Family of Bell-like Inequalities as Device-Independent Witnesses for Entanglement Depth

Yeong-Cherng Liang,[1,2,*] Denis Rosset,[3] Jean-Daniel Bancal,[4] Gilles Pütz,[3] Tomer Jack Barnea,[3] and Nicolas Gisin[3]
[1]Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland
[2]Department of Physics, National Cheng Kung University, Tainan 701, Taiwan
[3]Group of Applied Physics, University of Geneva, CH-1211 Geneva 4, Switzerland
[4]Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
(Received 8 December 2014; published 12 May 2015)

We present a simple family of Bell inequalities applicable to a scenario involving arbitrarily many parties, each of which performs two binary-outcome measurements. We show that these inequalities are members of the complete set of full-correlation Bell inequalities discovered by Werner-Wolf-Żukowski-Brukner. For scenarios involving a small number of parties, we further verify that these inequalities are facet defining for the convex set of Bell-local correlations. Moreover, we show that the amount of quantum violation of these inequalities naturally manifests the extent to which the underlying system is genuinely many-body entangled. In other words, our Bell inequalities, when supplemented with the appropriate quantum bounds, naturally serve as device-independent witnesses for entanglement depth, allowing one to certify genuine k-partite entanglement in an arbitrary $n \geq k$-partite scenario without relying on any assumption about the measurements being performed, or the dimension of the underlying physical system. A brief comparison is made between our witnesses and those based on some other Bell inequalities, as well as quantum Fisher information. A family of witnesses for genuine k-partite nonlocality applicable to an arbitrary $n \geq k$-partite scenario based on our Bell inequalities is also presented.

DOI: 10.1103/PhysRevLett.114.190401 PACS numbers: 03.65.Ud, 03.67.Mn

One of the most important no-go theorems in physics concerns the impossibility to reproduce all quantum mechanical predictions using *any* locally causal theory [1], a fact commonly referred to as Bell's theorem [2]. An important observation leading to this well-known result is that measurement statistics allowed by such theories must satisfy constraints in the form of an inequality, a "Bell inequality." Since these inequalities only involve experimentally accessible quantities, their violation—a manifestation of Bell nonlocality [3]—can be, and has been (modulo some arguably implausible loopholes [4]) empirically demonstrated (see, e.g., [3–5] and references therein).

Clearly, Bell inequalities played an instrumental role in the aforementioned discovery. Remarkably, they also find applications in numerous quantum information and communication tasks, e.g., in quantum key distribution involving untrusted devices [6–8], in the reduction of communication complexity [9], in the expansion of trusted random numbers [10,11], in certifying the Hilbert space dimension of physical systems [12,13], in self-testing [14–18] of quantum devices, and in witnessing [19–21] and quantifying [22–25] (multipartite) quantum entanglement using untrusted devices, etc. For a recent review on these and other applications, see [3].

Identifying interesting or useful Bell inequalities is nonetheless by no means obvious. For instance, the approach of solving for the complete set of *facet-defining* Bell inequalities for a given experimental scenario—though useful for the identification of non-Bell-local (hereafter nonlocal) correlations—typically produces a large number of inequalities with no apparent structure (see, however, [26] for some progress made on classifying Bell inequalities). In contrast, carefully constructed Bell inequalities, such as the *families* of two-party Bell inequalities considered in [27–30], have enabled us to conclude that certain correlations derived from maximally entangled states do not admit any *local content* [31], and that the prediction of quantum theory cannot be refined even when supplemented with hidden variables satisfying certain auxiliary assumptions [32].

Going beyond the bipartite scenario, the family of Mermin-Ardehali-Belinskiĭ-Klyshko (MABK) inequalities [33,34] is a prominent example of an interesting family of Bell inequalities, giving clear evidence that a macroscopic number of physical systems can still give rise to strongly nonclassical behavior (see also [35,36]). Moreover, a sufficiently strong violation of the n-partite MABK inequalities can also be used to certify the presence of genuine n-partite entanglement in a device-independent manner, i.e., without relying on any assumption about the measurement device or the Hilbert space dimension of the test systems (see, e.g., [19–21,37,38]).

