Quantum Algorithm for Universal Implementation of the Projective Measurement of Energy

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A projective measurement of energy (PME) on a quantum system is a quantum measurement determined by the Hamiltonian of the system. PME protocols exist when the Hamiltonian is given in advance. Unknown Hamiltonians can be identified by quantum tomography, but the time cost to achieve a given accuracy increases exponentially with the size of the quantum system. In this Letter, we improve the time cost by adapting quantum phase estimation, an algorithm designed for computational problems, to measurements on physical systems. We present a PME protocol without quantum tomography for Hamiltonians whose dimension and energy scale are given but which are otherwise unknown. Our protocol implements a PME to arbitrary accuracy without any dimension dependence on its time cost. We also show that another computational quantum algorithm may be used for efficient estimation of the energy scale. These algorithms show that computational quantum algorithms, with suitable modifications, have applications beyond their original context.

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Introduction.—The projective measurement of energy (PME) is a quantum counterpart of an ideal energy measurement in classical mechanics. A PME on a given system sets the system to an energy eigenstate and returns the corresponding energy eigenvalue. A PME alone has no effect on a system already in an energy eigenstate; thus, by repeating the same PME and observing that the outcomes remain unchanged, it can be used to confirm that the system remains in the initial energy eigenstate. These properties make the PME suitable for detecting small effects on a quantum system that is subject to an external influence such as a gravity wave [1] or thermal fluctuation [2–4].

In practice, a device that implements a quantum measurement must include a destructive component such as a photon detector. The PME, a nondestructive measurement, requires another quantum system as a "probe." The system (commonly referred to as the "target") interacts with the probe, and a direct measurement is performed only on the probe after the interaction (Fig. 1).

An implementation protocol of PME is known for systems for which the Hamiltonian H is given in advance [5]. The protocol chooses the interaction between the probe and target according to H, so that the two quantum systems are appropriately entangled. The entanglement ensures that the measurement on the probe sets the target to an energy eigenstate, and that the outcome of the measurement identifies the respective energy eigenvalue. The time needed to induce the entanglement can be made arbitrarily short by increasing the strength of the interaction. Thus, the PME of a known H can, in principle, be implemented instantaneously.

This protocol, however, does not take into account the time required to identify H. Let us estimate the time cost by analyzing quantum process tomography [6,7] on the time evolution of the system. Process tomography involves setting the target to various "test states" and measuring the expectation value of appropriate observables for each resulting state after the time evolution. A complete process tomography for a system described by a d-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ requires a number of observables, $O(d^2)$, equal to the number of parameters in the Hamiltonian [8].

An accurate estimation of the expectation values needs to accumulate sufficient statistics. Each use of the time evolution costs time t; hence, the total time cost for the tomography to achieve a given accuracy for a d-dimensional system scales, at least, at $O(d^2)$. This implies that, if H is unknown, the total implementation time for the PME via process tomography grows at least exponentially in the number of subsystems due to the exponential growth of the total dimensions for composite systems.

Tomography is required even if a PME is to be performed only once. It extracts enough information to identify all the eigenspaces and eigenvalues of H, so the dimension dependence is thus unavoidable. A single use of the PME, however, does not reveal the exact description of the energy eigenspaces or the whole energy spectrum. A more efficient PME protocol is needed.

To improve a PME protocol is to find a better quantum algorithm. Some quantum algorithms are known to provide an efficient solution to computational problems [9]. These algorithms, however, assume that the dynamics of a

