APPROXIMATION TECHNIQUES IN NETWORK INFORMATION THEORY

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

LE SY QUOC
15 August 2014

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Abstract

In the early years of information theory, Shannon and other pioneers in information theory set a high standard for future generations of information theorists by determining the exact fundamental limits in point-to-point communication and source coding problems. Extending their results to network information theory is important and challenging. Many problems in network information theory, such as characterizing the capacity regions for fundamental building blocks of a communication network, namely the broadcast channel, the interference channel and the relay channel, have been open problems for several decades. When exact solutions are elusive, progress can be made by seeking for approximate solutions first. The first contribution of the thesis is to obtain the approximate capacity region for the symmetric Gaussian interference channel in the presence of noisy feedback. The key approximation technique used to complete this task is the so-called linear deterministic model. It is found that when the feedback link strengths exceed certain thresholds, the performance of the interference channel starts to improve. The second contribution is on the understanding of the interference channel in the finite-blocklength regime. In the so-called strictly very strong interference regime, the normal approximation is used to obtain the approximate finite-blocklength fundamental limits of the Gaussian interference channel. It is found that, in this regime, the Gaussian interference still behaves like a pair of separate independent channels. The third contribution is a study of the finite-blocklength source coding problem with side information available at both the encoder and the decoder. It is found that the rate of convergence to the Shannon limit is governed by both the randomness of the information source and the randomness of the side information.

Contents

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Acronyms

IC	Interference channel
GIC	Gaussian interference channel
LD-IC	Linear deterministic interference channel
MAC	Multiple-access channel
i.i.d.	Independent and identically distributed
AWGN	Additive white Gaussian noise
SNR	Signal-to-noise ratio
INR	Interference-to-noise ratio
DM	Discrete and memoryless

Chapter 1

Introduction

1.1 Motivation

Information theory has played an important role in guiding communication engineers to design better communication systems in terms of speed, efficiency, reliability and robustness. Yet, many fundamental questions in designing better networks have been left unanswered for decades. For example, to determine the capacity of a two-user interference channel setting has been an open problem for more than 30 years. When exact answers are hard to find, it makes sense to obtain good approximations. This is the theme of this thesis.

The first aspect that we will consider is *feedback*. Feedback is in general very helpful in a communication network. Feedback allows communication nodes to learn about each other's transmitted signals, to manage interference due to simultaneous transmission and to cooperate with each other. Thus, the overall performance of the network may in general be improved with feedback. However, the feedback links may be affected by *noise*. Will *noisy feedback* still be helpful in boosting the performance of a communication network in general? If that is possible, how could a communication engineer quantify this performance gain to justify for the cost of building feedback links in a noisy environment? In another scenario, an application may be constrained by certain quality-of-service requirements. For example, in an emergency situation, delay in communication is not accepted and quick, effective communication is expected. In real-time multimedia streaming, sequences of multimedia frames are expected to reach a destination node within a specific delay. Nevertheless, most of results in information theory hold provided the duration of communication is very long. These results do not provide satisfactory answers in such *delay-constrained* communication settings. One may wonder how communication nodes can coexist in a short, finite duration of communication. How should a communications engineer compress and decompress an information source within a restricted number of symbols if both the encoder and the decoder share some side information? To find the exact answers to these questions is challenging. Instead, using approximation techniques, the thesis provides approximate answers to these questions.

■ 1.2 Thesis Overview

Chapter 2 provides a necessary background for the rest of the thesis. A reader who is familiar with concepts and topics in Chapter 2 can read any of the subsequent chapters without any loss of continuity.

Chapter 3 is devoted to obtain the approximate capacity region for the symmetric Gaussian interference channel in the presence of noisy feedback. The key approximation technique used to complete this task is the *linear deterministic model*, which excludes certain complexities of a Gaussian counterpart model yet possesses essential properties of this Gaussian model. Chapter 3 first focuses on determining the capacity region of the symmetric linear deterministic interference channel with noisy feedback. Based on the insights gained from working with linear deterministic interference channel, we tackle the symmetric Gaussian interference channel with noisy feedback.

Chapter 4 focuses on the understanding of the interference channel in a finite-blocklength communication. In the strictly very strong interference regime, this chapter uses *normal approximations* to obtain the approximate finite-blocklength capacity region of the Gaussian interference channel. The constituent dispersions, which characterize the rates of convergence to Shannon limits of direct links in the point-to-point communication setting, are found to also characterize the rate of convergence to Shannon limits in the interference channel.

Chapter 5 contains a preliminary study of the finite-blocklength source-coding problem with side information available at both the encoder and the decoder when the information source is discrete, stationary and memoryless. This chapter also uses normal approximations to approximate the finite-blocklength rate-distortion function in the presence of side information.

While all three Chapter 3,4 and 5 focus on the theme of approximation, there are other relations between the chapters. While Chapter 3 and Chapter 4 both focus on Gaussian interference channel, Chapter 3 considers Gaussian interference channel with noisy feedback and Chapter 4 considers Gaussian interference channel without feedback. While Chapter 4 and Chapter 5 both focus on second-order analysis, Chapter 4 works on second-order analysis for Gaussian interference channel and Chapter 5 works on second-order analysis for conditional rate-distortion. While the theory of chapter 3 is general in the sense that it is not restricted to any particular application, Chapter 4 and Chapter 5 cater to the need of delay-constrained applications.

The thesis ends with Chapter 6, where reflections on the thesis and suggestions for further avenues of research are found.

■ 1.3 Thesis Contributions

■ 1.3.1 On role of noisy feedback

- Chapter 3 in this thesis considers the impact of noise on the gain due to feedback. Specifically, as a stepping stone to characterize the capacity region for the two-user Gaussian interference channel with noisy feedback, the two-user linear deterministic interference channel with noisy feedback is considered. The capacity region for the symmetric linear deterministic interference channel with noisy feedback has been obtained. Noisy feedback has been shown to increase the capacity region of the symmetric linear deterministic interference channel with noisy feedback if and only if the amount of feedback level l is greater than a certain threshold l^*. Denote α as the normalized interference link gain with respect to the direct link gain. It is found that, excluding the moderately strong interference regime and the strong interference regime, i.e., $\frac{1}{2} \leq \alpha \leq 2$, in which even full feedback does not increase symmetric capacity, l^* is equal to the per-user symmetric capacity without feedback. Key ideas in the converse proof are novel converse outer bounds on weighted sum rates $2R_1 + R_2$ and $R_1 + 2R_2$ and on the sum rate $R_1 + R_2$. The novel outer bounds are tightened by specially defined auxiliary random variables. The key idea in the achievability proof is message splitting. Each transmitted message is split into a private message, a cooperative common message and a non-cooperative message. The sizes and positions of these messages need to be carefully designed to maximize the achievable rate region for both transmitters.

- The results and the techniques developed for this linear deterministic model are then applied to characterize inner bounds and outer bounds for the symmetric Gaussian IC with noisy feedback. In the achievability proof, we also use message splitting. The difficulty in message splitting is to design the power allocation scheme so that the achievable rate region for both transmitters is maximized. In principle, the transmitted power of the private information should be chosen such that the received power of the private information at non-intended receivers are below the noise level. The transmitted powers of non-cooperative messages and cooperative messages are governed by many factors: direct link strengths, interference link strengths and feedback link strengths. Intuitively, as feedback link strengths increase, the chance for cooperation increases. As a result, more power can be allocated to cooperative messages. The specially defined auxiliary random variables for the linear deterministic model helps us define corresponding auxiliary random variables for the Gaussian model so that the outer bounds can be tightened. Even though most of the techniques for the linear deterministic models can be lifted to be applied to the Gaussian model, the presence of Gaussian noise can lead to a complicated analysis, so careful use of lifted techniques is required. The performance gain due to noisy feedback is approximated in terms of the signal-to-noise ratios of the direct links, the interference links and the feedback links. The outer bounds have been shown to be at most 4.7 bits/s/Hz away from the achievable rate region. This result holds for a large range of the signal-to-noise ratio of the direct links.

1.3.2 On interference networks in the finite-blocklength regime

- Chapter 4 of this thesis characterizes the second-order coding rates of the Gaussian interference channel in the strictly very strong interference regime. In other words, we characterize the speed of convergence of rates of optimal block codes towards a boundary point of the capacity region. These second-order rates are expressed in terms of the average probability of error and variances of some modified information densities. These variances coincide with the dispersions of the constituent point-to-point Gaussian channels. Thus, the approximate finite-blocklength capacity region in the strictly very strong interference regime is obtained. Intuitively, in the strictly very strong interference regime, the interference caused by a non-intended transmitter can be decoded by a non-intended receiver. As a result, the Gaussian interference channel approximately behaves like a pair of separate channels in the finite-blocklength communication.

- In the achievability proof, Feinstein's Lemma is generalized to yield any achievable coding scheme for the Gaussian interference channel. In the converse proof, Verdú-Han Lemma is generalized. In the strictly very strong interference regime, the number of error events involved in the achievability proof is reduced and the forward bounds match the converse bounds up to the second-order term.

1.3.3 On the combined effect of side information and finite-blocklength communication on source coding

- Chapter 5 of this thesis obtains the second-order rate-distortion function of the source coding problem with side information available at both the encoder and the decoder. In other words, the finite-blocklength rate-distortion problem for this source coding is approximated. It is found that the rate of convergence to the Shannon limit is governed by both the randomness of the information source and the randomness of the side information.

- The key idea in the achievablity proof is a random coding bound, which allows us to deal with the information source random variable and the side information random variable jointly.

- The concept of D-tilted information density is found to be useful not only in the source coding problem without side information, but also useful in the source coding problem with side information. The method of types is very helpful in the second-order analysis of the source coding problem without side information. However, it is not easy to use the method of types in the second-order analysis of the source coding problem with side information.

1.4 Bibliographical Notes

The material in this thesis has been presented in parts at various conferences and submitted to various journals.

- The material in Chapter 3 was presented in [63, 64, 65] and was submitted to IEEE Transactions on Information Theory in Dec 2012 [66].
- The material in Chapter 4 was presented in [67, 68, 69] and was submitted to IEEE Transactions on Information Theory in Apr 2014 [70].
- The material in Chapter 5 was published as an NUS Technical Report.

Chapter 2

Background

IN this background chapter, we review some basic concepts and tools in information theory and probability theory, which lay the foundations for subsequent chapters. Interested readers who want to see the proofs of the theorems stated in this chapter are referred to texts in information theory such as [18, 19, 30, 125], and texts in probability theory such as [26, 83, 89]. In addition, we also briefly review the linear deterministic model [3].

2.1 Information theory

Information theory is a branch of applied mathematics, electrical engineering and computer science [18, 19, 30, 125]. It is generally believed that information theory was created when Shannon, in 1948, published his landmark paper titled *A Mathematical Theory of Communication* in the *Bell System Technical Journal* [96]. This paper contained ground-breaking concepts that changed the world. Shannon showed how information can be quantified and demonstrated that all information media can be unified. Information can exist in many forms such as texts, images, videos, electromagnetic waves. However, it can always be digitized. Information theory is not created by Shannon alone. It has been a product of crucial contributions made by many scientists, who have come from diverse fields, have been motivated by Shannon's revolutionary ideas and expanded upon them. Although information theory is mathematical in nature, it serves as a beacon of light for generations of communication engineers who have made great products for the world.

In 1948, Shannon made a prophecy that every white additive Gaussian noise (AWGN) has a capacity limit. In a layman language, it says it is mathematically impossible to get an error-free communication if the transmission rate is above the channel limit. On the other hand, it is mathematically possible to get an error-free communication if the transmission rate is below the channel limit. The noisy channel coding theorem does not tell a communication engineer how a code can be constructed. However, it predicts that reliable communication is possible. Indeed, the noisy channel coding theorem gave rise to the entire field of coding theory. Error-correcting codes are important contributions of coding theory. In error-correcting codes, redundancy are introduced into the digital representation of information at the encoder so that this information can be recovered at the decoder's side. For example, if you scratch the surface of any DVD, there is a high chance that this DVD can still play back perfectly. The spacecraft Mariner VI, in 1969, used Reed-Muller codes for communication in the exploration of Mars. At Neptune, which is 4.4 billion miles from the Earth, the spacecraft Voyager could transmit information back to the Earth at a rate of 21.6 kbits/s in 1979. The advances in microprocessors provided the computation power to realize many complicated coding schemes. In fact, 50 years after the publication of Shannon's landmark paper, turbo codes and LDPC codes are shown to iteratively achieve the capacity limit of the AWGN channel. In his landmark paper, Shannon also discussed source coding, which considers efficient representation of data. In 1952, David Huffman came up with Huffman code, which is optimal in the sense that its minimum expected length achieves the theoretical limit. Huffman code is still widely used in data compression standards such as JPEG, MP3, ZIP. Storage devices, such as hard drives and RAM, employ information theory concepts. Information theory has also strongly influenced the development of wireless systems and computer networks.

Information theory is essential not only in communication theory, but also in many other fields such as statistical inference and statistics [20, 61, 74], economics [50], physics [80]. However, in this thesis, we will only discuss information theory as a sub-topic in communication theory.

Next, we briefly review some concepts and tools in information theory.

■ 2.2 Measures of information for discrete random variables

There are various ways to measure information. One way to do so is to use Shannon entropy (we will call it entropy for short).

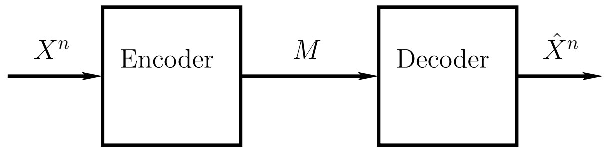

Figure 2.1. Lossless source compression system.

Definition 2.1. The entropy $H(X)$ of a discrete random variable X, taking values in a finite alphabet $\mathcal{X}$, with probability mass function $P_X(x)$ is defined as

$$H(X) \triangleq \sum_{x \in \mathcal{X}} P_X(x) \log_2 \frac{1}{P_X(x)}, \tag{2.1}$$

where the unit of information is called a *bit*. The unit of information is called a *nat* if the base of the logarithm used in the definition is e.

Unless otherwise stated, we will assume that all logarithms in this thesis are taken to base 2. In this definition, we adopt the convention that $0 \log 0 = 0$. Note that the smaller the probability $P_X(x)$ is, the larger the value of $\log_2 \frac{1}{P_X(x)}$ is. Intuitively, the more surprising the event $X = x$ is, the more information it contains. In other words, the entropy of a discrete random variable is a measure of uncertainty in that random variable.

Operationally, the entropy of the source $H(X)$ is a fundamental limit in source compression problems. Consider a scenario when a discrete memoryless stationary information source produces a sequence of random variables $X^n = (X_1, X_2, \ldots, X_n)$. The source is discrete in the sense that each X_i, for $i = 1, 2, \ldots, n$, only takes values from a finite source alphabet $\mathcal{X}$. The source is memoryless and stationary in the sense that the random variables X_i are independent and have the same distribution P_X. Given an observation of a sequence X^n, a communication engineer needs to encode this sequence into a binary codeword, so that at the destination, this sequence can be recovered given an observation of the corresponding binary codeword (see Figure 2.1). It is proven that, as the number of source letters n gets sufficiently large, the number of bits per source letter to complete this compression task, with arbitrarily small probability of error, can be made to be arbitrarily close to the entropy of the source $H(X)$ [7, 19, 96, 98].

Similarly to the above, we can define the joint entropy $H(X_1, X_2, \ldots, X_n)$ of a discrete random vector $(X_1, X_2, \ldots, X_n)$. Next, we define conditional entropy.

Definition 2.2. The conditional entropy $H(X|Y)$ of a discrete random variable X, taking values in a finite alphabet $\mathcal{X}$, given a discrete random variable Y, with joint probability mass function $P_{XY}(xy)$ is defined as

$$H(X|Y) \triangleq \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y), \tag{2.2}$$

where $H(X|Y = y)$ is the entropy of the random variable $X|Y = y$ and this entropy is defined in Definition 2.1.

Definition 2.3. Consider two discrete random variables X and Y, taking values in finite alphabet $\mathcal{X}$ and $\mathcal{Y}$ respectively, with joint probability mass function $P_{XY}(xy)$. The mutual information $I(X;Y)$ is defined as

$$I(X;Y) \triangleq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{XY}(xy) \log_2 \frac{P_{XY}(xy)}{P_X(x) P_Y(y)}. \tag{2.3}$$

Operationally, the mutual information $I(X;Y)$ is an important quantity in characterizing a fundamental limit in channel coding problems. Consider the

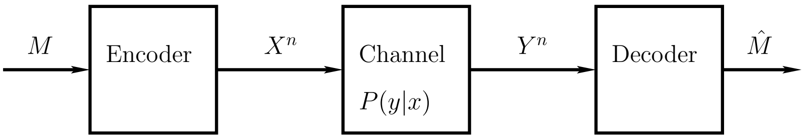

Figure 2.2. Discrete memoryless point-to-point channel.

scenario when a transmitter wants to transmit a message to a receiver through a discrete memoryless stationary channel $P_{Y|X}$ (see Figure 2.2). A communication engineer needs to design an encoder which encodes a message into a codeword X^n, which is then transmitted through the discrete memoryless channel in n channel uses. At the receiver's side, he needs to design a decoder which recovers the message based on the observation of the received signal Y^n. It is proven that, as the number of channel uses n becomes sufficiently large, the data rate that the channel can support, with arbitrarily small probability of error, can be chosen to be arbitrarily close to $\max_X I(X;Y)$ bits per channel use [25, 29, 96, 120].

Definition 2.4. Consider three discrete random variables X, Y and Z, with joint probability mass function $P_{XYZ}(xyz)$. The conditional mutual information $I(X;Y|Z)$ is defined as

$$I(X;Y|Z) \triangleq H(X|Z) - H(X|YZ). \tag{2.4}$$

Next, we state some important properties of entropy, conditional entropy, mutual information and conditional mutual information [18, 30].

Theorem 2.1. *Consider three discrete random variables X, Y and Z, with joint probability mass function $P_{XYZ}(xyz)$. We have*

(i) $H(X) \geq 0$.

(ii) $H(X) \leq \log |\mathcal{X}|$, *where* $|\mathcal{X}|$ *denotes the cardinality of the set* $\mathcal{X}$.

(iii) $H(XY) = H(X) + H(Y|X)$.

(iv) $I(X;Y|Z) \geq 0$.

(v) $H(X|Y) \leq H(X)$.

(vi) *If* X, Y *and* Z *form a Markov chain in that order, i.e.* $X \to Y \to Z$, *then* $I(X;Y) \geq I(X;Z)$. *This is commonly known as the data-processing inequality.*

Fano's inequality is very helpful in proving weak converses for many information-theoretic problems [18].

Theorem 2.2 (Fano's inequality). *Consider two discrete random variables* W *and* $\hat{W}$, *taking values in the alphabets* $\mathcal{W}$ *and* $\hat{\mathcal{W}}$, *with joint probability mass function* $P_{W\hat{W}}(w\hat{w})$. *Define* $P_e = \Pr(W \neq \hat{W})$. *We have*

$$H(W|\hat{W}) \leq 1 + P_e \log |\mathcal{W}|. \tag{2.5}$$

■ 2.3 Measures of information for continuous random variables

Sometimes, the source alphabet may not be discrete but continuous. We need a measure of information for such a source. In this section, we introduce the concept of *differential entropy* for continuous random variables [18].

Definition 2.5. A real-valued random variable X is said to be *continuous* if its cumulative distribution function $F_X(x) = \Pr(X \leq x)$ is continuous. Let

$f_X(x) = F'_X(x)$ when the derivative is defined. The function $f_X(x)$ is called the probability density function for X. The support set S for random variable X is the subset of $\mathcal{X}$, where $f_X(x) > 0$. The *differential entropy* $h(X)$ of the random variable X is defined as

$$h(X) = -\int_S f_X(x) \log f_X(x) dx. \tag{2.6}$$

Being different from entropy for discrete (finite) random variable which is always non-negative and finite [29], the differential entropy of a random variable can be negative or unbounded. Similarly, we can define differential entropy for a random vector. Next, we define conditional differential entropy.

Definition 2.6. Consider continuous random variables X and Y, with joint probability density function $f_{XY}(xy)$. The *conditional differential entropy* $h(X|Y)$ is defined as

$$h(X|Y) = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(xy) \log f_{X|Y}(x|y) dxdy. \tag{2.7}$$

Definition 2.7. Consider continuous random variables X and Y, with joint probability density function $f_{XY}(xy)$. The *mutual information* $I(X;Y)$ is defined as

$$I(X;Y) \triangleq h(X) + h(Y) - h(XY). \tag{2.8}$$

Differential entropy has many properties that are similar to that of entropy for discrete random variables.

Theorem 2.3. *Consider three continuous random variables X, Y and Z, with joint probability density function $f_{XYZ}(xyz)$. We have* *(i)* $h(X, Y) = h(X) + h(Y|X)$.

(ii) $I(X; Y|Z) \geq 0$.

(iii) $h(X|Y) \leq h(X)$. *Equality occurs if and only if* X *and* Y *are independent.*

(iv) *If* X, Y *and* Z *form a Markov chain in that order, i.e.* $X \to Y \to Z$, *then* $I(X; Y) \geq I(X; Z)$. *This is commonly known as the data-processing inequality.*

(v) $h(X + c) = h(X)$, *where* c *is any real-valued constant.*

(vi) $h(cX) = h(X) + \log |c|$, *where* c *is any real-valued constant.*

The following theorem presents an useful result. Over all distributions with the same covariance, the multivariate normal distribution maximizes the entropy.

Theorem 2.4. *Consider a random vector* $\boldsymbol{X} \in \mathbb{R}^k$, *with zero mean and covariance matrix* K. *We have* $h(\boldsymbol{X}) \leq \frac{1}{2} \log[(2\pi e)^k \det(K)]$. *Equality occurs if and only if* $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{0}, K)$.

■ 2.4 Measures of information for arbitrary random variables

The previously discussed measures of information for discrete and continuous random variables give a sufficient background for us to present our new results in the subsequent chapters. Readers, who are interested in rigorous definition of measure of information for arbitrary random variables, are referred to works by Kolmogorov [51], Pinsker [84], Gray [37].

2.5 Weakly typical sequences

Having defined measure of information, we are next going to review some useful tools in information theory. The concept of weakly typical sequences is useful in constructing achievability schemes.

Definition 2.8. Consider a sequence of random variables $X_1, X_2, \ldots$, which are independent and identically distributed according to $P_X(x)$. The weakly typical set $A_\epsilon^{(n)}(X)$ with respect to a probability distribution $P_X(x)$ is defined the set of n-tuples $(x_1, x_2, ..., x_n) \in \mathcal{X}^n$ satisfying

$$2^{-n(H(X)+\epsilon)} \leq P_{X_1 X_2 \ldots X_n}(x_1, x_2, ..., x_n) \leq 2^{-n(H(X)-\epsilon)}. \tag{2.9}$$

A weakly typical set has the following properties.

Theorem 2.5. *Consider a sequence of random variables $X^n = (X_1, X_2, \ldots, X_n)$, which are independent and identically distributed to $P_X(x)$. The weakly typical set $A_\epsilon^{(n)}(X)$ has the following properties.*

(i) For n sufficiently large, $\Pr\{X^n \in A_\epsilon^{(n)}(X)\} > 1 - \epsilon$.

(ii) $|A_\epsilon^{(n)}(X)| \leq 2^{n(H(X)+\epsilon)}$, *where $|A|$ is the cardinality of set A.*

(iii) For n sufficiently large, $|A_\epsilon^{(n)}(X)| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$.

One of the most popular decoding rules is the jointly weakly typical decoding rule, in which the codeword sequence is decoded as a sent sequence if it is jointly weakly typical with the received sequence. In this decoding rule, the concept of a jointly weakly typical set and its properties are important.

Definition 2.9. Consider a length-n sequence of random vectors (X^nY^n), which are independent and identically distributed according to $P_{XY}(xy)$, so that we have $P_{X^nY^n}(x^ny^n) = \prod_{i=1}^{n} P_{XY}(x_iy_i)$. The jointly weakly typical set $A_\epsilon^{(n)}(XY)$ with respect to a probability distribution $P_{XY}(xy)$ is the set of length-n sequences $(x^ny^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ satisfying

$$2^{-n(H(X)+\epsilon)} \leq P_{X^n}(x^n) \leq 2^{-n(H(X)-\epsilon)}, \tag{2.10}$$

$$2^{-n(H(Y)+\epsilon)} \leq P_{Y^n}(y^n) \leq 2^{-n(H(Y)-\epsilon)}, \tag{2.11}$$

$$2^{-n(H(XY)+\epsilon)} \leq P_{X^nY^n}(x^ny^n) \leq 2^{-n(H(XY)-\epsilon)}. \tag{2.12}$$

A jointly weakly typical set has the following properties [18]

Theorem 2.6. *Consider a length-n sequence of random vectors (X^nY^n), which are independent and identically distributed according to $P_{XY}(xy)$, so that we have $P_{X^nY^n}(x^ny^n) = \prod_{i=1}^{n} P_{XY}(x_iy_i)$. The jointly weakly typical set $A_\epsilon^{(n)}(XY)$ has the following properties.*

(i) For n sufficiently large, $\Pr\{(X^nY^n) \in A_\epsilon^{(n)}(XY)\} > 1 - \epsilon$.

(ii) $|A_\epsilon^{(n)}(XY)| \leq 2^{n(H(XY)+\epsilon)}$*, where $|A|$ is the cardinality of set A.*

(iii) Consider two random vectors $\tilde{X}^n$ and $\tilde{Y}^n$, which are independent and have the same marginals as that of $P_{X^nY^n}(x^ny^n)$. Then we have

$$\Pr(\{(\tilde{X}^n\tilde{Y}^n) \in A_\epsilon^{(n)}(XY)\}) \leq 2^{-n(I(X;Y)-3\epsilon)}. \tag{2.13}$$

When n is sufficiently large, we have

$$\Pr(\{(\tilde{X}^n\tilde{Y}^n) \in A_\epsilon^{(n)}(XY)\}) \geq 2^{-n(I(X;Y)+3\epsilon)}. \tag{2.14}$$

In network information theory problems, it is useful to make use of jointly typical sets which involves more than two random variables.

Definition 2.10. Consider a sequence of random vectors $(X^{(1)n}X^{(2)n}\ldots X^{(k)n})$, which are independent and identically distributed according to the probability distribution $P_{X^{(1)}X^{(2)}\ldots X^{(k)}}(x^{(1)}x^{(2)}\ldots x^{(k)})$, so that

$$P_{X^{(1)n}X^{(2)n}\ldots X^{(k)n}}(x^{(1)n}x^{(2)n}\ldots x^{(k)n}) = \prod_{i=1}^{n} P_{X^{(1)}X^{(2)}\ldots X^{(k)}}(x_i^{(1)}x_i^{(2)}\ldots x_i^{(k)}). \tag{2.15}$$

The jointly weakly typical set $A_\epsilon^{(n)}(X^{(1)}X^{(2)}\ldots X^{(k)})$ with respect to the probability distribution $P_{X^{(1)n}X^{(2)n}\ldots X^{(k)n}}(x^{(1)n}x^{(2)n}\ldots x^{(k)n})$ is the set of length-n sequences $(x^{(1)n}x^{(2)n}\ldots x^{(k)n}) \in \mathcal{X}^{(1)n} \times \ldots \times \mathcal{X}^{(k)n}$ satisfying

$$2^{-n(H(S)+\epsilon)} \leq P_{S^n}(s^n) \leq 2^{-n(H(S)-\epsilon)}, \tag{2.16}$$

where S is any subset of the set of random variables $\{X^{(1)}X^{(2)}\ldots X^{(k)}\}$.

A jointly typical set of a random vector has similar properties to that in Theorem 2.6. In addition, it has the following important property [18, Theorem 15.2.3].

Theorem 2.7. *Consider a sequence of random vectors $(X^{(1)n}X^{(2)n}\ldots X^{(k)n})$, which are independent and identically distributed according to the probability distribution $P_{X^{(1)}X^{(2)}\ldots X^{(k)}}(x^{(1)}x^{(2)}\ldots x^{(k)})$. Let S_1, S_2 and S_3 be three random vectors, which are arbitrary subsets of $\{X^{(1)}X^{(2)}\ldots X^{(k)}\}$. If random vector $\tilde{S}_1$ and* *random vector $\tilde{S}_2$ are conditionally independent given a random vector $\tilde{S}_3$, and these three random vectors have the same pairwise marginals as that of $(S_1S_2S_3)$, then we have*

$$\left|\frac{1}{n}\log \Pr(\tilde{S}_1^n, \tilde{S}_2^n\tilde{S}_3^n \in A_\epsilon^{(n)}(S_1S_2S_3)) - I(S_1;S_2|S_3)\right| < 6\epsilon, \tag{2.17}$$

for n sufficiently large.

■ 2.6 Results in probability theory

In this section, we review some results in probability theory, that we will use in subsequent chapters. We start with the well-known weak law of large numbers [89].

Theorem 2.8 (The weak law of large numbers)**.** *Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables, each having mean $\mathbb{E}(X_i) = \mu$ and finite variance. Then, for any $\epsilon > 0$, we have*

$$\Pr\left\{\left|\frac{X_1 + \dots + X_n}{n} - \mu\right| \geq \epsilon\right\} \to 0 \tag{2.18}$$

as $n \to \infty$.

The weak law of large numbers is essential in the proof of Theorem 2.5.

The central limit theorem is one of most remarkable results in probability theorem. In its simplest form, the central limit theorem is as follows [89].

Theorem 2.9 (The central limit theorem)**.** *Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables, each having* *mean μ and variance σ^2. Then, the distribution of*

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \tag{2.19}$$

tends to the standard normal as $n \to \infty$. That is, for any $-\infty < a < \infty$, we have

$$\Pr\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx \tag{2.20}$$

as $n \to \infty$.

There are other versions of central limit theorems which are not restricted to i.i.d. random variables or other technical conditions. In second-order asymptotics analysis, we are often interested in knowing the rate of convergence of the scaled sum $\frac{X_1+\cdots+X_n-n\mu}{\sqrt{n}\sigma}$ to the standard normal distribution. This rate of convergence is quantified by the Berry-Esséen theorem, which is presented next [26, Theorem 2, Chapter XVI. 5]

Theorem 2.10 (Berry-Esséen Theorem)**.** . *Let X_k, for $k = 1, 2, \ldots, n$ be independent random variables with $\mu_k = \mathbb{E}[X_k]$, $\sigma_k^2 = \mathsf{var}[X_k]$, $t_k = \mathbb{E}[|X_k - \mu_k|^3]$, $\sigma^2 = \sum_{k=1}^{n} \sigma_k^2$, and $T = \sum_{k=1}^{n} t_k$. Then for any $-\infty < \lambda < \infty$, we have*

$$\left|\Pr\left[\sum_{k=1}^{n}(X_k - \mu_k) \geq \lambda\sigma\right] - Q(\lambda)\right| \leq \frac{6T}{\sigma^3}. \tag{2.21}$$

The following theorem gives a variant of the multivariate Berry-Esséen Theorem [35] [9], which is a restatement of Corollary 38 in [118]. The theorem can be applied to random vectors which are independent, but not necessarily identically distributed. For i.i.d. random vectors, interested readers are referred to Bentkus's work [5].

Theorem 2.11. *Let $\mathbf{U}_1, \ldots, \mathbf{U}_n$ be independent, zero-mean random vectors in $\mathbb{R}^m$. Let $\mathbf{G}_n \triangleq \frac{1}{\sqrt{n}}(\mathbf{U}_1 + \ldots + \mathbf{U}_n)$, $V \triangleq \mathrm{cov}(\mathbf{G}_n)$, $t \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[\|\mathbf{U}_i\|_2^3]$ and let $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, V)$. Let $\mathbb{C}_m$ be the family of all convex, Borel measurable subsets of $\mathbb{R}^m$. Assume $V \succ 0$ and let the minimum eigenvalue of V be $\lambda_{\min}(V)$. Then, for all $n \in \mathbb{N}$, we have*

$$\sup_{\mathfrak{C} \in \mathbb{C}_m} |\Pr(\mathbf{G}_n \in \mathfrak{C}) - \Pr(\mathbf{Z} \in \mathfrak{C})| \leq \frac{254\sqrt{m}\, t}{\lambda_{\min}(V)^{3/2}\sqrt{n}}. \tag{2.22}$$

The following theorem provides a variant of the multivariate Berry-Esséen Theorem [35] [9], which is a restatement of Proposition 1 in [79]. The lemma can be applied to functions of sums of i.i.d. random vectors under certain conditions. This theorem is used in the direct proof of Theorem 4.1.

Theorem 2.12. *Let $\{\mathbf{U}_t \triangleq (U_{1t}, U_{2t}, \ldots, U_{at})\}_{t=1}^{\infty}$ be a sequence of zero-mean i.i.d. random vectors in $\mathbb{R}^a$ with $\mathbb{E}[\|\mathbf{U}_t\|_2^3]$ being finite. Consider a vector-valued function $\mathbf{g} : \mathbb{R}^a \to \mathbb{R}^b$. Denote $\mathbf{g}(\mathbf{u}) \triangleq [g_1(\mathbf{u}), g_2(\mathbf{u}), \ldots, g_b(\mathbf{u})]^T$. Assume that $\mathbf{g}(\mathbf{u})$ has continuous second-order partial derivatives in a neighbourhood of $\mathbf{u} = \mathbf{0}$. Denote the corresponding Jacobian matrix J at $\mathbf{u} = \mathbf{0}$ of $\mathbf{g}(\mathbf{u})$ as $J \in \mathbb{R}^{b \times a}$, whose components are defined as*

$$J_{ji} \triangleq \left.\frac{\partial g_j(\mathbf{u})}{\partial u_i}\right|_{\mathbf{u}=\mathbf{0}} \tag{2.23}$$

for $j \in \{1, 2, \ldots, b\}$, and $i \in \{1, 2, \ldots, a\}$. Let the random vector $\mathbf{Z}$ have distribution $\mathcal{N}(\mathbf{g}(\mathbf{0}), \frac{1}{n} J\, \mathrm{Cov}(\mathbf{U}_1)\, J^T)$. Then, for any convex Borel-measurable set $\mathcal{D}$ in $\mathbb{R}^b$, there exists a finite positive constant c such that

$$\left| \Pr\left[\mathbf{g}\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{U}_t\right) \in \mathcal{D}\right] - \Pr[\mathbf{Z} \in \mathcal{D}] \right| \leq \frac{c}{\sqrt{n}}. \tag{2.24}$$

2.7 Network information theory

Point-to-point information theory has been well developed and been shown to be useful to communication engineers to design point-to-point communication systems. However, the point-to-point information theory cannot help engineers design optimal communication networks which involve many transmitters and many receivers. There are many elements of a communication network that cannot be captured in the point-to-point model such as cooperation between users and interference. This leads to the need for network information theory, which has been one of the main foci in information theory for the past few decades. However, our understanding in this field is far from being complete. In this section, we briefly review the four basic building blocks of a communication network: the multiple-access channel, the broadcast channel, the interference channel and the relay channel.

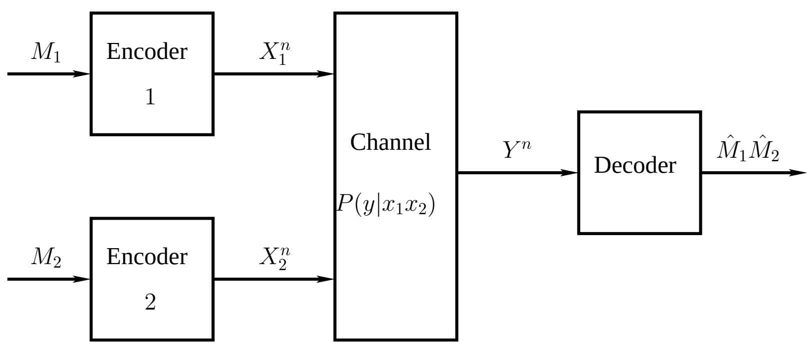

Figure 2.3. Multiple-access channel.

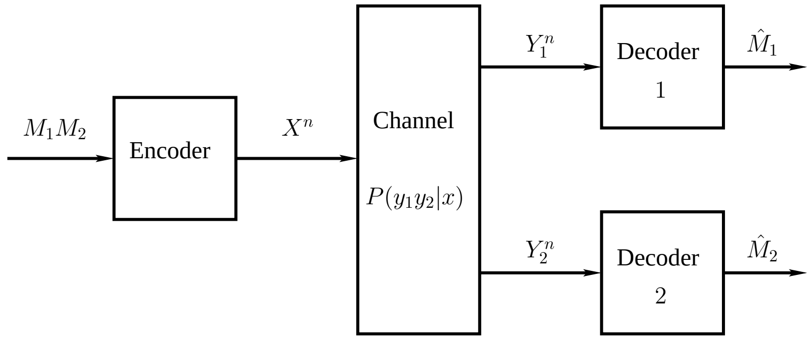

Figure 2.4. Broadcast channel.

■ 2.7.1 Multiple-access channel

The multiple-access channel consists of multiple transmitters communicating simultaneously with a single receiver (see Figure 2.3). The multiple-access channel capacity region was found by Ahlswede [1] and Liao [72].

■ 2.7.2 Broadcast channel

The broadcast channel consists of a single transmitter communicating with multiple receivers (see Figure 2.4). The capacity region of the broadcast channel has been found only for a few special cases. The capacity region for the degraded broadcast channel was found by Bergmans [8] and Gallager [31]. El Gamal [32] found the capacity region for a class of more capable broadcast channel. Körner and Marton [54] found the capacity region for the broadcast channels with degraded message sets. The largest achievable region for the broadcast channel was found by Marton [76], and a simpler proof was discovered by El Gamal and van der Meulen [33].

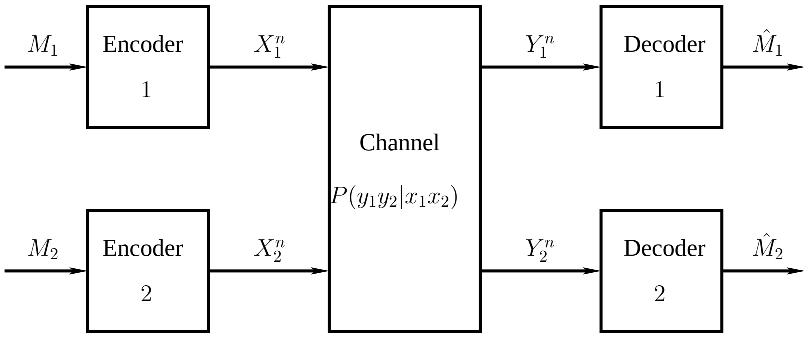

Figure 2.5. Interference channel.

2.7.3 Interference channel

The interference channel was first introduced by Ahlswede [1]. In the interference channel, there are two transmitters and two receivers. Each transmitter wants to transmit a message to its intended receiver only (see Figure 2.5). In the process of doing so, both transmitters interfere with each other. The capacity channel of the interference channel has been found for only a few cases. Carleial first showed that [14] interference is the same as no interference for the Gaussian interference channel with very strong interference. Sato [92, 93] found the capacity region for the interference channel with strong interference. The capacity region of degraded interference channel was found by Benzel [6]. The largest achievable rate region for the discrete memoryless interference channel was found by Han and Kobayashi [41].

2.7.4 Relay channel

The relay channel consists of one transmitter, one receiver and one relay node (see Figure 2.6). This channel was introduced by van der Meulen [21]. The

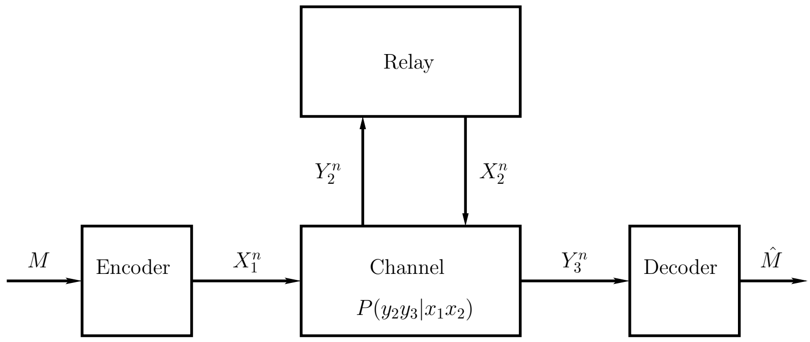

Figure 2.6. Relay channel.

capacity region for the degraded relay channel was determined by Cover and El Gamal [16]. The capacity region of the discrete memoryless relay channel is an open problem.

2.8 Linear deterministic model

The linear deterministic model was introduced by Avestimehr, Diggavi and Tse [3]. It is capable of capturing certain features of a communication network such as signal strength, superposition and broadcasting. The beauty of linear deterministic model is the strong connection between the linear deterministic model and the corresponding Gaussian model. Under certain circumstances, a capacity-achieving scheme in the deterministic model naturally suggests a scheme, which can achieve within a constant gap from the outer bounds, for the corresponding Gaussian model. In this section, we briefly review the linear deterministic model for the point-to-point channel and its connection with the corresponding Gaussian model.

Consider a point-to-point Gaussian channel, which is given by

$$Y = gX + Z, \tag{2.25}$$

where Y is the output of the channel, X is the input random variable, g is a fixed real-valued channel gain and $Z \sim \mathcal{N}(0,1)$ is Gaussian noise. The input satisfies the average power constraint $\mathbb{E}(|X|^2) \leq 1$. Thus, the channel gain g satisfies

$$|g| = \sqrt{\mathsf{SNR}}, \tag{2.26}$$

where SNR is the signal-to-noise ratio of the channel. It is well-known that the capacity of this point-to-point channel is $C_{\mathsf{AWGN}} = \frac{1}{2}\log(1 + \mathsf{SNR})$.

For simplicity, assume the input signal x has a peak power constraint of 1. It is also assumed that the background noise z also has a peak power constraint of 1. Denote the binary expansion of x as $\sum_{i=1}^{\infty} x(i)2^{-i}$. Denote the binary expansion of z as $\sum_{i=1}^{\infty} z(i)2^{-i}$. Expressing the received signal y in terms of the binary expansion of x and z, we have

$$y = 2^{\frac{1}{2}\log\mathsf{SNR}} \sum_{i=1}^{\infty} x(i)2^{-i} + \sum_{i=1}^{\infty} z(i)2^{-i}. \tag{2.27}$$

Define $n \triangleq \lceil \frac{1}{2}\log\mathsf{SNR} \rceil$. We have

$$y \approx 2^n \sum_{i=1}^{n} x(i)2^{-i} + \sum_{i=1}^{\infty} [x(i+n) + z(i)]2^{-i}. \tag{2.28}$$

Ignoring input signal bits below the noise level, we have

$$y \approx 2^n \sum_{i=1}^{n} x(i)2^{-i}. \tag{2.29}$$

This approximation equation motivates the definition of the linear deterministic model. Consider a transmitted signal $\hat{x}^q$, which is a binary vector of length q. The deterministic channel only passes the top n bits to the destination. Therefore, the output signal $\hat{y}^q$, which is also a binary vector of length q, and the input signal $\hat{x}^q$ are governed by the following linear deterministic model

$$\hat{y}^q = S^{q-n}\hat{x}^q, \tag{2.30}$$

where S is a $q \times q$ shifting matrix,

$$S \triangleq \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}. \tag{2.31}$$

The capacity of this linear deterministic point-to-point channel is n [3].

Linear deterministic model can also model superposition. Consider a Gaussian MAC channel, which is given by

$$Y = g_1 X_1 + g_2 X_2 + Z, \tag{2.32}$$

where Y is the output of the channel, X_1 and X_2 are the input random variables, g_1 and g_2 are fixed real-valued channel gains and $Z \sim \mathcal{N}(0, 1)$ is Gaussian noise. The input random variables satisfy the average power constraint $\mathbb{E}(|X_j|^2) \le 1$, for $j = 1, 2$. Thus, the channel gains g_j, for $j = 1, 2$, satisfy

$$|g_j| = \sqrt{\mathsf{SNR}_j}, \tag{2.33}$$

where SNR_j are the signal-to-noise ratios of the channel.

