WAVELET AND ITS APPLICATIONS

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Fan Zhitao

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Summary

Motivated from the dual Gramian analysis of shift-invariant frames in [94], we developed the dual Gramian analysis for frames in abstract Hilbert spaces. We show the dual Gramian analysis is still a powerful tool for the analysis of frames, e.g. to characterize a frame, to estimate the frame bounds, and to find the dual frames. The dual Gramian analysis can be easily extended to the analysis of dual (or bi-) frames by mixed dual Gramian analysis.

With the introduction of adjoint systems, the duality principle plays a key role in this analysis. The duality principle also lies in the core of the analysis of Gabor systems, by which we unify several classical identities, e.g. the Walnut representation, the Janssen/Wexler-Raz representation, and the Wexler-Raz biorthogonal relationship. Moreover, several dual Gabor window pairs are constructed from this duality viewpoint, especially the non-separable multivariate case with any order of smoothness.

For MRA wavelet frames, the (mixed) unitary extension principle can be viewed as the perfect reconstruction filter bank condition for sequences. The duality perspective leads to a new and simple way to construct filter banks, or tight/dual wavelet frames from a prescribed MRA. The new method reduces the construction to a constant matrix completion problem rather than the usual methods to complete matrices with trigonometric polynomial entries. The new construction guarantees the existence of multivariate tight/dual wavelet frames from a given refinement mask, with the constructed wavelets easily satisfying additional properties, e.g. small support, symmetric/anti-symmetric.

Several multivariate tight and dual wavelet frames from given refinable functions have been constructed.

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Chapter 1

Introduction

1.1 Background

Frame theory, peaking in the last few decades, has infused new life and energy to both theory and applications and many fascinating results are obtained. The notion of frame was first introduced by Duffin and Schaeffer to study nonharmonic Fourier series [46]. Like an orthonormal basis, a frame system guarantees a numerical stable reconstruction from the decomposition of an element in a Hilbert space, however the reconstruction is no longer unique since most of frame systems are redundant; in other words, they contain more elements than needed. For a given frame system, one of the natural systems that could be considered for the reconstruction is the canonical dual frame, which is the pre-image of the frame system under the frame operator. But due to the redundancy of the frame system, the choice of the reconstruction system is not unique. All the alternative systems that provide the perfect reconstruction are called dual frames. When the canonical dual frame coincides with the original frame system, one has the so-called tight frame. Tight frame systems, which contain the orthonormal basis as a special case, provide more flexibility in various properties than orthonormal basis. This additional flexibility is sometimes desirable in theoretical analysis and applications.

In real applications, e.g. image processing, systems with special structure should be considered. Given a signal $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-i\omega\cdot x}\, dx, \quad \omega \in \mathbb{R}^d,$$

which could be extended to $L_2(\mathbb{R}^d)$, exhibits the frequency content of a function. Local changes of the signal f will in general result in a global change of its Fourier transform and the information about time-localization of different frequencies cannot be easily interpreted from $\hat{f}$. The classical way to resolve this problem is the introduction of a compactly supported or fast decaying window $\phi \in L_2(\mathbb{R}^d)$, resulting in the *windowed (or short-time) Fourier transform*

$$V_\phi f(\omega, t) := \langle f, M^\omega E^t \phi \rangle = \int_{\mathbb{R}^d} f(x) e^{-i\omega\cdot x} \overline{\phi(x-t)}\, dx, \quad (\omega, t) \in \mathbb{R}^{2d}. \tag{1.1}$$

Here, we denote by E^t the translation operator and by M^t the modulation operator on $L_2(\mathbb{R}^d)$, i.e. $E^t f := f(\cdot - t)$ and $M^t f := e_t f$, where $e_t\colon x \mapsto e^{it\cdot x}$ and $x, t \in \mathbb{R}^d$. By a discrete sampling of its continuous time-frequency representation (1.1), one is thus led to considering the properties of the *irregular Gabor system (or Weyl-Heisenberg system)*

$$X = \{E^\gamma M^\eta \phi \colon (\gamma, \eta) \in \Lambda\},$$

where $\Lambda \subset \mathbb{R}^{2d}$ is some discrete set. The system can be used to analyze and study the numerical stable reconstruction of the signal from the discrete samples of its continuous time-frequency domain, or to characterize function spaces (see e.g. [48]). In addition to a good localization of the window, i.e. of the elements of X, a good simultaneous frequency localization of the elements of X is often important. This makes windows that are smooth, i.e. have fast decaying Fourier transform, desirable. But the Balian-

Low theorem, see e.g. [35, 52, 59, 60], sets some theoretical boundaries. If the shifts and modulations are lattices (see Chapter 3), then there do not exist windows with both good time and frequency localization that generate orthonormal bases X. However, there exist windows with excellent time-frequency localization, that generate frames and even tight frames, thus ensuring numerically stable and even perfect reconstruction. This makes Gabor systems an example of systems for which it becomes imperative to oversample, i.e. to move beyond orthonormal bases into the realm of frames. There is a vast literature on studying irregular Gabor systems. Most of the results concern perturbation and density theorems of the sampling sets, see e.g. [25, 50, 51, 58, 80, 103, 108].

The irregular Gabor system with time-frequency varying on lattices will be called *(regular) Gabor system*. The frame property of the regular Gabor system in one dimensional case has first been studied in [36]. In order to get a reconstruction from the decomposition by a Gabor frame, the dual frame is needed. The canonical dual frame of a Gabor frame remains a Gabor frame with the window to be the pre-image of the window function under the frame operator, since the frame operator commutes with the shift and modulation operator. By observing that the Gabor system with a separable shift and modulation lattice is shift-invariant, the analysis developed for shift-invariant systems in [94] could be applied and the analysis is done in arbitrary dimensions. The analysis in [94], namely the dual Gramian analysis, is based on a fiberized matrix representation of the frame operator by making use of the shift-invariant structure. This matrix representation is useful in several ways. One is to estimate the frame bounds by a simple matrix norm. Another is to introduce the adjoint system by a simple row-column relationship, which greatly simplifies the study of Gabor frames. Since studying the Riesz sequence property is generally easier than the frame property, the Gabor frame property could be transferred to the Riesz sequence property of the adjoint system, which is one consequence of duality principle. In particular, the characterization of tight frames can be simplified to an orthonormal sequence property of the adjoint system, and more generally, dual Gabor frames properties to a biorthogonal relationship of their adjoint systems. This biorthogonal relationship for characterizing dual Gabor frames is independently observed by [39, 70], where it is proved by the Wexler-Raz identity. The Wexer-Raz identity is essentially a representation of the frame operator, which is later generalized by Janssen [71]. Not only biorthogonality relationship simplifies the verification of dual Gabor frames, but also it makes the construction of dual Gabor windows painless [36]. Duality results for Gabor system with Λ being a general non-separable lattice are discussed in [53, 54].

A considerable body of literature on the construction of dual Gabor windows already exists. In [24], a construction for a dual window of a given compactly supported window, in particular a given B-spline which is a smooth piecewise polynomial function [2], is presented. The support of the dual window constructed in [24] is twice as large as that of the primary window. Moreover, the density of the modulation lattice depends on the support size of the primary window. Larger support of the primary window, i.e. in the B-spline case higher smoothness, forces a denser modulation lattice. Also note that in [22, 27, 77, 81] the authors construct dual windows that overcome the problem of support in [24]. The paper [81] gives several constructions of Gabor windows using spline functions and discusses the smoothness of the constructed windows and choice of lattices. However, it involves complicated symbolic computations, especially when the smoothness of the window is increased. The idea of [24] is generalized to higher dimensions in [28]. Similar to the one dimensional case, the support size of the dual window gets larger when the smoothness of the primary window increases, resulting in a denser modulation lattice.

Another system widely used in application is the wavelet system. The wavelet system is a discrete sample of the continuous wavelet transform, which is given as

$$Wf(a,b) := a^{-1/2} \int_{\mathbb{R}^d} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx, \quad (a,b) \in \mathbb{R}_{>0} \times \mathbb{R}^d,$$

for $f, \psi \in L_2(\mathbb{R}^d)$. Similar to windowed Fourier transform, the wavelet transform can also provide a time-frequency transform on the signal. But the windowed Fourier transform has the same window function at different time locations while the wavelet transform can change the window size, due to the dilation, adapted to the frequency of the signal. This makes the wavelet transform more suitable, e.g. to analyze transients in a signal. The *wavelet (or affine) system* is defined as

$$X = \{D^k E^j \psi : k \in \mathbb{Z}, j \in \mathbb{Z}^d, \psi \in \Psi\}$$

where D^k is the dilation operator: $D^k : f \mapsto 2^{kd/2} f(2^k \cdot)$ and $\Psi \subset L_2(\mathbb{R}^d)$ are called the *wavelets*. Again the system can be used to analyze the signal or study the numerical stable reconstruction of the signal from the discrete samples of its continuous time-frequency domain.

There are many works on the construction of wavelet orthonormal basis in $L_2(\mathbb{R})$, see e.g. [35] for some pioneer works. With the introduction of multiresolution analysis (MRA) by Mallat and Meyer [86, 88], most of the construction could be explained with a firm theoretical framework and it inspires more constructions. Wavelet orthonormal basis with bandlimited windows, i.e. of compact support in the Fourier domain, by Meyer is shown in [87] and with compactly supported windows by Daubechies is constructed in [35]. Symmetry of the wavelets is sometimes desirable in applications, but it has been proved by Daubechies that the only dyadic real symmetric orthonormal wavelet with compact support is the Haar wavelet [35]. In searching for symmetric wavelet windows, one way is to drop the single system assumption and the biorthogonal wavelet, i.e.

wavelet Riesz basis, is then studied in [33]. Wavelet frames, in particular tight wavelet frames, once again do not have such restriction on the symmetry of the window function.

Wavelet frames in $L_2(\mathbb{R})$ are first studied in [36] and the frame bounds are then estimated in [34]. Compared with Gabor frames, it is not easy to find the canonical dual wavelet frame, since the frame operator no longer commutes with the dilation operator in this case. Wavelet frames in $L_2(\mathbb{R}^d)$ are systematically studied in [96] by the dual Gramian analysis developed for the analysis of shift-invariant frames. But a wavelet system is not shift-invariant due to the negative and decreasing dilation. In [96], the quasi-affine system is introduced by oversampling the wavelet system, which is made shift-invariant and shares the same frame property as the wavelet system. The wavelet frame bounds can be easily estimated from a matrix norm and the tight frame property can be stated as a simple condition on the wavelet windows. Under MRA, the tight frame property on the windows can be further reduced to a condition on the masks, namely the unitary extension principle (UEP).

The construction of tight wavelet frames ever since attracts a lot of attention as UEP provides a useful tool but there is still no simple and unified algorithm to give constructions of all wavelets with desired properties. In particular, the multivariate wavelet construction becomes more difficult due to the increasing dimension. One simple refinable function that enjoys wide applications is the B-spline. Using the UEP, totally m wavelets could be constructed for a given B-spline function B_m, where m is the order of the B-spline, see e.g. [45]. There have been many other methods to construct univariate tight wavelet frames from B-splines. For example, by using the UEP and trigonometric polynomial matrix completion, the construction in [30] can give only two wavelets for B-splines of any order, and three if certain symmetry is imposed on the wavelets. Independent of which method or which B-spline function is used, the approximation order of the truncated tight wavelet frames constructed via the UEP from B-splines is never greater than two. Constructing spline tight wavelet frames of better approximation order leads to the discovery of the oblique extension principle (OEP), independently discovered in [32] and [38]. By using the OEP, spline tight wavelet frames with two or three wavelets are constructed in [38] with better approximation order than the ones constructed from UEP. In [63–65], interesting examples of symmetric tight wavelet frames with two or three wavelets are constructed by splitting a matrix of Laurant polynomials with symmetry.

The construction of non-separable multivariate tight wavelet frames by using refinable box splines first appeared in [91], where exponentially decaying orthonormal wavelets for dimension two or three are constructed. After the UEP was introduced, compactly supported tight wavelet frames from box splines was first constructed in [99]. The methods provided in [99] are applicable in general to box splines of any order, however, the support of the constructed wavelet can be large. There are also many other construction schemes of tight wavelet frames from box splines, see e.g. [18, 31, 61, 79]. The main challenge in these construction schemes is the completion of a trigonometric polynomial matrix with multivariables from one single row given by the refinement mask such that the matrix satisfies the UEP condition. For the case of nonnegative refinement masks, a new local construction scheme of tight wavelet frames is proposed in [20] which simplifies the problem from polynomial matrix completion to a constant matrix factorization.

The dual Gramian analysis can be easily extended to the analysis of dual (or bi-) systems, called the mixed dual Gramian analysis, and as a result a characterization of dual wavelet frames can be derived [97]. This characterization, under MRA, can as well be reduced to a sufficient condition on the masks, which is the mixed unitary extension principle (MEP). Compared with the UEP, the construction of wavelets based on the MEP is to complete two matrices, which gains more flexibility in mask design. Several one dimensional dual wavelet frames have been constructed in [33, 35, 37, 38, 73]. Construction of multivariate dual frames, similar to the multivariate tight frame construction, becomes increasingly difficult, since it involves the completion of two matrices with polynomial entries. As orthonormal bases are a special class of tight frames, the biorthogonal systems, i.e. Riesz basis and its dual, are a special class of dual frames. The literature has a rich history of biorthogonal wavelet constructions but lack dual wavelet frames constructions. Several biorthogonal wavelet construction based on box splines have been proposed in [72, 91, 93]. There are many multivariate biorthogonal wavelet constructions with high order of vanishing moment in [21]. Also note that the lifting scheme proposed in [109], which is essentially linked to biorthogonal wavelets, leads to several constructions of multivariate biorthogonal wavelets, see e.g. [55, 78, 106]. A multivariate dual wavelet frames construction via a projection method is proposed in [62].

Frames are proved to be effective in real applications. For example, tight wavelet frames have been implemented in many image restorations such as image inpainting [4, 41], image denoising [13, 57, 105], image deblurring [8, 9, 12], image demosaicing [85], and image enhencement [69]. Moreover, wavelet frame related algorithms are developed to solve medical and biological image processing problems, e.g. medical image segmentation [40, 110], X-ray computed tomography (CT) image reconstruction [43], and protein molecule 3D reconstruction from electron microscopy images [83]. Frame has more flexibility of designing appropriate filters for the need of applications. For example, the filters used for image restoration problems in [1, 11, 90] are learned from the image so that the filters have captured certain feature of the image and the transform gives a better sparse representation. In [74], Gabor frame filter banks are designed to get a high orientation selectivity that adapts to the geometry of image edges for sparse image approximation. Wavelet filters can be considered as a discrete approximation of certain differential operators. The tight wavelet frame based approach for image processing has close relationship with the PDE based approaches, whose connection to the total variation based approach is established in [6], to the Mumford-Shah model in [7], and to nonlinear evolution PDE models in [42].

1.2 Organization

The thesis mainly contributes to the development of the theory of dual Gramian analysis for frames in an abstract Hilbert space (chapter 3), and a few applications of the resulted core duality principle for Gabor frame analysis (chapter 4) and wavelets construction (chapter 5) in $L_2(\mathbb{R}^d)$. We give a short overview of the contents and contributions of each part.

- Chapter 2: The synthesis operator and the analysis operator in a Hilbert space will be defined. Various systems and their definitions will be reviewed. The synthesis operator and the analysis operator form two self-adjoint operators by composing in different orders, by which the characterization of systems is investigated. The synthesis operator and the analysis operator of two different systems form mixed operators, of which the properties are studied. In the end of this chapter, we consider the restriction of the coefficient space corresponding to the synthesis operator, which could be viewed as studying a special mixed operator.

- Chapter 3: The pre-Gramian matrix, the (mixed) Gramian matrix and the (mixed) dual Gramian matrix will be introduced. The links between the (infinite) matrices and the operators in Chapter 2 are established. In particular, the dual Gramian matrix, which is formed by only the elements of the system, is a matrix representation of the frame operator. There are several benefits by writing the operator in a matrix form, e.g., to find the canonical dual frame by a matrix inverse, and to estimate the frame bound by a matrix norm. The pre-Gramian matrix could be further simplified if systems exhibit special structure. For example, we show the fiber pre-Gramian matrix of a shift-invariant system introduced in [94] is a special realization of the abstract pre-Gramian matrix by choosing an adequate orthonormal basis. In [98] the fiber dual Gramian analysis for regular Gabor systems as a special class of shift-invariant systems is developed. While the emphasis of [98] is more on the dual Gramian analysis of single systems with only a few glimpse of dual frames, here we give a detailed mixed dual Gramian analysis of bi-systems.

- Chapter 4: A new system, namely adjoint system, can be easily defined from the matrix view point of the synthesis operator by a simple row and column relationship; as a result duality principle is derived. Part of the duality principle states that the frame property of the system is characterized by the Riesz sequence property of the adjoint system counterpart, which simplifies the study of frame greatly. We will see that all the dual frames could be characterized and parametrized by the adjoint systems. The duality principle also brings a new viewpoint of the perfect reconstruction filter banks in $\ell_2(\mathbb{Z}^d)$, which leads to a simple filter bank construction scheme involving only a constant matrix completion. We then show the dual Graman analysis for irregular Gabor systems in $L_2(\mathbb{R}^d)$ by choosing a Gabor orthonormal basis to best adapt to the structure of the system, and present the duality principle. For regular Gabor system, we will show how those classical identities, e.g. Walnut representation, Wexler-Raz/Janssen identities, Wexler-Raz biorthogonal relationship, could be a simple consequence of the dual Gramian analysis and the duality principle. We proposed several simple ways to construct dual Gabor windows based on the duality principle viewpoint, which have coinciding support and can achieve arbitrary smoothness of the windows.

- Chapter 5: The fiber dual Gramian analysis for wavelet frames in [96] is reviewed.

In particular, under a multiresolution analysis (MRA), we review the unitary extension principle (UEP) for tight wavelet frames and mixed unitary extension principle (MEP) for dual wavelet frames. The UEP, respectively the MEP, is indeed the perfect reconstruction condition for filter banks in $\ell_2(\mathbb{Z}^d)$ associated with the wavelet masks. This connection, with the construction scheme for filter banks resulted from duality principle, leads to a simple way of constructing tight and dual wavelet frames, which, in contrast to the existing constructions involving matrix completion with polynomials, only requires completing constant matrices. Especially, this greatly simplifies the task of finding multivariate tight or dual wavelet frames, and most importantly, guarantees the existence of multivariate tight or dual wavelet frames from any given refinement mask satisfying a weak condition. Several multivariate tight wavelet frames constructed from box spline and multivariate dual wavelet frame constructed from interplotary refinable functions will be shown. Finally, given a set of tight frame filter bank, as long as there is a low pass filter, we show that this filter bank corresponds to an MRA tight wavelet system in $L_2(\mathbb{R}^d)$ whose masks are derived from the filter bank.

1.3 Contributions

The contributions of the thesis include the following:

- Built up the dual Gramian analysis of a single system in a separable Hilbert space, and the mixed dual Gramian analysis for bi-systems.
- Made the connection of dual Gramian analysis proposed in the thesis and the dual Gramian analysis of shift-invariant system in [94].
- Introduced the adjoint systems and showed the power of duality principle in studying the frame properties of original systems in Hilbert spaces.

- Detailed mixed dual Gramian analysis for Gabor systems, and unified several classical identities of Gabor systems by the duality principle.
- Constructed Gabor windows with same compact support and arbitrary smoothness, in particular for high dimensional case.
- Proposed a new and simple scheme to construct perfect reconstruction filter banks by duality principle.
- Proved the existence of multivariate tight/dual wavelet frames, and proposed a new and simple way to construct tight/dual wavelet frames which only involves a completion of constant matrices.

Chapter 2

Hilbert space and operators

In this chapter, we review the basic notations of a Hilbert space and introduce several operators, in particular, the synthesis operator and the analysis operator, by which various systems are defined in the Hilbert space. The two different ways of composition of the synthesis and analysis operators lead to two self-adjoint operators, which are convenient in the characterization of different systems. The compositions of the synthesis and analysis operators from two different systems give the mixed operators, several of whose properties are investigated. Lastly, we will examine the synthesis operator on a sequence subspace, and the frame operator with a restriction on the sequence subspace is studied. Parts of this chapter could be found in e.g. [23, 35, 67, 68, 94, 113]. We summarize, make the notations consistent and provide a sketch of proof to make the thesis more self-contained.

2.1 Hilbert space and systems

A **Hilbert space** H is a complex inner product space which is complete with respect to the norm function induced by the inner product. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the induced norm is defined as $\|x\| := \langle x, x \rangle^{1/2}$ for $x \in H$. We only consider the separable Hilbert space in this thesis, which admits a countable orthonormal basis. An **orthonormal basis** $\mathcal{O}$ is a subset in H, of which the linear span is dense in H, each element has a unit norm and the elements are pairwise orthogonal, i.e. $\langle x, y\rangle = 0$ for any two distinct $x, y \in \mathcal{O}$. With a countable orthonormal basis, an infinite-dimensional separable Hilbert space is isometrically isomorphic to ℓ_2 which is the space of square summable sequences. A sequence X with a certain indexing in H is hereafter referred to as a **system**. An **operator** Λ from a Hilbert space H to another Hilbert space H' is a linear mapping with the domain being a subspace of H and range in H'. The operator is said to be **bounded** or **continuous** on H if there exists $B > 0$ such that $\|\Lambda h\| \leq B\|h\|$ for all $h \in H$.

Given a system X in H, we start with two operators that are naturally associated with this system. First let $\ell_2(X)$ be the space of square summable sequences indexed by X and $\ell_0(X)$ be the space of sequences with finite support. The **synthesis operator** of X is defined by

$$T_X : \ell_2(X) \to H : c \to \sum_{x\in X} c(x)x,$$

which is well defined on the dense subspace $\ell_0(X)$ of $\ell_2(X)$. The system X is called a **Bessel system** if T_X is bounded on $\ell_0(X)$, in which case we consider T_X as its unique continuous extension to a bounded operator on $\ell_2(X)$. The operator norm $\|T_X\|$ is called the **Bessel bound**.

A Bessel system X is called **fundamental** if its closed linear span is all of H and it is called **ℓ_2-independent** if T_X is injective. A Bessel system X is called a **Riesz sequence** if T_X is bounded below on $\ell_2(X)$, or equivalently there exist two positive constants $A \leq B$ such that

$$A\|c\| \leq \|\sum_{x\in X} c(x)x\| \leq B\|c\| \quad \text{for all } c \in \ell_2(X).$$

We denote the partial inverse of a bounded operator Λ as $\Lambda^\dagger$, namely, the inverse of the map Λ restricted on $(\ker \Lambda)^\perp$ to its range. Then for a Riesz sequence X, $\|T_X\|$ and $\|T_X^\dagger\|^{-1}$ will be called the **upper**, respectively **lower Riesz bound** of X. A Riesz sequence X is called a **Riesz basis** if in addition X is fundamental.

The **analysis operator** associated with the system X is defined as

$$T_X^* : H \to \ell_2(X) : h \mapsto \{\langle h, x\rangle\}_{x \in X},$$

which is the unique adjoint operator of T_X. System X is a Bessel system if and only if T_X^* is bounded and the Bessel bound is $\|T_X^*\|$. In addition, a Bessel system X is fundamental if and only if T_X^* is injective. A Bessel system is called a **frame** if T_X^* is bounded below on H, or equivalently there exist two positive constants $A \leq B$ such that

$$A\|f\| \leq \left(\sum_{x \in X} |\langle f, x\rangle|^2\right)^{1/2} \leq B\|f\| \quad \text{for all } f \in H.$$

In this case, $\|T_X^*\|$ and $\|(T_X^*)^\dagger\|^{-1}$ are the **upper**, respectively **lower frame bound** of X, and X is a **tight frame** if those two bounds coincide (with default value to be 1 throughout the thesis if no specification). The frame property guarantees the numerical stable reconstruction from the coefficients given by the analysis operator. Note that when X is a frame, it is already fundamental. We say that X forms a **frame sequence** if it is a frame for a closed subspace of H, or equivalently any of the following criteria is satisfied.

Proposition 2.1.1. *Let X be a Bessel system. Then the following are equivalent:*

(a) $\operatorname{ran} T_X$ *is closed.*

(b) T_X *is bounded below on* $(\ker T_X)^\perp$.

(c) T_X^* *is onto* $(\ker T_X)^\perp$.

(d) T_X^* *is bounded below on* $(\ker T_X^*)^\perp$.

Proof. First to show (a) implies (b). It is clear that T_X is a bounded and one-to-one linear mapping from $(\ker T_X)^\perp$ to $\operatorname{ran} T_X$. Since $\operatorname{ran} T_X$ is closed by (a), with [101, Corollary 2.12], the partial inverse

$$T_X^\dagger : \operatorname{ran} T_X \to (\ker T_X)^\perp$$

is bounded, i.e. there exists $M > 0$ such that $\|T_X^\dagger f\| \le M\|f\|$ for $f \in \operatorname{ran} T_X$. Since $f \in \operatorname{ran} T_X$, there exists $c \in (\ker T_X)^\perp$ such that $T_X c = f$. Hence

$$\|c\| = \|T_X^\dagger T_X c\| \le M\|T_X c\|$$

which says T_X is bounded below on $(\ker T_X)^\perp$.

To show (b) implies (a). For a given sequence $\{f_n\} \in \operatorname{ran} T_X$ converging to $f \in H$, to show $f \in \operatorname{ran} T_X$, i.e. there exists $c \in (\ker T_X)^\perp$ such that $T_X c = f$. For any $f_n \in \operatorname{ran} T_X$, there exists $c_n \in (\ker T_X)^\perp$ satisfying $T_X c_n = f_n$. Since T_X is bounded below on $(\ker T_X)^\perp$, then there exists $M > 0$ and

$$\|c_n - c_m\| \le \frac{1}{M}\|T_X c_n - T_X c_m\| = \frac{1}{M}\|f_n - f_m\|.$$

That the sequence $\{f_n\}$ is Cauchy implies that $\{c_n\}$ is Cauchy. Hence $\{c_n\}$ converges to a point $c \in (\ker T_X)^\perp$. Since T_X is bounded, we have $T_X c = f$ which says $f \in \operatorname{ran} T_X$.

Since $\overline{\operatorname{ran}} T_X^* = (\ker T_X)^\perp$, that T_X^* is onto $(\ker T_X)^\perp$ is equivalent to that $\operatorname{ran} T_X^*$ is closed. As a result, the proof for the equivalence of (c) and (d) is analogous to that for the equivalence of (a) and (b).

Now we show (a) implies (d), i.e. if $\operatorname{ran} T_X$ is closed, then T_X^* is bounded below on $(\ker T_X^*)^\perp$. Since $\ker T_X^* = (\operatorname{ran} T_X)^\perp$ and $\operatorname{ran} T_X$ is closed, then $\operatorname{ran} T_X = (\ker T_X^*)^\perp$. By open mapping theorem [101], the bounded linear operator $T_X : (\ker T_X)^\perp \to (\ker T_X^*)^\perp$ is an open map, i.e. let U be the unit ball in $(\ker T_X)^\perp$ and V be the unit ball in $(\ker T_X^*)^\perp$, then there exists $\delta > 0$ such that $\delta V \subset T_X U$. For any $f \in (\ker T_X^*)^\perp$, we have

$$\begin{aligned}\|T_X^* f\| &= \sup\{\langle c, T_X^* f\rangle, c \in U\} = \sup\{\langle T_X c, f\rangle, c \in U\} \\ &\geq \sup\{\langle f_0, f\rangle, f_0 \in \delta V\} = \delta\|f\|.\end{aligned}$$

Lastly we show that (d) implies (a), i.e. if T_X^* is bounded below on $(\ker T_X^*)^\perp$, say by $\delta > 0$, then $\operatorname{ran} T_X$ is closed. Let a sequence $\{f_n\} \subset \operatorname{ran} T_X$ converging to f, to show there exist c such that $T_X c = f$. Let $c_n \in (\ker T_X)^\perp$ such that $T_X c_n = f_n$. Note in addition that T_X^* is onto $(\ker T_X)^\perp$ since (d) and (c) are equivalent. Then

$$\begin{aligned}\|c_n - c_m\| &= \sup_{d \in (\ker T_X)^\perp : \|d\| \leq 1} |\langle c_n - c_m, d\rangle| = \sup_{h : \|T_X^* h\| \leq 1} |\langle c_n - c_m, T_X^* h\rangle| \\ &\leq \sup_{h : \|h\| \leq 1/\delta} |\langle c_n - c_m, T_X^* h\rangle| = \sup_{h : \|h\| \leq 1/\delta} |\langle T_X c_n - T_X c_m, h\rangle| \\ &= \sup_{h : \|h\| \leq 1/\delta} |\langle f_n - f_m, h\rangle| = \frac{1}{\delta}\|f_n - f_m\|,\end{aligned}$$

which shows $\{c_n\}$ is Cauchy since $\{f_n\}$ is Cauchy. So there is c such that $c_n \to c$, and by the continuity of T_X, we have $T_X c = f$. □

2.2 Self-adjoint operators

In this section, we will show that the two self-adjoint operators, $\mathcal{R} := T_X^* T_X$ and $\mathcal{S} := T_X T_X^*$, could be conveniently used to characterize various properties of a given system X. Moreover, we will see that the operator $\mathcal{R}$ is naturally linked to the linear independence property of the system X while $\mathcal{S}$, usually referred to as the frame operator, is linked to the redundancy property of the system.

Theorem 2.2.1. *Suppose system X is Bessel in H. Then*

(a) *System X is ℓ_2-independent if and only if $\mathcal{R}$ is injective.*

(b) *System X is fundamental if and only if $\mathcal{S}$ is injective.*

Proof. The proofs of (a) and (b) are analogous due to the similar structure of $\mathcal{R}$ and $\mathcal{S}$. We show in details the proof of (b). If Bessel system X is fundamental, then T_X^* is injective. The condition $\mathcal{S}f = T_X T_X^* f = 0$ implies $T_X^* f \in \ker T_X$. With $\ker T_X = (\operatorname{ran} T_X^*)^\perp$, we have $T_X^* f = 0$. The injectivity of T_X^* implies $f = 0$, which concludes that $\mathcal{S}$ is injective. Conversely, suppose $\mathcal{S}$ is injective. For $T_X^* f = 0$, we have $T_X T_X^* f = 0$. Then $f = 0$ by the injective of $\mathcal{S}$. So this gives that T_X^* is injective, and hence system X is fundamental. □

Theorem 2.2.2. *Let system X be a Bessel system in H. Then*

(a) *System X forms a Riesz sequence with lower bound A if and only if $\mathcal{R}$ is invertible and the inverse is bounded by A^{-2}.*

(b) *System X forms a frame with lower bound A if and only if $\mathcal{S}$ is invertible and the inverse is bounded by A^{-2}.*

Proof. We show the proof of (b). If Bessel system X is a frame, i.e. T_X^* is bounded below by A, then

$$A^2\|f\|^2 \leq \|T_X^* f\|^2 = \langle T_X T_X^* f, f\rangle \leq \|T_X^* T_X f\|\|f\|.$$

Since T_X^* is bounded, say by B, then we have in addition that

$$A^2\|f\| \leq \|\mathcal{S}f\| \leq B^2\|f\|.$$

So by [101, Theorem 12.12], $\mathcal{S}$ is invertible and the inverse bound can be obtained from the lower bound.

Conversely, if $\mathcal{S}$ is invertible, then T_X is onto H. By Proposition 2.1.1, T_X^* is bounded below on $(\ker T_X^*)^{\perp}$. Since $H = \operatorname{ran} T_X \subset (\ker T_X^*)^{\perp}$, thus $H = (\ker T_X^*)^{\perp}$. This gives that T_X^* is bounded below on H, or equivalently that X is a frame. □

Theorem 2.2.3. *Let X be a Bessel system. Then*

(a) *System X is an orthonormal sequence if and only if $\mathcal{R} = I$.*

(b) *System X is a tight frame if and only if $\mathcal{S} = I$.*

Proof. We only show the proof of (b). If $\mathcal{S} = I$, then

$$\langle f, f\rangle = \langle \mathcal{S} f, f\rangle = \langle T_X^* f, T_X^* f\rangle$$

Hence $\|T_X^* f\| = \|f\|$ which implies that X is a tight frame.