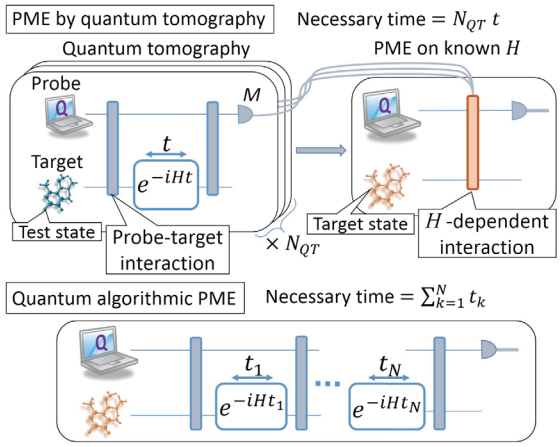

FIG. 1 (color online). Schematic diagrams of PME protocols on a system of unknown self-Hamiltonian H (labeled as "target.") The blue boxes $\exp(-iHt)$ denote the target being allowed to evolve for time t with $\hbar = 1$. M is a quantum measurement that returns a numerical outcome. The implementation time has a lower bound from the time required to induce the evolution of the target system. There is no limit, in principle, to the strength of the interaction induced on the system from outside. In the top protocol, H is identified by quantum (process) tomography, with at least $N_{\mathrm{QT}} = O(d^2)$ uses of the time evolution $\exp(-iHt)$ for d-dimensional systems. The quantum algorithmic PME (bottom) proposed in this Letter avoids quantum tomography, and all interactions are H independent.

quantum system can be "switched off" at will, which does not hold in this problem.

In this Letter, we introduce a more efficient PME protocol, and we show that we can remove the dimension dependence in the time cost for unknown Hamiltonians whose energy scale is given. Our protocol exploits a modified version of quantum phase estimation (QPE) [10]. Finally, we discuss an estimation protocol for the energy scale, based on an estimation of the trace of a unitary operator. We will show that another computational quantum algorithm, adapted from Ref. [11], performs more efficiently than a complete tomography.

Projective measurement by QPE.—QPE is designed so that each run returns a good estimate for some eigenvalue of a given unitary operator $U = \sum_{k=1}^{d} \exp(i\theta_k)|\theta_k\rangle\langle\theta_k|$ on $\mathcal{H} = \mathbb{C}^d$. Note that we assume $0 \le \theta_k < 2\pi$. For a given input state $|\theta_k\rangle$, the corresponding phase θ_k is estimated by QPE. An essential building block of QPE is a controlled-unitary operation C_U, which is a unitary gate that conditionally operates U on a d-dimensional target system, denoted by $\mathcal{H}_t = \mathbb{C}^d$, according to the state of an extra control qubit, denoted by $\mathcal{H}_c = \mathbb{C}^2$. Formally, the action of C_U on $\mathcal{H}_c \otimes \mathcal{H}_t$ is defined by $C_U|0\rangle|\varphi\rangle = |0\rangle|\varphi\rangle$ and $C_U|1\rangle|\varphi\rangle = |1\rangle U|\varphi\rangle$ for any $|\varphi\rangle \in \mathcal{H}_t$, where $\{|0\rangle, |1\rangle\}$ forms the computational basis of $\mathcal{H}_c$. To achieve an N-bit estimate, QPE uses N control qubits for applying $(C_U)^{2^{l-1}}$ between the lth control qubit for each $l \in \{1, \ldots, N\}$ and the target. We obtain an N-bit string $\{n_1, \ldots, n_N\}$ of outcomes by the final measurements on the N control qubits in the computational basis. When labeling $\mathbf{n}_N := \sum_{l=1}^{N} 2^{l-1} n_l$ and $f(\mathbf{n}_N) := \mathbf{n}_N/2^N$, the estimate of the phase is $\theta_k = 2\pi f(\mathbf{n}_N)$.

In the limit $N \to \infty$, $f(\mathbf{n}_N)$ can be regarded as a continuous variable f with $0 \le f \le 1$. For any θ_k, the probability $p_N[f(\mathbf{n}_N)|\theta_k]$ to obtain $\mathbf{n}_N$ for an initial state $|\theta_k\rangle$ approaches the delta function $\delta(f - \theta_k/2\pi)$ in distribution. The distance between $p_N[f(\mathbf{n}_N)|\theta_k]$ and $\delta(f - \theta_k/2\pi)$ is independent of d. At the same limit, the target is transformed to an eigenstate by a projection onto the corresponding eigenspace induced by the final measurements of QPE. Interested readers may refer to the Supplemental Material [12] for details of QPE.