For simplicity, assume the input signal x_j, for $j = 1, 2$, have peak power constraints of 1. It is also assumed that the background noise z also has a peak power constraint of 1. Denote the binary expansion of x_j, as $\sum_{i=1}^{\infty} x_j(i)2^{-i}$. Denote the binary expansion of z as $\sum_{i=1}^{\infty} z(i)2^{-i}$. Expressing the received signal y in terms of the binary expansion of x_j and z, we have

$$y = 2^{\frac{1}{2}\log \mathsf{SNR}_1} \sum_{i=1}^{\infty} x_1(i)2^{-i} + 2^{\frac{1}{2}\log \mathsf{SNR}_2} \sum_{i=1}^{\infty} x_2(i)2^{-i} + \sum_{i=1}^{\infty} z(i)2^{-i}. \tag{2.34}$$

Define $n_j \triangleq \lceil \frac{1}{2}\log \mathsf{SNR}_j \rceil$. We have

$$y = 2^{n_1} \sum_{i=1}^{n_1} x_1(i)2^{-i} + 2^{n_2} \sum_{i=1}^{n_2} x_2(i)2^{-i} + \sum_{i=1}^{\infty} [x_1(n_1+i) + x_2(n_2+i) + z(i)]2^{-i}. \tag{2.35}$$

Ignoring input signal bits below the noise level, we have

$$y \approx 2^{n_1} \sum_{i=1}^{n_1} x_1(i)2^{-i} + 2^{n_2} \sum_{i=1}^{n_2} x_2(i)2^{-i}. \tag{2.36}$$

This approximation equation motivates the definition of the corresponding linear deterministic model for the Gaussian MAC channel. Denote $q = \max(n_1, n_2)$. Consider a transmitted signals $\hat{x}_j^q$, which are binary vectors of length q, for $j = 1, 2$. Due to the strengths of signal links, the deterministic channel only passes the top n_j bits of the j-th transmitter to the destination. The output signal $\hat{y}^q$, which is also a binary vector of length q, and the input signal $\hat{x_j}^q$ are governed by the following linear deterministic model

$$\hat{y}^q = S^{q-n_1}\hat{x}_1^q \oplus S^{q-n_2}\hat{x}_2^q, \tag{2.37}$$

where S is a $q \times q$ shifting matrix defined similarly to (2.31). For further information, please see [3].

Chapter 3

On the Gaussian Interference Channel with Noisy Feedback

RECENT results have shown that feedback can significantly increase the capacity of interference networks. This chapter considers the impact of noise on the gain due to feedback. Specifically, this chapter studies the two-user Gaussian interference channel with noisy feedback. It is too hard to find the capacity region of the two-user Gaussian interference channel with noisy feedback. Instead, we aim to approximate the capacity region of this channel using the linear deterministic approach, which was introduced in the previous chapter, section 2.8.

3.1 Introduction

One of the most important issues for communication networks is that of interference management. Characterizing the capacity region of the two-user Gaussian interference channel (GIC) remains one of the fundamental unresolved problems in information theory. Recent breakthroughs in dealing with the capacity characterization of the GIC have made use of the linear deterministic interference channel (LD-IC) model [3, 11]. The main idea behind these works is that an appropriately defined LD model can serve as a good approximation to the Gaussian channel. By gaining valuable insights from studying the LD-IC, the proof techniques and ideas can be lifted over to the GIC. The capacity region of the GIC has been characterized to within 1-bit in [24].

There are many techniques to manage interference, such as treatment of interference as noise, interference alignment [12], and usage of feedback [102]. In this work, we focus on interference management via feedback. It is well known that, while feedback does not increase the capacity of the discrete memoryless point-to-point channel, it may enlarge the capacity region of multi-user channels. The fact that feedback enlarges the capacity region of the discrete memoryless multiple-access channel (MAC) was shown by Gaarder and Wolf [28]. Afterwards, Ozarow [82] found the capacity region of the two-user Gaussian MAC with noiseless feedback. Recently, Suh and Tse [102] obtained an interesting result that noiseless feedback can provide significant capacity gains for the GIC. To understand the usefulness of feedback for the interference channel, consider the very strong interference regime, in which the direct links are weaker than the cross (interference) links. In such a scenario, feedback can provide a substantial capacity gain by using the alternate path of $\mathrm{Tx}_1 \rightarrow \mathrm{Rx}_2 \rightarrow \mathrm{Tx}_2 \rightarrow \mathrm{Rx}_1$, i.e., the information intended from Tx_1 first reaches Rx_2, which is then received as feedback at Tx_2, which uses the strong cross (interference) link to reach the eventual destination at Rx_1. The approximate capacity region of the GIC with noiseless channel output feedback has been characterized [102] to within 2-bits. The results in [102] have been generalized to the case of the fully connected K-user IC [77], and the cyclic K-user IC [108].

Full and noiseless feedback is too much to ask for when the feedback link is not reliable. Vahid et al. considered an interesting generalization of [102] by studying the two-user GIC with rate-limited feedback [114]. Rate-limited feedback refers to a setting in which the receiver can utilize all the information it has received so far and feed back information over an orthogonal channel of finite capacity (bit-pipe). Several interesting results for the GIC with rate-limited feedback are obtained in [114].

While rate-limited feedback may be useful in scenarios in which the feedback links have good coding schemes to protect feedback signals from error, it places much complexity on the receiver's side. As a result, this model is not appropriate when the complexity of the feedback design is a concern. In order to take some of these issues into account, this chapter aims to investigate the model in which the feedback at transmitter j is a scaled and noisy (additive white Gaussian noise corrupted) version of the channel output at receiver j, for $j = 1, 2$. In particular, if the channel output at receiver j is Y_j, then the feedback to transmitter j is $Y_{F_j} = g_j Y_j + \tilde{Z}_j$, for $j = 1, 2$ (see Figure 3.1). With the eventual goal of understanding the capacity region of the GIC with noisy feedback, we present a linear deterministic model with noisy feedback. We show that the LD-IC with noisy feedback serves as a good approximation to the GIC with noisy feedback. First, we consider the linear deterministic interference channel with noisy feedback. Subsequently, we consider the Gaussian interference channel with noisy feedback, based on the insights that we gain from the linear deterministic model.

Other related work that studied multi-user channels with feedback includes [34, 49, 60, 62, 90, 107, 112, 113, 122]. [112] [122] found an achievable rate region for interference channel with generalized feedback, and their model can be reduced to many well-known multi-user channels, including ours. Note that all results for the general memoryless IC with generalized feedback can be immediately specialized to the IC with user cooperation by evaluating the bounds for independent noises, or to the IC with noisy feedback by evaluating the bounds for correlated noises. However, further optimization needs to be done to make the inner bound tight for our current problem. In [49], Jiang et al. established an achievable rate region for the interference channel with full noiseless feedback. [34] found outer bounds for interference channel with degraded noisy feedback. Additive white Gaussian noise (AWGN) MAC with imperfect feedback was studied in [62], which showed that the achievable rate region for MAC with even imperfect feedback is larger than that without feedback. Tandon and Ulukus in [107] derived outer bounds for the Gaussian MAC with noisy feedback and outer bounds for Gaussian interference channel with user cooperation. In [113], Tuninetti developed outer bounds on $R_1 + R_2$ for interference with generalized feedback. Existing literature on the IC with source cooperation [4, 13, 87, 117, 123] are different but also related to literature on the IC with noisy feedback. In both cases, each transmitter receives a noisy version of signals from the other transmitter. However, while the signal noises in the source cooperation model are independent of channel noises, the signal noises in the noisy feedback model may be correlated with the channel noises. Wang and Tse [117] characterized the capacity region, to within a constant number of bits, of the two-user Gaussian interference channel with conferencing transmitters.

3.1.1 Main contributions

The main contributions of this chapter are summarized as follows.

- In this chapter, we characterize the capacity region for the symmetric LD-IC with noisy feedback. We illustrate through numerous examples, that the sum-rate bounds derived in [63] alone are not sufficient to characterize the capacity region, and $2R_1 + R_2$ and $R_1 + 2R_2$ bounds are also necessary. Note that outer bounds are tightened with the help of specially defined auxiliary random variables. We show that noisy feedback increases the capacity region if and only if the amount of feedback level l is greater than a certain threshold l^*. It is found that, excluding the regime $\frac{1}{2} \leq \alpha \leq 2$ in which even full feedback does not increase symmetric capacity, l^* is equal to the per-user symmetric capacity without feedback.

- Based on results for the symmetric LD-IC with noisy feedback, we derive inner bounds and outer bounds for the symmetric Gaussian interference channel with noisy feedback. The outer bounds are shown to be at most 4.7 bits/s/Hz from the achievable rate region. As a corollary of this result, we also obtain a generalized-degree-of-freedom region for the symmetric

Gaussian IC with noisy feedback.

3.1.2 Chapter outline

The structure of this chapter is as follows.

- In section 4.2, we introduce the system models for the discrete memoryless interference channel with noisy feedback, the Gaussian interference channel with noisy feedback and the LD-IC with noisy feedback, then we formally state the problem.
- In section 3.3, we present the results and discussion for the symmetric linear deterministic interference channel with noisy feedback.
- In the subsequent section, we present the results and discussion for the symmetric Gaussian interference channel with noisy feedback.
- Finally, the chapter ends with a conclusion and the appendix, which contains proofs to results in the chapter.

3.2 System model

The two-user Gaussian interference channel with noisy feedback (see Figure 3.1), is defined by the following input-output relationships

$$Y_{1i} = h_{11}X_{1i} + h_{21}X_{2i} + Z_{1i}, \tag{3.1}$$

$$Y_{2i} = h_{12}X_{1i} + h_{22}X_{2i} + Z_{2i}, \tag{3.2}$$

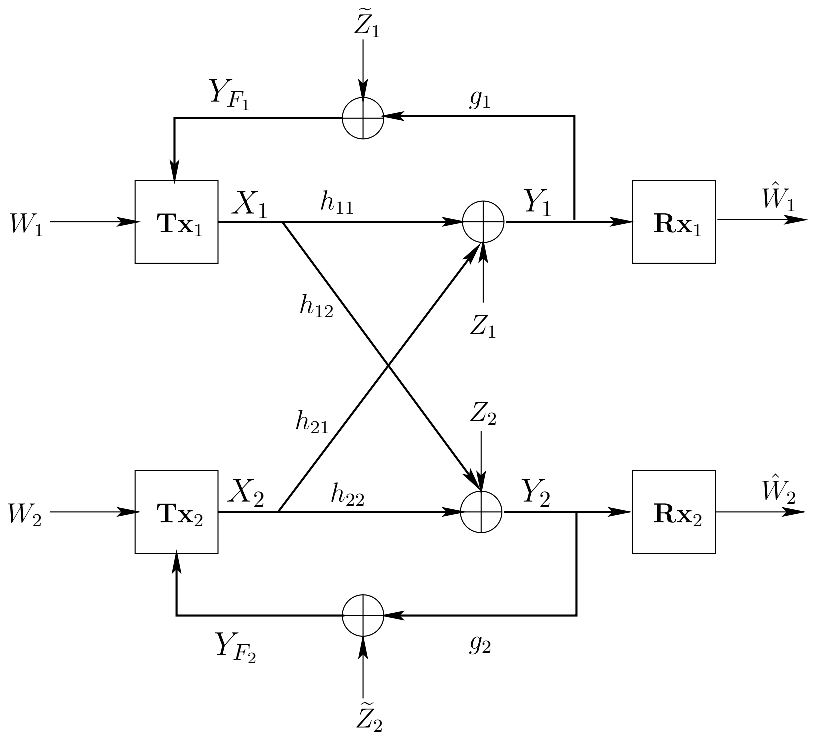

Figure 3.1. Gaussian IC with Noisy Feedback.

$$Y_{F_1 i} = g_1 Y_{1i} + \tilde{Z}_{1i}, \tag{3.3}$$

$$Y_{F_2 i} = g_2 Y_{2i} + \tilde{Z}_{2i}, \tag{3.4}$$

where X_{ji} denotes the signal sent by transmitter j, Y_{ji} denotes the output at receiver j, $Y_{F_j,i}$ denotes the feedback received at transmitter j, for $j = 1, 2$, at time i, for $i \in \{1, 2, ..., T\}$, and $\{Z_{ji}\}_{i=1}^{T}$ and $\{\tilde{Z}_{ji}\}_{i=1}^{T}$ are independent, additive white Gaussian noise processes with zero means and unit variances. The forward channel gains $\{h_{11}, h_{21}, h_{12}, h_{22}\}$ and the feedback channel gains $\{g_1, g_2\}$ are assumed to be constant and known at all terminals. Average unit power constraints are imposed at each transmitter. In other words, for a code of block length T, input sequences must satisfy $\frac{1}{T}\mathbb{E}(\sum_{i=1}^{T} |X_{ji}|^2) \leq 1$, for $j = 1, 2$.

Transmitter Tx_j, for $j = 1, 2$, wishes to communicate a message $m_j \in$

$\{1, 2, ..., M_j\} \triangleq \mathcal{W}_j$ to receiver Rx_j. It is assumed that W_1 and W_2 are independent. An (M_1, M_2, T, P_e) feedback code for the interference channel (IC) with noisy feedback consists of a sequence of encoding functions such that

$$X_{ji} = f_j^i(W_j, Y_{F_j 1}, Y_{F_j 2}, ..., Y_{F_j, i-1}) \tag{3.5}$$

where $X_{ij} \in \mathcal{X}_j$ for $j = 1, 2$, and $i = 1, 2, ..., T$, and two decoding functions such that

$$\hat{W}_j = d_{jT}(Y_j^T) \text{ for } j = 1, 2; \tag{3.6}$$

such that $\max\{P_{e,1T}, P_{e,2T}\} \leq P_e$, where $P_{e,1T}$ and $P_{e,2T}$ denote the average decoding error probabilities, which are computed as $P_{e,jT} = \Pr(\hat{W}_j \neq W_j)$. A rate pair (R_1, R_2) is achievable for the IC with noisy feedback if there exists an (M_1, M_2, T, P_e)-feedback code such that $P_e \to 0$ as $T \to \infty$ and $\frac{\log(M_1)}{T} \leq R_1$ and $\frac{\log(M_2)}{T} \leq R_2$. The capacity region of the IC with noisy feedback is defined as the closure of the set of all achievable rate pairs. With the goal of understanding the capacity region of the GIC with noisy feedback as defined above, we next describe the linear deterministic interference channel with noisy feedback.

Using the deterministic model in [3], a non-negative integer n_{kj} is used to represent the channel gain from transmitter Tx_k to receiver Rx_j and it is given by $n_{kj} = \lceil \log h_{kj}^2 \rceil^+$. Note that the effect of the Gaussian noise is captured by these representative numbers. Let q denote the maximum channel gains in the interference channel, i.e., $q = \max(n_{kj})$. Thus, the transmitted signal from transmitter k at the time i will have a maximum of q bits visible to any receiver. Denote $X_{ki} = [X_{ki}^1, ..., X_{ki}^q]^T \in F_2^q$, for $k = 1, 2$, where the leftmost bit is the

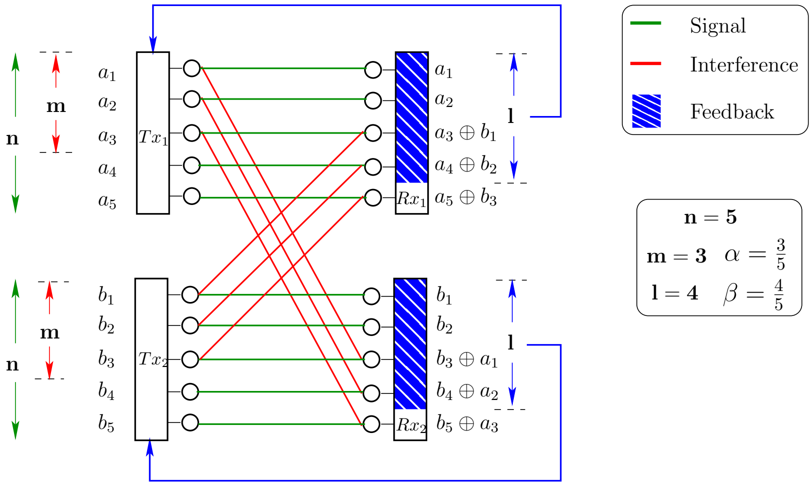

Figure 3.2. Symmetric Linear Deterministic IC with Noisy Feedback.

most significant bit and the rightmost bit is the least significant bit. In this linear model, the effect of interference between various signals is captured as the superposition of those signals. At the time i, the outputs at the receivers are given as

$$Y_{1i} = S^{q-n_{11}} X_{1i} \oplus S^{q-n_{21}} X_{2i}, \tag{3.7}$$

$$Y_{2i} = S^{q-n_{12}} X_{1i} \oplus S^{q-n_{22}} X_{2i}, \tag{3.8}$$

where S is the a square shift matrix of size q given by

$$S := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \tag{3.9}$$

and the operation is modulo 2 addition in F_2.

Next, we analyze the feedback links in the Gaussian interference channel. The feedback links are effectively equivalent to

$$Y_{F_1 i} = g_1 Y_{1i} + \tilde{Z}_{1i} \tag{3.10}$$

$$= g_{F_1}\frac{Y_{1i}}{\sqrt{h_{11}^2+h_{21}^2+2h_{11}h_{21}+1}}+\tilde{Z}_{1i}, \tag{3.11}$$

$$Y_{F_2 i} = g_2 Y_{2i}+\tilde{Z}_{2i} \tag{3.12}$$

$$= g_{F_2}\frac{Y_{2i}}{\sqrt{h_{12}^2+h_{22}^2+2h_{12}h_{22}+1}}+\tilde{Z}_{2i}, \tag{3.13}$$

where

$$g_{F_1} \triangleq g_1\sqrt{h_{11}^2+h_{21}^2+2h_{11}h_{21}+1}$$

$$g_{F_2} \triangleq g_2\sqrt{h_{12}^2+h_{22}^2+2h_{12}h_{22}+1}.$$

Using equations (3.10-3.13), we now model the corresponding feedback in the LD-IC model. The channel gains g_{F_j} for the feedback links can be represented by l_j, for $j = 1, 2$, where $l_j = \lceil \log g_{F_j}^2 \rceil^+$. Note that when $l_j = q$, this corresponds to the case of full feedback, which is the best kind of feedback that a system can get. Therefore, there is no need to consider the case $l_j > q$. It is thus sufficient to consider only the case $0 \leq l_j \leq q$. The feedback signals at the transmitters are given as

$$Y_{F_1 i} = S^{q-l_1}Y_{1i}, \quad Y_{F_2 i} = S^{q-l_2}Y_{2i}. \tag{3.14}$$

Effectively, via the feedback links, in time slot i, the transmitter j sees only the top l_j bits of the received signals Y_{ji}, for $j = 1, 2$ (see Figure 3.2) .

The chapter focuses on the symmetric LD-IC in which $m = n_{12} = n_{21}, n = n_{11} = n_{22}$, and $l = l_1 = l_2$, and the symmetric Gaussian IC with noisy feedback, where $h_{11} = h_{22}$, $h_{12} = h_{21}$ and $g_1 = g_2$. Define

$$\mathrm{SNR} \triangleq h_{11}^2 = h_{22}^2, \tag{3.15}$$

$$\mathrm{INR} \triangleq h_{21}^2 = h_{12}^2, \tag{3.16}$$

$$\begin{aligned}\mathrm{SNR}_F &\triangleq g_{F_1}^2 = g_1^2 \cdot (h_{11}^2 + h_{21}^2 + 2h_{11}h_{21} + 1) \\ &= g_{F_2}^2 = g_2^2 \cdot (h_{12}^2 + h_{22}^2 + 2h_{12}h_{22} + 1). \end{aligned} \tag{3.17}$$

Remark 3.1. Now we show how the Gaussian IC with noisy feedback is also related to, but different from, the Gaussian IC with source cooperation. Note

$$\begin{aligned} Y_{F_1 i} &= g_1 Y_{1i} + \tilde{Z}_{1i} \qquad (3.18)\\ &= g_1 h_{11} X_{1i} + g_1 h_{21} X_{2i} + g_1 Z_{1i} + \tilde{Z}_{1i}. \qquad (3.19) \end{aligned}$$

Transmitter 1 has access to its own codewords, so we will subtract the contribution from X_{1i}, and define a scaled version of the remaining part

$$Y'_{F_1 i} = \frac{1}{\sqrt{g_1^2+1}} g_1 h_{21} X_{2i} + \frac{1}{\sqrt{g_1^2+1}} (g_1 Z_{1i} + \tilde{Z}_{1i}). \tag{3.20}$$

Thus, the Gaussian IC with noisy feedback is also related to the Gaussian IC with source cooperation considered by Prabhakaran and Wiswanath [87] and others. However, there are differences. In the noisy feedback model, the noise $\frac{1}{\sqrt{g_1^2+1}}(g_1 Z_{1i} + \tilde{Z}_{1i})$ is correlated with Z_{1i}. In the source cooperation model, the cooperation noises are independent of the channel noises Z_{1i} and Z_{2i}. Operationally, in the noisy feedback model, receiver 1 at time slot i does not know the message X_{2i} clearly. Therefore, it sends a copy of the received message Y_{1i} via the feedback link. After receiving the noisy feedback $Y_{F_1 i}$ and subtracting its own message, transmitter 1 learns about transmitter 2's message X_{2i} through a noisy version $Y'_{F_1 i}$. On the other hand, in the source cooperation model, transmitter 2 knows exactly its own message and cooperates directly with transmitter 1. Without having to remove its own message, transmitter 1 learns directly about transmitter 2's message X_{2i} through a noisy version $Y'_{F_1 i}$.

■ 3.3 Symmetric deterministic IC with noisy feedback

As a stepping stone towards approximating the capacity region for the Gaussian IC with noisy feedback, we first consider the associated symmetric linear deterministic model.

■ 3.3.1 Capacity region

Given a triple (n, m, l), we denote the capacity region for symmetric LD-IC with noisy feedback by $\mathcal{C}^{\mathrm{N-FB}}(n, m, l)$, which is the set of all achievable rate pairs (R_1, R_2) with noisy feedback. We find it useful to define forward and feedback interference parameters respectively as follows

$$\alpha \triangleq \frac{m}{n}, \quad \beta \triangleq \frac{l}{n}. \tag{3.21}$$

The forward interference parameter α measures the normalized interference, whereas the feedback interference parameter β measures the normalized feedback. For the purpose of comparison with related work, we also define the normalized rates, with respect to n, as $R_j^* \triangleq \frac{R_j}{n}$, for $j = 1, 2$. Equivalent to $\mathcal{C}^{\mathrm{N-FB}}(n, m, l)$, the normalized capacity region $\mathcal{C}^{\mathrm{N-FB}}(\alpha, \beta)$ is the set of all achievable normalized rate pairs (R_1^*, R_2^*) with noisy feedback.

The capacity region for the symmetric LD-IC with noisy feedback is given by the following theorem.

Theorem 3.1. *The normalized capacity region $\mathcal{C}^{\mathrm{N-FB}}(\alpha, \beta)$ of the symmetric linear deterministic interference channel with noisy feedback, is the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy*

$$R_1^* \leq \max(1, \alpha), \tag{3.22}$$

$$R_2^* \leq \max(1, \alpha), \tag{3.23}$$

$$R_1^* \leq 1 + (\beta - 1)^+, \tag{3.24}$$

$$R_2^* \leq 1 + (\beta - 1)^+, \tag{3.25}$$

$$R_1^* + R_2^* \leq (1 - \alpha)^+ + \max(1, \alpha), \tag{3.26}$$

$$R_1^* + R_2^* \leq 2 \max[(1 - \alpha)^+, \alpha] + 2 \min[(1 - \alpha)^+, (\beta - \max(\alpha, (1 - \alpha)^+))^+], \tag{3.27}$$

$$\begin{aligned} 2R_1^* + R_2^* \leq\ & (1 - \alpha)^+ + \max(1, \alpha) + \max[\alpha, (1 - \alpha)^+] \\ & + \min[(1 - \alpha)^+, (\beta - \max(\alpha, (1 - \alpha)^+))^+], \end{aligned} \tag{3.28}$$

$$\begin{aligned} R_1^* + 2R_2^* \leq\ & (1 - \alpha)^+ + \max(1, \alpha) + \max[\alpha, (1 - \alpha)^+] \\ & + \min[(1 - \alpha)^+, (\beta - \max(\alpha, (1 - \alpha)^+))^+]. \end{aligned} \tag{3.29}$$

where $(\alpha)^+ \triangleq \max(0, \alpha)$.

Proof. One of the key ideas in the forward proof is the following lemma, which gives an achievable rate region for the general two-user discrete memoryless interference channel with noisy feedback. This lemma was derived in [122].

Lemma 3.1. *The capacity region of the two-user discrete memoryless interference channel with noisy feedback as defined above includes the set of* (R_1, R_2) *such that*

$$R_1 \leq \rho_1 + \kappa_2 + \rho_3 \tag{3.30}$$

$$R_2 \leq \kappa_1 + \rho_2 + \kappa_3 \tag{3.31}$$

$$R_1 \leq \kappa_6 \tag{3.32}$$

$$R_1 \leq \kappa_4 + \rho_1 \tag{3.33}$$

$$R_2 \leq \rho_6 \tag{3.34}$$

$$R_2 \leq \rho_4 + \kappa_1 \tag{3.35}$$

$$R_1 + R_2 \leq \kappa_2 + \rho_6 \tag{3.36}$$

$$R_1 + R_2 \leq \rho_2 + \kappa_6 \tag{3.37}$$

$$R_1 + R_2 \leq \kappa_1 + \rho_1 + \kappa_5 + \rho_2 \tag{3.38}$$

$$R_1 + R_2 \leq \kappa_1 + \rho_1 + \rho_5 + \kappa_2 \tag{3.39}$$

$$R_1 + R_2 \leq \kappa_1 + \rho_1 + \kappa_3 + \rho_3 \tag{3.40}$$

$$2R_1 + R_2 \leq \kappa_6 + \kappa_2 + \rho_3 + \rho_1 \tag{3.41}$$

$$2R_1 + R_2 \leq 2\rho_1 + \kappa_1 + \kappa_5 + \kappa_2 + \rho_3 \tag{3.42}$$

$$R_1 + 2R_2 \leq \rho_6 + \rho_2 + \kappa_3 + \kappa_1 \tag{3.43}$$

$$R_1 + 2R_2 \leq 2\kappa_1 + \rho_1 + \rho_5 + \rho_2 + \kappa_3, \tag{3.44}$$

over all joint distributions

$$p(u)p(u_1|u)p(u_2|u)p(v_1|u,u_1)p(v_2|u,u_2)p(x_1|u,u_1,v_1)$$

$$p(x_2|u,u_2,v_2)p(y_1y_2|x_1x_2)p(y_{F_1}|y_1)p(y_{F_2}|y_2), \tag{3.45}$$

where

$$\kappa_1 = I(U_2; Y_{F_1}|X_1, V_1, U_1, U) \tag{3.46}$$

$$\kappa_2 = I(X_1; Y_1|U, U_1, U_2, V_1, V_2) \tag{3.47}$$

$$\kappa_3 = I(X_1, V_2; Y_1|U, U_1, V_1, U_2) \tag{3.48}$$

$$\kappa_4 = I(X_1; Y_1|U, U_1, U_2, V_2) \tag{3.49}$$

$$\kappa_5 = I(X_1, V_2; Y_1|U, U_1, U_2) \tag{3.50}$$

$$\kappa_6 = I(U, U_2, V_2, X_1; Y_1) \tag{3.51}$$

$$\rho_1 = I(U_1; Y_{F_2}|UV_2U_2X_2) \tag{3.52}$$

$$\rho_2 = I(X_2; Y_2|U, U_1, U_2, V_1, V_2) \tag{3.53}$$

$$\rho_3 = I(X_2, V_1; Y_2|U, U_2, V_2, U_1) \tag{3.54}$$

$$\rho_4 = I(X_2; Y_2|U, U_2, U_1, V_1) \tag{3.55}$$

$$\rho_5 = I(X_2, V_1; Y_2|U, U_1, U_2) \tag{3.56}$$

$$\rho_6 = I(U, U_1, V_1, X_2; Y_2). \tag{3.57}$$

The details in applying Lemma 3.1 to do the forward proof are presented in subsection 3.6.2.

Remark 3.2. We sometimes use the notation convenience $p_{V|U}(v|u) = p(v|u)$ and

$p_V(v) = p(v)$, where the dropped subscripts are obvious by observation of the arguments used in the functions.

Remark 3.3. The lemma, just as related works in [122] [112] [102] [114], uses standard methods which combine three techniques: *block Markov encoding* [16], *backward decoding* [17], and Han-Kobayashi message splitting [41]. A message from each transmitter is split into three parts: private message, cooperative common message and non-cooperative common message.

A system with noisy feedback can perform no better than a system with full feedback. Thus, any outer bound that is applicable to the full feedback model, is also applicable to the noisy feedback model. Thus, for the proof of outer bounds for equations (3.22), (3.23) and (3.26), please refer to [102]. The outer bound for the equation (3.24) is a simple cut-set bound [18], that follows from the outer bound

$$R_1 \leq H(Y_1, Y_{F_2}|X_2), \tag{3.58}$$

which can be proved easily. Nevertheless, there is an alternative way to prove this outer bound. In the regime where $\alpha < 1$, this outer bound is inactive due to outer bound in (3.22); in the strong and very strong interference regimes, i.e. $\alpha \geq 1$, this outer bound follows from an interesting observation. The observation is that, when $\beta \leq 1$, feedback Y_{F_2} does not help as the feedback is a composition of X_2 and the top n bits of X_1, and when $\beta > 1$, feedback starts to help but there is some overlap as the top n of X_1 in this case is a mixture of Y_{F_2} and X_2. Thus, we will present, in the appendix, an alternative, slightly more complicated, proof, which might be of interest to some readers, based on this simple, but intriguing, observation. The outer bound on the equation (3.25) is proved similarly. In addition, we will present the rest of the converse proof for Theorem 3.1 in subsection 3.6.1. □

Next, we will compare the result for the noisy feedback model with related results for the no feedback model, the rate-limited feedback model and the full feedback model.

■ 3.3.2 Comparison with other feedback models

We recall here the capacity regions for the no feedback model, the rate-limited feedback model and the full feedback model. The normalized capacity region $\mathcal{C}^{\text{No-FB}}(\alpha)$ of the symmetric linear deterministic channel with no feedback model [11], in which $\beta = 0$, is given the set of non-negative rate pairs (R_1^*, R_2^*) that satisfy

$$\begin{aligned}
R_1^* &\le 1,\\
R_2^* &\le 1,\\
R_1^* + R_2^* &\le (1-\alpha)^+ + \max(1,\alpha),\\
R_1^* + R_2^* &\le 2\max[(1-\alpha)^+, \alpha],\\
2R_1^* + R_2^* &\le (1-\alpha)^+ + \max(1,\alpha) + \max[\alpha, (1-\alpha)^+],\\
R_1^* + 2R_2^* &\le (1-\alpha)^+ + \max(1,\alpha) + \max[\alpha, (1-\alpha)^+].
\end{aligned} \tag{3.59}$$

The normalized capacity region $\mathcal{C}^{\text{Full-FB}}(\alpha)$ of the full feedback model [102], in which $\beta = 1$, is given the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy

$$
\begin{aligned}
R_1^* &\leq \max(1, \alpha), \\
R_2^* &\leq \max(1, \alpha), \\
R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha).
\end{aligned} \tag{3.60}
$$

The normalized capacity region $\mathcal{C}^{\mathrm{RL-FB}}(\alpha, \beta')$ of the rate-limited feedback model found in [114], is equivalent to the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy

$$
\begin{aligned}
R_1^* &\leq \max(1, \alpha), \\
R_2^* &\leq \max(1, \alpha), \\
R_1^* &\leq 1 + \beta', \\
R_2^* &\leq 1 + \beta', \\
R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha), \\
R_1^* + R_2^* &\leq 2\max[(1-\alpha)^+, \alpha] + 2\min[(1-\alpha)^+, \beta'], \\
2R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+] + \min[(1-\alpha)^+, \beta'], \\
R_1^* + 2R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+] + \min[(1-\alpha)^+, \beta'].
\end{aligned} \tag{3.61}
$$

In contrast to that in the partial-feedback model, the receivers in a rate-limited feedback model, with feedback rate β', can feed back to the transmitters any function of the received outputs, even though β' in the rate-limited feedback model is also a normalized rate of feedback just like β in the partial-feedback model. Clearly, such encoding functions include sending back the top $n\beta'$ bits; and hence the capacity of our model is in general contained within the capacity region with the same amount of rate-limited feedback. Thus, when $\beta = \beta'$, the capacity regions for these four models always satisfy the following rule

$$\mathcal{C}^{\mathrm{No-FB}}(\alpha) \subseteq \mathcal{C}^{\mathrm{P-FB}}(\alpha, \beta) \subseteq \mathcal{C}^{\mathrm{RL-FB}}(\alpha, \beta') \subseteq \mathcal{C}^{\mathrm{Full-FB}}(\alpha). \tag{3.62}$$

The set inclusions here can be strict. We illustrate the results through examples.

Example 3.1. Consider a channel in which $n = 6, m = 2$ and $l = 5$. Figure 3.3 shows the capacity regions with no feedback, with full feedback, with rate-limited feedback of $l = 5$ bits, and with noisy feedback of $l = 5$ bits. Several interesting observations are worth making:

- The capacity region with full feedback coincides with that of rate-limited feedback of $l = 5$ bits.
- The sum capacity is 10 bits/channel-use for full, rate-limited and noisy feedback settings.
- Most importantly, the capacity region with noisy feedback is strictly contained in the capacity region with full feedback and rate-limited feedback. Previously, the capacity region for the model with full feedback did not require the bounds on $2R_1 + R_2$ and $R_1 + 2R_2$. On the other hand, it is here

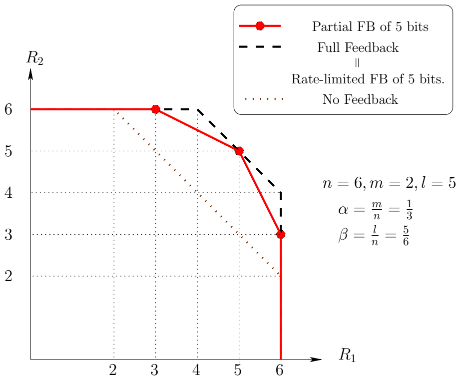

Figure 3.3. Capacity regions for $n = 6, m = 2$ and $l = 5$.

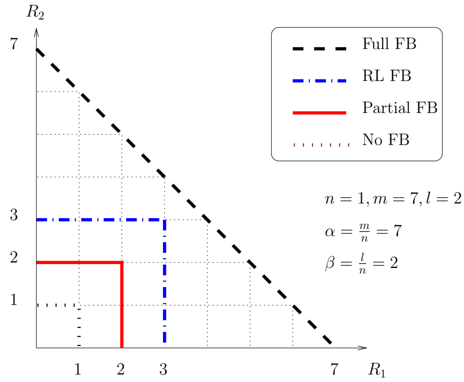

Figure 3.4. Capacity regions for $n = 1, m = 7$ and $l = 2$.

that we can clearly see the necessity of $2R_1 + R_2$ and $R_1 + 2R_2$ bounds in characterizing the exact capacity region when the feedback links are noisy.

Example 3.2. Consider another channel in which $n = 1, m = 7$ and $l = 2$. Figure 3.4 shows the capacity regions with no feedback, with full feedback, with rate-limited feedback of $l = 2$ bits, and with noisy feedback of $l = 2$ bits. Several interesting observations are worth making:

- All the set inclusions in (3.62) are strict. In other words, the capacity regions of the no feedback model, the noisy feedback model and the rate-limited feedback model are strictly included in that of the noisy feedback model, the rate-limited feedback model, and the full feedback model respectively.

- When $l = 2$, the capacity region of the noisy feedback model is strictly larger than that of the no feedback model. In fact, this holds as long as $l > 1$. Thus, we can partially observe the role the noisy feedback link plays in enlarging the capacity region. The capacity region of the noisy feedback model are characterized by not only the direct link strength n and the cross interference link strength m, but also the feedback link strength l.

As a direct result of Theorem 3.1, we have the following corollary.

Corollary 3.1. *The normalized sum rate $R_1^* + R_2^*$ of the noisy feedback model is the same as that of the no feedback model when $\beta \leq \beta_1^*$, where*

$$\beta_1^* = \begin{cases} \max(\alpha, (1-\alpha)^+) & \text{if } \alpha \leq 1, \\ 1 & \text{if } 1 < \alpha. \end{cases} \tag{3.63}$$

The normalized sum rate $R_1^* + R_2^*$ of the noisy feedback model is the same as that of the full feedback model when $\beta \geq \beta_2^*$, where

$$\beta_2^* = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } \alpha \leq 1, \\ \frac{\alpha}{2} & \text{if } 1 < \alpha. \end{cases} \tag{3.64}$$

The normalized sum rate $R_1^* + R_2^*$ as a function of β, for a fixed value of α, in different regimes, is illustrated in Figures 3.5, 3.6, 3.7, and 3.8. The normalized sum rate $R_1^* + R_2^*$ of the noisy feedback model is the same as that of the no feedback model when $\beta \leq \beta_1^*$, which is defined in Corollary 3.1. Notice that, excluding the case $\frac{2}{3} \leq \alpha \leq 2$, β_1^* is the per-user symmetric capacity for the no feedback model. The normalized sum rate $R_1^* + R_2^*$ of the noisy feedback model is strictly smaller than that of the rate-limited feedback model. The normalized sum rate $R_1^* + R_2^*$ for the noisy feedback model reaches saturation and achieves the same performance as that of the full feedback model when $\beta \geq \beta_2^*$. Notice that β_2^* is the per-user symmetric capacity of the full feedback model.

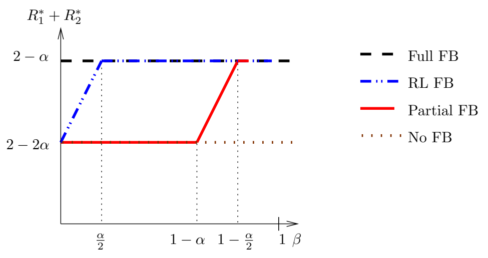

Figure 3.5. Normalized sum rate for $0 \leq \alpha < \frac{1}{2}$.

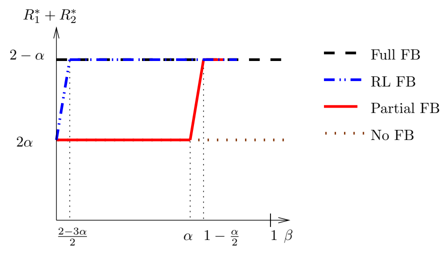

Figure 3.6. Normalized sum rate for $\frac{1}{2} \leq \alpha < \frac{2}{3}$.

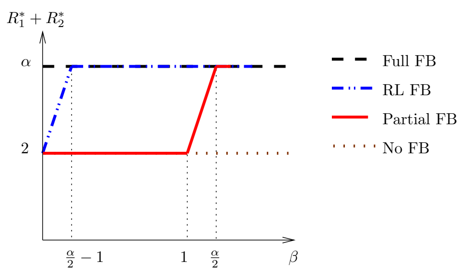

Figure 3.7. Normalized sum rate for $2 \leq \alpha$ and $\alpha < 4$.

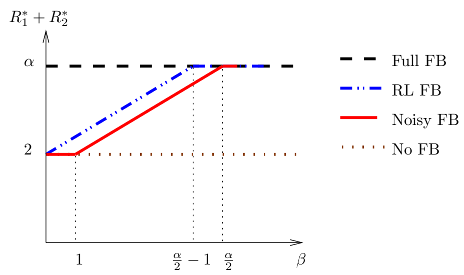

Figure 3.8. Normalized sum rate for $2 \leq \alpha$, and $4 \leq \alpha$.

Note that the normalized sum rate $R_1^* + R_2^*$ is not increased by any amount of feedback in the moderately strong interference regime, where $\frac{2}{3} \leq \alpha \leq 1$, and the strong interference regime, where $1 \leq \alpha \leq 2$.

As a direct result of Theorem 3.1, we have another corollary.

Corollary 3.2. *The capacity region of the noisy feedback model is increased by the noisy feedback if and only if $\beta \geq \beta_1^*$, where β_1^* is defined as in Corollary 3.1.*

In the following subsection, we present discussion on ideas of the achievability proof and the converse proof for Theorem 3.1.

3.3.3 Comparison with the linear deterministic IC models with source cooperation

In the literature, The Gaussian IC with source cooperation [87] indicates IC with generalized feedback where all noises are independent and jointly Gaussian. The difference between the Gaussian IC with source cooperation and the model in this chapter is explained in Remark 3.1. However, in the high-SNR linear deterministic model, there is no noise; hence the result for the case of source cooperation can be readily specialized to the noisy feedback case. For this reason, the result in [87] can be specialized to our scenario.

Denote β_{sc} as the channel gain between the sources in the paper [87]. First, we find the relationship between β in the noisy feedback model and β_{sc} in the source cooperation model. In the noisy feedback model, we have the restriction $\beta \leq \max(1, \alpha)$ as a receiver cannot feedback more than the amount of information it has received. In the source cooperation model, we have the restriction $\beta_{\mathsf{sc}} \leq \alpha$ as a transmitter does not need to see more bits of the interference signal than the intended receiver. Notice that what matters in these types of problems is how many interfering bits of X_2 are seen at receiver 1 and how many bits of X_1 are seen at receiver 2. Hence, we have

$$\beta = \beta_{\mathsf{sc}} + [1 - \alpha]^{+}.$$

Thus, from Theorem 1 in [87], we obtain from the following corollary.

Corollary 3.3. *The normalized sum-capacity region $\mathcal{C}^{\mathrm{N-FB}}(\alpha, \beta)$ of the symmetric linear deterministic interference channel with noisy feedback, is the set of* *non-negative normalized rate pairs* (R_1^*, R_2^*) *that satisfy*

$$R_1^* + R_2^* \leq 2\max(1-\alpha+(\beta-(1-\alpha)^+)^+, \alpha, (\beta-(1-\alpha)^+)^+), \tag{3.65}$$

$$R_1^* + R_2^* \leq \max(1,\alpha) + \max(1,\alpha,(\beta-(1-\alpha)^+)^+ - \alpha, \tag{3.66}$$

$$R_1^* + R_2^* \leq 2\max(1,(\beta-(1-\alpha)^+)^+), \tag{3.67}$$

$$R_1^* + R_2^* \leq 2\max(1,\alpha). \tag{3.68}$$

Note that the bounds in this corollary are exactly equivalent to our bounds on the sum rate $R_1^* + R_2^*$ in Theorem 3.1. In the case $\alpha \geq 1$, the bounds on $2R_1^* + R_2^*$ and $R_1^* + 2R_2^*$ are not active in characterizing the capacity region the symmetric LD IC with noisy feedback. Therefore, the sum capacity in [87, Theorem 1], together with the bounds on the individual rates (3.22-3.25), can lead the same result as that in Theorem 3.1.