Conversely, if X is a tight frame, for $f, g \in H$, we have

$$\|T_X^* f + T_X^* g\|^2 = \|f + g\|^2$$

Thus

$$\|T_X^* f\|^2 + 2\mathrm{Re}\,\langle T_X^* f, T_X^* g\rangle + \|T_X^* g\|^2 = \|f\|^2 + 2\mathrm{Re}\,\langle f, g\rangle + \|g\|^2,$$

which gives

$$\mathrm{Re}\,\langle T_X^* f, T_X^* g\rangle = \mathrm{Re}\,\langle f, g\rangle.$$

By taking if, ig instead of f, g and going through the same calculation, we have

$$\mathrm{Im}\,\langle T_X^* f, T_X^* g\rangle = \mathrm{Im}\,\langle f, g\rangle.$$

This shows

$$\langle T_X T_X^* f, g\rangle = \langle T_X^* f, T_X^* g\rangle = \langle f, g\rangle.$$

Since g is arbitrary, we then have $T_X T_X^* f = f$, i.e. $\mathcal{S} = I$. $\square$

2.3 Mixed operators

Suppose X and $Y = RX$ are Bessel systems in H where R denotes the indexing by system X, we now study the properties of systems X and Y. If $T_Y T_X^*$ or its adjoint $T_X T_Y^*$ is the identity of H, then X and Y are called a pair of **dual frames** in H, and Y is called a dual frame of X. Note that

$$\|h\|^2 = \langle T_Y T_X^* h, h\rangle = \langle T_X^* h, T_Y^* h\rangle \leq \|T_Y^*\|\|h\|\|T_X^* h\|$$

for any $h \in H$, which shows that X, and by symmetry of the situation also Y, is a frame in H.

Given a frame X for H, it is well known that X and $\mathcal{S}^{-1}X$, where $\mathcal{S} = T_X T_X^*$, are dual frames since $\mathcal{S}^{-1}$ is self-adjoint and therefore $T^*_{\mathcal{S}^{-1}X} = T_X^* \mathcal{S}^{-1}$. The system $\mathcal{S}^{-1}X$ is called the **canonical dual frame** of X. In particular, a tight frame has itself as the canonical dual frame. The canonical dual frame $\mathcal{S}^{-1}X$ is distinguished from any other dual frame RX by several properties. For example, $\mathcal{S}^{-1}X$ is the unique dual frame to make the projector $T^*_{RX} T_X$ an orthogonal projector, see [97]. Also, $\|T^*_{\mathcal{S}^{-1}X} f\| \leq \|T^*_{RX} f\|$ for any $f \in H$, see e.g. [35, 68]. Moreover, $\mathcal{S}^{-1}$ is the only self-adjoint operator among all dual frame maps R. That is, if X is a frame in H and RX is a dual frame, then RX is the canonical dual frame if and only if

$$\langle x, Rx'\rangle = \langle Rx, x'\rangle \quad \text{for all } x, x' \in X, \tag{2.1}$$

see [97]. The canonical dual frame can also be used to verify the independence properties of the system. Specifically, see [97], if X is a frame in H, then $\langle x, \mathcal{S}^{-1}x\rangle \leq 1$ for all $x \in X$

and X is a Riesz basis if and only if

$$\langle x, \mathcal{S}^{-1}x\rangle = 1 \quad \text{for all } x \in X. \tag{2.2}$$

In particular, a tight frame is an orthonormal basis if and only if all its elements have unit norm.

We now present some facts about the **mixed operators** $T_Y T_X^*$, $T_X T_Y^*$, $T_Y^* T_X$ and $T_X^* T_Y$. The first is in the spirit of the canonical dual frame.

Proposition 2.3.1. *Let X and $Y = RX$ be frames for H such that* $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$. *Then $T_Y T_X^*$ is boundedly invertible and $(T_Y T_X^*)^{-1} Y$ and X are dual frames.*

Proof. We have

$$\operatorname{ran} T_Y T_X^* = \operatorname{ran}(T_Y|_{\operatorname{ran} T_X^*}) = \operatorname{ran}(T_Y|_{\operatorname{ran} T_Y^*}) = \operatorname{ran}(T_Y|_{(\ker T_Y)^\perp}) = \operatorname{ran} T_Y = H.$$

To show the injectivity of $T_Y T_X^*$, let $f \in H$ such that $T_Y T_X^* f = 0$. Then $T_X^* f \in \ker T_Y = (\operatorname{ran} T_Y^*)^\perp = (\operatorname{ran} T_X^*)^\perp$ which gives $T_X^* f = 0$. Since X is a frame, T_X^* is injective, and thus $f = 0$, showing that $T_Y T_X^*$ is injective. A similar proof shows that $T_X T_Y^*$ is also invertible. Thus by open mapping theorem [101], $T_Y T_X^*$ and $T_X T_Y^*$ are boundedly invertible on H and, denoting $Q = (T_Y T_X^*)^{-1} R$, we have

$$T_{QX}^* h = \{\langle h, (T_Y T_X^*)^{-1} R x\rangle\}_{x \in X}$$

for any $h \in H$, i.e. $T_{QX}^* = T_Y^* (T_X T_Y^*)^{-1}$. Therefore, $T_X T_{QX}^*$ is the identity on H. $\square$

Proposition 2.3.2. *Let X and $Y = RX$ be Bessel systems in H such that* $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$. *Then the following are equivalent:*

(a) *X and Y are frames.*

(b) *$T_Y T_X^*$ and $T_X T_Y^*$ are bounded below.*

Proof. Suppose X and Y are frames in H. Then T_X is bounded below on $(\ker T_X)^\perp = \operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ and T_Y^* is bounded below on H. Thus, for every $h \in H$, we have

$$\|T_X^\dagger\|^{-1}\|(T_Y^*)^\dagger\|^{-1}\|h\| \leq \|T_X^\dagger\|^{-1}\|T_Y^* h\| \leq \|T_X T_Y^* h\|.$$

Hence $T_X T_Y^*$ is bounded below on H and similarly so is $T_Y T_X^*$. Conversely, since $T_X T_Y^*$ is bounded below together with T_X being bounded, then T_Y^* is bounded below which gives that Y is a frame. By the same argument X is a frame. □

Note that the assumption $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ in Proposition 2.3.1 and 2.3.2 is essential. There do exist frames X and $Y = RX$ such that $T_Y T_X^* = T_X T_Y^* = 0$ which are called orthogonal frames, see [76]. Note that $\operatorname{ran} T_X^* = \operatorname{ran} T_Y^*$ is not needed in Proposition 2.3.2 for (b) to imply (a) but (b) does not imply this condition. Indeed, the canonical dual of a frame X is the only Bessel system $R'X$ in H for which $T_{R'X} T_X^* = I$ and $\operatorname{ran} T_X^* = \operatorname{ran} T_{R'X}^*$, see [94].

Proposition 2.3.3. *Let X and $Y = RX$ be Bessel systems in H such that $\operatorname{ran} T_X = \operatorname{ran} T_Y$. Then the following are equivalent:*

(a) *X and Y are Riesz sequences.*

(b) *$T_Y^* T_X$ and $T_X^* T_Y$ are bounded below.*

Proof. If $T_Y^* T_X$ is bounded below and T_Y^* is bounded, then T_X is bounded below which implies X is a Riesz sequence. On the other hand, if X and Y are Riesz sequences, then T_X is bounded below and T_Y^* is bounded below on $(\ker T_Y^*)^\perp = (\ker T_X^*)^\perp$. Therefore $T_Y^* T_X$ is bounded below. □

2.4 Restricting coefficient space

One question about reducing the redundancy of a frame can be posed by asking whether every frame contains a Riesz basis. The question has a negative answer, since the vectors of a Riesz basis are necessarily bounded and bounded below away from zero in norm. Thus, if $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis, then $\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \ldots\}$ where $\frac{1}{\sqrt{n}}e_n$ appears n times is a tight frame which does not contain a Riesz basis. Counterexamples still exist if one does not allow frames which contain a subsequence converging to zero in norm [14, 104]. In other words, it is in general not possible to choose a coordinate subspace of the coefficient space $\ell_2(X)$ of a tight frame X, such that the restriction of T_X to this subspace becomes injective while still being onto. Here by a **coordinate subspace** of $\ell_2(X)$ we mean any subspace of the form $\overline{\operatorname{span}}\{e_x\}_{x\in Y}$ where $Y \subset X$ and $e_x \in \ell_2(X)$ is the standard unit vector given by $e_x(x') = \delta_{x,x'}$. That is, the coordinate subspaces are those subspaces that can be identified with $\ell_2(Y)$ for some $Y \subset X$.

The situation drastically changes if one considers arbitrary subspaces of the coefficient space. Given a Bessel system X, the question becomes whether there is some subspace $S \subset \ell_2(X)$ such that $T_X|_S$ is bounded below, i.e. such that

$$\|(T_X|_S)^{-1}\|^{-1}\|c\| \leq \left\| \sum_{x\in X} c(x)x \right\| \leq \|T_X|_S\|\|c\| \quad \text{for all } c \in S. \tag{2.3}$$

Note that a system X is a frame sequence if (2.3) holds for all $c \in (\ker T_X)^\perp$, that X contains a Riesz sequence if (2.3) holds for a coordinate subspace $\ell_2(X)$ and that X is a Riesz sequence if (2.3) holds for $S = \ell_2(X)$.

If X is a frame in H, then $T_X T_X^*$ is bounded below and onto, thus $T_X|_{\operatorname{ran} T_X^*}$ is bounded below and onto, and one can choose $S = \operatorname{ran} T_X^*$. Moreover, $\operatorname{ran} T_X^*$ is exactly the space of coefficients needed for X to span H, so we really may restrict our attention to precisely this subspace of $\ell_2(X)$. In effect, in this view the distinction between frame and Riesz property vanishes and in this sense one can always make a redundant system non-redundant by considering it on a smaller coefficient space. The more redundant a system is, the fewer coefficients one needs to represent the whole space since $\ker T_X$ gets larger while $\operatorname{ran} T_X^*$ gets smaller.

We now turn to the restriction of the coefficient space to some subspace of $\ell_2(X)$ through an orthogonal projection. That is, consider the operator $T_X P_S T_X^*$ where X is a Bessel system in H, S is a subspace of $\ell_2(X)$ and P_S is the orthogonal projection onto S. If $\operatorname{ran} T_X^* \subset S$, then $T_X P_S T_X^* = T_X T_X^*$ and one has the classical situation with T_X acting on the whole coefficient space $\operatorname{ran} T_X^*$. In general, decompose S into the orthogonal direct sum $S = S_1 \oplus S_2$, where $S_1 = S \cap \operatorname{ran} T_X^*$ and S_2 is the orthogonal complement of S_1 in S. Then $S_1 \subset \operatorname{ran} T_X^*$ and $S_2 \subset \ker T_X$, i.e. $T_X P_{S_2} T_X^*$ is identical to zero.

Operators of the form $T_X P_S T_X^*$ arise in many contexts. Let, say, X be a tight frame in H, let $Y \subset X$ and $M(Y) = \sum_{x \in Y} \langle \cdot, x \rangle x$. Then the mapping M defined on the power set of X is a simple example of a positive operator valued measure, a notion playing a major role in quantum information theory, describing generalized measurements [102]. In general, $M(Y)$ is not an orthogonal projection. Indeed, letting $P(Y)$ be the orthogonal projection of $\ell_2(X)$ onto the subspace $\overline{\operatorname{span}}\{e_x\}_{x \in Y}$, then $M(Y) = T_X P(Y) T_X^*$, which is an instance of Naimark's dilation theorem (see e.g. [89]). The mapping P defines a projection valued measure, describing the standard measurements in quantum theory. We now show that $T_X P_S T_X^*$ is an orthogonal projection whenever X is a tight frame for certain subspaces related to S.

Proposition 2.4.1. *Let X be a fundamental Bessel system in H and S a subspace of $\ell_2(X)$. Then the following are equivalent:*

(a) *$T_X T_X^*$ is the identity on $E_1 = \{f \in H : T_X^* f \in S\}$.*

(b) *$T_X T_X^*$ is the identity on $E_2 = \operatorname{ran} T_X|_S$.*

If (a) (or (b)) holds, then $T_X P_S T_X^*$ is an orthogonal projection with range E_1 and $E_1 = E_2$.

Proof. With S decomposed as above, notice that $T_X P_S T_X^* = T_X P_{S_1} T_X^* + T_X P_{S_2} T_X^* = T_X P_{S_1} T_X^*$. Hence, $\operatorname{ran} T_X|_S = \operatorname{ran} T_X|_{S_1}$ and $T_X T_X^*$ is the identity on E_1 if and only if it is the identity on $\{f \in H : T_X^* f \in S_1\}$. Therefore, assume as we may, that $S \subset \operatorname{ran} T_X^*$.

(a) $\Rightarrow$ (b): Note that $T_X T_X^*$ maps E_1 onto E_2 and is the identity on E_1. So these two sets coincide. (b) $\Rightarrow$ (a): If $T_X^* h \in S$, then $(T_X T_X^*) T_X T_X^* h = (T_X T_X^*) h$, i.e. $T_X T_X^* h - h \in \ker T_X T_X^*$. Since X is fundamental, hence $H = \overline{\operatorname{ran}} T_X = (\ker T_X^*)^\perp = (\ker T_X T_X^*)^\perp$, which gives $\ker T_X T_X^* = \{0\}$.

Now assume (a). As $T_X P_S T_X^*$ is self-adjoint it remains to show that it is the identity on its range. If $h \in E_1$, then $T_X P_S T_X^* h = T_X T_X^* h = h$ and thus $E_1 \subset \operatorname{ran} T_X P_S T_X^*$. It therefore remains to show $\operatorname{ran} T_X P_S T_X^* \subset E_1$. To this end, let $h \in \operatorname{ran} T_X P_S T_X^*$, say $h = T_X P_S T_X^* g$. Since $S \subset \operatorname{ran} T_X^*$, we have $P_S T_X^* g = T_X^* f$ for some $f \in E_1$. Therefore, by (b),

$$T_X^* h = T_X^* T_X P_S T_X^* g = T_X^* T_X T_X^* f = T_X^* f \in S,$$

i.e. $h \in E_1$. $\square$

Without the assumption on X to be fundamental, (b) does not imply (a). Take, say, $H = \mathbb{R}^3$ with the standard orthonormal basis $\{e_1, e_2, e_3\}$ and $X = \{e_1, e_2\}$. Let $S = \operatorname{ran} T_X^* = \mathbb{R}^2$. Then $T_X T_X^*$ is the identity on E_2 but not on $E_1 = \mathbb{R}^3$. Merely under the assumption on X being fundamental, E_1 in general does not coincide with E_2 (or its closure). Take $H = \mathbb{R}^2$ and $X = \sqrt{2/5}\{(0,1)^\top, (1,0)^\top, (1,1)^\top\}$. Let $S = \operatorname{span}\{(0,1,1)^\top\} \subset \operatorname{span}\{(1,0,1)^\top, (0,1,1)^\top\} = \operatorname{ran} T_X^*$. Then $E_1 = \operatorname{span}\{(1,0)^\top\}$ but $E_2 = \operatorname{span}\{(2,1)^\top\}$. Note also that in this example $T_X P_S T_X^*$ is the orthogonal projection onto E_2, i.e. in the above result, the assumption $T_X P_S T_X^*$ to be an orthogonal projection does not imply (a) (or (b)).

According to Proposition 2.4.1, the operator $T_X P_S T_X^*$ is a projection whenever $T_X T_X^*$ is the identity on certain subspaces of H, i.e. when X is a tight frame for certain subspaces of H. We now look at the weaker condition when $T_X T_X^*$ is bounded below on the aforementioned subspaces. Note however, that if S is a proper subspace of $\operatorname{ran} T_X^*$, we can choose a nonzero $h \in H$ such that $T_X^* h \in (\operatorname{ran} T_X^*) \ominus S$. Then $T_X P_S T_X^* h = 0$. Thus, in this case $T_X P_S T_X^*$ cannot be bounded below. However, if X is a fundamental Bessel system in H and $S \subset \operatorname{ran} T_X^*$, then $T_X T_X^*$ is bounded below on E_1 if and only if $(T_X T_X^*|_{E_1})^{-1}$ is bounded on E_2.

Chapter 3

Dual Gramian analysis

In this chapter, we will introduce the dual Gramian matrix of a given system in a Hilbert space and its corresponding analysis to the frame property of the system. For a given system X, the key to the dual Gramian analysis is to find a pre-Gramian matrix J_X that represents the synthesis operator T_X by only the elements of X and the corresponding adjoint J_X^* satisfies the following identity with a unitary operator $U : H \to \ell_2$:

$$\|T_X^* f\|^2 = \|J_X^* U f\|^2 = (Uf)^* \widetilde{G}_X (Uf), \quad \text{for } f \in H,$$

where $\widetilde{G}_X := J_X J_X^*$ denotes the dual Gramian. With this representation, one hopefully can characterize various properties of the system X in terms of its elements. The dual Gramian matrix can also be conveniently used for constructing the canonical dual frame, and estimating the frame bounds.

The matrix representations could be further simplified as soon as the given system exhibits some structure. In [94, 96–98] for general and particular shift-invariant systems in $L_2(\mathbb{R}^d)$, the synthesis and analysis operators are represented by a continuum of matrices, the so-called *fibers*, instead of just one matrix. Properties of the analysis and synthesis operators can then be characterized by properties of the fibers, which have to hold in a uniform way. Connection of abstract pre-Gramian and the fiber pre-Gramian matrices will be shown by designing an appropriate orthonormal basis adapting to the shift-invariant structure. Gabor systems are a particular well-structured class of shift-invariant systems and the mixed dual Gramian analysis, which is a generalization from single system analysis to dual (or bi-) systems analysis, is developed in complement to [98].

3.1 Definitions

Depending on the structure of the given system and the associated underlying Hilbert space, there are many ways to define a pre-Gramian matrix. In general, for a given system X of a Hilbert space H, the **pre-Gramian matrix** J_X of X associated with an orthonormal basis $\mathcal{O}$ in H is defined as

$$J_X := (\langle x, e\rangle)_{e\in\mathcal{O}, x\in X} \tag{3.1}$$

where the rows are indexed by $\mathcal{O}$ and the columns are indexed by X, and the (e, x)-entry is the inner product of x with e.

Note that matrix J_X is dependent on the orthonormal basis $\mathcal{O}$ chosen, and is infinite when either the index $\mathcal{O}$ or X is infinite. The matrix naturally defines an operator:

$$\ell_2(X) \to \ell_2(\mathcal{O}) : c \mapsto \left(\sum_{x\in X} c(x)\langle x, e\rangle\right)_{e\in\mathcal{O}}$$

which is well-defined on $\ell_0(X)$. In order for each entry $\sum_{x\in X} c(x)\langle x, e\rangle$ to be well defined for $c \in \ell_2(X)$, we need

$$\sum_{x\in X} |\langle x, e\rangle|^2 < \infty \tag{3.2}$$

for all $e \in \mathcal{O}$. We will use J_X to denote both this operator and the pre-Gramian matrix.

Note the condition (3.2) will always be satisfied for any choice of orthonormal basis if X is a Bessel system. The adjoint matrix

$$J_X^* = (\langle e, x\rangle)_{x\in X, e\in\mathcal{O}}$$

will also denote the corresponding operator on $\ell_2(\mathcal{O})$.

Let X be a system in H and assume (3.2) for X with respect to an orthonormal basis $\mathcal{O}$ of H. The **Gramian matrix** of X is defined as

$$G_X := J_X^* J_X = \left(\sum_{e\in\mathcal{O}} \langle x', e\rangle\langle e, x\rangle\right)_{x\in X, x'\in X} = (\langle x', x\rangle)_{x\in X, x'\in X}, \tag{3.3}$$

and its **dual Gramian matrix** (with respect to $\mathcal{O}$) as

$$\widetilde{G}_X := J_X J_X^* = \left(\sum_{x\in X} \langle x, e\rangle\langle e', x\rangle\right)_{e\in\mathcal{O}, e'\in\mathcal{O}}. \tag{3.4}$$

The entries of the Gramian matrix are well-defined since $\mathcal{O}$ is an orthonormal basis of H and the entries of the dual Gramian matrix are well-defined since X satisfies the condition (3.2). The last equality in (3.3), which follows again from the fact that $\mathcal{O}$ is an orthonormal basis of H, shows that this definition coincides with the traditional definition of Gramian matrix of a given system X. Hence, the definition of the Gramian matrix is independent of the choice of the orthonormal basis $\mathcal{O}$. If given another system RX satisfies (3.2) with respect to the same orthonormal basis $\mathcal{O}$, the **mixed Gramian matrix** of X and RX is defined as

$$G_{RX,X} := J_{RX}^* J_X = (\langle x', Rx\rangle)_{x\in X, x'\in X}, \tag{3.5}$$

and their **mixed dual Gramian matrix** (with respect to $\mathcal{O}$) as

$$\widetilde{G}_{RX,X} := J_{RX} J_X^* = \left(\sum_{x\in X} \langle Rx, e\rangle\langle e', x\rangle\right)_{e\in\mathcal{O}, e'\in\mathcal{O}}. \tag{3.6}$$

Note the (dual) Gramian matrix is Hermitian but the mixed (dual) Gramian matrix is generally not. The finite dimensional situation is a simple example to illustrate the matrices defined.

Example 3.1.1. *Let H be the finite dimensional Hilbert space $\mathbb{C}^m$ and let $\{e_i\}_{i=1}^m$ denote its canonical orthonormal basis. Let $X = \{x_k\}_{k=1}^n \subset \mathbb{C}^m$. Then, the pre-Gramian matrix of X is*

$$J_X = \begin{pmatrix} x_1(1) & \cdots & x_n(1) \\ \vdots & \ddots & \vdots \\ x_1(m) & \cdots & x_n(m) \end{pmatrix}$$

which is the matrix representation of the synthesis operator associated with X

$$T_X : \ell_2(X) \to \mathbb{C}^m : c \mapsto \sum_{k=1}^{n} c_k x_k.$$

Similarly, the adjoint matrix J_X^*

$$J_X^* = \begin{pmatrix} \overline{x_1(1)} & \cdots & \overline{x_1(m)} \\ \vdots & \ddots & \vdots \\ \overline{x_n(1)} & \cdots & \overline{x_n(m)} \end{pmatrix}$$

is the matrix representation of the analysis operator

$$T_X^* : \mathbb{C}^m \to \ell_2(X) : f \mapsto \{\langle f, x_k \rangle\}_{k=1}^n.$$

Its corresponding Gramian matrix and the dual Gramian matrix are

$$G_X = J_X^* J_X = (\langle x_{k'}, x_k \rangle)_{k,k'}, \quad \widetilde{G}_X = J_X J_X^* = \left(\sum_{k=1}^{n} x_k(j) \overline{x_k(j')} \right)_{j,j'}$$

which are the matrix representations of the linear operators $T_X^* T_X$ and $T_X T_X^*$ respec-

tively.

If in addition, there is another system $Y = \{y_k\}_{k=1}^n \subset \mathbb{C}^m$ *and* J_Y *is the associated pre-Gramian matrix, then the mixed Gramian matrix and mixed dual Gramian matrix are*

$$G_{X,Y} = J_X^* J_Y = (\langle y_{k'}, x_k\rangle)_{k,k'}\,, \quad \widetilde{G}_{X,Y} = J_X J_Y^* = \left(\sum_{k=1}^n x_k(j)\overline{y_k(j')}\right)_{j,j'}$$

which are then the matrix representations of the mixed operators $T_X^* T_Y$ *and* $T_X T_Y^*$ *respectively.*

3.2 Analysis

In order to link the (mixed) dual Gramian matrix $\widetilde{G}_X$ to the (mixed) frame operator, we need the synthesis operator corresponding to the orthonormal basis $\mathcal{O}$ of H, denoted as U, which is the unitary operator given by

$$U : \ell_2(\mathcal{O}) \to H : c \mapsto \sum_{e\in\mathcal{O}} c(e)e.$$

The adjoint operator counterpart is the analysis operator

$$U^* : H \to \ell_2(\mathcal{O}) : f \mapsto \{\langle f, e\rangle\}_{e\in\mathcal{O}}.$$

The unitary operator U maps the sequence space $\ell_2(\mathcal{O})$ to H and the adjoint operator U^* maps H to the sequence space $\ell_2(\mathcal{O})$. Using this unitary operator U, the link between the pre-Gramian matrix of X and the synthesis operator T_X of X is stated as follows.

Proposition 3.2.1. *Let* X *be a given system in* H *and let* $\mathcal{O}$ *be an orthonormal basis of* H*. Assume that* X *and* $\mathcal{O}$ *satisfy (3.2). Then we have*

$$T_X c = U J_X c \quad \textit{for any } c \in \ell_0(X), \tag{3.7}$$

and

$$T_X^* U d = J_X^* d \quad \textit{for any } d \in \ell_0(\mathcal{O}). \tag{3.8}$$

Consequently, X is a Bessel system if and only if J_X (or J_X^*) is bounded. The Bessel bound equals $\|J_X\| = \|J_X^*\|$. A Bessel system X is a Riesz sequence (resp. frame) if and only if J_X (resp. J_X^*) is bounded below.

Proof. For any $c \in \ell_0(X)$, we have

$$UJ_X c = \sum_{e\in\mathcal{O}} \sum_{x\in X} c(x)\langle x, e\rangle e = \sum_{x\in X} c(x) \sum_{e\in\mathcal{O}} \langle x, e\rangle e = \sum_{x\in X} c(x) x = T_X c.$$

In the above derivation, the sequence $(\sum_{x\in X} c(x)\langle x, e\rangle)_{e\in\mathcal{O}}$ is in $\ell_2(\mathcal{O})$ since $c \in \ell_0(X)$ and $\mathcal{O}$ is an orthonormal basis. The summation order can be changed because the summation indexed by X is finite for $c \in \ell_0(X)$.

To prove (3.8), for any $d \in \ell_0(\mathcal{O})$, we have

$$T_X^* U d = (\langle Ud, x\rangle)_{x\in X} = \left(\langle \sum_{e\in\mathcal{O}} d(e)e, x\rangle\right)_{x\in X} = \left(\sum_{e\in\mathcal{O}} d(e)\langle e, x\rangle\right)_{x\in X} = J_X^* d.$$

Notice that $J_X^* d \in \ell_2(X)$, because X and $\mathcal{O}$ satisfy (3.2). With the two relationships (3.7) and (3.8), the characterizations of various properties of the system X can be transferred from the synthesis operator T_X and the analysis operator T_X^* to the corresponding pre-Gramian matrix J_X and its adjoint J_X^*. Hence, the rest of the results follow from the definitions of Bessel systems, Riesz sequences or frames that are given in terms of the operator T_X or T_X^*. □

It is noted that the proof of (3.7) does not require the assumption (3.2). However, the assumption (3.2) makes the matrix-vector product $J_X c$ well-defined for any vector c in $\ell_2(X)$. As a result, the matrix J_X can be formally used to define an operator on $\ell_2(X)$, but it may not map to $\ell_2(\mathcal{O})$. The connection of (dual) Gramian matrix to operators $T_X^* T_X$ and $T_X T_X^*$ is established as follows.

Proposition 3.2.2. *Let X be a given system in H and let $\mathcal{O}$ be an orthonormal basis of H. Assume that X and $\mathcal{O}$ satisfy (3.2). Then we have*

$$\langle T_X c, T_X d\rangle = d^* G_X c \quad \text{for any } c, d \in \ell_0(X), \tag{3.9}$$

and

$$\langle T_X^* U c, T_X^* U d\rangle = d^* \widetilde{G}_X c \quad \text{for any } c, d \in \ell_0(\mathcal{O}). \tag{3.10}$$

Furthermore, X is a Bessel system if and only if the Gramian matrix G_X (resp. the dual Gramian matrix $\widetilde{G}_X$) defines a bounded operator of $\ell_2(X)$ (resp. $\ell_2(\mathcal{O})$). The Bessel bound equals to $\|G_X\|^{1/2} = \|\widetilde{G}_X\|^{1/2}$. If the system X is Bessel, we have

$$T_X^* T_X c = G_X c \quad \text{for any } c \in \ell_2(X),$$

$$U^* T_X T_X^* U d = \widetilde{G}_X d \quad \text{for any } d \in \ell_2(\mathcal{O}).$$

Proof. Let $c, d \in \ell_0(X)$. Then

$$\langle T_X c, T_X d\rangle = \langle \sum_{x' \in X} c(x') x', \sum_{x \in X} d(x) x\rangle = \sum_{x \in X} \overline{d(x)} \sum_{x' \in X} c(x') \langle x', x\rangle = d^* G_X c.$$

For $c, d \in \ell_0(\mathcal{O})$ we get

$$\begin{aligned}
\langle T_X^* U c, T_X^* U d\rangle &= \sum_{x \in X} \langle U c, x\rangle \langle x, U d\rangle \\
&= \sum_{e \in \mathcal{O}} \overline{d(e)} \sum_{e' \in \mathcal{O}} c(e') \sum_{x \in X} \langle e', x\rangle \langle x, e\rangle \\
&= d^* \widetilde{G}_X c.
\end{aligned}$$

□

From (3.10), by taking limit, we conclude that

$$\|T_X^* U c\|^2 = c^* \widetilde{G}_X c \quad \text{for arbitrary } c \in \ell_2(\mathcal{O}),$$

although both sides may equal infinity for some cases. In fact, it shows that both the upper bound and the lower bound of the operator T_X^*, which is equivalent to the frame property of X, can be characterized by the bounds of the nonnegative Hermitian matrix $\widetilde{G}_X$. Since the Bessel property has already been characterized by the upper bounds of $\widetilde{G}_X$ or G_X in Proposition 3.2.2, the following proposition characterizes the lower bound of frames and Riesz sequences in terms of the dual Gramian and Gramian matrices.

Proposition 3.2.3. *Let X be a Bessel system in H and let $\mathcal{O}$ be an orthonormal basis of H. Then*

(a) *X is ℓ_2-independent if and only if G_X is injective. X forms a Riesz sequence if and only if G_X has a bounded inverse and the lower Riesz bound is $\|G_X^{-1}\|^{-1/2}$. X is an orthonormal sequence if and only if $G_X = I$.*

(b) *X is fundamental if and only if $\widetilde{G}_X$ is injective. X is a frame if and only if $\widetilde{G}_X$ has a bounded inverse and the lower frame bound is $\|\widetilde{G}_X^{-1}\|^{-1/2}$. X is a tight frame if and only if $\widetilde{G}_X = I$.*

Proof. If the system X is Bessel, then by Proposition 3.2.2, we have

$$T_X^* T_X = G_X, \quad U^* T_X T_X^* U = \widetilde{G}_X.$$

Hence (a) and (b) follow immediately from the characterization by $T_X^* T_X$ and $T_X T_X^*$. $\square$

The properties of a frame can also be characterized by the Gramian matrix. In general, as an operator, the Gramian matrix has a non-trivial null set for a frame system. Thus, the analysis of frame properties via Gramian matrix involves the partial inverse and its boundedness, see [94] for the details of the characterization of a frame via Gramian matrix. While the Gramian matrix is very handy for studying Riesz and orthonormal properties of a system, the dual Gramian matrix is more convenient for studying the frame and tight frame properties of a system.

Note that when X is a Bessel system with an upper bound B, the summation $\sum_{x\in X}|\langle x,e\rangle|^2$ is uniformly bounded by B^2 for all $e\in\mathcal{O}$. Hence the condition (3.2) holds. Furthermore, the elements $\{\sum_{x\in X}|\langle x,e\rangle|^2, e\in\mathcal{O}\}$ form the diagonal entries of the dual Gramian matrix $\widetilde{G}_X$. Hence, the necessary condition for X being a tight frame is $\sum_{x\in X}|\langle x,e\rangle|^2=1$ for all $e\in\mathcal{O}$ and it becomes sufficient when X is a Bessel system with bound 1.