QPE and universal controllization.—The evolution of a target with Hamiltonian H for time t is given by the unitary operator $U(t) = \exp(-iHt)$, with $\hbar = 1$. It may appear that QPE on $U(t)$ readily implements a projection onto the eigenspace corresponding to the estimated phase of $U(t)$, which is also the desired PME of H up to the ambiguity due to the phase periodicity. QPE assumes that U is available in its quantum-controlled form, namely, C_U, but the time evolution operator is not. Adding a quantum control to a quantum gate—a task which, we call controllization—is not trivial when U is unknown. In this Letter we introduce universal controllization, a quantum subroutine that approximately implements controllization for an unknown U.

We introduce a d-dimensional ancillary system denoted by $\mathcal{H}_a = \mathbb{C}^d$ and define a unitary gate $W_U := C_S(\mathbb{I}_2 \otimes U \otimes \mathbb{I}_d)C_S$ on $\mathcal{H}_c \otimes \mathcal{H}_t \otimes \mathcal{H}_a$, where C_S is a unitary gate called the controlled-swap operation defined by $C_S|0\rangle|\psi\rangle|\phi\rangle = |0\rangle|\phi\rangle|\psi\rangle$ and $C_S|1\rangle|\psi\rangle|\phi\rangle = |1\rangle|\psi\rangle|\phi\rangle$, for any $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$, and $\mathbb{I}_k$ denotes the $k \times k$ identity matrix. We call W_U a classically conditioned quantum gate since it perfectly simulates C_U when the control qubit is in a state $|0\rangle$ or $|1\rangle$. However, W_U deviates from C_U for a general input state $|\eta\rangle = \alpha|0\rangle + \beta|1\rangle$ in the control. Since $W_U|\eta\rangle|\psi\rangle|\phi\rangle = \alpha|0\rangle|\psi\rangle U|\phi\rangle + \beta|1\rangle U|\psi\rangle|\phi\rangle$, the ancilla system is also entangled to the control and target systems and, thus, decoherence occurs in the control-target system in general. If we can prepare an eigenstate of U in the ancilla system, exact implementations of C_U are possible [17–19], but such implementations require some knowledge of U. Other known controllization schemes [20,21] also require the quantum gate to be at least partially known. Indeed, it has even been proven that an exact controllization is impossible within quantum mechanics [19,22].

These results are derived assuming that the input quantum gate is a black box. The unitary operator $U(t)$, on the other hand, has a tunable parameter, namely, the evolution duration t. We exploit this feature and a decoupling method [23] used in quantum information theory to asymptotically implement a *universal* controllization of $U(t)$. The implementation accuracy of our controllization

depends on the maximum difference between any two eigenvalues of H.

To reduce the decoherence by W_U, we need to make the resulting state of the ancilla depend as little as possible on the initial control-target state. Let us prepare the ancilla in the completely mixed state $\mathbb{1}_d/d$, so that the state of the ancilla remains the same, at least when the control qubit is in $|0\rangle$ or $|1\rangle$ for any given U. We consider the reduced map on the control-target system,

$$\Gamma_U[\rho] := \mathrm{Tr}_{\mathcal{H}_a}(W_U(\rho \otimes \mathbb{1}_d/d)W_U^\dagger), \tag{1}$$

where ρ is a density matrix on $\mathcal{H}_c \otimes \mathcal{H}_t$. We call the map Γ_U pseudocontrollization. For $\rho = |\eta\rangle\langle\eta| \otimes |\psi\rangle\langle\psi|$, we have

$$\begin{aligned}&\Gamma_U[|\eta\rangle\langle\eta| \otimes |\psi\rangle\langle\psi|] \\ &= C_U(|\eta\rangle\langle\eta| \otimes |\psi\rangle\langle\psi|)C_U^\dagger \\ &\quad + [\alpha\beta^* |0\rangle\langle 1| \otimes |\psi\rangle\langle\psi|(\gamma_U - 1)U^\dagger + \text{c.c.}],\end{aligned} \tag{2}$$