■ 3.3.4 Achievability

In the classical interference channel without feedback, the HK encoding scheme currently gives the best achievable rate region [41] [15]. It was proved in [24] [11] that the HK encoding scheme can achieve the capacity region of the linear deterministic interference channel with no feedback. In the HK encoding scheme, messages are split into two parts: common information and private information. However, splitting messages into two parts is not sufficient to account for the effect of feedback links on the capacity region of the interference channel with noisy feedback. Previous works have made used of more-than-two message

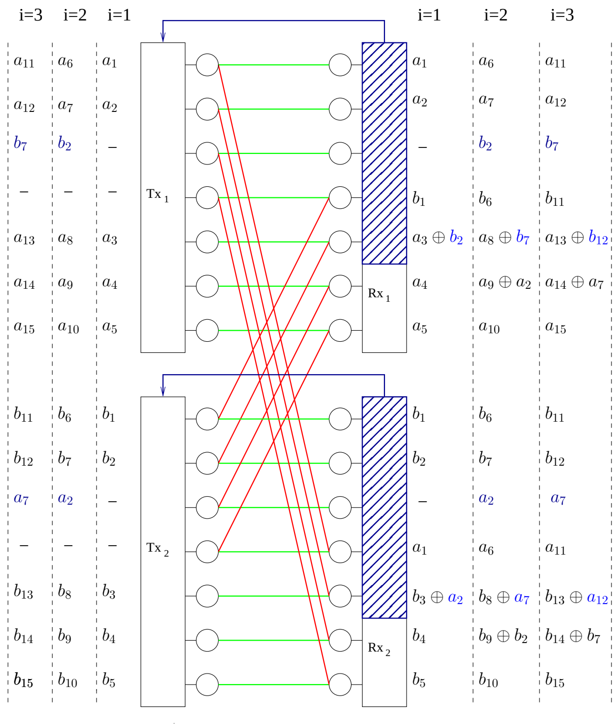

Figure 3.9. Encoding example for $(n = 7; m = 4; l = 5)$

splitting [112] [122] [114] [87]. The works [112] [122] developed an achievability scheme for a very generic model: IC with generalized feedback. It is true that IC with noisy feedback is a special case of IC with generalized feedback. Thus, any achievable scheme developed for IC with generalized feedback is also applicable for IC with noisy feedback. The remaining question is which choice of auxiliary random variables will obtain the optimal achievable rate region. Before answering this question, we will consider an example.

Example 3.3. Consider an example, in which $n = 7, m = 4$ and $l = 5$. In this example, we show an encoding scheme to achieve the point $(R_1, R_2) = (5, 5)$ in the achievable rate region. Without feedback, the maximum achievable sum rate is 8 bits per channel use. Here, we manage to obtain a sum rate of 10 bits per channel use through feedback. The encoding scheme is shown in Figure 3.9.

In the first time slot $t = 1$, each transmitter sends 5 fresh information bits as shown in the figure. With a feedback channel gain $l = 4$, Tx_1 sees only the top 5 bits, which are $a_1, a_2, -, b_1, a_3 \oplus b_2$, and hence it can recover b_2. In the second time slot $i = 2$, transmitter Tx_1 sends 5 new fresh information bits again and encodes b_2 at the third topmost signal level as shown in the figure. The third topmost signal level is chosen to ensure that the resolving signal bit b_2 is received cleanly at Tx_1. With the help of b_2, Rx_1 can resolve the interference in the previous time slot and decode a_3 successfully. Due to symmetry, the same encoding operation is carried out at Tx_2 and Rx_2. We can repeat this encoding scheme again for a duration of B time slots. It is easy to see that this scheme asymptotically achieves a sum rate $R_1 + R_2 = 10$ bits/channel use. Thus, the bound $R_1 + R_2 \leq 2m + 2(l - m)^+$ is active in this example and the encoding scheme has achieved the sum capacity in this regime.

A careful observation suggests, in each channel use, the message bits from a transmitter is categorized into three parts. For example, transmitter 1, in the second time slot when $t = 2$, has 3 private bits a_8, a_9, a_{10}, 2 cooperative common bit $(b_2, -)$, and the remaining 2 bits as non-cooperative common bits. This example suggests the size of the cooperate common message in general to be $(l - (n - m)^+)^+$ and the position of the cooperative common message to be within the top m bits of each transmitter.

A detailed choice of auxiliary random variables are shown in the proof in subsection 3.6.2.

Remark 3.4. Apart from the generic achievable scheme shown in subsection 3.6.2, we developed an alternative, more elementary achievable scheme, which is presented in subsection 3.6.3. That alternative scheme gives certain alternative points of view, which are not captured by the generic achievable scheme here.

3.3.5 Outer bounds

Consider the same example in Figure 3.9. Notice that for $l \leq 4$, the feedback link does not show any advantage over the situation without feedback. For example, in the first slot $t = 1$ when $l = 4$, even though transmitter 1 sees 4 bits $(a_1, a_2, -, b_1)$ via the feedback link, the knowledge of b_1 is redundant as no interference has appeared at receiver 1 yet. However, when $l = 5$, there is interference at $a_3 \oplus b_2$. Thus, we start to see the benefit of the feedback link. Notice that transmitter 1 always knows the top $n - m = 3$ bits of receiver 1. However, the benefit of feedback does not occur when l exceeds $n - m$. It only occurs when l exceeds m. This motivates us to define X_{top1} and X_{top2} in the converse. For more details, please refer to subsection 3.6.1.

3.4 Symmetric Gaussian interference channel with noisy feedback

With the results and techniques developed for the symmetric linear deterministic model, we are one step closer to approximating the capacity region for the symmetric Gaussian IC with noisy feedback. First, we derive the outer bounds, next we derive the inner bounds. Then, we show that the gap between the outer bounds and the inner bounds is a constant.

3.4.1 Outer bounds

Define

$$\alpha_G \triangleq \frac{\log \mathrm{INR}}{\log \mathrm{SNR}}. \tag{3.69}$$

The outer bounds for the symmetric Gaussian interference channel with noisy feedback is given by the following theorem.

Theorem 3.2. *The capacity region of the symmetric Gaussian interference channel with noisy feedback, is included by the set of non-negative pairs* (R_1, R_2), *for some* $0 \le \rho \le 1$, *satisfying*

$$R_1 \le \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \triangleq \psi_1 \tag{3.70}$$

$$R_2 \le \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \tag{3.71}$$

$$R_1 \le \frac{1}{2}\log\left(\mathrm{SNR} + 1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1} + 1\right) \triangleq \psi_2 \tag{3.72}$$

$$R_2 \le \frac{1}{2}\log\left(\mathrm{SNR} + 1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1} + 1\right) \tag{3.73}$$

$$\begin{aligned} R_1 + R_2 \le{}& \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1} + 1\right) \\ &+ \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \triangleq \psi_3 \end{aligned} \tag{3.74}$$

$$R_1 + R_2 \le \psi_4 \tag{3.75}$$

$$2R_1 + R_2 \le \psi_5 \tag{3.76}$$

$$R_1 + 2R_2 \le \psi_5, \tag{3.77}$$

where

$$\psi_4 \triangleq \begin{cases} \log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right)+\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 \\ \quad \textit{if } \frac{1}{2}\le\alpha_G<1, \\ \\ \log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\,.\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \\ \quad \textit{otherwise}, \end{cases} \tag{3.78}$$

$$\psi_5 \triangleq \begin{cases} \frac{1}{2}\log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3 \\ \quad +\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \\ \quad \textit{if } \frac{1}{2}\le\alpha_G<1, \\ \\ \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \\ \quad +\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \\ \quad \textit{otherwise}. \end{cases} \tag{3.79}$$

Proof. The bounds of (3.70),(3.71) and (3.74) were derived in [102]. Thus, it suffices to prove the bounds of (3.72),(3.75) and (3.76). The proof of (3.73) and (3.77) follow by symmetry. One of the key ideas in proving these outer bounds is to make use of the following auxiliary random variables. Inspired by the linear deterministic interference channel with partial feedback, we define

$$S_{2G} \triangleq \sqrt{\mathrm{INR}}X_2 + Z_1, \tag{3.80}$$

$$S_{1G} \triangleq \sqrt{\mathrm{INR}}X_1 + Z_2, \tag{3.81}$$

$$X_{top1G} \triangleq \begin{cases} \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2, & \frac{1}{2}\le\alpha_G\le 1 \\ 0, & \text{otherwise}, \end{cases} \tag{3.82}$$

$$X_{top2G} \triangleq \begin{cases} \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_2 + Z_1, & \frac{1}{2}\le\alpha_G\le 1 \\ 0, & \text{otherwise}. \end{cases} \tag{3.83}$$

Almost similarly to the proof of the outer bounds in Theorem 3.1, we can show the following lemma. However, there are subtle differences, which call for the right analysis so that we can deal with channel noises and feedback noises. The specially defined random variables X_{top1G} and X_{top2G} play a key role in obtaining the sum rate bound (3.85) and the weighted sum rate bound (3.86).

Lemma 3.2. *Consider the Gaussian IC with noisy feedback as defined in Section 4.2. The capacity region of the symmetric Gaussian IC with noisy feedback, is included by the set of non-negative pairs (R_1, R_2) satisfying*

$$R_1 \leq h(Y_1, Y_{F_2}|X_2) - h(Z_1) - h(\tilde{Z}_2), \tag{3.84}$$

$$\begin{aligned} R_1 + R_2 \leq\ & h(X_{top1G}|S_{2G}) + h(Y_{F_2}|X_2, X_{top1G}) + h(Y_2|S_{2G}, X_{top1G}) \\ & + h(X_{top2G}|S_{1G}) + h(Y_{F_1}|X_1, X_{top2G}) + h(Y_1|S_{1G}, X_{top2G}) \\ & - h(\tilde{Z}_2) - h(\tilde{Z}_1) - 2h(Z_2) - 2h(Z_1) \end{aligned} \tag{3.85}$$

$$\begin{aligned} 2R_1 + R_2 \leq\ & h(X_{top1G}|S_{2G}) + h(Y_{F_2}|X_2, X_{top1G}) + h(Y_2|S_{2G}, X_{top1G}) \\ & + h(Y_1|S_{1G}, X_2) + h(Y_1) - 2h(Z_1) - 2h(Z_2) - h(\tilde{Z}_2). \end{aligned} \tag{3.86}$$

The proof of Lemma 3.2 is presented in the appendix, subsection 3.6.5. Lemma 3.2 is tailor-made to deal with the effect of noisy feedback on the performance of the symmetric Gaussian interference channel.

Using Lemma 3.2, we can prove bounds (3.72),(3.75), (3.76) and (3.73). The details are presented in subsection 3.6.4. □

Remark 3.5. When $\mathrm{SNR}_F \to \infty$, the outer bounds of (3.72), (3.73), (3.76) and (3.77) are redundant.

Remark 3.6. At high SNR, the outer bounds here are equivalent to that in the full-feedback model [102]. At low SNR, the outer bounds here are slightly looser than that in the full-feedback model as we do not include the following cut-set outer bound [102]

$$R_1 \le h(Y_2|X_2) - h(Z_2) + h(Y_1|X_2, S_1) - H(Z_1) \tag{3.87}$$

$$\le \frac{1}{2}\log\left(1 + (1-\rho^2)\mathrm{INR}\right) + \frac{1}{2}\log\left(1 + \frac{(1-\rho^2)\mathrm{SNR}}{1+(1-\rho^2)\mathrm{INR}}\right). \tag{3.88}$$

However, it is beyond the scope of this chapter to tighten the constant-gap result, which is presented in Theorem 3.4, so we do not consider it here.

Remark 3.7. The symmetric Gaussian IC with noisy feedback and the symmetric Gaussian IC with rate-limited feedback [114] share the same bounds of (3.70), (3.71) and (3.74).

Remark 3.8. Theorem II.2 in the paper [113] gives a generic outer bound on the sum rate $R_1 + R_2$ for IC with generalized feedback. That outer bound is also applicable to our setting and potentially helpful in obtaining a better constant-gap result.

Remark 3.9. The outer bounds on the sum rate in Theorem 2 and Appendix IV in [24] possibly have competitive performance with our outer bounds in terms of quantifying the constant gap for the sum rate.

■ 3.4.2 Inner bounds

The inner bounds for the symmetric Gaussian interference channel with noisy feedback is given by the following theorem.

Theorem 3.3. *Given any real-valued number ρ such that $0 \le \rho \le 1$. The capacity region of the two-user symmetric Gaussian interference channel with* *noisy feedback includes the set of all non-negative pairs of* (R_1, R_2) *satisfying*

$$R_1 \leq \min(\tau_6, \tau_4 + \tau_1, \tau_1 + \tau_2 + \tau_3) \tag{3.89}$$

$$R_2 \leq \min(\tau_6, \tau_4 + \tau_1, \tau_1 + \tau_2 + \tau_3) \tag{3.90}$$

$$R_1 + R_2 \leq \min(\tau_2 + \tau_6, 2\tau_1 + \tau_5 + \tau_2, 2\tau_1 + 2\tau_3) \tag{3.91}$$

$$2R_1 + R_2 \leq \min(\tau_6 + \tau_2 + \tau_3 + \tau_1, 3\tau_1 + \tau_5 + \tau_2 + \tau_3) \tag{3.92}$$

$$R_1 + 2R_2 \leq \min(\tau_6 + \tau_2 + \tau_3 + \tau_1, 3\tau_1 + \tau_5 + \tau_2 + \tau_3) \tag{3.93}$$

where

$$\tau_6 \triangleq \frac{1}{2} \log \frac{\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR} \cdot \mathrm{INR}} + 1}{\mathrm{INR} \cdot P_p + 1} \tag{3.94}$$

$$\tau_5 \triangleq \frac{1}{2} \log \frac{\mathrm{SNR}(P_{nc} + P_p) + \mathrm{INR}(P_{nc} + P_p) + 1}{\mathrm{INR} \cdot P_p + 1} \tag{3.95}$$

$$\tau_4 \triangleq \frac{1}{2} \log \frac{\mathrm{SNR}(P_{nc} + P_p) + \mathrm{INR} \cdot P_p + 1}{\mathrm{INR} \cdot P_p + 1} \tag{3.96}$$

$$\tau_3 \triangleq \frac{1}{2} \log \frac{\mathrm{SNR} \cdot P_p + \mathrm{INR}(P_{nc} + P_p) + 1}{\mathrm{INR} \cdot P_p + 1} \tag{3.97}$$

$$\tau_2 \triangleq \frac{1}{2} \log \frac{\mathrm{SNR} \cdot P_p + \mathrm{INR} \cdot P_p + 1}{\mathrm{INR} \cdot P_p + 1} \tag{3.98}$$

$$\tau_1 \triangleq \frac{1}{2} \log \frac{\tau_{1n}}{\tau_{1d}} \tag{3.99}$$

$$\tau_{1n} \triangleq \frac{\mathrm{SNR}_F}{\mathrm{SNR} + \mathrm{INR} + 2\sqrt{\mathrm{SNR} \cdot \mathrm{INR}} + 1} \times [\mathrm{INR}(P_{cc} + P_{nc} + P_p) + 1] + 1$$

$$\tau_{1d} \triangleq \frac{\mathrm{SNR}_F}{\mathrm{SNR} + \mathrm{INR} + 2\sqrt{\mathrm{SNR} \cdot \mathrm{INR}} + 1} \times [\mathrm{INR}(P_{nc} + P_p) + 1] + 1,$$

for all power allocation schemes that satisfy

$$P_p + P_{cc} + P_{nc} = 1 - \rho, \tag{3.100}$$

and P_p*,* P_{cc} *and* P_{nc} *are non-negative.*

Proof. Theorem 3.3 is a direct corollary of Lemma 3.1. Choose (U, U_i, V_i, X_{ip}), for $i \in \{1,2\}$ as jointly Gaussian, independent random variables which satisfy

$$U \sim \mathcal{N}(0,\rho), \tag{3.101}$$

$$U_i \sim \mathcal{N}(0,P_{cc}), \tag{3.102}$$

$$V_i \sim \mathcal{N}(0,P_{nc}), \tag{3.103}$$

$$X_{ip} \sim \mathcal{N}(0,P_p), \tag{3.104}$$

$$P_{cc} + P_{nc} + P_p = 1 - \rho. \tag{3.105}$$

Set $X_i = U + U_i + V_i + X_{ip}$. With this choice of random variables, Theorem 3.3 is a direct corollary of Lemma 3.1. □

Theorem 3 plays a key role in obtaining the constant-gap result in Theorem 3.4. The major difficulty in using Theorem 3 is to choose the right power allocation scheme, so that we can get tight inner bounds.

■ 3.4.3 A constant gap between inner and outer bounds

Define

$$\delta_R \triangleq \delta_{R_1} \triangleq \delta_{R_2} \triangleq \min(\psi_1,\psi_2) - \min(\tau_6, \tau_4+\tau_1, \tau_1+\tau_2+\tau_3, \tau) \tag{3.106}$$

$$\begin{aligned}\delta_{2R} \triangleq \delta_{R_1+R_2} &\triangleq \min(\psi_3,\psi_4,2\psi_1,2\psi_2,\psi_1+\psi_2)\\ &\quad - \min(\tau_2+\tau_6, 2\tau_1+\tau_2+\tau_5, 2\tau_1+2\tau_3, 2\tau)\end{aligned} \tag{3.107}$$

$$\delta_{3R} \triangleq \delta_{2R_1+R_2} \triangleq \delta_{R_1+2R_2}$$

$$\triangleq \min(\psi_5, \psi_1 + \psi_3, \psi_1 + \psi_4, \psi_2 + \psi_3, \psi_2 + \psi_4, 3\psi_1, 3\psi_2)$$

$$- \min(\tau_1 + \tau_2 + \tau_3 + \tau_6, 3\tau_1 + \tau_2 + \tau_3 + \tau_5, 3\tau). \tag{3.108}$$

$$\delta \triangleq \max(\delta_R, \frac{1}{2}\delta_{2R}, \frac{1}{3}\delta_{3R}) \tag{3.109}$$

where τ is any achievable rate for any transmitter, using some achievability scheme. In words, δ_R, δ_{2R} and δ_{3R} are the possible gaps between the minimum of the set of the derived outer bounds and and the minimum of the set of the derived inner bounds for the individual rate, the sum rate $R_1 + R_2$, and the weighted sum rates $2R_1 + R_2$ and $R_1 + 2R_2$ respectively.

This is the main result in this chapter.

Theorem 3.4. *Outer bounds in Theorem 3.2 are no more than* 4.7 *bits/s/Hz away from the achievable rate region. More precisely, we have*

$$\delta \leq 4.7 \tag{3.110}$$

Proof. Refer to subsection 3.6.6. □

Remark 3.10. HK [41] scheme used two-message splitting, which was proved to be at most 1 bit/s/Hz away from the outer bounds [24]. The achievability scheme here makes use of three-message splitting. Prabhakaran and Viswanath [87] proposed three different encoding schemes, which are based on three-message splitting (which is the same as ours here), four-message splitting and mixture of these two schemes. Other advanced achievibility schemes developed in [122] for the IC with generalized feedback is also applicable to our model. We are not sure if our current inner bounds are sufficiently strict so it might be advantageous to use any alternative achievability schemes found in these related works to reduce the gap further. The large gap may be also due to the outer bounds. Further works need to be done to tighten the outer bounds. In addition, particular attention should be paid to the outer bounds, such as the bounds of (3.72), (3.73), (3.75 - 3.77), which are functions of the feedback link strength SNR_F. Furthermore, the gap is estimated based on a crude estimation method. A more refined technique should be employed to reduce the gap further. The work in [87] considered bounds for the symmetric Gaussian IC with source cooperation and obtained a gap of 10 bits/s/Hz for the sum rate $R_1 + R_2$, for real random variables. From the proof of this theorem, our gap for the sum rate only is $\delta_{2R} = 9.3$ bits/s/Hz. At high SNR, bounds in [87] and the bounds here will give the same result. However, at low SNR, ignoring the differences in estimation of the gap, our outer bounds on the sum rate seem to be slightly better for symmetric Gaussian IC with symmetric noisy feedback.

Define

$$\beta_G \triangleq \frac{\log \mathrm{SNR_F}}{\log \mathrm{SNR}}. \tag{3.111}$$

Next, define the generalized degrees of freedom as

$$d_1(\alpha_G, \beta_G) \triangleq \lim_{\mathrm{SNR}\to\infty} \frac{R_1(\mathrm{SNR}, \mathrm{INR}, \mathrm{SNR_F})}{\frac{1}{2}\log(1+\mathrm{SNR})}, \tag{3.112}$$

$$d_2(\alpha_G, \beta_G) \triangleq \lim_{\mathrm{SNR}\to\infty} \frac{R_2(\mathrm{SNR}, \mathrm{INR}, \mathrm{SNR_F})}{\frac{1}{2}\log(1+\mathrm{SNR})}. \tag{3.113}$$

As a result of Theorem 3.4, we obtain the following corollary, which gives the generalized-degree-of-freedom region of the symmetric Gaussian interference channel with noisy feedback.

Corollary 3.4. *For the symmetric Gaussian interference channel with noisy feedback, the generalized-degrees-of-freedom region is the set of non-negative pairs* (d_1, d_2) *that satisfy*

$$d_1 \leq \max(1, \alpha_G), \tag{3.114}$$

$$d_2 \leq \max(1, \alpha_G), \tag{3.115}$$

$$d_1 \leq \max(1, \beta_G), \tag{3.116}$$

$$d_2 \leq \max(1, \beta_G), \tag{3.117}$$

$$d_1 + d_2 \leq \max(1, \alpha_G) + (1 - \alpha_G)^+, \tag{3.118}$$

$$d_1 + d_2 \leq 2\max(1 - \alpha_G, \alpha_G) + 2(\beta_G - \max(1 - \alpha_G, \alpha_G))^+, \tag{3.119}$$

$$\begin{aligned} 2d_1 + d_2 \leq\ & (\beta_G - \max(1 - \alpha_G, \alpha_G))^+ \\ & + \max(1 - \alpha_G, \alpha_G) + (1 - \alpha_G)^+ + \max(1, \alpha_G), \end{aligned} \tag{3.120}$$

$$\begin{aligned} d_1 + 2d_2 \leq\ & (\beta_G - \max(1 - \alpha_G, \alpha_G))^+ \\ & + \max(1 - \alpha_G, \alpha_G) + (1 - \alpha_G)^+ + \max(1, \alpha_G). \end{aligned} \tag{3.121}$$

Remark 3.11. The generalized-degrees-of-freedom region for the symmetric Gaussian IC is the same as the capacity region for the symmetric LD-IC. Therefore, any set of remarks and observations that are applicable to the symmetric LD-IC, also applies directly to the generalized-degree-of-freedom region of the Gaussian IC.

3.4.4 Discussion on the asymmetric Gaussian interference channel with noisy feedback

The symmetric Gaussian interference channel with symmetric noisy feedback is only a special case of the asymmetric Gaussian interference channel with asymmetric noisy feedback. To approximate the asymmetric Gaussian interference channel with noisy feedback directly is a challenging task. Thus, it is beneficial to first find the capacity region for the asymmetric LD-IC with asymmetric noisy feedback. A keen reader would have noticed that the inner bounds and outer bounds developed in the proof of Theorem 1 (sections 3.6.1 and 3.6.2) are also applicable to the asymmetric LD-IC with asymmetric noisy feedback. However, in the outer bounds, we relied on carefully-defined auxiliary random variables X_{top1} and X_{top2} to optimally tighten the outer bounds. Similarly, in the inner bounds, we relied on the carefully-chosen random variables U_1 and U_2, in terms of the size and the location of the bits assigned to these two random variables with respect to X_1 and X_2 respectively, so that we can optimally maximize the inner bounds to the extent that the inner bounds match the outer bounds exactly. We are not sure if the current outer bounds and inner bounds are sufficient to determine the capacity region for the asymmetric LD-IC with noisy feedback. To choose optimal sets of random variables X_{topj}, U_j, for $j \in \{1, 2\}$, which enable us to determine the capacity region for asymmetric LD-IC with asymmetric noisy feedback, for different values of n_{ij} and l_j, for $i, j \in \{1, 2\}$, is a non-trivial problem. Therefore, to approximate the capacity region for the asymmetric Gaussian IC with asymmetric noisy feedback remains an open problem for now.

3.5 Conclusions

In this chapter, we have obtained the capacity region for the symmetric linear deterministic interference channel with noisy feedback. We have shown that noisy feedback increases the capacity region if and only if the amount of feedback level l is greater than a certain threshold l^*, and it is found that l^* is equal to the per-user symmetric capacity without feedback. One of the key ideas is a novel converse proof which includes outer bound on weighted sum rates $2R_1 + R_2$ and $R_1 + 2R_2$. Our novel outer bounds are tightened by specially defined auxiliary random variables. We have also illustrated through numerous examples, that the outer bounds on the sum rate R_1+R_2 derived in [63] alone are not sufficient to characterize the capacity region, and $2R_1 + R_2$ and $R_1 + 2R_2$ bounds are also necessary. The result and the techniques developed for this linear deterministic model are then applied to characterize inner bounds and outer bounds for the symmetric Gaussian IC with noisy feedback. The outer bounds are shown to be at most 4.7 bits/s/Hz away from the achievable rate region. As a corollary, the generalized-degree-of-freedom region, which approximates the capacity region of the symmetric Gaussian IC at high SNR, is found.

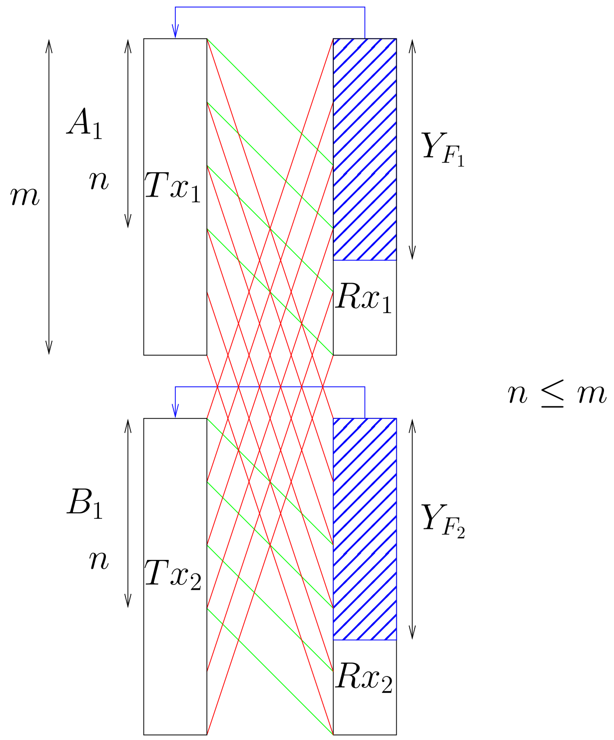

Figure 3.10. Illustration of A_1 and B_1 when $n \leq m$.

3.6 Appendix

3.6.1 Converse proof of Theorem 3.1

3.6.1.1 Bounds on R_1 and R_2

Now, the outer bounds of (3.24) and (3.25) on R_1 and R_2 respectively, are proved. When $0 \leq m < n$, we always have $R_j \leq n$, for $j = 1, 2$, as proved above. Thus, we only need to consider the case $n \leq m$. Consider Figure 3.10. Let A_1 denote the top n bits of transmitter 1, and let B_1 denote the top n bits of transmitter 2.

We have

$$
\begin{aligned}
TR_1 &= H(W_1) \\
&\overset{(a)}{=} H(W_1|W_2) \\
&= I(W_1; A_1^T Y_{F_2}^T|W_2) + H(W_1|W_2 A_1^T Y_{F_2}^T) \\
&\overset{(b)}{=} I(W_1; A_1^T Y_{F_2}^T|W_2) + H(W_1|W_2 A_1^T Y_{F_2}^T X_2^T Y_1^T) \\
&\overset{(c)}{\leq} H(A_1^T Y_{F_2}^T|W_2) + H(W_1|Y_1^T) \\
&\overset{(d)}{\leq} \sum_{i=1}^{T} H(A_{1i} Y_{F_2 i}|A_1^{i-1} Y_{F_2}^{i-1} W_2) + 1 + TP_e^T \\
&\overset{(e)}{\leq} \sum_{i=1}^{T} H(A_{1i} Y_{F_2 i}|X_{2i}) + 1 + TP_e^T, \qquad (3.122)
\end{aligned}
$$

where

(a) follows from the independence between W_1 and W_2;

(b) follows from the fact X_2^T is a function of $(W_2 Y_{F_2}^T)$, and Y_1^T is a function of $(A_1^T X_2^T)$;

(c) follows from the facts that $H(A_1^T Y_{F_2}^T|W_2 W_1) = 0$ and that conditioning reduces the entropy;

(d) follows from Fano's inequality; and

(e) follows from the fact that X_{2i} is a function of $(W_2 Y_{F_2}^{i-1})$.

We next bound the term $\sum_{i=1}^{T} H(A_{1i} Y_{F_2 i}|X_{2i})$ in (3.122).

Case 1: $0 \leq l \leq n$. For this case, $Y_{F_2 i}$ is a function of $(A_{1i} X_{2i})$. Thus, we have

$$\begin{aligned}
\sum_{i=1}^{T} H(A_{1i} Y_{F_2 i} | X_{2i}) &= \sum_{i=1}^{T} H(Y_{F_2 i} | X_{2i} A_{1i}) + \sum_{i=1}^{T} H(A_{1i} | X_{2i}) \\
&\leq 0 + nT. \qquad (3.123)
\end{aligned}$$

Case 2: $n \leq l \leq m$. In this case, A_{1i} is a function of $(Y_{F_2,i} X_{2i})$. We have

$$\begin{aligned}
\sum_{i=1}^{T} H(A_{1i} Y_{F_2 i} | X_{2i}) &= \sum_{i=1}^{T} H(Y_{F_2 i} | X_{2i}) + \sum_{i=1}^{T} H(A_{1i} | X_{2i} Y_{F_2 i}) \\
&\leq lT + 0. \qquad (3.124)
\end{aligned}$$

From both these cases, we conclude that $R_1 \leq n + (l - n)^+$. The inequality for R_2 can be proved in a similar manner.

■ 3.6.1.2 Bound on $R_1 + R_2$

Let S_{D_1} represent the top m bits of the first transmitter. When $m < n$, it will be the top m bits out of n bits. When $n < m$, it will represent all the bits from the first transmitter. Intuitively, S_{D_1} represents the m information bits that are visible at both receivers. Similarly, let S_{D_2} represent the top m bits for the second transmitter.

Furthermore, define X_{topj} as the top $\min(m, (2m-n)^+)$ bits of transmitter j. In other words, X_{topj} is the top $(2m-n)^+$ bits of transmitter j when $\frac{n}{2} \leq m \leq n$, the top m bits when $m \geq n$. No equivalent variable is defined in the case when

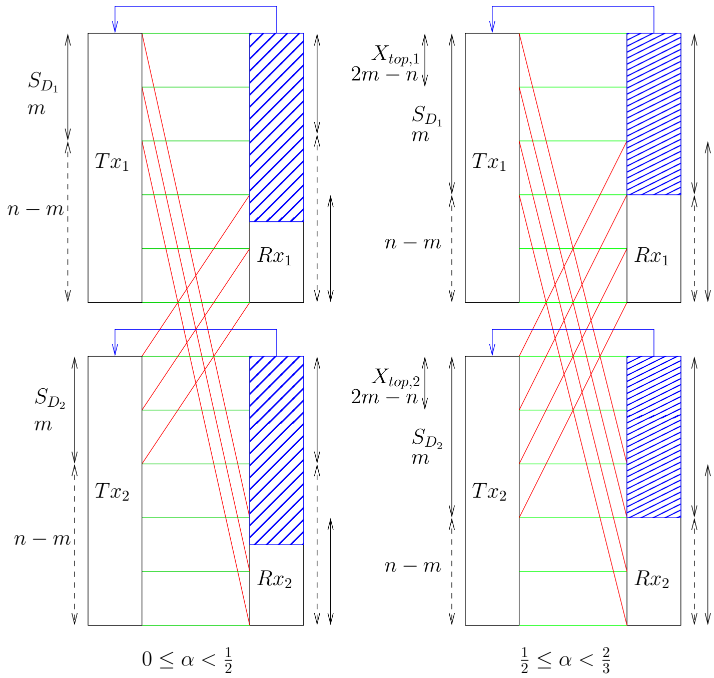

Figure 3.11. Illustration of S_{D_j} and $X_{top,j}$.

$m \le \frac{n}{2}$. These two random variables mainly serve to explain the bounds in the weak interference regime and the moderately strong interference regime.

It is worthwhile to give examples on these four random variables for ease of reading. Consider the case of the very weak interference regime where $0 < m \le \frac{n}{2}$. X_{topj} and S_{D_j}, for $j = 1, 2$, are illustrated in Figure 3.11. In this regime, S_{D_1} represents the top m bits of transmitter 1, and X_{top1} is a null region in this regime.

Consider a second example. Consider the case of the weak interference regime where $\frac{n}{2} \le m \le \frac{2n}{3}$. Again, X_{topj} and S_{D_j}, for $j = 1, 2$, are also illustrated in Figure 3.11. In this regime, S_{D_1} also represents the top m bits of transmitter 1. X_{top1} is the top $2m - n$ bits of transmitter 1.

In the proofs below, we make use of the following lemma.

Lemma 3.3.

$$I(S_{D_2}^T X_{top1}^T Y_{F_2}^T W_2; Y_{F_1}^T W_1) \leq H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2i}|X_{2i}X_{top1,i}) + H(X_{top1,i}|S_{D_2i})] \tag{3.125}$$

$$I(S_{D_1}^T X_{top2}^T Y_{F_1}^T W_1; Y_{F_2}^T W_2) \leq H(Y_{F_2}^T|W_2) + \sum_{i=1}^{T}[H(Y_{F_1i}|X_{1i}X_{top2,i}) + H(X_{top2,i}|S_{D_1i})]. \tag{3.126}$$

Proof.

$$\begin{aligned}
&I(S_{D_2}^T X_{top1}^T Y_{F_2}^T W_2; Y_{F_1}^T W_1) \\
&= I(W_2; Y_{F_1}^T W_1) + I(S_{D_2}^T X_{top1}^T Y_{F_2}^T; Y_{F_1}^T W_1|W_2) \\
&\overset{(a)}{=} H(Y_{F_1}^T|W_1) + H(Y_{F_2}^T S_{D_2}^T X_{top1}^T|W_2) \\
&\overset{(b)}{\leq} H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2i}|Y_{F_2}^{i-1} S_{D_2}^i X_{top1}^i W_2 X_{2i}) \\
&\quad + H(X_{top1,i}|Y_{F_2}^{i-1} S_{D_2}^i W_2 X_{2i}) + H(S_{D_2i}|Y_{F_2}^{i-1} W_2 X_{2i})] \\
&\overset{(c)}{=} H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2i}|X_{2i}X_{top1,i}) + H(X_{top1,i}|S_{D_2i})]
\end{aligned}$$

where

(a) follows from the fact that, given $(W_1 W_2)$, the entropy of any random variable is 0; and that W_1 is independent of W_2;

(b) comes from the fact that X_{2i} is a function of $(Y_{F_2}^{i-1} W_2)$; and

(c) follows from the fact S_{D_2i} is a function of X_{2i}.

The second part of the lemma is proved similarly to the above. □

We have

$$
\begin{aligned}
&T(R_1 + R_2 - p_e^T) \\
&\leq I(W_1; Y_1^T) + I(W_2; Y_2^T) \\
&\leq I(W_1; Y_1^T Y_{F_1}^T) + I(W_2; Y_2^T Y_{F_2}^T) \\
&= H(Y_1^T) + H(Y_{F_1}^T | Y_1^T) - H(Y_{F_1}^T | W_1) - H(Y_1^T | Y_{F_1}^T W_1) \\
&\quad + H(Y_2^T) + H(Y_{F_2}^T | Y_2^T) - H(Y_{F_2}^T | W_2) - H(Y_2^T | Y_{F_2}^T W_2) \\
&\stackrel{(a)}{=} H(Y_1^T) - H(S_{D_2}^T | Y_{F_1}^T W_1) + H(Y_2^T) - H(S_{D_1}^T | Y_{F_2}^T W_2) \\
&\quad - H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2) \\
&\stackrel{(b)}{=} H(Y_1^T) - H(S_{D_2}^T X_{top1}^T | Y_{F_1}^T W_1) + H(Y_2^T) - H(S_{D_1}^T X_{top2}^T | Y_{F_2}^T W_2) \\
&\quad - H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2) \\
&\leq H(Y_1^T) + [I(S_{D_2}^T X_{top1}^T ; Y_{F_1}^T W_1) - H(S_{D_2}^T X_{top1}^T)] \\
&\quad + [H(S_{D_2}^T X_{top1}^T | Y_2^T) - H(S_{D_2}^T X_{top1}^T | Y_2^T X_2^T)] \\
&\quad + H(Y_2^T) + [I(S_{D_1}^T X_{top2}^T ; Y_{F_2}^T W_2) - H(S_{D_1}^T X_{top2}^T)] \\
&\quad + [H(S_{D_1}^T X_{top2}^T | Y_1^T) - H(S_{D_1}^T X_{top2}^T | Y_1^T X_1^T)] \\
&\quad - H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2)
\end{aligned}
$$

$$= I(S_{D_2}^T X_{top1}^T; Y_{F_1}^T W_1) + H(Y_2^T | S_{D_2}^T X_{top1}^T)$$

$$- H(S_{D_2}^T X_{top1}^T | Y_2^T X_2^T)$$

$$+ I(S_{D_1}^T X_{top2}^T; Y_{F_2}^T W_2) + H(Y_1^T | S_{D_1}^T X_{top2}^T)$$

$$- H(S_{D_1}^T X_{top2}^T | Y_1^T X_1^T) - H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2)$$

$$\stackrel{(c)}{=} I(S_{D_2}^T X_{top1}^T; Y_{F_1}^T W_1) + H(Y_2^T | S_{D_2}^T X_{top1}^T)$$

$$+ I(S_{D_1}^T X_{top2}^T; Y_{F_2}^T W_2) + H(Y_1^T | S_{D_1}^T X_{top2}^T)$$

$$- H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2)$$

$$\stackrel{(d)}{\leq} I(S_{D_2}^T X_{top1}^T Y_{F_2}^T W_2; Y_{F_1}^T W_1) + H(Y_2^T | S_{D_2}^T X_{top1}^T)$$

$$+ I(S_{D_1}^T X_{top2}^T Y_{F_1}^T W_1; Y_{F_2}^T W_2) + H(Y_1^T | S_{D_1}^T X_{top2}^T)$$

$$- H(Y_{F_1}^T | W_1) - H(Y_{F_2}^T | W_2)$$

$$\stackrel{(e)}{\leq} \sum_{i=1}^{T} [H(Y_{2i} | S_{D_2 i} X_{top1,i}) + H(Y_{1i} | S_{D_1 i} X_{top2,i})$$

$$+ H(Y_{F_2 i} | X_{2i} X_{top1,i}) + H(Y_{F_1 i} | X_{1i} X_{top2,i}) + H(X_{top1,i} | S_{D_2 i})$$

$$+ H(X_{top2,i} | S_{D_1 i}),$$

where

(a) follows from the fact that Y_{F_j} is a function of Y_j for $j = 1, 2$;

(b) follows from the fact that X_{topj}^T is a function of X_j^T, which is in turn a function of $(Y_{F_j}^T W_j)$, for $j = 1, 2$;

(c) follows from the facts that $S_{D_j}^T$ is a function of X_j^T for $j = 1, 2$; and X_{top1}^T is a function of X_1^T, which is in turn a function of $(Y_2^T X_2^T)$, and vice versa;

(d) follows from the fact that side information increases the mutual information; and

(e) follows from Lemma 3.3.

Case 1: $0 \le m \le \frac{n}{2}$

We have

$$H(Y_{2i}|S_{D_2i}X_{top1,i}) = H(Y_{1i}|S_{D_1i}X_{top2,i}) \le n - m, \tag{3.127}$$

$$H(Y_{F_2i}|X_{2i}X_{top1,i}) = H(Y_{F_1i}|X_{1i}X_{top2,i}) \le (l - (n-m))^+, \tag{3.128}$$

$$H(X_{top1,i}|S_{D_2i}) = H(X_{top2,i}|S_{D_1i}) = 0. \tag{3.129}$$

Thus, we have $R_1 + R_2 \le 2(n-m) + 2[l-(n-m)]^+$.

Case 2: $\frac{n}{2} \le m \le n$

We have

$$H(Y_{2i}|S_{D_2i}X_{top1,i}) = H(Y_{1i}|S_{D_1i}X_{top2,i}) \le n - m, \tag{3.130}$$

$$H(Y_{F_2i}|X_{2i}X_{top1,i}) = H(Y_{F_1i}|X_{1i}X_{top2,i}) \le (l-m)^+, \tag{3.131}$$

$$H(X_{top1,i}|S_{D_2i}) = H(X_{top2,i}|S_{D_1i}) \le (2m-n)^+. \tag{3.132}$$

Thus, we have $R_1 + R_2 \le 2m + 2[l-m]^+$.

Case 3: $n \leq m$ We have

$$\begin{aligned} H(Y_{2i}|S_{D_2 i}X_{top1,i}) &= H(Y_{1i}|S_{D_1 i}X_{top2,i}) \\ &= H(Y_{F_2 i}|X_{2i}X_{top1,i}) \\ &= H(Y_{F_1 i}|X_{1i}X_{top2,i}) = 0, \end{aligned} \tag{3.133}$$

$$H(X_{top1,i}|S_{D_2 i}) = H(X_{top2,i}|S_{D_1 i}) \leq m. \tag{3.134}$$

Thus, we have $R_1 + R_2 \leq 2m$.

Combining the three cases, we have proved the fourth outer bound for $R_1 + R_2$.

■ 3.6.1.3 Bound on $2R_1 + R_2$ and $R_1 + 2R_2$

In this subsection, we focus on the proof for the upper bound on $2R_1 + R_2$. The proof for the bound on $R_1 + 2R_2$ follows in a similar manner.