Proposition 3.2.4. *Let X be a given system in H and let $\mathcal{O}$ be an orthonormal basis of H. Assume X is a Bessel system of H with bound 1. Then the system X is a tight frame if and only if*

$$\sum_{x\in X}|\langle x,e\rangle|^2=1 \quad \textit{for all } e\in\mathcal{O}. \tag{3.11}$$

Proof. The necessity part is easy to see, as each element $\sum_{x\in X}|\langle x,e\rangle|^2$ is one of the diagonal entries of $\widetilde{G}_X$. For the sufficiency part, consider the sequence $c\in\ell_2(\mathcal{O})$ whose e'-th element has value 1 and others have value 0. Then $\widetilde{G}_X c$ gives the e'-th column of matrix $\widetilde{G}_X$. By Proposition 3.2.2 and the fact that X is a Bessel system with bound 1, we have $\|\widetilde{G}_X c\|\le 1$. Moreover,

$$\begin{aligned}
\|\widetilde{G}_X c\|^2 &= \|\{\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle\}_{e\in\mathcal{O}}\|^2=\left(\sum_{x\in X}|\langle x,e'\rangle|^2\right)^2+\sum_{e\in\mathcal{O}\setminus\{e'\}}|\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle|^2\\
&= 1+\sum_{e\in\mathcal{O}\setminus e'}|\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle|^2,
\end{aligned}$$

which implies that

$$\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle=0, \quad \text{for } e\in\mathcal{O}\setminus\{e'\},$$

Hence the dual Gramian matrix $\widetilde{G}_X=I$, and therefore X is a tight frame by Proposition 3.2.3. □

As a direct application of Proposition 3.2.4, an orthonormal sequence X is clearly a Bessel system with bound 1 and it becomes an orthonormal basis if it satisfies the additional condition (3.11) for some orthonormal basis $\mathcal{O}$, i.e. X is also fundamental. For a general Bessel system X with bound B, following the same argument as Proposition 3.2.4, the condition $\sum_{x\in X}|\langle x,e\rangle|^2 = B^2$ for all $e \in \mathcal{O}$ implies that the system is a tight frame with bound B, i.e. $\sum_{x\in X}|\langle f,x\rangle|^2 = B^2\|f\|^2$ for all $f \in H$.

Given another system RX in H which is indexed by X, the mixed (dual) Gramian matrices are connected to $T^*_{RX}T_X$ and $T_{RX}T^*_X$ as follows.

Proposition 3.2.5. *Let X and RX be systems in H both of which satisfy (3.2) with respect to an orthonormal basis $\mathcal{O}$. Suppose the mixed Gramian $G_{RX,X}$ and mixed dual Gramian $\widetilde{G}_{RX,X}$ are both defined with respect to $\mathcal{O}$ and let U be the synthesis operator of $\mathcal{O}$. Then*

$$\langle T_X c, T_{RX} d\rangle = d^* G_{RX,X} c \quad \text{for all } c,d \in \ell_0(X), \tag{3.12}$$

and

$$\langle T^*_X U c, T^*_{RX} U d\rangle = d^* \widetilde{G}_{RX,X} c \quad \text{for all } c,d \in \ell_0(\mathcal{O}). \tag{3.13}$$

Further, if X and RX are Bessel systems, then $G_{RX,X}$ defines a bounded operator on $\ell_2(X)$ and $\widetilde{G}_{RX,X}$ defines a bounded operator on $\ell_2(\mathcal{O})$. If X and RX are Bessel systems, then

$$T^*_{RX} T_X c = G_{RX,X} c \quad \text{for all } c \in \ell_2(X), \tag{3.14}$$

and

$$U^* T_{RX} T^*_X U c = \widetilde{G}_{RX,X} c \quad \text{for all } c \in \ell_2(\mathcal{O}). \tag{3.15}$$

Since the mixed (dual) Gramian matrix is not Hermitian, the upper bound of the matrix may not give the upper bound of the operators T_X and T_{RX}. As a result, the mixed (dual) Gramian does not give the characterization of Bessel property. The relationship (3.15) implies the following characterization of dual frames X and RX.

Proposition 3.2.6. *Let X and RX be Bessel systems for H. Then X and RX are dual frames for H if and only if $\widetilde{G}_{RX,X} = I$ on $\ell_2(\mathcal{O})$.*

3.3 The canonical dual frame

In this section, we demonstrate the convenience brought by the dual Gramian analysis in the construction of the canonical dual frame from a given frame or in the construction of tight frames. Proposition 3.2.2 implies that the dual Gramian matrix makes the computation of the canonical dual frame feasible, as shown in the following proposition.

Proposition 3.3.1. *Let X be a frame in H with frame bounds A, B and let U be the synthesis operator of an orthonormal basis of H. The system $U\widetilde{G}_X^{-1}U^*X$ is a frame with bounds B^{-1}, A^{-1}, and is the canonical dual frame of X.*

For the finite dimensional case, the canonical dual frame can be easily computed by a matrix inverse, and see e.g. [49] for a connection of canonical dual frame and matrix pseudo-inverse. As one can use the Gramian matrix to construct an orthonormal basis from a Riesz basis, we can use the dual Gramian matrix to construct a tight frame from a frame. Let $\mathcal{S}^{-1/2}$ denote the inverse of the positive square root of $\mathcal{S}$.

$$f = \mathcal{S}^{-1/2}\mathcal{S}\mathcal{S}^{-1/2}f = \mathcal{S}^{-1/2}\sum_{x\in X}\langle \mathcal{S}^{-1/2}f, x\rangle x = \sum_{x\in X}\langle f, \mathcal{S}^{-1/2}x\rangle \mathcal{S}^{-1/2}x.$$

Thus, $\mathcal{S}^{-1/2}X$ forms a tight frame, usually referred to as the *canonical tight frame*.

Proposition 3.3.2. *Let X be a frame in H and let U be the synthesis operator of an orthonormal basis of H. Let $\widetilde{G}_X^{-1/2}$ denote the inverse of the positive square root of $\widetilde{G}_X$. Then, the system $U\widetilde{G}_X^{-1/2}U^*X$ forms the canonical tight frame.*

The following example illustrates that the computation of the dual frame becomes straightforward for the finite dimensional case.

Example 3.3.3. *Let H be the finite dimensional Hilbert space $\mathbb{C}^n$. Let $X = \{f_k\}_{k=1}^m$ be a frame in $\mathbb{C}^n$. The dual Gramian matrix $\widetilde{G}_X$ is self-adjoint and positive definite hence invertible. Then $\{\widetilde{G}_X^{-1} f_k\}_{k=1}^m$ forms the (canonical) dual frame of system $\{f_k\}_{k=1}^m$.*

Example 3.3.3 can be extended to the construction of tight frames. Let $\widetilde{G}_X^{-1/2}$ denote the inverse of the positive square root of $\widetilde{G}_X$ which can be found, for example, by a unitary diagonalization of the positive definite matrix $\widetilde{G}_X$. Then $\{\widetilde{G}_X^{-1/2} f_k\}_{k=1}^m$ is a tight frame.

Note the property of being a Riesz basis can be verified by evaluating the inner products of the elements of X and its corresponding canonical dual frame.

Corollary 3.3.4. *Let X be a frame in H and let U be the synthesis operator of an orthonormal basis of H. Then $\langle U^*x, \widetilde{G}_X^{-1} U^*x\rangle \leq 1$ for all $x \in X$. Moreover, X is a Riesz basis if and only if $\langle U^*x, \widetilde{G}_X^{-1} U^*x\rangle = 1$ for all $x \in X$.*

3.4 Frame bounds estimation

Dual Gramian analysis can be used to estimate the frame bounds by using matrix norm inequalities. Let $\mathcal{I}$ be a countable index set, and let M be a complex valued nonnegative Hermitian matrix with its rows and columns indexed by $\mathcal{I}$. The matrix M can be viewed as an operator from $\ell_2(\mathcal{I})$ to $\ell_2(\mathcal{I})$. We use the following inequality to estimate $\|M\|$:

$$\sup_{i\in\mathcal{I}} \left(\sum_{j\in\mathcal{I}} |M(i,j)|^2 \right)^{1/2} \leq \|M\| \leq \sup_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}} |M(i,j)|.$$

The left inequality is easy to see by the definition of matrix norm and using the canonical orthonormal basis of $\ell_2(\mathcal{I})$. The right inequality is by that fact that $\|M\| \leq$

$\sqrt{\|M\|_1\|M\|_\infty}$, see e.g. [56]. Since M is Hermitian, then $\|M\|_1 = \|M\|_\infty$. Together with Proposition 3.2.2, we give an estimate of the Bessel bound of a given system X.

Proposition 3.4.1. *Let X be a system in a Hilbert space H satisfying (3.2) with respect to an orthonormal basis $\mathcal{O}$ of H .*

(a) *Let*

$$\widetilde{B}_1 : e \mapsto \sum_{e'\in\mathcal{O}} |\sum_{x\in X} \langle e', x\rangle\langle x, e\rangle|.$$

Then X is a Bessel system whenever $\sup_{e\in\mathcal{O}} \widetilde{B}_1(e) < \infty$ and its Bessel bound is not larger than $(\sup_{e\in\mathcal{O}} \widetilde{B}_1(e))^{1/2}$.

(b) *Assume that X is a Bessel system, let*

$$\widetilde{B}_2 : e \mapsto \left(\sum_{e'\in\mathcal{O}} |\sum_{x\in X} \langle e', x\rangle\langle x, e\rangle|^2\right)^{1/2}.$$

Then $K = (\sup_{e\in\mathcal{O}} \widetilde{B}_2(e))^{1/2} < \infty$ and the Bessel bound is not smaller than K.

The lower frame bound can be obtained when the dual Gramian matrix is diagonally dominant. Recall that for a Hermitian diagonally dominant matrix M,

$$\|M^{-1}\| \le \sup_{i\in\mathcal{I}} \left(|M(i,i)| - \sum_{j\in\mathcal{I}\setminus i} |M(i,j)|\right)^{-1}$$

by $\|M^{-1}\| \le \|M^{-1}\|_\infty$ since M is Hermitian, and the bound of $\|M^{-1}\|_\infty$ see e.g. [111]. This leads to the following proposition.

Proposition 3.4.2. *Let X be a system in Hilbert space H satisfying (3.2) with respect to an orthonormal basis $\mathcal{O}$ of H. Let*

$$\tilde{b}_1 : e \mapsto \left(\sum_{x\in X} |\langle e, x\rangle|^2 - \sum_{e'\neq e} |\sum_{x\in X} \langle e', x\rangle\langle x, e\rangle|\right)^{-1}.$$

Then X is a frame whenever $\sup_{e\in\mathcal{O}} \tilde{b}_1(e) < \infty$ *and the lower frame bound is not smaller than* $(\sup_{e\in\mathcal{O}} \tilde{b}_1(e))^{-1/2}$.

Similarly, Riesz bounds can be estimated by using the Gramian matrix G_X and we omit the details here.

3.5 Shift-invariant system and fiber matrices

If the system is shift-invariant in $L_2(\mathbb{R}^d)$, for a suitable basis, the pre-Gramian matrix exhibits a strong block-wise structure. It then can be simplified to the fiber pre-Gramian matrices of shift-invariant systems introduced in [94]. Before proceeding, we recall the relevant notations and facts about lattices. Let $K \subset \mathbb{R}^d$ be a **lattice**, i.e. the image of $\mathbb{Z}^d$ under some invertible linear map $A_K : \mathbb{R}^d \to \mathbb{R}^d$. The **volume** of K is $|K| := |\det A_K|$ and its **dual lattice** is $\widetilde{K} := \{\tilde{k} \in \mathbb{R}^d : \tilde{k}\cdot k \in 2\pi\mathbb{Z}, \forall k \in K\}$, which implies $|K||\widetilde{K}| = (2\pi)^d$. We denote the **fundamental domain** of K by Ω_K, i.e. a subset of $\mathbb{R}^d$, whose K-shifts form an essentially partition of $\mathbb{R}^d$. The Lebesgure measure of Ω_K is $|K|$. The **density** of a pair (K,L) of lattices in $\mathbb{R}^d$ is $\mathrm{den}(K,L) := \frac{(2\pi)^d}{|K||L|}$ and the **adjoint** of (K,L) is $(\widetilde{L},\widetilde{K})$. Then $\mathrm{den}(K,L)\mathrm{den}(\widetilde{L},\widetilde{K}) = 1$.

For a general (K-)shift-invariant system $X = E_{\Phi,K} := \{E^k\varphi\colon \varphi\in\Phi, k\in K\}$ with $\Phi \subset L_2(\mathbb{R}^d)$, we choose

$$\{((2\pi)^d|\widetilde{K}|)^{-1/2} E^k M^{-\tilde{k}} \widehat{\chi}_{\Omega_{\widetilde{K}}}(-\cdot) : k\in K, \tilde{k}\in\widetilde{K}\}$$

as the orthonormal basis and the pre-Gramian matrix of X is

$$\begin{aligned} J_X &= ((2\pi)^d|\widetilde{K}|)^{-1/2}(\langle E^{k'}\phi, E^k M^{-\tilde{k}}\widehat{\chi}_{\Omega_{\widetilde{K}}}(-\cdot)\rangle)_{(k,\tilde{k})\in K\times\widetilde{K},(k',\varphi)\in K\times\Phi} \\ &= ((2\pi)^d|\widetilde{K}|)^{-1/2}(\langle E^{\tilde{k}}\hat{\varphi}, M^{k'-k}\chi_{\Omega_{\widetilde{K}}}\rangle)_{(k,\tilde{k})\in K\times\widetilde{K},(k',\phi)\in K\times\Phi} \end{aligned}$$

where we have used the Fourier transform properties such as $(E^k f)^\wedge = M^{-k}\hat{f}$, $(M^k f)^\wedge = E^k\hat{f}$ and $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$ for $f,g \in L_2(\mathbb{R}^d)$. For $\tilde{k} \in \widetilde{K}$ and $l \in L$ fixed, J_X consists of repeated blocks of the Fourier sequence of $\hat{\varphi}(\cdot - \tilde{k})$ on $\Omega_{\widetilde{K}}$. The abstract pre-Gramian is therefore linked to the **fiber pre-Gramian matrices** of the (K-)shift-invariant system X, which have been introduced in [94] as the infinite matrices

$$\mathcal{J}_X(\omega) := |K|^{-1/2}\left(\hat{\varphi}(\omega - \tilde{k})\right)_{\tilde{k}\in\widetilde{K},\varphi\in\Phi} \tag{3.16}$$

indexed by $\omega \in \mathbb{R}^d$.

Theorem 3.5.1. *Let $\Phi \subset L_2(\mathbb{R}^d)$, $K \subset \mathbb{R}^d$ be a lattice and $X = \{E^k\varphi\colon \varphi \in \Phi, k \in K\}$. Let $c \in \ell_0(X) = \ell_0(K \times \Phi)$. Then*

$$\mathcal{J}_X(\omega)\hat{c}(\omega) = (J_X c)^\wedge(\omega) \qquad \text{for a.e. } \omega \in \Omega_{\widetilde{K}},$$

with respect to the orthonormal basis $\{((2\pi)^d|\widetilde{K}|)^{-1/2}E^k M^{-\tilde{k}}\chi_{\Omega_{\widetilde{K}}}(-\cdot)\colon k \in K, \tilde{k} \in \widetilde{K}\}$.

Proof. Let $c \in \ell_0(X)$. For almost every $\omega \in \Omega_{\widetilde{K}}$, we have

$$\begin{aligned}
\mathcal{J}_X(\omega)\widehat{c}(\omega) &= |K|^{-1/2}\left(\sum_{\varphi\in\Phi}\hat{\varphi}(\omega - \tilde{k})\widehat{c_\varphi}(\omega)\right)_{\tilde{k}\in\widetilde{K}} \\
&= |K|^{-1/2}|\widetilde{K}|^{-1}\left(\sum_{\varphi\in\Phi}\sum_{k'\in K}c_\varphi[k']e^{-ik'\cdot\omega}\sum_{k\in K}\langle E^{\tilde{k}}\hat{\varphi}, M^k\chi_{\Omega_{\widetilde{K}}}\rangle e^{ik\cdot\omega}\right)_{\tilde{k}\in\widetilde{K}} \\
&= ((2\pi)^d|\widetilde{K}|)^{-1/2}\left(\sum_{k\in K}\sum_{\varphi\in\Phi}\sum_{k'\in K}c_\varphi[k']\langle E^{\tilde{k}}\hat{\varphi}, M^{k-k'}\chi_{\Omega_{\widetilde{K}}}\rangle e^{-ik\cdot\omega}\right)_{\tilde{k}\in\widetilde{K}}.
\end{aligned}$$

The last term is the Fourier series of the sequence

$$J_X c = ((2\pi)^d|\widetilde{K}|)^{-1/2}\left(\sum_{\varphi\in\Phi}\sum_{k'\in K}c_\varphi[k']\langle E^{\tilde{k}}\hat{\varphi}, M^{k-k'}\chi_{\Omega_{\widetilde{K}}}\rangle\right)_{k\in K,\tilde{k}\in\widetilde{K}}$$

at ω. $\square$

The fiber pre-Gramian matrices $\mathcal{J}_X(\omega)$ introduced in [94] are used to study shift-invariant systems through the representations

$$((T_X c)^\wedge(\omega - \tilde{k}))_{\tilde{k}\in\widetilde{K}} = |K|^{1/2}\mathcal{J}_X(\omega)\hat{c}(\omega), \tag{3.17}$$

$$(T_X^* f)^\wedge(\omega) = |K|^{-1/2}\mathcal{J}_X^*(\omega)(\hat{f}(\omega - \tilde{k}))_{\tilde{k}\in\widetilde{K}} \tag{3.18}$$

which hold for all $c \in \ell_0(X)$, $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \Omega_{\widetilde{K}}$. By the relationships (3.17) and (3.18), the study of properties of the operators T_X and T_X^* are then transferred to that of a continuum of simple structured matrices which are only defined by the generators of the system. In [94], the Gramian matrix $\mathcal{G}_X(\omega) = \mathcal{J}_X^*(\omega)\mathcal{J}_X(\omega)$ and the dual Gramian matrix $\widetilde{\mathcal{G}}_X(\omega) = \mathcal{J}_X(\omega)\mathcal{J}_X^*(\omega)$ (with the weak condition $\sum_{\varphi\in\Phi}|\hat{\varphi}(w)|^2 < \infty$ for a.e. $w \in \mathbb{R}^d$ for it to be well-defined) are introduced. Those matrices are used to decompose the operators $T_X^* T_X$ and $T_X T_X^*$ in the Fourier transform domain into simple fibers which are then used to characterize various properties of shift-invariant systems. More specifically, define

$$\Lambda(\omega) := \|\widetilde{\mathcal{G}}_X(\omega)\|, \qquad \lambda(\omega) := \|\widetilde{\mathcal{G}}_X(\omega)^{-1}\|$$

as the operator norms of $\widetilde{\mathcal{G}}_X(\omega)$ and $\widetilde{\mathcal{G}}_X(\omega)^{-1}$ at each $\omega \in \mathbb{T}^d$, and $\lambda(\omega)$ is ∞ if $\widetilde{\mathcal{G}}_X(\omega)$ is not invertible. The shift-invariant system X is Bessel if and only if the function Λ is essentially bounded on $\mathbb{T}^d$. Moreover, the Bessel bound is $\|\Lambda\|_{L_\infty}^{1/2}$. When X is Bessel, this system is a frame if and only if the function λ is essentially bounded on $\mathbb{T}^d$. The lower frame bound is $\|\lambda\|_{L_\infty}^{-1/2}$. The system is a tight frame if and only if $\Lambda(w) = \lambda(w) = 1$, or equivalently, $\widetilde{\mathcal{G}}_X(w) = I$ for a.e. $w \in \mathbb{T}^d$.

Similarly, one can use the Gramian matrix to investigate the Bessel and Riesz properties of a shift-invariant system. The interested reader is referred to [94] for more details on the fiberization technique and the (dual) Gramian analysis for shift-invariant systems.

As observed in [98], Gabor systems are the special case of shift-invariant systems with Φ being modulations of the window $\phi \in L_2(\mathbb{R}^d)$. Given $\phi \in L_2(\mathbb{R}^d)$ and two lattices K, L, the system

$$(K,L)_\phi := \{E^k M^l \phi \colon k \in K, l \in L\}$$

is called the *(regular) Gabor system* generated by ϕ. This system, being the collection of all K-shifts of the set $\{M^l \phi : l \in L\}$, is K-shift-invariant. In particular the fiber pre-Gramian matrices of a Gabor system $X = (K,L)_\phi$ are

$$\mathcal{J}_X(\omega) = |K|^{-1/2} \left(\hat{\phi}(\omega - \tilde{k} - l)\right)_{\tilde{k} \in \widetilde{K}, l \in L} \tag{3.19}$$

for $\omega \in \mathbb{R}^d$.

The statements in this section hold verbatim for Gabor shift-invariant systems. The analysis of Gabor systems by using dual Gramian analysis is shown in [98], which however mainly focuses on single systems. In the next section we present the mixed dual Gramian analysis for dual Gabor systems.

3.6 Mixed dual Gramian analysis for Gabor systems

The matrices $\mathcal{J}_X(\omega)$ represent the synthesis operator T_X, see (3.17), while their matrix adjoints, denoted as $\mathcal{J}_X^*(\omega)$, represent the analysis operator T_X^*, see (3.18). Consequently the mixed operator $T_Y T_X^*$ can be represented by $\mathcal{J}_Y(\omega)\mathcal{J}_X^*(\omega)$ and $T_Y^* T_X$ can be represented by $\mathcal{J}_Y^*(\omega)\mathcal{J}_X(\omega)$. These fiberized representations of $T_Y T_X^*$ (and $T_Y^* T_X$) transfer the study of these operators to the study of the family of simpler fiber matrices, for which the respective properties have to hold uniformly in ω. Precisely, if $\phi, \psi \in L_2(\mathbb{R}^d)$

are such that $X=(K,L)_\phi$ and $Y=(K,L)_\psi$ are Bessel systems, the *mixed dual Gramian fiber matrices* are

$$\widetilde{\mathcal{G}}_{Y,X}(\omega) := \mathcal{J}_Y(\omega)\mathcal{J}_X^*(\omega) = |K|^{-1}\left(\sum_{l\in L}\hat{\psi}(\omega-\tilde{k}-l)\overline{\hat{\phi}(\omega-\tilde{k}'-l)}\right)_{\tilde{k},\tilde{k}'\in\widetilde{K}}, \tag{3.20}$$

where $\omega\in\mathbb{R}^d$. Then

$$((T_Y T_X^* f)^\wedge(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}} = \widetilde{\mathcal{G}}_{Y,X}(\omega)(\hat{f}(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}} \tag{3.21}$$

for all $f\in L_2(\mathbb{R}^d)$ and a.e. $\omega\in\mathbb{R}^d$. Note that for any $\phi,\psi\in L_2(\mathbb{R}^d)$ the entries of the mixed dual Gramians are locally integrable, and therefore a.e. finite, regardless of the Bessel assumption on X and Y. The Bessel assumption, however, ensures that the matrix product $\widetilde{\mathcal{G}}_{Y,X}(\omega)(\hat{f}(\omega-\tilde{k}))_{\tilde{k}\in\widetilde{K}}$ for any $f\in L_2(\mathbb{R}^d)$ has a.e. finite values, since the relevant series converge absolutely a.e..

The image of a Gabor system under the unitary Fourier transform is again a Gabor system, with the role of the shift and modulation lattice interchanged. Thus the Bessel, Riesz, frame and dual frame properties can be equivalently studied through the Fourier transform counterparts $\widehat{X} := (L,K)_{\hat{\phi}}$ and $\widehat{Y} := (L,K)_{\hat{\psi}}$ of the Bessel systems X and Y. Their mixed dual Gramian fiber matrices are

$$\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega) = \mathcal{J}_{\widehat{Y}}(\omega)\mathcal{J}_{\widehat{X}}^*(\omega) = (2\pi)^{2d}|L|^{-1}\left(\sum_{k\in K}\psi(-\omega+\tilde{l}+k)\overline{\phi(-\omega+\tilde{l}'+k)}\right)_{\tilde{l},\tilde{l}'\in\widetilde{L}} \tag{3.22}$$

and

$$((T_{\widehat{Y}}T_{\widehat{X}}^*\hat{f})^\wedge(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^d\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega)(f(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}}$$

holds for all $f\in L_2(\mathbb{R}^d)$ and a.e. $\omega\in\mathbb{R}^d$. On the other hand

$$((T_{\widehat{Y}}T_{\widehat{X}}^*\hat{f})^\wedge(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^d((T_Y T_X^* f)^{\wedge\wedge}(\omega-\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^{2d}((T_Y T_X^* f)(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}}$$

and therefore

$$((T_Y T_X^* f)(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}} = (2\pi)^{-d}\widetilde{\mathcal{G}}_{\widehat{Y},\widehat{X}}(\omega)(f(-\omega+\tilde{l}))_{\tilde{l}\in\widetilde{L}} \tag{3.23}$$

for all $f \in L_2(\mathbb{R}^d)$ and a.e. $\omega \in \mathbb{R}^d$. We refer to (3.21) as the representation of the Gabor frame operator in Fourier domain and (3.23) as the representation in time domain.

Switching the order of multiplication of the fiber pre-Gramian matrices leads to the *mixed Gramian fiber matrices*

$$\mathcal{G}_{Y,X}(\omega) := \mathcal{J}_Y^*(\omega)\mathcal{J}_X(\omega) = |K|^{-1}\left(\sum_{\tilde{k}\in\widetilde{K}} \overline{\hat{\psi}(\omega-\tilde{k}-l)}\hat{\phi}(\omega-\tilde{k}-l')\right)_{l,l'\in L} \tag{3.24}$$

and

$$\mathcal{G}_{\widehat{Y},\widehat{X}}(\omega) = \mathcal{J}_{\widehat{Y}}^*(\omega)\mathcal{J}_{\widehat{X}}(\omega) = (2\pi)^{2d}|L|^{-1}\left(\sum_{\tilde{l}\in\widetilde{L}} \overline{\psi(-\omega+k+\tilde{l})}\phi(-\omega+k'+\tilde{l})\right)_{k,k'\in K} \tag{3.25}$$

for $\omega \in \mathbb{R}^d$, which can be used for fiberized representations of the mixed operator $T_Y^* T_X$. If X, Y are Bessel and $c \in \ell_2(K \times L)$, then

$$(T_Y^* T_X c)^\wedge(\omega) = \mathcal{G}_{Y,X}(\omega)\hat{c}(\omega) \tag{3.26}$$

for a.e. $\omega \in \mathbb{R}^d$. By using the Fourier transformed systems, one may get another fiber representation of the operator $T_Y^* T_X$, which has a similar form as (3.26). Since the Fourier system does not help to get a simple real domain representation and the representation needs dedicated attentions to the lattice changes, the exact representation will not be spelled out here.

The abstract Gramian matrix of two regular Gabor systems X and Y already has the simple form

$$G_{Y,X} = \left(\langle E^k M^l \psi, E^{k'} M^{l'} \phi\rangle\right)_{(k,l)\in K\times L,(k',l')\in K\times L}$$

and might be of some interest. If for example X and Y are Bessel systems, then $T_Y^* T_X$ is the identity if and only if $G_{Y,X}$ is the identity, i.e. if and only if X and Y biorthonormal. However, the fiber Gramian matrices (3.24) and (3.25) reveal more information of the operator $T_Y^* T_X$ on each fiber.

Chapter 4

Duality principle

While the synthesis operator characterizes the linear independent property of a system, the analysis operator is more suitable for the study of the frame (redundant) property. Due to this dual nature, one may consider two systems X and Y such that the synthesis operators are the adjoint operators of each other, i.e. $T_X = T_Y^*$. Such a system Y will be called an adjoint system of X. The relationship of the operators is rather crude and it is not clear how to find an adjoint system. If one, however, has a matrix representation of the synthesis operator, the relationship between a given system and its adjoint system will be reduced to a column-row relationship. The simple principle around which the results in this chapter revolve is thus provided by the matrix representations and the duality of the adjoint operators.

Duality Principle. *The systems X and X^* are adjoint to each other if for some matrix representation of the synthesis operator of X, the columns can be associated with X while the rows can be associated with X^*. Consequently, the analysis (resp. synthesis) properties of X are characterized by the synthesis (resp. analysis) properties of X^*.*

The trivial case of course is the one of finite systems in finite dimensions. If $X = \{x_k\}_{k=1}^n \subset \mathbb{C}^m$, then with respect to the standard orthonormal bases T_X is given by the matrix

$$J_X = \begin{pmatrix} x_1(1) & \cdots & x_n(1) \\ \vdots & \ddots & \vdots \\ x_1(m) & \cdots & x_n(m) \end{pmatrix},$$

and T_X^* by its adjoint matrix J_X^*. Thus, a possible adjoint system of X is given by

$$X^* = \{(x_k(i))_{k=1,\ldots,n} \colon i = 1, \ldots, m\} \subset \mathbb{C}^n,$$

i.e. by the rows of J_X.

In this chapter, we will introduce adjoint systems based on the pre-Gramian matrix defined in the previous chapter. As a result the duality principle is derived, i.e. the dual Gramian matrix of X equals the Gramian matrix of its adjoint system X^*. An immediate consequence of duality principle is that the frame property could be equally studied from the Riesz sequence property of the adjoint system counterpart. The adjoint systems of a given frame could as well be used to characterize and parametrize all the dual frames. Moreover, from the duality viewpoint and the special structure of the adjoint system associated with a filter bank in $\ell_2(\mathbb{Z}^d)$, we propose a simple construction scheme for perfect reconstruction filter banks by easily completing constant matrices. The synthesis operator of a system with a special structure may admit a different matrix representation. As seen in Section 3.5, the abstract dual Gramian analysis could be simplified for the shift-invariant system to fiber dual Gramian analysis. But the underlying duality principle is the same, no matter how technical the matrix representation of the operator is.

For systems with special structure, more appropriate orthonormal basis could be designed to take advantage of the structure. We will define the adjoint systems for irregular Gabor systems by using a Gabor orthonormal basis and derive the duality principle. In general, the adjoint systems defined through the abstract pre-Gramian matrix may not have the same structure as the original system. But for the regular Gabor system, if using the fiber pre-Gramian matrix representation instead, an adjoint system could again be realized as a Gabor system. The fiber dual Gramian analysis of Gabor systems and the duality principle straightforwardly lead to several classical representations of the mixed frame operator, e.g. Walnut representation, Wexler-Raz/Janssen representation, and the Wexler-Raz biorthogonal relationship for dual frames. As already observed in [98], the duality principle on each fiber is also the unifying theme behind the numerous painless constructions of Gabor windows and we use it to explicitly construct dual Gabor windows. The dual window pairs whose construction we outline have coinciding support and can achieve arbitrary smoothness. Most importantly, the method is easily generalized to high dimensions.

4.1 Adjoint system and duality principle

By writing the synthesis operator in the pre-Gramian matrix form (3.1), we observe the given system X forms the columns of the matrix. We define the rows of the pre-Gramian matrix to be a new system, called adjoint system. The Gramian matrix of the adjoint system will be the dual Gramian matrix of the original system, which is the core of duality principle.

Definition 4.1.1. *Let X be a system in a Hilbert space H satisfying (3.2) with respect to an orthonormal basis $\mathcal{O}$, and let J_X be the corresponding pre-Gramian matrix defined in (3.1). A system X^* in a Hilbert space H' is called an **adjoint system** of X if*

(a) *there exists an orthonormal basis $\mathcal{O}'$ of H' such that X^* and $\mathcal{O}'$ satisfy (3.2),*

(b) *the corresponding pre-Gramian matrix J_{X^*} of X^* associated with $\mathcal{O}'$ is the adjoint* *matrix of the pre-Gramian matrix J_X (up to unitary equivalence), i.e.*

$$J_{X^*} = UJ_X^*V, \tag{4.1}$$

where U and V are two unitary operators. [1]

For a given system X, there are many ways to construct a pre-Gramian matrix that is the same as the synthesis operator up to unitary equivalence. The definition of adjoint systems can be adapted to any pre-Gramian matrix of the synthesis operator of the original system, which leads to different ways to define an adjoint system. The following example shows that the R-dual sequence defined in [15] is indeed an adjoint system of a given system (see also [16, 26, 29]).