where $\gamma_U = \mathrm{Tr}[U]/d$. The second term in Eq. (2) acts as a kind of phase damping noise on the control-target system. The factor $\gamma_U - 1$ determines the deviation of the reduced map Γ_U from the ideal controllization. We define the coherence factor $a_U := |\gamma_U|$ and a phase factor $e^{i\varphi_U} := \gamma_U/|\gamma_U|$. Notice that $1 - a_U \le |\gamma_U - 1|$. Thus, the phase damping noise is minimized if we regard Γ_U as an approximation of $C_{U'}$ for $U' = e^{-i\varphi_U}U$. In a sense, Γ_U implements a noisy controlled-unitary operation, where the magnitude of the noise is determined by a positive quantity $1 - a_U$.

We further reduce the dependence of the ancilla on the initial control-target state by the use of a set $\{\sigma_r\}$ of unitary operations on the ancilla such that

$$\frac{1}{d^2}\sum_r \sigma_r W_U(\rho_{\mathrm{tot}} \otimes \mathbb{1}_d/d)W_U^\dagger \sigma_r^\dagger = \Gamma_U[\rho_{\mathrm{tot}}] \otimes \mathbb{1}_d/d. \tag{3}$$

Note that the ancilla is "refreshed" to the completely mixed state only by operations on the ancilla. (Such a random operation has been extensively applied to questions in quantum communication [23].) We divide $W_{U(t)}$ into m repetitions of $W_{U(t/m)}$, each followed by the refreshing operation (3). Here, m fixes the refresh rate. The strength of the noise after each refreshing operation is $O(1/m^2)$. Thus the total effect of the noise scales $O(m \times [1/m^2]) = O(1/m)$, which vanishes in the asymptotic limit of $m \to \infty$ [12]. This phenomenon is mathematically analogous to the quantum Zeno effect [24].

We call this asymptotic implementation of a controlled-unitary operation including the repeated refreshing operation, universal controllization. For finite m, the universal controllization approximates the controlled-unitary operation $C_{U^{[m]}(t)}$, where $U^{[m]}(t) = e^{-im\varphi_{U(t/m)}}U(t)$. With $m \to \infty$, $\exp(im\varphi_{U(t/m)})$ converges to $\exp(-i\mathrm{Tr}[H]t/d)$. In a sense, the universal controllization fixes the reference point of the energy of H so that $\mathrm{Tr}[H] = 0$. A more detailed discussion of the universal controllization is presented in Ref. [12].

PME by universal controllization.—A perfect PME for a system with a Hamiltonian H is distinguished from other quantum operations by two properties. First, the system remains in the same eigenstate when a PME is applied consecutively. Second, the outcomes of the consecutive measurements are all precisely equal to E_k. The probability density $p(E|E_k)$ of obtaining E as the outcome must be the delta function $\delta(E; E_k) := \delta(E - E_k)$. Conversely, the only measurement satisfying these properties is a perfect PME.

A subtlety is that perfect PMEs for H and for $H - \lambda\mathbb{1}$ should be considered equivalent, since two Hamiltonians with different reference points of energy are physically equivalent. A measurement scheme is regarded as a perfect PME for H if $p(E|E_k) = \delta(E; E_k - \lambda)$, as long as λ is independent of k.

Our PME protocol uses QPE on the time evolution operator $U(t)$ with $C_{U^{[m]}(t)}$ implemented by the universal controllization. Here, the control qubits and ancilla of the universal controllization serve as the probe. The probe-target interaction is used to perform $W_{U(t/m)}$, the refreshing operations, and quantum Fourier transformation. The lower figure in Fig. 1 provides a conceptual diagram.

In the ideal case of $m \to \infty$ and $N \to \infty$, the modified QPE implements the projective measurement defined by the spectral decomposition of $\tilde{U}(t) = \exp(-i\tilde{H}t)$, where $\tilde{H} := H - \mathrm{Tr}[H]\mathbb{1}$. The outcome f gives $-\tilde{E}_k t \pmod{2\pi}$ for some energy eigenvalue $\tilde{E}_k$ of $\tilde{H}$.

