(1+\sigma\mu)(1+\tau_b)}$,

$$\frac{\partial \rho_1^{\tau_b}}{\partial \tau_b} = \frac{(\sigma-1)\mu(-1)}{(1+\sigma\mu)(1+\tau_b)^2} \leq 0. \tag{4.18}$$

With no population growth, at the fixed points, the average capital stock for each time period must be equals to the expected wealth holdings of a family of lineage i for each time period. From equations (C.12), (C.14), (4.5) and (4.1), the expected after-tax bequest plus government transfers for an agent of lineage i is constant and independent of the tax-rate.

$$\begin{aligned} sign\frac{\partial CV\left[\frac{b_{t+1}^i}{Y_{t+1}} + \frac{g_{t+1}}{Y_{t+1}}\right]}{\partial \tau_b} &= sign\frac{\partial Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial \tau_b} \\ &= sign\sigma_l^2 \frac{\phi_1^{*2}}{\sigma^2} \frac{(-2)(1+\tau_b)}{\left[(1+\tau_b)^2 - \phi_1^{*2}\right]^2} \leq 0. \end{aligned} \tag{4.19}$$

■

This extends Bossmann et al. (2007) findings that redistribution via estate taxes can reduce intragenerational after-tax wealth inequality with transfers, and increases intergenerational mobility beyond the neoclassical framework to cover endogenous growth with Solow, Romer, or Period-2 growth regimes. Furthermore, with finite variance of labor productivity draws, I can utilise the time series properties to explore the intergenerational mobility, via the covariance or correlations, that is absent in Wan and Zhu (2012).

As noted by Bossmann et al. (2007), this result depends on the assumption of the utility functions. With logarithmic preferences and "warm-glow" bequest motive used in this Chapter and by Bossmann et al. (2007) and Wan and Zhu (2012) in their discussion of estate taxes, the inheritance taxes do not affect the aggregate macroeconomic variables as the aggregate savings, hence aggregate wealth, remains unchanged. Thus, this result does not apply to CES utility functions when the constant elasticity of substitution is > 1, as in García-Peñalosa and Turnovsky (2007) or García-Peñalosa and Turnovsky (2011), or Appendix B.1. of Bossmann et al. (2007). Similarly, this result does not apply when savings decisions are optimized on the combined family income across two or more generations (e.g. Becker and Tomes (1979) and Davies and Kuhn (1991)), or on life-cycle income with mortality influence (e.g. Atkinson (1980)).

4.2.3 Impact of inheritance tax rate in the presence of correlated productivity shocks

Let $l_t^i = \bar{l}_t + \nu(l_{t-1}^i - \bar{l}_{t-1}) + \varepsilon_t^i$, as in Chapter 1 and Bossmann et al. (2007).

Lineage bequest evolution with correlated shocks at Solow or Romer steady states

Corresponding to equation (1.83) of Chapter 1,

$$\frac{1}{\Omega_5}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{1}{\Omega_5}\left(\phi_1^{\tau_b} + \nu\right)\frac{b_t^i}{Y_t} - \frac{1}{\Omega_5}\left(-\phi_1^{\tau_b}\nu\right)\frac{b_{t-1}^i}{Y_{t-1}} - \frac{\delta_4}{\Omega_5}(1-\nu) = \varepsilon_t^i \sim i.i.d.(0, \sigma_l^2). \tag{4.20}$$

where $\Omega_5 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma}$, $\delta_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma}\left[1+\frac{\tau_b}{(1+\tau_b)}\frac{\beta\mu}{1+\beta(1+\mu)}\sigma\Omega_3\right]$, $\phi_1^{\tau_b} = \frac{\phi_1^*}{1+\tau_b}$ and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. We can infer from Proposition 1.15 that:

$$Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right] = \frac{\sigma_l^2\phi_1^{\tau_b 2}}{\sigma^2(1-\phi_1^{\tau_b 2})(1-\nu^2)}\left(\frac{1+\phi_1^{\tau_b}\nu}{1-\phi_1^{\tau_b}\nu}\right); \tag{4.21}$$

$$\rho\left[\frac{b_{t+1}^i}{Y_{t+1}}, \frac{b_t^i}{Y_t}\right] = \rho_1 = \frac{\phi_1^{\tau_b}+\nu}{1+\phi_1^{\tau_b}\nu}. \tag{4.22}$$

Proposition 4.3. *For an economy with mean reverting correlated productivity shocks, at either Solow or Romer steady states, the higher the inheritance tax, the lower the correlation of intergenerational bequest in the same dynasty, the higher the intergenerational mobility;*

furthermore, the cross sectional inequality of after-tax inherited wealth plus government transfer is lower the higher the inheritance tax.

Proof.

$$\begin{aligned}\frac{\partial\rho_1^{\tau_b}}{\partial\tau_b} &= \frac{\partial\rho_1^{\tau_b}}{\partial\phi_1^{\tau_b}}\frac{\partial\phi_1^{\tau_b}}{\partial\tau_b}\\ &= \underbrace{\frac{1-\nu^2}{(1+\phi_1^{\tau_b}\nu)^2}}_{>0}\times\underbrace{\frac{\phi_1^*(-1)}{(1+\tau_b)^2}}_{\leq 0}\leq 0.\end{aligned} \tag{4.23}$$

$$\begin{aligned}sign\frac{\partial CV\left[\frac{b_{t+1}^i}{Y_{t+1}}+\frac{g_{t+1}}{Y_{t+1}}\right]}{\partial\tau_b} &= sign\frac{\partial Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial\tau_b}\\ &= sign\underbrace{\frac{\partial Var\left[\frac{b_{t+1}^i}{Y_{t+1}}\right]}{\partial\phi_1^{\tau_b}}}_{\geq 0}\underbrace{\frac{\partial\phi_1^{\tau_b}}{\partial\tau_b}}_{\leq 0}\leq 0.\end{aligned} \tag{4.24}$$

■

Lineage bequest evolution with correlated shocks for period-2 cycles

Corresponding to equation (1.114) of Chapter 1,

$$\frac{1}{\Omega_5}\frac{b^i_{t+1}}{Y_{t+1}} - \left(\phi_1^{\tau_b 2} + \nu^2\right)\frac{1}{\Omega_5}\frac{b^i_{t-1}}{Y_{t-1}} - \left(-\phi_1^{\tau_b 2}\nu^2\right)\frac{1}{\Omega_5}\frac{b^i_{t-3}}{Y_{t-3}}$$
$$- \left(1+\phi_1^{\tau_b}\right)\left(\frac{\delta_4}{\Omega_5} - \frac{\nu^2\delta_4}{\Omega_5}\right) = \varepsilon^i_t + \left(\phi_1^{\tau_b} + \nu\right)\varepsilon^i_{t-1} + \phi_1^{\tau_b}\nu\varepsilon^i_{t-2} \quad (4.25)$$

where $\Omega_5 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma}$, $\delta_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)](1+\tau_b)\sigma}\left[1+\frac{\tau_b}{(1+\tau_b)}\frac{\beta\mu}{1+\beta(1+\mu)}\sigma\Omega_3\right]$, $\phi_1^{\tau_b} = \frac{\phi_1^*}{1+\tau_b}$ and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. We can infer from Proposition 1.20 that:

$$Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right] = \sigma_l^2\left(\frac{\phi_1^{\tau_b}}{\sigma}\right)^2\left[\frac{1}{(1-\nu^4)(1-\nu^2\phi_1^{\tau_b 2})(1-\phi_1^{\tau_b 4})}\right]$$
$$\times[1+\phi_1^{\tau_b 2} + \nu(2\phi_1^* + \phi_1^{\tau_b 3}) + \nu^2(1+2\phi_1^{\tau_b 2} + \phi_1^{*4})\ldots$$
$$+\nu^3(\phi_1^{\tau_b} + 2\phi_1^{\tau_b 3} - \phi_1^{\tau_b 5}) + \nu^4(\phi_1^{\tau_b 2} + \phi_1^{\tau_b 4}) + \nu^5(-\phi_1^{\tau_b 3})]. \quad (4.26)$$

Proposition 4.4. *For an economy with mean reverting correlated productivity shocks, over period-2 cycles, the cross sectional inequality of after-tax inherited wealth plus government transfer is lower the higher the inheritance tax.*

Proof.

$$sign\frac{\partial CV\left[\frac{b^i_{t+1}}{Y_{t+1}} + \frac{g_{t+1}}{Y_{t+1}}\right]}{\partial\tau_b} = sign\frac{\partial Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]}{\partial\tau_b}$$
$$= sign\underbrace{\frac{\partial Var\left[\frac{b^i_{t+1}}{Y_{t+1}}\right]}{\partial\phi_1^{\tau_b}}}_{\geq 0}\underbrace{\frac{\partial\phi_1^{\tau_b}}{\partial\tau_b}}_{\leq 0} \leq 0. \quad (4.27)$$

■

However, with correlated shocks and period-2 cycles, the impact of taxes on lineage mobility is less clear. From equations (A.34) and (A.39) in Chapter 1, we can infer that with inheritance tax:

$$\gamma(1) = \sigma_l^2 \frac{\left(\phi_1^{\tau_b} + \nu\right)}{(1-\nu^2)\,(1-\phi_1^{\tau_b 2})(1-\nu\phi_1^{\tau_b})}; \tag{4.28}$$

$$\begin{aligned}\gamma(0) = \sigma_l^2 &\left[\frac{1}{(1-\nu^4)(1-\nu^2\phi_1^{\tau_b 2})(1-\phi_1^{\tau_b 4})}\right] \\ &\times \Big[1+\phi_1^{\tau_b 2} + \nu(2\phi_1^{\tau_b} + \phi_1^{\tau_b 3}) + \nu^2(1+2\phi_1^{\tau_b 2} + \phi_1^{\tau_4})\dots \\ &+ \nu^3(\phi_1^{\tau_b} + 2\phi_1^{\tau_b 3} - \phi_1^{\tau_b 5}) + \nu^4(\phi_1^{\tau_b 2} + \phi_1^{\tau_b 4}) + \nu^5(-\phi_1^{\tau_b 3}).\end{aligned} \tag{4.29}$$

As with (1.134), if $0 < \nu < 1$,

$$\frac{\partial \rho_1^{\tau_b}}{\partial \phi_1^{\tau_b}} = \frac{1}{[\gamma(0)]^2}\left[\underbrace{\gamma(0)\frac{\partial\gamma(1)}{\partial\phi_1^{\tau_b}}}_{\geq 0} - \underbrace{\gamma(1)\frac{\partial\gamma(0)}{\partial\phi_1^{\tau_b}}}_{\geq 0}\right] \gtreqless 0, \tag{4.30}$$

$$\Rightarrow \frac{\partial \rho_1^{\tau_b}}{\partial \tau_b} = \underbrace{\frac{\partial \rho_1^{\tau_b}}{\partial \phi_1^{\tau_b}}}_{\gtreqless 0}\underbrace{\frac{\partial \phi_1^{\tau_b}}{\partial \tau_b}}_{\leq 0} \gtreqless 0. \tag{4.31}$$

In summary, with logarithmic preference and "joy of giving", redistribution via estate taxes reduces cross-sectional inequality in the presence of both i.i.d. and mean-reverting correlated productivity shocks, for both Solow and Romer steady states and period-2 growth cycles. Under neoclassical framework, Wan and Zhu (2012) show this result for the case of i.i.d. shocks, while Bossmann et al. (2007) derive the result for both i.i.d. shocks and mean-reverting correlated shocks. In addition, redistribution via estate taxes can also increase intergenerational mobility under i.i.d. shocks for both Solow and Romer steady states and period-2 growth cycles, hence extending the coverage of the mobility result from Bossmann et al. (2007). Wan and Zhu (2012) do not discuss intergenerationl lineage mobility as they relax the finite variance assumption of the labor productivity shocks, which consequently restricts their results to cross-sectional intra-generational inequality comparisons.