Example 4.1.2 (*[15]*)**.** *Let $X = \{f_k\}_{k\in\mathbb{N}}$ be a system indexed by natural number $\mathbb{N}$ in a Hilbert space H satisfying (3.2) with respect to an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Suppose $\{h_k\}_{k\in\mathbb{N}}$ is another orthonormal basis of H and define $X' = \{g_i := \sum_{k\in\mathbb{N}}\langle f_k, e_i\rangle h_k\}_{i\in\mathbb{N}}$. Then X' is indeed an adjoint system of X. Firstly, the system X' satisfies (3.2) since*

$$\sum_{i\in\mathbb{N}} |\langle g_i, h_k\rangle|^2 = \sum_{i\in\mathbb{N}} |\langle f_k, e_i\rangle|^2 < \infty \qquad \textit{for all } k \in \mathbb{N}.$$

Secondly, it is easy to see that

$$J_{X'} = (\langle g_i, h_k\rangle)_{k,i} = (\langle f_k, e_i\rangle)_{k,i} = \overline{J_X^*}.$$

From the matrix point of view, the columns of J_X correspond to the original system X while the rows of J_X form an adjoint system in $\ell_2(\mathbb{N})$, which is then mapped to the system X' in H by the unitary map corresponding to the orthonormal basis $\{h_k\}_{k\in\mathbb{N}}$. The definition for adjoint system by R-dual sequence works fine due to the same cardinality

[1] Note that one might also consider the matrix transpose of the pre-Gramian in (4.1) without introducing essential changes to the discussion that follows.

of the system X and the orthonormal basis $\{h_k\}_{k\in\mathbb{N}}$ of H. But indeed the adjoint system may not always lie in the same Hilbert space, e.g. a finite system in a finite Hilbert space, see Example 4.1.7.

With the definition of adjoint system, observe that

$$\widetilde{G}_X = J_X J_X^* = V J_{X^*}^* U U^* J_{X^*} V^* = V G_{X^*} V^*,$$

i.e. the dual Gramian matrix of system X is (unitarily equivalent to) the Gramian matrix of the adjoint system X^*, which is the essence of duality principle. This observation could be easily generalized to two systems. Given a system RX in H, then

$$\widetilde{G}_{RX,X} = J_{RX} J_X^* = V J_{(RX)^*}^* U U^* J_{X^*} V^* = V G_{(RX)^*,X^*} V^*$$

and

$$G_{RX,X} = J_{RX}^* J_X = U^* J_{(RX)^*} V^* V J_{X^*}^* U = U^* \widetilde{G}_{(RX)^*,X^*} U.$$

which says that the mixed dual Gramian matrix of two systems X and RX is (unitarily equivalent to) the mixed Gramian matrix of their adjoint systems X^* and $(RX)^*$. Note that whenever we consider several systems and their corresponding adjoint systems, we will always assume that all of them are satisfying the respective adjoint relationship (4.1) with respect to the same orthonormal bases and unitary operators. Since the mixed (dual) Gramian matrix is reduced to (dual) Gramian matrix when $R = I$, we formulate the following the central result of this section, the duality principle, for two systems.

Theorem 4.1.3. *Let X, RX be systems in H satisfying (3.2) with respect to an orthonormal basis, and X^*, resp. $(RX)^*$, be adjoint systems of X, resp. RX (with respect to the same orthonormal bases and unitaries). Then, up to unitary equivalence, $\widetilde{G}_{RX,X}$ is equal to $G_{(RX)^*,X^*}$ and $G_{RX,X}$ is equal to $\widetilde{G}_{(RX)^*,X^*}$.*

Note that the assumption on the existence of the adjoint systems in this duality principle puts a restriction on the systems for which it can be applied. Both have to satisfy (3.2) (for the same orthonormal basis), for which it is sufficient that both systems are Bessel systems. In the case that R is the identity, the duality principle results several duality statements about a single system and its adjoint. Note the requirement for taking the same orthonormal basis for both X and RX will vanish since $RX = X$ in this case.

Theorem 4.1.4. *Let X be a given system in H, and suppose that X^* is an adjoint system of X in H' as defined in Definition 4.1.1. Then*

(a) *A system X is Bessel in H if and only if its adjoint system X^* is Bessel in H' with the same Bessel bound.*

(b) *A Bessel system X is ℓ_2-independent if and only if its adjoint system X^* is Bessel and fundamental.*

(c) *A system X forms a frame in H if and only if its adjoint system X^* forms a Riesz sequence in H'. The frame bounds of X coincide with the Riesz bounds of X^*.*

(d) *A system X forms a tight frame in H if and only if its adjoint system X^* forms an orthonormal sequence in H'.*

Since the adjoint system of the adjoint system is the original system itself, the role of X and X^* in the above theorem is interchangeable. The duality principle for the sequence pair in Example 4.1.2, i.e. the sequence $\{f_k\}_{k\in\mathbb{N}}$ and its R-dual sequence $\{g_i\}_{i\in\mathbb{N}}$ in [15], follows immediately from Theorem 4.1.4. Indeed, understanding the results in [15] from the viewpoint of dual Gramian analysis is one of the motivations of this thesis. As another direct consequence of duality principle, we get the following characterization of dual frames.

Theorem 4.1.5. *Suppose X and RX are Bessel systems in H. Systems X and RX are dual frames if and only if X^* is biorthonormal to $(RX)^*$.*

Note that the above biorthonormality is with respect to the natural indexing of the vectors of X^* and $(RX)^*$ by means of the orthonormal basis chosen for the representation of the pre-Gramian matrices J_X and J_{RX}. Moreover, the following is also a consequence of the duality principle.

Example 4.1.6. *Consider the setting of [15], i.e. let $X = \{f_i\}_{i\in\mathbb{N}}$ be a frame for H and $\{e_i\}_{i\in\mathbb{N}}$ and $\{h_i\}_{i\in\mathbb{N}}$ be two orthonormal bases of H. Denote $\mathcal{S} = T_X T_X^*$. For each $j \in \mathbb{N}$, define*

$$g_j = \sum_{i\in\mathbb{N}} \langle f_i, e_j\rangle h_i,$$

and

$$g_j^* = \sum_{i\in\mathbb{N}} \langle \mathcal{S}^{-1} f_i, e_j\rangle h_i.$$

As seen in Example 4.1.2 $X' = \{g_j\}_{j\in\mathbb{N}}$ is an adjoint system of X. The system $\{g_j^*\}_{j\in\mathbb{N}}$ is the corresponding adjoint system of $\mathcal{S}^{-1}X$ and therefore

$$\langle g_i^*, g_j\rangle = \delta_{i,j} \quad \textit{for all } i, j \in \mathbb{N}$$

follows from Theorem 4.1.3 since $\mathcal{S}^{-1}X$ is the canonical dual frame of X.

In Example 4.1.2 and Example 4.1.6, the Hilbert spaces of the original system and its adjoint system are the same. But the Hilbert space H' might be different from H as we will see next in a finite dimensional Hilbert space.

Example 4.1.7. *Proceeding with Example 3.1.1. Let $X = \{x_k\}_{k=1}^n \subset \mathbb{C}^m$. The pre-Gramian matrix is*

$$J_X = \begin{pmatrix} x_1(1) & \cdots & x_n(1) \\ \vdots & \ddots & \vdots \\ x_1(m) & \cdots & x_n(m) \end{pmatrix}.$$

By Definition 4.1.1, the rows of J_X form an adjoint system of X. Notice that the rows *are elements in $\mathbb{C}^n$, a space maybe different from $\mathbb{C}^m$. The duality principle for the finite case can be understood in terms of matrix terminology. The columns are fundamental (equivalent to be a frame) if and only if the rows are linearly independent (equivalent to be a Riesz sequence). The columns form a tight frame if and only if the rows form an orthonormal sequence. The row and column relationship for finite matrices is well studied, see e.g. [23, 107].*

Given another system $Y=\{y_k\}_{k=1}^n$ in $\mathbb{C}^m$, and let J_Y be the pre-Gramian matrix. Hence the duality principle also implies that the columns of J_X and J_Y, i.e. X and Y, are dual frames if and only if the rows of J_X and J_Y are biorthonormal, i.e. $\sum_{k=1}^n x_k(i)\overline{y_k(j)}=\delta_{ij}$ for $i,j=1,2,\dots,m$.

As another application of Definition 4.1.1 to the mixed frame operator, we observe a relationship between the mixed frame operators of systems and their adjoint systems. Let X, $Y=RX$ and Z be Bessel systems in H, and $\mathcal{O}$ be an orthonormal basis of H. Then the obvious matrix relationship

$$J_X J_Y^* J_Z = (J_Z^* J_Y J_X^*)^*$$

holds, where all pre-Gramians are being considered with respect to $\mathcal{O}$. If X^*, Y^*, Z^* are adjoint systems of X, Y, Z, respectively, all with respect to the same orthonormal basis $\mathcal{O}'$ and unitaries U, V, then

$$J_X J_Y^* J_Z = V(J_{Z^*} J_{Y^*}^* J_{X^*})^* U. \tag{4.2}$$

Note that $J_X J_Y^* J_Z$ is the pre-Gramian matrix of the system $X'=T_X T_Y^* Z$ while $J_{Z^*} J_{Y^*}^* J_{X^*}$ is the pre-Gramian matrix of the system $Y'=T_{Z^*} T_{Y^*}^* X^*$. Equation (4.2) implies that the systems X' and Y' are adjoint systems as in Definition 4.1.1. The result is summarized as the following.

Theorem 4.1.8. *If* $X, Y = RX, Z$ *are Bessel systems in* H*, and* X^**,* Y^**,* Z^* *are their respective adjoint systems (with respect to the same orthonormal bases and unitaries), then the system* $T_X T_Y^* Z$ *is an adjoint system of* $T_{Z^*} T_{Y^*}^* X^*$*.*

4.2 Adjoint system and dual frames

The set of dual frames not only could be characterized by a biorthogonal relationship of the adjoint systems, but also can be parametrized in terms of the adjoint systems.

Proposition 4.2.1. *Let* X *be a frame for* H *and* RX *indexed by* X *in* H *is a Bessel system. Denote* $\mathcal{S} = T_X T_X^*$*. Let* X^**,* $(RX)^*$ *and* $(\mathcal{S}^{-1}X)^*$ *be adjoint systems of* X*,* RX *and* $\mathcal{S}^{-1}X$ *respectively (with respect to the same orthonormal bases and unitaries). Then*

(a) $(\mathcal{S}^{-1}X)^*$ *is a system in* $\operatorname{ran} T_{X^*}$*,*

(b) RX *is a dual frame of* X *if and only if* $(RX)^*$ *is an additive perturbation of* $(\mathcal{S}^{-1}X)^*$ *by vectors from* $(\operatorname{ran} T_{X^*})^\perp$ *whose collection is a Bessel system.*

Proof. (a) We have $\ker T_X = \ker T_{\mathcal{S}^{-1}X}$ so in particular $\ker J_X = \ker J_{\mathcal{S}^{-1}X}$, i.e. $(\operatorname{ran} J_X^*)^\perp = (\operatorname{ran} J_{\mathcal{S}^{-1}X}^*)^\perp$ and in turn $\operatorname{ran} J_X^* = \operatorname{ran} J_{\mathcal{S}^{-1}X}^*$. Therefore, $\operatorname{ran} U^* J_{X^*} V^* = \operatorname{ran} U^* J_{(\mathcal{S}^{-1}X)^*} V^*$ with unitaries U and V and thus $\operatorname{ran} J_{X^*} = \operatorname{ran} J_{(\mathcal{S}^{-1}X)^*}$.

(b) Suppose RX is a dual frame for X, i.e. $J_{RX} J_X^* = I$. Then $(J_{RX} - J_{\mathcal{S}^{-1}X}) J_X^* = 0$ since $J_{\mathcal{S}^{-1}X} J_X^* = I$. Taking adjoint yields

$$\operatorname{ran}(J_{RX}^* - J_{\mathcal{S}^{-1}X}^*) \subset \ker J_X = (\operatorname{ran} J_X^*)^\perp,$$

and thus

$$\operatorname{ran}(U^*(J_{(RX)^*} - J_{(\mathcal{S}^{-1}X)^*})V^*) \subset (\operatorname{ran} U^* J_{X^*} V^*)^\perp,$$

i.e.

$$\operatorname{ran}(J_{(RX)^*} - J_{(\mathcal{S}^{-1}X)^*}) \subset (\operatorname{ran} J_{X^*})^\perp.$$

Conversely, suppose the vectors of $(RX)^*$ are perturbations of the vectors of $(\mathcal{S}^{-1}X)^*$ by certain elements of $(\operatorname{ran} T_{X^*})^\perp$ whose collection is a Bessel system. If $y \in (\operatorname{ran} T_{X^*})^\perp$, then y is orthogonal to every vector of the system X^*. Since $(\mathcal{S}^{-1}X)^*$ is biorthonormal to X^* by the duality principle Theorem 4.1.4, this implies that $(RX)^*$ is biorthonormal to X^*. In turn RX and X are dual frames. $\square$

Alternatively the dual frames of a given frame could be parametrized even without referring to the adjoint systems (see [84]). The canonical dual frame $\mathcal{S}^{-1}X$ of X can now be characterized by norm minimization properties analogous to the Gabor frame case (see [59, Proposition 7.6.2]), in which the canonical dual window is characterized among all dual windows as the one of minimal norm and as the window that among all dual windows is closest to the primary window. The simple proof is precisely as for this special case (see also [15, Proposition 20]).

Proposition 4.2.2. *Let X be a frame for H and RX be a dual frame of X. Denote $\mathcal{S} = T_X T_X^*$. Index all adjoints by the orthonormal basis $\mathcal{O}$ with respect to which the pre-Gramians J_X and J_{RX} are represented, for example $X^* = \{X_e^*\}_{e\in\mathcal{O}}$. Then the following are equivalent:*

(a) $RX = \mathcal{S}^{-1}X$.

(b) *If $R'X$ is a dual frame of X, then*

$$\|(RX)_e^*\| < \|(R'X)_e^*\|$$

whenever $(RX)_e^* \neq (R'X)_e^*$.

(c) *If $R'X$ is a dual frame of X, then*

$$\left\| \frac{(RX)_e^*}{\|(RX)_e^*\|} - \frac{X_e^*}{\|X_e^*\|} \right\| < \left\| \frac{(R'X)_e^*}{\|(R'X)_e^*\|} - \frac{X_e^*}{\|X_e^*\|} \right\|$$

whenever $(RX)^*_e \neq (RX)^*_e$.

Proof. For the first equivalence note that by Proposition 4.2.1, every vector of $(R'X)^*$ is the sum of two orthogonal vectors, say $(R'X)^*_e = (\mathcal{S}^{-1}X)^*_e + y_e$, and thus

$$\|(R'X)^*_e\|^2 = \|(\mathcal{S}^{-1}X)^*_e\|^2 + \|y_e\|^2 \geq \|(\mathcal{S}^{-1}X)^*_e\|^2.$$

The second equivalence can be established by noting that the biorthonormality of $(RX)^*$ and X^* implies

$$\left\| \frac{(RX)^*_e}{\|(RX)^*_e\|} - \frac{X^*_e}{\|X^*_e\|} \right\|^2 = 2 - \frac{2}{\|(RX)^*_e\| \|X^*_e\|}.$$ □

With the notion of the adjoint system, the properties of the dual systems characterized by the mixed frame operator can now be phrased in terms of the mixed dual Gramian matrix. For example, finding a dual frame can be done by finding a matrix inverse.

Corollary 4.2.3. *Let X and $Y = RX$ be frames for H such that $\overline{span}\{X^*\} = \overline{span}\{Y^*\}$. Let $\mathcal{O}$ be an orthonormal basis for H and U be the synthesis operator of $\mathcal{O}$. Then $\widetilde{G}_{Y,X}$ is boundedly invertible and $U\widetilde{G}^{-1}_{Y,X}U^*Y$ and X are dual frames.*

In the following corollary, the verification of the mixed operators to be bounded below in Proposition 2.3.2 is transferred to the mixed dual Gramian matrices.

Corollary 4.2.4. *Let X and $Y = RX$ be Bessel systems in H such that $\overline{span}\{X^*\} = \overline{span}\{Y^*\}$, then the following are equivalent:*

(a) *X and Y are frames.*

(b) *$\widetilde{G}_{X,Y}$ and $\widetilde{G}_{Y,X}$ are bounded below.*

By the duality principle Theorem 4.1.4, two Bessel systems X and $Y = RX$ are frames if and only if X^* and Y^* are Riesz sequences. Now $\widetilde{G}_{X,Y} = G_{X^*,Y^*}$ and $\widetilde{G}_{X,Y} = G_{Y^*,X^*}$

up to unitaries. Thus, by Corollary 4.2.4, if $\overline{\text{span}}\{X^*\} = \overline{\text{span}}\{Y^*\}$, then X^* and Y^* are Riesz sequences if and only if G_{X^*,Y^*} and G_{Y^*,X^*} are bounded below. Replacing X^* by X and Y^* by Y yields the following. If $\overline{\text{span}}\{X\} = \overline{\text{span}}\{Y\}$, then X and Y are Riesz sequences if and only if $G_{X,Y}$ and $G_{Y,X}$ are bounded below. This is Proposition 2.3.3 stated in terms of Gramian matrices. In this sense, Proposition 2.3.2 and 2.3.3 can be considered as one.

4.3 Duality for filter banks

One of the realizations of the abstract Hilbert spaces is the discrete $\ell_2(\mathbb{Z}^d)$ space, and using the dual Gramian analysis and the duality principle to study filter banks in this space reveals a simple filter banks construction scheme. Filter banks are the implementation of N-shift-invariant systems in $\ell_2(\mathbb{Z}^d)$ of the form

$$X = X(a,N) := \{(a_l(n-Nk))_{n\in\mathbb{Z}^d} \colon l \in \mathbb{Z}_r, k \in \mathbb{Z}^d\}, \tag{4.3}$$

where $a = \{a_l\}_{l\in\mathbb{Z}_r} \subset \ell_2(\mathbb{Z}^d)$ is a set of filters, $N \in \mathbb{N}$ is the (sub)sampling rate, $r \in \mathbb{N}$ is the number of channels and $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$. The associated synthesis operator

$$T_X \colon \ell_2(\mathbb{Z}_r \times \mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d) \colon c \mapsto \sum_{l\in\mathbb{Z}_r} (\uparrow_N c(l,\cdot)) * a_l$$

is given by upsampling followed by discrete convolutions. Here, for fixed $l \in \mathbb{Z}_r$, $\uparrow_N c(l,k)$ is equal to $c(l,N^{-1}k)$ if N divides all entries of $k \in \mathbb{Z}^d$ and is equal to 0 otherwise. The transform given by the synthesis operator of X is called a synthesis filter bank. The analysis operator

$$T_X^* \colon \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}_r \times \mathbb{Z}^d) \colon c \mapsto (\downarrow_N (c * \overline{a_l(-\cdot)})(k))_{(l,k)\in\mathbb{Z}_r\times\mathbb{Z}^d}$$

is composed of discrete convolutions followed by downsampling by the factor N, i.e. $\downarrow_N d(k) = d(Nk)$ for $k \in \mathbb{Z}^d$. This transform is called an analysis filter bank. A filter bank consists of an analysis and synthesis filter bank with equal number of channels and the same sampling rate but, in general, with respect to different filters. If $Y = X(b, N)$ for $b = \{b_l\}_{l\in\mathbb{Z}_r} \subset \ell_2(\mathbb{Z}^d)$, then the pair X and Y, or more precisely the mixed operator $T_Y T_X^*$, is called a **perfect reconstruction filter bank**, whenever X and Y are dual frames in $\ell_2(\mathbb{Z}^d)$. A tight frame filter bank is a perfect reconstruction filter bank with coinciding analysis and synthesis filters, i.e. $X = Y$. We now use the abstract pre-Gramian analysis to study the frame properties of filter banks. Due to the finite number of filters, X satisfies (3.2) with respect to the canonical orthonormal basis of $\ell_2(\mathbb{Z}^d)$ and the pre-Gramian matrix of X is

$$J_X = (a_l(n - Nk))_{n\in\mathbb{Z}^d,(l,k)\in\mathbb{Z}_r\times\mathbb{Z}^d}.$$

Given a second system

$$Y = X(b, N) := \{(b_l(n - Nk))_{n\in\mathbb{Z}^d} \colon l \in \mathbb{Z}_r, k \in \mathbb{Z}^d\}, \tag{4.4}$$

associated with a set of filters $b = \{b_l\}_{l\in\mathbb{Z}_r} \subset \ell_2(\mathbb{Z}^d)$. The mixed dual Gramian matrix of X and Y is

$$\widetilde{G}_{Y,X} = J_Y J_X^* = \left(\sum_{l=0}^{r-1} \sum_{k\in\mathbb{Z}^d} \overline{a_l(n' - Nk)} b_l(n - Nk) \right)_{n,n'\in\mathbb{Z}^d}.$$

In particular, two Bessel systems X and Y are dual frames if and only if $\widetilde{G}_{Y,X} = I$ on $\ell_2(\mathbb{Z}^d)$.

We now show how the duality principle can provide a significant simplification of perfect reconstruction filter banks construction. Its generality and simplicity makes the constructed filters flexible enough to be useful in design problems that meet many conditions. For example, in Section 5.2, we will refine it to meet additional constraints and to result in a simple dual and tight MRA-wavelet frame construction. As reasonable for the design problem, we now restrict ourselves to finitely supported filters, also referred to as finite impulse response (FIR) filters. Besides making the filter bank systems automatically Bessel, all information on the systems can now be written in well-structured finite matrices in terms of the filters. Since the columns of the pre-Gramian matrix J_X are formed by the system, a reordering of the columns of the pre-Gramian is equivalent to the reordering of the system. Based on the masks $\{a_l\}_{l=0}^{r-1}$ which are assumed to be finitely supported here, the columns of J_X are reordered by grouping different a_l's (omitting the shift) together so that the pre-Gramian matrix J_X is formed by shifts of a small block matrix (or a permutation of the columns of it) given by

$$A = \begin{pmatrix} a_0(n_1) & a_0(n_2) & \cdots & a_0(n_m) \\ a_1(n_1) & a_1(n_2) & \cdots & a_1(n_m) \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1}(n_1) & a_{r-1}(n_2) & \cdots & a_{r-1}(n_m) \end{pmatrix} \tag{4.5}$$

where $n_i \in \mathbb{Z}^d,\ i = 1, 2, \ldots, m$, is the coordinate of the masks and m is the maximum number of the supports of masks $\{a_l\}_{l=0}^{r-1}$. To be specific, for one dimension case, by reordering the columns of J_X or equivalently reordering the rows of J_X^*, the matrix J_X^* can be expressed as a block-wise matrix generated by the N-shifts of the block matrix $\overline{A}$, i.e.

$$J_X^*(n,k) = \begin{cases} \overline{A_{k-Nn}} & \text{if } 1 \le |k - Nn| \le m \\ 0 & \text{else} \end{cases},$$

where A_j denotes the j-th column of A, $j = 1, \ldots, m$. In other words, each block of the matrix J_X^* is the same as the block matrix $\overline{A}$ shifted to the right by N:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \overline{A_1} & \overline{A_2} & \overline{A_3} & \overline{A_4} & \cdot & \cdot & \cdot & \overline{A_m} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \overline{A_1} & \overline{A_2} & \overline{A_3} & \overline{A_4} & \cdot & \cdot & \overline{A_{m-1}} & \overline{A_m} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \overline{A_1} & \overline{A_2} & \overline{A_3} & \cdot & \cdot & \cdot & \overline{A_{m-1}} & \overline{A_m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Thus, the columns of J_X^*, or the adjoint system X^* of X (4.4), is formed by concatenating different masks entries that lie in the same $N\mathbb{Z}$-coset of an index n. That is,

$$X^* = \{(a_l(n))_{(l,n)\in\mathbb{Z}_r\times\Omega_j} : j \in \mathbb{Z}^d\},$$

with $\Omega_j := j + N\mathbb{Z}^d$ and each element of the adjoint system being the concatenation of the filters entries indexed by the $N\mathbb{Z}^d$-coset of an index. For higher dimensions, it is still true that each element in X^* is formed by concatenating different masks entries that lie in the same $N\mathbb{Z}^d$-coset of an index.

Suppose B is the matrix defined in (4.5) associated with filters $\{b_l\}_{l=0}^{r-1}$ and m in this case is the maximum number of the support of $\{a_l, b_l\}_{l=0}^{r-1}$. Essentially, this block-wise observation on the pre-Gramian matrix reduces the infinitely many biorthonormal requirements on the rows of J_X and J_Y where $Y = X(b,N)$ to finitely many conditions on the matrices A and B.

Theorem 4.3.1. *Let $X = X(a,N)$ and $Y = X(b,N)$, for FIR filters $a = \{a_l\}_{l=0}^{r-1}$ and $b = \{b_l\}_{l=0}^{r-1}$ in $\ell_2(\mathbb{Z}^d)$ and $N \in \mathbb{N}$. Then X and Y are dual frames in $\ell_2(\mathbb{Z}^d)$, provided that*

$$\sum_{l=0}^{r-1} \overline{a_l(n)} b_l(n') = 0, \tag{4.6}$$

$$\sum_{l=0}^{r-1}\sum_{n\in\Omega_j}\overline{a_l(n)}b_l(n)=1 \tag{4.7}$$

for all $n,n'\in\mathbb{Z}^d$ *with* $n\neq n'$ *and all* $j\in\mathbb{Z}^d/N\mathbb{Z}^d$*. The system* X *is a tight frame when* $a_l=b_l$ *for* $l=0,\ldots,r-1$*.*

Proof. Let $n\in\mathbb{Z}^d$. If $n'\neq n$, then (4.6) implies

$$\sum_{l=0}^{r-1}\sum_{k\in\mathbb{Z}^d}\overline{a_l(n'-Nk)}b_l(n-Nk)=\sum_{k\in\mathbb{Z}^d}\sum_{l=0}^{r-1}\overline{a_l(n'-Nk)}b_l(n-Nk)=0,$$

while for $n'=n$, (4.7) implies

$$\sum_{l=0}^{r-1}\sum_{k\in\mathbb{Z}^d}\overline{a_l(n-Nk)}b_l(n-Nk)=\sum_{l=0}^{r-1}\sum_{k\in\Omega_n}\overline{a_l(k)}b_l(k)=1.$$

Thus the masks satisfy $\widetilde{G}_{Y,X}=I$. □

This condition leads to the following result on the construction of perfect reconstruction filter banks. Given any linearly independent FIR filters defining some analysis filter bank, it provides large degrees of freedom in designing a synthesis filter bank via a matrix inversion.

Construction 4.3.2. *Let* $A=(a_l(n_j))_{l\in\mathbb{Z}_r,j\in\mathbb{Z}_r}\in\mathbb{C}^{r\times r}$ *be invertible and* $M\in\mathbb{C}^{r\times r}$ *be a diagonal matrix with diagonal* $(c(n_0),\ldots,c(n_{r-1}))$ *such that*

$$\sum_{n\in\Omega_j}c(n)=1$$

for every $j\in\mathbb{Z}^d/N\mathbb{Z}^d$*. Let* $B=(b_l(n_j))_{l\in\mathbb{Z}_r,j\in\mathbb{Z}_r}=(A^*)^{-1}M$*. Then the filters* $a=\{a_l\}_{l=0}^{r-1}\subset\ell_2(\mathbb{Z}^d)$ *and* $b=\{b_l\}_{l=0}^{r-1}\subset\ell_2(\mathbb{Z}^d)$ *defined by* A *and* B *generate dual frames* $X(a,N)$ *and* $X(b,N)$ *in* $\ell_2(\mathbb{Z}^d)$*.*

Proof. By construction $A^*B = M$. Thus the n-th column of A is orthogonal to the m-th column of B whenever $n \neq m$. Moreover, $\sum_{l=0}^{r}\sum_{n\in\Omega_j}\overline{a_l(n)}b_l(n) = \sum_{n\in\Omega_j} c(n) = 1$ for every $j \in \mathbb{Z}^d/N\mathbb{Z}^d$. The claim therefore follows from Theorem 4.3.1 since the systems are Bessel due to the finite support of the filters. □

Remark. Note let A be the matrix defined in (4.5) associated with masks $\{a_l\}_{l=0}^{r-1}$ and B associated with masks $\{b_l\}_{l=0}^{r-1}$. Theorem 4.3.1 says $A^*B = M$ where M is a diagonal matrix with the diagonal entries c (which has the same indexing as the masks) satisfying $\sum_{n\in\Omega_j} c(n) = 1$ for all $j \in \mathbb{Z}^d/N\mathbb{Z}^d$. This assumption on the filters is strong and Construction 4.3.2 based on this theorem hence is limited in the sense that the proposed construction could not cover all the possible constructions by using $\widetilde{G}_{Y,X} = I$. The condition $\widetilde{G}_{Y,X} = I$ indeed can still be written as a condition $A^*B = M$ but the matrix M is not necessarily a diagonal matrix.

Under stronger conditions on the matrices one can derive tight frame filter banks. If in Construction 4.3.2 one starts out with a unitary matrix A and a diagonal matrix M which in addition has only nonnegative entries, then the rows of the matrix $AM^{1/2}$ define a tight frame filter bank.