We have

$$\begin{aligned} &T(2R_1 + R_2 - p_e^T) \\ &\leq 2I(W_1; Y_1^T) + I(W_2; Y_2^T) \\ &\leq I(W_1; Y_1^T Y_{F_1}^T) + I(W_1; Y_1^T Y_{F_2}^T|W_2) + I(W_2; Y_2^T Y_{F_2}^T) \\ &= H(Y_1^T) + H(Y_{F_1}^T|Y_1^T) - H(Y_{F_1}^T|W_1) - H(Y_1^T|Y_{F_1}^T W_1) \end{aligned}$$

$$+ H(Y_{F_2}^T|W_2) + H(Y_1^T|Y_{F_2}^T W_2) - H(Y_1^T Y_{F_2}^T|W_2 W_1)$$

$$+ H(Y_2^T) + H(Y_{F_2}^T|Y_2^T) - H(Y_{F_2}^T|W_2) - H(Y_2^T|Y_{F_2}^T W_2)$$

$$\stackrel{(a)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(Y_1^T|Y_{F_1}^T W_1)$$

$$+ H(Y_1^T|Y_{F_2}^T W_2) + H(Y_2^T) - H(Y_2^T|Y_{F_2}^T W_2)$$

$$\stackrel{(b)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T|Y_{F_1}^T W_1)$$

$$+ H(Y_1^T|Y_{F_2}^T W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T W_2)$$

$$\stackrel{(c)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T X_{top1}^T|Y_{F_1}^T W_1)$$

$$+ H(Y_1^T|Y_{F_2}^T W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T W_2)$$

$$\stackrel{(d)}{\leq} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T X_{top1}^T|Y_{F_1}^T W_1)$$

$$+ H(Y_1^T S_{D_1}^T|Y_{F_2}^T W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T W_2)$$

$$\stackrel{(e)}{\leq} H(Y_1^T) - H(Y_{F_1}^T|W_1) - [H(S_{D_2}^T X_{top1}^T)$$

$$- I(S_{D_2}^T X_{top1}^T; Y_{F_1}^T W_1)] + [H(S_{D_1}^T|Y_{F_2}^T W_2)$$

$$+ H(Y_1^T|S_{D_1}^T Y_{F_2}^T W_2)] + H(Y_2^T S_{D_2}^T X_{top1}^T) - H(S_{D_1}^T|Y_{F_2}^T W_2)$$

$$= H(Y_1^T) - H(Y_{F_1}^T|W_1) + I(S_{D_2}^T X_{top1}^T; Y_{F_1}^T W_1)$$

$$+ H(Y_1^T|S_{D_1}^T Y_{F_2}^T W_2) + H(Y_2^T|S_{D_2}^T X_{top1}^T)$$

$$\stackrel{(f)}{\leq} H(Y_1^T) - H(Y_{F_1}^T|W_1) + I(S_{D_2}^T X_{top1}^T Y_{F_2}^T W_2; Y_{F_1}^T W_1)$$

$$+ H(Y_1^T|S_{D_1}^T Y_{F_2}^T W_2) + H(Y_2^T|S_{D_2}^T X_{top1}^T)$$

$$
\begin{aligned}
&\stackrel{(g)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) + [I(W_2; Y_{F_1}^T W_1) \\
&\quad + I(S_{D_2}^T X_{top1}^T Y_{F_2}^T; Y_{F_1}^T W_1|W_2)] \\
&\quad + H(Y_1^T|S_{D_1}^T Y_{F_2}^T W_2) + H(Y_2^T|S_{D_2}^T X_{top1}^T) \\
&\stackrel{(h)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) + [(H(Y_{F_1}^T|W_1) \\
&\quad + H(Y_{F_2}^T S_{D_2}^T X_{top1}^T|W_2)] \\
&\quad + H(Y_1^T|S_{D_1}^T Y_{F_2}^T W_2) + H(Y_2^T|S_{D_2}^T X_{top1}^T) \\
&\stackrel{(i)}{\leq} \sum_{i=1}^{T} [H(Y_{1i}) + H(Y_{F_2 i}|Y_{F_2}^{i-1} S_{D_2}^i X_{top1}^i W_2 X_{2i}) \\
&\quad + H(S_{D_2 i}|Y_{F_2}^{i-1} W_2 X_{2i}) + H(X_{top1,i}|Y_{F_2}^{i-1} S_{D_2}^i W_2) \\
&\quad + H(Y_{1i}|S_{D_1 i} S_{D_2 i}) + H(Y_{2i}|S_{D_2 i} X_{top1,i})] \\
&\stackrel{(j)}{=} \sum_{i=1}^{T} [H(Y_{1i}) + H(Y_{F_2 i}|X_{2i} X_{top1,i}) + H(X_{top1,i}|S_{D_2 i}) \\
&\quad + H(Y_{1i}|S_{D_1 i} S_{D_2 i}) + H(Y_{2i}|S_{D_2 i} X_{top1,i})],
\end{aligned}
$$

where

(a) follows from the facts that $H(Y_1^T Y_{F_2}^T|W_2 W_1) = 0$, $H(Y_{F_1}^T|Y_1^T) = 0$, and $H(Y_{F_2}^T|Y_2^T) = 0$;

(b) follows from the fact that X_{ji} is a function of $(Y_{F_j}^{i-1} W_j)$, for $j = 1, 2$;

(c) follows from the fact that X_{top1}^T is a function of X_1^T, which is in turn a function of $(W_1 Y_{F_1}^{T-1})$.This is the crucial step;

(d) follows from the fact that side information increases the entropy;

(e) follows from the fact that side information increases the entropy;

(f) follows from the fact side information increases the mutual information;

(g) follows from the fact that $S_{D_2}^T$ is a function of $(Y_{F_2}^T W_2)$;

(h) follows from the fact given $(W_1 W_2)$, the entropy of any random variable is 0;

(i) follows from the fact that X_{2i} is a function of $(Y_{F_2}^{i-1} W_2)$; and

(j) follows from the fact that $S_{D_2 i}$ is a function of X_{2i}.

Case 1: $0 \leq m \leq \frac{n}{2}$

We have

$$H(Y_{1i}) \leq n, \tag{3.135}$$

$$H(Y_{F_2 i}|X_{2i} X_{top1,i}) \leq (l-(n-m))^+, \tag{3.136}$$

$$H(X_{top1,i}|S_{D_2 i}) = 0, \tag{3.137}$$

$$H(Y_{1i}|S_{D_1 i} S_{D_2 i}) = H(Y_{2i}|S_{D_2 i} X_{top1,i}) \leq n-m. \tag{3.138}$$

Thus, we have $2R_1 + R_2 \leq 3n - 2m + [l-(n-m)]^+$.

Case 2: $\frac{n}{2} \leq m \leq n$

We have

$$H(Y_{1i}) \leq n, \tag{3.139}$$

$$H(Y_{F_2 i}|X_{2i} X_{top1,i}) \leq (l-m)^+, \tag{3.140}$$

$$H(X_{top1,i}|S_{D_2 i}) \leq (2m-n)^+, \tag{3.141}$$

$$H(Y_{1i}|S_{D_1 i} S_{D_2 i}) = H(Y_{2i}|S_{D_2 i} X_{top1,i}) \leq n-m. \tag{3.142}$$

Thus, we have $2R_1 + R_2 \leq 2n + [l-m]^+$.

Case 3: $n \leq m$

We have

$$H(Y_{1i}) = H(Y_{2i}|S_{D_2 i}X_{top1,i}) \leq m \quad (3.143)$$

$$H(Y_{F_2 i}|X_{2i}X_{top1,i}) = H(Y_{1i}|S_{D_1 i}S_{D_2 i}) = H(Y_{2i}|S_{D_2 i}X_{top1,i}) = 0. \quad (3.144)$$

Thus, we have $2R_1 + R_2 \leq 2m$.

Combining the three cases, we have proved the bound on $2R_1 + R_2$.

3.6.2 Forward proof of Theorem 3.1

In this subsection, we apply Lemma 3.1 and present an encoding scheme, for the symmetric linear deterministic interference channel with noisy feedback.

3.6.2.1 Achievable rate region for the symmetric linear deterministic interference channel with noisy feedback

Now, we apply this lemma to construct a generic encoding scheme, and to find the corresponding achievable rate region for the symmetric deterministic interference channel with noisy feedback. Denote $X_{j,CC}, X_{j,NCC}$ and $X_{j,P}$ as column vectors of size $\max(n,m)$ bits, for $j,k \in \{1,2\}$ and $j \neq k$. We let

$$U = \emptyset \quad (3.145)$$

$$U_j = U \oplus X_{j,CC} \quad (3.146)$$

$$V_j = U_j \oplus X_{j,NCC} \tag{3.147}$$

$$X_j = V_j \oplus X_{j,P}. \tag{3.148}$$

$X_{j,CC}$, $X_{j,NCC}$, $X_{j,P}$ contain the interfering common message, the non-interfering common message and private message, respectively, for the transmitter j. Consider Figure 3.12, which illustrates the generic encoding scheme. The interfering common message and the non-interfering common message are restricted to the top area of m bits, and the private message is restricted to the bottom area of $(n-m)^+$ bits. That means in the strong and very strong interference regimes when $n < m$, no private message is encoded. Intuitively, this should be the case as any transmitted signal from any transmitter j will be received by both receivers anyway. As interfering common message causes interference to the non-intended receiver, it needs to be fed back via the feedback link so that interference can be resolved. Thus, the achievable rate of the interfering common message depends directly on the feedback link strength. Hence, we propose an adaptive encoding scheme that varies according the strength of the feedback link. Here, we choose the size of the interfering common message of the transmitter j to be upper-bounded by m . Once the codeword U_j for the cooperative common message $X_{j,CC}$ has been constructed, we construct the codeword V_j which depends on the non-interfering common message $X_{j,NCC}$ and U_j. The non-interfering common message can either contain fresh information bits, or fedback signals, which needs to be relayed again for resolving interference, or null information. Furthermore, the non-interfering common message only occupies positions in the top area of m bits, which has not been taken by the interfering common information. Finally, the codeword X_j for the transmitter j depends on the private

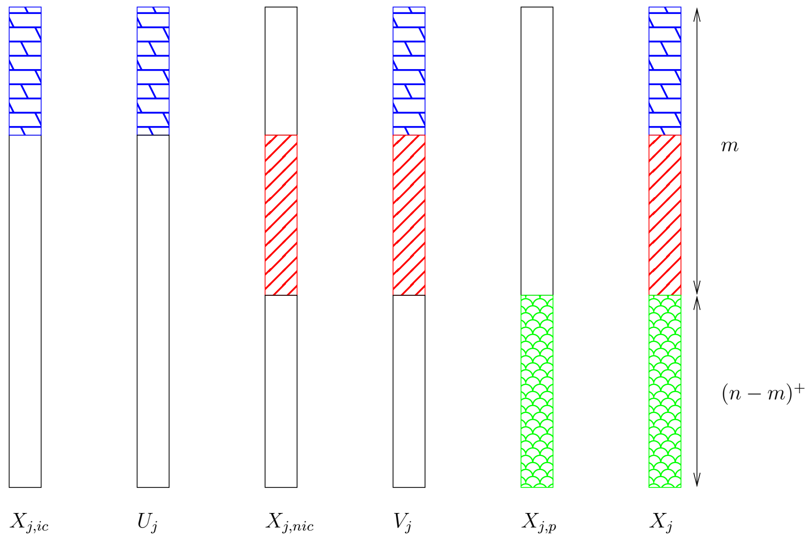

Figure 3.12. Generic encoding

message $X_{j,P}$ and V_j. We will show that the optimal achievable rate region matches the outer bound region.

For readers' convenience and for ease of calculation, we illustrate the encoding schemes case by case.

3.6.2.2 Very weak interference: $m \le \frac{1}{2}n$

We consider 2 cases.

Case 1: $l \le n - m$.

Set

- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

We have

$$\rho_1 = \kappa_1 = I(U_2; Y_{F_1}|X_1) = 0, \tag{3.149}$$

$$\rho_2 = \kappa_2 = H(Y_1|V_1, V_2) = n - m, \tag{3.150}$$

$$\rho_3 = \kappa_3 = H(Y_1|V_1, U_2) = n - m, \tag{3.151}$$

$$\rho_4 = \kappa_4 = H(Y_1|U_1, V_2) = n, \tag{3.152}$$

$$\rho_5 = \kappa_5 = H(Y_1|U_1, U_2) = n, \tag{3.153}$$

$$\rho_6 = \kappa_6 = H(Y_1) = n. \tag{3.154}$$

Applying Lemma 3.1, the following region is achievable

$$R_1 \leq n, \tag{3.155}$$

$$R_2 \leq n, \tag{3.156}$$

$$R_1 + R_2 \leq 2n - m, \tag{3.157}$$

$$R_1 + R_2 \leq 2(n - m), \tag{3.158}$$

$$2R_1 + R_2 \leq 3n - 2m, \tag{3.159}$$

$$R_1 + 2R_2 \leq 3n - 2m. \tag{3.160}$$

Case 2: $n - m \leq l$.

In this case, the feedback link helps to increase the rate of interfering common message. Set

- $X_{1CC} = X_{2CC} = (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = m - (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits, right below the interfering common message's region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

Applying Lemma 3.1, the following region is achievable

$$R_1 \leq n, \tag{3.161}$$

$$R_2 \leq n, \tag{3.162}$$

$$R_1 + R_2 \leq 2n - m, \tag{3.163}$$

$$R_1 + R_2 \leq 2(n - m) + 2(l - (n - m)^+))^+, \tag{3.164}$$

$$2R_1 + R_2 \leq 3n - 2m + (l - (n - m)^+)^+, \tag{3.165}$$

$$R_1 + 2R_2 \leq 3n - 2m + (l - (n - m)^+)^+. \tag{3.166}$$

Thus, we have shown the achievability of the capacity region in Theorem 3.1 in the very weak interference regime.

The calculation in other regimes are similar. Thus, we will only show the assignment of bits to the random variables, and leave it the readers that these bit assignments allow us to achieve the capacity region in Theorem 3.1.

■ 3.6.2.3 Weak and moderately strong interference: $\frac{1}{2}n \leq m \leq n$

We consider 2 cases.

Case 1: $l \leq m$.

This is the case of weak feedback link, thus, feedback link cannot help to resolve interference at receivers. No interfering common message should be sent. Set

- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

Case 2: $m \leq l$. The rate of interfering common message should be chosen carefully to make use of the strong feedback links. Set

- $X_{1CC} = X_{2CC} = (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = m - (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits, right below the interfering common message's region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

3.6.2.4 Strong and very strong interference: $n \leq m$

Note that no private information is sent in these regimes. We consider two cases.

Case 1: $l \leq n$. Set

- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = 0$.

Case 2: $n \leq l$. Set

- $X_{1CC} = X_{2CC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = 0$;
- $X_{1P} = X_{2P} = 0$.

3.6.3 2^{nd} achievability proof of Theorem 3.1

In this subsection, we present the second achievability proof of Theorem 3.1. From the outer bounds in subsection 3.6.1, we can determine the corner points for each of five regimes. If we can show those corner points are achievable, the capacity region is established for that particular regime. This is the approach we take in this subsection. This proof gives us insight into the encoding scheme at corner points of the capacity region.

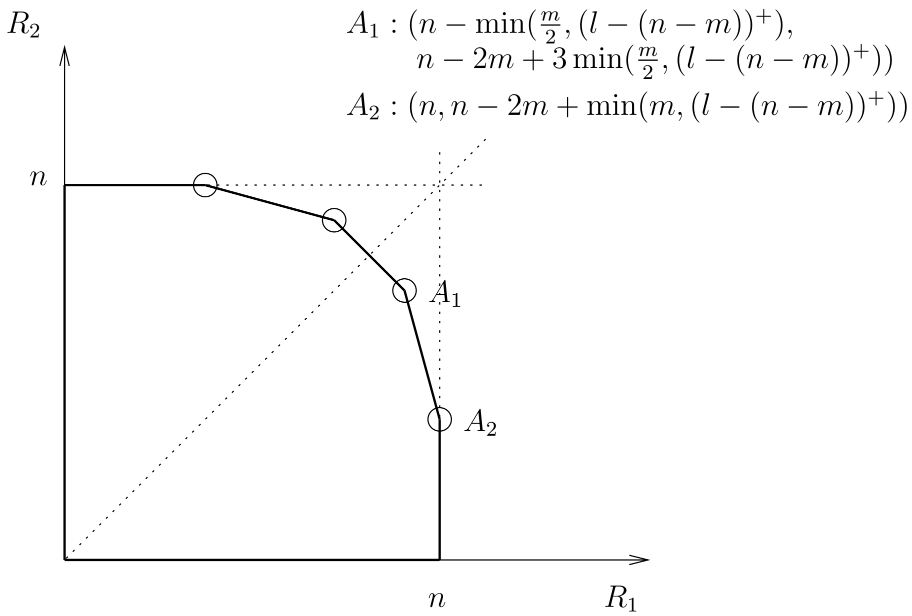

Figure 3.13. Capacity region for LD-IC for $\alpha \in [0, \frac{1}{2}]$.

The capacity region in this case is shown in Figure 3.13.

It is trivial to show that the points $(0,0)$, $(n,0)$ and $(0,n)$ are achievable. Due to symmetry, we just need to show that the two points $(n, n-2m+\min(m,(l-(n-m))^+))$ and $(n-\min(\frac{m}{2},(l-(n-m))^+), n-2m+3\min(\frac{m}{2},(l-(n-m))^+))$ are achievable.

■ 3.6.3.1 Very-Weak Interference: $\alpha \in [0, \frac{1}{2}]$

Firstly, we are going to show how to achieve the point $(n, n-2m+\min(m,(l-(n-m))^+))$. The encoding scheme is shown in Figure 3.14. The set of n bits at transmitter Tx_1, for each channel use, is divided into 5 encoding regions, $A_1, A_2, ..., A_5$, of respective sizes $\frac{m}{2}, \frac{m}{2}, n-2m, \frac{m}{2}, \frac{m}{2}$. A similar partition is done at Tx_2.

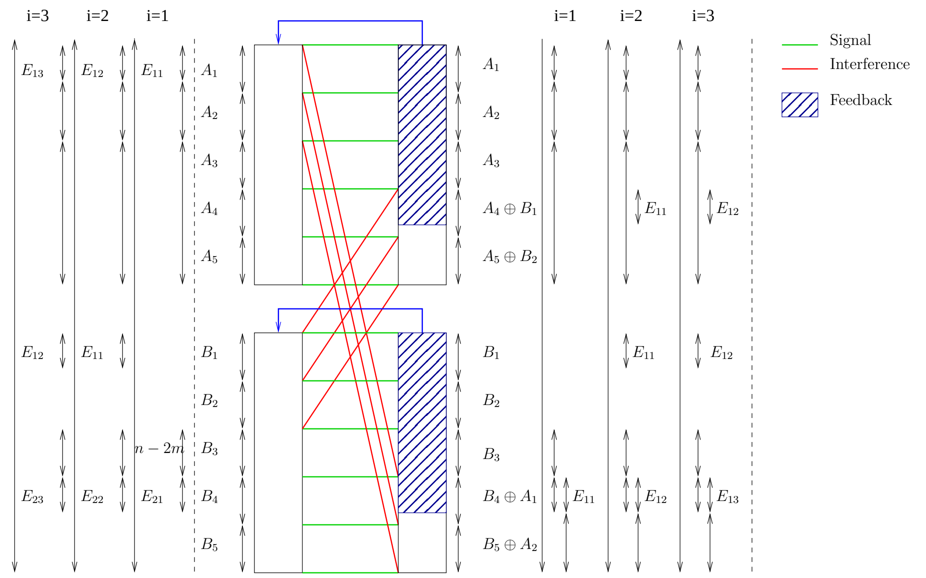

Figure 3.14. Encoding for corner point $(n, n-2m+\min(m,(l-(n-m))^+))$.

For every time slot, transmitter 1 always transmits n fresh information bits. In the first time slot, transmitter 2 transmits $(n-2m)+E_{21}$ fresh information bits in the regions B_3 and (B_4, B_5) as shown in the diagram. The size of E_{21} is $\min(m,(l-(n-m))^+)$. Receiver 2 feeds back the top l bits, which include $E_{21} \oplus E_{11}$. Thus, at the end of the first time slot, transmitter 2 can decode E_{11}. In the second time slot, transmitter 2 relays E_{11} bits in the region (B_1, B_2). The rest of the operations are similar to that in the first time slot. Notice that E_{11} this time does not cause interference to receiver 1 as receiver 1 has already received those bits in the first time slot. At the same time, E_{11} is received cleanly at receiver 2. As a result, receiver 2 can decode the information E_{21} transmitted in the first time slot. By repeating those operations for all time slots, the corner point $(n, n-2m+\min(m,(l-(n-m))^+))$ is achievable.

Next, we are going to show how to achieve the corner point $(n-\min(\frac{m}{2},(l-(n-m))^+), n-2m+3\min(\frac{m}{2},(l-(n-m))^+))$. The encoding scheme is shown in

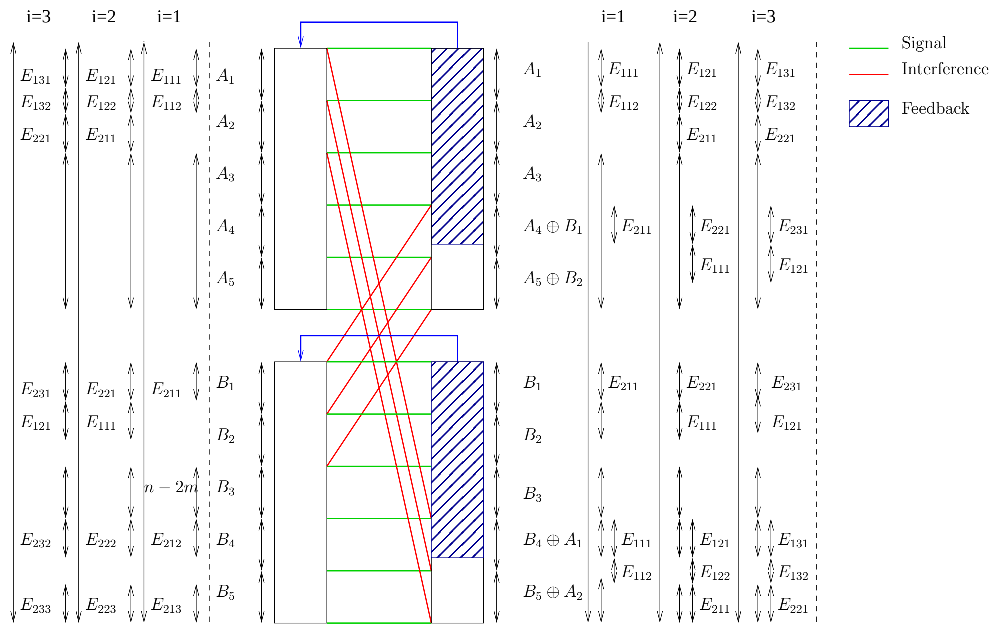

Figure 3.15. Encoding for corner point $(n - \min(\frac{m}{2}, (l-(n-m))^+), n-2m+3\min(\frac{m}{2}, (l-(n-m))^+))$.

Figure 3.15. In the first time slot, transmitter 1 encodes $n-\min(\frac{m}{2}, (l-(n-m))^+)$ fresh information bits which cover all of the regions (A_1, A_3, A_4, A_5) and partially cover the region A_2. The bottom of the region A_2, of the size $\min(\frac{m}{2}, (l-(n-m))^+)$ is left empty. Transmitter 2 transmits $n-2m+3\min(\frac{m}{2}, (l-(n-m))^+)$ bits in the regions B_3, E_{211}, E_{212}, and E_{213}. Notice that $|E_{111}| = |E_{21j}| = \min(\frac{m}{2}, (l-(n-m))^+)$, for $j = 1, 2, 3$. At the end of the first time slot, via the feedback link, transmitter 1 can receive E_{211}, and transmitter 2 can receive E_{111}. In the second time slot, besides repeating the operation in the previous time slot, transmitter 1 relays the information in E_{211}. In this time slot, receiver 1 can receive E_{211} cleanly, thus receiver 1 can resolve the interference in the previous time slot and decode those corresponding corrupted bits in the region A_4 in the previous time slot. Notice that E_{211} does not cause interference to receiver 2 as it was received perfectly by receiver 2 in the previous time slot already. Besides repeating operation in the previous time slot, transmitter 2 relays the

information in E_{111}. Similarly, E_{111} is received cleanly by receiver 2, and thus helps receiver 2 resolve interference in the previous time slot. By repeating those operations, we can achieve the corner point $(n - \min(\frac{m}{2}, (l-(n-m))^+), n - 2m + 3\min(\frac{m}{2}, (l-(n-m))^+))$.

3.6.3.2 Weak Interference: $\alpha \in [\frac{1}{2}, \frac{2}{3}]$

The outer bound for the capacity region in this regime has a similar shape to the previous regime, but has a different set of corner points. To show that this region is achievable, we are going to show that all the corner points are achievable. Trivially, the points $(0,0)$, $(n,0)$ and $(0,n)$ are achievable. Due to symmetry, we just need to show that the two points $(n, \min(\frac{2n-3m}{2}, (l-m)^+))$ and $(2(n-m) - \min(\frac{2n-3m}{2}, (l-m)^+), 2(2m-n) + \min(\frac{2n-3m}{2}, (l-m)^+))$ are achievable.

Firstly, we are going to show how the corner point $(n, \min(\frac{2n-3m}{2}, (l-m)^+))$ is achieved. The encoding scheme is shown in Figure 3.16. The set of n bits at the transmitter Tx_1 is divided into 5 regions A_1, A_2, A_3, A_4 and A_5. The sizes of the 5 regions are $2m-n, 2n-3m, 2m-n, 2n-3m$ and $2m-n$ respectively. A similar partition is done at transmitter Tx_2.

The story in this regime is similar to that of the very weak regime, except for the sizes and the places of encoding regions. In all time slots, transmitter 1 always transmits n fresh bits. Transmitter 2 only transmits E_{21} fresh bits in B_4. Note $|E_{21}| = \min(\frac{2n-3m}{2}, (l-m)^+)$. In the second time slot, transmitter

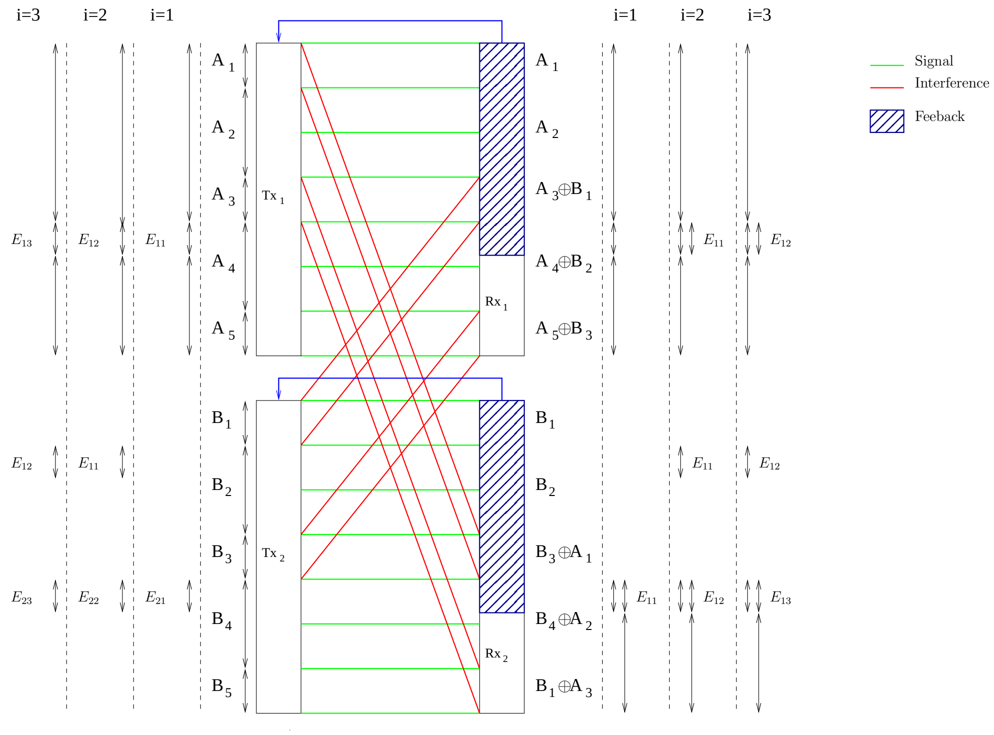

Figure 3.16. Encoding for corner point $(n, \min(\frac{2n-3m}{2}, (l-m)^+))$.

2 relays the bits in E_{11} in the region B_2, besides sending a new batch of E_{22} bits. By repeating those operations, the corner point $(n, \min(\frac{2n-3m}{2}, (l-m)^+))$ is achieved.

Next, we are going to show how to achieve the corner point $(2(n-m) - \min(\frac{2n-3m}{2}, (l-m)^+), 2(2m-n)+\min(\frac{2n-3m}{2}, (l-m)^+))$. The encoding scheme is shown in Figure 3.17. In the first time slot, transmitter 1 transmits $(n - m) - \min(\frac{2n-3m}{2}, (l-m)^+)$ fresh bits (A_1, A_2), and another $n-m$ fresh bits in the region (A_4, A_5). No information is encoded in the region A_3. Transmitter 2 encodes $(2m-n)$ fresh bits in B_1, E_{211} fresh information bits in B_2, $E_{212}+E_{213}$ bits in the region B_4, and $(2m-n)$ fresh bits in the region B_5. In the second time slot, besides sending new information as in the first time slot, transmitter 1 relays E_{211} in the region A_2. Besides sending new information, transmitter 2 relays E_{11}

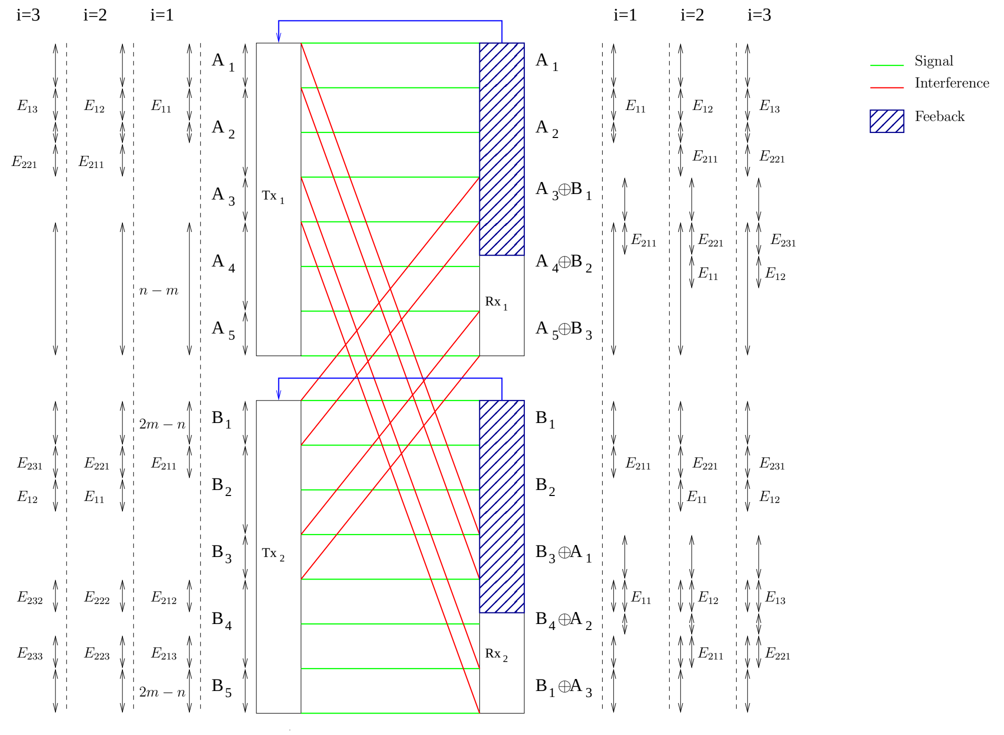

Figure 3.17. Encoding for corner point $(2(n-m)-\min(\frac{2n-3m}{2},(l-m)^{+}),2(2m-n)+\min(\frac{2n-3m}{2},(l-m)^{+}))$.

in the region B_2. By repeating those operations again for all other time slots, we can achieve the corner point $(2(n-m)-\min(\frac{2n-3m}{2},(l-m)^{+}),2(2m-n)+\min(\frac{2n-3m}{2},(l-m)^{+}))$.

■ 3.6.3.3 Moderately Strong Interference: $\alpha \in [\frac{2}{3},1]$

Case 1: $l<m$

In this case, the capacity region is the same as that of the region without any feedback. The encoding schemes for this case are shown in [11].

Case 2: $m \le l \le n$

The corner points in this case are given by $(0,0),(\min(n,l),0),(0,\min(n,l))$, $(m+(l-m)^{+},2(n-m)-(l-m)^{+})$ and $(2(n-m)-(l-m)^{+},m+(l-m)^{+})$.

The encoding schemes for the first three points are trivial. In the rest of this subsection, we will show how to achieve the corner point $(m + (l - m)^+, 2(n - m) - (l - m)^+)$. The scheme to achieve $(2(n - m) - (l - m)^+, m + (l - m)^+)$ is similar. There are 2 sub-cases to consider here.

Sub-case 2.1: $3(n - m) \leq l$.

The encoding scheme for this sub-case is shown in Figure 3.18. In the first time slot, transmitter 1 transmits $(2m - n) + (l - m)^+$ fresh bits in the top region, and $n - m$ fresh bits in the bottom region. Thus, it sends a total of l fresh bits. Transmitter 2 transmits $(n - m) - (l - m)^+$, or $n - l$ fresh bits in the top region, and $n - m$ fresh bits in the bottom region. Thus, it sends a total of $2(n - m) - (l - m)$ fresh bits. At the receiving sides, E_{11} causes interference to receiver 2, and F_{21} causes interference to receiver 2. Via the feedback link, transmitter 1 can decode F_{21}, and transmitter 1 can decode E_{11}.

In the second time slot, besides sending fresh information as in the first time slot, transmitter 1 relays F_{21} in the middle gap as shown in the Figure 3.18, and transmitter 2 relays E_{11}. No matter what value l takes, receiver 2 can always decode E_{11}, thus can resolve the interference in the first time slot. As a result, it can receive all of $2(n - m) - (l - m)$ fresh bits intended for itself in the first time slot. Notice in this sub-case, we have an inequality that always holds: $n - l \leq 3m - 2n$. Thus, $(n - l) + |E_{11}| \leq (3m - 2n) + (l - m)$. Therefore, the bits F_{21} are always received cleanly in this sub-case. With those bits, receiver 2 can resolve the interference in the first time slot. Those operations are repeated over

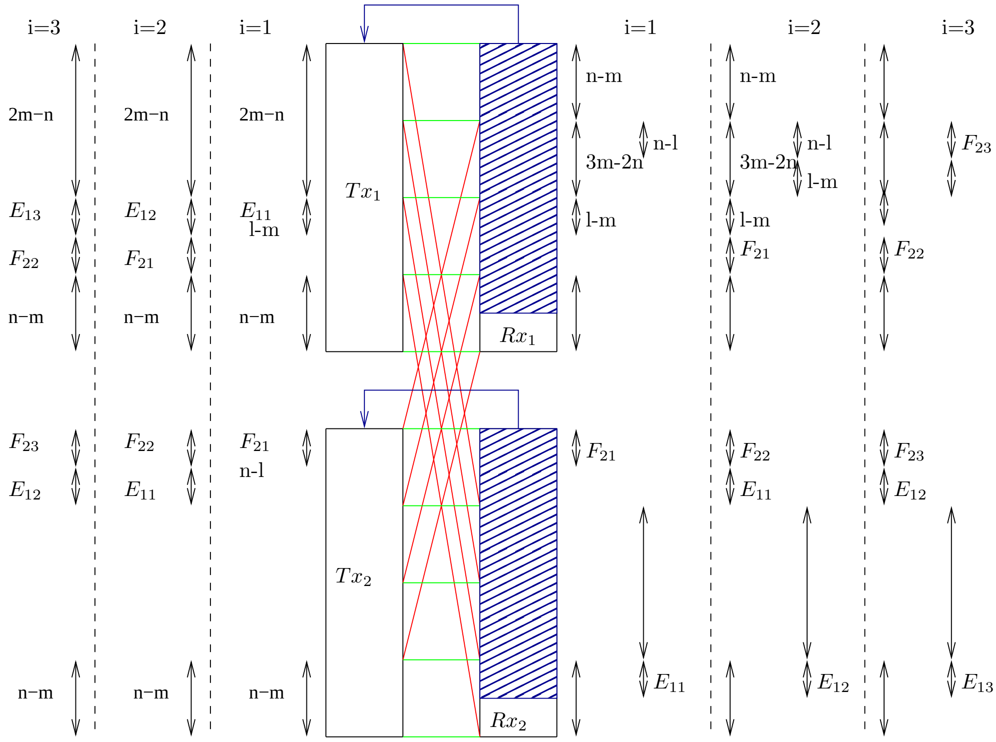

Figure 3.18. Encoding schemes: $\frac{3n}{2} \le m \le n, m \le l, 3(n-m) \le l$

time. The corner point $(m + (l - m)^+, 2(n - m) - (l - m)^+)$ is asymptotically achievable.

Sub-case 2.2: $l < 3(n - m)$.

The encoding schemes in this case are similar to the first sub-case. Transmitter 1 sends l fresh information bits, transmitter 2 sends $2(n - m) - (l - m)$ fresh bits in the same regions every time slot. From the second time slot onwards, transmitter 2 relays the bits in the region E_{11}. However, there is a slight variation. Transmitter 1 relays in part, or in whole, the bits in the region F_{21}, which have not been decoded by receiver 1 yet, depending on the relative value of l. The encoding scheme is illustrated by Figure 3.19.

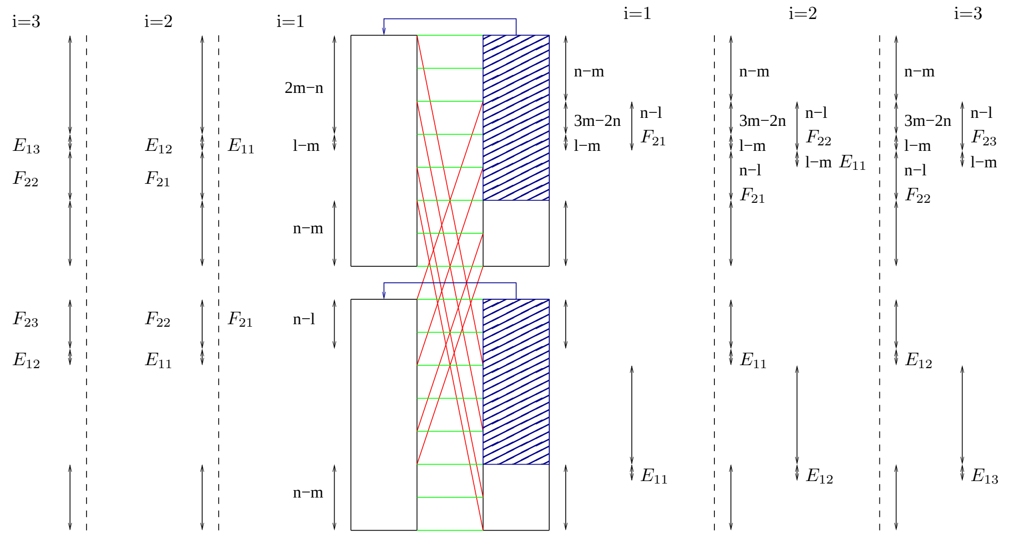

Figure 3.19. Encoding schemes: $\frac{3n}{2} \leq m \leq n, m \leq l, l < 3(n-m)$

Using this strategy, receiver 2 can always decode the bits in the region E_{11}; thus, it can resolve interference caused by transmitter 1 and achieve a rate of $2(n-m)-(l-m)$ fresh bits per channel use asymptotically. It can be shown that receiver 1 always achieves a rate of l bits per channel use.

■ 3.6.3.4 Strong Interference: $\alpha \in [1,2]$

Case 1: $l \leq n$

The corner points in this case are given by $(0,0)$, $(n,0)$, $(0,n)$, $(n,m-n)$ and $(m-n,n)$. The encoding schemes to achieve the first three corner points are trivial. The encoding schemes to achieve the last two points are the same as the encoding schemes without feedback, and are shown in [11].

Case 2: $n \leq l$

The corner points in this case are $(0,0)$, $(\min(m,l),0)$, $(0,\min(m,l))$, $(l,m-l)$

and $(m-l,l)$. The encoding schemes for the first three points are trivial. In the rest of this subsection, we will show how to achieve the corner point $(l,m-l)$. The scheme to achieve $(m-l,l)$ is similar. There are 2 sub-cases to consider here.

Sub-case 2.1: $n \le l$ and $l \le 2m-2n$

Assume $l \le m$; otherwise, the result is trivial. The encoding scheme is shown in Figure 3.20. In the first time slot, transmitter 1 sends l fresh bits in the top region. Transmitter 2 sends F_{21} in the top region, where $|F_{21}| = 2n-m$. In addition, it sends $2m-2n-l$ bits in the middle region, such that there is a small gap of $l-n$ bits. Thus, transmitter 2 sends a total of $m-l$ fresh information bits. At the end of the first time slot, out of l bits sent from transmitter 1, receiver 1 can receive n intended bits cleanly. It cannot receive E_{11} directly yet. Receiver

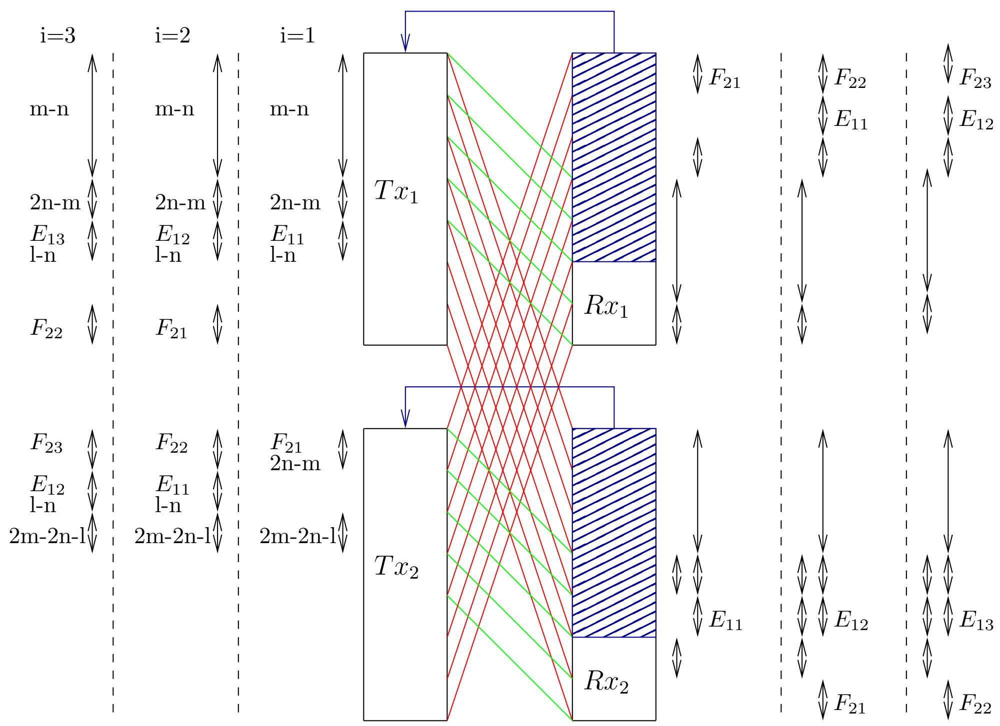

Figure 3.20. Encoding scheme: $n \le m \le 2n, n \le l \le 2m-2n$

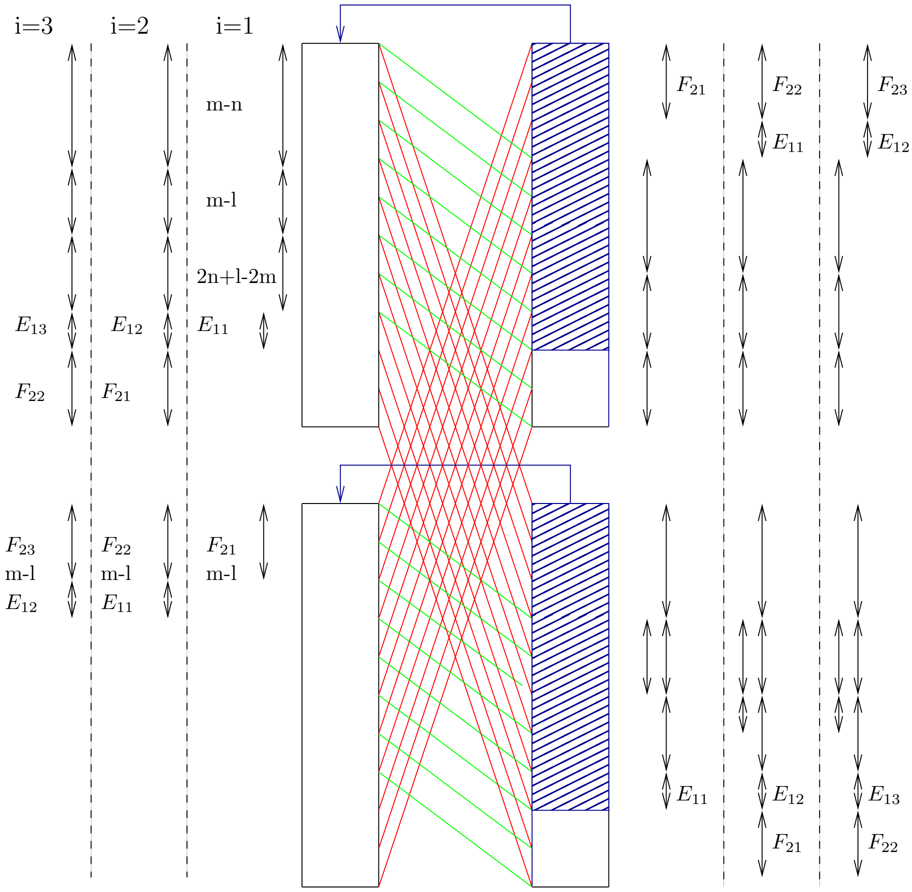

Figure 3.21. Encoding scheme: $n \leq m \leq 2n, n \leq l, 2m - 2n \leq l$

2 can receive $2m - 2n - l$ bits cleanly and it cannot receive the bits in F_{21} due to interference. Via the feedback link of strength l bits, transmitter 2 can decode E_{11}, and transmitter 1 can decode F_{21}. In the second time slot, besides sending new l fresh information, transmitter 1 relays F_{21} in the bottom region. Besides sending new $m - l$ fresh information bits, transmitter 2 relays E_{11} as shown in the diagram. Subsequently, receiver 1 can recover E_{11}, and receiver 2 can recover F_{21} at the end of the second time slot. Those operations are repeated over time. Asymptotically, the corner point $(l, m - l)$ is achieved.