4.3 The impact of innovation subsidies

4.3.1 Motivation for exploring changes in innovation subsidies

Data sourced from OECD (2011, pp. 29, Figure 3)[1] and National Science Foundation, National Center for Science and Engineering Statistics (2013, Table 2, in 2005 constant USD)[2] suggest a trend-growth in Research and Development expenditure from both the public and private sectors. The empirics are complemented by theoretical developments of R&D-based growth models in the last 2 decades, led by Romer (1990) and Aghion and Howitt (1992). In these models, innovations incur a fixed cost. To promote growth and reduce aggregate economic fluctuations arising from cycles, the role of innovation subsidies in Matsuyama (1999) has been studied by Aloi and Lasselle (2007) and Li and Zhang (2014). Aloi and Lasselle (2007) explore a lump-sum fixed cost subsidies with lumpsum tax on the young, without bequest or envy motive, by relax-

[1] OECD (2011).
Retrieved from: http://www.oecd.org/els/soc/49499779.pdf, on November 9, 2014.

[2] National Science Foundation, National Center for Science and Engineering Statistics (2013).
Retrieved from: http://www.nsf.gov/statistics/nsf14304/, on July 1, 2014.

ing current period capital constraint[3]. On the other hand, Li and Zhang (2014) consider a proportional innovation cost and intermediate goods' price subsidies with consumption tax, without bequest or envy motive, while maintaining the current period capital constraint, respecting the "time to build" spirit. In this chapter, I consider imposing a lump-sum fixed cost subsidies, with relaxation of the current period capital constraint as in Aloi and Lasselle (2007), but with the added bequest motive to study how innovation subsidies can have distributional effects. The advantage of this approach is the existence of an exit strategy with the subsidies as the economy approaches the Romer steady state. However, unlike Li and Zhang (2014), this approach has a disadvantage as it does not address the static inefficiency of the monopoly pricing of newly innovated intermediate goods. To avoid other distortionary distributional effect, I shall fund this innovation subsidy by a uniform output or value added tax, instead of a lump-sum tax on the young as in Aloi and Lasselle (2007). This helps to isolate effect of fiscal policy from bequest and envy (for extensions of Chapter 2) motives, and to maintain tractability for distribution analysis across regimes.

[3]By relaxing the capital constraint on innovation using the current tax revenue directly in Aloi and Lasselle (2007), effectively, it leads to a qualified recovery of the Romer (1990) model, where the final goods are used for intermediate production and innovation, resulting in no transitional dynamics and cycles. This is a deviation from Matsuyama (1999), where accumulated capital instead of final goods is used for the production of intermediate goods and innovation, providing the channel for growth through cycles.

4.3.2 Impact of innovation subsidies

Agent's optimization

Suppose the government now levies a uniform tax at a flat rate τ_t on both labor income and capital income (akin to a value-added tax instead of a lumpsum tax on young as in Aloi and Lasselle (2007)), to promote innovations under an otherwise Solow regime over period-2 cycles. The first period budget constraint for individual from lineage i at time t is:

$$c_t^i + s_t^i = b_t^i + (1-\tau_t)w_t^i \equiv y_t^i. \tag{4.32}$$

The second period budget constraint changes to:

$$(1-\tau_{t+1})R_{t+1}s_t^i = d_{t+1}^i + b_{t+1}^i. \tag{4.33}$$

Substituting the new budget constraints into the household optimization, and working through as with Appendix C.1, we get the optimal choices of an agent of lineage i:

$$b_{t+1}^i = \frac{\beta\mu(1-\tau_{t+1})}{1+\beta(1+\mu)}R_{t+1}y_t^i; \tag{4.34}$$

$$d_{t+1}^i = \frac{(1-\tau_{t+1})}{1+\beta(1+\mu)}R_{t+1}y_t^i; \tag{4.35}$$

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}y_t^i; \tag{4.36}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)}y_t^i. \tag{4.37}$$

Production with fixed cost subsidies as suggested by Aloi and Lasselle (2007)

The production sector is based on Matsuyama (1999). The production function of the final goods is unchanged from Chapter 1:

$$Y_t = \hat{A}(L)^{\frac{1}{\sigma}} \left\{ \int_0^{N_t} [x_t(z)]^{1-\frac{1}{\sigma}} dz \right\}, \tag{4.38}$$

where $\hat{A}$ is the total factor productivity; $x_t(z)$ denotes the intermediate input of variety z in period t; $\sigma \in (1, \infty)$ is the direct partial elasticity of substitution between each pair of intermediate goods; and $[0, N_t]$ is the range of intermediate available at period t. Since the final goods sector is perfectly competitive, we can derive the factor prices by their marginal products.

$$p_t^c = \left(1 - \frac{1}{\sigma}\right) \hat{A}(L)^{\frac{1}{\sigma}} (x_t^c)^{-\frac{1}{\sigma}}; \tag{4.39}$$

$$p_t^m = \left(1 - \frac{1}{\sigma}\right) \hat{A}(L)^{\frac{1}{\sigma}} (x_t^m)^{-\frac{1}{\sigma}}; \tag{4.40}$$

$$w_t = \frac{1}{\sigma} \left(\frac{Y_t}{L}\right). \tag{4.41}$$

From equations (4.39) and (4.40), the relative demand for "old" and "new" intermediates can be expressed as:

$$\frac{x_t^c}{x_t^m} = \left(\frac{p_t^c}{p_t^m}\right)^{-\sigma} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma}. \tag{4.42}$$

Innovation of new intermediates is driven by the existence of 1-period monopoly profits, specifically:

$$\pi_t = p_t^m x_t^m - R_t(a x_t^m + F + T_{t-1}), \tag{4.43}$$

where $T_{t-1} < 0$ is a lumpsum innovation subsidy set by the government at the end of period $t-1$. With one-period monopoly power and free entry, the demand for each intermediate input are:

$$x_t^c = \frac{1}{a}\theta\sigma F\left(1+\frac{T_{t-1}}{F}\right); \tag{4.44}$$

$$x_t^m = \frac{1}{a}(\sigma-1)F\left(1+\frac{T_{t-1}}{F}\right). \tag{4.45}$$

The resource constraint on capital in period t is:

$$K_{t-1} = N_{t-1}ax_t^c + (N_t - N_{t-1})(ax_t^m + F + T_{t-1}). \tag{4.46}$$

From the relative prices of intermediate inputs (4.42), demand for the intermediates (4.44) and (4.45), and the resource constraint on capital (4.46), working through as in Chapter 1, the dynamics for innovation is:

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \max\left\{0, \theta\left(\frac{k_{t-1}}{1+\frac{T_{t-1}}{F}} - 1\right)\right\}; \tag{4.47}$$

where $k_t \equiv \frac{K_t}{\theta\sigma F N_t}$, $\theta \equiv \left[1-\frac{1}{\sigma}\right]^{1-\sigma}$, $\theta \in [1,e]$, $e = 2.71828...$, and θ is increasing with σ. The critical point for profitable innovation is $k_{cr} \equiv 1+\frac{T_{t-1}}{F}$.

Equation (4.38) can be written as:

$$Y_t = \hat{A}(L)^{\frac{1}{\sigma}}\left[N_{t-1}(x_t^c)^{1-\frac{1}{\sigma}} + (N_t - N_{t-1})(x_t^m)^{1-\frac{1}{\sigma}}\right]. \tag{4.48}$$

Substituting the dynamics of innovation (4.47) into (4.48), the total output is:

$$Y_t = \begin{cases} \widetilde{A}_{t-1}\left[1+\frac{T_{t-1}}{F}\right]^{\frac{1}{\sigma}}[k_{t-1}]^{-\frac{1}{\sigma}}K_{t-1} & \text{if} \quad k_{t-1} \leq 1+\frac{T_{t-1}}{F} \equiv k_{cr} \\ \widetilde{A}_{t-1}K_{t-1} & \text{if} \quad k_{t-1} \geq k_{cr} \end{cases} \tag{4.49}$$

where $\widetilde{A}_{t-1} \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{\frac{1}{\sigma}}\left[1+\frac{T_{t-1}}{F}\right]^{-\frac{1}{\sigma}}$ and $k_{cr} \equiv 1+\frac{T_{t-1}}{F}$.