What about the possibility of identifying genuine k-partite entanglement in an n-partite scenario with $n > k > 2$? This is the question of entanglement depth [39], or, equivalently, non-k producibility [40] (see also [41]), which both seek to identify the extent to which many-body entanglement is present in a multipartite quantum system. It is worth noting that genuine many-body entanglement is known to be essential, e.g., in achieving extreme spin squeezing [39] and high sensitivity in some general metrology tasks [42]. For well-calibrated or trusted [43] measurement devices, there exist few criteria [39–42,44–48] to certify such many-body entanglement. For example, an entanglement depth larger than 28 was recently demonstrated [48] using such a witness. However, the possibility of certifying—in a device-independent manner—genuine k-partite entanglement in an arbitrary $n \geq k > 2$-partite scenario has so far remained elusive.

Here, we show that such robust certification in a scenario involving arbitrarily many parties is indeed possible, e.g., by using a novel family of n-partite Bell inequalities as well as the characterized quantum violation of these inequalities by quantum states, assuming only k-partite entanglement. Moreover, we show that, together with the appropriate bounds, these inequalities can also be used to witness genuine k-partite nonlocality [49] in an arbitrary n-partite scenario (with $n \geq k$). Since genuine k-partite entanglement is a prerequisite for the presence of genuine k-partite quantum nonlocality [50,51], witnesses for such multipartite nonlocality are also witnesses for entanglement depth. Let us stress, however, that our family of device-independent witnesses for entanglement depth *does not* rely on the detection of such genuine multipartite nonlocality.

A novel family of n-partite Bell inequalities.—Consider a Bell-type experiment involving n spatially separated parties (labeled by $i \in \{1, 2, \ldots, n\}$), each of them performing two binary-outcome measurements. We denote the measurement setting of the ith party by $x_i \in \{0, 1\}$, and the corresponding measurement outcome by $a_i = \pm 1$. The correlation between these measurement outcomes can be summarized succinctly using the collection of joint conditional probability distributions $\{P(\vec{a}|\vec{x})\}$ where $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$ are n-component vectors describing, respectively, the combination of measurement outcomes and measurement settings. In terms of the n-partite full correlators $E_n(\vec{x}) = \sum_{a_1,a_2,\ldots,a_n} \prod_{i=1}^n a_i P(\vec{a}|\vec{x})$, our family of n-partite Bell inequalities $\mathcal{I}_n$ reads as

$$\mathcal{I}_n\colon\ \mathcal{S}_n = 2^{1-n}\left[\sum_{\vec{x}\in\{0,1\}^n} E_n(\vec{x})\right] - E_n(\vec{1}_n) \overset{\mathcal{L}}{\leq} 1, \qquad (1)$$

where $\vec{1}_n = (1, \ldots, 1)$ is an n-bit string of ones and $\mathcal{L}$ signifies that the inequality holds for a locally causal theory. For $n = 2$, inequality (1) is the Clauser-Horne-Shimony-Holt Bell inequality [52]; for $n = 3$, it is equivalent to the seventh tripartite inequality of [53,54]. For general n, we show [54] that $\mathcal{I}_n$ defines a facet [59] of the n-partite full-correlation polytope characterized by Werner-Wolf-Żukowski-Brukner [60,61], thus being a member of the 2^{2^n} Bell inequalities discovered therein. For $n \leq 8$, we further verify numerically that $\mathcal{I}_n$ corresponds to a facet of the polytope of locally causal correlations—a property which we conjecture to hold true for general n.

From [60], it thus follows that the maximal quantum violation of $\mathcal{I}_n$ (denoted by $\mathcal{S}_n^{\mathcal{Q},*}$) is attainable if each party measures the ± 1-outcome observables [60] $A_{x_i=0} = \cos\alpha\,\sigma_x + \sin\alpha\,\sigma_y$ and $A_{x_i=1} = \cos(\varphi_i + \alpha)\,\sigma_x + \sin(\varphi_i + \alpha)\,\sigma_y$ for some judiciously chosen α, $\varphi_i \in [0, 2\pi]$ on the n-partite Greenberger-Horne-Zeilinger (GHZ) state [62] $|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$. For $n \leq 8$, we certified using a converging hierarchy of semidefinite programs [63,64] that $\mathcal{S}_n^{\mathcal{Q},*}$ can be achieved by further setting $\varphi_1 = \varphi_2 = \cdots = \varphi_n = \phi_n$ and $\alpha = -[(n-1)/2n]\phi_n$ for some $\phi_n \in [0, \pi/2]$. Explicitly, this ansatz gives the quantum value

$$\mathcal{S}_n^{\mathcal{Q}}(\phi_n) = 2\cos^{n+1}\frac{\phi_n}{2} - \cos\left(\frac{n+1}{2}\phi_n\right), \qquad (2a)$$

where the explicit analytic values of ϕ_n (for $n \leq 7$) leading to $\mathcal{S}_n^{\mathcal{Q},*}$ can be found in [54] (see Table I for the corresponding value of $\mathcal{S}_n^{\mathcal{Q},*}$). For larger values of n, the above observation and further numerical evidence lead us to conjecture that

$$\mathcal{S}_n^{\mathcal{Q},*} = \max_{\phi_n} \mathcal{S}_n^{\mathcal{Q}}(\phi_n). \qquad (2b)$$