4.4 Irregular Gabor systems

The abstract pre-Gramian matrix can be defined for any system, but it may be simpler in case the system exhibits a structure that can be exploited by choosing an appropriate orthonormal basis. In this section, we discuss the duality principle for irregular Gabor systems in $L_2(\mathbb{R}^d)$ with no assumption on the structure of the sampling set. Given a countable subset $\Lambda \subset \mathbb{R}^{2d}$, consider the irregular Gabor system

$$X = \{E^\gamma M^\eta \phi \colon (\gamma,\eta) \in \Lambda\}$$

generated by the window function $\phi \in L_2(\mathbb{R}^d)$. Any orthonormal basis of $L_2(\mathbb{R}^d)$ can be used for the dual Gramian analysis of this system. Given the modulation and translation structure of X, a natural choice is

$$\{|\widetilde{K}|^{-1/2} M^k E^{\tilde{k}} \chi_{\Omega_{\widetilde{K}}} \colon k \in K, \tilde{k} \in \widetilde{K}\} \tag{4.8}$$

for some lattice $K \subset \mathbb{R}^d$. The resulting pre-Gramian is

$$J_X = |\widetilde{K}|^{-1/2} (e^{i\eta\cdot(\tilde{k}-\gamma)} \langle M^{\eta-k} E^{\gamma-\tilde{k}} \phi, \chi_{\Omega_{\widetilde{K}}} \rangle)_{(k,\tilde{k})\in K\times\widetilde{K}, (\gamma,\eta)\in\Lambda}.$$

The pre-Gramian matrix seems complicated but it has a blockwise structure. For example, if $\tilde{k}, \gamma$ are fixed, the submatrix indexed by (k, η) is formed by the Fourier sequence of $E^{\gamma-\tilde{k}}\phi$ on $\Omega_{\widetilde{K}}$. Moreover, if ϕ has, say, compact support, then J_X is a sparse matrix, composed of infinitely many infinite band-matrix blocks. In case the rows of J_X are square summable, i.e. if

$$\sum_{(\gamma,\eta)\in\Lambda} |\langle M^{\eta-k} E^{\gamma-\tilde{k}} \phi, \chi_{\Omega_{\widetilde{K}}} \rangle|^2 < \infty \quad \text{for all } k \in K, \tilde{k} \in \widetilde{K}, \tag{4.9}$$

the entries of the dual Gramian

$$\widetilde{G}_{X,X} = J_X J_X^* = |\widetilde{K}|^{-1} \left(\sum_{(\gamma,\eta)\in\Lambda} \langle M^{\eta-k} E^{\gamma-\tilde{k}} \phi, \chi_{\Omega_{\widetilde{K}}} \rangle \langle \chi_{\Omega_{\widetilde{K}}}, M^{\eta-k'} E^{\gamma-\tilde{k}'} \phi \rangle \right)_{(k,\tilde{k}),(k',\tilde{k}')\in K\times\widetilde{K}}$$

of X are well-defined. The condition (4.9) is weak in the sense that it is always satisfied if system X is Bessel. The rows of J_X define an adjoint system X^*

$$X^* = \{(|\widetilde{K}|^{-1/2} e^{i\eta\cdot(\tilde{k}-\gamma)} \langle M^{\eta-k} E^{\gamma-\tilde{k}} \phi, \chi_{\Omega_{\widetilde{K}}} \rangle)_{(\gamma,\eta)\in\Lambda} \colon k \in K, \tilde{k} \in \widetilde{K}\} \tag{4.10}$$

in $\ell_2(\Lambda)$ if in addition the columns of J_X are square summable, i.e. that

$$\sum_{(k,\tilde{k})\in K\times\widetilde{K}} |\langle M^{\eta-k}E^{\gamma-\tilde{k}}\phi, \chi_{\Omega_{\widetilde{K}}}\rangle|^2 < \infty \quad \text{for all } (\gamma,\eta)\in\Lambda. \tag{4.11}$$

The system X is being considered in $\ell_2(K\times\widetilde{K})$ via its coefficient sequences with respect to the chosen orthonormal basis, while its adjoint system X^* is being considered in $\ell_2(\Lambda)$, i.e. as a system of sequences indexed by the original system X itself. In accordance with Definition 4.1.1, any image of X^* under a unitary map into a separable Hilbert space is as well an adjoint system of X. To get an adjoint system in $L_2(\mathbb{R}^d)$, one may for example use the orthonormal basis (4.8) to unitarily map X^* into $L_2(\mathbb{R}^d)$. That is, identifying Λ with $K\times\widetilde{K}$ via a bijection $\Lambda\to K\times\widetilde{K}\colon(\gamma,\eta)\mapsto(R\gamma,R\eta)$, one may consider the adjoint system $\{f_{k,\tilde{k}}\}_{(k,\tilde{k})\in K\times\widetilde{K}}\subset L_2(\mathbb{R}^d)$, where

$$f_{k,\tilde{k}} := |\widetilde{K}|^{-1}\sum_{(\gamma,\lambda)\in\Lambda} e^{i\lambda\cdot(\tilde{k}-\gamma)}\langle M^{\lambda-k}E^{\gamma-\tilde{k}}\phi,\chi_{\Omega_{\widetilde{K}}}\rangle M^{R\gamma}E^{R\eta}\chi_{\Omega_{\widetilde{K}}}. \tag{4.12}$$

Though the expression (4.12) is complicated, it is nothing but a reconstruction by the Gabor orthonormal basis (4.8) using sequences in ℓ_2. By using the same orthonormal basis (4.8), it is easy to see that the pre-Gramian matrix of this adjoint system (4.12) is the same as that of the system in sequence space (4.10), and moreover the dual Gramian of X is equal to the Gramian of X^*, i.e.

$$\widetilde{G}_X = G_{X^*}, \tag{4.13}$$

which is the duality principle for irregular Gabor systems. This duality principle has the following consequence for irregular Gabor systems generated by a single window.

Theorem 4.4.1. *Let $X=\{E^\gamma M^\eta\phi\colon(\gamma,\eta)\in\Lambda\}$ be an irregular Gabor system and $K\subset\mathbb{R}^d$ be a lattice such that (4.9) and (4.11) hold. Further, let X^* be the $\ell_2(\Lambda)$ system*

(4.10) *or its image under some unitary map in some Hilbert space. Then:*

(a) *The system X is a Bessel system if and only if X^* is a Bessel system, in which case the Bessel bounds coincide.*

(b) *If X is Bessel, then it is fundamental if and only if X^* is ℓ_2-independent.*

(c) *The system X is a frame if and only if X^* is a Riesz sequence, in which case the frame bounds and Riesz bounds coincide. In particular, X is a tight frame if and only if X^* is an orthonormal sequence.*

Given a second window, i.e. a second system $Y = \{E^\gamma M^\eta \psi \colon (\gamma, \eta) \in \Lambda\}$, and considering J_Y and Y^* with respect to the same orthonormal basis (4.8) (and under the same unitary map as for X^*), then we have the duality principle for two irregular Gabor systems

$$\widetilde{G}_{X,Y} = G_{X^*,Y^*}, \tag{4.14}$$

which yields the following characterization of dual frames.

Theorem 4.4.2. *Suppose $X = \{E^\gamma M^\eta \phi \colon (\gamma, \eta) \in \Lambda\}$ and $Y = \{E^\gamma M^\eta \psi \colon (\gamma, \eta) \in \Lambda\}$ are Bessel systems. Let X^* and Y^* be their respective adjoint systems with respect to* (4.8) *(see* (4.10)*) or their image under some unitary map (see* (4.12)*). Then X and Y are dual frames if and only if X^* and Y^* are biorthonormal.*

The adjoint system is defined for the particular orthonormal basis (4.8) and the duality principle is stated. While any other orthonormal basis is possible, we have already seen in Section 3.5 that in case the shifts and modulations of a Gabor system are regular lattices, the abstract pre-Gramian with an appropriate choice of orthonormal basis could be simplified to the fiber matrices of shift-invariant systems. Moreover, this allows for adjoint systems, which, back in the original space $L_2(\mathbb{R}^d)$, not only have a closed form expression, but also have a Gabor structure closely tied to that of the original system. In [53, 54], a Gabor system defined on an adjoint lattice for a general non-separable lattice is introduced and duality principle is discussed, but the analysis is somewhat technical.

4.5 Duality principle for (regular) Gabor systems

Definition 4.1.1 for adjoint systems is most general in the sense that one can always consider the rows of the pre-Gramian of the original system as a new system, which will lead to the duality between the two systems whenever those rows and columns are square summable. An adjoint system of a shift-invariant system, can be defined through Definition 4.1.1 and the fiber pre-Gramian matrices, but in general it may not be possible to give it in an explicit form, i.e. as a shift-invariant system.

Denoting the columns of the fiber pre-Gramian matrix $\mathcal{J}_X(\omega)$ in (3.16) of a shift-invariant system X as a system X_ω at each $\omega \in \Omega_{\widetilde{K}}$, the fiberization technique transfers the properties of X to properties of the collection of systems $\{X_\omega\}_{\omega\in\Omega_{\widetilde{K}}}$, see [94]. For example, a shift-invariant system is a frame if and only if the systems X_ω are frame systems uniformly for a.e. $\omega \in \Omega_{\widetilde{K}}$ with the same frame bounds. On each fiber, the adjoint system of X_ω can be introduced as the collection of rows of $\mathcal{J}_X(\omega)$ and the duality principle can be discussed. In particular, the collection of all adjoint systems of the systems $\{X_\omega\}_{\omega\in\Omega_{\widetilde{K}}}$ can be considered as an adjoint system of the given shift-invariant system. The adjoint system defined this way may not have an explicit form as a shift-invariant system. In case of a given Gabor shift-invariant system X, it is possible, as we will review below, to find another Gabor system Y, such that all of its fiber system Y_ω are adjoint systems of the respective fiber systems X_ω. In other words, on each fiber, $\mathcal{J}_X(\omega) = \mathcal{J}^*_Y(\omega)$ (modulo a complex conjugation). However, the fiber systems and the corresponding duality principle discussed on each fiber is still of some interest. This weak formulation of the duality principle is very convenient for the construction of tight/dual Gabor windows detailed in Section 4.7. It is also the foundation for the tight (resp. dual) wavelet frame construction through the UEP (resp. MEP), see (5.2) below.

Note the matrices $\mathcal{G}_{Y,X}(\omega)$ in (3.24) and $\widetilde{\mathcal{G}}_{Y,X}(\omega)$ in (3.20) have very similar structure. By changing the lattices from (K,L) to $(\widetilde{L},\widetilde{K})$, the mixed Gramian fiber matrices $\mathcal{G}_{Y,X}(\omega)$ of the two new systems are then essentially the same as the mixed dual Gramian matrices $\widetilde{\mathcal{G}}_{Y,X}(\omega)$ of the original systems. On the pre-Gramian level this means the columns of the pre-Gramian of the original system are the rows of the pre-Gramian of the new system. With this in mind the *adjoint system* of the Gabor system $X = (K,L)_\phi$ has been defined in [98] (up to the scalar factor) as

$$X^* = (\mathrm{den}(K,L))^{1/2}(\widetilde{L},\widetilde{K})_\phi. \tag{4.15}$$

The $(l,\tilde{k}) \in (L,\widetilde{K})$ entry of $\mathcal{J}_{X^*}(\omega)$ is $|K|^{-1/2}\hat{\phi}(\omega - l - \tilde{k})$, hence

$$\mathcal{J}_{X^*}(\omega) = \overline{\mathcal{J}_X^*(\omega)} \tag{4.16}$$

for all $\omega \in \mathbb{R}^d$. On each fiber level, this is the analogy to the abstract Definition 4.1.1. Given another Gabor systems $Y = (K,L)_\psi$, one observes

$$\widetilde{\mathcal{G}}_{X,Y}(\omega) = \mathcal{J}_X(\omega)\mathcal{J}_Y^*(\omega) = \overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Y^*}(\omega)} = \overline{\mathcal{G}_{X^*,Y^*}(\omega)},$$

for all $\omega \in \mathbb{R}^d$, which is the duality principle for Gabor systems.

Theorem 4.5.1. *Let $X = (K,L)_\phi$ and $Y = (K,L)_\psi$ be Bessel systems and X^*, Y^* their respective adjoint systems defined in (4.15). Then $\widetilde{\mathcal{G}}_{X,Y}(\omega) = \overline{\mathcal{G}_{X^*,Y^*}(\omega)}$ for all $\omega \in \mathbb{R}^d$. Consequently, X and Y are dual frames if and only if X^* and Y^* are biorthonormal.*

The roles of X and X^* are interchangable since $X^{**} = X$. If $X = Y$, the duality principle says that the dual Gramian matrices of X are (up to complex conjugation) equal to the Gramian matrices of its adjoint system X^*. This is the essence of the duality principle derived in [98] for single systems. In the case of single systems, the Bessel asssumption on the system is no longer needed. Moreover, the (dual) Gramian matrices are Hermitian, the boundedness of the relevant operators, i.e. the Bessel property, can also be characterized through the (dual) Gramian fiber matrices. The duality principles for single systems therefore are (see [98, Theorem 2.2]): the system X is Bessel if and only if X^* is Bessel, in which case the Bessel bounds coincide; if X is Bessel, then it is fundamental if and only if X^* is ℓ_2-independent; the system X is a frame if and only if X^* is a Riesz sequence, in which case the frame bounds of X coincide with the Riesz bounds of X^*; in particular, X is a tight frame if and only if X^* is an orthonormal sequence.

In the abstract setting, a simple matrix relation reveals the relationship Theorem 4.1.8 between the mixed frame operator of systems and their adjoints. For three Gabor Bessel systems $X=(K,L)_\phi$, $Y=(K,L)_\psi$ and $Z=(K,L)_g$, both $X'=T_YT_Z^*X$ and $Y'=T_{X^*}T_{Z^*}^*Y^*$ are Gabor systems with windows $T_YT_Z^*\phi$ and $(\mathrm{den}(K,L)^{1/2})T_{X^*}T_{Z^*}^*\psi$, respectively. As in Theorem 4.1.8, the new system X' is an adjoint system of Y'.

Theorem 4.5.2. *Suppose $X=(K,L)_\phi$, $Y=(K,L)_\psi$ and $Z=(K,L)_g$ are Bessel systems, with adjoint systems X^*, Y^*, Z^* defined in (4.15). Then $T_YT_Z^*X$ is an adjoint system of $T_{X^*}T_{Z^*}^*Y^*$. In particular, the windows of the two systems coincide, i.e.*

$$T_YT_Z^*\phi = T_{X^*}T_{Z^*}^*\psi. \tag{4.17}$$

Proof. As observed in [98], (4.17) follows from the matrix identity

$$\mathcal{J}_Y(\omega)\mathcal{J}_Z^*(\omega)\mathcal{J}_X(\omega) = (\mathcal{J}_X^*(\omega)\mathcal{J}_Z(\omega)\mathcal{J}_Y^*(\omega))^* = (\overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)})^*.$$

By (3.21) the (0,0)th entry of $\mathcal{J}_Y(\omega)\mathcal{J}_Z^*(\omega)\mathcal{J}_X(\omega)$ is equal to $(T_Y T_Z^* \phi)^\wedge(\omega)$ for a.e. ω. On the other hand, the (0,0)th entry of $(\overline{\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)})^*$ equals the (0,0)th entry of $\mathcal{J}_{X^*}(\omega)\mathcal{J}_{Z^*}^*(\omega)\mathcal{J}_{Y^*}(\omega)$, which is equal to $(T_{X^*}T_{Z^*}^*\psi)^\wedge(\omega)$ for a.e. ω. □

Remark. The abstract pre-Gramian can be used to define adjoint systems of a Gabor system $(K, L)_\phi$. The question whether $(\widetilde{L}, \widetilde{K})_\phi$ can be defined through this abstract pre-Gramian setting is also asked in [15] by using the R-dual sequences. The question does not necessarily arise if one considers the row-column relationship of different matrix representations of the synthesis operator. A system and its R-dual sequence are adjoint via the abstract pre-Gramian representation, while $(K, L)_\phi$ and $(\widetilde{L}, \widetilde{K})_\phi$ are adjoint via the fiber pre-Gramian representation. Different matrix representations can be particularly tailored to make best use of a specific structure of the given system. Note that adjoints systems defined through the abstract pre-Gramian directly yield a statement about the synthesis operators of the respective systems to be adjoint operators (up to a unitary transform). The fiber adjoint relation between $(K, L)_\phi$ and $(\widetilde{L}, \widetilde{K})_\phi$ in general makes finding the connection of operators much more involved since the domain and target spaces become more subtle, interested reader is referred to Section 3.2 of [98] for more discussions. However, the relationship on fiber level is sufficed to give the characterization of system properties of interest, e.g. Bessel, Riesz sequence and frame.

4.6 Duality identities for Gabor systems

The essence of several important classical identities for dual Gabor frames is the duality principle. We briefly show how three classical results, the Walnut representation, the Wexler-Raz biorthogonality relation and the Janssen/Wexler-Raz identity, are as well merely reformulations and certain aspects of the fiberized dual Gramain matrix representation of the Gabor frame operator and the duality principle.

Throughout this section, let $X = (K,L)_\phi$ and $Y = (K,L)_\psi$ be Bessel Gabor systems. The Walnut representation is the fiberized representation of the Gabor frame operator in time domain (3.23) evaluated at $\tilde{l} = 0$, which says that

$$T_Y T_X^* f = |\widetilde{L}| \sum_{\tilde{l} \in \widetilde{L}} \sum_{k \in K} E^k \psi \overline{E^{\tilde{l}+k}\phi} E^{\tilde{l}} f$$

for all $f \in L_2(\mathbb{R}^d)$. This representation has first been proposed for rectangular lattices in [112] where it is shown under the technical condition that the windows ϕ, ψ belong to the Wiener space

$$\{g \in L_\infty(\mathbb{R}^d) \colon \sum_{n \in \mathbb{Z}^d} \|g E^n \chi_{[0,1]^d}\|_\infty < \infty\}.$$

This condition implies that X and Y are Bessel systems (see [98, Corollary 3.26]). The Bessel condition is the natural and weaker condition for the fiberized representation (3.23) to hold.

With the notion of adjoint system, the mixed frame operator can be written as

$$T_Y T_Z^* \phi = T_{X^*} T_{Z^*}^* \psi. \tag{4.18}$$

which holds for Bessel systems X, Y and $Z = (K,L)_g$ and their respective adjoints X^*, Y^* and Z^* as defined in (4.15). This is known as Wexler-Raz or Janssen identity. This duality identity, as shown in Theorem 4.5.2, is part of $T_Y T_Z^* X$ and $T_{X^*} T_{Z^*}^* Y^*$ being adjoint systems of each other.

Given two Gabor Bessel systems X and Y, they are dual frames if and only if

$$(\mathrm{den}(K,L))\langle \psi, E^{\tilde{l}} M^{\tilde{k}} \phi \rangle = \delta_{\tilde{l}0}\delta_{\tilde{k}0} \tag{4.19}$$

for all $\tilde{l} \in \widetilde{L}$ and $\tilde{k} \in \widetilde{K}$. This however, known as Wexler-Raz biorthogonality relations, is only one aspect of the duality principle Theorem 4.5.1 which establishes $\widetilde{\mathcal{G}}_{X,Y}(\omega) = \overline{\mathcal{G}_{X^*,Y^*}(\omega)}$ and under the Bessel assumption leads to the statement about the systems. The Wexler-Raz biorthogonality relation is picking out one case of this more general duality identity, the case that the dual Gramian fibers are the identity if and only if the Gramian fibers of their adjoint systems are. The statement that those matrices are actually equal, however implies many other interesting dualities, as we have already indicated above. The Wexler-Raz biorthogonality relations have independently been proved in various places, e.g. the approach in [39, 70] is via the the Janssen/Wexler-Raz identity (4.18).

4.7 Dual Gabor windows construction

In this section, we explicitly construct windows for dual Gabor frames. Of course any given Gabor frame and its canonical dual frame, which is generated by the image of the primary window under the inverse of the frame operator, are dual frames (see e.g. [98]). Though the canonical dual has the minimal norm property mentioned earlier, alternate dual windows might be more desirable in terms of, e.g., support size and smoothness.

Given two Bessel Gabor systems $X = (K,L)_\phi$ and $Y = (K,L)_\psi$, we have seen in Section 3.6 how the mixed Gabor frame operator $T_Y T_X^*$ can be represented in Fourier domain (3.21) and time domain (3.23). The systems X and Y are dual frames if this operator is equal to the identity, i.e. if the rows of the pre-Gramians (3.19) of X and Y

are biorthonormal at almost every fiber, and the representations (3.21) and (3.23) result in the following characterization of dual window pairs.

Proposition 4.7.1 ([98]). *If $\phi, \psi \in L_2(\mathbb{R}^d)$ are such that the Gabor systems $(K, L)_\phi$ and $(K, L)_\psi$ are Bessel systems, then $(K, L)_\phi$ and $(K, L)_\psi$ are dual frames if and only if one (and therefore both) of the following conditions holds:*

$$\sum_{k \in K} E^k \phi E^{k+\tilde{l}} \overline{\psi} = |\widetilde{L}|^{-1} \delta_{\tilde{l},0} \quad \text{for all } \tilde{l} \in \widetilde{L}, \tag{4.20}$$

and

$$\sum_{l \in L} E^l \hat{\phi} E^{\tilde{k}+l} \overline{\hat{\psi}} = |K| \delta_{\tilde{k},0} \quad \text{for all } \tilde{k} \in \widetilde{K}. \tag{4.21}$$

Note that (4.20) is equivalent to the biorthogonality relationship (4.19) while (4.21) is equivalent to the Fourier version. Note the duality principle Theorem 4.5.1 says the fiber mixed dual Gramian matrix of two Bessel systems is equal to the fiber mixed Gramian matrix of the adjoint systems. Proposition 4.7.1 only states the special case when the mixed dual Gramian matrix is the identity. This is then also equivalent to the fiber mixed Gramian matrix of the adjoint systems being the identity, or the biorthogonality relationship (4.19). Proposition 4.7.1 transfers the dual frame property of the columns to the birorthogonal relationship of the rows on each fiber, which greatly simplifies the construction of windows. In case the two windows ϕ and ψ coincide, a construction scheme based on this duality point of view has already been sketched in [98] for tight Gabor frame windows construction in $L_2(\mathbb{R}^d)$, which has also been observed in [36] as the "painless" way to construct tight Gabor frames for $L_2(\mathbb{R})$. Based on Proposition 4.7.1, we propose the following for constructing dual Gabor frame windows.

Proposition 4.7.2. *Let $\phi, \psi \in L_2(\mathbb{R}^d)$ be compactly supported and bounded, such that $\sum_{k \in K} E^k(\phi\overline{\psi}) = 1$. Choose a lattice L such that $\widetilde{L}$ is sparse enough to ensure that the* *support of $E^{\tilde{l}}\psi$ is disjoint from the support of ϕ for all $\tilde{l} \in \widetilde{L} \setminus \{0\}$. Then the Gabor systems $(K,L)_{c\phi}$ and $(K,L)_{c\psi}$, where $c = |\widetilde{L}|^{-1/2}$, are dual frames.*

Proof. Since ϕ and ψ are compactly supported and bounded, both Gabor systems are Bessel systems, see [94, Corollary 1.6.3]. Moreover, the assumption on the functions to be compactly supported implies that the sum in (4.20) only has finitely many nonzero terms. By the choice of L, condition (4.20) reduces to one condition, namely for the case $\tilde{l} = 0$. This condition is met after scaling, since $\phi\overline{\psi}$ is a partition of unity with respect to K. $\square$

Note that the shift lattice K is determined by the shift size in the partition of unity. One could however use some dilation to restate the results for arbitrary sizes of the shift lattice. An analogue result can be formulated, starting from (4.21) instead of (4.20), i.e. by changing the roles of the lattices, the partition of unity in Proposition 4.7.2 might also be used to contruct bandlimited dual windows. We illustrate the construction, starting from different classes of partitions of unity.

Constructed partition of unity. Partitions of unity characterize orthonormal refinable functions for multiresolution analysis and one may draw this from the classical orthonormal wavelet constructions for Gabor window constructions.

Example 4.7.3. *In [87], Meyer constructed a function*

$$h(x) = \cos\left[\frac{\pi\beta}{2}\left(\frac{3|x|}{2\pi} - 1\right)\right]$$

for $x \in \mathbb{R}$, where β is some C^k or C^∞-function for which $\beta(x) = 0$ if $x \leq 0$ and $\beta(x) = 1$ if $x \geq 1$, and

$$\beta(x) + \beta(1-x) = 1 \tag{4.22}$$

for all $x \in \mathbb{R}$. *Note that this implies* $h(x) = 1$ *for* $|x| \leq 2\pi/3$ *and* $h(x) = 0$ *for* $|x| \geq 4\pi/3$. *Moreover, the regularity of* h *is the same as the regularity of* β. *Since* β *satisfies* (4.22), *one gets*

$$\sum_{k\in 2\pi\mathbb{Z}} |h(x+k)|^2 = 1.$$

Therefore, $g = \sqrt{3/(8\pi)}h$ *is the window of a tight Gabor frame with respect to* $(2\pi\mathbb{Z}, \frac{3}{4}\mathbb{Z})$. *For a factorization of* $|g|^2$ *into two different functions one can get dual Gabor windows.*

Factoring piecewise polynomials. One family of functions whose shifts form a partition of unity are B-splines. The B-spline B_m of order $m \in \mathbb{N}$ is inductively given by $B_1 = \chi_{[0,1]}$ and $B_{m+1} = B_m * B_1$. Since $\sum_{k\in\mathbb{Z}} B_m(x+k) = 1$, if one factors B_m into two bounded functions supported in $[0,m]$, say $B_m = \phi\psi$, then for any $a \geq m$ the functions $a^{-1/2}\phi$ and $a^{-1/2}\psi$ are dual Gabor windows with respect to $(\mathbb{Z}, 2\pi a^{-1}\mathbb{Z})$. Since B-splines are nonnegative, one may also take their square root as window function for a tight frame. This idea has also been used in [74] to construct discrete tight Gabor frames, which exhibit good orientation selectivity and are very useful in various image processing problems.

Example 4.7.4. *Starting from the linear B-spline* B_2, *we get that* $2^{-1/2}B_2$ *and* $2^{-1/2}\chi_{[0,2]}$ *are dual Gabor windows with respect to* $(\mathbb{Z}, \pi\mathbb{Z})$. *The cubic B-spline*

$$B_4(x) = \frac{1}{6}\begin{cases} x^3 & \text{if } 0 \leq x < 1 \\ -3x^3 + 12x^2 - 12x + 4 & \text{if } 1 \leq x < 2 \\ 3x^3 - 24x^2 + 60x - 44 & \text{if } 2 \leq x < 3 \\ (4-x)^3 & \text{if } 3 \leq x < 4 \\ 0 & \text{else} \end{cases},$$

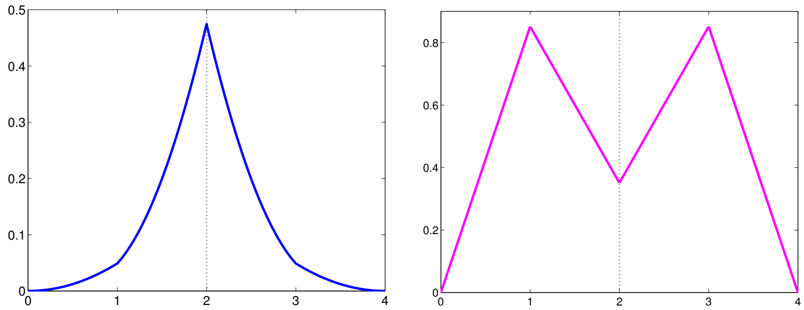


Fig. 4.1 Primary and dual Gabor windows constructed from the cubic B-spline in Example 4.7.4.

can be factored into a piecewise linear and a piecewise quadratic polynomial:

$$\tilde{\phi}(x) = \frac{1}{6}\begin{cases} (a-1)^{-1}x^2 & \textit{if } 0 \le x < 1 \\ x^2 + (3a-12)x + 4a^{-1} & \textit{if } 1 \le x < 2 \\ x^2 + (3b-24)x + 44b^{-1} & \textit{if } 2 \le x < 3 \\ (3-b)^{-1}(4-x)^2 & \textit{if } 3 \le x < 4 \\ 0 & \textit{else} \end{cases}, \quad \tilde{\psi}(x) = \begin{cases} (a-1)x & \textit{if } 0 \le x < 1 \\ -x+a & \textit{if } 1 \le x < 2 \\ x-b & \textit{if } 2 \le x < 3 \\ (3-b)(4-x) & \textit{if } 3 \le x \le 4 \\ 0 & \textit{else} \end{cases},$$

where a (resp. b) is the (only) real solution of the second (resp. third) cubic equation in B_4. Thus $2^{-1}\tilde{\phi}$ and $2^{-1}\tilde{\psi}$ are dual Gabor windows with respect to $(\mathbb{Z}, \frac{\pi}{2}\mathbb{Z})$ (see Figure 4.1).

In principle this method can be used on B-splines of higher order by solving higher order polynomial equations in order to get factorizations. In general, however, those solutions will be numerical in nature and have no explicit closed form. Moreover, using this method one can only guarantee continuity but no higher order smoothness of the windows. In order to construct smoother windows, we now turn to alternative constructions.

Trigonometric polynomials. Starting from the identity $\cos^2 x + \sin^2 x = 1$, and restricting ourselves to functions of compact support, let $h = \cos^2(\cdot\,\pi/2)\chi_{[-1,1]}$. Then

$h(x)+h(x-1)=1$ for all $x\in[0,1]$ and the integer shifts of h are a partition of unity. Factoring h into two functions, e.g. $\cos(\cdot\,\pi/2)\chi_{[-1,1]}$ and $\cos(\cdot\,\pi/2)\chi_{[-1,1]}$, we can get two symmetric and continuous dual Gabor windows with respect to $(\mathbb{Z},\pi\mathbb{Z})$. In order to get windows with higher smoothness, we now improve this construction by leveraging on the idea for the construction of pseudo-spline wavelet masks [44], namely that higher powers can result in higher smoothness. That is, we now start from the identity

$$1=\left(\cos^2\left(\frac{\pi x}{2}\right)+\sin^2\left(\frac{\pi x}{2}\right)\right)^{2m-1}, \tag{4.23}$$

for any given nonnegative integer m. Define h in a similar fashion as above by cutting off the first m-terms of the binomial expansion of (4.23), i.e. let

$$h(x)=\cos^{2m}\left(\frac{\pi x}{2}\right)\sum_{j=0}^{m-1}\binom{2m-1}{j}\cos^{2(m-1-j)}\left(\frac{\pi x}{2}\right)\sin^{2j}\left(\frac{\pi x}{2}\right)\chi_{[-1,1]}(x). \tag{4.24}$$

As above integer shifts of h are a partition of unity. Taking, say, $l,\tilde{l}\in\mathbb{N}$ such that $l+\tilde{l}=m$, h can be factored into the two functions

$$\phi(x)=\cos^{2l}\left(\frac{\pi x}{2}\right)\chi_{[-1,1]}(x),$$

and

$$\psi(x)=\cos^{2\tilde{l}}\left(\frac{\pi x}{2}\right)\sum_{j=0}^{m-1}\binom{2m-1}{j}\cos^{2(m-j-1)}\left(\frac{\pi x}{2}\right)\sin^{2j}\left(\frac{\pi x}{2}\right)\chi_{[-1,1]}(x).$$

Then for any $a\geq|\operatorname{supp}h|=2$, the functions $a^{-1/2}\phi$ and $a^{-1/2}\psi$ are dual Gabor windows with respect to $(\mathbb{Z},2\pi a^{-1}\mathbb{Z})$. Note that the larger m is chosen, the smoother one can make the two functions.

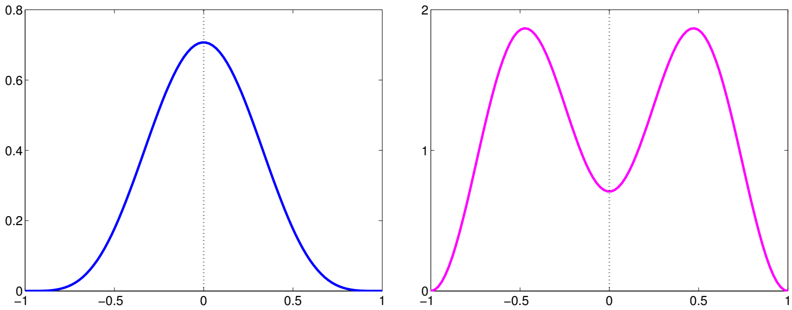


Fig. 4.2 Primary and dual Gabor windows constructed from trigonometric polynomials in Example 4.7.5.

Example 4.7.5. *Let $l = 2$ nad $\tilde{l} = 1$. Let*

$$\phi(x) = \cos^4\left(\frac{\pi x}{2}\right)\chi_{[-1,1]}(x),$$

and

$$\psi(x) = \cos^2\left(\frac{\pi x}{2}\right)\left(\cos^4\left(\frac{\pi x}{2}\right) + 5\cos^2\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right) + 10\sin^4\left(\frac{\pi x}{2}\right)\right)\chi_{[-1,1]}(x).$$

Then $2^{-1/2}\phi$ and $2^{-1/2}\psi$ are dual Gabor windows with respect to $(\mathbb{Z}, \pi\mathbb{Z})$. The window functions are shown in Figure 4.2.

In [77] and [22], the authors are motivated from the solutions of

$$\hat{a}(\pi\omega)\overline{\hat{b}(\pi\omega)} + \hat{a}(\pi(\omega+1))\overline{\hat{b}(\pi(\omega+1))} = 1, \tag{4.25}$$

to construct dual windows of equal support size. Several of their windows coincide with ours, which is not surprising since the trigonometric polynomials we construct from (4.24), solve (4.25) [44]. However, our method is easier than solving the polynomial equation (4.25) directly and moreover can be easily generalized to higher dimensions.