The government's budget constraint

I assume that the policy makers follow the simple stabilization principle as suggested by Aloi and Lasselle (2007):

$$T_{t-1} = \kappa(k^{**} - k_{t-1}) \text{ if } k_{t-1} \leq k^{**}, \tag{4.50}$$

where $\kappa \leq 0$, whose magnitude represents the size of government intervention, and k^{**} represents the Romer steady state. This intervention will promote the innovation of new intermediate products, to bring the aggregate economy towards the Romer steady state in the long-run. The government runs a balanced budget at all times, with:

$$\tau_t Y_t \equiv \tau_t \left(Lw_t + R_t K_{t-1}\right) = -(N_t - N_{t-1})T_{t-1}. \tag{4.51}$$

General equilibrium

From aggregate savings equation by integrating individual agent's savings (4.36), budget balanced (4.51), together with market clearing conditions - (1.24), (1.25), (1.26) and (1.27) :

$$\begin{aligned} S_t = K_t &= \Omega_1 \Omega_3 Y_t - \Omega_1 \Omega_3 \tau_t Y_t \\ &= \Omega_1 \Omega_3 Y_t - \Omega_1 \Omega_3 [-(N_t - N_{t-1})T_{t-1}], \end{aligned} \tag{4.52}$$

where $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$ and $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$. Substituting (4.47) and (4.49) into (4.52), the law of motion of capital is:

$$K_t = \Omega_1\Omega_3\widetilde{A}_{t-1} max\left[\left(1+\frac{T_{t-1}}{F}\right)^{\frac{1}{\sigma}} k_{t-1}^{\frac{1}{\sigma}}, 1\right] K_{t-1} + \Omega_1\Omega_3 T_{t-1} max\left[0, \theta N_{t-1}\left(\frac{k_{t-1}}{1+\frac{T_{t-1}}{F}} - 1\right)\right], \quad (4.53)$$

$$K_t = \begin{cases} \Omega_1\Omega_3\widetilde{A}_{t-1}\left[1+\frac{T_{t-1}}{F}\right]^{\frac{1}{\sigma}}[k_{t-1}]^{-\frac{1}{\sigma}} K_{t-1} & \text{if} \quad k_{t-1} \leq 1+\frac{T_{t-1}}{F} \equiv k_{cr} \\ \Omega_1\Omega_3\widetilde{A}_{t-1}K_{t-1} & \text{if} \quad k_{t-1} \geq k_{cr}, \end{cases} \quad (4.54)$$

where $\widetilde{A}_{t-1} \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{\frac{1}{\sigma}}\left[1+\frac{T_{t-1}}{F}\right]^{-\frac{1}{\sigma}}$ and $k_{cr} \equiv 1+\frac{T_{t-1}}{F}$. Dividing both sides by $\theta\sigma F N_t$, the aggregate dynamics of the system can be characterized by the equilibrium path for k_t:

$$k_t = \begin{cases} G\left(k_{t-1}\right)^{1-\frac{1}{\sigma}} & \text{if} \quad k_{t-1} \leq k_{cr}, \\ \frac{G\left(1+\frac{T_{t-1}}{F}\right)^{-\frac{1}{\sigma}} k_{t-1}}{1+\theta\left(\frac{k_{t-1}}{1+\frac{T_{t-1}}{F}}-1\right)} + \frac{\Omega_1\Omega_3 T_{t-1}}{\theta\sigma F}\left(1-\frac{1}{1+\theta\left(\frac{k_{t-1}}{1+\frac{T_{t-1}}{F}}-1\right)}\right) & \text{if} \quad k_{t-1} \geq k_{cr}, \end{cases} \quad (4.55)$$

where $G \equiv \Omega_1\Omega_3 A$, $\Omega_1 \equiv \frac{\beta(1+\mu)}{1+\beta(1+\mu)}$, $\Omega_3 \equiv \frac{1+\sigma\mu}{\sigma(1+\mu)}$, and $A \equiv \frac{\hat{A}}{a}(\frac{aL}{\theta\sigma F})^{\frac{1}{\sigma}}$.

From agent's optimal bequest (4.34), budget balanced (4.51), together with market clearing conditions - (1.24), (1.25), (1.26) and (1.27), the law of motion of bequest for a lineage i is:

$$\begin{aligned} b^i_{t+1} &= \frac{(1-\tau_{t+1})R_{t+1}\beta\mu}{[1+\beta(1+\mu)]}\left[y^i_t\right] \\ &= \frac{(1-\tau_{t+1})R_{t+1}\beta\mu}{[1+\beta(1+\mu)]}\left[b^i_t+(1-\tau_t)l^i_t w_t\right] \\ &= \frac{(1-\tau_{t+1})R_{t+1}\beta\mu}{[1+\beta(1+\mu)]}\left[b^i_t+(1-\tau_t)l^i_t \frac{1}{\sigma}Y_t\right]. \end{aligned} \tag{4.56}$$

Normalizing by the final output,

$$\begin{aligned} \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\tau_{t+1})R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]}\frac{b^i_t}{Y_t} - \frac{(1-\tau_{t+1})(1-\tau_t)R_{t+1}\beta\mu}{G_{t+1}\sigma[1+\beta(1+\mu)]} \\ = \frac{(1-\tau_{t+1})(1-\tau_t)R_{t+1}\beta\mu}{G_{t+1}\sigma[1+\beta(1+\mu)]}[l^i_t-\bar{l}_t], \end{aligned} \tag{4.57}$$

$$\begin{aligned} \Rightarrow \frac{b^i_{t+1}}{Y_{t+1}} - \frac{(1-\tau_{t+1})(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{b^i_t}{Y_t} - \frac{(1-\tau_{t+1})(1-\tau_t)(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3\sigma(1+\tau_b)[1+\beta(1+\mu)]} \\ = \frac{(1-\tau_{t+1})(1-\tau_t)(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma}[l^i_t-\bar{l}_t]. \end{aligned} \tag{4.58}$$

Proposition 4.5. *From period-2 cycles without innovation subsidies, by shifting to the Romer steady state, using the policy rule* (4.50)*:*

(i) lineage mobility is unchanged when comparing the Romer steady state to the onset, as the mobility only increases during transition;

(ii) inequality of inherited wealth is unchanged if shocks are i.i.d., but if shocks are correlated, cross-sectional inequality of the ratio of bequests to total output under period-2 cycles is greater (less) that under steady states if the correlation of mean reverting productivity shocks, $\nu \geq (<) [\phi_1^{*3}+\nu^2(\phi_1^*+\phi_1^{*5})+\nu^4(\phi_1^{*3})]$.[4]

[4]See Aloi and Lasselle (2007) for a discussion on the aggregate welfare treatment. Here, the transitional impact on mobility and distributional impact on cross-sectional inequality are highlighted.

Proof. First, by inspecting the AR(1) coefficient of (4.58), the lineage mobility when transitioning to the Romer steady state is higher if the innovation subsidy is funded by a uniform output tax. Second, from chapter 1, Propositions 1.10 and 1.21, a cross-sectional wealth inequality comparison can be made, in the presence of innovation subsidy policy (4.50) that stablizes period-2 cycles by moving the economy towards the Romer steady state. ■

In summary, pro-growth stabilization and welfare innovation subsidies as suggested by Aloi and Lasselle (2007) do not have long-run impact on mobility or inequality if the labor productivity shocks are i.i.d. in nature. In this chapter, by funding this fixed cost innovation subsidies with a value-added tax (instead of a lumpsum tax on young as in Aloi and Lasselle (2007)) , this policy has an added advantage of enhancing the lineage mobility by reducing the persistence (autocorrelation coefficient) of the law of motion of bequest during the transition to the Romer steady state. However, if the labor productivity shocks are correlated, there is a long-run impact on cross-sectional inequality as the economy moves from a period-2 cycle to the Romer steady state, and the impact depends on the relative magnitude of the propagation via the autoregressive coefficient of the law of motion of bequest and the magnitude of the labor productivity correlation.

4.4 Conclusion - impact of fiscal policies on mobility and inequality

In this chapter, an increase in inheritance tax decreases cross-sectional inequality, under i.i.d. and correlated shocks for both growth regimes and period-2 cycles, strengthening the results from Bossmann et al. (2007). Furthermore, an increase in inheritance tax also increases intergenerational lineage mobility under i.i.d. shocks. In addition, the use of innovation subsidy policy rule proposed by Aloi and Lasselle (2007), which stabilizes period 2 cycles, could level the playing field, by decreasing the persistence of inherited wealth during the transition to the Romer steady state, if funded by a uniform value-added tax. However, when stabilising the peiod-2 cycles with innovation subsidy, there can be a long-run impact on the cross-sectional wealth inequality if the labor productivity shocks are correlated.

It is also clear we need a combination of fiscal policies to meet the multi-dimensional goals of growth, mobility and inequality, and consumption externality. The potential tradeoffs between mobility, cross sectional inequality and growth are highlighted by this OLG model. Mobility-inequality tradeoffs are often hidden in infinitely lived or 2-way altruistic models, where policies that improve cross-sectional inequality distributions tend to imply that all agents in the economy will benefit. When agents have a finite life, the timing, duration, and nature of fiscal policy matters.

Finally, one should also bear in mind the political economy aspects of fiscal policies. There exist distinct differences in the normative recommendations depending on one's philosophical stance, as illustrated by the following two quotes from Rawls (2009) and Hayek (1976) respectively:

"...no one knows his place in society, his class position or social status; nor does he know his fortune in the distribution of natural assets and abilities, his intelligence and strength, and the like." Rawls (2009, pp. 118);

"For in such a system in which each is allowed to use his knowledge for his own purposes the concept of 'social justice' is necessarily empty and meaningless, because in it nobody's will can determine the relative incomes of the different people, or prevent that they be partly dependent on accident. 'Social justice' can be given a meaning only in a directed or 'command' economy (such as an army) in which the individuals are ordered what to do; and any particular conception of 'social justice' could be realized only in such a centrally directed system. It presupposes that people are guided by specific directions and not by rules of just individual conduct. Indeed, no system of rules of just individual conduct, and therefore no free action of the individuals, could produce results satisfying any principle of distributive justice." Hayek (1976, pp. 330).