Indeed, for sufficiently large n, this maximum value over ϕ_n is well approximated by setting $\phi_n = 2\pi/n$, thus giving $\max_{\phi_n} \mathcal{S}_n^{\mathcal{Q}}(\phi_n) \overset{n\to\infty}{\to} 3$, i.e., the algebraic maximum of $\mathcal{S}_\infty$. [The algebraic maximum of $\mathcal{S}_n$ is the maximal value of $\mathcal{S}_n$ attainable by all legitimate conditional probability distributions. As $n \to \infty$, the quantum violation (2) is thus as strong as that allowed by, for instance, signaling correlations.]

Entanglement depth and k producibility.—To see how $\mathcal{I}_n$, or, more precisely, its quantum violation, can witness entanglement depth, let us now briefly recall the notion of k producibility [40]: An n-partite pure state $|\psi\rangle = \bigotimes_{j=1}^m |\varphi_j\rangle$ is said to be k producible if all of its constituent states $|\varphi_j\rangle$ are *at most* k-partite. Analogously, a mixed state ρ is said to be k producible if it can be written as a convex mixture of k-producible pure states; the set of k-producible quantum states is, thus, convex. Evidently, the production of a k-producible state only requires (up to) k-partite entanglement. In the following, we say that a quantum state has an

TABLE I. Summary of the maximal quantum violation and the critical visibility $v_{n,1}^{\mathrm{Ent}}$, i.e., the infimum of v_n in Eq. (4) before the mixture stops violating $\mathcal{I}_n$. Also included in the table is the algebraic maximum of $\mathcal{I}_n$, denoted by $\mathcal{S}_n^*$.

n	2	3	4	5	6	7	8	∞
$\mathcal{S}_n^{\mathcal{Q},*}$	$\sqrt{2}$	5/3	1.8428	1.9746	2.0777	2.1610	2.2299	3
$v_{n,1}^{\mathrm{Ent}}$	$1/\sqrt{2}$	3/5	0.5427	0.5064	0.4813	0.4627	0.4485	1/3
$\mathcal{S}_n^*$	2	5/2	2.7500	2.8750	2.9375	2.9688	2.9844	3

entanglement depth of k if it is k producible but not $(k-1)$ producible.

A family of device-independent witnesses for entanglement depth.—It is well known that the observed Bell-inequality violation of a quantum state ρ immediately implies that ρ is entangled [50], and, hence, has an entanglement depth of 2 or higher. Moreover, from the convexity of the set of k-producible quantum states, we see that—when there is no restriction on the Hilbert space dimension—the set of correlations due to k-producible quantum states is also convex. In particular, since k producibility implies k' producibility for all $k' \geq k$, one expects that quantum states having a larger entanglement depth may also lead to a stronger violation of any given n-partite Bell inequality (e.g., $\mathcal{I}_n$): This is the central intuition behind what we call device-independent witnesses for entanglement depth (DIWEDs), a violation of which implies some lower bound on the entanglement depth of the underlying state. To this end, let us denote by $\mathcal{S}^{\mathcal{Q},*}_{k\text{-pr.}}$ the maximal quantum violation of $\mathcal{I}_n$ attainable by n-partite quantum states having an entanglement depth of k. In general, one may expect $\mathcal{S}^{\mathcal{Q},*}_{k\text{-pr.}}$ to depend on both n and k, but the algebraic structure of $\mathcal{S}_n$, cf. Eq. (1), allows us to show otherwise.

Theorem 1: The maximal possible quantum violation of $\mathcal{I}_n$ by k-producible quantum states, $\mathcal{S}^{\mathcal{Q},*}_{k\text{-pr.}}$, is independent of n and equals to $\mathcal{S}^{\mathcal{Q},*}_{k}$, the maximal possible quantum violation of $\mathcal{I}_k$.