The idea to use higher powers in order to improve smoothness has already been used [73] for constructing wavelet masks satisfying interpolatory conditions and higher order smoothness properties. A generalization to multidimensions has for example been considered in [72]. There, one starts as well from a partition of unity, namely let $a \in \ell_0(\mathbb{Z}^d)$ satisfy the *interpolatory condition*

$$\sum_{\nu \in \mathbb{Z}_2^d} \hat{a}(\omega + \pi\nu) = 1 \quad \text{for all } \omega \in \mathbb{R}^d, \tag{4.26}$$

where $\mathbb{Z}_2^d := \mathbb{Z}^d / 2\mathbb{Z}^d$ and $\hat{a}(\omega) = \sum_{n \in \mathbb{Z}^d} a(n) e^{-in\cdot\omega}$. After raising to some positive integer power, this interpolatory condition is being factored in [72] to construct higher order smoothness interpolatory functions in high dimensions. Here we will use the factoring to derive dual windows. The technique of [72] is as follows. The formal Laurent polynomial P corresponding to the mask $\hat{a}$ as

$$P(z) = \sum_{n \in \mathbb{Z}^d} a(n) z^n.$$

Defining

$$P_\nu(z) = P(z \exp(-\pi i \nu)), \quad \nu \in \mathbb{Z}_2^d, \ |z| = 1,$$

the interpolatory condition (4.26) is equivalent to

$$\sum_{\nu \in \mathbb{Z}_2^d} P_\nu(z) = 1 \quad \text{for all } |z| = 1. \tag{4.27}$$

This in turn implies

$$\left(\sum_{\nu \in \mathbb{Z}_2^d} P_\nu(z) \right)^{mN} = \sum_{|\gamma| = mN} \left(C_{mN}^{\gamma} \prod_{\mu \in \mathbb{Z}_2^d} P_\mu^{\gamma_\mu}(z) \right) = 1 \quad \text{for all } |z| = 1,$$

for any positive integers m and N, where C_{mN}^{γ} are the multinomial coefficients. Letting

$m=2^d$ and $N\in\mathbb{N}$, this interpolatory condition for P is being factored in [72, Theorem 2.3.] as follows.[2] Define

$$G_0=\{\gamma\in\mathbb{N}_0^m : |\gamma|=mN, \gamma_0>N \text{ and } \gamma_0>\gamma_\nu, \nu\in\mathbb{Z}_2^d\setminus\{0\}\},$$

$$G_j=\{\gamma\in\mathbb{N}_0^m : |\gamma|=mN, \gamma_0>N, \gamma_0\geq\gamma_\nu, \nu\in\mathbb{Z}_2^d\setminus\{0\}, \text{ with exactly } j \text{ equalities}\},$$

for $j=1,\dots,m-2$, and

$$H=\sum_{j=0}^{m-2}\frac{1}{j+1}\left(\sum_{\gamma\in G_j} C_{mN}^{\gamma} P^{\gamma_0-1}\prod_{\nu\in\mathbb{Z}_2^d\setminus\{0\}} P_\nu^{\gamma_\nu}\right)+C_{mN}^{(N,\dots,N)}\prod_{\nu\in\mathbb{Z}_2^d} P_\nu^N.$$

Then ([72, Theorem 2.3]) proves that the product PH satisfies the interpolatory condition (4.27). The following example shows one particular construction of dual Gabor frames based on this result.

Example 4.7.6. *A possible $a\in\ell_0(\mathbb{Z}^d)$ to satisfy the interpolatory condition (4.26) is given by*

$$\hat{a}(\omega)=\frac{1}{2}\left(\cos\left(\frac{\omega_1}{2}\right)\cos\left(\frac{\omega_2}{2}\right)\cos\left(\frac{\omega_1+\omega_2}{2}\right)\right)^2(5-\cos(\omega_1)-\cos(\omega_2)-\cos(\omega_1+\omega_2)),$$

see [72]. The corresponding Laurant polynomial is

$$P(z)=\frac{1}{128}((z_1+z_1^{-1})(z_2+z_2^{-1})(z_1z_2+(z_1z_2)^{-1}))^2\left(5-\frac{z_1^2+z_1^{-2}}{2}-\frac{z_2^2+z_2^{-2}}{2}-\frac{(z_1z_2)^2+(z_1z_2)^{-2}}{2}\right),$$

where $z_1=e^{-i\omega_1/2}$ and $z_2=e^{-i\omega_2/2}$. Applying [72, Theorem 2.3] with $m=4$ and $N=1$ yields

$$H=P(P^2+4P(P_{\nu_1}+P_{\nu_2}+P_{\nu_3})+12(P_{\nu_1}P_{\nu_2}+P_{\nu_2}P_{\nu_3}+P_{\nu_1}P_{\nu_3})+3(P_{\nu_1}^2+P_{\nu_2}^2+P_{\nu_3}^2)+24P_{\nu_1}P_{\nu_2}P_{\nu_3})$$

[2] The factor 2^d is an artifact of the dyadic dilations we use. The construction in [72] works for general dilation matrices and 2^d is being replaced by the determinant of the dilation matrix.

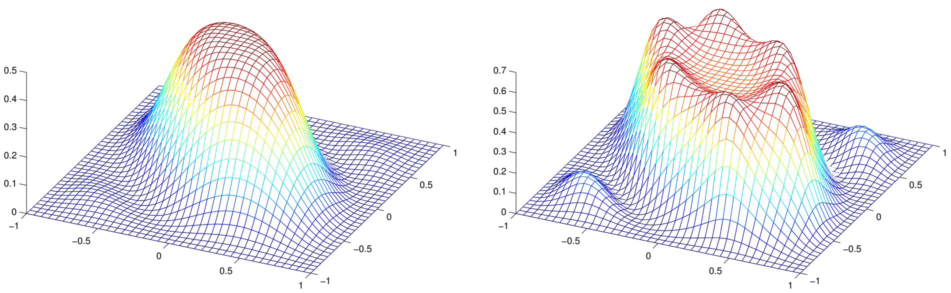


Fig. 4.3 Primary and dual Gabor windows of Example 4.7.6.

where $\nu_1, \nu_2, \nu_3 \in \mathbb{Z}_2^d \setminus \{0\}$ *are the* 3 *coset elements. Therefore, defining*

$$\phi(x) = \begin{cases} P(e^{-i\pi x/2}) & \text{if } x \in [-1,1]^2 \\ 0 & \text{else} \end{cases}, \quad \psi(x) = \begin{cases} H(e^{-i\pi x/2}) & \text{if } x \in [-1,1]^2 \\ 0 & \text{else} \end{cases},$$

and choosing the lattices $(\mathbb{Z}^2, \pi\mathbb{Z}^2)$*, the conditions of Proposition 4.7.1 are satisfied for the windows* $2^{-1}\phi$ *and* $2^{-1}\psi$*. The graphs of the dual windows are shown in Figure 4.3.*

Chapter 5

Wavelet systems: Tight and dual frames

The dual Gramian analysis established for shift-invariant systems in [94] is used in [96] for studying wavelet frames with the assistant of the shift-invariant quasi-affine system, which shares the same (tight) frame property of the wavelet system. Thus, the dual Gramian analysis can be carried on for the quasi-affine system to obtain a complete characterization of (tight) frame property of its corresponding wavelet system in terms of its generators. The dual Gramian analysis of a single wavelet system can be easily generalized (with certain conditions) to the mixed dual Gramian analysis of dual wavelet systems, see [97]. Under the condition of the wavelet systems being Bessel, the tight frame property of a wavelet system can be considered as the special case of dual frames when the two systems coincide.

In [96, 97], the characterization of wavelet frames via the (mixed) dual Gramian analysis is applied to a special class of wavelet frames which are generated from a multiresolution analysis (MRA). Then, with some additional conditions, the huge (mixed) dual Gramian matrix can be factored through the MRA such that the dual Gramian matrix is reduced to a finite order matrix generated by the masks of the refinable function and wavelets. This leads to the unitary extension principle (UEP) for tight wavelet frames first presented in [96] and the mixed unitary extension principle (MEP) for dual wavelet frames in [97]. The UEP (resp. MEP) greatly simplifies the construction of tight (resp. dual) wavelet frames, particularly for the univariate case.

The UEP (resp. MEP) also connects the study of the tight (resp. dual) frame property of an entire wavelet system to the study of one-level perfect reconstruction property of a filter bank in $\ell_2(\mathbb{Z}^d)$. This connection, with the perfect reconstruction filter bank construction scheme in Section 4.3 resulted from duality principle, leads to a new and simple construction scheme of multivariate tight/dual wavelet frames from given refinable functions with many desired properties. For example, the supports of the constructed wavelet frames are small, which are not larger than that of the associated refinable function in the MRA. All wavelets are symmetric or anti-symmetric. The number of wavelets is relatively small compared to, e.g., the number of wavelets obtained from the tensor product of univariate B-spline framelets in [96]. Moreover, compared to the existing construction schemes that involve completing matrices with polynomial entries, the proposed construction scheme only requires completing constant matrices. The construction scheme also guarantees the existence of multivariate tight/dual MRA wavelet frames with compact support for $L_2(\mathbb{R}^d)$ starting from a given MRA with weak condition.

The tensor product of univariate B-spline wavelet frames has been widely used in many image restoration tasks, e.g., see [4, 5, 8–10, 12, 13, 41, 57, 105]. Using tensor product tight frames is convenient for the computation of frame decomposition and reconstruction, but it may be limited for certain applications in image processing since many types of images are non-separable multi-dimensional data. So far, the existing non-separable tight wavelet frames are not as widely used as the tensor product B-spline tight wavelet frames. Possible reasons are the construction process is tedious and wavelets may lack certain desired properties including small support, symmetry/anti-symmetry, and relatively small number of wavelets. We hope that the simple construction scheme and the examples of the multivariate box spline tight wavelet frames constructed in this thesis will inspire some new applications that benefit from the nice properties of multivariate box spline tight wavelet frames. The construction of dual frames, which involves completing two constant matrices, is easier than completing a unitary matrix for a single system in the sense that one can design one matrix (i.e. the primary wavelets) and simply find the other by a matrix inversion. With this gained flexibility and simplicity, one may choose to apply dual wavelet frames in adequate signal processing tasks. In addition, compared to the construction of tight wavelet frames using only refinement masks with nonnegative entries, the construction of dual wavelet frames now can start from refinement masks consisting of negative entries. The construction can, for example, be applied to any mask of a compactly supported stable refinable function whose integer shifts form a partition of unity.

So far, we only show, starting from an MRA in $L_2(\mathbb{R}^d)$, how to make use of the filter banks construction to obtain tight/dual wavelet frames. Lastly, we will show, given a set of tight frame filter bank, as long as there is a low pass filter, this filter bank corresponds to an MRA tight wavelet system in $L_2(\mathbb{R}^d)$ whose masks are derived from the filter bank.

5.1 Wavelet frames

A **wavelet system** $X \subset L_2(\mathbb{R}^d)$ is the collection of functions of the form

$$X = X(\Psi) := \bigcup_{k \in \mathbb{Z}} D^k E(\Psi) \tag{5.1}$$

where Ψ is a finite subset of $L_2(\mathbb{R}^d)$, $E(\Psi)$ is the set of the integer translations of the functions in Ψ, and D^k is the dilation operator $D^k : f \mapsto 2^{kd/2} f(2^k\cdot)$. Since wavelet systems are not shift-invariant due to dilations of decreasing and negative powers, quasi-affine systems are introduced in [96] to be able to apply the dual Gramian analysis for shift-invariant systems [94]. For a given wavelet system X, the **quasi-affine system** X^q is the shift-invariant system generated by replacing $D^k E^j \psi(\cdot)$ by the functions

$$2^{kd/2} E^{\gamma} D^k \psi(\cdot - j)$$

at each dilation level $k < 0$, for all $\psi \in \Psi$ and $j \in \mathbb{Z}^d$, where each entry of $\gamma \subset \mathbb{Z}^d$ takes values in $\{0, 1, 2, \ldots, 2^{-k} - 1\}$. The dual Gramian matrix of this shift-invariant system X^q at $\omega \in \mathbb{T}^d := [-\pi, \pi]^d$ is

$$\widetilde{\mathcal{G}}_{X^q}(\omega) = \left(\sum_{\psi \in \Psi} \sum_{k=\kappa(\alpha-\beta)}^{\infty} \hat{\psi}(2^k(\omega+\alpha)) \overline{\hat{\psi}(2^k(\omega+\beta))} \right)_{\alpha,\beta \in 2\pi\mathbb{Z}^d}, \tag{5.2}$$

where κ denotes the dyadic valuation

$$\kappa : \mathbb{R}^d \to \mathbb{Z}^d : \omega \mapsto \inf\{k \in \mathbb{Z} : 2^k \omega \in 2\pi\mathbb{Z}^d\},$$

see [96]. Given a mapping $R : \Psi \to L_2(\mathbb{R}^d)$, the mixed dual Gramian matrix of the shift-invariant system X^q and $(RX)^q$ at $\omega \in \mathbb{T}^d$ is

$$\widetilde{\mathcal{G}}_{X^q,(RX)^q}(\omega) = \left(\sum_{\psi \in \Psi} \sum_{k=\kappa(\alpha-\beta)}^{\infty} \hat{\psi}(2^k(\omega+\alpha)) \overline{\widehat{R\psi}(2^k(\omega+\beta))} \right)_{\alpha,\beta \in 2\pi\mathbb{Z}^d}. \tag{5.3}$$

The fiber matrices $\widetilde{\mathcal{G}}_{X^q}(\omega)$ represent the frame operator of X^q while the matrices $\widetilde{\mathcal{G}}_{X^q,(RX)^q}(\omega)$ represent the mixed frame operator of X^q and $(RX)^q$. In case $R = I$, the mixed dual Gramian matrix $\widetilde{\mathcal{G}}_{X^q,(RX)^q}$ is reduced to the dual Gramian matrix $\widetilde{\mathcal{G}}_{X^q}$. It is proven in [96] that the wavelet system X is a frame if and only if the quasi-affine system X^q is a frame and these two systems have the same frame bounds. Therefore, the frame property of wavelet system X is completely characterized by the dual Gramian matrix $\widetilde{\mathcal{G}}_{X^q}$. Particularly, the wavelet system X forms a tight frame if and only if the quasi-affine system X^q forms a tight frame, i.e. the wavelet system X is a tight frame if and only if $\widetilde{\mathcal{G}}_{X^q}(w)$ is the identity matrix for a.e. $\omega \in \mathbb{T}^d$. In fact, using the same method as Section 3.4, we can obtain many wavelet frame bounds estimates via the dual Gramian matrix. Furthermore, the oversampling theory for the wavelet frame can also be obtained by the observation that the submatrix of the dual Gramian matrix of the wavelet system still preserves the same operator bounds as the dual Gramian matrix, see [96, 100] for more details. For a general map $R : \Psi \to L_2(\mathbb{R}^d)$, it has been proved in [97] that X and RX are dual frames if and only if both systems are Bessel systems and $\widetilde{\mathcal{G}}_{X^q,(RX)^q}(\omega)$ is the identity matrix for a.e. $\omega \in \mathbb{T}^d$.

When the wavelet system is generated by an MRA and under some further mild assumption, the (mixed) dual Gramian matrix defined in (5.2) and (5.3) can be factored through the MRA to a finite order matrix. This results in the UEP for characterizing MRA-based tight wavelet frames and the MEP for characterizing MRA-based dual wavelet frames. Recall that a function $\phi \in L_2(\mathbb{R}^d)$ is called a **refinable function** if

$$\hat{\phi}(2\cdot) = \hat{a}_0\hat{\phi} \tag{5.4}$$

for some $a_0 \in \ell_2(\mathbb{Z}^d)$ where $\hat{a}_0$ is the Fourier series of a_0. The sequence a_0 or its Fourier series $\hat{a}_0$ is called the **refinement mask** of ϕ. Now let $\phi \in L_2(\mathbb{R}^d)$ and $V_0 \subset L_2(\mathbb{R}^d)$ be the closed linear span of $E(\phi)$ and $V_k := D^k(V_0)$ for $k \in \mathbb{Z}$. The sequence of subspaces $\{V_k\}_{k\in\mathbb{Z}}$ is called an MRA if (i) $V_k \subset V_{k+1}$; (ii) $\cup_k V_k$ is dense in $L_2(\mathbb{R}^d)$ and (iii) $\cap_k V_k = \{0\}$. For $\{V_k\}_{k\in\mathbb{Z}}$ to be an MRA, it is sufficient that $\phi \in L_2(\mathbb{R}^d)$ is a compactly supported refinable function with $\hat{\phi}(0) = 1$ (see e.g. [75]). With such an MRA in hand, the wavelets

$\Psi = \{\psi_l\}_{l=1}^r \subset L_2(\mathbb{R}^d)$ are then defined as

$$\hat{\psi}_l(2\cdot) = \hat{a}_l \hat{\phi} \tag{5.5}$$

for some $a_l \in \ell_2(\mathbb{Z}^d)$. The sequence a_l or its Fourier series $\hat{a}_l$ is called a **wavelet mask** and the function $\psi_l \in \Psi$ is called a **wavelet**. For simplicity, we assume, in all of what follows, that the refinable function ϕ is compactly supported with $\hat{\phi}(0) = 1$ (as a result, the refinement mask is finitely supported) and that all wavelet masks are finitely supported. The UEP for MRA tight wavelet frames in [96] is stated as follows.

Theorem 5.1.1 ([96]). *Let ϕ be a compactly supported refinable function with $\hat{\phi}(0) = 1$ and the refinement mask a_0 is finitely supported. Let $\Psi = \{\psi_l\}_{l=1}^r$ be the wavelets with the finite supported wavelet masks $\{a_l\}_{l=1}^r$. If, for a.e. $w \in \mathbb{R}^d$ and $\nu \in \{0,\pi\}^d$,*

$$\sum_{l=0}^r \hat{a}_l(w)\overline{\hat{a}_l(w+\nu)} = \delta_{\nu,0}, \tag{5.6}$$

then the wavelet system $X(\Psi)$ is a tight frame.

The MEP for MRA dual wavelet frames is stated as follows.

Theorem 5.1.2 ([97]). *Let ϕ_a and ϕ_b be compactly supported refinable functions with $\hat{\phi}_a(0) = \hat{\phi}_b(0) = 1$ and finitely supported refinement masks a_0 and b_0. Let $\{a_l\}_{l=1}^r$, resp. $\{b_l\}_{l=1}^r$, be the masks of a wavelet system X derived from ϕ_a, resp. Y derived from ϕ_b. Then X and Y are dual frames, provided they are Bessel systems and*

$$\sum_{l=0}^r \hat{a}_l(\omega)\overline{\hat{b}_l(\omega+\nu)} = \delta_{\nu,0}, \tag{5.7}$$

for any $\nu \in \{0,\pi\}^d$ and a.e. $\omega \in \mathbb{T}^d$.

In the MEP, it is crucial to assume that the two wavelet systems X and Y are Bessel. If in addition all wavelet masks have first order vanishing moments, i.e. $\sum_{k\in\mathbb{Z}} a_l(k) = 0$

for all $l = 1, \dots r$, then the wavelet system $X(\Psi)$ is a Bessel system (see e.g. [61]; or [95] for a dual Gramian argument under an additional mild smoothness condition on the refinable function). However, for the dual Gramian analysis of a single system which is Hermitian, the strong assumption that the system is a Bessel system is no longer needed. Moreover, due to the Hermitian property of the matrix, the dual Gramian can be used for the characterization of Bessel property under a mild condition on the refinable function, see [94].

Note that, one can associate with the wavelet system $X(\Psi)$ the matrix

$$\mathcal{H}_X(\omega) = \begin{pmatrix} \hat{a}_0(\omega+\nu_1) & \hat{a}_1(\omega+\nu_1) & \dots & \hat{a}_r(\omega+\nu_1) \\ \hat{a}_0(\omega+\nu_2) & \hat{a}_1(\omega+\nu_2) & \dots & \hat{a}_r(\omega+\nu_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_0(\omega+\nu_{2^d}) & \hat{a}_1(\omega+\nu_{2^d}) & \dots & \hat{a}_r(\omega+\nu_{2^d}) \end{pmatrix}, \tag{5.8}$$

generated by the masks $\{a_l\}_{l=0}^r$, where $\omega \in \mathbb{T}^d$ and $\{\nu_1, \dots, \nu_{2^d}\} = \{0, \pi\}^d$, and $\mathcal{H}_Y$ is the same matrix associated with $Y = RX$. Under the conditions on the refinable function, the UEP states that a wavelet system $X(\Psi)$ is a tight frame system if $\mathcal{H}_X(w)\mathcal{H}_X^*(w) = I$ for a.e. $w \in \mathbb{T}^d$, while the MEP says that X and Y are dual wavelet frames whenever they are Bessel systems and $\mathcal{H}_X(\omega)\mathcal{H}_Y^*(\omega) = I$ for a.e. $\omega \in \mathbb{T}^d$. In other words, under the MRA assumption and some additional mild conditions, the infinite (mixed) dual Gramian matrix defined in (5.2) can be factored to the finite order matrix $\mathcal{H}_X$ (and $\mathcal{H}_Y$), and the tight (resp. dual) wavelet frame property for two wavelet Bessel systems is reduced to a condition on $\mathcal{H}_X\mathcal{H}_X^*$ (resp. $\mathcal{H}_X\mathcal{H}_Y^*$). The interested reader is referred to [96, 97] for more details.

In addition, the UEP (resp. MEP) condition also reduces the tight (resp. dual) frame property of wavelet systems of infinite levels to a single-level perfect reconstruction property of a filber bank for ℓ_2-sequences. In fact, the UEP (resp. MEP) is exactly the dual Gramian conidtion for the tight (resp. dual) frame property of systems formulated by the masks in the sequence space. Recall that the system $X = X(\{2^{d/2}a_l\}_{l=0}^r, 2)$ defined in (4.4) is a 2-shift-invariant system. As a result the fiber dual Gramian analysis in [94] can be applied. Indeed the matrix $\mathcal{H}_X$ (5.8) is the fiber pre-Gramian matrix of this system which links to the operator T_X and T_X^* as follows:

$$((T_X c)^\wedge(\omega+\nu))_{\nu\in\{0,\pi\}^d} = 2^{d/2}\mathcal{H}_X(\omega)(\hat{c}_l(2\omega))_{l\in\mathbb{Z}_r}, \text{ for } c \in \ell_0(X),$$

$$(T_X^* c)^\wedge(\omega) = 2^{-d/2}\mathcal{H}_X^*(\omega/2)(\hat{c}(\omega/2+\nu))_{\nu\in\{0,\pi\}^d}, \text{ for } d \in \ell_0(\mathbb{Z}_r\times\mathbb{Z}^d).$$

The fiber mixed dual Gramian matrix is then $\widetilde{\mathcal{G}}_{X,Y} = \mathcal{H}_X\mathcal{H}_Y^*$ where $Y = X(\{2^{d/2}b_l\}_{l=0}^r, 2)$. Since the filters are FIR, X and Y are Bessel systems and thus $\widetilde{\mathcal{G}}_{Y,X}(\omega)$ is the identity for a.e. $\omega \in \mathbb{R}^d$, i.e. the MEP condition (5.7), if and only if the (time-domain) mixed dual Gramian matrix $\widetilde{G}_{Y,X}$ is the identity, i.e.

$$2^d\sum_{l=0}^r\sum_{k\in\mathbb{Z}^d} a_l(n+2k+\ell)\overline{b_l(2k+\ell)} = \delta_{n,0} \quad \text{for any } n,\ell \in \mathbb{Z}^d. \tag{5.9}$$

Both are the necessary and sufficient conditions for the filter bank systems X and Y to be dual frames. Besides (5.7), the sufficiency condition provided by Theorem 5.1.2 contains the important second part that X and Y have to be Bessel systems. This however, as reviewed above, can be guaranteed by working with wavelet masks that have first order vanishing moments. In summary, if the wavelet masks $\{a_l, b_l\}_{l=1}^r$ have first order vanishing moments, then Theorem 5.1.2 leads to a much simpler sufficient condition for MRA-based wavelet systems to be dual wavelet frames. Namely, the generated wavelet systems $X(\Psi)$ and $X(\tilde{\Psi})$ are dual wavelet frames for $L_2(\mathbb{R}^d)$, whenever the filter bank systems $X = X(\{2^{d/2}a_l\}_{l=0}^r, 2)$ and $Y = X(\{2^{d/2}b_l\}_{l=0}^r, 2)$ are dual frames in $\ell_2(\mathbb{Z}^d)$. Similarly, in the case that $Y = X$, the fiber dual Gramian $\widetilde{\mathcal{G}}_X(\omega)$ is the identity for a.e. $\omega \in \mathbb{R}^d$, i.e. the UEP condition (5.6), if and only if the dual Gramian matrix $\widetilde{G}_X$ is the identity, i.e.

$$2^d \sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} a_l(n+2k+\ell)\overline{a_l(2k+\ell)} = \delta_{n,0} \quad \text{for any } n,\ell \in \mathbb{Z}^d. \tag{5.10}$$

In other words, the wavelet system $X(\Psi)$ is a tigh wavelet frame for $L_2(\mathbb{R}^d)$, whenever the filter bank system $X = X(\{2^{d/2}a_l\}_{l=0}^r, 2)$ is a tight frame in $\ell_2(\mathbb{Z}^d)$.

In the next section we will use this connection and adapt the duality principle construction for perfect reconstruction filter banks to meet the vanishing moment requirements on the masks. In doing so, we give a simple matrix inversion scheme for constructing multivariate dual wavelet frames, as well as a unitary matrix completion scheme for tight wavelet frames, with a prescribed MRA as yet another application of the duality principle.

5.2 Tight/dual wavelet frames construction via constant matrix completion

Having established a link between the MEP (resp. UEP) and the perfect reconstruction of filter banks, we propose a simple construction method for multivariate dual (resp. tight) wavelet frames for a given MRA in terms of a constant matrix completion scheme. This is done by fine tuning the filter bank Construction 4.3.2 to meet the extra requirement of first order vanishing moments. Our freedom to achieve this, lies in the appropriate choice of the diagonal matrix involved. In this section, we mainly describe the proposed construction of dual wavelet frames, and the construction of tight wavelet frames can be considered as the special case when the two systems coincide but with an additional condition on the refinement mask. The construction starts from a real-valued refinement mask a_0 satisfying

$$\sum_{n\in\Omega_j} a_0(n) = 2^{-d}, \tag{5.11}$$

for all $j \in \mathbb{Z}^d/2\mathbb{Z}^d$, where $\Omega_j = (2\mathbb{Z}^d + j)\cap\operatorname{supp}(a_0)$. This condition on the refinement mask is a rather mild requirement. Condition (5.11) is equivalent to $\hat{a}_0(0) = 1$ and $\hat{a}_0(j\pi) = 0$ for $j \in (\mathbb{Z}^d/2\mathbb{Z}^d)\backslash\{0\}$. If $\hat{a}_0(0) = 1$, then (5.11) holds provided that the cascade algorithm for a_0 converges in $L_2(\mathbb{R}^d)$ for any compactly supported initial function whose integer shifts are a partition of unity, see [82]. Recall that the *cascade algorithm* for the refinement mask a_0 is the sequence $\phi_n = 2^d \sum_{k\in\mathbb{Z}^d} a_0(k)\phi_{n-1}(2\cdot -k)$, $n \in \mathbb{N}$, where ϕ_0 is some compactly supported function. If for example a refinable function is stable, i.e. its integer shifts form a Riesz sequence, and its integer shifts form a partition of unity, then its refinement mask satisfies (5.11). Refinement masks that satisfy (5.11) include masks of box splines, of certain butterfly subdivision schemes or of the interpolation function of [92]. We will use those in examples below.

Construction 5.2.1. *Suppose the finitely supported real-valued refinement mask a_0 satisfies* (5.11)*.*

- ***Step 1*** *(initialization): Define the first row of a matrix A by collecting the nonzero entries of a_0. Let M be the diagonal matrix with the first row of A as its diagonal.*

- ***Step 2*** *(primary wavelet masks): Complete the matrix A to be an invertible square matrix, each of whose remaining rows has entries summing to zero.*

- ***Step 3*** *(dual wavelet masks): Define $\tilde{A} = AM^{-1}$ and $B = (\tilde{A}^*)^{-1}$.*

If a_0 has, say, m nonzero entries, then A is an $m\times m$ matrix whose first row contains the nonzero entries of the refinement mask a_0. This defines a one-to-one correspondence between the support of a_0 and $\{1, \ldots, m\}$. Via this correspondence (or more precisely by reversing the procedure the matrix in (4.5) is derived from given masks), the remaining rows of A define $m-1$ finitely supported d-dimensional wavelet masks, whose support is contained in the support of a_0. By the same correspondence, the first row of B defines a d-dimensional refinement mask and the remaining rows of B define d-dimensional wavelet masks with first order vanishing moments, the support of which is contained in the support of a_0. This is implied by the following lemma.

Lemma 5.2.2. *Let A and B be the matrices derived in Construction 5.2.1. Then the first rows of A and B coincide and each of their remaining rows sums to* 0*.*

Proof. That all but the first row of A sum to zero is a requirement in Construction 5.2.1. Every entry of the first row of $\tilde{A}$ is equal to 1 and $B\tilde{A}^* = I$. Therefore, the entries of the first row of B sum to 1 and the entries of each remaining row of B sum to 0. If $\tilde{B} = BM^{-1}$, then $\tilde{B}A^* = I$, where $\tilde{B}$ is uniquely determined since A^* is invertible by construction. Since the entries of the first column of A^* sum to 1 and the entries of each remaining column of A^* sum to zero by construction, it follows that each entry of the first row of $\tilde{B}$ is equal to 1. This implies that the first rows of A and B coincide. $\square$

Remark. (1) Construction 5.2.1 is a special way to construct adjoint systems of the original system X generated by the masks a_0, which is inspired from the idea of the connection between the MEP and perfect reconstruction filter banks. There is a lot of freedom to construct matrix A with only its first row provided, which allows us to construct wavelet masks with desired properties. For example, if the refinement mask a_0 has certain symmetric properties, one may impose extra symmetric conditions on the matrix extension to generate wavelet masks with the same symmetries, as we will see later in the examples. (2) Construction 5.2.1 is only one possible scheme to obtain matrices A and B that satisfy the conditions specified in Theorem 4.3.1. One may consider a matrix A with more rows than columns, i.e. there are more wavelets.

Construction 5.2.1 contains the minimal number of wavelet masks among all the possible constructions using Theorem 4.3.1.

Construction 5.2.1 is best possible, on the other hand, in the sense that it cannot be improved to yield two different MRAs generated by real refinement masks for the primary and dual wavelets. Indeed, suppose a finitely supported mask satisfies (5.11) and its nonzero entries define the diagonal of a diagonal matrix M. The crux of the construction is to factor $M = A^*B$, with A and B such that their first rows $\mathbf{a}$ and $\mathbf{b}$ each have entries summing to one, while each of their remaining rows has entries summing to zero. Now, if A_0 and B_0 are the submatrices derived from A and B by deleting their first rows, then $A_0^*B_0 = M - \mathbf{a}^\top\mathbf{b}$. Letting $\mathbf{1}$ the constant one vector and $\mathbf{0}$ the constant zero vector, then $A_0^*B_0\mathbf{1}^\top = A_0^*\mathbf{0}^\top = \mathbf{0}^\top$, while $(M - \mathbf{a}^\top\mathbf{b})\mathbf{1}^\top = \mathrm{diag}(M)^\top - \mathbf{a}^\top$. Thus $\mathrm{diag}(M) = \mathbf{a}$. Similarly, multiplying $\mathbf{1}$ from the left, $\mathrm{diag}(M) = \mathbf{b}$ follows. Consequently $\mathbf{a} = \mathbf{b}$, i.e. such a construction cannot produce different primary and dual refinement masks, no matter what the number of wavelets is.

Now let $X(\Psi)$ be the wavelet system generated from the masks determined by A and let $X(\tilde{\Psi})$ be the wavelet system generated from the masks determined by B. The above argument shows that both wavelet systems $X(\Psi)$ and $X(\tilde{\Psi})$ are derived from the same underlying MRA. Our main result is that $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames.