Sub-case 2.2: $n \leq l$ and $2m - 2n \leq l$

Assume $l \leq m$; otherwise, the result is trivial. The encoding scheme is shown in Figure 3.21. Notice that $(m-l)+(m-n) \leq n$, which makes the encoding scheme feasible. In the first time slot, transmitter 1 sends l fresh bits in the top region; transmitter 2 sends F_{21} in the top region, where $|F_{21}| = m-l$. At the end of the first time slot, out of l bits sent from transmitter 1, receiver 1 can receive n intended bits cleanly. It cannot receive E_{11} directly yet. At the end of the first time slot, receiver 2 cannot receive the bits in F_{21} directly yet as F_{21} is corrupted by interference from transmitter 1. In the second time slot, besides sending new l fresh information, transmitter 1 relays F_{21} in the bottom region. Besides sending new $m-l$ fresh information bits, transmitter 2 relays E_{11} as shown in the diagram. Subsequently, receiver 1 can recover E_{11}, and receiver 2 can recover F_{21} at the end of the second time slot. Operations are repeated over time. Asymptotically, the corner point $(l, m-l)$ is achieved.

3.6.3.5 Very Strong Interference: $\alpha \in [2, \infty)$

Case 1: $2l < m$

The corner points in this case are given by $(0,0)$, $(n+(l-n)^+, 0)$, $(0, n+(l-n)^+)$ and $(n+(l-n)^+, n+(l-n)^+)$. The achievable scheme for the first three points are trivial. The achievable scheme for the last corner point is given in the paper [63]. We focus on the next case which is more interesting.

Case 2: $m < 2l$

The corner points in this case are given by $(0,0)$, $(\min(m, n+(l-n)^+), 0)$,

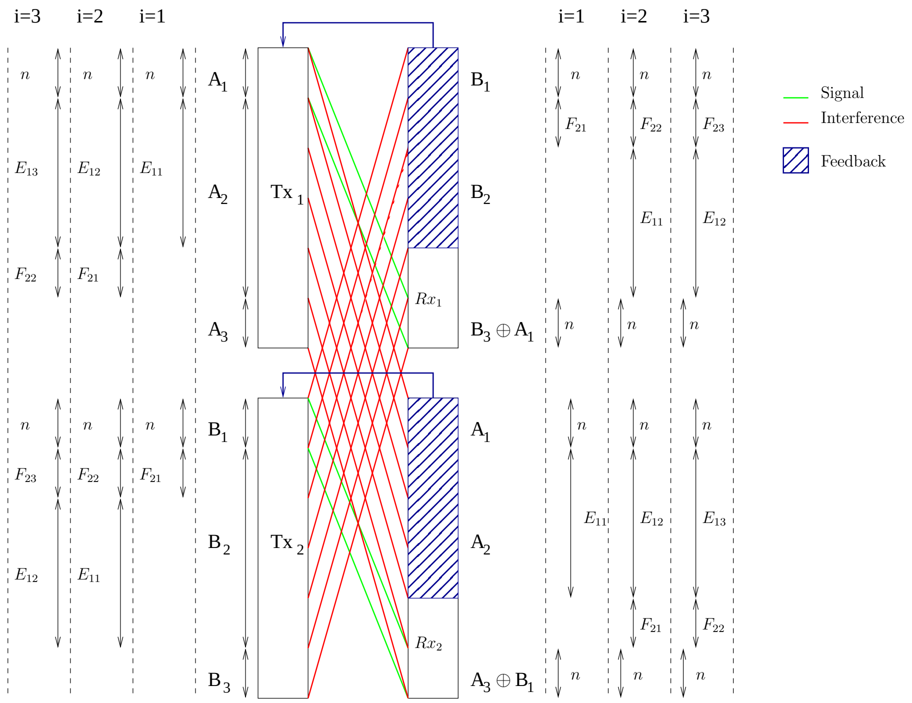

Figure 3.22. Encoding scheme for the corner point $(l, m-l)$

$(0, \min(m, n + (l-n)^+))$, $(l, m-l)$ and $(m-l, l)$. We are going to show how to achieve the point $(l, m-l)$. The encoding scheme to achieve this corner point is shown in Figure 3.22. The set of m bits at transmitter Tx_1 is partitioned into 3 regions A_1, A_2 and A_3 with respective sizes $n, m-2n$ and n. A similar partition is done at transmitter 2.

In the first time slot, transmitter 1 transmits a total of l fresh bits, n of which are encoded in A_1, and $(l-n)$ of which are encoding in (A_2, A_3). Note $|E_{11}| = l - n$. For the second transmitter, there will be two sub-cases. Consider the first sub-case when $m - l \geq n$. In this sub-case, in the first time slot, n fresh information bits are encoded in the region B_1, and F_{11} bits are encoded in the next region as shown in Figure 3.22. Note $|F_{11}| = m - l - n$. In the second sub-case, $m - l < n$, there is a slight difference from the first sub-case. In this sub-case, only $m - l$ bits are encoded in the region B_1.

Operations in subsequent time slots between these two sub-case are similar; thus we only discuss the first sub-case in detail here. Via the feedback link, transmitter 2 can recover E_{11} easily. Notice $m - l < l$, thus transmitter 1 can recover F_{11} too with a feedback level of l bits.

In the second time slot, besides encoding l fresh bits, transmitter 1 relays F_{11} bits in the region A_2. Besides encoding $m-l$ fresh bits, transmitter 2 relays E_{11} . Due to the very strong interference link, receiver 1 can receive E_{11} cleanly, and receiver 2 can receive F_{11} cleanly. Similar operations are done in the subsequent time. By this approach, we can achieve the corner point $(l, m - l)$.

Due to symmetry, the encoding scheme for the last point is similar to the above.

3.6.4 Proof of Theorem 3.2

Lemma 3.2 is used in the proof of Theorem 3.2. We are going to upper-bound the mutual informations in Lemma 3.2 and simplify them. We have

$$h(Y_1|X_2) - h(Z_1) \leq \frac{1}{2}\log[\mathrm{SNR}(1-\rho^2)+1] \tag{3.167}$$

$$\leq \frac{1}{2}\log(\mathrm{SNR}+1). \tag{3.168}$$

We can also show that

$$h(Y_{F_2}|Y_1, X_2) - h(\tilde{Z}_2)$$

$$= h(g_2(h_{12}X_1 + Z_2) + \tilde{Z}_2|h_{11}X_1 + Z_1, X_2) - h(\tilde{Z}_2) \tag{3.169}$$

$$\leq h(g_2(h_{12}X_1 + Z_2) + \tilde{Z}_2 | h_{11}X_1 + Z_1) - h(\tilde{Z}_2) \tag{3.170}$$

$$\leq \frac{1}{2}\log\left(\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\sqrt{\mathrm{SNR.INR}}+1}\cdot(\mathrm{SNR}+\mathrm{INR}+1)}{\mathrm{SNR}+1}+1\right) \tag{3.171}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1}+1\right). \tag{3.172}$$

From equations (3.84), (3.168) and (3.172), we can prove the validity of the bound (3.72).

Next, we are going to prove the bound (3.75).

Case 1: $\frac{1}{2} \leq \alpha_G < 1$

We have

$$h(X_{top1G}|S_{2G}) - h(Z_2) \leq h(X_{top1G}) - h(Z_2) \tag{3.173}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right). \tag{3.174}$$

Next, we have

$$h(Y_2|S_{2G}, X_{top1G}) - h(Z_2) \tag{3.175}$$

$$= h\left(h_{21}X_1 + h_{22}X_2 + Z_2 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(Z_2) \tag{3.176}$$

$$= \log\sqrt{\frac{\mathrm{SNR}}{\mathrm{INR}}} - h(Z_2)$$
$$+ h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \sqrt{\mathrm{INR}}X_2 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) \tag{3.177}$$

$$= \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 - Z_1 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right)$$
$$- h(Z_2) \tag{3.178}$$

$$\leq \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 - Z_1 \middle| \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(Z_2) \tag{3.179}$$

$$= \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{1}{2}\log\left(\frac{\frac{\mathrm{INR}^3}{\mathrm{SNR}^2} + 2\frac{\mathrm{INR}^2}{\mathrm{SNR}} + 1 - 2\frac{\mathrm{INR}^2\sqrt{\mathrm{INR}}}{\mathrm{SNR}\sqrt{\mathrm{SNR}}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}} + 1}\right) \tag{3.180}$$

$$\leq \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{1}{2}\log 3. \tag{3.181}$$

Next, we have

$$h(Y_{F_2}|X_2, X_{top1G}) - h(\tilde{Z}_2)$$

$$= h\left(g_2(h_{12}X_1 + Z_2) + \tilde{Z}_2 \middle| X_2, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(\tilde{Z}_2) \tag{3.182}$$

$$\leq h\left(g_2\sqrt{\mathrm{INR}}\cdot X_1 + g_2 Z_2 + \tilde{Z}_2 \middle| \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(\tilde{Z}_2) \tag{3.183}$$

$$\leq \frac{1}{2}\log\left(g_2^2\frac{\mathrm{INR} + \frac{\mathrm{INR}^2}{\mathrm{SNR}} - 2\frac{\mathrm{INR}\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}} + 1} + 1\right) \tag{3.184}$$

$$\leq \frac{1}{2}\log\left(g_2^2\frac{\mathrm{INR} + \frac{\mathrm{INR}^2}{\mathrm{SNR}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}}} + 1\right) \tag{3.185}$$

$$= \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR} + \mathrm{INR} + 2\sqrt{\mathrm{SNRINR}} + 1}\cdot\frac{\mathrm{SNR} + \mathrm{INR}}{\mathrm{INR}} + 1\right) \tag{3.186}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}} + 1\right) \tag{3.187}$$

Combining equations (3.85), (3.174),(3.181) and (3.187), and using symmetry, we have proved the first half of the bound (3.75).

Case 2: $\alpha_G \notin [\frac{1}{2}, 1]$

In this case, due to the definition of X_{topjG}, the equation (3.85) is equivalent to

$$R_1 + R_2 \leq h(Y_{F_2}|X_2) + h(Y_2|S_{2G}) + h(Y_{F_1}|X_1) + h(Y_1|S_{1G})$$
$$- h(\tilde{Z}_2) - h(\tilde{Z}_1) - h(Z_2) - h(Z_1). \tag{3.188}$$

Next, we have

$$h(Y_{F_2}|X_2) - h(\tilde{Z}_2)$$
$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}\cdot[\mathrm{INR}(1-\rho^2)+1]+1\right) \tag{3.189}$$
$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+1}\cdot(\mathrm{INR}+1)+1\right). \tag{3.190}$$

Next, we have

$$h(Y_1|S_{1G}) - h(Z_1) \tag{3.191}$$
$$= \frac{1}{2}\log\left([\mathrm{INR}^2(1-\rho^2)+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1]\cdot\frac{1}{\mathrm{INR}+1}\right) \tag{3.192}$$
$$\leq \frac{1}{2}\log\left([\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1]\cdot\frac{1}{\mathrm{INR}+1}\right). \tag{3.193}$$

Combining equations (3.188), (3.190), and (3.193), and using symmetry, we have proved the last half of the bound (3.75).

Next, we are going to prove the validity of the bound (3.76).

We have

$$h(Y_1|X_2, S_{1G}) - h(Z_1) \leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) \tag{3.194}$$
$$h(Y_1) - h(Z_1) \leq \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). \tag{3.195}$$

Case 1: $\frac{1}{2} \leq \alpha_G < 1$

Combining equations (3.86), (3.194), (3.195), (3.174), (3.181) and (3.187), we have proved the bound (3.76) for this case.

Case 2: $\alpha_G \notin [\frac{1}{2}, 1]$

Combining equations (3.86), (3.194), (3.195), (3.188), (3.190) and (3.193), we have proved the bound (3.76) for this remaining case.

■ 3.6.5 Proof of Lemma 3.2

The proof of the bound (3.84) in Lemma 3.2 is trivial. In this subsection, we will only prove the bound (3.86). The proof of the bound (3.85) contains no new ideas and can be proved similarly to the proof in Theorem 3.1 and the proof of (3.86). Before proving the bound of (3.86), we need to prove a lemma.

Lemma 3.4.

$$\begin{aligned} I(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) \\ \leq \sum_{i=1}^{T} [h(X_{top1G,i}|S_{2G,i}) - h(Z_{2i}) \\ + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_{F_1}^T|W_1), \end{aligned} \tag{3.196}$$

$$\begin{aligned} I(S_{1G}^T, X_{top2G}^T, Y_{F_1}^T, W_1; Y_{F_2}^T, W_2) \\ \leq \sum_{i=1}^{T} [h(X_{top2G,i}|S_{1G,i}) - h(Z_{1i}) \\ + h(Y_{F_1 i}|X_{1i}, X_{top2G,i}) - h(\tilde{Z}_{2i}) - h(\tilde{Z}_{1i})] + h(Y_{F_2}^T|W_2). \end{aligned} \tag{3.197}$$

Proof.

$$I(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1)$$

$$= I(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T; Y_{F_1}^T, W_1|W_2) + I(W_2; Y_{F_1}^T, W_1)$$

$$
\begin{aligned}
&= h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|W_2) - h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|Y_{F_1}^T, W_1, W_2) \\
&\quad + h(Y_{F_1}^T|W_1) + h(W_1) - h(W_1|W_2) - h(Y_{F_1}^T|W_1, W_2) \\
&\overset{(a)}{=} h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|W_2) - h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|Y_{F_1}^T, W_1, W_2) \\
&\quad + h(Y_{F_1}^T|W_1) - h(Y_{F_1}^T|W_1, W_2) \\
&= h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|W_2) - h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, Y_{F_1}^T|W_1, W_2) + h(Y_{F_1}^T|W_1)
\end{aligned} \tag{3.198}
$$

where

(a) follows from the fact that W_1 and W_2 are independent.

Next we have an upper bound on the first term of equation (3.198).

$$
\begin{aligned}
&h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T|W_2) \\
&= \sum_{i=1}^{T} [h(S_{2G,i}, X_{top1G,i}, Y_{F_2 i}|S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, W_2)] \\
&\overset{(a)}{=} \sum_{i=1}^{T} [h(S_{2G,i} X_{top1G,i}, Y_{F_2 i}|X_{2i}, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, W_2)] \\
&\overset{(b)}{\leq} \sum_{i=1}^{T} [h(S_{2G,i}, X_{top1G,i}, Y_{F_2 i}|X_{2i})] \\
&= \sum_{i=1}^{T} [h(S_{2G,i}|X_{2i}) + h(X_{top1G,i}|X_{2i}, S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, S_{2G,i}, X_{top1G,i})] \\
&\leq \sum_{i=1}^{T} [h(Z_{1i}|X_{2i}) + h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i})] \\
&\overset{(c)}{=} \sum_{i=1}^{T} [h(Z_{1i}) + h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i})]
\end{aligned} \tag{3.199}
$$

where

(a) follows from the fact that X_{2i} is a function of $(Y_{F_2}^{i-1}, W_2)$;

(b) follows from the fact that more conditioning reduces the entropy; and

(c) follows from the fact that Z_{1i} is independent of X_{2i}.

Next we manipulate the second term of equation (3.198).

$$
\begin{aligned}
& h(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, Y_{F_1}^T | W_1, W_2) \\
& = \sum_{i=1}^{T} [h(S_{2G,i}, X_{top1G,i}, Y_{F_2,i}, Y_{F_1,i} | W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1})] \\
& \overset{(a)}{=} \sum_{i=1}^{T} [h(S_{2G,i}, X_{top1G,i}, Y_{F_2,i}, Y_{F_1,i} | W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i})] \\
& = \sum_{i=1}^{T} [h(Z_{1,i}, Z_{2,i}, Y_{F_2,i}, Y_{F_1,i} | W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i})] \\
& \overset{(b)}{=} \sum_{i=1}^{T} [h(Z_{1,i}) + h(Z_{2,i}) \\
& \quad + h(Y_{F_2,i}, Y_{F_1,i} | W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i}, Z_{1,i}, Z_{2,i})] \\
& = \sum_{i=1}^{T} [h(Z_{1,i}) + h(Z_{2,i}) \\
& \quad + h(\tilde{Z}_{2,i}, \tilde{Z}_{1,i} | W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i}, Z_{1,i}, Z_{2,i})] \\
& \overset{(c)}{=} \sum_{i=1}^{T} [h(Z_{1,i}) + h(Z_{2,i}) + h(\tilde{Z}_{2,i}) + h(\tilde{Z}_{1,i})]
\end{aligned}
\tag{3.200}
$$

where

(a) follows from the fact that X_{ji} is a function of $(Y_{F_j}^{i-1}, W_j)$, for $j = 1, 2$;

(b) follows from the fact that Z_{1i} and Z_{2i} are independent of each other and independent of $(W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i})$; and

(c) follows from the fact that $\tilde{Z}_{1i}$ and $\tilde{Z}_{2i}$ are independent of each other and independent of $(W_1, W_2, S_{2G}^{i-1}, X_{top1G}^{i-1}, Y_{F_2}^{i-1}, Y_{F_1}^{i-1}, X_{1i}, X_{2i}, Z_{1,i}, Z_{2,i})$.

Combining (3.198), (3.199) and (3.200), we have proved the lemma. □

Now, we are going to prove the bound (3.86). We have

$$
\begin{aligned}
&T(2R_1 + R_2 - p_e^T) \\
&\leq 2I(W_1; Y_1^T) + I(W_2; Y_2^T) \\
&\leq I(W_1; Y_1^T, Y_{F_1}^T) + I(W_1; Y_1^T, Y_{F_2}^T | W_2) + I(W_2; Y_2^T, Y_{F_2}^T) \\
&= h(Y_1^T) + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_1^T | Y_{F_1}^T, W_1) \\
&\quad + h(Y_{F_2}^T | W_2) - h(Y_{F_2}^T | W_2, W_1) + I(W_1; Y_1^T | Y_{F_2}^T, W_2) \\
&\quad + h(Y_2^T) + h(Y_{F_2}^T | Y_2^T) - h(Y_{F_2}^T | W_2) - h(Y_2^T | Y_{F_2}^T, W_2) \\
&= h(Y_1^T) - h(Y_1^T | Y_{F_1}^T, W_1) + I(W_1; Y_1^T | Y_{F_2}^T, W_2) + h(Y_2^T) - h(Y_2^T | Y_{F_2}^T, W_2) \\
&\quad + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T) \\
&= h(Y_1^T) - h(S_{2G}^T | Y_{F_1}^T, W_1) + I(W_1; Y_1^T | Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T | Y_{F_2}^T, W_2) \\
&\quad + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T) \\
&\overset{(a)}{=} h(Y_1^T) - h(S_{2G}^T, Z_2^T | Y_{F_1}^T, W_1, X_1^T) + h(Z_2^T | Y_{F_1}^T, W_1, X_1^T S_{2G}^T) \\
&\quad + I(W_1; Y_1^T | Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T | Y_{F_2}^T, W_2) \\
&\quad + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)
\end{aligned}
$$

$$\stackrel{(b)}{\leq} h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T | Y_{F_1}^T, W_1)$$

$$+ I(W_1; Y_1^T | Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T | Y_{F_2}^T, W_2)$$

$$+ h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)$$

$$\stackrel{(c)}{\leq} h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T | Y_{F_1}^T, W_1)$$

$$+ I(W_1; Y_1^T, S_{1G}^T | Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T | Y_{F_2}^T, W_2)$$

$$+ h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T, | Y_2^T)$$

$$= h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T | Y_{F_1}^T, W_1) + h(Y_1^T, S_{1G}^T | Y_{F_2}^T, W_2)$$

$$- h(Y_1^T, S_{1G}^T | Y_{F_2}^T, W_2, W_1) + h(Y_2^T)$$

$$- h(S_{1G}^T | Y_{F_2}^T, W_2) + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)$$

$$\stackrel{(d)}{\leq} h(Y_2^T) - h(S_{2G}^T, X_{top1G}^T | Y_{F_1}^T, W_1) + h(Y_1^T | S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - h(Z_1^T, Z_2^T)$$

$$+ h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)$$

$$\stackrel{(e)}{\leq} h(Y_2^T) + [I(S_{2G}^T, X_{top1G}^T; Y_{F_1}^T, W_1) - h(S_{2G}^T, X_{top1G}^T)]$$

$$+ [h(S_{2G}^T, X_{top1G}^T | Y_2^T) - h(S_{2G}^T, X_{top1G}^T | Y_2^T, X_2^T, X_1^T)]$$

$$+ h(Y_1^T | S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - h(Z_1^T) - h(Z_2^T)$$

$$+ h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1) - h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)$$

$$= I(S_{2G}^T, X_{top1G}^T; Y_{F_1}^T, W_1) + h(Y_2^T | S_{2G}^T, X_{top1G}^T)$$

$$- h(Z_1^T, Z_2^T | Y_2^T, X_2^T, X_1^T) + h(Y_1^T | S_{1G}^T, Y_{F_2}^T, W_2)$$

$$+ h(Y_1^T) - h(Z_1^T) - h(Z_2^T) + h(Y_{F_1}^T | Y_1^T) - h(Y_{F_1}^T | W_1)$$

$$- h(Y_{F_2}^T | W_2, W_1) + h(Y_{F_2}^T | Y_2^T)$$

$$\overset{(f)}{\leq} I(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) + h(Y_2^T|S_{2G}^T, X_{top1G}^T)$$

$$+ h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T)$$

$$+ h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)$$

$$\overset{(g)}{\leq} \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(Z_{2i})$$

$$- h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_{F_1}^T|W_1)$$

$$+ h(Y_2^T|S_{2G}^T, X_{top1G}^T)$$

$$+ h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T)$$

$$+ h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)$$

$$\overset{(h)}{\leq} \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2Gi}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(Z_{2i})$$

$$- h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_2^T|S_{2G}^T, X_{top1G}^T)$$

$$+ h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T)$$

$$+ h(\tilde{Z}_1^T|Y_1^T) - h(Y_{F_2}^T|W_2, W_1, Y_2^T) + h(\tilde{Z}_2^T|Y_2^T)$$

$$= \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2Gi}) + h(Y_{F_2 i}|X_{2,i}, X_{top1G,i}) - 2h(Z_{1i}) - 2h(Z_{2i})]$$

$$+ h(Y_2^T|S_{2G}^T, X_{top1G}^T) + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2, X_2^T)$$

$$+ h(Y_1^T) - h(\tilde{Z}_2^T|W_2, W_1, Y_2^T)$$

$$\leq \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i})$$

$$+ h(Y_{2i}|S_{2G,i}, X_{top1G,i}) + h(Y_{1i}|S_{1G,i}, X_{2i}) + h(Y_{1i})$$

$$- 2h(Z_{1i}) - 2h(Z_{2i}) - h(\tilde{Z}_{2i})]$$

where

(a) follows from the fact that X_1^T is a function of $(Y_{F_1}^T, W_1)$;

(b) follows from the fact X_{top1G}^T is a function of Z_2^T and X_1^T, which is in turn a function of $(Y_{F_1}^T, W_1)$;

(c) follows from the fact, more side information increases the mutual information;

(d) follows from

$$\begin{aligned} h(Y_1^T, S_{1G}^T | Y_{F_2}^T, W_2, W_1) &\geq h(Y_1^T, S_{1G}^T | Y_{F_2}^T, W_2, W_1, X_1^T) \\ &= h(Z_1^T, Z_2^T); \end{aligned}$$

(e) follows from the fact more conditioning reduces the entropy;

(f) follows from the fact, more side information increases the mutual information;

(g) follows from utilization of Lemma 3.4; and

(h) follows from the fact more conditioning reduces the entropy.

■ 3.6.6 Proof of Theorem 3.4

The strategy to prove this theorem is that we need to carefully choose the right power allocations P_p, P_{nc}, P_{cc} such that the achievable rate region approximates the capacity region within a constant gap.

When $\mathrm{INR} < 1$, by treating interference as noise and not using any feedback, each receiver can achieve a rate of

$$\frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right). \tag{3.201}$$

We have

$$\begin{aligned}
&\psi_1 - \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) \\
&= \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) - \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) && (3.202)\\
&\leq \frac{1}{2}\log(3\mathrm{SNR}+3) - \frac{1}{2}\log\left(\frac{1}{2}+\frac{\mathrm{SNR}}{2}\right) && (3.203)\\
&= \frac{1}{2}\log 3 + \frac{1}{2} = 1.3 \text{ bits} && (3.204)
\end{aligned}$$

Thus, $\delta_R \leq 1.3$.

Next, we have

$$\begin{aligned}
\psi_3 - 2\cdot\frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) &= \psi_1 - \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) && (3.205)\\
&\leq \frac{1}{2}\log 3 + \frac{1}{2} = 1.3 \text{ bits}. && (3.206)
\end{aligned}$$

Thus, $\delta_{2R} \leq 1.3$.

Subsequently,

$$\psi_3 + \psi_1 - 3\cdot\frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) \leq 2(\frac{1}{2}\log 3 + \frac{1}{2}) = 2.6 \text{ bits}. \tag{3.207}$$

Thus, $\delta_{3R} \leq 2.6$.

Therefore, we have $\delta = 1.3$ and the outer bounds in Theorem 3.2 is within 1.3 bits/s/Hz away from the achievable rate region. Thus, our main focus in this subsection now is to quantify the gaps for $\mathrm{INR} \geq 1$.

Notice that Theorem 3.3 holds for all ρ, satisfying $0 \leq \rho \leq 1$. Therefore, in all the power allocations below, we simply choose $\rho = 0$ to obtain inner bounds that are easily to be dealt with. In the outer bounds, we cannot choose ρ, therefore, all outer bounds still involves ρ.

- **Case 1**: $1 \leq \alpha_G$

  - Sub-case 1.1: $\mathrm{SNR}_F < \mathrm{SNR}$

    Choose $P_p = 0, P_{cc} = 0, P_{nc} = 1$. With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{3.208}$$

$$\tau_5 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) \geq \frac{1}{2}\log\mathrm{INR} \tag{3.209}$$

$$\tau_4 = \frac{1}{2}\log(\mathrm{SNR} + 1) \tag{3.210}$$

$$\tau_3 = \frac{1}{2}\log(\mathrm{INR} + 1) \geq \frac{1}{2}\log\mathrm{INR} \tag{3.211}$$

$$\tau_2 = \tau_1 = 0. \tag{3.212}$$

    Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR} + 1} + 1\right) \tag{3.213}$$

$$\leq \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log(2). \tag{3.214}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \tag{3.215}$$

$$\leq \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right). \tag{3.216}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \tag{3.217}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+5\mathrm{INR}+4}{\mathrm{INR}+1}\right) + \frac{1}{2}\log\left(\mathrm{SNR}_F+1\right) \tag{3.218}$$

$$+ \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \tag{3.219}$$

$$\leq \frac{1}{2}\log\left(\mathrm{INR}+4\right) + \frac{1}{2}\log\left(\mathrm{SNR}_F+1\right) + \frac{1}{2} + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right). \tag{3.220}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = 0 \tag{3.221}$$

$$\psi_2 - (\tau_4+\tau_1) \leq \frac{1}{2} \tag{3.222}$$

$$\psi_1 - (\tau_1+\tau_2+\tau_3) \leq \frac{1}{2}\log(5\mathrm{INR}) - \frac{1}{2}\log(\mathrm{INR}) \tag{3.223}$$

$$= \frac{1}{2}\log(5) = 1.2. \tag{3.224}$$

Thus, $\delta_R \leq 1.2$.

Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{1}{2}, \tag{3.225}$$

$$\psi_3 - (2\tau_1 + \tau_2 + \tau_5) \leq \frac{1}{2} + \frac{1}{2}\log(5\mathrm{INR}) - \frac{1}{2}\log(\mathrm{INR}) \tag{3.226}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{3.227}$$

$$\psi_3 - (2\tau_1 + 2\tau_3) \leq \frac{1}{2} + \frac{1}{2}\log(5\mathrm{INR}) - \frac{1}{2}\log(\mathrm{INR}^2) \tag{3.228}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{3.229}$$

Thus, $\delta_{2R} \leq 1.7$.

Next, we have

$$(\psi_1 + \psi_3) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) = (\psi_1 - \tau_6) + (\psi_3 - \tau_3) \tag{3.230}$$

$$\leq \frac{1}{2} + \frac{1}{2}\log(5\mathrm{INR}) - \frac{1}{2}\log(\mathrm{INR}) \tag{3.231}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{3.232}$$

$$(\psi_1 + \psi_3) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5)$$

$$\leq [\frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1)$$

$$+ \frac{1}{2}\log(2) + \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1)] \tag{3.233}$$

$$- [\frac{1}{2}\log(\mathrm{INR}) + \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1)] \tag{3.234}$$

$$\leq [\log 3(\mathrm{SNR} + \mathrm{INR} + 1) + \frac{1}{2}\log(2) + \log(5\mathrm{INR})]$$

$$- [\frac{1}{2}\log(\mathrm{INR}) + \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1)] \tag{3.235}$$

$$= \frac{1}{2}\log(30) = 2.5 \tag{3.236}$$

Thus, $\delta_{3R} \leq 2.5$.

Therefore, with current power allocation, in this sub-case, the achievable region is at most $\delta = 1.2$ bits/s/Hz from the outer bounds.

– Sub-case 1.2: $\mathrm{SNR} \leq \mathrm{SNR}_F \leq \mathrm{INR}$

Choose $P_p = 0, P_{cc} = 1, P_{nc} = 0$. With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{3.237}$$

$$\tau_5 = \tau_4 = \tau_3 = \tau_2 = 0 \tag{3.238}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\sqrt{\mathrm{SNR.INR}}+1}\cdot(\mathrm{INR}+1)+1}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}+1} \tag{3.239}$$

$$\geq \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{5\mathrm{INR}}\cdot\mathrm{INR}}{2} \tag{3.240}$$

$$= \frac{1}{2}\log\mathrm{SNR}_F - \frac{1}{2}\log 10 \tag{3.241}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR}+1) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1}+1\right) \tag{3.242}$$

$$\leq \frac{1}{2}\log(3\mathrm{SNR}_F). \tag{3.243}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)$$

$$+\frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) \tag{3.244}$$

$$\leq \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right). \tag{3.245}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = 0 \tag{3.246}$$

$$\psi_2 - (\tau_4 + \tau_1) \leq \frac{1}{2}\log 30 = 2.5 \tag{3.247}$$

$$\psi_2 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log 30 = 2.5. \tag{3.248}$$

Thus, $\delta_R \leq 2.5$.

Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{1}{2}, \tag{3.249}$$

$$2\psi_2 - (2\tau_1 + \tau_2 + \tau_5) \leq \log(30) = 4.9 \tag{3.250}$$

$$2\psi_2 - (2\tau_1 + 2\tau_3) \leq \log(30) = 4.9 \tag{3.251}$$

Thus, $\delta_{2R} \leq 4.9$.

Next, we have

$$(\psi_2 + \psi_3) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log(60) = 3.0 \tag{3.252}$$

$$(3\psi_2) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log(27000) = 7.4 \tag{3.253}$$

Thus, $\delta_{3R} \leq 7.4$.

We have $\delta = 2.5$. Therefore, with current power allocation, in this sub-case, the achievable region is at most 2.5 bits/s/Hz from the outer bounds.

- **Case 2**: $\frac{1}{2} \leq \alpha_G \leq 1$

– Sub-case 2.1: $\text{SNR}_F \leq \text{INR}$

Choose $P_p = \frac{1}{\text{INR}}, P_{nc} = 1 - P_p, P_{cc} = 0$. With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) - \frac{1}{2} \tag{3.254}$$

$$\tau_5 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 1) - \frac{1}{2} \tag{3.255}$$

$$\tau_4 = \frac{1}{2}\log(\text{SNR} + 2) - \frac{1}{2} \geq \frac{1}{2}\log(\text{SNR} + 1) - \frac{1}{2} \tag{3.256}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1\right) - \frac{1}{2} \geq \frac{1}{2}\log(\text{INR}) - \frac{1}{2} \tag{3.257}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + 2\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) - \frac{1}{2} \tag{3.258}$$

$$\tau_1 = 0. \tag{3.259}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\text{SNR} + 1) + \frac{1}{2}\log\left(\frac{\text{SNR}_F}{\text{SNR} + 1} + 1\right) \tag{3.260}$$

$$\leq \frac{1}{2}\log(\text{SNR} + 1) + \frac{1}{2}\log(2). \tag{3.261}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR} + 1} + 1\right) + \log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) \tag{3.262}$$

$$\leq \frac{1}{2}\log\left(2\frac{\text{SNR}}{\text{INR}}\right) + \log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1). \tag{3.263}$$

$$\psi_4 = \log\left(\frac{\text{INR}^2}{\text{SNR}} + 1\right) + \log\left(\frac{\text{SNR}_F}{\text{INR}} + 1\right) + \log\left(\frac{\text{SNR}}{\text{INR}}\right) + \log 3 \tag{3.264}$$

$$\leq \log\left(\frac{2\text{INR}^2}{\text{SNR}}\right) + \log(2) + \log\left(\frac{\text{SNR}}{\text{INR}}\right) + \log 3 \tag{3.265}$$

$$= \log 12 + \log \text{INR} \tag{3.266}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)$$
$$+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3$$
$$+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)$$
$$+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{3.267}$$
$$\leq \frac{1}{2}\log\left(\frac{2\mathrm{INR}^2}{\mathrm{SNR}}\right)+\frac{1}{2}\log(2)$$
$$+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3 \tag{3.268}$$
$$+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)$$
$$+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{3.269}$$
$$\overset{(a)}{=} \frac{1}{2}\log 24+\frac{1}{2}\log\mathrm{SNR}$$
$$+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{3.270}$$
$$\overset{(b)}{\leq} \frac{1}{2}\log 24+\frac{1}{2}\log\mathrm{SNR}+\frac{1}{2}\log 3(\mathrm{SNR}+\mathrm{INR}+1) \tag{3.271}$$

where, depending on our need, we use either the form (a) or the form (b).

Now, the gap can be quantified easily.

$$\psi_1-\tau_6=\frac{1}{2} \tag{3.272}$$

$$\psi_2-(\tau_4+\tau_1)\leq 1 \tag{3.273}$$

$$\psi_2-(\tau_1+\tau_2+\tau_3)\leq\left(\frac{1}{2}\log(2\mathrm{SNR})+\frac{1}{2}\right)$$
$$-\left(\frac{1}{2}\log(\mathrm{INR})-\frac{1}{2}+\frac{1}{2}\log(\frac{\mathrm{SNR}}{\mathrm{INR}})-\frac{1}{2}\right) \tag{3.274}$$
$$=\frac{1}{2}\log(16)=2. \tag{3.275}$$

Thus, $\delta_R \leq 2$.

Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{3}{2}, \tag{3.276}$$

$$\begin{aligned}\psi_3 - (2\tau_1 + \tau_2 + \tau_5) &\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right) \\ &\quad + \log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \\ &\quad - \left(\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)\right. \\ &\quad \left. - \frac{1}{2} + \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) - \frac{1}{2}\right) \qquad (3.277) \\ &\leq \frac{1}{2}\log(2) + \log 3(\mathrm{SNR} + \mathrm{INR} + 1) \\ &\quad - \left(-\frac{1}{2} + \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) - \frac{1}{2}\right) \qquad (3.278) \\ &= \frac{1}{2}\log(24) = 2.3 \qquad (3.279)\end{aligned}$$

$$\begin{aligned}\psi_4 - (2\tau_1 + 2\tau_3) &\leq [\log 12 + \log(\mathrm{INR})] - [\log(\mathrm{INR}) - 1] \qquad (3.280) \\ &= \log(24) = 4.6 \qquad (3.281)\end{aligned}$$

Thus, $\delta_{2R} \leq 4.6$.

Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log 24 + \frac{3}{2} = 3.8 \tag{3.282}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log 72 + \frac{3}{2} = 4.6 \tag{3.283}$$

Thus, $\delta_{3R} \leq 4.6$.

Therefore, with current power allocation, in this sub-case, We have $\delta = 2.3$ and the achievable region is at most 2.3 bits/s/Hz from the outer bounds.

– Sub-case 2.2: $\mathrm{INR} \leq \mathrm{SNR}_F \leq \mathrm{SNR}$

Choose $P_p = \frac{1}{\mathrm{INR}}, P_{nc} = \frac{\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} - P_p, P_{cc} = \frac{\mathrm{INR}\cdot\mathrm{SNR}_F}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}$.

With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR.INR}}+1) - \frac{1}{2} \tag{3.284}$$

$$\tau_5 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} + \frac{\mathrm{INR.SNR}}{\mathrm{INR.SNR}_F+\mathrm{SNR}} + 1\right) - \frac{1}{2} \tag{3.285}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right) - \frac{1}{2} \tag{3.286}$$

$$\tau_4 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} + 2\right) - \frac{1}{2} \tag{3.287}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right) - \frac{1}{2} \tag{3.288}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{\mathrm{INR.SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} + 1\right) - \frac{1}{2} \tag{3.289}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{3.290}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + 2\right) - \frac{1}{2} \tag{3.291}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{3.292}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\mathrm{INR}+1)+1}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+1)+1} \tag{3.293}$$

$$\geq \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{5\mathrm{SNR}}(\mathrm{INR})}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F}+1)+1} \tag{3.294}$$

$$\geq \frac{1}{2}\log\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\mathrm{INR}) - \frac{1}{2}\log 15 \tag{3.295}$$

Next, we simplify some outer bounds first.

$$
\begin{aligned}
\psi_3 &= \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) && (3.296)\\
&\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). && (3.297)
\end{aligned}
$$

$$
\begin{aligned}
\psi_4 &= \log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right)+\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 && (3.298)\\
&\leq \log\left(\frac{2\mathrm{INR}^2}{\mathrm{SNR}}\right)+\log\left(\frac{2\mathrm{SNR}_F}{\mathrm{INR}}\right)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 && (3.299)\\
&= \log 12+\log\mathrm{SNR}_F && (3.300)
\end{aligned}
$$

$$
\begin{aligned}
\psi_5 &= \frac{1}{2}\log\left(\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)\\
&\quad+\frac{1}{2}\log 3+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) && (3.301)\\
&\leq \frac{1}{2}\log\left(\frac{2\mathrm{INR}^2}{\mathrm{SNR}}\right)+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}_F}{\mathrm{INR}}\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)\\
&\quad+\frac{1}{2}\log 3+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) && (3.302)\\
&= \frac{1}{2}\log 24+\frac{1}{2}\log\frac{\mathrm{SNR}\cdot\mathrm{SNR}_F}{\mathrm{INR}}\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) && (3.303)
\end{aligned}
$$

Now, the gap can be quantified easily.

$$\psi_1-\tau_6=\frac{1}{2} \tag{3.304}$$

$$\psi_1 - (\tau_4 + \tau_1) \le \frac{1}{2}\log(5\text{SNR}) - \left[\frac{1}{2}\log\frac{\text{SNR}_F}{\text{SNR}}(\text{INR}) - \frac{1}{2}\log 15 + \frac{1}{2}\log(\frac{\text{SNR}^2}{2\text{INR}\cdot\text{SNR}_F}) - \frac{1}{2}\right] \tag{3.305}$$

$$= \frac{1}{2}\log(150) + \frac{1}{2} = 4.1 \tag{3.306}$$

$$\psi_1 - (\tau_1 + \tau_2 + \tau_3) \le \frac{1}{2}\log(75) + 1 = 4.1 \tag{3.307}$$

Thus, $\delta_R \le 4.1$.

Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \le \frac{3}{2} \tag{3.308}$$

$$\psi_4 - (2\tau_1 + \tau_2 + \tau_5) = \log(180) + 1 = 8.5 \tag{3.309}$$

$$\psi_4 - (2\tau_1 + 2\tau_3) \le \log(180) + 1 = 8.5 \tag{3.310}$$

Thus, $\delta_{2R} \le 8.5$.

Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \le \frac{1}{2}\log 360 + \frac{3}{2} = 5.7 \tag{3.311}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \le \frac{1}{2}\log 405000 + \frac{3}{2} = 10.8 \tag{3.312}$$

Thus, $\delta_{3R} \le 10.8$.

Therefore, with current power allocation, in this sub-case, We have $\delta = 4.3$ and the achievable region is at most 4.3 bits/s/Hz from the outer bounds.

- **Case 3**: $0 \le \alpha_G \le \frac{1}{2}$

– Sub-case 3.1: $\mathrm{SNR}_F \leq \frac{\mathrm{SNR}}{\mathrm{INR}}$

Choose $P_p = \frac{1}{\mathrm{INR}}, P_{nc} = 1 - P_p, P_{cc} = 0$. With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) - \frac{1}{2} \tag{3.313}$$

$$\tau_5 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) - \frac{1}{2} \tag{3.314}$$

$$\tau_4 = \frac{1}{2}\log(\mathrm{SNR} + 2) - \frac{1}{2} \geq \frac{1}{2}\log(\mathrm{SNR} + 1) - \frac{1}{2} \tag{3.315}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + \mathrm{INR} + 1\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{3.316}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + 2\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{3.317}$$

$$\tau_1 = 0. \tag{3.318}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR} + 1} + 1\right) \tag{3.319}$$

$$\leq \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log(2). \tag{3.320}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR} + 1} + 1\right) + \log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{3.321}$$

$$\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1). \tag{3.322}$$

$$\psi_4 = \log\left(\frac{\mathrm{INR}^2 + \mathrm{SNR} + 2\mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1}{\mathrm{INR} + 1}\right)$$

$$+ \log\left(\frac{\mathrm{SNR}_F(\mathrm{INR} + 1)}{\mathrm{SNR} + \mathrm{INR} + 1} + 1\right) \tag{3.323}$$

$$\leq \log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right) + \log(3) = \log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(21) \tag{3.324}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2 + \mathrm{SNR} + 2\mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1}{\mathrm{INR}+1}\right)$$
$$+ \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1} + 1\right)$$
$$+ \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1} + 1\right)$$
$$+ \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{3.325}$$
$$\leq \frac{1}{2}\log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right) + \frac{1}{2}\log(3)$$
$$+ \frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)$$
$$+ \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{3.326}$$
$$\leq \frac{1}{2}\log 42 + \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}^2}\right)$$
$$+ \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1)$$
$$\leq \frac{1}{2}\log 42 + \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}^2}\right) + \frac{1}{2}\log 3(\mathrm{SNR}+\mathrm{INR}+1) \tag{3.327}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = \frac{1}{2} \tag{3.328}$$

$$\psi_2 - (\tau_4 + \tau_1) \leq 1 \tag{3.329}$$

$$\psi_2 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log(16) = 2. \tag{3.330}$$

Thus, $\delta_R \leq 2$.

Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{3}{2}, \tag{3.331}$$

$$\psi_3 - (2\tau_1 + \tau_2 + \tau_5) \leq \frac{1}{2}\log(24) = 2.3 \tag{3.332}$$

$$\psi_4 - (2\tau_1 + 2\tau_3) \leq \log(21) + 1 = 5.4 \tag{3.333}$$

Thus, $\delta_{2R} \leq 5.4$.

Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log 42 + \frac{3}{2} = 4.2 \tag{3.334}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log 126 + \frac{3}{2} = 5.0 \tag{3.335}$$

$$\tag{3.336}$$

Thus, $\delta_{3R} \leq 5.0$.

Therefore, with current power allocation, in this sub-case, We have $\delta = 2.7$ and the achievable region is at most 2.7 bits/s/Hz from the outer bounds.