The full proof of the theorem is provided in [54]. Here, let us show that $\mathcal{S}^{\mathcal{Q},*}_{k\text{-pr.}} \geq \mathcal{S}^{\mathcal{Q},*}_{k}$. Consider n parties sharing the quantum state $|\mathrm{GHZ}_k\rangle \otimes |0\rangle^{\otimes n-k}$ with the first k parties performing the *optimal* local measurements leading to $\mathcal{S}^{\mathcal{Q},*}_{k}$ while the rest of the parties always measure the trivial observable $\mathbb{1}$. It then follows from Eq. (1) and Born's rule that the quantum value of $\mathcal{S}_n$ becomes $\mathcal{S}^{\mathcal{Q},*}_{k}$. Since this is only one particular choice of quantum strategy, we must have $\mathcal{S}^{\mathcal{Q},*}_{k\text{-pr.}} \geq \mathcal{S}^{\mathcal{Q},*}_{k}$. For instance, it is conceivable that with nontrivial local measurements on $|\mathrm{GHZ}_2\rangle^{\otimes 2}$, a stronger violation of $\mathcal{I}_4$ could be obtained. Theorem 1, however, dictates that this intuition is false. Indeed, the proof of the theorem [54] suggests that to achieve the strongest quantum violation of $\mathcal{I}_n$ by k-producible quantum states, we should employ the above strategy of generating optimal nonlocal correlation for only k of the parties, while leaving the rest of the $n-k$ parties with trivial correlations.

The above theorem, together with the respective values of $\mathcal{S}^{\mathcal{Q},*}_{k}$ [cf. Eq. (2) and Table I], then provides us with a family of DIWEDs,

$$\mathcal{I}^k_n\colon\ 2^{1-n}\sum_{\vec{x}\in\{0,1\}^n} E_n(\vec{x}) - E_n(\vec{1}_n) \overset{k\text{-producible states}}{\leq} \mathcal{S}^{\mathcal{Q},*}_{k}. \tag{3}$$

Since the upper bound $\mathcal{S}^{\mathcal{Q},*}_{k}$ holds for *all* n-partite, k-producible quantum states of arbitrary Hilbert space dimensions and arbitrary binary-outcome measurements performed by each party, the witness is device independent in the sense that any observed violation of $\mathcal{I}^k_n$ by ρ implies that ρ is at least genuinely $(k+1)$-partite entangled, i.e., it has an entanglement depth of at least $k+1$, regardless of the details of the measurement devices and the Hilbert space dimensions. For instance, a measured quantum value of $\mathcal{I}_n$ that is greater than $\sqrt{2}$ and $\frac{5}{3}$ (cf. Table I) immediately implies, respectively, the presence of genuine tripartite and quadripartite entanglement, regardless of the total number of parties n. For the noisy GHZ state

$$\rho(v_n) = v_n|\mathrm{GHZ}_n\rangle\langle\mathrm{GHZ}_n| + (1-v_n)\frac{\mathbb{1}_{2^n}}{2^n}, \tag{4}$$

where $\mathbb{1}_{2^n}$ is the identity operator acting on $\mathbb{C}^{2n}$, such quantum violations then translate to the critical visibility of $v_n > v^{\mathrm{Ent}}_{n,k} = \mathcal{S}^{\mathcal{Q},*}_{k}/\mathcal{S}^{\mathcal{Q},*}_{n}$ required for the device-independent certification of genuine $(k+1)$-partite entanglement via $\mathcal{I}^k_n$ (Table I).

Let us emphasize again that the certification of genuine $(k+1)$-partite entanglement via $\mathcal{I}^k_n$ does not rely on the detection of genuine $(k+1)$-partite nonlocality [49,51]. Indeed, as we show in [54], the witnesses for genuine multipartite nonlocality [49] corresponding to $\mathcal{I}_n$ read as

$$\mathcal{I}^{k,\mathrm{NL}}_n\colon\ 2^{1-n}\sum_{\vec{x}\in\{0,1\}^n} E_n(\vec{x}) - E_n(\vec{1}_n) \overset{\mathcal{NS}_{n,k}}{\leq} 3 - 2^{2-k}, \tag{5}$$

where $\mathcal{NS}_{n,k}$ signifies that the inequality holds for arbitrary n-partite correlations that are k producible [51] (when assuming only nonsignaling [65,66] resources within each group). Interestingly, as with quantum entanglement, the right-hand side of inequality (5) is simply the algebraic maximum of $\mathcal{I}_k$, which is achievable by a general k-partite nonsignaling correlation. [Since $\mathcal{S}_n$ only involves a linear combination of full correlators, inequality (3) also holds true even if we consider, instead, k-producible Svetlichny [67] (signaling) correlations; see [51].] For $n \leq 8$, the explicit values of these algebraic maxima ($\mathcal{S}^*_n = 3 - 2^{2-n}$) are clearly higher than the corresponding quantum bounds (see Table I). Thus, witnessing genuine k-partite entanglement via $\mathcal{I}^k_n$ does not rely on the detection of genuine k-partite nonlocality.