$\tilde{E}_k$ cannot be uniquely determined from f for general t due to the periodicity of the phase function $\exp(i\theta)$. Let us restrict t so that $\tilde{E}_k \in (\pi/t, -\pi/t)$, namely,

$$\Delta_{\max} t \le \pi/2, \tag{4}$$

where $\Delta_{\max} = \max_{k,l}|\tilde{E}_k - \tilde{E}_l|$. The energy eigenvalues are uniquely determined by

$$E[f] = \begin{cases} -2\pi f/t & f \in [0, \frac{1}{2}) \\ -(2\pi f - 2\pi)/t & f \in [\frac{1}{2}, 1) \end{cases}. \tag{5}$$

Recall that the probability distribution of f is the delta function $\delta(f - \theta_k/2\pi)$. Thus, $p(E|E_k) = \delta(E; E_k - \mathrm{Tr}[H])$, which is the desired function. The projection onto the corresponding energy eigenspace is already guaranteed by QPE.

For finite m and N, we continue to choose t according to Eq. (4) and estimate E_k by Eq. (5) with f replaced by $f(\mathbf{n}_N)$. The implemented measurement is an approximation of a PME. A target initially in an energy eigenstate $|E_k\rangle$ results in the same state at the end of the scheme. One of the conditions for a perfect PME is still satisfied. Thus, the

accuracy of the scheme is determined by how close $p(E|E_k)$ for each $|E_k\rangle$ simulates a delta function $\delta(E; E_k - \lambda)$.

Implementation accuracy and time cost.—Recall that $p_N[f(\mathbf{n}_N)|\theta_k]$ in QPE needs to approach the delta function $\delta(f - \theta_k/2\pi)$ in $N \to \infty$ to achieve the projective measurement determined by U. If each C_U in QPE is replaced by the adapted classical controllization (i.e., substituted by W_U and an ancilla), $p_N[f(\mathbf{n}_N)|\theta_k]$ does not converge to the delta function unless $a_U = 1$.

Let us denote by $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ the probability distribution of $f(\mathbf{n}_N)$ for a given m, N, and initial state $|E_k\rangle$. For a finite m, the universal controllization approximately controllizes $U' = \exp[-i(Ht - m\varphi_{U(t/m)}\mathbb{1})]$. In this case, each run of the approximated QPE provides an estimate for the eigenvalue corresponding to $|E_k\rangle$, which is $\theta'_k = -E_k t + m\varphi_{U(t/m)} \pmod{2\pi}$. When N increases, the deviation of $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ from $p_N[f(\mathbf{n}_N)|\theta'_k]$ caused by the controllization error prevents the function from converging to a delta function [see Fig. 2(a)]. The deviation can be bounded by $|p_N^{[m]}[f(\mathbf{n}_N)|E_k] - p_N[f(\mathbf{n}_N)|\theta'_k]| \le \epsilon$ for any $\epsilon > 0$ when m is set to

$$m \ge (\Delta_{\max} t)^2 N 2^{N-3}/\epsilon, \tag{6}$$

as shown in Ref. [12] [see Fig. 2(b) for examples].

For a given refresh rate m, each universal controllization makes m uses of $W_{U(t/m)}$, where the total evolution duration $(t/m) \times m = t$ is independent of m. Hence, $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ can be brought arbitrarily close to $p_N[f(\mathbf{n}_N)|\theta'_k]$ without increasing the time cost. The distribution $p_N[f(\mathbf{n}_N)|\theta'_k]$ is not a delta function for any finite N, even with perfect controlled-unitary operations C_U (i.e., infinite m). The cost doubles for each control qubit added, but the distance between $p_N[f(\mathbf{n}_N)|\theta'_k]$ and the delta function $\delta(E; \tilde{E}_k)$ is independent of the dimension of the target. Hence, the implementation accuracy of the PME can be improved without any dimension dependence.