Theorem 5.2.3. *Suppose the real-valued refinement mask $a_0 \in \ell_2(\mathbb{Z}^d)$ is of finite support satisfying (5.11), and the corresponding refinable function $\phi \in L_2(\mathbb{R}^d)$ is supposed to be compactly supported with $\widehat{\phi}(0) = 1$. Then the masks derived by Construction 5.2.1 satisfy the MEP condition (5.7), and the wavelet systems $X(\Psi)$ and $X(\tilde{\Psi})$ generated by those masks are dual wavelet frames in $L_2(\mathbb{R}^d)$. The number of Ψ (or $\tilde{\Psi}$) is one less than the size of the support of a_0. Moreover, the support of the derived masks is no larger than the support of a_0 and if the support of ϕ is convex, then the support of the primary and dual wavelets is no larger than the support of ϕ.*

Proof. Let $\{a_l\}_{l=0}^r$ and $\{b_l\}_{l=0}^r$ be the masks derived from Construction 5.2.1. To use the MEP, $X(\Psi)$ and $X(\tilde{\Psi})$ need to be Bessel systems. This is ensured by the finite support of all masks and the first order vanishing moments of all wavelet masks guaranteed by Lemma 5.2.2. It remains to verify (5.7) of the MEP. By construction $A^*B = M$. This implies that the n-th column of A is orthogonal to the m-th column of B whenever $n \neq m$. Moreover, together with (5.11), it implies

$$\sum_{l=0}^{r} \sum_{n\in\Omega_j} \overline{a_l(n)} b_l(n) = \sum_{n\in\Omega_j} (A^*B)(n,n) = \sum_{n\in\Omega_j} a_0(n) = 2^{-d},$$

for every $j \in \mathbb{Z}^d/2\mathbb{Z}^d$. Thus the masks satisfy (4.6) and (4.7) of Theorem 4.3.1, which implies the MEP (5.9) and therefore (5.7). By construction, $\operatorname{supp}(a_l) \subset \operatorname{supp}(a_0) \subset \operatorname{supp}(\phi)$ for $l = 1,\dots,r$. If $\operatorname{supp}(\phi)$ is convex, then $\operatorname{supp}(\phi(2\cdot+k)) \subset \frac{1}{2}\operatorname{supp}(\phi) + \frac{1}{2}\operatorname{supp}(\phi) = \operatorname{supp}(\phi)$ for any $k \in \operatorname{supp}(a_0)$. The primary wavelets are given by

$$\psi_l = 2^d \sum_{k\in\mathbb{Z}^d} a_l(k)\phi(2\cdot+k)$$

and thus $\operatorname{supp}(\psi_l) \subset \operatorname{supp}(\phi)$ for $l = 1,\dots,r$. The same arguments hold for the dual wavelets. □

For any finitely supported refinement mask, one can always find matrices satisfying Step 2 of Construction 5.2.1. This implies the following existence result.

Theorem 5.2.4. *For any MRA of $L_2(\mathbb{R}^d)$, derived from a real-valued refinement mask satisfying (5.11) with the corresponding refinable function satisfying $\widehat{\phi}(0) = 1$, there exist dual wavelet frames with the following properties:*

(a) *the number of primary and dual wavelets is one less than the size of the support of the refinement mask,*

(b) *the support of the all wavelet masks is contained in the support of the refinement mask,*

(c) *the support of all wavelets is contained in the support of the refinable function, whenever the refinable function has convex support.*

Proof. It only remains to note that Step 2 of Construction 5.2.1 can be executed for any finitely supported refinement mask. Indeed, if, as above, A is to be an $m \times m$ matrix, then its first row is not in $(\mathrm{span}\{\mathbf{1}\})^{\perp}$, where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^m$. Each of the remaining $m-1$ rows has to have entries summing to zero and one can choose any $m-1$ linear independent vectors of the $(m-1)$-dimensional space $(\mathrm{span}\{\mathbf{1}\})^{\perp}$ to complete A to be an invertible matrix. $\square$

When the case $A = B$, the construction idea of dual wavelet frames can be easily reduced to the construction of tight wavelet frames but with a few distinctions. In the case of $A = B$, one does not have the freedom to complete one matrix first but rather to complete matrix A satisfying the desired property $A^*A = M$. By defining $\tilde{A} = AM^{-1/2}$, it is to complete a matrix satisfying $\tilde{A}^*\tilde{A} = I$ which says $\tilde{A}$ is a unitary matrix. Since we have to take square root on the matrix M, we need to assume an extra assumption on the refinement mask, i.e. the entries of a_0 are nonnegative. Suppose mask a_0 satisfy (5.11) and all entries of a_0 are nonnegative. Then we have the following construction scheme for tight wavelet frames.

Construction 5.2.5. *Suppose we have a refinement mask a_0 with only nonnegative entries and satisfying (5.11).*

- ***Step 1*** *(initialization): Align the nonzero entries of a_0 as a row vector $\mathbf{a}$ and define a normalized vector $\tilde{a}_0$ with $\|\tilde{a}_0\| = 1$ by taking the square root of $\mathbf{a}$. Let M be the diagonal matrix with $\mathbf{a}$ as its diagonal.*

- ***Step 2*** *(orthogonal matrix extension): Construct an orthogonal matrix $\tilde{A}$ with the first row being $\tilde{a}_0$.*

- ***Step 3*** *(restoration): Define the matrix $A = \tilde{A}M^{1/2}$.*

Remark. Compare to Construction 5.2.1 for dual wavelet frames, Construction 5.2.5 for tight wavelet frames is more stringent. Construction 5.2.5 needs to complete a unitary matrix, while Construction 5.2.1 still has the freedom to construct one matrix A with the minimal requirement that A is invertible. One may use the differences of the two constructions adapted to various practical signal processing tasks.

Analogous to Theorem 5.2.3 for dual wavelet frames, Construction 5.2.5 leads to the following result for tight wavelet frames.

Theorem 5.2.6. *Suppose the refinement mask $a_0 \in \ell_2(\mathbb{Z}^d)$ with nonnegative entries is of finite support satisfying (5.11), and the corresponding refinable function $\phi \in L_2(\mathbb{R}^d)$ is supposed to be compactly supported with $\widehat{\phi}(0) = 1$. The masks derived from Construction 5.2.5 satisfy the UEP condition (5.10), and the wavelet system $X(\Psi)$ generated by the corresponding masks forms a tight frame in $L_2(\mathbb{R}^d)$. The number of Ψ is one less than the size of the support of a_0. the support of the derived masks is no larger than the support of a_0 and if the support of ϕ is convex, then the support of the primary and dual wavelets is no larger than the support of ϕ.*

Proof. The orthogonality of the columns of A is guaranteed by Step 2 and Step 3. Moreover, we have

$$\sum_{l=0}^{m-1}\sum_{n\in\Omega_j}|a_l(n)|^2 = \sum_{l=0}^{m-1}\sum_{n\in\Omega_j} a_0(n)|\tilde{a}_l(n)|^2 = \sum_{n\in\Omega_j} a_0(n) = 2^{-d},$$

where the assumption of the entries a_0 to be nonnegative is used. According to Theorem 4.3.1, the masks $\{a_l\}_{l=0}^{m-1}$ generated by Construction 5.2.5 satisfy the UEP (5.10).

Thus, by the UEP, the wavelet system $X(\Psi)$ generated by the wavelets defined from these wavelet masks forms a tight wavelet frame in $L_2(\mathbb{R}^d)$. $\square$

For any finitely supported refinement mask, one can always find unitary matrices satisfying Step 2 of Construction 5.2.5. This implies the following existence result.

Theorem 5.2.7. *For any MRA of $L_2(\mathbb{R}^d)$, derived from a nonnegative refinement mask satisfying (5.11) with the corresponding refinable function satisfying $\widehat{\phi}(0) = 1$, there exist tight wavelet frames with the following properties:*

(a) *the number of wavelets is one less than the size of the support of the refinement mask,*

(b) *the support of the all wavelet masks is contained in the support of the refinement mask,*

(c) *the support of all wavelets is contained in the support of the refinable function, whenever the refinable function has convex support.*

In the existing construction schemes, the construction of a compactly supported dual or tight wavelet frame from a given refinement mask is mainly to solve a problem of completing unitary matrices with trigonometric polynomial entries. In contrast, Construction 5.2.1 and 5.2.5 are only a problem of completing constant matrices. As a result, the construction of dual and tight wavelet frames is greatly simplified in our scheme. Such a simplification is very helpful to the construction of multivariate dual and tight wavelet frames from refinable function, e.g. box splines, with desired properties as we will show in the next two sections. Tight wavelet frames have a wide application in the last decades, for which we show the construction first.

5.3 Multivariate tight wavelet frame from box splines

Given a set of directions $\{\xi_j\}_{j=1}^{n} \subset \mathbb{Z}^d$ with multiplicity m_j for each ξ_j, the Fourier transform of the **box spline** ϕ associated with the given directions is defined by

$$\hat{\phi}(\omega) = \prod_{j=1}^{n} \left(\frac{1 - e^{-i\xi_j \cdot \omega}}{i\xi_j \cdot \omega} \right)^{m_j}.$$

Let L be the minimal number of directions $\{\xi_{j_k}\}_{k=1}^{L}$ whose removal from this set cannot span $\mathbb{R}^d$ anymore, then the corresponding box spline ϕ lies in $C^{L-2}(\mathbb{R}^d)$. The box spline ϕ is refinable and the refinement mask is given by

$$\hat{a}_0(\omega) = \prod_{j=1}^{n} \left(\frac{1 + e^{-i\xi_j \cdot \omega}}{2} \right)^{m_j}.$$

The entries of the refinement mask a_0 are nonnegative and a_0 satisfies (5.11). The interested reader is referred to [3] for a detailed introduction to box splines.

In the following examples, all multivariate wavelet frames constructed by Construction 5.2.5 have the following properties: the supports of wavelets are not larger than that of the box spline, wavelets and their masks are either symmetric or anti-symmetric, and the number of wavelets constructed is one less than the size of the support of the refinement mask.

Example 5.3.1. *Considering the linear bivariate box spline with the following three directions:*

$$\{\xi_1, \xi_2, \xi_3\} = \{(1,0)^\top,\ (0,1)^\top,\ (1,1)^\top\}.$$

The multiplicity $m_j = 1$ for all j. The graph of the function is plotted in (a) of Figure 5.1.

The refinement mask of this box spline is

$$a_0 = \frac{1}{8}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{5.12}$$

Construction 5.2.5 gives the following six wavelet masks:

$$\tfrac{1}{8}\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \tfrac{1}{8}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \tfrac{1}{8}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

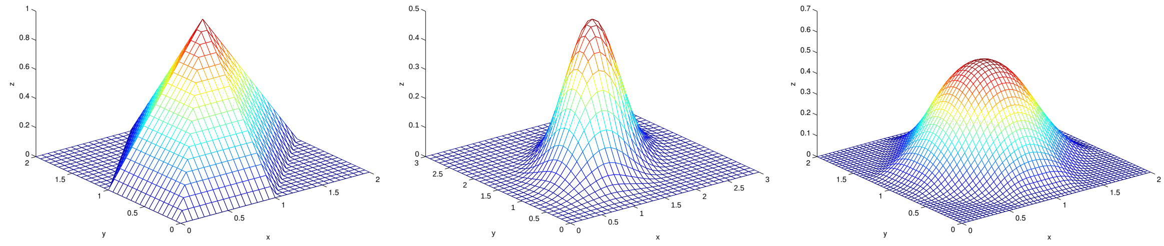
$$\tfrac{\sqrt{3}}{12}\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \tfrac{\sqrt{6}}{24}\begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 2 \\ -1 & -1 & 0 \end{pmatrix}, \quad \tfrac{\sqrt{2}}{8}\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

See Figure 5.2 for the graphs of the six corresponding wavelets. It is seen that the supports of the wavelet masks (resp. wavelets) are not larger than the support of the refinement mask (resp. box spline). All wavelets are either symmetric or anti-symmetric. As a comparison, the number of bivariate wavelets obtained by the tensor product of linear B-spline wavelets in [96] is eight and they have larger support. The number of bivariate wavelets constructed in [31] is seven and their supports are the same as the support of box spline. There are six wavelets in the construction of [79]. Seven or six wavelets with the same support as the box spline are constructed in [20].

Example 5.3.2. *Considering the box spline with the following three directions*

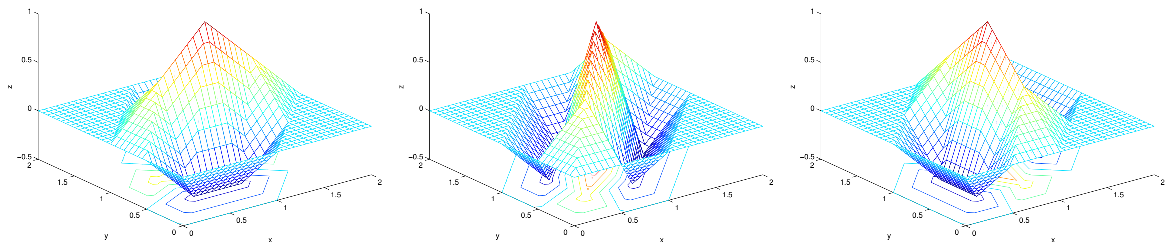
$$\{\xi_1, \xi_2, \xi_3\} = \{(1,0)^\top,\ (0,1)^\top,\ (1,1)^\top\},$$

with multiplicities $m_j = 2$ for all j. The graph of the function is plotted in (b) of Fig-

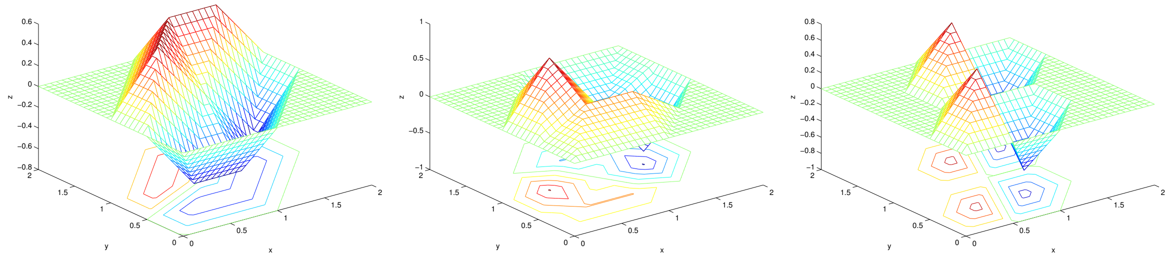


(a) ϕ_1 (b) ϕ_2 (c) ϕ_3

Fig. 5.1 Graphs of refinable box splines used in the constructions: (a) the box spline in Example 5.3.1; (b) the box spline in Example 5.3.2; (c) the box spline in Example 5.3.3.

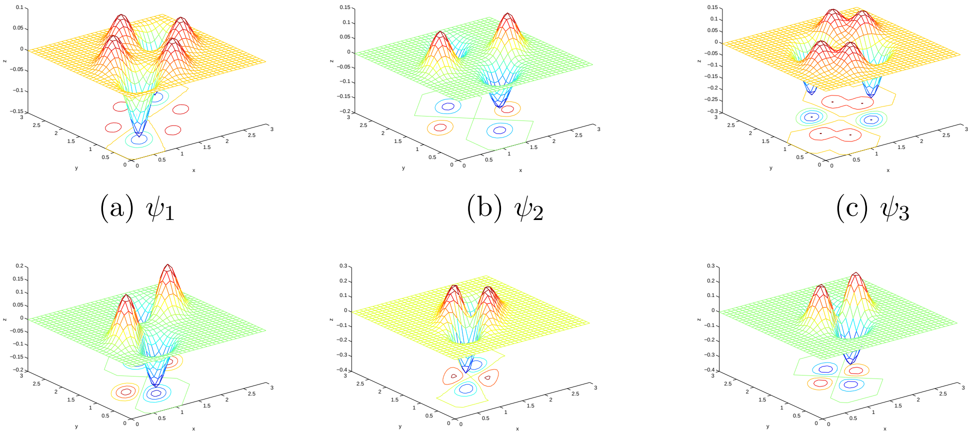


(a) ψ_1 (b) ψ_2 (c) ψ_3



(d) ψ_4 (e) ψ_5 (f) ψ_6

Fig. 5.2 Graphs of the six wavelets constructed from box spline of three directions with multiplicity one in Example 5.3.1.



(a) ψ_1 (b) ψ_2 (c) ψ_3

(d) ψ_4 (e) ψ_5 (f) ψ_6

Fig. 5.3 Graphs of the first six wavelets constructed from box spline of three directions with multiplicity two in Example 5.3.2.

ure 5.1. The refinement mask is

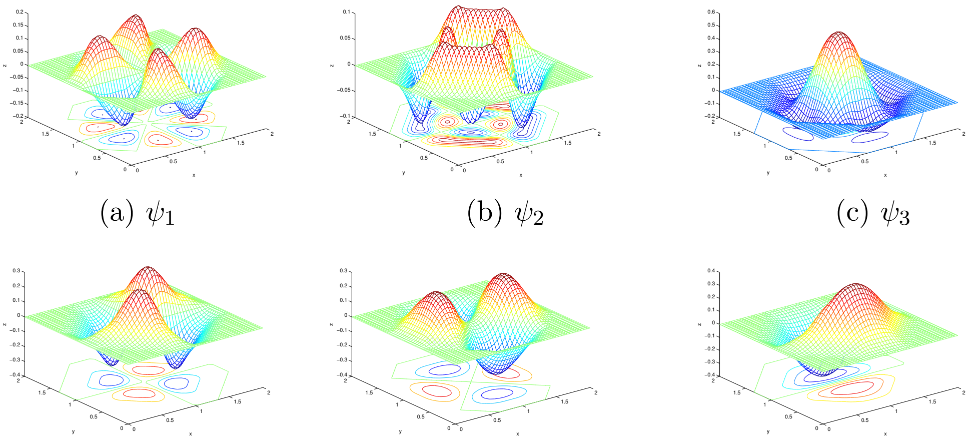
$$a_0 = \frac{1}{64}\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 1 & 6 & 10 & 6 & 1 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{pmatrix}. \tag{5.13}$$

Construction 5.2.5 gives 18 *wavelet masks (see Appendix A.1). See Figure 5.3 for the plots of the first six wavelets. The same box spline is also used in [79] to generate seven wavelets, whose explicit expressions are provided in [19].*

Example 5.3.3. *Considering the bivariate box spline in* $\mathbb{R}^2$ *with the following four directions:*

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \{(0,1)^\top,\ (1,0)^\top,\ (1,1)^\top,\ (1,-1)^\top\}$$

with multiplicity $m_j = 1$ *for all* j*. The graph of the function is plotted in (c) of Figure 5.1.*



(a) ψ_1 (b) ψ_2 (c) ψ_3

(d) ψ_4 (e) ψ_5 (f) ψ_6

Fig. 5.4 Graphs of the first six wavelets constructed from box spline of four directions with multiplicity one in Example 5.3.3.

The refinement mask is

$$a_0 = \frac{1}{16}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{5.14}$$

Construction 5.2.5 gives 11 *wavelet masks (see Appendix A.2) and see Figure 5.4 for the graphs of the first six wavelets. Using the same bivariate box spline,* 15 *wavelets are constructed using the method proposed in [31] and six wavelets with larger support are constructed using the method in [79].*

Example 5.3.4. *Considering the refinable function the box spline in* $\mathbb{R}^3$ *with the following four directions:*

$$(\xi_1,\xi_2,\xi_3,\xi_4) = \{(1,0,0)^\top,\ (0,1,0)^\top,\ (0,0,1)^\top,\ (1,1,1)^\top\}.$$

with multiplicity 1 *for each direction. The refinement mask is*

$$a_0 = \frac{1}{16}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.15}$$

The above matrix is the 3D matrix aligned slice by slice by the x-coordinate. Construction 5.2.5 gives 14 *wavelets (see Appendix A.3). When using the tensor product of univariate wavelets to construct trivariate wavelets, e.g. linear B-spline and its two wavelets, it will produce totally* 26 *wavelets. As a comparison, only* 14 *wavelets are produced with their supports no larger than the support of the box spline. The reduced number of wavelets and the relative small support of wavelet masks could benefit the applications of tight wavelet frames in high-dimensional data, in terms of both computational efficiency and memory utilization efficiency.*

5.4 Multivariate dual wavelet frame construction

Construction 5.2.1 can start from any given finitely supported real-valued refinement mask satisfying (5.11) and we now illustrate it by some examples. Example 5.4.1 is based on a piecewise linear box spline. The refinement mask used in Example 5.4.2 is derived from the butterfly subdivision scheme [47], while Example 5.4.3 starts from an interpolatory refinable function derived from a box spline [92]. Note that the latter two examples use interpolatory refinement masks containing negative entries, which cannot be used for the multivariate tight wavelet construction in Constructon 5.2.5. In all the examples, the primary wavelet masks are defined based on discrete first or second order difference operators along certain directions. Again, all dual multivariate wavelet frames constructed by Construction 5.2.1 have the following properties: the supports of primary or dual wavelets are not larger than that of the box spline, wavelets and their masks are either symmetric or anti-symmetric, and the number of primary or dual wavelets constructed is one less than the size of the support of the refinement mask.

Example 5.4.1. *Starting from the box spline of the three directions* $\{(1,0)^\top,(0,1)^\top,(1,1)^\top\}$ *with the mask* a_0 (5.12), *we choose the* 6 *primary wavelet masks*

$$\frac{1}{2}\begin{pmatrix}0&1&0\\0&0&0\\0&-1&0\end{pmatrix},\quad \frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad \frac{1}{2}\begin{pmatrix}0&0&0\\1&0&-1\\0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&-1&0\\0&2&0\\0&-1&0\end{pmatrix},\quad \frac{1}{4}\begin{pmatrix}0&0&-1\\0&2&0\\-1&0&0\end{pmatrix},\quad \frac{1}{4}\begin{pmatrix}0&0&0\\-1&2&-1\\0&0&0\end{pmatrix}.$$

These masks correspond to wavelets with certain directions (see Figure 5.5). The dual refinement mask again is a_0 *while the dual wavelets obtained from Construction 5.2.1 have the following masks (see Figure 5.6 for the dual wavelets)*

$$\frac{1}{8}\begin{pmatrix}0&1&0\\0&0&0\\0&-1&0\end{pmatrix},\quad \frac{1}{8}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad \frac{1}{8}\begin{pmatrix}0&0&0\\1&0&-1\\0&0&0\end{pmatrix},$$

$$\frac{1}{16}\begin{pmatrix}0&-3&1\\1&2&1\\1&-3&0\end{pmatrix},\quad \frac{1}{16}\begin{pmatrix}0&1&-3\\1&2&1\\-3&1&0\end{pmatrix},\quad \frac{1}{16}\begin{pmatrix}0&1&1\\-3&2&-3\\1&1&0\end{pmatrix}.$$

Example 5.4.2. *The butterfly subdivision scheme, widely used in computer graphics, has first been proposed in [47]. If this subdivision scheme is applied on a regular grid*

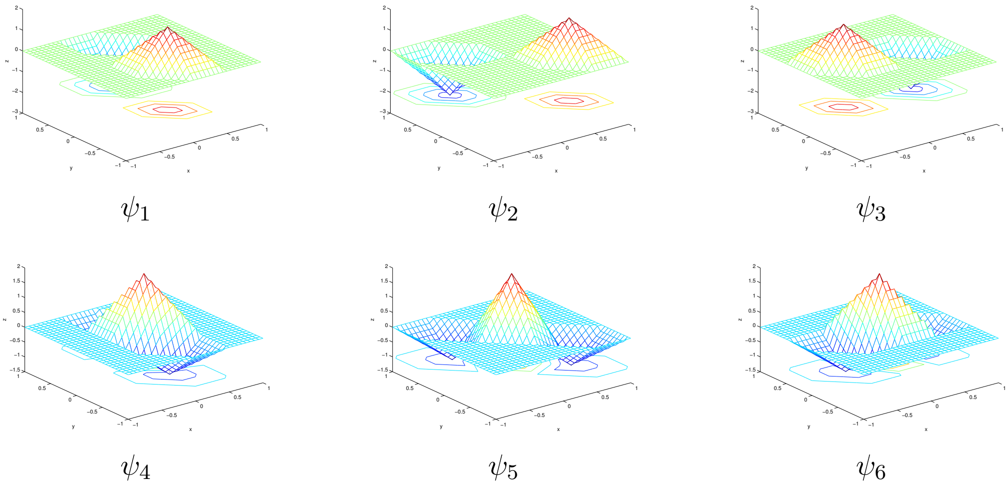


Fig. 5.5 The primary wavelets of Example 5.4.1.

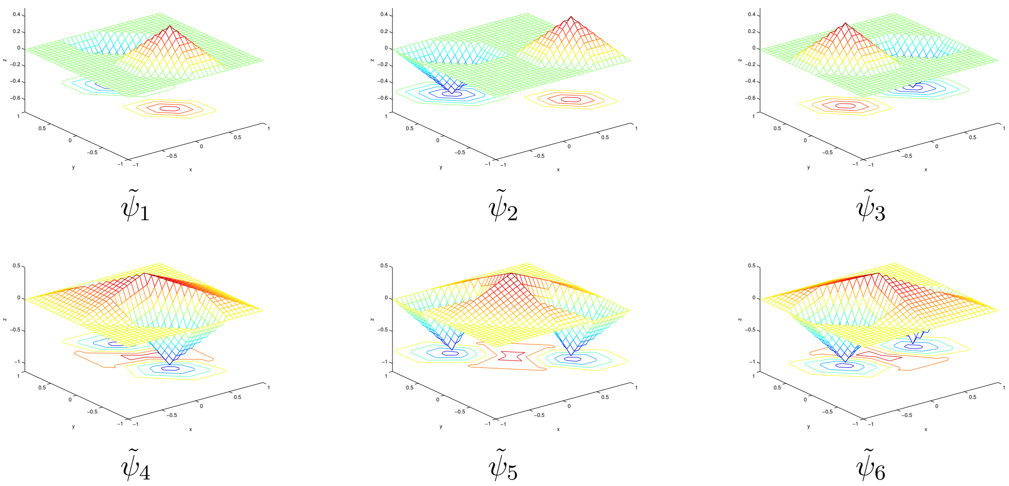


Fig. 5.6 The dual wavelets of Example 5.4.1.

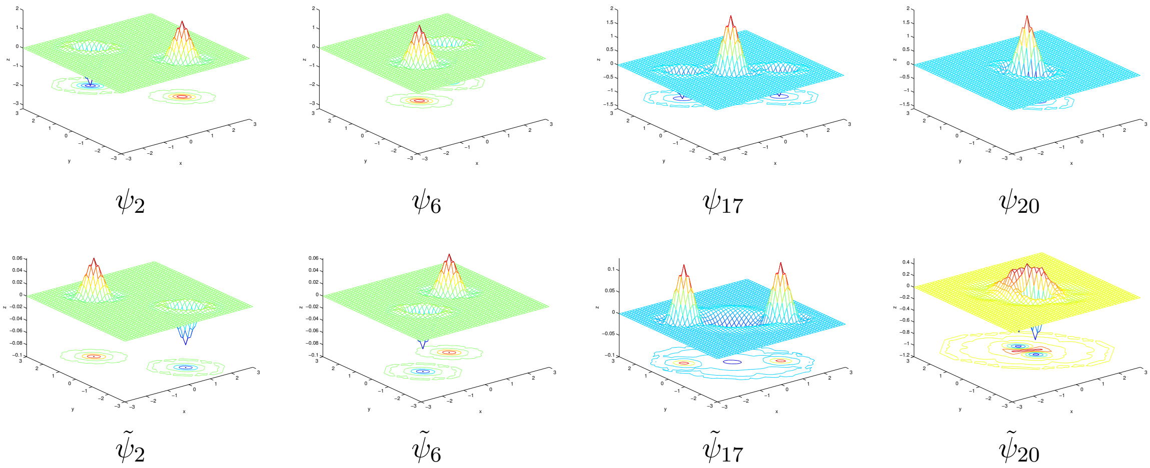


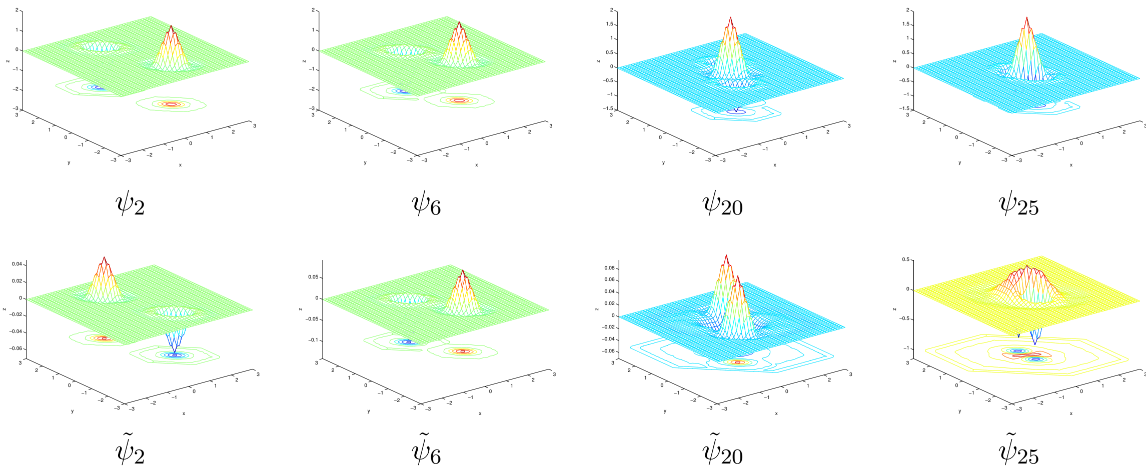
Fig. 5.7 Some primary and dual wavelets of Example 5.4.2.

with both coordinates indexed by integers, it corresponds to the refinement mask

$$a_0 = \frac{1}{64}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This mask satisfies condition (5.11) *and hence can be used in Construction 5.2.1. In total we construct* 24 *primary and* 24 *dual wavelets (see Appendix B.1). The graphs of some of the wavelets are plotted in Figure 5.7. Since the support of the refinement mask* a_0 *is large, the primary wavelets can cover a wide range of directions.*

Example 5.4.3. *Several interpolatory refinable functions have been constructed in [92] by using box splines. The mask of the interpolatory refinable function constructed using*



ψ_2 ψ_6 ψ_{20} ψ_{25}

$\tilde{\psi}_2$ $\tilde{\psi}_6$ $\tilde{\psi}_{20}$ $\tilde{\psi}_{25}$

Fig. 5.8 Some primary and dual wavelets of Example 5.4.3.

the box spline of the three directions $\{(1,0)^\top,(0,1)^\top,(1,1)^\top\}$ *with multiplicity* 2 *is*

$$a_0 = \frac{1}{256}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -3 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -3 & 6 & 33 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix},$$

which satisfies condition (5.11) *and hence can be used in Construction 5.2.1. In total we construct* 30 *primary wavelets and* 30 *dual wavelets (see Appendix B.2). The support of the masks derived in this example is larger and hence more directions can be covered by the wavelets. See Figure 5.8 for the graphs of some of the wavelets.*

5.5 Filter banks revisited

In the wavelet literature one usually constructs tight or dual wavelet frames for given MRAs, i.e. the refinable function and its mask are already prescribed. We have for example done so in Section 5.2, using the connection between tight/dual MRA wavelet frames and filter banks as described in Section 5.1. Here we consider a different perspective and ask whether for a given filter bank, regardless of how it is constructed, there is an underylying MRA wavelet frame system in $L_2(\mathbb{R}^d)$ whose masks are the given filter bank. In general this is a hard question unless one likes to go to Sobolev spaces, see e.g. [66]. However, when the filter bank satisfies the UEP condition, the answer is positive.

Let $\{a_l\}_{l=0}^r$ be a filter bank of finitely supported filters. Suppose this filter bank satisfies the UEP condition for subsampling rate 2, i.e.

$$2^d \sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} a_l(n+2k+\ell)\overline{a_l(2k+\ell)} = \delta_{n,0} \quad \text{for any } n,\ell \in \mathbb{Z}^d,$$

or equivalently in Fourier domain

$$\sum_{l=0}^{r} \hat{a}_l(\omega)\overline{\hat{a}_l(\omega+\nu)} = \delta_{\nu,0} \tag{5.16}$$

for all $\nu \in \{0,\pi\}^d$ and a.e. $\omega \in \mathbb{T}^d$. By (5.16), we have $r \geq 2^d$ and $\mathcal{H}_X(\omega)$ in (5.8) can be extended to a unitary matrix for a.e. $\omega \in \mathbb{T}^d$. In particular, the norm of any column of this matrix is at most one, i.e.

$$\sum_{\nu\in\{0,\pi\}^d} |\hat{a}_l(\omega+\nu)|^2 \leq 1 \tag{5.17}$$

for a.e. $\omega \in \mathbb{T}^d$ and all $l = 0,\ldots,r$.