– Sub-case 3.2: $\frac{\text{SNR}}{\text{INR}} \leq \text{SNR}_F \leq \text{SNR}$

Choose $P_p = \frac{1}{\text{INR}}, P_{nc} = \frac{\text{SNR}}{\text{INR}\cdot\text{SNR}_F + \text{SNR}} - P_p, P_{cc} = \frac{\text{INR}\cdot\text{SNR}_F}{\text{INR}\cdot\text{SNR}_F + \text{SNR}}$.

With this power allocation, from Theorem 3.3 we have

$$\tau_6 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) - \frac{1}{2} \tag{3.337}$$

$$\tau_5 = \frac{1}{2}\log\left(\frac{\text{SNR}^2}{\text{INR.SNR}_F + \text{SNR}} + \frac{\text{INR}\cdot\text{SNR}}{\text{INR.SNR}_F + \text{SNR}} + 1\right) - \frac{1}{2} \tag{3.338}$$

$$\geq \frac{1}{2}\log\left(\frac{\text{SNR}^2}{2\text{INR}\cdot\text{SNR}_F}\right) - \frac{1}{2} \tag{3.339}$$

$$\tau_4 = \frac{1}{2}\log\left(\frac{\text{SNR}^2}{\text{INR}\cdot\text{SNR}_F + \text{SNR}} + 2\right) - \frac{1}{2} \tag{3.340}$$

$$\geq \frac{1}{2}\log\left(\frac{\text{SNR}^2}{2\text{INR}\cdot\text{SNR}_F}\right) - \frac{1}{2} \tag{3.341}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + \frac{\text{INR}\cdot\text{SNR}}{\text{INR.SNR}_F + \text{SNR}} + 1\right) - \frac{1}{2} \tag{3.342}$$

$$\geq \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) - \frac{1}{2} \tag{3.343}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + 2\right) - \frac{1}{2} \tag{3.344}$$

$$\geq \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) - \frac{1}{2} \tag{3.345}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\text{SNR}_F}{\text{SNR}+\text{INR}\sqrt{\text{SNR}\cdot\text{INR}+1}}(\text{INR}+1)+1}{\frac{\text{SNR}_F}{\text{SNR}+\text{INR}\sqrt{\text{SNR}\cdot\text{INR}+1}}(\frac{\text{INR}\cdot\text{SNR}}{\text{INR}\cdot\text{SNR}_F+\text{SNR}}+1)+1} \tag{3.346}$$

$$\geq \frac{1}{2}\log\frac{\frac{\text{SNR}_F}{5\text{SNR}}(\text{INR})}{\frac{\text{SNR}_F}{\text{SNR}}(\frac{\text{INR}\cdot\text{SNR}}{\text{INR}.\text{SNR}_F}+1)+1} \tag{3.347}$$

$$\geq \frac{1}{2}\log\frac{\text{SNR}_F}{\text{SNR}}(\text{INR}) - \frac{1}{2}\log 15 \tag{3.348}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\text{SNR}+1) + \frac{1}{2}\log\left(\frac{\text{SNR}_F}{\text{SNR}+1}+1\right) \tag{3.349}$$

$$\leq \frac{1}{2}\log(2\text{SNR}) + \frac{1}{2}\log(2). \tag{3.350}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}+1}+1\right) + \log(\text{SNR}+\text{INR}+2\rho\sqrt{\text{SNR}\cdot\text{INR}}+1) \tag{3.351}$$

$$\leq \frac{1}{2}\log\left(2\frac{\text{SNR}}{\text{INR}}\right) + \log(\text{SNR}+\text{INR}+2\rho\sqrt{\text{SNR}\cdot\text{INR}}+1). \tag{3.352}$$

$$\psi_4 = \log\left(\frac{\text{INR}^2+\text{SNR}+2\text{INR}+2\rho\sqrt{\text{SNR}\cdot\text{INR}}+1}{\text{INR}+1}\right)$$

$$+\log\left(\frac{\text{SNR}_F(\text{INR}+1)}{\text{SNR}+\text{INR}+1}+1\right) \tag{3.353}$$

$$\leq \log\left(\frac{7\text{SNR}}{\text{INR}}\right) + \log\left(\frac{3\text{SNR}_F\text{INR}}{\text{SNR}}\right) \tag{3.354}$$

$$\leq \log(\text{SNR}_F) + \log(21) \tag{3.355}$$

$$\begin{aligned}
\psi_5 &= \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right)\\
&\quad+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right)\\
&\quad+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \qquad (3.356)\\
&\le \frac{1}{2}\log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log\left(\frac{3\mathrm{SNR}_F\mathrm{INR}}{\mathrm{SNR}}\right)\\
&\quad+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \qquad (3.357)\\
&\le \frac{1}{2}\log 42+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F\mathrm{SNR}}{\mathrm{INR}}\right)\\
&\quad+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \qquad (3.358)
\end{aligned}$$

Now, the gap can be quantified easily.

$$\psi_1-\tau_6=\frac{1}{2} \qquad (3.359)$$

$$\psi_2-(\tau_4+\tau_1)\le\frac{1}{2}\log(60)+1=4.0 \qquad (3.360)$$

$$\psi_2-(\tau_1+\tau_2+\tau_3)\le\frac{1}{2}\log(60)+1=4.0 \qquad (3.361)$$

Thus, $\delta_R \le 4.0$.

Next, we have

$$\psi_3-(\tau_2+\tau_6)\le\frac{3}{2} \qquad (3.362)$$

$$\psi_3-(2\tau_1+\tau_2+\tau_5)=\frac{1}{2}\log(2250)+\frac{3}{2}=7.1 \qquad (3.363)$$

$$\psi_4-(2\tau_1+2\tau_3)\le\log(315)+1=9.3 \qquad (3.364)$$

Thus, $\delta_{2R} \le 9.3$.

Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log 630 + \frac{3}{2} = 6.2 \tag{3.365}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log 708750 + 2 = 11.7 \tag{3.366}$$

$$\tag{3.367}$$

Thus, $\delta_{3R} \leq 11.7$.

Therefore, with current power allocation, in this sub-case, We have $\delta = 4.7$ the achievable region is at most 4.7 bits/s/Hz from the outer bounds.

In conclusion, we have proved that the outer bounds are at most 4.7 bits/s/Hz from the achievable rate region.

Chapter 4

A Case Where Interference Does Not Affect Dispersion

MANY results in information theory are asymptotic in the sense that the number of channel uses grow without bound. For example, as the number of channel uses n becomes sufficiently large, the maximum data rate that a point-to-point channel can support, with arbitrarily small probability of error, is arbitrarily close to $\max_X I(X;Y)$ bits per channel use, where X denotes the input random variable and Y denotes the output random variable. However, in some applications, communication systems need to operate in short blocklengths due to delay constraints. How do these results change if we require the communication systems to operate at a fixed finite number of blocklengths? It is not easy to answer this question precisely. Recent works have made use of Gaussian approximation to provide approximate answers. In this chapter, using normal approximation, we approximate the maximum data rates that a Gaussian interference channel can support, when it operates in the strictly very strong interference regime, the blocklength is fixed and finite, and the probability of error is allowed to be non-vanishing. It is shown that, in the second-order analysis, the Gaussian interference channel behaves as a pair of independent point-to-point channels in the strictly very strong interference regime. In other words, interference does not affect dispersions of the constituent channels in this special case. This result extends Carleial's result [14].

4.1 Introduction

Recently, the study of second-order coding rates for fixed error probabilities has become an increasingly prominent research topic in network information theory because the analysis provides key insights into the (delay-constrained) performance of the communication systems in the finite blocklength regime [85]. Strassen [101], Hayashi [43], and Polyanskiy, Poor and Verdú [85] characterized the second-order coding rate of the discrete memoryless (DM) point-to-point channel and the additive white Gaussian noise (AWGN) point-to-point channel. The result can be summarized as follows. If $M^*(n, \epsilon, \mathsf{SNR})$ denotes the maximum number of codewords that can be transmitted over n uses of a discrete-time AWGN channel with signal-to-noise ratio SNR and average error probability no larger than $\epsilon \in (0, 1)$, then, it was shown by [85] and [106] that

$$\log M^*(n, \epsilon, \mathsf{SNR}) = n\mathsf{C}(\mathsf{SNR}) + \sqrt{n\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon) + \frac{1}{2}\log n + O(1) \tag{4.1}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian, and the *Gaussian capacity* $\mathsf{C}(\mathsf{SNR})$ and *Gaussian dispersion* functions $\mathsf{V}(\mathsf{SNR})$ are respectively defined as

$$\mathsf{C}(\mathsf{SNR}) \triangleq \frac{1}{2}\log(1 + \mathsf{SNR}) \text{ nats per channel use,} \tag{4.2}$$

and

$$\mathsf{V}(\mathsf{SNR}) \triangleq \frac{\mathsf{SNR}(\mathsf{SNR}+2)}{2(\mathsf{SNR}+1)^2} \text{ nats}^2 \text{ per channel use.} \tag{4.3}$$

The sum of the first two terms of equation (1), $n\mathsf{C}(\mathsf{SNR}) + \sqrt{n\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon)$, is called the *normal approximation* to the logarithm of the size of the optimal codebooks $\log M^*(n, \epsilon, \mathsf{SNR})$. Since it has been shown that the normal approximation is a good proxy to the finite blocklength fundamental limits [85] at moderate blocklengths, the result can be interpreted as follows: If a system designer desires to use a Gaussian communication channel up to n times with a tolerable average error probability not exceeding ϵ, the maximum number of nats of information he can communicate is roughly $n\mathsf{C}(\mathsf{SNR}) + \sqrt{n\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon)$. Thus, for $\epsilon < 0.5$, the backoff from the Shannon limit (Gaussian capacity) is $\sqrt{\mathsf{V}(\mathsf{SNR})/n}\,\Phi^{-1}(1-\epsilon)$ (a positive quantity). The constraint on the blocklength is motivated by real-world, delay-constrained applications such as real-time multimedia streaming. In such applications, the communication data is usually divided into a stream of packets, which have to arrive at their desired destinations within a certain acceptable, and usually short, delay.

The quantities $\mathsf{C}(\mathsf{SNR})$ and $\mathsf{V}(\mathsf{SNR})$ are respectively the expectation and the conditional variance of an appropriately defined information density random variable. These are information-theoretic quantities that characterize the information transmission capability of the channel. In fact, $\mathsf{V}(\mathsf{SNR})$, coined the "dispersion" by Polyanskiy-Poor-Verdú [85], is a channel-dependent quantity that characterizes the speed at which the rates of capacity-achieving codes converge to the Shannon limit. The *second-order coding rate*, a term coined by Hayashi [42, 43], is a different, but related, object. It is the coefficient of the $\sqrt{n}$ term in (4.1), namely $\sqrt{\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon)$. More precisely, the (κ, ϵ)-second-order coding rate $L^*(\kappa, \epsilon) \in \mathbb{R}$ is the maximum L for which there exists a sequence of length-n block codes of sizes M_n and error probabilities asymptotically not exceeding ϵ such that

$$\log M_n \geq n\kappa + \sqrt{n}L + o\big(\sqrt{n}\big). \tag{4.4}$$

If $\kappa < \mathsf{C}(\mathsf{SNR})$, then it can be seen by the direct part of the coding theorem for the AWGN channel that $L^*(\kappa, \epsilon) = \infty$. If the *strong converse* holds (and for the AWGN channel it does [126]), then for all $\kappa > \mathsf{C}(\mathsf{SNR})$, the (κ, ϵ)-second-order coding rate $L^*(\kappa, \epsilon) = -\infty$. Hence, the only non-trivial case is the phase-transition point $\kappa = \mathsf{C}(\mathsf{SNR})$. Hayashi's result is that [43]

$$L^*(\mathsf{C}(\mathsf{SNR}), \epsilon) = \sqrt{\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon), \tag{4.5}$$

which implies the set of real numbers L satisfying

$$L \leq \sqrt{\mathsf{V}(\mathsf{SNR})}\Phi^{-1}(\epsilon), \tag{4.6}$$

is second-order achievable, i.e., there exists a sequence of length-n block codes, with average error probabilities not exceeding ϵ asymptotically, and fixed sizes M_n, such that (4.4) holds.

Note that second-order coding rates can be negative depending on ϵ. Since the problem we are solving in this chapter is a multi-terminal one, we focus

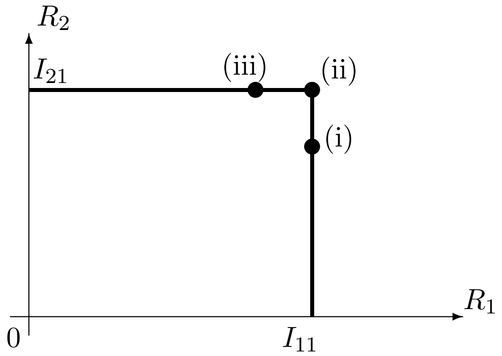

Figure 4.1. Illustration of the capacity region of the Gaussian IC with very strong interference. The signal-to-noise ratios $S_j = h_{jj}^2 P_j$ and $I_{11} = \mathsf{C}(S_1)$ and $I_{21} = \mathsf{C}(S_2)$.

on characterization of the *set of* achievable second-order coding rates (L_1, L_2), which is a subset of the real plane.

■ 4.1.1 Prior Work

Following the pioneering works in [42, 101], there have been many follow-up works for various point-to-point models [43, 45, 47, 85, 110, 124], for source coding [48, 53, 56, 59], for joint source-channel coding [58, 116], and for coding with side-information [118]. However, it is not trivial to generalize these results from the single- to the multi-user setting. Thus far, there have been only a few second-order works for multi-user settings. Hence, the understanding is far from being complete. Initial efforts focused on *global achievable dispersions*[38] for the DM multiple-access channel (MAC) [46, 78, 79, 105], for the DM asymmetric broadcast channel [105], and for the DM interference channel (IC) [67]. However, as pointed out by Haim et al. [38], global dispersion analysis has certain drawbacks such as the failure to precisely capture the nature of convergence to the boundary of the capacity region, the inability in characterizing the deviation from a specific point on the boundary and the difficulty in obtaining conclusive second-order results. To overcome these weaknesses, Haim et al. [38] proposed *local dispersion analysis*. Tan-Kosut [105] and Nomura-Han [81] characterized the second-order optimal rate region (the set of achievable second-order coding rates for fixed error probability ϵ and a fixed point on the optimal rate region) for distributed source coding, i.e., the Slepian-Wolf problem [100]. While it is possible to obtain tight second-order converse bounds for distributed source coding, it is challenging to do similarly for channel coding problems such as the DM-MAC. This is due in part to the union over independent input distributions. Scarlett-Tan [95] recently obtained the second-order capacity region for the Gaussian MAC with degraded message sets. The degradedness of the message sets makes it possible to avoid certain difficulties to get a tight converse by appealing to the reductions similar to the method of types. The local second-order capacity region for the Gaussian MAC with non-degraded message sets is an open problem.

■ 4.1.2 Main Contributions

In this chapter, we study the local dispersions of the Gaussian IC in the strictly very strong interference regime. Carleial showed that the capacity region of the very strong Gaussian IC (which includes the strictly very strong Gaussian IC) is a rectangle [14], as shown in Figure 4.1. We characterize the so-called *second-order capacity region*, which we briefly explain here. We fix a point (κ_1, κ_2) lying on the boundary of the capacity region. We also fix an admissible error probability $\epsilon \in (0, 1)$. We then characterize the set of pairs (L_1, L_2) for which there exists a sequence of blocklength-n codes with M_{jn} codewords, and average error probabilities not exceeding ϵ asymptotically, such that

$$\log M_{jn} \geq n\kappa_j + \sqrt{n}L_j + o(\sqrt{n}), \tag{4.7}$$

for $j = 1, 2$. The converse is proved using a generalized version of Verdú-Han Lemma [43, 44, 115], which involves only two error events. The direct part is proved using a generalized version of Feinstein's lemma [25], which involves four error events. The condition of being in the strictly very strong interference regime reduces the number of error events involved in the direct part, thus allowing the converse to match the direct part. Our key contribution is the determination of the set of second-order rate pairs (L_1, L_2), which characterize the rate of convergence of optimal (first-order) rates to a particular point (κ_1, κ_2) lying on the boundary of the capacity region. One of the interesting observations is that, if (κ_1, κ_2) is the corner point of the rectangular capacity region (case (ii) in Figure 4.1), then the set of all such $(L_1, L_2) \in \mathbb{R}^2$ is given by

$$\Phi\left(-\frac{L_1}{\sqrt{V_1}}\right)\Phi\left(-\frac{L_2}{\sqrt{V_2}}\right) \geq 1 - \epsilon, \tag{4.8}$$

where $V_j \triangleq \mathsf{V}(\mathsf{SNR}_j)$ is the effective Gaussian dispersion of the channel from the j^{th} transmitter to the j^{th} receiver, i.e., V_j is equal to (4.3) evaluated at signal-to-noise ratio SNR_j. An illustration of the (L_1, L_2) region is provided in Figure 4.2. We see from (4.8) that the two channels appear to operate independently of each other. Indeed $\Phi\left(-L_j/\sqrt{V_j}\right)$ is asymptotically the probability of correct detection of the j^{th}-channel where the number of codewords for the j^{th} codebook is given by M_{jn}. Intuitively, the inequality in (4.8) says that the system does not make an error if and only if both channels do not err. Just as Carleial [14] showed that in the very strong interference regime the capacities of the constituent channel are not reduced, in the *strictly* very strong interference regime, our main result shows that *the dispersions* V_1 *and* V_2 *remain unchanged* and there is no cross-correlation between the two channels in the sense of (4.8).

We emphasize that apart from Scarlett-Tan's work [95], this is the only work that completely characterizes the local dispersions for a channel-type network information theory problem. Furthermore, this is the first work which characterizes the local dispersions for a channel-type network information theory problem, where input distributions are of the product form.

■ 4.1.3 Chapter Organization

This chapter is organized as follows.

- The system model is introduced and the problem is formulated in Section 4.2.
- Next, the main result of the chapter is stated and discussed in Section 4.3.
- Future works are then discussed in Section 4.4.
- All proofs are deferred to the appendix of this chapter.

■ 4.2 System model and problem formulation

The two-user Gaussian interference channel (IC) is defined by the following input-output relationships

$$Y_{1i} = h_{11}X_{1i} + h_{21}X_{2i} + Z_{1i}, \tag{4.9}$$

$$Y_{2i} = h_{12}X_{1i} + h_{22}X_{2i} + Z_{2i}, \tag{4.10}$$

where X_{ji} denotes the signal sent by transmitter j (Tx_j in short), Y_{ji} denotes the output at receiver j (Rx_j in short), for $j = 1, 2$, at time i, for $i \in \{1, 2, ..., n\}$, and $\{Z_{ji}\}_{i=1}^n$ are independent (across time and between users at a fixed time), additive white Gaussian noise processes with zero means and unit variances. Denote the input alphabets as $\mathcal{X}_j^n$, and the output alphabets as $\mathcal{Y}_j^n$. Denote the transitional probability $P_{Y_1^n Y_2^n|X_1^n X_2^n}(y_1^n y_2^n|x_1^n x_2^n)$ as $W^n(y_1^n y_2^n|x_1^n x_2^n)$ for conciseness. Denote the Y_1- and Y_2-marginals of W as W_1 and W_2 respectively. The forward channel gains $\{h_{11}, h_{21}, h_{12}, h_{22}\}$ are assumed to be positive constants and known at all terminals. Transmitter Tx_j, for $j = 1, 2$, wishes to communicate a message $S_j \in \{1, 2, ..., M_{jn}\}$ to receiver Rx_j. It is assumed that the messages S_1 and S_2 are independent, and uniformly distributed on their respective message sets $\mathcal{W}_j \triangleq \{1, 2, ..., M_{jn}\}$, for $j = 1, 2$. We use nats as the units of information.

Define the *feasible set* of channel inputs

$$\mathcal{F}_{jn} \triangleq \left\{ x_j^n \in \mathcal{X}_j^n \;\middle|\; \sum_{k=1}^{n} x_{jk}^2 \le nP_j \right\} \tag{4.11}$$

for positive numbers $P_j, j = 1, 2$. P_1 and P_2 are the upper bounds on the average powers of the codewords. An $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$*-code for the Gaussian IC* consists of two encoding functions

$$f_{jn} : \mathcal{W}_j \to \mathcal{F}_{jn} \tag{4.12}$$

and two decoding functions

$$g_{jn} : \mathcal{Y}_j^n \to \hat{\mathcal{W}}_j \text{ for } j = 1, 2, \tag{4.13}$$

where the *average probability of error* is defined as

$$\epsilon_n \triangleq \Pr\left(\hat{S}_1 \neq S_1 \text{ or } \hat{S}_2 \neq S_2\right). \tag{4.14}$$

In the spirit of the works on second-order asymptotics [42, 43, 81, 95, 105], we define the second-order capacity region as follows.

Definition 4.1. Fix any two non-negative numbers κ_1 and κ_2. A real-valued pair (L_1, L_2) is said to be $(\kappa_1, \kappa_2, \epsilon)$-*achievable* [1] if there exists a sequence of $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$-codes such that

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon, \tag{4.15}$$

and

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_{jn} - n\kappa_j) \geq L_j \tag{4.16}$$

for $j = 1, 2$. The $(\kappa_1, \kappa_2, \epsilon)$-*second-order capacity region* of the IC $\mathcal{L}(\kappa_1, \kappa_2, \epsilon) \subset \mathbb{R}^2$ is defined as the closure of the set of all $(\kappa_1, \kappa_2, \epsilon)$-achievable rate pairs (L_1, L_2).

Definition 4.2. The IC is said to have a *very strong interference* if

$$h_{22}^2 \leq \frac{h_{21}^2}{1 + h_{11}^2 P_1} \text{ and } h_{11}^2 \leq \frac{h_{12}^2}{1 + h_{22}^2 P_2}. \tag{4.17}$$

[1] We note that it is more precise to define a pair being $(P_1, P_2, \kappa_1, \kappa_2, \epsilon)$-achievable. However, we omit the dependence on (P_1, P_2) as (P_1, P_2) are fixed throughout the chapter.

The IC is said to have a *strictly very strong interference* if both inequalities in (4.17) are strict.

Example 4.1. Consider a Gaussian IC, where $P_1 = P_2 = 1$, $h_{11} = h_{22} = 1$, $h_{21} = 3$, and $h_{12} = 4$. This is an example of a Gaussian IC in the strictly very strong interference regime. Clearly, there are uncountably many such examples as long as the interference link gains h_{21} and h_{12} are sufficiently large compared to the direct link gains h_{11} and h_{22} and the admissible powers P_1 and P_2.

Definition 4.3. Recall the definition of the Gaussian capacity function $\mathsf{C}(\cdot)$ in (4.2). Define the following first-order quantities

$$I_{11} \triangleq \mathsf{C}(h_{11}^2 P_1), \qquad I_{12} \triangleq \mathsf{C}(h_{11}^2 P_1 + h_{21}^2 P_2), \tag{4.18}$$

$$I_{21} \triangleq \mathsf{C}(h_{22}^2 P_2), \qquad I_{22} \triangleq \mathsf{C}(h_{22}^2 P_2 + h_{12}^2 P_1), \tag{4.19}$$

$$\mathbf{I}_{\mathrm{c}} \triangleq [I_{11}\ I_{21}]^T, \qquad \mathbf{I}_{\mathrm{d}} \triangleq [I_{11}\ I_{21}\ I_{12}\ I_{22}]^T. \tag{4.20}$$

The vectors $\mathbf{I}_{\mathrm{c}}$ and $\mathbf{I}_{\mathrm{d}}$ characterize the first-order regions that are obtained naturally from converse and direct bounds respectively. The non-asymptotic bounds that we evaluate also yield these first-order vectors.

Carleial [14] proved that the capacity region $\mathcal{C}$ of the Gaussian IC in the very strong interference regime is given by

$$\mathcal{C} = \left\{(R_1, R_2) \in \mathbb{R}_+^2 \mid R_1 \leq I_{11},\ R_2 \leq I_{21}\right\}. \tag{4.21}$$

A certain set of information densities plays an important role for the IC [15, 41, 67]. However, in dealing with channels with cost constraints, modified information densities [43, 79] offer certain advantages in the evaluation of non-asymptotic bounds as $n \to \infty$.

Definition 4.4. Fix a joint distribution

$$P_{Y_1^n Y_2^n X_1^n X_2^n}(y_1^n y_2^n x_1^n x_2^n) = P_{X_1^n}(x_1^n) P_{X_2^n}(x_2^n) W_1^n(y_1^n | x_1^n x_2^n) W_2^n(y_2^n | x_1^n x_2^n). \quad (4.22)$$

Given two auxiliary (conditional) output distributions $Q_{Y_1^n|X_2^n}$ and $Q_{Y_1^n}$ [2], define the *modified information densities*

$$\tilde{\imath}_{11}^n(X_1^n X_2^n Y_1^n) \triangleq \log \frac{W_1^n(Y_1^n | X_1^n X_2^n)}{Q_{Y_1^n|X_2^n}(Y_1^n | X_2^n)}, \quad (4.23)$$

$$\tilde{\imath}_{12}^n(X_1^n X_2^n Y_1^n) \triangleq \log \frac{W_1^n(Y_1^n | X_1^n X_2^n)}{Q_{Y_1^n}(Y_1^n)}. \quad (4.24)$$

We will often use the shorthands $\tilde{\imath}_{11}^n$ and $\tilde{\imath}_{12}^n$. Furthermore, the dependencies of $\tilde{\imath}_{11}^n$ and $\tilde{\imath}_{12}^n$ on the channel W_1^n and the output distributions $Q_{Y_1^n|X_2^n}$ and $Q_{Y_1^n}$ will be suppressed for the sake of brevity.

Similarly, given two auxiliary output distributions $Q_{Y_2^n|X_1^n}$ and $Q_{Y_2^n}$, we define $\tilde{\imath}_{21}^n(X_1^n X_2^n Y_2^n)$ and $\tilde{\imath}_{22}^n(X_1^n X_2^n Y_2^n)$.

$$\tilde{\imath}_{21}^n(X_1^n X_2^n Y_2^n) \triangleq \log \frac{W_2^n(Y_2^n | X_1^n X_2^n)}{Q_{Y_2^n|X_1^n}(Y_2^n | X_1^n)}, \quad (4.25)$$

$$\tilde{\imath}_{22}^n(X_1^n X_2^n Y_2^n) \triangleq \log \frac{W_2^n(Y_2^n | X_1^n X_2^n)}{Q_{Y_2^n}(Y_2^n)}. \quad (4.26)$$

In addition, we define

$$\tilde{\mathbf{i}}_{\mathrm{c}}^n(X_1^n X_2^n Y_1^n Y_2^n) \triangleq [\tilde{\imath}_{11}^n \quad \tilde{\imath}_{21}^n]^T \quad (4.27)$$

$$\tilde{\mathbf{i}}_{\mathrm{d}}^n(X_1^n X_2^n Y_1^n Y_2^n) \triangleq [\tilde{\imath}_{11}^n \quad \tilde{\imath}_{21}^n \quad \tilde{\imath}_{12}^n \quad \tilde{\imath}_{22}^n]^T. \quad (4.28)$$

[2] In the following, we will refer to $Q_{Y_1^n|X_2^n}$ and $Q_{Y_1^n}$ collectively as output distributions, dropping the qualifier *conditional*, for the sake of brevity.

Remark 4.1. Note that the idea of modified information density was first introduced by Hayashi and Nagaoka in [44] in quantum information theory. The paper [43] introduced this idea in non-quantum information theory.

Definition 4.5. Recall the definition of the Gaussian dispersion function $\mathsf{V}(\cdot)$ in (4.3). Define the second-order quantities

$$V_1 \triangleq \mathsf{V}(h_{11}^2 P_1), \quad \text{and} \quad V_2 \triangleq \mathsf{V}(h_{22}^2 P_2). \tag{4.29}$$

Note that $h_{jj}^2 P_j$ is the signal-to-noise ratio of the direct channel from Tx_j to Rx_j and $\mathsf{V}(h_{jj}^2 P_j)$ is the corresponding dispersion. Also, the expectation and the conditional covariance of the random vector $\tilde{\mathbf{i}}_{\mathrm{c}}(X_1 X_2 Y_1 Y_2)$ are $\mathbf{I}_{\mathrm{c}}$ and $\mathrm{diag}([V_1, V_2])$ respectively if $(X_1, X_2) \sim \mathcal{N}(\mathbf{0}, \mathrm{diag}([P_1, P_2]))$, $Q_{Y_1|X_2}(\cdot|x_2) = \mathcal{N}(h_{21}x_2, h_{11}^2 P_1 + 1)$ and $Q_{Y_2|X_1}(\cdot|x_1) = \mathcal{N}(h_{12}x_1, h_{22}^2 P_2 + 1)$.

The following is the cumulative distribution function of a standard Gaussian distribution

$$\Phi(t) \triangleq \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \, \mathrm{d}u. \tag{4.30}$$

The inverse of Φ is defined as $\Phi^{-1}(\epsilon) \triangleq \sup\{t \in \mathbb{R} \,|\, \Phi(t) \leq \epsilon\}$.

In this chapter, we aim to characterize the $(\kappa_1, \kappa_2, \epsilon)$-capacity region of the Gaussian IC in the strictly very strong interference regime, i.e., we determine $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ for any $(\kappa_1, \kappa_2) \in [0, \infty)^2$ and $\epsilon \in (0, 1)$.

4.3 Main result

The main result of this chapter is summarized in the following theorem. See Figure 4.1 for an illustration of the different cases.

Theorem 4.1. *For any $0 < \epsilon < 1$, the $(\kappa_1, \kappa_2, \epsilon)$-second-order capacity region for the strictly very strong Gaussian interference channel in the following special cases is given by*

i) When $\kappa_1 = I_{11}$ and $\kappa_2 < I_{21}$ (vertical boundary),

$$\mathcal{L}(\kappa_1, \kappa_2, \epsilon) = \left\{ (L_1, L_2) \in \mathbb{R}^2 \middle| \Phi\left(\frac{L_1}{\sqrt{V_1}} \right) \leq \epsilon \right\}; \tag{4.31}$$

ii) When $\kappa_1 = I_{11}$ and $\kappa_2 = I_{21}$ (corner point),

$$\mathcal{L}(\kappa_1, \kappa_2, \epsilon) = \left\{ (L_1, L_2) \in \mathbb{R}^2 \middle| \Phi\left(-\frac{L_1}{\sqrt{V_1}} \right) \Phi\left(-\frac{L_2}{\sqrt{V_2}} \right) \geq 1 - \epsilon \right\}; \tag{4.32}$$

iii) When $\kappa_1 < I_{11}$ and $\kappa_2 = I_{21}$ (horizontal boundary),

$$\mathcal{L}(\kappa_1, \kappa_2, \epsilon) = \left\{ (L_1, L_2) \in \mathbb{R}^2 \middle| \Phi\left(\frac{L_2}{\sqrt{V_2}} \right) \leq \epsilon \right\}. \tag{4.33}$$

Proof. This theorem is proved in the appendix of this chapter. □

Example 4.2. We visualize the result of case (ii) of Theorem 4.1 via an example. Consider a Gaussian IC where the dispersions are equal, i.e., $V_1 = V_2$, and the average error probability $\epsilon = 0.001$. Clearly, by choosing h_{12} and h_{21} sufficiently large, we can guarantee that the Gaussian IC is in the strictly very

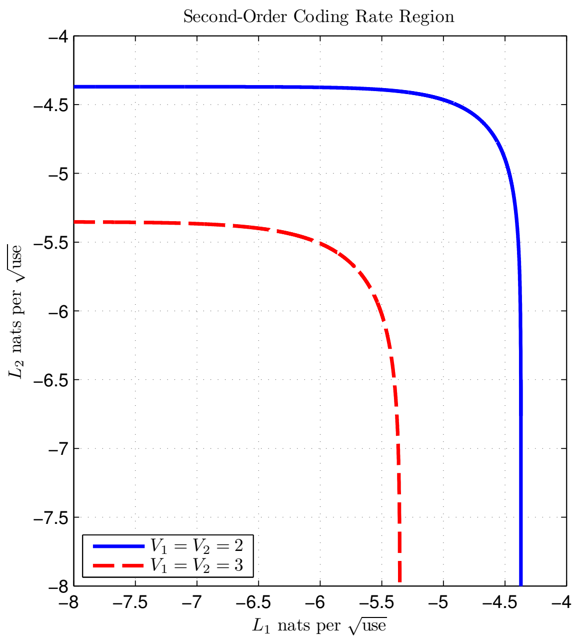

Figure 4.2. The second-order capacity region $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ of case 2 when $\epsilon = 0.001$
.

strong interference regime (see Example 4.1). The second-order capacity region $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ of case (ii) where $(\kappa_1, \kappa_2) = (I_{11}, I_{21})$ is illustrated in Figure 4.2. Because $\epsilon < 1/2$, the second-order capacity region $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ lies entirely in the third quadrant of $\mathbb{R}^2$. Due to the fact that $V_1 = V_2$, the second-order capacity region $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ for case (ii) is also symmetric about the line $L_1 = L_2$.

■ 4.3.1 Remarks Concerning Theorem 4.1

1. The result can be generalized to any $(\kappa_1, \kappa_2) \in [0, \infty)^2$. If (κ_1, κ_2) is in the interior of $\mathcal{C}$, then it can be shown that $\mathcal{L}(\kappa_1, \kappa_2, \epsilon) = \mathbb{R}^2$. If (κ_1, κ_2) is in the exterior of $\mathcal{C}$, then $\mathcal{L}(\kappa_1, \kappa_2, \epsilon) = \emptyset$. This implies the *strong converse*. Thus, the strong converse, which was hitherto not established for the Gaussian IC with very strong interference, is a by-product of our

analyses. The only interesting cases, in which (κ_1, κ_2) is on the boundary of the capacity region, are presented in Theorem 4.1.

2. In case (i), the $(\kappa_1, \kappa_2, \epsilon)$-capacity region depends on ϵ and V_1 only. This region is more succinctly described as

$$L_1 \le \sqrt{V_1}\Phi^{-1}(\epsilon), \quad \text{and} \quad L_2 \in \mathbb{R}. \tag{4.34}$$

Note that $\sqrt{V_1}\Phi^{-1}(\epsilon)$ is exactly the second-order coding rate of the AWGN channel between transmitter Tx_1 and receiver Rx_1 when there is no interference from transmitter Tx_2 [43]. The fact that user 2's parameters do not feature in (4.34) is because $\kappa_2 < I_{21}$. Note that $\kappa_2 < I_{21}$ implies that Tx_2 operates at a rate strictly below the capacity of the second channel I_{21}. In this case, the second channel operates in the large-deviations (error exponents) regime so the second constraint is not featured in our dispersion analysis. This is because the error probability is exponentially small in this regime. See [38, 81, 95, 105]. By symmetry, case (iii) is similar to case (i).

3. In case (ii), the $(\kappa_1, \kappa_2, \epsilon)$-second-order capacity region is a function of ϵ and *both* V_1 and V_2 because we are operating at rates near the *corner point* of $\mathcal{C}$. The two constraints on the rates come into play in the characterization of $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$. Roughly speaking, $\Phi(-L_j/\sqrt{V_j})$ is the probability that the j^{th}-decoder decodes correctly if the number of codewords of the j^{th}-user is

$$M_{jn} = \big\lfloor \exp\big(n\kappa_j + \sqrt{n}L_j + o(\sqrt{n})\big)\big\rfloor. \tag{4.35}$$

Thus, the product $\Phi(-L_1/\sqrt{V_1})\Phi(-L_2/\sqrt{V_2})$, which is constrained to be larger than $1-\epsilon$ in (4.32), is the probability that *both* messages are decoded correctly assuming that both channels operate *independently*. More explicitly, using the definition of the error probability criterion in (4.14), we have that

$$\Pr\left(\hat{S}_1 = S_1 \text{ and } \hat{S}_2 = S_2\right) \geq 1 - \epsilon. \tag{4.36}$$

Assuming independence, this means that

$$\Pr\left(\hat{S}_1 = S_1\right) \Pr\left(\hat{S}_2 = S_2\right) \geq 1 - \epsilon. \tag{4.37}$$

Denoting $o(1)$ as a sequence that tends to zero as the blocklength grows, we observe that

$$\Pr\left(\hat{S}_j = S_j\right) = \Phi\left(-\frac{L_j}{\sqrt{V_j}}\right) + o(1) \tag{4.38}$$

if (4.35) holds (a result by Hayashi [43, Thm. 4]). In this way, we recover the main result in (4.32). Since $V_1 = \mathsf{V}(h_{11}^2 P_1)$ and $V_2 = \mathsf{V}(h_{22}^2 P_2)$ are the dispersions of the point-to-point Gaussian channels without interference, this is exactly analogous to Carleial's result for Gaussian ICs with very strong interference [14]. In other words, in this regime, the channel dispersions of the constituent channels are not affected. This explains the title of the chapter—namely that in this very special scenario, interference does not affect (reduce) the dispersions of the constituent channels. In addition, no cross dispersion terms are present in (4.32) unlike other network problems [81, 95, 105]. This is due to the independence of the noises Z_{1i} and Z_{2i} as well as the strictly very strong interference assumption.

4. One of the input distributions that achieves the capacity, error exponent, dispersion and even the third-order coding rate of the Gaussian point-to-point channel [85, 97, 106], is the uniform distribution on the power sphere.

MolavianJazi-Laneman [79] derived global achievable dispersions for the two-user Gaussian MAC using uniform distributions on power spheres. In this work, we also use the *uniform input distributions on power spheres*. It is not easy to use the *cost constrained ensemble* in [95] as that input distribution is more suited to, for example, superposition coding.

5. The proof of the direct part makes use of a generalized version of Feinstein's lemma [25], which involves four error events. We also use the *central limit theorem for functions* by MolavianJazi and Laneman [79] to "lift" the problem to a higher dimension, in fact 10-dimensional Euclidean space, ensuring that the i.i.d. version of the multivariate Berry-Esséen theorem [9, 35, 118] may be employed. The converse makes use of a generalized version of Verdú-Han Lemma [43, 44, 115], which involves only two error events. At a high level, we use the strictly very strong interference condition to reduce the number of error events in the direct part, so that it matches the converse.

6. Finally, it is somewhat surprising that in the converse, even though we must ensure that the transmitter outputs are independent, we do not need to use the wringing technique, invented by Ahlswede [2] and used originally to prove that the DM-MAC admits a strong converse. This is due to Gaussianity which allows us to show that the first- and second-order statistics of a certain set of information densities are independent of x_1^n and x_2^n on power spheres. See (4.43)-(4.44).

■ 4.4 Conclusion

In this work, we characterized the second-order coding rates of the Gaussian interference channel in the strictly very strong interference regime. The strictly very strong interference assumption reduces the number of error events in the direct part so that it matches the converse. It would be interesting to find the second-order capacity region in the other regimes. New non-asymptotic achievability and converse bounds are needed for other cases. In particular, it is intriguing to see what the second-order capacity region for the interference channel in the strong interference regime is. Note that in the strong interference regime, the interference channel behaves like a pair of MACs but unfortunately the second-order capacity region for the MAC remains unknown [46, 79, 95, 105]. The achievability scheme in this work is also applicable to the interference channel in the strong interference regime. A non-trivial problem here is to derive a tighter converse than that prescribed by Lemma 4.1 to be evaluated assuming only strong interference.

■ 4.5 Appendix to chapter 4

■ 4.5.1 Proof of Theorem 4.1: Converse Part

In this sub-section, we present the converse proof of Theorem 4.1. By a standard $n \leftrightarrow n+1$ argument [97, Sec. X] [85, Lem. 39], we may assume that the power constraints are satisfied with equality. We first start with an non-asymptotic bound, which is a generalized version of Verdú-Han Lemma [115, Lem. 4] [43, 44]. Verdú-Han introduced this kind of lemma without the modified information density in the point-to-point channel. Hayashi [43] introduced this kind of lemma with the modified information density in the point-to-point channel. Here, we generalize this kind of lemma in the interference channel. The proof of this lemma is given in sub-section 4.5.5.

Lemma 4.1. *For every $n \in \mathbb{N}$, for every $\gamma > 0$, and for any auxiliary output distributions $Q_{Y_1^n|X_2^n}$ and $Q_{Y_2^n|X_1^n}$, every $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$-code for the Gaussian IC satisfies*

$$\epsilon_n \geq \Pr(\tilde{i}_{11}^n(X_1^n X_2^n Y_1^n) \leq \log M_{1n} - n\gamma$$

$$\text{or } \tilde{i}_{21}^n(X_1^n X_2^n Y_2^n) \leq \log M_{2n} - n\gamma) - 2e^{-n\gamma}, \tag{4.39}$$

where $\tilde{i}_{11}$ and $\tilde{i}_{21}$ are modified information densities defined in (4.23) and (4.24) respectively and X_j^n is uniformly distributed over the j^{th} codebook and so $\|X_j^n\|^2 = nP_j$ with probability one.

Remark 4.2. Intuitively, the proof of Lemma 4.1 relies on the fact that a system with help of a genie, which provides the transmitted information of transmitter 2 to decoder 1, and the transmitted information from transmitter 1 to decoder 2, will always do no worse than a system without help from a genie.

Fix any pair of rates (κ_1, κ_2) on the boundary of $\mathcal{C}$ in (4.21). Consider any second-order pair (L_1, L_2) that is $(\kappa_1, \kappa_2, \epsilon)$-achievable for the Gaussian IC. This implies that there exists a sequence of $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$-codes satisfying (4.16).

By the definition of $\liminf$, for any $\beta > 0$, there exists an integer N_β such that for all $n > N_\beta$

$$\log M_{jn} - n\kappa_j \geq \sqrt{n}(L_j - \beta). \tag{4.40}$$

Let $\mathcal{L}_{\text{eq}}(\kappa_1, \kappa_2, \epsilon)$ be the $(\kappa_1, \kappa_2, \epsilon)$-second-order capacity region of the IC with equal power constraints, i.e. each codeword x_j^n satisfies $\sum_{k=1}^n x_{jk}^2 = nP_j$ for $j = 1, 2$. As mentioned above, it can be shown that (cf. [85, Lem. 39]) $\mathcal{L}_{\text{eq}}(\kappa_1, \kappa_2, \epsilon) = \mathcal{L}(\kappa_1, \kappa_2, \epsilon)$. Therefore, in this converse proof, it is sufficient to assume equal power constraints.

Define the auxiliary output distributions

$$\hat{Q}_{Y_1|X_2}(y_1|x_2) \triangleq \mathcal{N}(y_1; h_{21}x_2, h_{11}^2 P_1 + 1) \tag{4.41}$$

$$\hat{Q}_{Y_2|X_1}(y_2|x_1) \triangleq \mathcal{N}(y_2; h_{12}x_1, h_{22}^2 P_2 + 1). \tag{4.42}$$

These are the conditional output distributions of the Gaussian IC when the inputs are $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$.

Choose the conditional output distributions $Q_{Y_1^n|X_2^n}$ and $Q_{Y_2^n|X_1^n}$ in Lemma 4.1, respectively as the n-fold products of $\hat{Q}_{Y_1|X_2}(y_1|x_2)$ and $\hat{Q}_{Y_2|X_1}(y_2|x_1)$, which are defined above. Next, choose $\gamma = \frac{\log n}{2n}$. Let V_{c} be the 2×2 diagonal matrix with V_1 and V_2 along its diagonals.

Next, we have the following lemma whose proof is presented in full in subsection 4.5.3.

Lemma 4.2. *For all x_1^n and x_2^n satisfying $\|x_j^n\|^2 = nP_j$ we have*

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{ck}(x_{1k}x_{2k}Y_{1k}Y_{2k})\right] = \mathbf{I}_{\mathrm{c}}, \quad \textit{and} \tag{4.43}$$

$$\operatorname{cov}\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{ck}(x_{1k}x_{2k}Y_{1k}Y_{2k})\right] = V_{\mathrm{c}}, \tag{4.44}$$

where $\tilde{\mathbf{i}}_{ck}$ is the random vector with components given by (4.23) and (4.24).