Comparison with some other witnesses for entanglement depth.—Given the intimate connection [40] between k producibility and m separability, one expects that DIWEDs can also be constructed from other multipartite Bell inequalities whose m-separability properties are well studied. (A pure state is m separable if it can be written as the tensor product of m constituent pure states. The definition for mixed states proceeds analogously. Thus, an m-separable state is also k producible for some $k \geq \lceil n/m \rceil$.) Indeed, investigations [34,68] on the MABK inequalities have culminated in the following characterization [37,69]: The maximal possible quantum violation of the n-partite MABK inequality by n-partite, m-separable states ($m < n$) consisting of L unentangled subsystems is [37,69] $2^{(n+L-2m+1)/2}$. Hence, for $n \leq 5$, the MABK inequalities give the following DIWEDs [54]:

$$\mathcal{M}_n^k\colon\ 2^{(1-n)/2} \sum_{\vec{x}\in\{0,1\}^n} \cos\left[\frac{\pi}{4}(1-n+2\mathbf{x})\right] \times E(\vec{x}) \overset{k\text{-producible states}}{\leq} 2^{(k-1)/2}, \tag{6}$$

where $\mathbf{x} = \sum_i x_i$ and we have made use of the compact representation of the MABK inequality obtained in [36]. Unfortunately, for $n \geq 6$, except for $k = 2$ and $k = n - 1$, the inequality given in Eq. (6) generally does not hold for k-producible states [54].

To compare the strength of $\mathcal{I}_n^k$ and $\mathcal{M}_n^k$ in witnessing entanglement depth, we numerically optimized the quantum violation of these witnesses for the GHZ state $|\mathrm{GHZ}_n\rangle$, the n-partite W state [70] $|W_n\rangle = (1/\sqrt{n})(|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle)$, as well as the n-partite one-dimensional cluster states [71] $|C_n^-\rangle = \prod_{i=1}^{n-1} CZ_{i,i+1}|+\rangle^{\otimes n}$ and $|C_n^o\rangle = CZ_{n,1}^{1-\delta_{n,2}}|C_n^-\rangle$ where $CZ_{i,j} = \mathrm{diag}(1,1,1,-1)$ is the controlled Z gate acting on the ith and the jth qubit, and $\delta_{i,j}$ is the Kronecker delta. A comparison between the best quantum violations found [54] and the respective bounds associated with the witnesses, cf. Eq. (3) and Eq. (6), then allows us to obtain a lower bound for the entanglement depth of these states (see Table II). [Note that all these states are known have an entanglement depth of n (see, e.g., [72,73]).] Interestingly, the entanglement depth of $|W_n\rangle$ is better witnessed by $\mathcal{M}_n^k$, whereas that of $|C_n^o\rangle$ is better witnessed by $\mathcal{I}_n^k$. Moreover, for $|C_n^o\rangle$, the lower bounds originating from both DIWEDs even outperform those obtained from the non-device-independent witnesses based on quantum Fisher information [42].

TABLE II. Lower bounds on entanglement depth certifiable by the violation of DIWEDs $\mathcal{I}_n^k$, $\mathcal{M}_n^k$ and quantum Fisher information (QFI) [74]. Integers in the top row give n (the number of parties), whereas all integers underneath are the respective lower bounds on entanglement depth for the quantum state given in the leftmost column, using the witness(es) indicated in the second column. A tight lower bound is marked with an asterisk (*).