Appendix A

Proofs of chapter 1

A.1 Solutions to agents' optimization

The problem of agent i born at time t is:

$$\underset{c_t^i,\, d_{t+1}^i}{Max}\; U_t\left(c_t^i, d_{t+1,}^i b_{t+1,}^i\right) \equiv \ln c_t^i + \beta\left[\ln d_{t+1}^i + \mu \ln b_{t+1}^i\right] \tag{A.1}$$

$$\text{subject to:} \qquad c_t^i + s_t^i = b_t^i + w_t^i \equiv y_t^i,$$

$$\text{and} \qquad R_{t+1} s_t^i = d_{t+1}^i + b_{t+1}^i.$$

At time $t+1$, an old agent maximizes his second period utility, defined as:

$$V \equiv \ln d_{t+1}^i + \mu \ln b_{t+1}^i, \tag{A.2}$$

by choosing his second period consumption and bequest for his immediate off-spring, given his second period budget constraint. The second period budget constraint for individual i is:

$$R_{t+1} s_t^i = d_{t+1}^i + b_{t+1}^i, \tag{A.3}$$

where R_{t+1} is the equilibrium gross rate of return on capital at time $t+1$. Substituting (A.3) into (A.2), and differentiating the latter with respect to d_{t+1}^i for the first order condition, the optimal choices for an old agent are:

$$d_{t+1}^{i} = \frac{1}{1+\mu}(R_{t+1}s_{t}^{i}); \tag{A.4}$$

$$b_{t+1}^{i} = \frac{\mu}{1+\mu}(R_{t+1}s_{t}^{i}). \tag{A.5}$$

By substituting these choices, rewrite the second period value function as:

$$V\left(R_{t+1}s_{t}^{i}\right) \equiv (1+\mu)\ln(R_{t+1}) + (1+\mu)\ln\left(s_{t}^{i}\right) + \ln\left(\frac{1}{1+\mu}\right) + \mu\ln\left(\frac{\mu}{1+\mu}\right). \tag{A.6}$$

At time t, subject to the first period budget constraint, the young agent chooses consumption and savings to maximize his utility:

$$\ln(c_{t}^{i}) + \beta V\left(R_{t+1}s_{t}^{i}\right). \tag{A.7}$$

The first period budget constraint for individual i is:

$$c_{t}^{i} + s_{t}^{i} = b_{t}^{i} + w_{t}^{i} \equiv y_{t}^{i}, \tag{A.8}$$

where $w_{t}^{i} = w_{t}l_{t}^{i}$; w_{t} is the equilibrium wage rate for each unit of labor productivity; s_{t}^{i} is the amount saved when young; and y_{t}^{i} is the lifetime resource of an individual i for agent i, born at time t. Substituting (A.8) and (A.6) into (A.7), and differentiating the latter with respect to s_{t}^{i} for the first order condition:

$$\frac{1}{b_{t}^{i} + w_{t}l_{t}^{i} - s_{t}^{i}} = \frac{\beta(1+\mu)}{s_{t}^{i}}, \tag{A.9}$$

which implies that the optimal savings for agent i as a function of his lifetime resource is:

$$s_{t}^{i} = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}y_{t}^{i}. \tag{A.10}$$

Substituting (A.10) into (A.4), (A.5), and (A.8), the optimal choices of an agent i are:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} y_t^i; \tag{A.11}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)} y_t^i; \tag{A.12}$$

$$d_{t+1}^i = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} y_t^i; \tag{A.13}$$

$$b_{t+1}^i = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} y_t^i; \tag{A.14}$$

and the optimal choices of the average agent are:

$$\bar{s}_t = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} \bar{y}_t \equiv \Omega_1 \bar{y}_t; \tag{A.15}$$

$$\bar{c}_t = \frac{1}{1+\beta(1+\mu)} \bar{y}_t; \tag{A.16}$$

$$\bar{d}_{t+1} = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} \bar{y}_t; \tag{A.17}$$

$$\bar{b}_{t+1} = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)} \bar{y}_t \equiv R_{t+1}\Omega_2 \bar{y}_t. \tag{A.18}$$

A.2 Derivation of law of motion for period-2 cycle with correlated shocks

Let $l_t^i = \bar{l}_t + \nu(l_{t-1}^i - \bar{l}_{t-1}) + \varepsilon_t^i$, as with Davies and Kuhn (1991), Bossmann et al. (2007) and Wan and Zhu (2012), where $\bar{l}_{t1} = \bar{l}_{t-1} = 1$, $0 < \nu < 1$,and $\varepsilon_t^i \sim$ i.i.d. $(0, \sigma_l^2)$.

$$\left(l_t^i - \bar{l}_t\right) = \nu(l_{t-1}^i - \bar{l}_{t-1}) + \varepsilon_t^i. \tag{A.19}$$

From equation (1.41),

$$\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^* \frac{b_t^i}{Y_t} - \frac{\delta_1}{\Omega_4} = [l_t^i - \bar{l}_t], \tag{A.20}$$

where $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}\frac{1}{\sigma} = \delta_1$, and $\phi_1^* = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$. From the mean reverting shocks,

$$\left(l_t^i - \bar{l}_t\right) - \nu(l_{t-1}^i - \bar{l}_{t-1}) = \varepsilon_t^i, \text{and} \quad (A.21)$$

$$\nu\left(l_{t-1}^i - \bar{l}_{t-1}\right) - \nu^2(l_{t-2}^i - \bar{l}_{t-2}) = \nu\varepsilon_{t-1}^i, \quad (A.22)$$

$$\left(l_t^i - \bar{l}_t\right) - \nu(l_{t-1}^i - \bar{l}_{t-1}) + \nu\left(l_{t-1}^i - \bar{l}_{t-1}\right) - \nu^2(l_{t-2}^i - \bar{l}_{t-2}) = \varepsilon_t^i + \nu\varepsilon_{t-1}^i. \quad (A.23)$$

Substituting (A.20) into (A.23), we get:

$$\left(\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b_t^i}{Y_t} - \frac{\delta_1}{\Omega_4}\right) - \nu^2\left(\frac{1}{\Omega_4}\frac{b_{t-1}^i}{Y_{t-1}} - \frac{1}{\Omega_4}\phi_1^*\frac{b_{t-2}^i}{Y_{t-2}} - \frac{\delta_1}{\Omega_4}\right) = \varepsilon_t^i + \nu\varepsilon_{t-1}^i$$

$$\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{\nu^2}{\Omega_4}\frac{b_{t-1}^i}{Y_{t-1}} = \phi_1^*\left(\frac{1}{\Omega_4}\frac{b_t^i}{Y_t} - \frac{\nu^2}{\Omega_4}\frac{b_{t-2}^i}{Y_{t-2}}\right) + \frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4} + \varepsilon_t^i + \nu\varepsilon_{t-1}^i \quad (A.24)$$

By iterative substitution, we get an ARMA(4,2) process for each of the fixed point in period 2 cycle:

$$\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \frac{\nu^2}{\Omega_4}\frac{b_{t-1}^i}{Y_{t-1}} - \phi_1^*\left[\phi_1^*\left(\frac{1}{\Omega_4}\frac{b_{t-1}^i}{Y_{t-1}} - \frac{\nu^2}{\Omega_4}\frac{b_{t-3}^i}{Y_{t-3}}\right) + \frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4} + \varepsilon_{t-1}^i + \nu\varepsilon_{t-2}^i\right] - \frac{\delta_1}{\Omega_4} + \frac{\nu^2\delta_1}{\Omega_4} = \varepsilon_t^i + \nu\varepsilon_{t-1}^i \quad (A.25)$$

$$\frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - \left(\phi_1^{*2} + \nu^2\right)\frac{1}{\Omega_4}\frac{b_{t-1}^i}{Y_{t-1}} - \left(-\phi_1^{*2}\nu^2\right)\frac{1}{\Omega_4}\frac{b_{t-3}^i}{Y_{t-3}} - (1+\phi_1^*)\left(\frac{\delta_1}{\Omega_4} - \frac{\nu^2\delta_1}{\Omega_4}\right) = \varepsilon_t^i + (\phi_1^* + \nu)\,\varepsilon_{t-1}^i + \phi_1^*\nu\varepsilon_{t-2}^i \quad (A.26)$$

Define $z_{t+1}^{cmi} = \frac{1}{\Omega_4}\frac{b_{t+1}^i}{Y_{t+1}} - (1+\phi_1^*)(1-\nu^2)\left(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_2^{cm}-\phi_4^{cm}}\right)$. Lineage dynamics of period-2 cycles with mean reverting productivity shocks from (A.26) can be expressed as follows:

$$z_{t+1}^{cmi} - \phi_2^{cm} z_{t-1}^{cmi} - \phi_4^{cm} z_{t-3}^{cmi} = \left[l_t^i - \bar{l}_t\right] - \nu^2 \left[l_{t-2}^i - \bar{l}_{t-2}\right]$$

$$= \varepsilon_t^i + \left(\phi_1^* + \nu\right) \varepsilon_{t-1}^i + \phi_1^* \nu \varepsilon_{t-2}^i \quad \text{(A.27)}$$

where $\phi_2^{cm} = \left(\phi_1^{*2} + \nu^2\right)$, $\phi_4^{cm} = \left(-\phi_1^{*2}\nu^2\right)$, $\delta_z^{cm} = \left(1 + \phi_1^*\right)\left(1 - \nu^2\right)\left(\frac{\frac{1}{\Omega_4}\delta_1}{1-\phi_2^{cm}-\phi_4^{cm}}\right)$, $\phi_1^* = \frac{R_{t+1}\beta\mu}{G_{t+1}[1+\beta(1+\mu)]} = \frac{(1-\frac{1}{\sigma})\beta\mu}{\Omega_1\Omega_3[1+\beta(1+\mu)]}$, $\Omega_4 = \frac{(1-\frac{1}{\sigma})\beta\mu}{\sigma\Omega_1\Omega_3[1+\beta(1+\mu)]} = \delta_1$, and $\varepsilon_t^i, \varepsilon_{t-1}^i, \varepsilon_{t-2}^i$ $\sim$ i.i.d.$(0, \sigma_l^2)$.

If the economy is at Romer regime at time $t+1$ and $t-1$, and Solow regime at time t and $t-2$, for each of the fixed point, the asymptotic cross-sectional distribution can be derived from the lineage bequest evolution.