This lemma is the crux of the converse proof. Note that the covariance matrix in (4.44) is diagonal and this results in the decoupling of the events in the corner point case given by (4.32). The diagonal nature of (4.44) arises, in part, from the independence of the noises Z_{1i} and Z_{2i} for each time $i = 1, \ldots, n$.

Let $t_{\mathrm{c}} \triangleq \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[\|\tilde{\mathbf{i}}_{ck}(x_{1k}x_{2k}Y_{1k}Y_{2k})\|^3]$ be the third absolute moment and $\phi_{\mathrm{c}} \triangleq \frac{254\sqrt{2}t_{\mathrm{c}}}{\lambda_{\min}(V_{\mathrm{c}})^{3/2}}$, where $\lambda_{\min}(V_{\mathrm{c}})$ is the minimum eigenvalue of V_{c}. Define the rate pair $\mathbf{R}_{\mathrm{c}} \triangleq [\frac{\log M_{1n}}{n}, \frac{\log M_{2n}}{n}]^T$. Note that $V_{\mathrm{c}} \succ 0$ because the channel gains and powers are all positive. Also $t_{\mathrm{c}} < \infty$ from [95, App. A]. Thus, ϕ_{c} is finite. Define

$$\Psi\big([t_1, t_2]; \mathbf{m}, \mathbf{\Sigma}\big) \triangleq \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{\Sigma})\,\mathrm{d}\mathbf{u} \tag{4.45}$$

as the bivariate generalization of the Gaussian cumulative distribution function. Then we have

$$\begin{aligned}\Delta(x_1^n, x_2^n) &\triangleq \Pr\left(\frac{1}{n}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{ck}(x_{1k}x_{2k}Y_{1k}Y_{2k}) > \mathbf{R}_{\mathrm{c}} - \gamma\mathbf{1}\right)\\ &= \Pr\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{ck} - \sqrt{n}\mathbf{I}_{\mathrm{c}} > \sqrt{n}(\mathbf{R}_{\mathrm{c}} - \mathbf{I}_{\mathrm{c}} - \gamma\mathbf{1})\right)\\ &\overset{(a)}{\leq} \Psi(-\sqrt{n}(\mathbf{R}_{\mathrm{c}} - \mathbf{I}_{\mathrm{c}} - \gamma\mathbf{1}); \mathbf{0}, V_{\mathrm{c}}) + \frac{\phi_{\mathrm{c}}}{\sqrt{n}}\end{aligned}$$

$$\overset{(b)}{\leq} \Psi(-\sqrt{n}(\mathbf{R}_{\mathrm{c}} - \mathbf{I}_{\mathrm{c}}); \mathbf{0}, V_{\mathrm{c}}) + O\left(\frac{\log n}{\sqrt{n}}\right), \tag{4.46}$$

where

(a) follows from the application of a variant of the multivariate Berry-Esséen Theorem, which is stated in Theorem 2.11; and

(b) follows from Taylor expansion of the function $\Psi(\mathbf{t}; \mathbf{0}, V_{\mathrm{c}})$, which is differentiable with respect to $\mathbf{t}$.

From Lemma 4.1, we have

$$\begin{aligned}\epsilon_n &\geq 1 - \Pr\left(\frac{1}{n}\tilde{\mathbf{i}}_{\mathrm{c}}^n(X_1^n X_2^n Y_1^n Y_2^n) > \mathbf{R}_{\mathrm{c}} - \gamma\mathbf{1}\right) - 2e^{-n\gamma} \\ &= 1 - \mathbb{E}\left[\Delta(X_1^n, X_2^n)\right] - 2e^{-n\gamma}. \end{aligned} \tag{4.47}$$

Note that $e^{-n\gamma} = \frac{1}{\sqrt{n}}$. Combining (4.46) and (4.47), we have

$$\begin{aligned}\epsilon_n &\geq 1 - \Psi(-\sqrt{n}(\mathbf{R}_{\mathrm{c}} - \mathbf{I}_{\mathrm{c}}); \mathbf{0}, V_{\mathrm{c}}) - O\left(\frac{\log n}{\sqrt{n}}\right) - \frac{2}{\sqrt{n}} \\ &\overset{(a)}{\geq} 1 - \Psi\left(\begin{bmatrix}\sqrt{n}(I_{11} - \kappa_1) - L_1 + \beta \\ \sqrt{n}(I_{21} - \kappa_2) - L_2 + \beta\end{bmatrix}; \mathbf{0}, V_{\mathrm{c}}\right) - O\left(\frac{\log n}{\sqrt{n}}\right) - \frac{2}{\sqrt{n}} \end{aligned} \tag{4.48}$$

where

(a) holds for all $n > N_\beta$ and follows because $\mathbf{t} \mapsto \Psi(\mathbf{t}; \mathbf{0}, V_{\mathrm{c}})$ is monotonically increasing in $\mathbf{t}$ and (4.40).

We now consider three different cases.

Case 1: When $\kappa_1 = I_{11}$ and $\kappa_2 < I_{21}$

For any fixed L_2, if $\kappa_2 < I_{21}$, we have $\sqrt{n}(I_{21}-\kappa_2)-L_2+\beta \to +\infty$. Thus, the second term on the right hand side (RHS) of (4.48) converges to $\Psi\left(-L_1+\beta;0,V_1\right) = \Phi\big(\frac{-L_1+\beta}{\sqrt{V_1}}\big)$. Taking $\limsup$ on both sides of (4.48), and using (4.40), we have

$$\epsilon \geq \limsup_{n\to\infty} \epsilon_n \geq 1-\Phi\left(\frac{-L_1+\beta}{\sqrt{V_1}}\right). \tag{4.49}$$

Since this is true for any $\beta > 0$, we may let $\beta \downarrow 0$ and deduce that

$$\Phi\left(\frac{L_1}{\sqrt{V_1}}\right) \leq \epsilon. \tag{4.50}$$

Case 1 is proved.

Case 2: When $\kappa_1 = I_{11}$ and $\kappa_2 = I_{21}$

In this case, the second term on the RHS of (4.48) converges to $\Psi([-L_1+\beta,-L_2+\beta]^T;0,V_c)$. The rest of the arguments are similar to that in case 1. Note that because V_c is diagonal,

$$\Psi\left([-L_1,-L_2]^T;0,V_c\right) = \Phi\left(-\frac{L_1}{\sqrt{V_1}}\right)\Phi\left(-\frac{L_2}{\sqrt{V_2}}\right). \tag{4.51}$$

Case 3: When $\kappa_1 < I_{11}$ and $\kappa_2 = I_{21}$

By symmetry, case 3 is proved similarly to case 1.

■ 4.5.2 Proof of Theorem 4.1: Direct Part

In this sub-section, we present the achievability proof of Theorem 4.1. The following non-asymptotic bound, a generalized version of Feinstein's lemma [25], will be employed in the proof. The proof of this lemma is given in sub-section 4.5.6.

Lemma 4.3. *Fix a joint distribution satisfying (4.22). For any $n \in \mathbb{N}$, any $\gamma > 0$, and any auxiliary output distributions $Q_{Y_1^n|X_2^n}$, $Q_{Y_1^n}$, $Q_{Y_2^n|X_1^n}$ and $Q_{Y_2^n}$, there exists an $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$-code for the Gaussian IC, such that*

$$\epsilon_n \leq \Pr(\mathcal{E}_{11} \cup \mathcal{E}_{12} \cup \mathcal{E}_{21} \cup \mathcal{E}_{22}) + Ke^{-n\gamma} + P_{X_1^n}(\mathcal{F}_{1n}^c) + P_{X_2^n}(\mathcal{F}_{2n}^c) \tag{4.52}$$

where

$$\mathcal{E}_{11} \triangleq \{\tilde{i}_{11}^n(X_1^n X_2^n Y_1^n) \leq \log M_{1n} + n\gamma\} \tag{4.53}$$

$$\mathcal{E}_{21} \triangleq \{\tilde{i}_{21}^n(X_1^n X_2^n Y_2^n) \leq \log M_{2n} + n\gamma\} \tag{4.54}$$

$$\mathcal{E}_{12} \triangleq \{\tilde{i}_{12}^n(X_1^n X_2^n Y_1^n) \leq \log M_{1n} M_{2n} + n\gamma\} \tag{4.55}$$

$$\mathcal{E}_{22} \triangleq \{\tilde{i}_{22}^n(X_1^n X_2^n Y_2^n) \leq \log M_{1n} M_{2n} + n\gamma\}, \tag{4.56}$$

and

$$K \triangleq K_{11} + K_{12} + K_{21} + K_{22}, \tag{4.57}$$

$$K_{11} \triangleq \sup_{x_2^n, y_1^n} \frac{P_{Y_1^n|X_2^n}(y_1^n|x_2^n)}{Q_{Y_1^n|X_2^n}(y_1^n|x_2^n)}, \qquad K_{12} \triangleq \sup_{y_1^n} \frac{P_{Y_1^n}(y_1^n)}{Q_{Y_1^n}(y_1^n)}, \tag{4.58}$$

$$K_{21} \triangleq \sup_{x_1^n, y_2^n} \frac{P_{Y_2^n|X_1^n}(y_2^n|x_1^n)}{Q_{Y_2^n|X_1^n}(y_2^n|x_1^n)}, \qquad K_{22} \triangleq \sup_{y_2^n} \frac{P_{Y_2^n}(y_2^n)}{Q_{Y_2^n}(y_2^n)}. \tag{4.59}$$

Remark 4.3. In fact, this lemma holds not just for Gaussian ICs, but for general ICs.

Remark 4.4. The presence of the Radon-Nikodym derivatives K_{ij} in (4.57)–(4.59) is the price to pay for the luxury of using the auxiliary output distributions. This version of generalized Feinstein is different from the earlier versions (cf. [115, Thm. 1]) in that the information densities in this lemma involve auxiliary output distributions that can be chosen. This technique was similarly employed in [43, 44, 79]. By choosing the appropriate auxiliary output distributions and input distributions, we can show that the inner bound to $\mathcal{L}(\kappa_1, \kappa_2, \epsilon)$ coincides with the outer bound.

First, we present the achievability proof for case 1.

Case 1: When $\kappa_1 = I_{11}$ and $\kappa_2 < I_{21}$

Fix any pair (L_1, L_2) satisfying

$$\Phi\left(\frac{L_1}{\sqrt{V_1}}\right) \leq \epsilon. \tag{4.60}$$

Let the number of codewords in the j^{th} codebook be

$$M_{nj} = \lfloor \exp\left(n\kappa_j + \sqrt{n}L_j + n^{1/4}\beta\right) \rfloor \tag{4.61}$$

for $j = 1, 2$, and a fixed $\beta > 0$. It is clear that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_{jn} - n\kappa_j) \geq L_j. \tag{4.62}$$

Therefore, in order to show that (L_1, L_2) is $(\kappa_1, \kappa_2, \epsilon)$-achievable, it suffices to show the existence of a sequence of $(M_{1n}, M_{2n}, n, \epsilon_n, P_1, P_2)$-codes such that $\limsup_{n\to\infty} \epsilon_n \leq \epsilon$. For this, we define an appropriate input distribution to be used in Lemma 4.3, which is going to be applied in this sub-section. Inspired by [79, 106], we define the input distributions to be uniform on the respective power shells, i.e.

$$P_{X_j^n}(x_j^n) \triangleq \frac{\delta(\|x_j^n\| - \sqrt{nP_j})}{A_n(\sqrt{nP_j})}, \tag{4.63}$$

for $j = 1, 2$ and where $\delta(\cdot)$ is the Dirac delta and $A_n(r) \triangleq \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}$ is the surface area of a sphere in $\mathbb{R}^n$ with radius r. With this choice, we have $P_{X_1^n}(\mathcal{F}_{1n}^c) + P_{X_2^n}(\mathcal{F}_{2n}^c) = 0$, i.e. the power constraints are satisfied with probability 1.

Define the output distributions

$$\hat{Q}_{Y_1}(y_1) \triangleq \mathcal{N}(y_1; 0, h_{11}^2 P_1 + h_{12}^2 P_2 + 1) \tag{4.64}$$

$$\hat{Q}_{Y_2}(y_2) \triangleq \mathcal{N}(y_2; 0, h_{12}^2 P_1 + h_{22}^2 P_2 + 1) \tag{4.65}$$

$$\hat{Q}_{Y_1|X_2}(y_1|x_2) \triangleq \mathcal{N}(y_1; h_{21}x_2, h_{11}^2 P_1 + 1) \tag{4.66}$$

$$\hat{Q}_{Y_2|X_1}(y_2|x_1) \triangleq \mathcal{N}(y_2; h_{12}x_1, h_{22}^2 P_2 + 1). \tag{4.67}$$

These are the output distributions of the Gaussian IC when the inputs are $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$.

Choose the auxiliary output distributions $Q_{Y_1^n}(y_1^n)$, $Q_{Y_2^n}(y_2^n)$, $Q_{Y_1^n|X_2^n}(y_1^n|x_2^n)$ and $Q_{Y_2^n|X_1^n}(y_2^n|x_1^n)$ in Lemma 4.3 to be the n-fold memoryless extensions of $\hat{Q}_{Y_1}(y_1)$, $\hat{Q}_{Y_2}(y_2)$, $\hat{Q}_{Y_1|X_2}(y_1|x_2)$ and $\hat{Q}_{Y_2|X_1}(y_2|x_1)$ respectively, the distributions of which are given in (4.64-4.67). With this choice of auxiliary output distributions, the value of K in Lemma 4.3 is shown in the following lemma to be bounded.

Lemma 4.4. *For n sufficiently large, K_{11}, K_{21}, K_{12} and K_{22} are finite . Thus, K in (4.57) is also finite.*

This lemma is proved in sub-section 4.5.4.

Define

$$\alpha_{11} \triangleq 1 + h_{11}^2 P_1, \quad \alpha_{12} \triangleq 1 + h_{11}^2 P_1 + h_{21}^2 P_2, \tag{4.68}$$

$$\alpha_{21} \triangleq 1 + h_{22}^2 P_2, \quad \alpha_{22} \triangleq 1 + h_{12}^2 P_1 + h_{22}^2 P_2. \tag{4.69}$$

We have

$$\begin{aligned}
\tilde{i}_{11}^n &= \log \frac{W_1^n(Y_1^n | X_1^n X_2^n)}{Q_{Y_1^n|X_2^n}(Y_1^n|X_2^n)} && (4.70)\\
&= \frac{n}{2}\log(1 + h_{11}^2 P_1) + \frac{\sum_{k=1}^n (Y_{1k} - h_{21}X_{2k})^2}{2(1 + h_{11}^2 P_1)} \\
&\quad - \frac{\sum_{k=1}^n (Y_{1k} - h_{11}X_{1k} - h_{21}X_{2k})^2}{2} && (4.71)\\
&= \frac{n}{2}\log(1 + h_{11}^2 P_1) + \frac{\sum_{k=1}^n (Z_{1k} + h_{11}X_{1k})^2}{2(1 + h_{11}^2 P_1)} - \frac{\sum_{k=1}^n (Z_{1k})^2}{2} && (4.72)\\
&= nI_{11} + \frac{1}{2\alpha_{11}}[(\alpha_{11} - 1)(n - \|Z_1^n\|^2) + 2h_{11}\langle X_1^n, Z_1^n\rangle], && (4.73)
\end{aligned}$$

where $\langle a^n, b^n \rangle$ denotes the inner product between a^n and b^n.

Similarly, it can be shown that the other three modified information densities can be expressed as

$$\begin{aligned}
\tilde{i}_{21}^n &= nI_{21} + \frac{1}{2\alpha_{21}}[(\alpha_{21} - 1)(n - \|Z_2^n\|^2) + 2h_{22}\langle X_2^n, Z_2^n\rangle] \\
\tilde{i}_{12}^n &= nI_{12} + \frac{1}{2\alpha_{12}}[(\alpha_{12} - 1)(n - \|Z_1^n\|^2) \\
&\quad + 2h_{11}h_{21}\langle X_2^n, X_1^n\rangle + 2h_{11}\langle X_1^n, Z_1^n\rangle + 2h_{21}\langle X_2^n, Z_1^n\rangle] \\
\tilde{i}_{22}^n &= nI_{22} + \frac{1}{2\alpha_{22}}[(\alpha_{22} - 1)(n - \|Z_2^n\|^2) \\
&\quad + 2h_{22}h_{12}\langle X_2^n, X_1^n\rangle + 2h_{22}\langle X_2^n, Z_2^n\rangle + 2h_{12}\langle X_1^n, Z_2^n\rangle]. && (4.74)
\end{aligned}$$

Next, we use the *central limit theorem for functions* technique proposed by MolavianJazi-Laneman [79] to transform these modified information densities into functions of sums of independent random vectors. Let $T_j^n \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{n\times n})$, for $j = 1, 2$, be standard Gaussian random vectors that are independent of each other and of the noises Z_j^n. Note that the input distribution in (4.63) results in $X_{jk} = \sqrt{nP_j}\frac{T_{jk}}{\|T_j^n\|}$, for $k \in \{1, \dots, n\}$. Indeed, $\|X_j^n\|^2 = nP_j$ with probability one. Now consider the length-10 random vector

$$\mathbf{U}_k \triangleq (\{U_{j1k}\}_{j=1}^4, \{U_{j2k}\}_{j=1}^4, U_{9k}, U_{10k}), \tag{4.75}$$

where

$$\begin{aligned}
U_{11k} &\triangleq 1 - Z_{1k}^2, & U_{21k} &\triangleq h_{11}\sqrt{P_1}T_{1k}Z_{1k}, \\
U_{31k} &\triangleq h_{21}\sqrt{P_2}T_{2k}Z_{1k}, & U_{41k} &\triangleq h_{11}h_{21}\sqrt{P_1P_2}T_{1k}T_{2k}, \\
U_{12k} &\triangleq 1 - Z_{2k}^2, & U_{22k} &\triangleq h_{22}\sqrt{P_2}T_{2k}Z_{2k}, \\
U_{32k} &\triangleq h_{12}\sqrt{P_1}T_{1k}Z_{2k}, & U_{42k} &\triangleq h_{12}h_{22}\sqrt{P_1P_2}T_{1k}T_{2k}, \\
U_{9k} &\triangleq T_{1k}^2 - 1, & U_{10k} &\triangleq T_{2k}^2 - 1.
\end{aligned} \tag{4.76}$$

It is easy to verify that $\mathbf{U}_k$ is i.i.d. across all channel uses $k \in \{1, \dots, n\}$, and $\mathbb{E}(\mathbf{U}_k) = 0$ and $\mathbb{E}(\|\mathbf{U}_k\|^3)$ is finite. The covariance matrix of $\mathbf{U}_1$ is given by

$$\mathrm{Cov}(\mathbf{U}_1) = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_{11}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{44} & 0 & 0 & 0 & \alpha_{48} & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_{21}-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_{77} & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{48} & 0 & 0 & 0 & \alpha_{88} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}, \tag{4.77}$$

where

$$\alpha_{33} \triangleq h_{21}^2 P_2 \tag{4.78}$$

$$\alpha_{44} \triangleq h_{11}^2 h_{21}^2 P_1 P_2 \tag{4.79}$$

$$\alpha_{48} \triangleq P_1 P_2 h_{11} h_{21} h_{12} h_{22} \tag{4.80}$$

$$\alpha_{77} \triangleq h_{12}^2 P_1 \tag{4.81}$$

$$\alpha_{88} \triangleq h_{12}^2 h_{22}^2 P_1 P_2. \tag{4.82}$$

Note that $\alpha_{11} + \alpha_{33} = \alpha_{12}$ and $\alpha_{21} + \alpha_{77} = \alpha_{22}$.

Define the functions $\tau_{11}, \tau_{12} : \mathbb{R}^{10} \to \mathbb{R}$ as follows

$$\tau_{11}(\mathbf{u}) \triangleq (\alpha_{11} - 1)u_{11} + \frac{2u_{21}}{\sqrt{1+u_9}} \tag{4.83}$$

$$\tau_{12}(\mathbf{u}) \triangleq (\alpha_{12} - 1)u_{11} + \frac{2u_{21}}{\sqrt{1+u_9}} + \frac{2u_{31}}{\sqrt{1+u_{10}}} + \frac{2u_{41}}{\sqrt{1+u_9}\sqrt{1+u_{10}}}, \tag{4.84}$$

for receiver 1. Similarly, define $\tau_{21}(\mathbf{u})$ and $\tau_{22}(\mathbf{u})$ for receiver 2 as follows

$$\tau_{21}(\mathbf{u}) \triangleq (\alpha_{21} - 1)u_{12} + \frac{2u_{22}}{\sqrt{1+u_{10}}} \tag{4.85}$$

$$\tau_{22}(\mathbf{u}) \triangleq (\alpha_{22} - 1)u_{12} + \frac{2u_{22}}{\sqrt{1+u_{10}}} + \frac{2u_{32}}{\sqrt{1+u_9}} + \frac{2u_{42}}{\sqrt{1+u_9}\sqrt{1+u_{10}}}. \tag{4.86}$$

Denote

$$\tau(\mathbf{u}) \triangleq [\tau_{11}(\mathbf{u}), \tau_{21}(\mathbf{u}), \tau_{12}(\mathbf{u}), \tau_{22}(\mathbf{u})]^T. \tag{4.87}$$

It can be shown that, for $l \in \{11, 12, 21, 22\}$,

$$\tilde{i}_l^n = nI_l + \frac{n}{2\alpha_l}\tau_l\left(\frac{1}{n}\sum_{k=1}^{n}\mathbf{U}_k\right). \tag{4.88}$$

Denote the diagonal matrix

$$\Lambda \triangleq \mathrm{diag}\left(\frac{1}{\alpha_{11}}, \frac{1}{\alpha_{21}}, \frac{1}{\alpha_{12}}, \frac{1}{\alpha_{22}}\right). \tag{4.89}$$

We have

$$\frac{1}{\sqrt{n}}\tilde{\mathbf{i}}_{\mathrm{d}}^{n} - \sqrt{n}\,\mathbf{I}_{\mathrm{d}} = \frac{\sqrt{n}}{2}\Lambda\tau\left(\frac{1}{n}\sum_{k=1}^{n}\mathbf{U}_k\right). \tag{4.90}$$

Note that $\tau(\mathbf{0}) = \mathbf{0}$ and the vector function $\tau(\mathbf{u})$ has continuous second-order derivatives in all neighbourhood of $\mathbf{u} = \mathbf{0}$. Therefore, the vector function $\tau(\mathbf{u})$ satisfies the conditions given in Theorem 2.12. The Jacobian matrix $J_\tau(\mathbf{u})$ of $\tau(\mathbf{u})$ with respect to $\mathbf{u}$, calculated at $\mathbf{u} = \mathbf{0}$, is given by

$$J_\tau(\mathbf{0}) = \begin{bmatrix} \alpha_{11}-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{21}-1 & 2 & 0 & 0 & 0 & 0 \\ \alpha_{12}-1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{22}-1 & 2 & 2 & 2 & 0 & 0 \end{bmatrix}. \tag{4.91}$$

Next, by Theorem 2.12, we have that the random vector $\frac{1}{\sqrt{n}}\tilde{\mathbf{i}}_{\mathrm{d}}^{n} - \sqrt{n}\,\mathbf{I}_{\mathrm{d}}$ converges in distribution to a zero-mean Gaussian with covariance matrix V_{d}, which is given by

$$V_{\mathrm{d}} = \frac{1}{n}\cdot\frac{n}{4}\cdot\Lambda J_\tau(\mathbf{0})\mathrm{Cov}(\mathbf{U}_1)[J_\tau(\mathbf{0})]^T\Lambda \tag{4.92}$$

$$= \begin{bmatrix} V_1 & 0 & V_{\mathrm{d}13} & 0 \\ 0 & V_2 & 0 & V_{\mathrm{d}24} \\ V_{\mathrm{d}13} & 0 & V_{\mathrm{d}33} & V_{\mathrm{d}34} \\ 0 & V_{\mathrm{d}24} & V_{\mathrm{d}34} & V_{\mathrm{d}44} \end{bmatrix} \tag{4.93}$$

where

$$V_{\mathrm{d}13} \triangleq \mathsf{V}(h_{11}^2P_1, h_{11}^2P_1 + h_{21}^2P_2) \tag{4.94}$$

$$V_{\mathrm{d}24} \triangleq \mathsf{V}(h_{22}^2P_2, h_{22}^2P_2 + h_{12}^2P_1) \tag{4.95}$$

$$V_{\mathrm{d}33} \triangleq \mathsf{V}(h_{11}^2P_1 + h_{21}^2P_2) + \frac{h_{11}^2P_1h_{21}^2P_2}{(h_{11}^2P_1 + h_{21}^2P_2 + 1)^2} \tag{4.96}$$

$$V_{\mathrm{d}44} \triangleq \mathsf{V}(h_{22}^2 P_2 + h_{12}^2 P_1) + \frac{h_{12}^2 P_1 h_{22}^2 P_2}{(h_{12}^2 P_1 + h_{22}^2 P_2 + 1)^2} \tag{4.97}$$

$$V_{\mathrm{d}34} \triangleq \frac{h_{12} h_{11} P_1 h_{21} h_{22} P_2}{(h_{11}^2 P_1 + h_{21}^2 P_2 + 1)(h_{12}^2 P_1 + h_{22}^2 P_2 + 1)}. \tag{4.98}$$

Thus, V_d has the form

$$V_\mathrm{d} = \begin{bmatrix} V_1 & 0 & * & * \\ 0 & V_2 & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}. \tag{4.99}$$

In the above, the $*$'s represent entries that are inconsequential for the purposes of subsequent analyses.

Define the length-4 rate vector

$$\mathbf{R}_\mathrm{d} \triangleq \left[\frac{\log M_{1n}}{n}, \frac{\log M_{2n}}{n}, \frac{\log(M_{1n} M_{2n})}{n}, \frac{\log(M_{1n} M_{2n})}{n}\right]^T. \tag{4.100}$$

Appealing to Lemma 4.3, with $\gamma = \frac{\log n}{2n}$, we have

$$\begin{aligned} \epsilon_n &\leq 1 - \Pr\left(\frac{1}{\sqrt{n}} \tilde{\mathbf{i}}_\mathrm{d}^n (X_1^n X_2^n Y_1^n Y_2^n) > \sqrt{n}(\mathbf{R}_\mathrm{d} + \gamma \mathbf{1})\right) - K e^{-n\gamma} \\ &\leq 1 - \Pr\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\tilde{\mathbf{i}}_{\mathrm{d}k} - \mathbf{I}_\mathrm{d}) > \sqrt{n}\,(\mathbf{R}_\mathrm{d} - \mathbf{I}_\mathrm{d} + \gamma \mathbf{1})\right) - \frac{K}{\sqrt{n}} \\ &\overset{(a)}{\leq} 1 - \Psi\left(-\sqrt{n}\,(\mathbf{R}_\mathrm{d} - \mathbf{I}_\mathrm{d} + \gamma \mathbf{1})\,; \mathbf{0}, V_\mathrm{d}\right) - O\left(\frac{1}{\sqrt{n}}\right) \\ &\overset{(b)}{\leq} 1 - \Psi(-\sqrt{n}(\mathbf{R}_\mathrm{d} - \mathbf{I}_\mathrm{d}); \mathbf{0}, V_\mathrm{d}) + O\left(\frac{\log n}{\sqrt{n}}\right), \end{aligned} \tag{4.101}$$

where

(a) follows from a variant of the multivariate Berry-Esséen theorem, which is stated in Theorem 2.12; and

(b) follows from Taylor expanding $\mathbf{t} \mapsto \Psi(\mathbf{t}; \mathbf{0}, V_{\mathrm{d}})$.

Due to the strictly very strong interference assumption (Definition 4.2),

$$h_{22}^2 P_2 + 1 < \frac{h_{21}^2 P_2 + h_{11}^2 P_1 + 1}{h_{11}^2 P_1 + 1}. \tag{4.102}$$

Thus, $I_{11} + I_{21} < I_{12}$. Similarly, we have $I_{11} + I_{21} < I_{22}$. Therefore, as $n \to \infty$, we have

$$-\sqrt{n}(\mathbf{R}_{\mathrm{d}} - \mathbf{I}_{\mathrm{d}}) = -\sqrt{n} \begin{bmatrix} \kappa_1 + \frac{L_1}{\sqrt{n}} + \frac{\beta}{n^{3/4}} - I_{11} \\ \kappa_2 + \frac{L_2}{\sqrt{n}} + \frac{\beta}{n^{3/4}} - I_{21} \\ \kappa_1 + \kappa_2 + \frac{L_1}{\sqrt{n}} + \frac{L_2}{\sqrt{n}} + 2\frac{\beta}{n^{3/4}} - I_{12} \\ \kappa_1 + \kappa_2 + \frac{L_1}{\sqrt{n}} + \frac{L_2}{\sqrt{n}} + 2\frac{\beta}{n^{3/4}} - I_{22} \end{bmatrix} \to \begin{bmatrix} -L_1 \\ +\infty \\ +\infty \\ +\infty \end{bmatrix}. \tag{4.103}$$

Thus,

$$\Psi(-\sqrt{n}(\mathbf{R}_{\mathrm{d}} - \mathbf{I}_{\mathrm{d}}); \mathbf{0}, V_{\mathrm{d}}) \to \Psi(-L_1; 0, V_1) = \Phi\left(-\frac{L_1}{\sqrt{V_1}}\right). \tag{4.104}$$

Taking lim sup on both sides of (4.101), we have

$$\limsup_{n\to\infty} \epsilon_n \le 1 - \Phi\left(-\frac{L_1}{\sqrt{V_1}}\right) = \Phi\left(\frac{L_1}{\sqrt{V_1}}\right) \le \epsilon, \tag{4.105}$$

where the final inequality follows the choice of L_1 in (4.60). This completes the proof of the direct part for Case 1.

Case 2: When $\kappa_1 = I_{11}$ and $\kappa_2 = I_{21}$

In this case, we have

$$\Psi(-\sqrt{n}(\mathbf{R}_{\mathrm{d}} - \mathbf{I}_{\mathrm{d}}); \mathbf{0}, V_{\mathrm{d}}) \to \Psi([-L_1 \;\; -L_2]^T; 0, V_{\mathrm{c}}) \tag{4.106}$$

because the second and third entries in (4.103) tend to $+\infty$ (by the strictly very strong interference assumption) while the first and fourth entries tend to L_1 and

L_2 respectively. Thus, as mentioned previously, only the $(1,1),(1,2),(2,1)$ and $(2,2)$ entries in V_{d}, defined in (4.99), are required. Note that V_{c} is a sub-matrix of V_{d} (in the $[1:2,1:2]$ position). Furthermore, by the fact that V_{c} is diagonal, the relation in (4.51) also holds. The rest of the arguments are similar to case 1.

Case 3: When $\kappa_1 < I_{11}$ and $\kappa_2 = I_{21}$

By symmetry, case 3 is proved similarly to case 1.

4.5.3 Proof of Lemma 4.2

We have, for $k \in \{1,2,\ldots,n\}$,

$$\tilde{\imath}_{11k}(x_{1k}x_{2k}Y_{1k}) = \frac{1}{2}\log(h_{11}^2P_1+1) + \frac{(Y_{1k}-h_{21}x_{2k})^2}{2(1+h_{11}^2P_1)} - \frac{(Y_{1k}-h_{11}x_{1k}-h_{21}x_{2k})^2}{2}. \tag{4.107}$$

In this case, $\tilde{\imath}_{11k}(x_{1k}x_{2k}Y_{1k})$ has the same statistics as

$$g_{11}(Z_{1k}) = \frac{1}{2}\log(h_{11}^2P_1+1) + \frac{(Z_{1k}+h_{11}x_{1k})^2}{2(1+h_{11}^2P_1)} - \frac{Z_{1k}^2}{2}. \tag{4.108}$$

Using this expression, we have

$$\mathbb{E}[\tilde{\imath}_{11k}(x_{1k}x_{2k}Y_{1k})] = \frac{1}{2}\log(h_{11}^2P_1+1) + \frac{1+h_{11}^2x_{1k}^2}{2(1+h_{11}^2P_1)} - \frac{1}{2}, \tag{4.109}$$

$$\mathrm{var}[\tilde{\imath}_{11k}(x_{1k}x_{2k}Y_{1k})] = \frac{h_{11}^4P_1^2+2h_{11}^2x_{1k}^2}{2(1+h_{11}^2P_1)^2}. \tag{4.110}$$

Therefore,

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n}\tilde{\imath}_{11k}(x_{1k}x_{2k}Y_{1k})\right] = \frac{1}{2}\log(h_{11}^2P_1+1) + \frac{n+h_{11}^2\|x_1^n\|^2}{2n(1+h_{11}^2P_1)} - \frac{1}{2} \tag{4.111}$$

$$= I_{11}. \tag{4.112}$$

Next, we have

$$\mathrm{var}\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{i}_{11k}(x_{1k}x_{2k}Y_{1k})\right] \overset{(a)}{=} \frac{1}{n}\sum_{k=1}^{n}\mathrm{var}\left[\tilde{i}_{11k}(x_{1k}x_{2k}Y_{1k})\right] \tag{4.113}$$

$$= \frac{1}{n}\cdot\frac{nh_{11}^4P_1^2 + 2h_{11}^2\|x_1^n\|^2}{2(1+h_{11}^2P_1)^2} \tag{4.114}$$

$$= V_1, \tag{4.115}$$

where (a) follows from the mutual independence of Z_{1k}'s.

Similarly, $\tilde{i}_{21k}(x_{1k}x_{2k}Y_{2k})$ for $k \in \{1, 2, ..., n\}$ has the same statistics as

$$g_{21}(Z_{2k}) = \frac{1}{2}\log(h_{22}^2P_2 + 1) + \frac{(Z_{2k} + h_{22}x_{2k})^2}{2(1+h_{22}^2P_2)} - \frac{Z_{2k}^2}{2}, \tag{4.116}$$

and its statistics are given by

$$\mathbb{E}[\tilde{i}_{21}(x_{1k}x_{2k}Y_{2k})] = \frac{1}{2}\log(h_{22}^2P_2 + 1) + \frac{1 + h_{22}^2x_{2k}^2}{2(1+h_{22}^2P_2)} - \frac{1}{2}, \tag{4.117}$$

$$\mathrm{var}[\tilde{i}_{21}(x_{1k}x_{2k}Y_{2k})] = \frac{h_{22}^4P_2^2 + 2h_{22}^2x_{2k}^2}{2(1+h_{22}^2P_2)^2}. \tag{4.118}$$

Similarly, we can find the mean and the variance of the sum of these information densities, yielding

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{\mathrm{c}k}(x_{1k}x_{2k}Y_{1k}Y_{2k})\right] = \mathbf{I}_{\mathrm{c}}, \tag{4.119}$$

$$\mathrm{cov}\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{\mathbf{i}}_{\mathrm{c}k}(x_{1k}x_{2k}Y_{1k}Y_{2k})\right] = V_{\mathrm{c}}. \tag{4.120}$$

Interestingly, because Z_{1j} is independent of Z_{2k}, we have

$$\mathrm{cov}\left[\tilde{i}_{11j}(x_{1j}x_{2j}Y_{1j}), \tilde{i}_{21k}(x_{1k}x_{2k}Y_{2k})\right] = 0, \tag{4.121}$$

for all $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$. This leads directly to the diagonal covariance matrix in (4.120). The lemma is proved.

■ 4.5.4 Proof of Lemma 4.4

Similar to [85, Lem. 61] and [79, Prop. 3], we can prove that K_{11} and K_{21} are upper bounded by a constant when n is sufficiently large.

The marginal conditional output distribution $P_{Y_1^n|X_2^n}$ induced by feeding the input distributions, given in (4.63), into the Gaussian IC can be shown to be

$$P_{Y_1^n|X_2^n}(y_1^n|x_2^n) = \frac{1}{2\pi^{n/2}}\Gamma\left(\frac{n}{2}\right) e^{-nh_{11}^2P_1/2}e^{-\|y_1^n-h_{21}x_2^n\|^2/2} \\ \times \frac{I_{n/2-1}(\|y_1^n - h_{21}x_2^n\|\sqrt{nP_1}h_{11})}{(\|y_1^n - h_{21}x_2^n\|\sqrt{nP_1}h_{11})^{n/2-1}}, \tag{4.122}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind and v-th order. The marginal distribution $P_{Y_2^n|X_1^n}$ has a similar form to the above.

We have

$$D_{11}(y_1^n|x_2^n) \triangleq \frac{P_{Y_1^n|X_2^n}(y_1^n|x_2^n)}{Q_{Y_1^n|X_2^n}(y_1^n|x_2^n)} \\ = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)[2e^{-h_{11}^2P_1}(1+h_{11}^2P_1)]^{n/2}e^{-\frac{h_{11}^2P_1\|y_1^n-h_{21}x_2^n\|^2}{2(1+h_{11}^2P_1)}} \\ \times \frac{I_{n/2-1}(\|y_1^n - h_{21}x_2^n\|\sqrt{nP_1}h_{11})}{(\|y_1^n - h_{21}x_2^n\|\sqrt{nP_1}h_{11})^{n/2-1}}. \tag{4.123}$$

Note that the gamma function $\Gamma(\cdot)$ can take different forms. Using Binet's first formula for $\log\Gamma(z)$ [23, Chap. 1], we have

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}\mathrm{d}t. \tag{4.124}$$

Note that the fourth term converges to 0 as $z \to \infty$. Thus, we can upper-bound

$\Gamma\left(\frac{n}{2}\right)$ by

$$\Gamma\left(\frac{n}{2}\right) \leq \left(\frac{n}{2}-\frac{1}{2}\right)\log\frac{n}{2}-\frac{n}{2}+\frac{1}{2}\log(2\pi)+c_n \tag{4.125}$$

where $\{c_n\}_{n=1}^{\infty}$ is a sequence of numbers that converges to 0.

From Prokhorov's work [88] and [85, Lem. 61], when k is even we can upper-bound the modified Bessel function as

$$z^{-k}I_k(z) \leq \sqrt{\frac{\pi}{8}}(k^2+z^2)^{-1/4}(k+\sqrt{k^2+z^2})^{-k}e^{\sqrt{k^2+z^2}}. \tag{4.126}$$

Note that $I_{n/2-1}(\cdot) < I_{n/2-3/2}(\cdot)$. When n is odd, an upper bound is obtained by replacing $I_{n/2-1}(\cdot)$ by $I_{n/2-3/2}(\cdot)$. Thus, it is sufficient to consider the upper bound on $D(y_1^n|x_2^n)$ when n is even.

After some manipulations, we can show that

$$D_{11}(y_1^n|x_2^n) \leq \exp\left[c_{11}+c_n+\frac{n}{2}\phi_{\xi,P_1,n}\left(\frac{\|y_1^n-h_{21}x_2^n\|^2}{n}\right)\right], \tag{4.127}$$

where

$$c_{11} \triangleq \log\frac{1}{2}+\log\sqrt{\frac{\pi}{8}}+\frac{1}{2}\log(2\pi) \tag{4.128}$$

$$\begin{aligned}\phi_{\xi,P_1,n}(z) \triangleq{}& \log\left(2(1+h_{11}^2P_1)e^{-(1+h_{11}^2P_1)}\right)-\frac{h_{11}^2P_1z}{h_{11}^2P_1+1}+\sqrt{\xi^2+4h_{11}^2P_1z}\\ &-\xi\log\left(\xi+\sqrt{\xi^2+4h_{11}^2P_1z}\right)-\frac{1-\xi}{2}\log\left(\sqrt{\xi^2+4h_{11}^2P_1z}\right)\end{aligned} \tag{4.129}$$

$$\xi \triangleq \frac{n/2-1}{n/2}. \tag{4.130}$$

Note that

$$\lim_{n\to\infty}\phi_{\xi,P_1,n}(z)=\phi_{P_1}(z), \tag{4.131}$$

where

$$\phi_{P_1}(z) \triangleq \log\left(2(1+h_{11}^2P_1)e^{-(1+h_{11}^2P_1)}\right) - \frac{h_{11}^2P_1z}{h_{11}^2P_1+1} + \sqrt{1+4h_{11}^2P_1z} - \log\left(1+\sqrt{1+4h_{11}^2P_1z}\right). \tag{4.132}$$

It can be shown that $\phi_{P_1}(z) \leq 0$. Equality occurs when $z = 1+h_{11}^2P_1$. Therefore, we have K_{11} is upper bounded by a constant, when n is sufficiently large. Similarly, we can shown that K_{21} is upper bounded by a constant when n is sufficiently large.

It is hard to derive a closed-form expression for the output distribution $P_{Y_1^n}$ induced by the input distributions in (4.63) and the IC. However, we can characterize the distribution of $B^n \triangleq h_{11}X_1^n + h_{21}X_2^n$ (see [79, Equations (137-151)]). We have

$$P_{B^n}(b^n) = \begin{cases} 0 & \text{if } \|b^n\| \leq |h_{11}\sqrt{nP_1} - h_{21}\sqrt{nP_2}| \\ 0 & \text{if } \|b^n\| \geq |h_{11}\sqrt{nP_1} + h_{21}\sqrt{nP_2}| \\ \phi_{\mathrm{B}}(b^n) & \text{otherwise,} \end{cases} \tag{4.133}$$

where

$$\phi_{\mathrm{B}}(b^n) \triangleq \frac{1}{h_{21}^n}\sqrt{\frac{P_2}{\pi P_1}}\frac{h_{21}}{h_{11}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\frac{1}{S_n(\sqrt{nP_2})}\frac{1}{\|b^n\|} \times\left(1-\left(\frac{\|b^n\|^2+n(h_{11}^2P_1-h_{21}^2P_2)}{2h_{11}\sqrt{nP_1}\|b^n\|}\right)^2\right)^{(n-3)/2} \tag{4.134}$$

$$\cos\theta_0 \triangleq \frac{\|b^n\|^2+n(h_{11}^2P_1-h_{21}^2P_2)}{2h_{11}\sqrt{nP_1}\|b^n\|}. \tag{4.135}$$

Define the auxiliary input distribution

$$Q_{B^n}(b^n) \triangleq N(b^n; \mathbf{0}, (h_{11}^2P_1+h_{21}^2P_2)\mathbf{I}_{n\times n}). \tag{4.136}$$

If this distribution is used as an input for the channel $Y_1^n = B^n + Z_1^n$, the corresponding output distribution is $Q_{Y_1^n}$. If it can be proved that

$$K'_{12} \triangleq \sup_{b^n} \frac{P_{B^n}(b^n)}{Q_{B^n}(b^n)} \tag{4.137}$$

is uniformly bounded when n is sufficiently large, then, for any y_1^n, we have

$$\begin{aligned} P_{Y_1^n}(y_1^n) &= \int_{\mathbb{R}^n} P_{B^n}(b^n) P_{Y_1^n|B^n}(y_1^n|b^n)\mathrm{d}b^n \\ &\le \int_{\mathbb{R}^n} K'_{12} Q_{B^n}(b^n) P_{Y_1^n|B^n}(y_1^n|b^n)\mathrm{d}b^n \\ &= K'_{12} Q_{Y_1^n}(y_1^n). \end{aligned} \tag{4.138}$$

Therefore, $K_{12} \le K'_{12}$. That is, K_{12} is uniformly bounded when n is sufficiently large. Now, we prove the finiteness of K'_{12}. Define

$$D_{12}(b^n) \triangleq \frac{P_{B^n}(b^n)}{Q_{B^n}(b^n)}. \tag{4.139}$$

Next, by simple algebraic manipulations, it can be shown that

$$D_{12}(b^n) \le \exp\left[c_{12} + c_n + \rho_{12n}\left(\frac{\|b^n\|^2}{n}\right)\right] \tag{4.140}$$

where

$$c_{12} \triangleq \log\left(\frac{P_2}{\sqrt{\pi P_1}}\frac{h_{21}}{h_{11}}\right) + \frac{\log(2\pi)}{2} \tag{4.141}$$

$$\begin{aligned} \rho_{12n}(z) \triangleq& -\frac{\log z}{n} + \log\frac{h_{11}^2 P_1 + h_{21}^2 P_2}{e h_{21}^2 P_2} + \frac{z}{h_{11}^2 P_1 + h_{21}^2 P_2} \\ &+ \frac{n-3}{n}\log\left(1 - \frac{(z + h_{11}^2 P_1 - h_{21}^2 P_2)^2}{4 h_{11}^2 P_1 z}\right), \end{aligned} \tag{4.142}$$

and where $\{c_n\}$ is a sequence converging to 0, and

$$|h_{11}\sqrt{nP_1} - h_{21}\sqrt{nP_2}| < z < |h_{11}\sqrt{nP_1} + h_{21}\sqrt{nP_2}|. \tag{4.143}$$

Note that

$$\lim_{n\to\infty} \rho_{12n}(z) = \rho_{12}(z), \tag{4.144}$$

where

$$\rho_{12}(z) \triangleq \log \frac{h_{11}^2 P_1 + h_{21}^2 P_2}{e h_{21}^2 P_2} + \frac{z}{h_{11}^2 P_1 + h_{21}^2 P_2} + \log\left(1 - \frac{(z + h_{11}^2 P_1 - h_{21}^2 P_2)^2}{4 h_{11}^2 P_1 z}\right). \tag{4.145}$$

It can be shown that $\rho_{12}(z) \leq 0$. Equality occurs at $z = h_{11}^2 P_1 + h_{21}^2 P_2$. Thus, we can conclude that K_{12}' is upper bounded by a constant when n is sufficiently large. Similarly, K_{22} can be proved to be upper bounded by a constant for n sufficiently large.