Assume one of the filters in the filter bank, say a_0, is a low-pass filter, i.e. $\hat{a}_0(0) = 1$. Then (5.16) automatically implies $\hat{a}_l(0) = 0$ for $l = 1,\ldots,r$ and (5.17) implies $\hat{a}_0(\nu) = 0$

for $\nu \in \{0,\pi\}^d \backslash \{0\}$. As long as we show that the low-pass filter a_0 defines a refinable function $\phi \in L_2(\mathbb{R}^d)$, then $\{\psi_l\}_{l=1}^r$ defined in (5.5) by the filters $\{a_l\}_{l=1}^r$ and this refinable function ϕ generate a tight MRA-wavelet frame system in $L_2(\mathbb{R}^d)$. Define

$$\hat{\phi}(\omega) := \prod_{j=1}^{\infty} \hat{a}_0(2^{-j}\omega), \quad \omega \in \mathbb{R}^d.$$

It is clear that ϕ is a compactly supported refinable distribution. Using (5.17), one can prove that ϕ is a compactly supported refinable function in $L_2(\mathbb{R}^d)$ with refinement mask a_0. For completeness, we outline the proof which is contained in [17] for the univariate case.

Consider the cascade algorithm defined by

$$\hat{f}_n(\omega) = \hat{a}_0(2^{-1}\omega)\hat{f}_{n-1}(2^{-1}\omega) = \prod_{j=1}^{n} \hat{a}_0(2^{-j}\omega)\hat{f}_0(2^{-n}\omega), \quad \omega \in \mathbb{R}^d,$$

with $\hat{f}_0 = \chi_{\mathbb{T}^d}$. The pointwise limit $\hat{\phi}$ of $\{\hat{f}_n\}_{n\in\mathbb{N}}$ clearly satisfies the refinement equation $\hat{\phi}(2\cdot) = \hat{a}_0\hat{\phi}$. That $\phi \in L_2(\mathbb{R}^d)$ is guaranteed by the UEP, more precisely by (5.17), which implies that $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L_2(\mathbb{R}^d)$. Indeed,

$$\|\hat{f}_n\|^2 = \int_{2^n(0,\pi)^d} \prod_{j=1}^{n-1} |\hat{a}_0(2^{-j}\omega)|^2 \sum_{\nu\in\{0,\pi\}^d} |\hat{a}_0(2^{-n}\omega+\nu)|^2 \, d\omega \le \|\hat{f}_{n-1}\|^2$$

for all $n \ge 1$, thus $\|\hat{f}_n\| \le \|\hat{f}_0\| = (2\pi)^d$ for all $n \ge 1$ by induction. Since $\{\hat{f}_n\}_{n\in\mathbb{N}}$ converges pointwise to $\hat{\phi}$, Fatou's lemma implies $\|\hat{\phi}\| \le \liminf_{n\to\infty} \|\hat{f}_n\| < \infty$. Thus $\phi \in L_2(\mathbb{R}^d)$.

In summary, we have the following result for this section.

Theorem 5.5.1. *Suppose a given FIR filter bank satisfies the UEP condition* (5.6)*, or equivalently* (5.10)*. If one of the filters is a low pass filter, then there exists an MRA tight wavelet frame in* $L_2(\mathbb{R}^d)$ *whose underlying MRA is derived from this low pass filter* *and the wavelet masks are the rest of the filters in the filter bank.*

Appendix A

Tight wavelet frame masks

A.1 Wavelet masks of Example 5.3.2

$$\frac{\sqrt{12}}{96}\begin{pmatrix} 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \end{pmatrix}, \frac{1}{16}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \frac{\sqrt{6}}{48}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \frac{\sqrt{2}}{16}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\sqrt{2}}{16}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{\sqrt{6}}{16}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{24}\begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{pmatrix}, \frac{\sqrt{3}}{72}\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & -3 & -3 & 2 \\ 1 & -3 & 0 & -3 & 1 \\ 2 & -3 & -3 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{pmatrix},$$

$$\frac{\sqrt{15}}{576}\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 1 & 6 & -54 & 6 & 1 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{pmatrix}, \frac{\sqrt{12}}{96}\begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 \end{pmatrix}, \frac{1}{16}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \frac{\sqrt{6}}{48}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\sqrt{2}}{16}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \frac{\sqrt{2}}{16}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{\sqrt{6}}{16}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\sqrt{2}}{48}\begin{pmatrix} 0 & 0 & 1 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & \sqrt{6} & -\sqrt{6} & -\sqrt{2} \\ 1 & \sqrt{6} & 0 & -\sqrt{6} & -1 \\ \sqrt{2} & \sqrt{6} & -\sqrt{6} & -\sqrt{2} & 0 \\ 1 & -\sqrt{2} & -1 & 0 & 0 \end{pmatrix}, \frac{\sqrt{12}}{96}\begin{pmatrix} 0 & 0 & 1 & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & 0 & -1 \\ -\sqrt{2} & 0 & 0 & \sqrt{2} & 0 \\ 1 & \sqrt{2} & -1 & 0 & 0 \end{pmatrix},$$

$$\frac{1}{48}\begin{pmatrix} 0 & 0 & 1 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -2\sqrt{6} & 2\sqrt{6} & -\sqrt{2} \\ 1 & -2\sqrt{6} & 0 & 2\sqrt{6} & -1 \\ \sqrt{2} & -2\sqrt{6} & 2\sqrt{6} & -\sqrt{2} & 0 \\ 1 & -\sqrt{2} & -1 & 0 & 0 \end{pmatrix}.$$

A.2 Wavelet masks of Example 5.3.3

$$\frac{\sqrt{2}}{16}\begin{pmatrix} & 1 & -1 & \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ & -1 & 1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & 1 & -1 & \\ 1 & -2 & 2 & -1 \\ -1 & 2 & -2 & 1 \\ & -1 & 1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & -1 & -1 & \\ -1 & 2 & 2 & -1 \\ -1 & 2 & 2 & -1 \\ & -1 & -1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & 1 & -1 & \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ & -1 & 1 & \end{pmatrix},$$

$$\frac{\sqrt{2}}{16}\begin{pmatrix} & -1 & -1 & \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ & -1 & -1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & 1 & 1 & \\ 1 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ & -1 & -1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & 1 & -1 & \\ 1 & -2 & 2 & -1 \\ 1 & -2 & 2 & -1 \\ & 1 & -1 & \end{pmatrix}, \frac{1}{16}\begin{pmatrix} & 1 & 1 & \\ 1 & -2 & -2 & 1 \\ -1 & 2 & 2 & -1 \\ & -1 & -1 & \end{pmatrix},$$

$$\frac{1}{16}\begin{pmatrix} & 1 & -1 & \\ 1 & 2 & -2 & -1 \\ 1 & 2 & -2 & -1 \\ & 1 & -1 & \end{pmatrix}, \frac{\sqrt{2}}{16}\begin{pmatrix} & 1 & -1 & \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ & 1 & -1 & \end{pmatrix}, \frac{\sqrt{2}}{16}\begin{pmatrix} & -1 & -1 & \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ & 1 & 1 & \end{pmatrix}.$$

A.3 Wavelet masks of Example 5.3.4

$$\frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \frac{\sqrt{2}}{16}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\frac{\sqrt{2}}{16}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right), \frac{1}{16}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & -2 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\frac{\sqrt{14}}{56}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{array}\right), \frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \frac{1}{8}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\frac{\sqrt{2}}{16}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \frac{\sqrt{3}}{24}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\frac{\sqrt{42}}{336}\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & -4 & -4 & 0 & 3 & 3 \\ -3 & -3 & 0 & 4 & 0 & -4 & 0 & 3 & 3 \\ -3 & -3 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \end{array}\right).$$

Appendix B

Dual wavelet frame masks

B.1 Wavelet masks of Example 5.4.2

Primary wavelet masks:

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&2&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&2&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&2&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&2&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&2&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}.$$

Dual wavelet masks:

$$\frac{1}{64}\begin{pmatrix}0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\end{pmatrix},\frac{1}{64}\begin{pmatrix}0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\end{pmatrix},\frac{1}{64}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{32}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{64}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{64}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{32}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{8}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{8}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{32}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{64}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{8}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{1024}\begin{pmatrix}0&0&0&0&33&1&0\\0&0&1&0&-2&0&1\\0&1&-2&-8&-8&-2&1\\0&0&-8&-16&-8&0&0\\1&-2&-8&-8&-2&1&0\\1&0&-2&0&1&0&0\\0&1&33&0&0&0&0\end{pmatrix},\frac{1}{1024}\begin{pmatrix}0&0&0&0&1&33&0\\0&0&1&0&-2&0&1\\0&1&-2&-8&-8&-2&1\\0&0&-8&-16&-8&0&0\\1&-2&-8&-8&-2&1&0\\1&0&-2&0&1&0&0\\0&33&1&0&0&0&0\end{pmatrix},\frac{1}{1024}\begin{pmatrix}0&0&0&0&1&1&0\\0&0&33&0&-2&0&1\\0&1&-2&-8&-8&-2&1\\0&0&-8&-16&-8&0&0\\1&-2&-8&-8&-2&1&0\\1&0&-2&0&33&0&0\\0&1&1&0&0&0&0\end{pmatrix},\frac{1}{512}\begin{pmatrix}0&0&0&0&-1&-1&0\\0&0&-1&0&-30&0&-1\\0&-1&2&8&8&2&-1\\0&0&8&16&8&0&0\\-1&2&8&8&2&-1&0\\-1&0&-30&0&-1&0&0\\0&-1&-1&0&0&0&0\end{pmatrix},$$

$$\frac{1}{1024}\begin{pmatrix}0&0&0&0&1&1&0\\0&0&1&0&-2&0&33\\0&1&-2&-8&-8&-2&1\\0&0&-8&-16&-8&0&0\\1&-2&-8&-8&-2&1&0\\33&0&-2&0&1&0&0\\0&1&1&0&0&0&0\end{pmatrix},\frac{1}{1024}\begin{pmatrix}0&0&0&0&1&1&0\\0&0&1&0&-2&0&1\\0&33&-2&-8&-8&-2&1\\0&0&-8&-16&-8&0&0\\1&-2&-8&-8&-2&33&0\\1&0&-2&0&1&0&0\\0&1&1&0&0&0&0\end{pmatrix},\frac{1}{512}\begin{pmatrix}0&0&0&0&-1&-1&0\\0&0&-1&0&2&0&-1\\0&-1&-30&8&8&2&-1\\0&0&8&16&8&0&0\\-1&2&8&8&-30&-1&0\\-1&0&2&0&-1&0&0\\0&-1&-1&0&0&0&0\end{pmatrix},\frac{1}{128}\begin{pmatrix}0&0&0&0&-1&-1&0\\0&0&-1&0&2&0&-1\\0&-1&2&-24&8&2&-1\\0&0&8&16&8&0&0\\-1&2&8&-24&2&-1&0\\-1&0&2&0&-1&0&0\\0&-1&-1&0&0&0&0\end{pmatrix},$$

$$
\frac{1}{128}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & -24 & 2 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & -24 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{512}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & -30 & -1 \\ 0 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & -30 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{1024}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -2 & -8 & -8 & -2 & 33 \\ 0 & 0 & -8 & -16 & -8 & 0 & 0 \\ 33 & -2 & -8 & -8 & -2 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{128}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ 0 & 0 & -24 & 16 & -24 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

B.2 Wavelet masks of Example 5.4.3

Primary wavelet masks:

$$\frac{1}{2}\begin{pmatrix}0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&2&0&0&0\\0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&2&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&-1&0\\0&0&0&2&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&2&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&2&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{4}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-1&2&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}.$$

Dual wavelet masks:

$$\frac{1}{256}\begin{pmatrix}0&0&0&-1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&-3&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&3&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&-3&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&3&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&-3&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&3&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{128}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&3&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-3&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&-3\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\3&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-3&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&3&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{128}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&3&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&-3&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&33&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-33&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&33&0&0\\0&0&0&0&0&0&0\\0&0&-33&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{128}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&3&0\\0&0&0&0&0&0&0\\0&-3&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},$$

$$\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-3\\0&0&0&0&0&0&0\\3&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{1}{256}\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&33&0&-33&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix},\frac{-1}{2^{14}}\begin{pmatrix}0&0&0&-129&-3&-3&-1\\0&0&-3&0&6&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&6&0&-3&0&0\\-1&-3&-3&-129&0&0&0\end{pmatrix},\frac{-3}{2^{14}}\begin{pmatrix}0&0&0&-1&-131&-3&-1\\0&0&-3&0&6&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&6&0&-3&0&0\\-1&-3&-131&-1&0&0&0\end{pmatrix},$$

$$\frac{-3}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-131&-1\\0&0&-3&0&6&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&6&0&-3&0&0\\-1&-131&-3&-1&0&0&0\end{pmatrix},\frac{-1}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-3&-129\\0&0&-3&0&6&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&6&0&-3&0&0\\-129&-3&-3&-1&0&0&0\end{pmatrix},\frac{-3}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-131&0&6&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&6&0&-131&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},\frac{3}{2^{13}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-3&0&-122&0&-3\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-3&0&-122&0&-3&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},$$

$$\frac{-3}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-3&0&6&0&-131\\0&-3&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-3&0\\-131&0&6&0&-3&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},\frac{-3}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-3&0&6&0&-3\\0&-131&6&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&6&-131&0\\-3&0&6&0&-3&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},\frac{3}{2^{13}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-3&0&6&0&-3\\0&-3&-122&33&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&33&-122&-3&0\\-3&0&6&0&-3&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},\frac{33}{2^{14}}\begin{pmatrix}0&0&0&-1&-3&-3&-1\\0&0&-3&0&6&0&-3\\0&-3&-6&-95&33&6&-3\\-1&0&33&64&33&0&-1\\-3&6&33&-95&6&-3&0\\-3&0&6&0&-3&0&0\\-1&-3&-3&-1&0&0&0\end{pmatrix},$$

$$\frac{33}{2^{14}}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & -6 & 33 & -95 & 6 & -3 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -3 & 6 & -95 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix}, \frac{3}{2^{13}}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & -122 & -3 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -3 & -122 & 33 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix}, \frac{-3}{2^{14}}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -131 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -131 & 6 & 33 & 33 & -6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix}, \frac{-1}{2^{14}}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -3 \\ -129 & 0 & 33 & 64 & 33 & 0 & -129 \\ -3 & 6 & 33 & 33 & -6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{33}{2^{14}}\begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -3 \\ -1 & 0 & -95 & 64 & -95 & 0 & -1 \\ -3 & 6 & 33 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

References

[1] C. Bao, H. Ji, and Z. Shen. Convergence analysis for iterative data-driven tight frame construction scheme. *Applied and Computational Harmonic Analysis*, 38(3):510–523, 2015.

[2] C. De Boor. *A Pratical Guide to Splines.* Springer-Verlag, New York, 1978.

[3] C. De Boor, K. Höllig, and S. D. Riemenschneider. *Box Splines.* Springer-Verlag, New York, 1993.

[4] J. Cai, R. Chan, and Z. Shen. A framelet-based image inpainting algorithm. *Applied and Computational Harmonic Analysis*, 24:131–149, 2008.

[5] J. Cai, R. Chan, and Z. Shen. Simultaneous cartoon and texture inpainting. *Inverse Problem and Imaging*, 4:379–395, 2010.

[6] J. Cai, B. Dong, S. Osher, and Z. Shen. Image restoration: total variation, wavelet frames, and beyond. *Journal of the American Mathematical Society*, 25:1033–1089, 2012.

[7] J. Cai, B. Dong, and Z. Shen. Image restoration: a wavelet frame based model for piecewise smooth functions and beyond. Preprint.

[8] J. Cai, H. Ji, C. Liu, and Z. Shen. Blind motion deblurring from a single image using sparse approximation. In *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2009.

[9] J. Cai, H. Ji, C. Liu, and Z. Shen. Blind motion deblurring using multiple images. *Journal of Computational Physics*, 228:5057–5071, 2009.

[10] J. Cai, H. Ji, C. Liu, and Z. Shen. High-quality curvelet-based motion deblurring using an image pair. In *IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2009.

[11] J. Cai, H. Ji, Z. Shen, and G. B. Ye. Data-driven tight frame construction and image denoising. *Applied and Computational Harmonic Analysis*, 37(1):89–105, 2014.

[12] J. Cai, S. Osher, and Z. Shen. Linearized Bregman iteration for frame-based image deblurring. *SIAM Journal on Imaging Sciences*, 2:226–252, 2009.

[13] J. Cai, S. Osher, and Z. Shen. Split Bregman and frame based image restoration. *Multiscale Modeling and Simulation: A SIAM interdisciplinary Journal*, 8(2):337–369, 2009.

[14] P. G. Casazza and O. Christensen. Frames containing a Riesz basis and preservation of this property under perturbations. *SIAM Journal on Mathematical Analysis*, (29)1:266–278, 1998.

[15] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles in frame theory. *Journal of Fourier Analysis and Application*, 10(4):383–408, 2004.

[16] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles, localization of frames, and Gabor theory. *Wavelets XI. Proceedings of SPIE*, 5914:389–397, 2005.

[17] A. Chai and Z. Shen. Deconvolution: A wavelet frame approach. *Numerische Mathematik*, 106:529–587, 2007.

[18] M. Charina, C. K. Chui, and W. J. He. Tight frames of compactly supported multivariate multi-wavelets. *Journal of Computational and Applied Mathematics*, 233:2044–2061, 2010.

[19] M. Charina, M. Putinar, C. Scheiderer, and J. Stöckler. Multivariate tight wavelet frames. *Constructive Approximation*, 38:253–276, 2013.

[20] M. Charina and J. Stöckler. Tight wavelet frames for irregular multiresolution analysis. *Applied and Computational Harmonic Analysis*, 25(1):98–113, 2008.

[21] D. R. Chen, B. Han, and D. Riemenschneider. Construction of multivariate biorthogonal wavelets with arbitrary vanishing moments. *Advances in Computational Mathematics*, 13:131–165, 2000.

[22] L. H. Christensen and O. Christensen. Construction of smooth compactly supported windows generating dual pairs of Gabor frames. *Asian-European Journal of Mathematics*, 6, 2013.

[23] O. Christensen. *An Introduction to Frames and Riesz Bases*, volume Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2003.

[24] O. Christensen. Pairs of dual Gabor frame generators with compact support and desired frequency localization. *Applied and Computational Harmonic Analysis*, 20:403–410, 2006.

[25] O. Christensen, S. Favierand, and F. Zó. Irregular wavelet frames and Gabor frames. *Approximation Theory and its Applications*, 17(3):90–101, 2001.

[26] O. Christensen, H. O. Kim, and R. Y. Kim. On the duality principle by Casazza, Kutyniok, and Lammers. *Journal of Fourier Analysis and Applications*, 17:640–655, 2011.

[27] O. Christensen, H. O. Kim, and R. Y. Kim. On entire functions restricted to intervals, partition of unities, and dual Gabor frames. *Applied and Computational Harmonic Analysis*, 38(1):72–86, 2015.

[28] O. Christensen and R. Y. Kim. Pairs of explicitly given dual Gabor frames in $L_2(\mathbb{R}^d)$. *Journal of Fourier Analysis and Applications*, 12:243–255, 2006.

[29] O. Christensen, X. Z. Xiao, and Y. C. Zhu. Characterizing R-duality in Banach spaces. *Acta Mathematica Sinica, English Series*, 29(1):75–84, 2013.

[30] C. K. Chui and W. J. He. Compactly supported tight frames associated with refinable functions. *Applied and Computational Harmonic Analysis*, 8(3):293–319, 2000.

[31] C. K. Chui and W. J. He. Construction of multivariate tight frames via Kronecker products. *Applied and Computational Harmonic Analysis*, 11(2):305–312, 2001.

[32] C. K. Chui, W. J. He, and J. Stöckler. Compactly supported tight and sibling frames with maximum vanishing moments. *Applied and Computational Harmonic Analysis*, 13(3):224–262, 2002.

[33] A. Cohen, I. Daubechies, and J. C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Communication on Pure and Applied Mathematics*, 45:485–560, 1992.

[34] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Transactions on Information Theory*, 36(5):961–1005, 1990.

[35] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, CBMS-NSF Regional Conference Series in Applied Mathematics, 1992.

[36] I. Daubechies, A. Grossman, and Y. Meyer. Painless non-orthogonal expansions. *Journal of Mathematical Physics*, 45:1271–1283, 1986.

[37] I. Daubechies and B. Han. Pairs of dual wavelet frames from any two refinable functions. *Constructive Approximation*, 20:325–352, 2004.

[38] I. Daubechies, B. Han, A. Ron, and Z. Shen. Framelets: MRA-based constructions of wavelet frames. *Applied and Computational Harmonic Analysis*, 14(1):1–46, 2003.

[39] I. Daubechies, H. J. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *Journal of Fourier Analysis and Applications*, 1:437–478, 1994.

[40] B. Dong, A. Chien, and Z. Shen. Frame based segmentation for medical images. *Communications in Mathematical Sciences*, 9(2):551–559, 2010.

[41] B. Dong, H. Ji, J. Li, Z. Shen, and Y. Xu. Wavelet frame based blind image inpainting. *Applied and Computational Harmonic Analysis*, 32(2):268–279, 2012.

[42] B. Dong, Q. Jiang, and Z. Shen. Image restoration: wavelet frame shrinkage, nonlinear evolution functions and beyond. Preprint.

[43] B. Dong, J. Li, and Z. Shen. X-ray CT image reconstruction via wavelet frame based regularization and Radon domain inpainting. *Journal of Scientific Computing*, 54(2-3):333–349, 2013.

[44] B. Dong and Z. Shen. Pseudo-splines, wavelets and framelets. *Applied and Computational Harmonic Analysis*, 11:78–104, 2007.

[45] B. Dong and Z. Shen. MRA-based wavelet frames and applications. *IAS Lecture Notes Series, Summer Program on "The Mathematics of Image Processing", Park City Mathematics Institute*, 19, 2010.

[46] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Transactions of the American Mathematical Society*, 72:341–366, 1952.

[47] N. Dyn, D. Levine, and J. Gregory. A butterfly subdivision scheme for surface interpolation with tension control. *ACM Transactions on Graphics*, 9:160–169, 1990.

[48] H. G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Conference of Constructive Function Theory*, volume 19 of Rocky Mountain J. Math., pages 113–126, 1989.

[49] H. G. Feichtinger. Pseudo-inverse matrix methods for signal reconstruction from partial data. In *SPIE-Conf., Visual Comm. and Image Proc. Boston*, pages 766–772, 1991.

[50] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *Journal of Functional Analysis*, 86(2):307–340, 1989.

[51] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, part II. *Monatshefte für Mathematik*, 108(2-3):129–148, 1989.

[52] H. G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *Journal of Functional Analysis*, 146:464–495, 1997.

[53] H. G. Feichtinger and W. Kozek. Quantization of TF-lattice invariant operators on elementary LCA groups. In *Gabor analysis and algorithms: Theory and applications*, pages 233–266. Springer, 1998.

[54] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for gabor analysis. In *Gabor analysis and algorithms: Theory and applications*, pages 123–170. Springer, 1998.

[55] S. S. Goh, Q. T. Jiang, and T. Xia. Construction of biorthogonal multiwavelets using the lifting scheme. *Applied and Computational Harmonic Analysis*, 9:336–352, 2000.

[56] G. H. Golub and C. F. V. Loan. *Matrix Computations*, volume 3. Johns Hopkins University Press, 1996.

[57] Z. Gong, Z. Shen, and K. C. Toh. Image restoration with mixed or unknown noises. *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, 12(2):458–487, 2014.

[58] K. Gröchenig. Irregular sampling of wavelet and short-time Fourier transforms. *Constructive Approximation*, 9(2-3):283–297, 1993.

[59] K. Gröchenig. *Foundations of Time-frequency Analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston, 2001.

[60] K. Gröchenig, D. Han, C. Heil, and G. Kutyniok. The Balian-Low theorem for symplectic lattices in higher dimensions. *Applied and Computational Harmonic Analysis*, 13:169–176, 2002.

[61] B. Han. Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix. *Journal of Computational and Applied Mathematics*, 155(1):43–67, 2003.

[62] B. Han. The projection method for multidimensional framelet and wavelet analysis. *Mathematical Modeling of Natural Phenomena*, 9(5):83–110, 2014.

[63] B. Han and Q. Mo. Splitting a matrix of Laurent polynomials with symmetry and its application to symmetric framelet filter banks. *SIAM Journal on Matrix Analysis and Applications*, 26(1):97–124, 2004.

[64] B. Han and Q. Mo. Tight wavelet frames generated by three symmetric B-spline functions with high vanishing moments. *Proceedings of the American Mathematical Society*, 132(1):77–86, 2004.

[65] B. Han and Q. Mo. Symmetric MRA tight wavelet frames with three generators and high vanishing moments. *Applied and Computational Harmonic Analysis*, 18(1):67–93, 2005.

[66] B. Han and Z. Shen. Dual wavelet frames and Riesz bases in Sobolev spaces. *Constructive Approximation*, 29(3):369–406, 2009.

[67] C. Heil. *A Basis Theory Primer*, volume Applied and Numerical Harmonic Analysis. Basel:Birkhäuser, 2011.

[68] C. Heil and D. F. Walnut. Continuous and discrete wavelet transforms. *SIAM Review*, 31:628–666, 1989.

[69] L. Hou, H. Ji, and Z. Shen. Recovering over-/underexposed regions in photographs. *SIAM Journal on Imaging Sciences*, 6(4):2213–2235, 2013.

[70] A. J. E. M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *Journal of Fourier Analysis and Applications*, 1:403–436, 1994.

[71] A. J. E. M. Janssen. Representations of Gabor frame operators. In *Twentieth Century Harmonic Analysis - A Celebration*, pages 73–101. Springer, 2001.

[72] H. Ji, S. D. Riemenschneider, and Z. Shen. Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets. *Studies in Applied Mathematics*, 102:173–204, 1999.

[73] H. Ji and Z. Shen. Compactly supported (bi)orthogonal wavelets by interpolatory refinable functions. *Advances in Computational Mathematics*, 11:81–104, 1999.

[74] H. Ji, Z. Shen, and Y. Zhao. Directional frames for image recovery: multi-scale finite discrete Gabor frames. Preprint.

[75] R. Q. Jia and Z. Shen. Multiresolution and wavelets. *Proceedings of the Edinburgh Mathematical Society*, 37:271–300, 1994.

[76] H. O. Kim, R. Y. Kim, J. K. Lim, and Z. Shen. A pair of orthogonal frames. *Journal of Approximation Theory*, 147(2):196–204, 2007.

[77] I. Kim. *Gabor frames with trigonometric spline dual windows.* PhD thesis, University of Illinois at Urbana-Champaign, 2011.

[78] J. Kovacevic and W. Sweldens. Wavelet families of increasing order in arbitrary dimensions. *IEEE Transactions on image processing*, 9:480–496, 2000.

[79] M. J. Lai and J. Stöckler. Construction of multivariate compactly supported tight wavelet frames. *Applied and Computational Harmonic Analysis*, 21(3):324–348, 2006.

[80] H. J. Landau. On the density of phase-space expansions. *IEEE Transactions on Information Theory*, 39:1152–1156, 1993.

[81] R. S. Laugensen. Gabor dual spline windows. *Applied and Computational Harmonic Analysis*, 27:180–194, 2009.

[82] W. Lawton, S. L. Lee, and Z. Shen. Convergence of multidimensional cascade algorithm. *Numerische Mathematik*, 78:427–438, 1998.

[83] M. Li, Z. Fan, H. Ji, and Z. Shen. Wavelet frame based algorithm for 3D reconstruction in electron microscopy. *SIAM Journal on Scientific Computing*, 36(1):45–69, 2014.

[84] S. D. Li. On general frame decompositions. *Numerical Functional Analysis and Optimization*, 16:9–10, 1995.

[85] J. Liang, J. Li, Z. Shen, and X. Zhang. Wavelet frame based color image demosaicing. *Inverse Problems and Imaging*, 7(3):777–794, 2013.

[86] S. G. Mallat. A theory for multiresolution signal decomposition: the wavelet representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11:674–693, 1989.

[87] Y. Meyer. Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs. *Séminaire Bourbaki*, 662:209–223, 1985-1986.

[88] Y. Meyer. Ondelettes et fonctions splines. *Séinaire EDP, Ecole Polytechnique, Paris*, 1986.

[89] V. Paulsen. *Completely Bounded Maps and Operator Algebras.* Cambridge University Press, 2012.

[90] Y. Quan, H. Ji, and Z. Shen. Data-driven multi-scale non-local wavelet frame construction and image recovery. *Journal of Scientific Computing*, pages 1–23, 2014.

[91] S. D. Riemenschneider and Z. Shen. Box splines, cardinal series, and wavelets. *Approximation Theory and Functional Analysis*, pages 133–149, 1991.

[92] S. D. Riemenschneider and Z. Shen. Multidimensional interpolatory subdivision schemes. *SIAM Journal on Numerical Analysis*, 34:2357–2381, 1997.

[93] S.D. Riemenschneider and Z. Shen. Wavelets and pre-wavelets in low dimensions. *Journal of Approximation Theory*, 71(1):18–38, 1992.

[94] A. Ron and Z. Shen. Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$. *Canadian Journal of Mathematics*, 47:1051–1094, 1995.

[95] A. Ron and Z. Shen. Gramian analysis of affine basis and affine frames. In C. K. Chui and L. L. Schumaker, editors, *Approximation Theory VIII-Vol2, Wavelets and Multilevel Approximation*, pages 375–382, New Jersey, 1995. World Scientific Publishing.

[96] A. Ron and Z. Shen. Affine system in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator. *Journal of Functional Analysis*, 148:408–447, 1997.

[97] A. Ron and Z. Shen. Affine systems in $L_2(\mathbb{R}^d)$ II: dual systems. *Journal of Fourier Analysis and applications*, 3:617–637, 1997.

[98] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. *Duke Mathematical Journal*, 89:237–282, 1997.

[99] A. Ron and Z. Shen. Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$. *Mathematics of Computation*, 67(221):191–207, 1998.

[100] A. Ron and Z. Shen. Generalized shift-invariant systems. *Constructive Approximation*, 22:1–45, 2005.

[101] W. Rudin. *Functional Analysis.* McGraw Hill, 1991.

[102] F. E. Schroeck. *Quantum Mechanics on Phase Space.* Springer, 1996.

[103] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. *Bulletin of the American Mathematical Society*, 26(2):322–328, 1992.

[104] K. Seip. On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$. *Journal of Functional Analysis*, 130:131–160, 1995.

[105] Z. Shen, K. C. Toh, and S. Yun. An accelerated proximal gradient algorithm for frame-based image restoration via the balanced approach. *SIAM Journal on Imaging Sciences*, 4(2):573–596, 2011.

[106] D. Stahl. *Multivariate polynomial interpolation and the lifting scheme with an application to scattered data approximation.* PhD thesis, Technische Universität Kaiserslautern, 2013.

[107] G. Strang. *Introduction to Linear Algebra.* Wellesley-cambridge Press, 3rd ed. edition, 2003.

[108] W. Sun and X. Zhou. Irregular wavelet/Gabor frames. *Applied and Computational Harmonic Analysis*, 13(1):63–76, 2002.

[109] W. Sweldens. The lifting scheme: a new philosophy in biorthogonal wavelet construction. In *SPIE's 1995 International Symposium on Optical Science, Engineering, and Instrumentation*, volume 2569, pages 68–79, 1995.

[110] C. Tai, X. Zhang, and Z. Shen. Wavelet frame based multiphase image segmentation. *SIAM Journal on Imaging Sciences*, 6(4):2521–2546, 2013.

[111] J. M. Varah. A lower bound for the smallest singular value of a matrix. *Linear Algebra and its Applications*, 11(1):3–5, 1975.

[112] D. J. Walnut. Continuity properties of the Gabor frame operator. *Journal of Mathematical Analysis and Applications*, 165(2):479–504, 1992.

[113] R. Young. *An Introduction to Nonharmonic Fourier Series*, volume Pure and Applied Mathematics 93. Academic Press Inc., 1980.