A.3 Addendum to proof of proposition 1.20

With covariance-stationarity, we can compute the covariances and correlations of the lineage bequest evolution. To simplify notation, superscripts are dropped, and the time index on the moving average innovations are rescaled, by defining $u_t^i \equiv \varepsilon_{t-1}^i$.

$$z_t = \phi_2 z_{t-2} + \phi_4 z_{t-4} + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}. \quad \text{(A.28)}$$

$$\gamma(0) = E[z_t z_t] = \phi_2 E[z_t z_{t-2}] + \phi_4 E[z_t z_{t-4}] + E[z_t u_t] + \theta_1 E[z_t u_{t-1}] + \theta_2 E[z_t u_{t-2}]$$

$$= \phi_2\gamma(2) + \phi_4\gamma(4) + \sigma_l^2$$

$$+ \theta_1 E[(\phi_2 z_{t-2} + \phi_4 z_{t-4} + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})u_{t-1}]$$

$$+ \theta_2 E[(\phi_2 z_{t-2} + \phi_4 z_{t-4} + u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2})u_{t-2}]$$

$$\gamma(0) = \phi_2\gamma(2) + \phi_4\gamma(4) + \sigma_l^2 + \theta_1^2\sigma_l^2 + \theta_2^2\sigma_l^2. \tag{A.29}$$

$$\begin{aligned}
\gamma(1) = E[z_{t-1}z_t] = \phi_2 E[z_{t-1}z_{t-2}] + \phi_4 E[z_{t-1}z_{t-4}] + E[z_{t-1}u_t] \\
+ \theta_1 E[z_{t-1}u_{t-1}] + \theta_2 E[z_{t-1}u_{t-2}] \\
= \phi_2\gamma(1) + \phi_4\gamma(3) \\
+ E[(\phi_2 z_{t-3} + \phi_4 z_{t-5} + u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3})u_t] \\
+ \theta_1 E[(\phi_2 z_{t-3} + \phi_4 z_{t-5} + u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3})u_{t-1}] \\
+ \theta_2 E[(\phi_2 z_{t-3} + \phi_4 z_{t-5} + u_{t-1} + \theta_1 u_{t-2} + \theta_2 u_{t-3})u_{t-2}]
\end{aligned}$$

$$\gamma(1) = \phi_2\gamma(1) + \phi_4\gamma(3) + 0 + \theta_1\sigma_l^2 + \theta_1\theta_2\sigma_l^2. \tag{A.30}$$

$$\begin{aligned}
\gamma(2) = E[z_{t-2}z_t] = \phi_2 E[z_{t-2}z_{t-2}] + \phi_4 E[z_{t-2}z_{t-4}] + E[z_{t-2}u_t] \\
+ \theta_1 E[z_{t-2}u_{t-1}] + \theta_2 E[z_{t-2}u_{t-2}] \\
= \phi_2\gamma(0) + \phi_4\gamma(2) \\
+ E[(\phi_2 z_{t-4} + \phi_4 z_{t-4} + u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4})u_t] \\
+ \theta_1 E[(\phi_2 z_{t-4} + \phi_4 z_{t-6} + u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4})u_{t-1}] \\
+ \theta_2 E[(\phi_2 z_{t-4} + \phi_4 z_{t-6} + u_{t-2} + \theta_1 u_{t-3} + \theta_2 u_{t-4})u_{t-2}]
\end{aligned}$$

$$\gamma(2) = \phi_2\gamma(0) + \phi_4\gamma(2) + 0 + 0 + \theta_2\sigma_l^2. \tag{A.31}$$

$$\gamma(3) = E[z_{t-3}z_t] = \phi_2 E[z_{t-3}z_{t-2}] + \phi_4 E[z_{t-3}z_{t-4}] + E[z_{t-3}u_t]$$

$$+ \theta_1 E[z_{t-3}u_{t-1}] + \theta_2 E[z_{t-3}u_{t-2}]$$

$$= \phi_2\gamma(1) + \phi_4\gamma(1)$$

$$+ E[(\phi_2 z_{t-5} + \phi_4 z_{t-7} + u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5})u_t]$$

$$+ \theta_1 E[(\phi_2 z_{t-5} + \phi_4 z_{t-7} + u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5})u_{t-1}]$$

$$+ \theta_2 E[(\phi_2 z_{t-5} + \phi_4 z_{t-7} + u_{t-3} + \theta_1 u_{t-4} + \theta_2 u_{t-5})u_{t-2}]$$

$$\gamma(3) = \phi_2\gamma(1) + \phi_4\gamma(1) + 0 + 0 + 0. \tag{A.32}$$

$$\gamma(4) = E[z_{t-4}z_t] = \phi_2 E[z_{t-4}z_{t-2}] + \phi_4 E[z_{t-4}z_{t-4}] + E[z_{t-4}u_t]$$

$$+ \theta_1 E[z_{t-4}u_{t-1}] + \theta_2 E[z_{t-4}u_{t-2}]$$

$$= \phi_2\gamma(2) + \phi_4\gamma(0)$$

$$+ E[(\phi_2 z_{t-6} + \phi_4 z_{t-8} + u_{t-4} + \theta_1 u_{t-5} + \theta_2 u_{t-6})u_t]$$

$$+ \theta_1 E[(\phi_2 z_{t-6} + \phi_4 z_{t-8} + u_{t-4} + \theta_1 u_{t-5} + \theta_2 u_{t-6})u_{t-1}]$$

$$+ \theta_2 E[(\phi_2 z_{t-6} + \phi_4 z_{t-8} + u_{t-4} + \theta_1 u_{t-5} + \theta_2 u_{t-6})u_{t-2}]$$

$$\gamma(4) = \phi_2\gamma(2) + \phi_4\gamma(0) + 0 + 0 + 0. \tag{A.33}$$

Substituting $\gamma(3)$ from (A.32) into $\gamma(1)$ from (A.30), we get:

$$\begin{aligned}
\gamma(1) &= \phi_2\gamma(1) + \phi_4\left[\phi_2\gamma(1) + \phi_4\gamma(1)\right] + \theta_1\sigma_l^2 + \theta_1\theta_2\sigma_l^2 \\
&= \sigma_l^2 \frac{\theta_1 + \theta_1\theta_2}{1 - \phi_2 - \phi_2\phi_4 - \phi_4^2} = \sigma_l^2 \frac{\theta_1 + \theta_1\theta_2}{(1 - \phi_2 - \phi_4)(1 + \phi_4)} \\
&= \sigma_l^2 \frac{(\phi_1^* + \nu)(1 + \phi_1^*\nu)}{[1 - \left(\phi_1^{*2} + \nu^2\right) - \left(-\phi_1^{*2}\nu^2\right)](1 - \phi_1^{*2}\nu^2)} \\
&= \sigma_l^2 \frac{(\phi_1^* + \nu)}{(1 - \nu^2)(1 - \phi_1^{*2})(1 - \nu\phi_1^*)},
\end{aligned} \tag{A.34}$$

$$\gamma(3) = (\phi_2 + \phi_4)\gamma(1) = \sigma_l^2 \frac{(\phi_1^* + \nu)(\phi_1^{*2} + \nu^2 - \phi_1^{*2}\nu^2)}{(1-\nu^2)(1-\nu\phi_1^*)(1-\phi_1^{*2})}. \tag{A.35}$$

From (A.31) and (A.33), we get:

$$\gamma(2) = \frac{\phi_2\gamma(0) + \theta_2\sigma_l^2}{1-\phi_4}, \tag{A.36}$$

$$\gamma(4) = \frac{\phi_2^2\gamma(0) + \phi_2\theta_2\sigma_l^2}{1-\phi_4} + \phi_4\gamma(0). \tag{A.37}$$

Substituting (A.36) and (A.37) into (A.29), we get:

$$\gamma(0) = \frac{\phi_2^2\gamma(0) + \phi_2\theta_2\sigma_l^2}{1-\phi_4} + \frac{\phi_4\phi_2^2\gamma(0) + \phi_4\phi_2\theta_2\sigma_l^2}{1-\phi_4} + \phi_4^2\gamma(0) + \sigma_l^2 + \theta_1^2\sigma_l^2 + \theta_2^2\sigma_l^2,$$

$$\begin{aligned}\Rightarrow (1-\phi_4)\gamma(0) &= \phi_2^2\gamma(0) + \phi_4\phi_2^2\gamma(0) + (1-\phi_4)\phi_4^2\gamma(0)\\ &\quad + \phi_2\theta_2\sigma_l^2 + \phi_4\phi_2\theta_2\sigma_l^2 + (1-\phi_4)\left(\sigma_l^2 + \theta_1^2\sigma_l^2 + \theta_2^2\sigma_l^2\right).\end{aligned}$$

$$\begin{aligned}\gamma(0) &= \sigma_l^2 \frac{\phi_2\theta_2(1+\phi_4) + (1-\phi_4)\left(1+\theta_1^2+\theta_2^2\right)}{(1-\phi_4) - \phi_2^2(1+\phi_4) - (1-\phi_4)\phi_4^2}\\ &= \sigma_l^2 \frac{\phi_2\theta_2(1+\phi_4) + (1-\phi_4)\left(1+\theta_1^2+\theta_2^2\right)}{(1+\phi_4)(1-\phi_4-\phi_2)(1-\phi_4+\phi_2)}\\ &= \sigma_l^2 \frac{\left(1+\theta_1^2+\theta_2^2+\phi_2\theta_2\right) - \phi_4\left(1+\theta_1^2+\theta_2^2-\phi_2\theta_2\right)}{(1+\phi_4)(1-\phi_4-\phi_2)(1-\phi_4+\phi_2)}.\end{aligned} \tag{A.38}$$

$$\begin{aligned}\gamma(0) = \sigma_l^2 &\left[\frac{1}{(1-\nu^4)(1-\nu^2\phi_1^{*2})(1-\phi_1^{*4})}\right]\\ &\times [1 + \phi_1^{*2} + \nu(2\phi_1^* + \phi_1^{*3}) + \nu^2(1 + 2\phi_1^{*2} + \phi_1^{*4})\\ &+ \nu^3(\phi_1^* + 2\phi_1^{*3} - \phi_1^{*5}) + \nu^4(\phi_1^{*2} + \phi_1^{*4}) + \nu^5(-\phi_1^{*3})].\end{aligned} \tag{A.39}$$

$$\gamma(k) = E[z_{t-k}z_t] = \phi_2 E[z_{t-k}z_{t-2}] + \phi_4 E[z_{t-k}z_{t-4}] + E[z_{t-k}u_t]$$

$$+ \theta_1 E[z_{t-k}u_{t-1}] + \theta_2 E[z_{t-k}u_{t-2}] \quad \forall k \geq 3$$

$$= \phi_2\gamma(k-2) + \phi_4\gamma(k-4) = \left(\phi_1^{*2} + \nu^2\right)\gamma(k-2) + \left(-\phi_1^{*2}\nu^2\right)\gamma(k-4) \quad \forall k \geq 3. \tag{A.40}$$

From the covariances, we can obtain the autocorrelations, hence the lineage mobility, using $\rho_k = \frac{\gamma(k)}{\gamma(0)}$ $\forall k$.

Appendix B

Proofs of chapter 2

B.1 Solutions to agents' optimization

As proved in Alvarez-Cuadrado and Long (2012). The optimization behavior of the average agent, as well as agent of lineage i, are restated here, with some changes to accommodate the continuum of agents of measure 1, for completeness.

At time $t+1$, an old agent maximizes his second period utility, defined as:

$$V \equiv \ln(d^i_{t+1} - \xi \gamma \bar{d}_{t+1}) + \mu \ln(b^i_{t+1}), \tag{B.1}$$

by choosing his second period consumption and bequest for his immediate offspring, given his second period budget constraint. The second period budget constraint for individual i is:

$$R_{t+1} s^i_t = d^i_{t+1} + b^i_{t+1}, \tag{B.2}$$

where R_{t+1} is the equilibrium gross rate of return on capital at time $t+1$.