$\|\psi\rangle$	Witness(es)	2	3	4	5	6	7
$\|\mathrm{GHZ}_n\rangle$	$\mathcal{I}_n^k$, $\mathcal{M}_n^k$, QFI	2*	3*	4*	5*	6*	7*
$\|W_n\rangle$	$\mathcal{I}_n^k$	2*	3*	2	2	2	2
$\|W_n\rangle$	$\mathcal{M}_n^k$, QFI	2*	3*	3	3	3	3
$\|C_n^-\rangle$	$\mathcal{I}_n^k$, $\mathcal{M}_n^k$	2*	3*	2	2	2	2
$\|C_n^-\rangle$	QFI	2*	3*	2	3	2	3
$\|C_n^o\rangle$	$\mathcal{I}_n^k$	2*	3*	2	2	2	2
$\|C_n^o\rangle$	$\mathcal{M}_n^k$	2*	3*	2	1	1	1
$\|C_n^o\rangle$	QFI	2*	3*	1	1	1	1

Discussion.—Obviously, for any given n and k, the results of Table II highlight the fact that the set of correlations arising from k-producible quantum states cannot be fully characterized by any single DIWED, as one would expect from the studies of conventional, non-device-independent entanglement witnesses. (In general, each of these sets can only be fully characterized by an infinite number of such linear witnesses [51].) Nonetheless, one may ask if there exists a better DIWED, e.g., one that witnesses the actual entanglement depth of $|C_n^o\rangle$. To answer this, or, more generally, the question of whether some observed correlation $\{P(\vec{a}|\vec{x})\}$ could have come from a k-producible quantum state, the hierarchy of semidefinite programs proposed in [24] turns out to be well suited. For completeness, we include the explicit form of these semidefinite programs in [54]. Using this technique, it was found in [75] that all the 23 306 quadripartite Bell-like inequalities obtained therein are also legitimate DIWEDs for an entanglement depth of 2, and some even for an entanglement depth of 3. Moreover, our numerical optimizations show that some of these inequalities can further be used to certify, device independently, the genuine 4-partite entanglement present in $|W_4\rangle$, $|C_4^-\rangle$, and $|C_4^o\rangle$. Is it then always possible to find an appropriate DIWED to certify the entanglement depth of *any* pure entangled quantum state? Given the strong connection between nonlocality and pure entangled states (see, e.g., [76–79]), we are optimistic that the answer to the above question is positive.

Let us now comment on some other possibilities for future work. Naturally, a question that stems from our results is the typicality of Bell inequalities that are naturally suited for witnessing entanglement depth, in the sense of Theorem 1. To this end, we show [54] that the family of DIWEDs given in Eq. (3) actually belongs to an even more general family of DIWEDs, $\mathcal{I}_n^k(\gamma)$, such that $\mathcal{I}_n^k(2)$ gives Eq. (3). The usefulness of this more general family of DIWEDs, however, remains to be investigated. Note, also, that apart from $\gamma = 2$, none of the Bell inequalities corresponding to $\mathcal{I}_n^k(\gamma)$ define a facet of the local polytope for general n. In contrast, as we show here, the combination of full correlators given by $\mathcal{S}_n$, cf. Eq. (1), are natural both in the characterization of the set of locally causal correlations as well as the set of correlations allowed by k-producible quantum states, for arbitrary $k > 1$.

On the other hand, since the DIWEDs of Eq. (3) involve the expectation value of 2^n different combinations of measurement settings, measuring these expectation values using only local measurement is already experimentally challenging for moderate values of n (although this scaling is still favorable compared with doing a full-state tomography of an n-qubit state). Hence, to maintain the possibility of witnessing entanglement depth in a robust manner (e.g., without being susceptible to intrinsic systematic uncertainties [43]), it is worth looking for other (families of) Bell inequalities where the corresponding DIWEDs only involve few expectation values but which may still share features of $\mathcal{I}_n$ given in Theorem 1. The families of Bell-like inequalities presented in [21,80,81] are some possible starting points for such an investigation, and the numerical techniques detailed in [54] will be useful for this purpose. Note also that for any given positive integer k, Theorem 2 of [51] allows us to extend any given witness for $n \geq k$ parties to one for *arbitrarily* many parties while preserving the number of expectation values that need to be measured experimentally. From an experimental perspective, it will also be highly desirable to identify DIWEDs that only involve few-body correlators (cf. Bell inequalities given in [82]), a problem that we leave for future research.

This work is supported by the Swiss NCCR "Quantum Science and Technology," the CHIST-ERA DIQIP, the ERC Grant No. 258932, the Singapore National Research Foundation (partly through the Academic Research Fund Tier 3MOE2012-T3-1-009) and the Singapore Ministry of Education. We gratefully acknowledge Florian Fröwis for enlightening discussion and for sharing his computation of a lower bound on entanglement depth using quantum Fisher information.