Quantum algorithmic estimation of the energy scale.—We showed the existence of our PME protocol under the assumption that $\Delta_{\max}$ is known. The assumption can be relaxed to knowing an upper bound on $\Delta_{\max}$. The bound may be estimated by quantum (process) tomography, but the tomography requires that a prior distribution of H is given. For a certain prior distribution, it is possible to estimate the bound by measuring the coherence factor a_U. We observe that a_U approaches 1 as the product $\Delta_{\max} t$ decreases to 0. Thus, when a_U is estimated to be close to 1, it is possible that $\Delta_{\max} t$ is sufficiently small. While this is not true for some Hamiltonians, the probability of such an "error" decreases exponentially in the dimension d of the target for a particular class of prior distribution [12]. Hence, we can reliably estimate a_U.

To estimate a_U, we modify the quantum algorithm presented in Ref. [11]. The original algorithm outputs

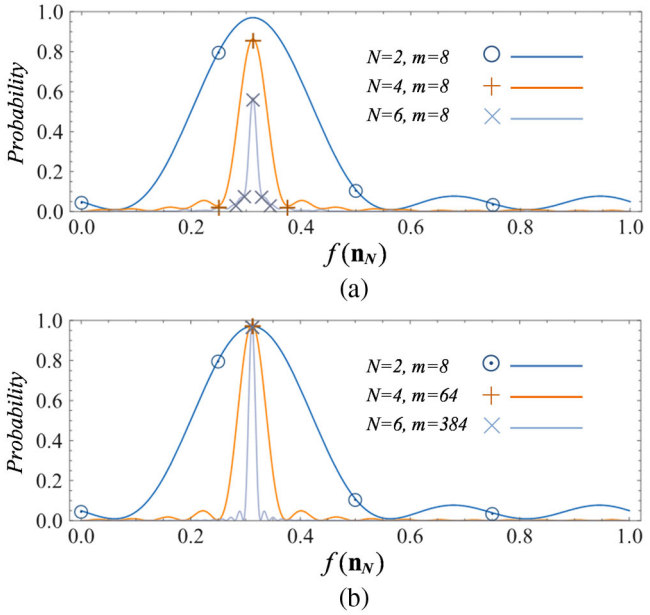

FIG. 2 (color online). Plots of probability distributions $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ and their envelope functions for the target Hamiltonian $H = -\sum_{\lambda=0}^{3} |E_0^{(\lambda)}\rangle\langle E_0^{(\lambda)}| + |E_1\rangle\langle E_1|$, $t = 0.225\pi$, and setting $E_k = E_1$. Each marker represents $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ of finding each outcome by a single round of the PME scheme. (a) $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ for $N = 2, 4, 6$ for a fixed refreshing rate $m = 8$. (b) $p_N^{[m]}[f(\mathbf{n}_N)|E_k]$ for $N = 2, 4, 6$ where each value of m is adaptively chosen as the smallest integer satisfying $m \ge (\Delta_{\max} t)^2 N 2^{N-3}/\epsilon$ and target error ϵ is set to 0.25. In all cases, markers corresponding to probabilities less than one tenth of the target error (0.025) are omitted for clarity.

the trace $\mathrm{Tr}[U]$ of an input unitary U, provided that the corresponding C_U is available. In our problem, we replace C_U with W_U. With this modification, the original algorithm returns $|\mathrm{Tr}[U]|^2$ [12]; thus, we obtain a_U, because $a_U^2 = |\mathrm{Tr}[U]|^2/d^2$. Clearly, this modified algorithm estimates a_U much more efficiently than process tomography.

Conclusion.—In this Letter, we presented an implementation protocol for a projective measurement of energy on a system driven by an unknown Hamiltonian with a given energy scale. The implementation time cost of the protocol is independent of the dimension of the system, unlike the protocol based on quantum process tomography. The protocol is based on a computational quantum algorithm called quantum phase estimation. We introduced universal controllization to make the computational algorithm executable without suppressing the evolution of the target system. Another computational quantum algorithm is shown to be effective in estimating the energy scale with a suitable modification. This motivates the search for further applications of quantum algorithms outside their original computational context.

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Note added.—Recently, we were notified that our algorithm for calculating $|\mathrm{Tr}[U(t)]|$ based on DQC1 has been independently discovered by Thompson *et al.* in Ref. [25]. We thank these authors for drawing their work to our attention.

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