Substituting (B.2) into (B.1), and differentiating the latter with respect to

d_{t+1}^i for the first order condition, the optimal choices for an old agent are:

$$d_{t+1}^i = \frac{1}{1+\mu}(R_{t+1}s_t^i + \mu\xi\gamma\bar{d}_{t+1}); \tag{B.3}$$

$$b_{t+1}^i = \frac{\mu}{1+\mu}(R_{t+1}s_t^i - \xi\gamma\bar{d}_{t+1}). \tag{B.4}$$

By substituting these choices, rewrite the second period value function as:

$$V\left(R_{t+1}s_t^i\right) \equiv (1+\mu)\ln\left(R_{t+1}s_t^i - \xi\gamma\bar{d}_{t+1}\right) + \ln\left(\frac{1}{1+\mu}\right) + \mu\ln\left(\frac{\mu}{1+\mu}\right). \tag{B.5}$$

At time t, subject to the first period budget constraint, the young agent chooses consumption and savings to maximize his utility:

$$\ln(c_t^i - \gamma\bar{c}_t) + \beta V\left(R_{t+1}s_t^i\right). \tag{B.6}$$

The first period budget constraint for individual i is:

$$c_t^i + s_t^i = b_t^i + w_t^i \equiv y_t^i, \tag{B.7}$$

where $w_t^i = w_t l_t^i$; w_t is the equilibrium wage rate for each unit of labor productivity; s_t^i is the amount saved when young; and y_t^i is the lifetime resource of an individual i for agent i, born at time t.

Substituting (B.7) into (B.6), and differentiating the latter with respect to c_t^i for the first order condition:

$$\frac{1}{c_t^i - \gamma\bar{c}_t} = \frac{\beta R_{t+1}(1+\mu)}{R_{t+1}s_t^i - \xi\gamma\bar{d}_{t+1}}, \tag{B.8}$$

which implies the optimal choices for a young agent are:

$$c_t^i = \frac{1}{\beta R_{t+1}(1+\mu)}\left(R_{t+1}s_t^i - \xi\gamma\bar{d}_{t+1}\right) + \gamma\bar{c}_t; \tag{B.9}$$

$$s_t^i = \frac{\beta(1+\mu)\left(y_t^i - \gamma\bar{c}_t\right) + \xi\gamma\frac{\bar{d}_{t+1}}{R_t+1}}{[1+\beta(1+\mu)]}. \tag{B.10}$$

B.1.1 Optimal behavior of the average agent

For the average agent, his lifetime budget constraint is given by:

$$\bar{y}_t = \int_0^1 b_t^i\, di + \int_0^1 w_t^i\, di = \bar{b}_t + \bar{w}_t. \tag{B.11}$$

From the second period budget constraint (B.2), and the optimal choices of an old agent i, given by (B.3) and (B.4), the optimal second period choices for the average agent are:

$$\bar{d}_{t+1} = \frac{1}{1+\mu(1-\xi\gamma)} R_{t+1}\bar{s}_t; \tag{B.12}$$

$$\bar{b}_{t+1} = \frac{\mu(1-\xi\gamma)}{1+\mu(1-\xi\gamma)} R_{t+1}\bar{s}_t. \tag{B.13}$$

Substituting the optimal savings of a young agent i into the first order condition (B.8), the consumption when young of the average agent is given by:

$$\bar{c}_t = \left(\frac{1-\xi\gamma}{1-\gamma}\right)\frac{1}{\beta[1+\mu(1-\xi\gamma)]}\bar{s}_t. \tag{B.14}$$

To obtain the optimal choices of the average agent as a function of his lifetime resources, we use the second period budget constraint (B.2), and the optimal choices of an old agent, (B.3) and (B.4). Thus, the optimal choices of the average agent are:

$$\bar{s}_t = \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t \equiv \Omega_1\bar{y}_t; \tag{B.15}$$

$$\bar{c}_t = \frac{(1-\xi\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\bar{y}_t; \tag{B.16}$$

$$\overline{d}_{t+1} = \frac{R_{t+1}\beta(1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\overline{y}_t; \tag{B.17}$$

$$\overline{b}_{t+1} = \frac{R_{t+1}\beta\mu(1-\gamma)(1-\xi\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}\overline{y}_t \equiv R_{t+1}\Omega_2\overline{y}_t. \tag{B.18}$$

B.1.2 Optimal behavior of an agent from lineage i

With the optimal choices of the average agent as the reference, using (B.10), for the $i-$th agent of the same generation,

$$s_t^i[1+\beta(1+\mu)] = \beta(1+\mu)y_t^i - \frac{(1+\mu)(1-\xi\gamma)\gamma-\xi\gamma(1-\gamma)}{(1-\xi\gamma)}\beta\overline{c}_t. \tag{B.19}$$

Substituting (B.16) into (B.19), the optimal saving decision is:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}\left[y_t^i - \frac{(1+\mu)(1-\xi\gamma)\gamma-\xi\gamma(1-\gamma)}{(1+\mu)\{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}}\overline{y}_t\right]. \tag{B.20}$$

From (B.3), (B.4), (B.8), (B.16) and (B.17), the optimal choices of an agent i are:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)}\left[y_t^i - \phi_s\overline{y}_t\right]; \tag{B.21}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)}\left[y_t^i + \phi_c\overline{y}_t\right]; \tag{B.22}$$

$$d_{t+1}^i = \frac{R_{t+1}\beta}{1+\beta(1+\mu)}\left[y_t^i + \phi_d\overline{y}_t\right]; \tag{B.23}$$

$$b_{t+1}^i = \frac{R_{t+1}\beta\mu}{1+\beta(1+\mu)}\left[y_t^i - \phi_b\overline{y}_t\right]; \tag{B.24}$$

where

$$\phi_s = \frac{(1+\mu)(1-\xi\gamma)\gamma-\xi\gamma(1-\gamma)}{(1+\mu)\{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}}; \tag{B.25}$$

$$\phi_c = \frac{\beta[(1+\mu)(1-\xi\gamma)\gamma - \xi\gamma(1-\gamma)]}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}; \tag{B.26}$$

$$\phi_d = \frac{\xi\gamma\mu(1-\gamma)\beta + \xi\gamma(1-\gamma) - (1-\xi\gamma)\gamma}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}; \tag{B.27}$$

$$\phi_b = \frac{(1-\xi\gamma)\gamma + \xi\gamma(1-\gamma)\beta}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}. \tag{B.28}$$

B.2 Addendum to proof of propositions 2.1 & 2.2

Since

$$\widetilde{\Omega}_1 \equiv \frac{\beta[1+\mu(1-\xi\gamma)](1-\gamma)}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]},$$

$$\frac{\partial\widetilde{\Omega}_1}{\partial\gamma} = \overbrace{\frac{\beta}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]}}^{\geq 0} \times \left\{\frac{(1-\gamma)[1+\mu(1-\xi\gamma)][\xi+\beta\xi(1-\gamma)\mu+\beta[1+\mu(1-\xi\gamma)]}{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]} - \xi(1-\gamma)\mu - [1+\mu(1-\xi\gamma)]\right\}. \tag{B.29}$$

If

$$RHS \geq LHS, \text{ then } \frac{\partial\widetilde{\Omega}_1}{\partial\gamma} \leq 0,$$

where:

$$LHS \equiv (1-\gamma)[1+\mu(1-\xi\gamma)][\xi+\beta\xi(1-\gamma)\mu+\beta[1+\mu(1-\xi\gamma)] \tag{B.30}$$

$$RHS \equiv \xi(1-\gamma)\mu + [1+\mu(1-\xi\gamma)] \times \{(1-\xi\gamma)+(1-\gamma)\beta[1+\mu(1-\xi\gamma)]\}. \tag{B.31}$$

Solving,

$$\begin{aligned} RHS - LHS &= 1 + \mu + \xi^2\gamma^2\mu - \xi - 2\xi\gamma\mu \\ &= 1 + \mu(1 + \xi^2\gamma^2 - 2\xi\gamma) - \xi \\ &= \underbrace{1 - \xi}_{\geq 0} + \underbrace{\mu(1 - \xi\gamma)^2}_{\geq 0} \\ &\geq 0 \Rightarrow \frac{\partial \widetilde{\Omega}_1}{\partial \gamma} \leq 0. \end{aligned} \tag{B.32}$$

$$\frac{\partial \widetilde{\Omega}_1}{\partial \xi} = \underbrace{\frac{\beta(1-\gamma)}{(1-\xi\gamma) + (1-\gamma)\beta[1+\mu(1-\xi\gamma)]}}_{\geq 0} \underbrace{\left\{ \frac{[1+\mu(1-\xi\gamma)][\gamma + \beta(1-\gamma)\mu\gamma]}{(1-\xi\gamma) + (1-\gamma)\beta[1+\mu(1-\xi\gamma)]} - \gamma\mu \right\}}_{\geq 0}.$$

Since,

$$\begin{aligned} &[1+\mu(1-\xi\gamma)][\gamma+\beta(1-\gamma)\mu\gamma] - \gamma\mu\{(1-\xi\gamma) + (1-\gamma)\beta[1+\mu(1-\xi\gamma)\} \\ &= \gamma + \beta(1-\gamma)\gamma\mu + \gamma\mu(1-\xi\gamma) + \beta(1-\gamma)(1-\xi\gamma)\gamma\mu^2 \dots \\ &-\gamma\mu + \gamma^2\mu\xi - \beta(1-\gamma)\gamma\mu - \beta(1-\gamma)(1-\xi\gamma)\gamma\mu^2 \\ &= \gamma \geq 0 \Rightarrow \frac{\partial \widetilde{\Omega}_1}{\partial \xi} \leq 0. \end{aligned} \tag{B.33}$$

Recall,

$$\widetilde{\Omega}_3 \equiv \frac{1+\sigma\mu(1-\xi\gamma)}{\sigma[1+\mu(1-\xi\gamma)]}, \tag{B.34}$$

$$\begin{aligned} \frac{\partial \widetilde{\Omega}_3}{\partial \gamma} &= \frac{-\sigma\mu\xi}{\sigma[1+\mu(1-\xi\gamma)]} + [1+\sigma\mu(1-\xi\gamma)]\frac{\sigma\mu\xi}{\sigma[1+\mu(1-\xi\gamma)]^2} \\ &= \frac{\sigma\mu\xi}{\sigma[1+\mu(1-\xi\gamma)]}\left[\frac{-\sigma - \sigma\mu + \sigma\mu\xi\gamma + 1 + \sigma\mu - \sigma\mu\xi\gamma)}{\sigma[1+\mu(1-\xi\gamma)]}\right] \\ &= \frac{\overbrace{\sigma\mu\xi}^{\geq 0}\overbrace{(1-\sigma)}^{\leq 0}}{\underbrace{\sigma[1+\mu(1-\xi\gamma)]^2}_{\geq 0}} \leq 0 \text{ since } \sigma \geq 1. \end{aligned} \tag{B.35}$$