*ycliang@mail.ncku.edu.tw

[1] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 2004).
[2] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[4] J.-Å. Larsson, J. Phys. A **47**, 424003 (2014).
[5] M. Giustina *et al.*, Nature (London) **497**, 227 (2013); B. Christensen *et al.*, Phys. Rev. Lett. **111**, 130406 (2013).
[6] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[7] J. Barrett, L. Hardy, and A. Kent, Phys. Rev. Lett. **95**, 010503 (2005); A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007).
[8] B.W. Reichardt, F. Unger, and U. Vazirani, Nature (London) **496**, 456 (2013); U. Vazirani and T. Vidick, Phys. Rev. Lett. **113**, 140501 (2014).
[9] H. Buhrman, R. Cleve, and W. van Dam, SIAM J. Comput. **30**, 1829 (2001); Č. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, Phys. Rev. Lett. **92**, 127901 (2004).
[10] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature (London) **464**, 1021 (2010).
[11] R. Colbeck, PhD thesis, University of Cambridge, 2006 [arXiv:0911.3814]; R. Colbeck and A. Kent, J. Phys. A **44**, 095305 (2011).
[12] N. Brunner, S. Pironio, A. Acín, N. Gisin, A. A. Méthot, and V. Scarani, Phys. Rev. Lett. **100**, 210503 (2008).
[13] M. Navascués, G. de la Torre, and T. Vértesi, Phys. Rev. X **4**, 011011 (2014).
[14] D. Mayers and A. Yao, Quantum Inf. Comput. **4**, 273 (2004).
[15] M. McKague, T. H. Yang, and V Scarani, J. Phys. A **45**, 455304 (2012).
[16] T. H. Yang and M. Navascués, Phys. Rev. A **87**, 050102 (2013).
[17] T. H. Yang, T. Vértesi, J.-D. Bancal, V. Scarani, and M. Navascués, Phys. Rev. Lett. **113**, 040401 (2014).
[18] X. Wu, Y. Cai, T. H. Yang, H. N. Le, J.-D. Bancal, and V. Scarani, Phys. Rev. A **90**, 042339 (2014); K. F. Pál, T. Vértesi, and M. Navascués, *ibid.* **90**, 042340 (2014).
[19] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Phys. Rev. Lett. **106**, 250404 (2011).
[20] K. F. Pál and T. Vértesi, Phys. Rev. A **83**, 062123 (2011).
[21] J.-D. Bancal, C. Branciard, N. Brunner, N. Gisin, and Y.-C. Liang, J. Phys. A **45**, 125301 (2012).
[22] F. Verstraete and M. M. Wolf, Phys. Rev. Lett. **89**, 170401 (2002).
[23] Y.-C. Liang, T. Vértesi, and N. Brunner, Phys. Rev. A **83**, 022108 (2011).
[24] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Phys. Rev. Lett. **111**, 030501 (2013).
[25] G. Tóth, T. Moroder, and O. Gühne, Phys. Rev. Lett. **114**, 160501 (2015).
[26] D. Rosset, J.-D. Bancal, and N. Gisin, J. Phys. A **47**, 424022 (2014).
[27] P. M. Pearle, Phys. Rev. D **2**, 1418 (1970).
[28] S. L. Braunstein and C. M. Caves, Ann. Phys. (N.Y.) **202**, 22 (1990).
[29] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).
[30] J. Barrett, A. Kent, and S. Pironio, Phys. Rev. Lett. **97**, 170409 (2006).
[31] A. Elitzur, S. Popescu, and D. Rohrlich, Phys. Lett. A **162**, 25 (1992).
[32] R. Colbeck and R. Renner, Nat. Commun. **2**, 411 (2011).
[33] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); S. M. Roy and V. Singh, *ibid.* **67**, 2761 (1991); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A. V. Belinskiĭ and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).
[34] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).
[35] Y.-C. Liang, N. Harrigan, S. D. Bartlett, and T. Rudolph, Phys. Rev. Lett. **104**, 050401 (2010).
[36] J. J. Wallman, Y.-C. Liang, and S. D. Bartlett, Phys. Rev. A **83**, 022110 (2011).
[37] K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. Lett. **89**, 260401 (2002).
[38] Y.-C. Liang, F. J. Curchod, J. Bowles, and N. Gisin, Phys. Rev. Lett. **113**, 130401 (2014).
[39] A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001).
[40] O. Gühne, G. Tóth, and H. J. Briegel, New J. Phys. **7**, 229 (2005).
[41] M. Seevinck and J. Uffink, Phys. Rev. A **65**, 012107 (2001).
[42] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, Phys. Rev. A **85**, 022321 (2012); G. Tóth, *ibid.* **85**, 022322 (2012).