By symmetry,

$$\frac{\partial \widetilde{\Omega}_3}{\partial \xi} = \frac{\overbrace{\sigma\mu\gamma}^{\geq 0}\overbrace{(1-\sigma)}^{\leq 0}}{\underbrace{\sigma[1+\mu(1-\xi\gamma)]^2}_{\geq 0}} \leq 0 \text{ since } \sigma \geq 1. \tag{B.36}$$

Appendix C

Proofs of chapter 4

C.1 Solutions to agents' optimization with inheritance tax

The government's budget constraint

Suppose the government now levies an inheritance tax at a flat rate τ_b on all inheritance, to fund lumpsum transfers to the young, as a new redistributive policy. All young agents receives the same lumpsum subsidy, g_t. The government runs a balanced budget at all times.

$$g_t = \tau_b \int_0^1 b_t^i \, di = \tau_b \bar{b}_t. \tag{C.1}$$

Agent's optimization

The problem of agent i born at time t is:

$$\underset{c_t^i,\, d_{t+1}^i}{Max} \; U_t \left(c_t^i, d_{t+1,}^i b_{t+1,}^i\right) \equiv \ln c_t^i + \beta \left[\ln d_{t+1}^i + \mu \ln b_{t+1}^i\right] \tag{C.2}$$

$$\text{subject to:} \qquad c_t^i + s_t^i = b_t^i + w_t^i + g_t \equiv y_t^i,$$

$$\text{and} \qquad R_{t+1} s_t^i = d_{t+1}^i + (1+\tau_b) b_{t+1}^i.$$

At time $t+1$, an old agent maximizes his second period utility, defined as:

$$V \equiv \ln d_{t+1}^{i} + \mu \ln b_{t+1}^{i}, \tag{C.3}$$

by choosing his second period consumption and bequest for his immediate offspring, given his second period budget constraint. The second period budget constraint for individual i is:

$$R_{t+1} s_t^i = d_{t+1}^i + (1+\tau_b) b_{t+1}^i, \tag{C.4}$$

where R_{t+1} is the equilibrium gross rate of return on capital at time $t+1$. Substituting (C.4) into (C.3), and differentiating the latter with respect to b_{t+1}^i for the first order condition, the optimal choices for an old agent are:

$$d_{t+1}^i = \frac{1}{1+\mu}(R_{t+1} s_t^i); \tag{C.5}$$

$$b_{t+1}^i = \frac{\mu}{(1+\mu)(1+\tau_b)}(R_{t+1} s_t^i). \tag{C.6}$$

By substituting these choices, rewrite the second period value function as:

$$V\left(R_{t+1} s_t^i\right) \equiv (1+\mu)\ln(R_{t+1}) + (1+\mu)\ln\left(s_t^i\right) + \ln\left(\frac{1}{1+\mu}\right) + \mu \ln\left(\frac{\mu}{(1+\mu)(1+\tau_b)}\right). \tag{C.7}$$

At time t, subject to the first period budget constraint, the young agent chooses consumption and savings to maximize his utility:

$$\ln(c_t^i) + \beta V\left(R_{t+1} s_t^i\right). \tag{C.8}$$

The first period budget constraint for individual i is:

$$c_t^i + s_t^i = b_t^i + w_t^i + g_t \equiv y_t^i, \tag{C.9}$$

where $w_t^i = w_t l_t^i$; w_t is the equilibrium wage rate for each unit of labor productivity; s_t^i is the amount saved when young; g_t is the flat amount of redistributed tax from the government; and y_t^i is the lifetime resource of an individual i for agent i, born at time t. Substituting (C.9) and (C.7) into (C.8), and differentiating the latter with respect to s_t^i for the first order condition:

$$\frac{1}{b_t^i + w_t l_t^i + g_t - s_t^i} = \frac{\beta\,(1+\mu)}{s_t^i}, \tag{C.10}$$

which implies that the optimal savings for agent i as a function of his lifetime resource is:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} y_t^i. \tag{C.11}$$

Substituting (C.11) into (C.5), (C.6), and (C.9), the optimal choices of an agent i are:

$$s_t^i = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} y_t^i; \tag{C.12}$$

$$c_t^i = \frac{1}{1+\beta(1+\mu)} y_t^i; \tag{C.13}$$

$$d_{t+1}^i = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} y_t^i; \tag{C.14}$$

$$b_{t+1}^i = \frac{R_{t+1}\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)} y_t^i; \tag{C.15}$$

and the optimal choices of the average agent are:

$$\bar{s}_t = \frac{\beta(1+\mu)}{1+\beta(1+\mu)} \bar{y}_t \equiv \Omega_1 \bar{y}_t; \tag{C.16}$$

$$\bar{c}_t = \frac{1}{1+\beta(1+\mu)} \bar{y}_t; \tag{C.17}$$

$$\bar{d}_{t+1} = \frac{R_{t+1}\beta}{1+\beta(1+\mu)} \bar{y}_t; \tag{C.18}$$

$$\bar{b}_{t+1} = \frac{R_{t+1}\beta\mu}{[1+\beta(1+\mu)](1+\tau_b)} \bar{y}_t \equiv R_{t+1}\Omega_2 \bar{y}_t. \tag{C.19}$$

Bibliography

Abel, A. B. (1990). Asset prices under habit formation and catching up with the joneses. *American Economic Review*, 80(2):38–42.

Acemoglu, D. and Autor, D. (2012). What does human capital do? a review of goldin and katz's the race between education and technology. *Journal of Economic Literature*, 50(2):426–463.

Acemoglu, D. and Robinson, J. A. (2001). A theory of political transitions. *American Economic Review*, 91(4):938–963.

Acemoglu, D. and Robinson, J. A. (2014). The rise and fall of general laws of capitalism. *Journal of Economic Perspectives, forthcoming*.

Aghion, P. and Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, 60:323–351.

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109(3):659–684.

Aloi, M. and Lasselle, L. (2007). Growth and welfare effects of stabilizing innovation cycles. *Journal of Macroeconomics*, 29(4):806–823.

Alonso-Carrera, J., Caballé, J., and Raurich, X. (2008). Estate taxes, consumption externalities, and altruism. *Journal of Public Economics*, 92(7):1751–1764.

Alpizar, F., Carlsson, F., and Johansson-Stenman, O. (2005). How much do we care about absolute versus relative income and consumption? *Journal of Economic Behavior & Organization*, 56(3):405–421.

Alvarez-Cuadrado, F. and Long, N. V. (2012). Envy and inequality. *Scandinavian Journal of Economics*, 114(3):949–973.

Andreoni, J. (1989). Giving with impure altruism: applications to charity and ricardian equivalence. *Journal of Political Economy*, 97(6):1447–1458.

Arrow, K. J. and Dasgupta, P. S. (2009). Conspicuous consumption, inconspicuous leisure. *Economic Journal*, 119(541):F497–F516.

Atkinson, A. B. (1980). Inheritance and the redistribution of wealth. *Public Policy and the Tax System, London: Allen and Urwin*.

Bakshi, G. S. and Chen, Z. (1996). The spirit of capitalism and stock-market prices. *American Economic Review*, 86(1):133–157.

Banerjee, A. V. and Duflo, E. (2003). Inequality and growth: what can the data say? *Journal of economic growth*, 8(3):267–299.

Bean, M. A. (2001). *Probability: the science of uncertainty with applications to* *investments, insurance, and engineering*, volume 6. American Mathematical Soc.

Becker, G. S., Murphy, K. M., and Werning, I. (2005). The equilibrium distribution of income and the market for status. *Journal of Political Economy*, 113(2):282–310.

Becker, G. S. and Tomes, N. (1979). An equilibrium theory of the distribution of income and intergenerational mobility. *Journal of Political Economy*, 87(6):1153–1189.

Benabou, R. (2000). Unequal societies: Income distribution and the social contract. *American Economic Review*, 90(1):96–129.

Benhabib, J., Bisin, A., and Zhu, S. (2011). The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica*, 79(1):123–157.

Benhabib, J., Bisin, A., and Zhu, S. (2014a). The distribution of wealth in the blanchard–yaari model. *Macroeconomic Dynamics*, pages 1–16.

Benhabib, J., Bisin, A., and Zhu, S. (2014b). The wealth distribution in bewley models with investment risk. Technical report, National Bureau of Economic Research.

Bental, B. and Peled, D. (1996). The accumulation of wealth and the cyclical generation of new technologies: A search theoretic approach. *International Economic Review*, 37(3):687–718.

Berg, A., Ostry, J. D., and Zettelmeyer, J. (2012). What makes growth sustained? *Journal of Development Economics*, 98(2):149–166.

Bossmann, M., Kleiber, C., and Wälde, K. (2007). Bequests, taxation and the distribution of wealth in a general equilibrium model. *Journal of Public Economics*, 91(7):1247–1271.

Bouzahzah, M., De la Croix, D., and Docquier, F. (2002). Policy reforms and growth in computable olg economies. *Journal of economic dynamics and control*, 26(12):2093–2113.

Brandt, A. (1986). The stochastic equation $y_{n+1} = a_n y_n + b_n$ with stationary coefficients. *Advances in Applied Probability*, 18(1):211–220.

Campbell, J. Y. and Cochrane, J. H. (1995). By force of habit: A consumption-based explanation of aggregate stock market behavior. Technical report, National Bureau of Economic Research.

Carroll, C. D., Overland, J., and Weil, D. N. (2000). Saving and growth with habit formation. *American Economic Review*, 90(3):341–355.

Carroll, C. D. and Weil, D. N. (1994). Saving and growth: a reinterpretation. In *Carnegie-Rochester Conference Series on Public Policy*, volume 40, pages 133–192. Elsevier.

Charles, K. K. and Hurst, E. (2002). The correlation of welath across generations. Technical report, National Bureau of Economic Research.

Charles, K. K., Hurst, E., and Roussanov, N. (2007). Conspicuous consumption and race. Technical report, National Bureau of Economic Research.

Conlisk, J. (1974). Can equalization of opportunity reduce social mobility? *American Economic Review*, 64(1):80–90.

Constantinides, G. M. (1990). Habit formation: A resolution of the equity premium puzzle. *Journal of political Economy*, 98(3):519–543.

Costinot, A., Vogel, J., and Wang, S. (2012). Global supply chains and wage inequality. *American Economic Review: Papers & Proceedings*, 102(3):396–401.

Costinot, A., Vogel, J., and Wang, S. (2013). An elementary theory of global supply chains. *Review of Economic Studies*, 80(1):109–144.

Davies, J. B. (1986). Does redistribution reduce inequality? *Journal of Labor Economics*, 4(4):538–559.

Davies, J. B. and Kuhn, P. J. (1991). A dynamic model of redistribution, inheritance, and inequality. *Canadian Journal of Economics*, 24(2):324–344.

Davies, J. B., Zhang, J., and Zeng, J. (2005). Intergenerational mobility under private vs. public education. *Scandinavian Journal of Economics*, 107(3):399–417.

De Botton, A. (2008). *Status anxiety*. Vintage.

De Nardi, M. (2004). Wealth inequality and intergenerational links. *Review of Economic Studies*, 71(3):743–768.

De Nardi, M. (2015). Quantitative models of wealth inequality: A survey. Technical report, National Bureau of Economic Research.

Di Tella, R. and MacCulloch, R. (2006). Some uses of happiness data in economics. *Journal of Economic Perspectives*, 20(1):25–46.

Dynan, K. E., Skinner, J., and Zeldes, S. P. (2004). Do the rich save more? *Journal of Political Economy*, 112(2):397–444.

Easterly, W. (2007). Inequality does cause underdevelopment: Insights from a new instrument. *Journal of Development Economics*, 84(2):755–776.

Francois, P. and Lloyd-Ellis, H. (2003). Animal spirits through creative destruction. *American Economic Review*, 93(3):530–550.

García-Peñalosa, C. and Turnovsky, S. J. (2007). Growth, income inequality, and fiscal policy: What are the relevant trade-offs? *Journal of Money, Credit and Banking*, 39(2-3):369–394.

García-Peñalosa, C. and Turnovsky, S. J. (2011). Taxation and income distribution dynamics in a neoclassical growth model. *Journal of Money, Credit and Banking*, 43(8):1543–1577.

Gardini, L., Sushko, I., and Naimzada, A. K. (2008). Growing through chaotic intervals. *Journal of Economic Theory*, 143(1):541–557.

Garicano, L. and Rossi-Hansberg, E. (2006). Organization and inequality in a knowledge economy. *Quarterly Journal of Economics*, 121(4):1383–1435.

Gershman, B. (2014). The two sides of envy. *Journal of Economic Growth*, 19:407–438.

Halvorsen, E., Thoresen, T. O., and Bø, E. E. (2012). Inheritance as a disincentive to labor effort. In *11th journées Louis-André Gérard-Varet*.

Hamilton, J. D. (1994). *Time series analysis*. Princeton university press Princeton.

Hayek, F. A. (1976). *The mirage of social justice*, volume 2. Routledge.

Holtz-Eakin, D., Joulfaian, D., and Rosen, H. S. (1993). The carnegie conjecture: Some empirical evidence. *Quarterly Journal of Economics*, 108:413–435.

Huggett, M. (1997). The one-sector growth model with idiosyncratic shocks: Steady states and dynamics. *Journal of monetary economics*, 39(3):385–403.

IMF Staff from Fiscal Affairs Department, supervised by Sanjeev Gupta and Michael Keen (2014). Fiscal policy and income inequality.

Jones, C. I. and Kim, J. (2014). A schumpeterian model of top income inequality. Technical report, National Bureau of Economic Research.

Joulfaian, D. and Wilhelm, M. O. (1994). Inheritance and labor supply. *Journal of Human Resources*, 29(4):1205–1234.

Jouvet, P.-A., Pestieau, P., and Ponthiere, G. (2010). Longevity and environmental quality in an olg model. *Journal of Economics*, 100(3):191–216.

Krueger, D., Perri, F., Pistaferri, L., and Violante, G. L. (2010). Cross-sectional facts for macroeconomists. *Review of Economic Dynamics*, 13(1):1–14.

Levy, F. and Temin, P. (2007). Inequality and institutions in 20th century america. Technical report, National Bureau of Economic Research.

Levy, M. (2003). Are rich people smarter? *Journal of Economic Theory*, 110(1):42–64.

Li, B. and Zhang, J. (2014). Subsidies in an economy with endogenous cycles over investment and innovation regimes. *Macroeconomic Dynamics*, 18:1351–1382.

Lindquist, M. J. (2004). Capital–skill complementarity and inequality over the business cycle. *Review of Economic Dynamics*, 7(3):519–540.

Ljungqvist, L. and Uhlig, H. (2000). Tax policy and aggregate demand management under catching up with the joneses. *American Economic Review*, 90(3):356–366.

Marcet, A., Obiols-Homs, F., and Weil, P. (2007). Incomplete markets, labor supply and capital accumulation. *Journal of Monetary Economics*, 54(8):2621–2635.

Matsuyama, K. (1999). Growing through cycles. *Econometrica*, 67(2):335–347.

Matsuyama, K. (2001). Growing through cycles in an infinitely lived agent economy. *Journal of Economic Theory*, 100(2):220–234.

Matsuyama, K. (2004). Financial market globalization, symmetry-breaking, and endogenous inequality of nations. *Econometrica*, 72(3):853–884.

Matsuyama, K. (2007). Credit traps and credit cycles. *American Economic Review*, 97(1):503–516.

Matsuyama, K. (2013). Endogenous ranking and equilibrium lorenz curve across (ex ante) identical countries. *Econometrica*, 81(5):2009–2031.

Mazumder, B. (2005). Fortunate sons: New estimates of intergenerational mobility in the united states using social security earnings data. *Review of Economics and Statistics*, 87(2):235–255.

Mitra, T. (2001). A sufficient condition for topological chaos with an application to a model of endogenous growth. *Journal of Economic Theory*, 96(1):133–152.

Moldovanu, B., Sela, A., and Shi, X. (2007). Contests for status. *Journal of Political Economy*, 115(2):338–363.

National Science Foundation, National Center for Science and Engineering Statistics (2013). National patterns of r&d resources: 2011-12 data update; detailed statistical tables nsf 14-304; available at http://www.nsf.gov/statistics/nsf14304/.

Neumark, D. and Postlewaite, A. (1998). Relative income concerns and the rise in married women's employment. *Journal of public Economics*, 70(1):157–183.

Nolan, B., Salverda, W., Checchi, D., Marx, I., McKnight, A., Tóth, I. G., and van de Werfhorst, H. G. (2014). *Changing Inequalities and Societal Impacts in Rich Countries: Thirty Countries' Experiences*. Oxford University Press.

OECD (2011). Divided we stand: Why inequality keeps rising.

Piketty, T. (2011). On the long run evolution of inheritance-france 1820-2050. *Quarterly Journal of Economics*, 126:1071–1131.

Piketty, T. (2014). *Capital in the Twenty-first Century*. Harvard University Press.

Piketty, T. and Zucman, G. (2014). Capital is back: Wealth-income ratios in rich countries, 1700-2010. *Quarterly Journal of Economics*, 129(3):1255–1310.

Quadrini, V. (2000). Entrepreneurship, saving, and social mobility. *Review of Economic Dynamics*, 3(1):1–40.

Rawls, J. (2009). *A theory of justice*. Harvard university press.

Rayo, L. (2013). Monopolistic signal provision. *BE Journal of Theoretical Economics*, 13(1):27–58.

Rivera-Batiz, L. A. and Romer, P. M. (1991). Economic integration and endogenous growth. *Quarterly Journal of Economics*, 106(2):531–555.

Romer, P. (1990). Endogenous technological change. *Journal of Political Economy*, 98:S71–S102.

Schneider, D. (2011). The labor share: A review of theory and evidence. *SFB 649 Economic Risk, Discussion Paper 2011*, 69.

Solon, G. (1992). Intergenerational income mobility in the united states. *American Economic Review*, 82(3):393–408.

Solon, G. (2002). Cross-country differences in intergenerational earnings mobility. *Journal of Economic Perspectives*, 16(3):59–66.

Solon, G. (2004). A model of intergenerational mobility variation over time and place. In Corak, M., editor, *Generational income mobility in North America and Europe*. Cambridge University Press, Cambridge, U.K.

Sornette, D. (2006). *Critical phenomena in natural sciences: chaos, fractals, selforganization and disorder: concepts and tools*. Springer Science & Business.

Stachurski, J. (2009). *Economic dynamics: theory and computation*. MIT Press.

Stuart, A. and Ord, J. K. (1994). *Kendall's advanced theory of statistics. Vol. I. Distribution theory*. Arnold, London.

Turnovsky, S. J. and Monteiro, G. (2007). Consumption externalities, production externalities, and efficient capital accumulation under time non-separable preferences. *European Economic Review*, 51(2):479–504.

Van de Stadt, H., Kapteyn, A., and Van de Geer, S. (1985). The relativity of utility: Evidence from panel data. *Review of Economics and Statistics*, 67(2):179–187.

Veblen, T. (1899). *The theory of the leisure class*. New York: The New American Library.

Wälde, K. (2005). Endogenous growth cycles. *International Economic Review*, 46(3):867–894.

Wan, J. and Zhu, S. (2012). Bequests, estate taxes, and wealth distributions. Technical report, Mimeo, NUS.

Wei, W. W.-S. (2006). *Time series analysis (2nd Ed.)*. Pearson International publ.

Weiss, Y. and Fershtman, C. (1998). Social status and economic performance::: A survey. *European Economic Review*, 42(3-5):801–820.

Wolff, E. N. (1992). Changing inequality of wealth. *American Economic Review*, 82(2):552–558.

World Bank (2013). The world bank group goals - end extreme poverty and promote shared prosperity.

Zhang, J. (2005). Income ranking and convergence with physical and human capital and income inequality. *Journal of Economic Dynamics and control*, 29(3):547–566.

Zhu, S. (2013a). Comparisons of stationary distributions of linear models. *Economics Letters*, 119(2):221–223.

Zhu, S. (2013b). Existence of equilibrium in an incomplete market model with endogenous labor supply. *Mimeo, NUS*.

Zhu, S. (2014). Bequest motives, estate taxes, and wealth distributions in becker-tomes models with investment risk. *Mimeo, NUS*.

Zimmerman, D. J. (1992). Regression toward mediocrity in economic stature. *American Economic Review*, 82(3):409–429.