[43] D Rosset, R. Ferretti-Schöbitz, J.-D. Bancal, N. Gisin, and Y.-C. Liang, Phys. Rev. A **86**, 062325 (2012).
[44] G. A. Durkin and C. Simon, Phys. Rev. Lett. **95**, 180402 (2005).
[45] M. Seevinck and J. Uffink, Phys. Rev. A **78**, 032101 (2008).
[46] F. Levi and F. Mintert, Phys. Rev. Lett. **110**, 150402 (2013); A. A. Valido, F. Levi, and F. Mintert, Phys. Rev. A **90**, 052321 (2014).
[47] M. Huber, M. Perarnau-Llobet, and J. I. de Vicente, Phys. Rev. A **88**, 042328 (2013).
[48] B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Phys. Rev. Lett. **112**, 155304 (2014).
[49] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Phys. Rev. Lett. **109**, 070401 (2012); J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Phys. Rev. A **88**, 014102 (2013).
[50] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[51] F. J. Curchod, N. Gisin, and Y.-C. Liang, Phys. Rev. A **91**, 012121 (2015).
[52] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[53] C. Śliwa, Phys. Lett. A **317**, 165 (2003).
[54] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.114.190401, which includes Refs. [55–58], for a detailed derivation of all the results presented in this Letter and the explicit forms of semidefinite programs relevant to the device-independent characterization of k producibility.
[55] Y.-C. Wu, P. Badziag, and M. Żukowski, Phys. Rev. A **79**, 022110 (2009); Y.-C. Wu, M. Żukowski, J.-L. Chen, and G.-C. Guo, Phys. Rev. A **88**, 022126 (2013).
[56] Y.-C. Liang, PhD thesis, University of Queensland, 2008 [arXiv:0810.5400].
[57] Y.-C. Liang and A. C. Doherty, Phys. Rev. A **75**, 042103 (2007).
[58] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[59] B. Grunbaum, *Convex Polytopes* (Springer Science, New York, 1967).
[60] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).
[61] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[62] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), pp. 69–72; N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
[63] M. Navascués, S. Pironio, and A. Acín, Phys. Rev. Lett. **98**, 010401 (2007); New J. Phys. **10**, 073013 (2008); S. Pironio, M. Navascués, and A. Acín, SIAM J. Optim. **20**, 2157 (2010).
[64] A. C. Doherty, Y.-C. Liang, B. Toner, and S. Wehner in *Proceedings of the 23rd IEEE Conference on Computational Complexity* (IEEE Computer Society, College Park, MD, 2008), pp. 199–210.
[65] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).
[66] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A **71**, 022101 (2005).
[67] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).
[68] R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000); D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002); J. Uffink, *ibid.* **88**, 230406 (2002).
[69] S. Yu, Z.-B. Chen, J.-W. Pan, and Y.-D. Zhang, Phys. Rev. Lett. **90**, 080401 (2003).
[70] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[71] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
[72] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A **69**, 062311 (2004).
[73] G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005).
[74] F. Fröwis (private communication).
[75] F. J. Curchod, Y.-C. Liang, and N. Gisin, J. Phys. A **47**, 424014 (2014).
[76] N. Gisin and A. Peres, Phys. Lett. A **162**, 15 (1992).
[77] S. Popescu and D. Rohrlich, Phys. Lett. A **166**, 293 (1992).
[78] S. Yu, Q. Chen, C. Zhang, C. H. Lai, and C. H. Oh, Phys. Rev. Lett. **109**, 120402 (2012).
[79] Q. Chen, S. Yu, C. Zhang, C. H. Lai, and C. H. Oh, Phys. Rev. Lett. **112**, 140404 (2014).
[80] J.-Å. Larsson and J. Semitecolos, Phys. Rev. A **63**, 022117 (2001).
[81] B. Grandjean, Y.-C. Liang, J.-D. Bancal, N. Brunner, and N. Gisin, Phys. Rev. A **85**, 052113 (2012).
[82] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, and A. Acín, Science **344**, 1256 (2014).