■ 4.5.5 Proof of Lemma 4.1

Given the joint distribution in (4.22), denote two of the marginal distributions as $P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n)$ and $P_{Y_2^n X_1^n X_2^n}(y_2^n x_1^n x_2^n)$, and denote two of the conditional distributions as $P_{X_1^n|X_2^n}(x_1^n|x_2^n)$, and $P_{X_2^n|X_1^n}(x_2^n|x_1^n)$, where

$$P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n) \triangleq \sum_{y_2^n} P_{Y_2^n Y_1^n X_1^n X_2^n}(y_2^n y_1^n x_1^n x_2^n), \tag{4.146}$$

$$P_{X_1^n|X_2^n}(x_1^n|x_2^n) \triangleq \frac{\sum_{y_2^n y_1^n} P_{Y_2^n Y_1^n X_1^n X_2^n}(y_2^n y_1^n x_1^n x_2^n)}{P_{X_2^n}(x_2^n)}, \tag{4.147}$$

and the remaining distributions are defined similarly.

Define the decoding regions

$$D_{1s_1} \triangleq \{y_1^n \in \mathcal{Y}_1^n | g_{1n}(y_1^n) = s_1\} \tag{4.148}$$

$$D_{2s_2} \triangleq \{y_2^n \in \mathcal{Y}_2^n | g_{2n}(y_2^n) = s_2\} \tag{4.149}$$

$$D'_{1s_1} \triangleq \{(y_1^n y_2^n) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n | y_1^n \in D_{1s_1}\} \tag{4.150}$$

$$D'_{2s_2} \triangleq \{(y_1^n y_2^n) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n | y_2^n \in D_{2s_2}\}, \tag{4.151}$$

where $s_1 \in \{1, 2, \ldots, M_{1n}\}$ and $s_2 \in \{1, 2, \ldots, M_{2n}\}$.

The decoding functions g_{jn} and the encoding functions f_{jn}, for $j = 1, 2$, in this proof, are defined in the section for problem formulation.

Note that

$$\frac{W_1^n(y_1^n|x_1^n x_2^n)}{Q_{Y_1^n|X_2^n}(y_1^n|x_2^n)} = \frac{P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n)}{Q_{Y_1^n X_2^n}(y_1^n x_2^n) P_{X_1^n|X_2^n}(x_1^n|x_2^n)} \tag{4.152}$$

$$\stackrel{(a)}{=} \frac{P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n)}{Q_{Y_1^n X_2^n}(y_1^n x_2^n) P_{X_1^n}(x_1^n)} \tag{4.153}$$

$$\stackrel{(b)}{=} M_{1n} \frac{P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n)}{Q_{Y_1^n X_2^n}(y_1^n x_2^n)}, \tag{4.154}$$

where

(a) follows from the fact that X_1^n and X_2^n are independent; and

(b) follows from the fact that $P_{X_1^n}(x_1^n) = \frac{1}{M_{1n}}$ for all x_1^n in the first codebook.

Similarly, we have

$$\frac{W_2^n(y_2^n|x_1^n x_2^n)}{Q_{Y_2^n|X_1^n}(y_2^n|x_1^n)} = M_{2n} \frac{P_{Y_2^n X_1^n X_2^n}(y_2^n x_1^n x_2^n)}{Q_{Y_2^n X_1^n}(y_2^n x_1^n)}. \tag{4.155}$$

Define

$$B_{1s_1s_2} \triangleq \left\{ y_1^n \in \mathcal{Y}_1^n \middle| \frac{P_{Y_1^n X_1^n X_2^n}(y_1^n f_{1n}(s_1) f_{2n}(s_2))}{Q_{Y_1^n X_2^n}(y_1^n f_{2n}(s_2))} \le e^{-n\gamma} \right\} \tag{4.156}$$

$$B'_{1s_1s_2} \triangleq \{(y_1^n y_2^n) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n | y_1^n \in B_{1s_1s_2}\} \tag{4.157}$$

$$B_{2s_1s_2} \triangleq \left\{ y_2^n \in \mathcal{Y}_2^n \middle| \frac{P_{Y_2^n X_1^n X_2^n}(y_2^n f_{1n}(s_1) f_{2n}(s_2))}{Q_{Y_2^n X_1^n}(y_2^n f_{1n}(s_1))} \le e^{-n\gamma} \right\} \tag{4.158}$$

$$B'_{2s_1s_2} \triangleq \{(y_1^n y_2^n) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n | y_2^n \in B_{2s_1s_2}\}, \tag{4.159}$$

where $s_1 \in \{1, 2, \ldots, M_{1n}\}$ and $s_2 \in \{1, 2, \ldots, M_{2n}\}$.

Define

$$G_1 \triangleq \left\{ (x_1^n x_2^n y_1^n y_2^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}_1^n \times \mathcal{Y}_2^n \middle| \frac{P_{Y_1^n X_1^n X_2^n}(y_1^n x_1^n x_2^n)}{Q_{Y_1^n X_2^n}(y_1^n x_2^n)} \le e^{-n\gamma} \right\} \tag{4.160}$$

$$G_2 \triangleq \left\{ (x_1^n x_2^n y_2^n y_2^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}_1^n \times \mathcal{Y}_2^n \middle| \frac{P_{Y_2^n X_1^n X_2^n}(y_2^n x_1^n x_2^n)}{Q_{Y_2^n X_1^n}(y_2^n x_1^n)} \le e^{-n\gamma} \right\}, \tag{4.161}$$

where $s_1 \in \{1, 2, \ldots, M_{1n}\}$ and $s_2 \in \{1, 2, \ldots, M_{2n}\}$.

In order to prove this lemma, it suffices to prove

$$P_{X_1^n X_2^n Y_1^n Y_2^n}(G_1 \cup G_2) \le \epsilon_n + 2e^{-n\gamma}. \tag{4.162}$$

We are going to prove the validity of this inequality. We have

$$P_{X_1^n X_2^n Y_1^n Y_2^n}(G_1 \cup G_2) \tag{4.163}$$

$$= \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{1s_1s_2} \cup B'_{2s_1s_2}) \tag{4.164}$$

$$= \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} [P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), (B'_{1s_1s_2} \cup B'_{2s_1s_2}) \cap (D_{1s_1} \times D_{2s_2})^c)$$

$$+ P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), (B'_{1s_1s_2} \cup B'_{2s_1s_2}) \cap (D_{1s_1} \times D_{2s_2}))] \tag{4.165}$$

$$\leq \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} [P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), (D_{1s_1} \times D_{2s_2})^c)$$

$$+ P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), (B'_{1s_1s_2} \cup B'_{2s_1s_2}) \cap (D_{1s_1} \times D_{2s_2}))] \tag{4.166}$$

$$\leq \epsilon_n + \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} [P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{1s_1s_2} \cap (D_{1s_1} \times D_{2s_2}))$$

$$+ P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{2s_1s_2} \cap (D_{1s_1} \times D_{2s_2}))]. \tag{4.167}$$

Next, we upper-bound the second and third terms. We have

$$\sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{1s_1s_2} \cap (D_{1s_1} \times D_{2s_2})) \tag{4.168}$$

$$\leq \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{1s_1s_2} \cap D'_{1s_1}) \tag{4.169}$$

$$= \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \sum_{(y_1^n y_2^n) \in B'_{1s_1s_2} \cap D'_{1s_1}} P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2) y_1^n y_2^n) \tag{4.170}$$

$$= \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \sum_{y_1^n \in B_{1s_1s_2} \cap D_{1s_1}} P_{X_1^n X_2^n Y_1^n}(f_{1n}(s_1) f_{2n}(s_2) y_1^n) \tag{4.171}$$

$$\stackrel{(a)}{\leq} \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \sum_{y_1^n \in B_{1s_1s_2} \cap D_{1s_1}} Q_{X_2^n Y_1^n}(f_{2n}(s_2) y_1^n) e^{-n\gamma} \tag{4.172}$$

$$\leq \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \sum_{y_1^n \in D_{1s_1}} Q_{X_2^n Y_1^n}(f_{2n}(s_2) y_1^n) e^{-n\gamma} \tag{4.173}$$

$$= \sum_{s_2=1}^{M_{2n}} Q_{X_2^n}(f_{2n}(s_2)) e^{-n\gamma} \tag{4.174}$$

$$\leq e^{-n\gamma}, \tag{4.175}$$

where (a) follows from the definition of $B_{1s_1s_2}$.

Similarly to the above, we can show that

$$\sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} P_{X_1^n X_2^n Y_1^n Y_2^n}(f_{1n}(s_1) f_{2n}(s_2), B'_{2s_1s_2} \cap (D_{1s_1} \times D_{2s_2})) \leq e^{-n\gamma}. \quad (4.176)$$

Thus, we have proved the lemma.

4.5.6 Proof of Lemma 4.3

First, we consider the case without cost constraints. Define the sets

$$T_{j1} \triangleq \left\{(x_1^n x_2^n y_j^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}_j^n | \tilde{i}_{j1}^n > \log M_{jn} + n\gamma\right\} \quad (4.177)$$

$$T_{j2} \triangleq \left\{(x_1^n x_2^n y_j^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}_j^n | \tilde{i}_{j2}^n > \log M_{1n} M_{2n} + n\gamma\right\} \quad (4.178)$$

$$T_j = T_{j1} \cap T_{j2}, \quad (4.179)$$

where the modified information densities $\tilde{i}_{j1}^n$ and $\tilde{i}_{j2}^n$ are defined in (4.23) and (4.24).

a) *Codebook generation*

Fix a joint distribution $P_{X_1^n}(x_1^n) P_{X_2^n}(x_2^n)$. Generate M_{jn} codewords $f_{jn}(s_j)$, for $s_j \in \{1, 2, ..., M_{jn}\}$, and $j = 1, 2$. We denote the random codewords $f_{jn}(s_j)$ as $X_j^n(s_j)$ in the proof of this lemma.

b) *Encoding rules at transmitters*:

To transmit message s_j, transmitter j sends the codewords $X_j^n(s_j)$.

c) *Decoding rules at receivers*

Upon receiving an output y_1^n, receiver 1 finds the unique message $\hat{s_1}$ such that

$$(x_1^n(\hat{s_1})x_2^n(\hat{s}_2)y_1^n) \in T_1^n \tag{4.180}$$

for some $\hat{s_2}$. An error is declared otherwise. This decoding rule is also known as *simultaneous non-unique decoding rule* [22, Section 6.2]. The decoding rule at receiver 2 is defined similarly to the above.

d) *Calculation of probability of error*

For ease of presentation, we define the event, for $j = 1, 2$,

$$E_{js_1s_2} \triangleq \{((X_1^n(s_1)X_2^n(s_2)Y_j^n) \in T_j^n\}. \tag{4.181}$$

Decoding errors at receiver 1 is bounded as

$$\frac{1}{M_{1n}M_{2n}} \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \left[\Pr(E_{1s_1s_2}^c) + \Pr\left(\bigcup_{s_1' \neq s_1, \text{ any } s_2\prime} E_{1s_1's_2'} \right) \right] \tag{4.182}$$

$$\overset{(a)}{=} \Pr(E_{111}^c) + \Pr\left(\bigcup_{s_1' \neq 1, \text{ any } s_2\prime} E_{1s_1's_2'} \right) \tag{4.183}$$

$$\overset{(b)}{\leq} \Pr(E_{111}^c) + \sum_{s_1' \neq 1} \Pr(E_{1s_1'1}) + \sum_{s_1' \neq 1, s_2' \neq 1} \Pr(E_{1s_1's_2'}), \tag{4.184}$$

where

(a) follows from the symmetry of the codebooks, and

(b) follows from the union rule.

Next, we bound the second term in the equation right above.

$$\sum_{s_1'\neq 1} \Pr(E_{1s_1'1}) = (M_{1n}-1)\Pr(\{(X_1^n(s_1')X_2^n(1)Y_1^n) \in T_1\}) \tag{4.185}$$

$$\overset{(a)}{=} (M_{1n}-1) \sum_{(x_1^n x_2^n y_1^n)\in T_1} P_{X_1^n}(x_1^n) P_{X_2^n Y_1^n}(x_2^n y_1^n) \tag{4.186}$$

$$\leq (M_{1n}-1) \sum_{(x_1^n x_2^n y_1^n)\in T_{11}} P_{X_1^n}(x_1^n) P_{X_2^n Y_1^n}(x_2^n y_1^n) \tag{4.187}$$

$$\leq (M_{1n}-1) \sum_{(x_1^n x_2^n y_1^n)\in T_{11}} K_{11} P_{X_1^n}(x_1^n) Q_{X_2^n Y_1^n}(x_2^n y_1^n) \tag{4.188}$$

$$\overset{(b)}{\leq} (M_{1n}-1) \times \sum_{(x_1^n x_2^n y_1^n)\in T_{11}} K_{11} P_{X_1^n}(x_1^n) P_{X_2^n}(x_2^n) W_1(y_1^n|x_2^n x_1^n) e^{-n\gamma}\frac{1}{M_{1n}} \tag{4.189}$$

$$\leq K_{11}e^{-n\gamma} \tag{4.190}$$

where

(a) follows from the fact that $X_1^n(s_1')$ and $(X_2^n(1), Y_1^n)$ are independent, when message pair $(1,1)$ are transmitted by transmitters, and

(b) follows from the definition of the set T_{11}.

Similarly, we can show that

$$\sum_{s_1'\neq 1} \Pr(E_{1s_1's_2'}) \leq K_{12}e^{-n\gamma}. \tag{4.191}$$

Similarly, we can upper-bound the decoding error events at receiver 2 by

$$\frac{1}{M_{1n}M_{2n}} \sum_{s_1=1}^{M_{1n}} \sum_{s_2=1}^{M_{2n}} \left[\Pr(E_{2s_1s_2}^c) + \Pr\left(\bigcup_{s_2'\neq s_2,\ \text{any } s_1\prime} E_{2s_1's_2'} \right) \right] \tag{4.192}$$

$$\leq \Pr(E_{211}^c) + (K_{21}+K_{22})e^{-n\gamma}. \tag{4.193}$$

Therefore, we have

$$\epsilon_n \leq \Pr(E_{111}^c \cup E_{211}^c) + (K_{11} + K_{12})e^{-n\gamma} + (K_{21} + K_{22})e^{-n\gamma} \tag{4.194}$$

$$= \Pr(\mathcal{E}_{11} \cup \mathcal{E}_{12} \cup \mathcal{E}_{21} \cup \mathcal{E}_{22}) + Ke^{-n\gamma}. \tag{4.195}$$

In the case where the cost constraint is imposed, we have

$$\epsilon_n \leq \Pr(\mathcal{E}_{11} \cup \mathcal{E}_{12} \cup \mathcal{E}_{21} \cup \mathcal{E}_{22}) + Ke^{-n\gamma} + P_{X_1^n}P_{X_2^n}(\{X_1^n \notin \mathcal{F}_{1n} \cup X_2^n \notin \mathcal{F}_{2n}\}). \tag{4.196}$$

Thus, we have proved the lemma.

Chapter 5

Second-order Rate-Distortion Function for Source Coding with Side Information

THE class of source coding problems with side information is important as it can model many practical problems. Consider a scenario when a source wants to transmit a high-resolution image to a receiver who happens to have a low-resolution version of the same image. In another example, the source may be a piece of music contaminated by a background noise source and the intended receiver has already had samples of the background noise. This chapter focuses on the approximation of the finite-blocklength rate-distortion function for the source coding problem with side information available at both the encoder and the decoder.

5.1 Introduction

In lossless source coding, the Shannon entropy of a source is, on average, the minimum number of bits required to represent a given source [96]. In lossy source coding, the rate-distortion function (which, in this chapter, is more specifically called the rate-distortion function without side information) plays the role of the Shannon entropy [98]. The rate-distortion function without side information is the minimum number of bits per symbol required to reconstruct a given source with the probability of excess distortion being asymptotically small, or with an average distortion that does not exceed a specified upper bound.

The rate-distortion problem without side information can be extended to the case when the side information is available at both the encoder and the decoder [7, 36], only causally available at the decoder [119], or non-causally available at the decoder (i.e., Wyner-Ziv problem) [121]. The rate-distortion function for stationary-ergodic sources with side information was found in [71]. The rate-distortion function for mixed types of side information (i.e., a mixture of some side information known at both the encoder and the decoder and some known only at the decoder) was evaluated in [27]. For memoryless sources, delayed side information at the decoder does not improve the rate-distortion function. However, this is not the case for sources with memory [99]. The authors of [73] considered source coding with side information, and with distortion measures as functions of side information.

All the results shown above hold provided the blocklength, i.e., the number of source symbols, is allowed to grow without bound. However, some applications are required to operate with short blocklengths due to delay or complexity constraints at the destination. Thus, it is of high interest to characterize the finite blocklength rate-distortion function, i.e., the minimum number of bits per symbol that is required to reconstruct a source at a given fixed blocklength. This is, in general, a difficult task, and thus, we focus on approximating this quantity.

5.1.1 Related Works

Strassen [101] obtained the second-order coding rate for almost lossless source coding without side information. Recently, Hayashi [42] considered second-order coding rate for fixed-length source coding and showed that the outputs of fixed-length source codes are not uniformly distributed (debunking Han's folklore theorem [40] in the second-order sense). The second-order analysis is closely related to the method of information spectrum [42]. In particular, the second-order analysis of the source coding can be derived by the combination of the central limit theorem and the method of information spectrum introduced by Han [39]. Kostina and Verdú [56] and Ingber and Kochman [48] characterized the dispersion of lossy source coding problem without side information. When the source is stationary and memoryless, they showed that the finite blocklength rate-distortion function without side information $R_{\mathsf{noSI}}(n, D, \epsilon)$ can be approximated as

$$R_{\mathsf{noSI}}(n, D, \epsilon) = R_{\mathsf{noSI}}(D) + \frac{\sqrt{V_{\mathsf{noSI}}(D)}}{n} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right), \tag{5.1}$$

where $R_{\mathsf{noSI}}(D)$ is the rate-distortion function without side information, $V_{\mathsf{noSI}}(D)$ is the *dispersion* that characterizes the convergence rate to the Shannon limit $R_{\mathsf{noSI}}(D)$, n is the blocklength, D is the excess distortion threshold, and ϵ is the upper bound on the probability that the distortion exceeds D. The rate-distortion problem may also be studied from the moderate deviations perspective

[103] and the fundamental limit there is also dependent on $V_{\mathsf{noSI}}(D)$. Achievable second-order coding rates for the Wyner-Ahlswede-Korner problem of almost-lossless source coding with rate-limited side-information, the Wyner-Ziv problem of lossy source coding with side-information at the decoder and the Gelfand-Pinsker problem of channel coding with non-causal state information available at the decoder were established in [118]. The paper [53] studied second-order coding rates for the fixed-to-variable lossless compression. For other related works in the study of fixed error asymptotics, the reader is referred to [104].

■ 5.1.2 Main Contributions

This chapter focuses on the analysis and approximation of the finite blocklength rate-distortion function for source coding with side information available at both the encoder and the decoder. The contributions of this chapter are stated below.

- A non-asymptotic achievability bound is established for the problem of lossy source coding with side information available at both the encoder and the decoder.
- We establish the second-order coding rate for the discrete memoryless source with a side information variable taking values in a finite alphabet. As a corollary, we obtain the second-order coding rate for the case when the source alphabet, the reconstruction alphabet and the side information alphabet are finite and the distortion measure is the Hamming distance.
- We establish the second-order coding rate for Gaussian source with Gaus-

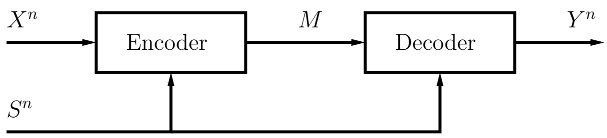

Figure 5.1. Source coding with side information

sian side information and the squared-error distortion measure.

- When the source has memory, we establish the second-order coding rate for the case where the sequence of source and side information variables jointly forms a time-homogeneous Markov chain.

5.2 Problem formulation and definitions

Let $\mathcal{X}$ be the source alphabet, let $\mathcal{Y}$ be the reproduction alphabet, and let $\mathcal{S}$ be the side information alphabet. The random variables X, Y and S follow the distribution

$$P_{YXS}(yxs) = P_{Y|XS}(y|xs)P_{X|S}(x|s)P_S(s). \tag{5.2}$$

We use a single-letter fidelity criterion to measure the distortion between the source sequence x^n and the reproducing sequence y^n, i.e.,

$$d(x^n, y^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i), \tag{5.3}$$

where $d : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{R}_+$, for $n \in \mathbb{N}$, is a bounded real-valued non-negative distortion function.

Definition 5.1. An (M_n, n, D, ϵ_n)-code for the source coding system with side information (see Figure 5.1) consists of an encoding function

$$\phi_n : \mathcal{X}^n \times \mathcal{S}^n \to \mathcal{M}_n \triangleq \{1, 2, \ldots, M_n\}, \tag{5.4}$$

and a decoding function

$$\psi_n : \mathcal{M}_n \times \mathcal{S}^n \to \mathcal{Y}^n, \tag{5.5}$$

such that the probability of excess distortion satisfies

$$\Pr\{d[X^n, \psi_n(\phi_n(X^n, S^n), S^n)] > D\} \leq \epsilon_n. \tag{5.6}$$

An (M_n, n, D, ϵ_n)-code, which is defined as shown above, is called a D-*semifaithful* code in the rate-distortion literature [127, 128].

Definition 5.2. A rate R is defined to be (ϵ, D)-*achievable* if there exists a sequence of (M_n, n, D, ϵ_n)-codes satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n \leq R, \tag{5.7}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon. \tag{5.8}$$

In contrast to the above definition, the following definition is non-asymptotic.

Definition 5.3. A rate R is defined to be (ϵ, D, n)-*achievable* if there exists a $(\lfloor \exp(nR) \rfloor, n, D, \epsilon_n)$-code. The (ϵ, D, n) finite blocklength rate-distortion function $R(\epsilon, D, n)$ is defined as the infimum of the set of all (ϵ, D, n)-achievable rates.

The following definition defines the quantity of interest in this chapter.

Definition 5.4. A number $L \in \mathbb{R}$ is defined to be *second-order* (ϵ, D, κ)-*achievable* if there exists a sequence of (M_n, n, D, ϵ_n)-codes satisfying

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_n - n\kappa) \leq L, \tag{5.9}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon. \tag{5.10}$$

The (ϵ, D, κ) *second-order rate-distortion function* $L^*(\epsilon, D, \kappa)$ is defined as the infimum of the set of all second-order (ϵ, D, κ)-achievable rates.

The aim of this chapter is to characterize the (ϵ, D, κ) second-order rate-distortion function $L^*(\epsilon, D, \kappa)$ for source coding with side information available at both the encoder and the decoder.

Before presenting the main result, we state some definitions that will be used throughout this chapter.

Definition 5.5. Fix the distribution of XS as P_{XS}. Define the rate-distortion function with side information as

$$R(X; D|S) = \min_{P_{Y|XS}} I(X; Y|S), \tag{5.11}$$

where the minimum is taken over the set of all marginal conditional distributions $P_{Y|XS}$ satisfying

$$P_{Y|XS}(y|xs) \geq 0 \qquad \text{for all } (y, x, s), \tag{5.12}$$

$$\sum_{y\in\mathcal{Y}} P_{Y|XS}(y|xs) = 1, \tag{5.13}$$

$$\sum_{s\in\mathcal{S}, x\in\mathcal{X}, y\in\mathcal{Y}} P_{Y|XS}(y|xs)P_{X|S}(x|s)P_S(s)d(x, y) \leq D. \tag{5.14}$$

To make the dependence on the distribution P_{XS} explicit, we sometimes also denote $R(X; D|S)$ as $R(P_{X|S}, D|P_S)$. Assume the distribution that achieves the minimum in (5.11) is unique. When there is no side information, i.e., $S = \emptyset$, we recover the rate-distortion function without side information denoted as $R(X; D)$ or $R(P_X, D)$.

When the excess distortion criterion is employed, we have the following first-order result for the source coding problem with side information [7] (i.e., the conditional rate-distortion problem [36]),

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} R(\epsilon, D, n) = R(X; D|S). \tag{5.15}$$

In order to characterize the second-order rate-distortion function, we state the following definitions. The notion of information densities will play an important role in characterizing the second-order rate-distortion function. In fact, in order to deal with the constraints inherent in the rate-distortion problem, the concept of D-tilted information densities, which was introduced in [57], is useful.

Definition 5.6. Define the conditional information densities as follows:

$$i_{X;Y|S}(x; y|s) \triangleq \log \frac{P_{XY|S}(xy|s)}{P_{Y|S}(y|s) P_{X|S}(x|s)}, \quad \text{and} \tag{5.16}$$

$$i_{X|S}(x|s) \triangleq i_{X;X|S}(x; x|s). \tag{5.17}$$

Note that $i_{X|S}$ is also known as the conditional self-information.

Definition 5.7. Define the conditional D-tilted information density as follows:

$$j_{X|S}(x, D|s) \triangleq \log \frac{1}{\mathbb{E}[\exp\{\lambda^* D - \lambda^* d(x, Y^*)\}|S = s]} \tag{5.18}$$

where $P_{Y^*|XS}$ is the distribution that achieves the minimum in (5.5), the expectation is taken with respect to the induced output distribution $P_{Y^*|S}(y|s) =$

$\sum_x P_{Y^*|XS}(y|x,s)P_{X|S}(x|s)$, and λ^* is defined as

$$\lambda^* \triangleq -\frac{dR(P_{X|S}, D|P_S)}{dD}. \tag{5.19}$$

Remark 5.1. In this definition, the conditional D-tilted information density has a built-in feature which takes the distortion constraint into consideration.

The conditional D-tilted information density $j_{X|S}(x, D|s)$ has some important properties which can be found in [57]. We review them here.

Lemma 5.1. *The conditional D-tilted information density $j_{X|S}(x, D|s)$ has the following properties.*

1. $j_{X|S}(x, D|s) = i_{X;Y*|S}(x; y|s) + \lambda^* d(x, y) - \lambda^* D$.

2. $R(X; D|S) = \mathbb{E}[j_{X|S}(X, D|S)]$.

3. *For any $P_{Y|S}$ where $X \to S \to Y$, we have* $\mathbb{E}[\exp\{\lambda^* d - \lambda^* d(X, Y) + j_{X|S}(X, D|S)\}] \leq 1$.

In the achievability proof of the conditional rate-distortion problem, the following concept is important.

Definition 5.8. Given a source sequence $x^n \in \mathcal{X}^n$, define the D-ball $B_D(x^n)$ around this sequence as

$$B_D(x^n) \triangleq \{y^n \in \mathcal{Y}^n | d(x^n, y^n) \leq D\}. \tag{5.20}$$

The following is the cumulative distribution function of a standard Gaussian distribution

$$\Phi(t) \triangleq \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\, \mathrm{d}u. \tag{5.21}$$

The complementary cumulative distribution function is $Q(t) \triangleq 1 - \Phi(t)$. Since these functions are monotonic, they admit inverses, which we will denote as Φ^{-1} and Q^{-1}.

■ 5.3 Non-Asymptotic Bounds

In this section, we first present a non-asymptotic achievability bound.

Lemma 5.2 (Achievability). *For every $P_{\bar{Y}^n|S^n}$, there exists an (M_n, D, n, ϵ_n)-code such that*

$$\epsilon_n \leq \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)|S^n)^M]\} \tag{5.22}$$

where the inner expectation is w.r.t. $P_{X^n|S^n=s^n}$, the outer expectation is w.r.t. P_{S^n}, and we have

$$P_{\bar{Y}^n X^n S^n} = P_{\bar{Y}^n|S^n} P_{X^n|S^n} P_{S^n}. \tag{5.23}$$

Proof. Given each side information sequence $S^n = s^n$, we construct a reconstruction codebook $\mathcal{C}(s^n)$, which consists of M random reconstruction sequences $\{Y^n(m, s^n)\}_{m=1}^{M}$. Each of the sequence $Y^n(m, s^n)$, for $m \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$, is generated independently according to an arbitrary distribution $P_{\bar{Y}^n|S^n=s^n}$, which satisfies equation (5.23). Choose a sub-code (ϕ_n, ψ_n), the encoder and decoder of which are defined as

$$\phi_n(x^n, s^n) = \text{arg} \min_{m \in \mathcal{M}} d(x^n, Y^n(m, s^n)), \tag{5.24}$$

$$\psi_n(m, s^n) = Y^n(m, s^n). \tag{5.25}$$

The average probability of error of this sub-code is given by

$$\begin{aligned}
\bar{\epsilon}(s^n) &= \mathbb{E}[1\{\min_{m \in \mathcal{M}} d(X^n, Y^n(m, s^n)) > D\} | S^n = s^n] && (5.26)\\
&= \mathbb{E}\left[\prod_{m=1}^{M} 1\{d(X^n, Y^n(m, s^n)) > D\} | S^n = s^n\right] && (5.27)\\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{m=1}^{M} 1\{d(X^n, Y^n(m, s^n)) > D\} | X^n\right] \middle| S^n = s^n\right] && (5.28)\\
&= \mathbb{E}\left[\prod_{m=1}^{M} \mathbb{E}[1\{d(X^n, \bar{Y}^n) > D\} | X^n] | S^n = s^n\right] && (5.29)\\
&= \mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)) | S^n = s^n)^M] && (5.30)
\end{aligned}$$

where equation (5.29) follows from the independence of reconstruction sequences.

Taking the average over all sub-codes, we have the average probability of error is

$$\begin{aligned}
\bar{\epsilon} &= \sum_{s^n \in \mathcal{S}^n} P_{S^n}(s^n) \bar{\epsilon}(s^n) && (5.31)\\
&= \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)) | S^n)^M]\}. && (5.32)
\end{aligned}$$

By the random coding argument, there exists an (M_n, D, n, ϵ_n)-code such that

$$\epsilon_n \leq \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)) | S^n)^M]\}. \tag{5.33}$$

This concludes the proof. □

Next, we relax the bound in Lemma 5.2 to obtain the following lemma, which turns out to be more amenable to asymptotic evaluations.

Lemma 5.3. *For any γ_n, β_n, and δ_n, there exists an (M_n, D, n, ϵ_n)-code such that*

$$\epsilon_n \leq \Pr[j_{X^n|S^n}(X^n, D|S^n) > \log \gamma_n - \log \beta_n - \lambda_n^* \delta_n]$$
$$+ \mathbb{E}[\mathbb{E}[|1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n]|^+|S^n]]$$
$$+ e^{-\frac{M}{\gamma_n}} \mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D|S^n)))|S^n]\}, \tag{5.34}$$

where $P_{Y^*|XS}$ achieves the minimum in (5.5), and $P_{Y^{n*}|X^nS^n}$ is the n-th order product distribution of $P_{Y^*|XS}$.

This lemma is proved in section 5.8.1.

The following lemma, which plays an important part in the converse, was derived in [57].

Lemma 5.4. *Any (M_n, n, D, ϵ_n)-code for the lossy source coding system with side information satisfies*

$$\epsilon_n \geq \sup_{\gamma > 0} \{\Pr[j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \gamma] - \exp(-\gamma)\}. \tag{5.35}$$

Remark 5.2. It is non-obvious how this non-asymptotic bound should be applied to get a converse that is optimal in the second-order sense. If we follow the approach and the intuition in the dual problem, which is the problem of finding the dispersion for the point-to-point channel with state information available at both the encoder and the decoder [111], we should first partition the side information into different type classes and derive the converse for each type class separately. However, this approach does not work for the source coding with side information. This is because source coding with side information is an optimization problem with constraint whether the point-to-point channel with state is an optimization problem without constraint. The presence of the excess distortion constraint makes the analysis difficult.

■ 5.4 Discrete memoryless source with i.i.d. side information

In this section, we consider the discrete memoryless source. Assume that the source alphabet $\mathcal{X}$, the reproduction alphabet $\mathcal{Y}$, and the side information alphabet $\mathcal{S}$ are finite. The source coding system is memoryless and stationary in the sense that

$$P_{X^n S^n}(x^n s^n) = \prod_{i=1}^{n} P_{XS}(x_i s_i). \tag{5.36}$$

Before presenting the main results of this section, we define an important quantity.

Definition 5.9. Define the variance V of the conditional D-tilted information density $j_{X|S}(X, D|S)$ with respect to P_{XS} as

$$V \triangleq \mathsf{var}(j_{X|S}(X, D|S)) \tag{5.37}$$

$$= \sum_{x \in \mathcal{X}, s \in \mathcal{S}} P_{XS}(xs)[j_{X|S}(x, D|s)]^2 - [R(X; D|S)]^2. \tag{5.38}$$

Next, we present the first main result of this chapter.

Theorem 5.1. *The second-order rate-distortion function $L^*(\epsilon, D, R(X;D|S))$ for the discrete memoryless source coding with side information is given by*

$$L^*(\epsilon, D, R(X;D|S)) = \sqrt{V} Q^{-1}(\epsilon). \tag{5.39}$$

Let us mention that the *dispersion* [56] is an operational quantity that is closely related to the second-order coding rate. It characterizes the speed at which the rate of optimal codes converge to the first-order fundamental limit. For conditional rate-distortion, we may define the dispersion V_{dps} as

$$V_{\mathsf{dps}} \triangleq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \left(\frac{\sqrt{n}(R(\epsilon, D, n) - R(X;D|S))}{Q^{-1}(\epsilon)} \right)^2. \tag{5.40}$$

From Theorem 5.1, we observe that the operational quantity V_{dps} is equal to the information quantity V.

Let $V_s \triangleq \mathsf{var}(j_{X|S}(X, D|S) \,|\, S = s)$ be the dispersion[1] of the source $X_s \sim P_{X|S}(\cdot|s)$. Now notice that by the law of total variance, V can be decomposed as

$$V = \mathbb{E}\big[\mathsf{var}(j_{X|S}(X, D|S) \,|\, S)\big] + \mathsf{var}\big[\mathbb{E}(j_{X|S}(X, D|S) \,|\, S)\big] \tag{5.41}$$

$$= \mathbb{E}[V_S] + \mathsf{var}[R(P_{X|S}(\cdot|S), D)]. \tag{5.42}$$

The first term represents the randomness of the source weighted by the probability mass function of the side information, while the second term represents the randomness of the side information in terms of the constituent rate-distortion functions.

[1]Note that term *dispersion* [56] here refers to the unconditional rate-distortion problem. This should not cause any confusion in the sequel.

Theorem 5.1 is proved in subsection 5.8.2. One of the key ideas in the achievability proof of Theorem 5.1 is to apply the random coding bound (Lemma 5.2) in the asymptotic evaluation. The key idea in the converse proof of Theorem 5.1 is to make use of the non-asymptotic converse bound (Lemma 5.4) in the asymptotic evaluation.

We illustrate this theorem through an example.

Example 5.1. Consider the case when the source alphabet $\mathcal{X}$, the reconstruction alphabet $\mathcal{Y}$ and the side information alphabet S are binary $\{0, 1\}$. The distortion function is the Hamming distance function $d(x, y) = 1\{x \neq y\}$. Assume $P_S(1) = a$, $P_S(0) = 1 - a$, $P_X(1) = b$ and $P_X(0) = 1 - b$, for $0 < a, b < 1$. Assume $P_{X|S}(1|0) = P_{X|S}(1|1) = c$, and $P_{X|S}(0|0) = P_{X|S}(0|1) = 1 - c$, for $0 < c < \frac{1}{2}$. It can shown that

$$j_{X|S}(x, D|s) = i_{X|S}(x|s) - H(D) \tag{5.43}$$

if $0 < D < c$, and 0 if $D \geq c$. Note that the conditional D-tilted information density in this case is independent of the marginal distributions P_X and P_S. Next, we have

$$R(X; D|S) = H(X|S) - H(D) \tag{5.44}$$

$$= H(c) - H(D) \tag{5.45}$$

if $0 < D < c$, and 0 if $D \geq c$. Here $H(D)$ is the entropy of a Bernoulli(D) source.

In this example, we can show that

$$V = c(1 - c) \log^2 \frac{1 - c}{c}, \tag{5.46}$$

which is simply the dispersion of a Bernoulli(c) source.

In general, we have the following corollary.

Corollary 5.1. *The second-order rate-distortion function $L^*(\epsilon, D, R(X;D|S))$ for the binary source with binary side information and Hamming distortion function is given by*

$$L^*(\epsilon, D, R(X;D|S)) = \sqrt{\mathsf{var}[i_{X|S}(X|S)]}Q^{-1}(\epsilon). \tag{5.47}$$

■ 5.4.1 Remarks concerning Theorem 5.1

1. The relationship between the rate-distortion function $R(P_{X|S}(\cdot|s), D)$, in which the side information is fixed, and the conditional rate-distortion function $R(P_{X|S}, D|P_S)$ is given by the following lemma [36].

   Lemma 5.5. *We have*

   $$R(P_{X|S}, D|P_S) = \inf_{\{d_s\}_{s\in\mathcal{S}}\in\mathcal{D}} \sum_{s\in\mathcal{S}} P_S(s) R(P_{X|S}(\cdot|s), d_s), \tag{5.48}$$

   where the set $\mathcal{D}$ is defined as

   $$\mathcal{D} = \left\{ \{d_s\}_{s\in\mathcal{S}} \,\middle|\, \sum_{s\in\mathcal{S}} P_S(s) d_s = D, d_s \geq 0 \right\}. \tag{5.49}$$

   Intuitively, any achievable code, that is optimal in the first-order sense, for the conditional rate-distortion problem can be thought of as a combination of sub-codes for sub-channels with the side information $S = s$ and the excess distortion d_s. The total distortion D is the P_S-convex combination of the constituent excess distortions d_s. However, this intuition is no longer true in the second-order sense. The dispersion for conditional source coding is not simply a convex combination of dispersions for the sub-systems of source coding with the side information given. Secondly, for each sub-system of source coding with side information given, the optimal threshold d_s changes with s. These facts make the second-order analysis difficult. Note that Ingber-Kochman [48] used the method of types (similarly to the technique used in Marton's covering lemma [75]) to perform a second-order (dispersion) analysis for the rate-distortion problem without side information. We attempted to adapt their technique for our setting but it was not straightforward to generalize their method to the conditional rate-distortion problem at hand. This is because Lemma 5.6 intuitively suggests to treat X and S *jointly* to obtain the second-order rate-distortion function $L^*(\epsilon, D, R(P_{X|S}, D|P_S))$. However, if the method of types is used, the relationship in Lemma 5.5 restricts us to treat X conditioning on $S = s$ first, in the achievability proof, in order to obtain the first-order term. However, this method leads to a different (and, in fact, inferior) second-order term. The key idea in the random coding bound in Lemma 5.3 is that we need to treat X and S *jointly*, not separately.

2. One of the challenges in this problem is to find an achievable scheme that is optimal in the second-order sense. An achievable scheme that allows us to do so is presented in Lemma 5.2. The next challenge is how we should use this non-asymptotic bound to obtain the achievability bound that is second-order optimal. Similarly to the reason given in the previous point, we cannot directly use the technique given in the well-known paper by

Kostina-Verdu [56] because in our paper side information is involved. The technique given in [56] makes use of hypothesis testing, and it is difficult to generalize this approach to the case with side information. Neither could we directly use the technique given by Ingber-Kochman [47]. Our solution is to first relax Lemma 5.2 to obtain Lemma 5.3.

3. The proof of Theorem 5.1 can be used to characterize $L^*(\epsilon, D, \kappa)$ when $\kappa \neq R(X; D|S)$. We have

$$L^*(\epsilon, D, \kappa) = \begin{cases} +\infty & \kappa < R(X; D|S) \\ \sqrt{V} Q^{-1}(\epsilon) & \kappa = R(X; D|S) \\ -\infty & \kappa > R(X; D|S) \end{cases} \tag{5.50}$$

The first statement above (for the case $\kappa < R(X; D|S)$) implies the strong converse for conditional rate-distortion. The strong converse for unconditional rate-distortion for discrete memoryless sources is already well known (e.g., [19, Chapter 7]).

4. From Theorem 5.1, we deduce that there exists a sequence of (M_n, n, D, ϵ_n)-codes for the source coding system with side information such that its rate is

$$\frac{1}{n} \log M_n = R(X; D|S) + \sqrt{\frac{V}{n}} Q^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right) \tag{5.51}$$

and its asymptotic probability of excess distortion satisfies

$$\epsilon_n \leq \epsilon + o(1). \tag{5.52}$$

It is observed that V characterizes the rate of convergence to the first-order rate-distortion function $R(X; D|S)$.

5. In order to compute V, it is noted that the gradient of $R(X; D|S)$ plays an important role.

Definition 5.10. For each $a \in \mathcal{X}, b \in \mathcal{S}$, define

$$R'(P_{X|S}(a|b), D|P_S(b)) \triangleq \frac{dR(P_{\bar{X}|\bar{S}}, D|P_{\bar{S}})}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}}. \tag{5.53}$$

The function $R(P_{\bar{X}|\bar{S}}, D|P_{\bar{S}})$ can be thought of as that of $|\mathcal{X}||\mathcal{S}|$ variables. By stacking up $|\mathcal{X}||\mathcal{S}|$ partial derivatives as defined in Definition 5.10, we form the gradient $\nabla R(P_{XS})$ of $R(P_{\bar{X}|\bar{S}}, D|P_{\bar{S}})$ evaluated at P_{XS}. The joint distribution P_{XS} can be regarded as a length-$|\mathcal{X}||\mathcal{S}|$ vector that sums to one.

Even though the conditional D-tilted information density $j_{X|S}(X, D|S)$ is useful in characterizing the second-order rate-distortion function, it is not easy to compute. The task of computing V is made easier by the following lemma.

Lemma 5.6. *For any $a \in \mathcal{X}$ and $b \in \mathcal{S}$, we have*

$$j_{X|S}(a, D|b) = R'(P_{X|S}(a|b), D|P_S(b)). \tag{5.54}$$

Proof. We have

$$\begin{aligned}
R'(P_{X|S}(a|b), D|P_S(b)) &= \frac{dR(P_{\bar{X}|\bar{S}}, D|P_{\bar{S}})}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}} && (5.55)\\
&= \frac{d\mathbb{E}[j_{\bar{X}|\bar{S}}(\bar{X}, D|\bar{S})]}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}} && (5.56)\\
&= \frac{d\left[\sum_{x,s} P_{\bar{X}\bar{S}}(xs) j_{\bar{X}|\bar{S}}(x, D|s)\right]}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}} && (5.57)\\
&= j_{\bar{X}|\bar{S}}(a, D|b) + \frac{d\mathbb{E}[j_{\bar{X}|\bar{S}}(X, D|S)]}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}}. && (5.58)
\end{aligned}$$

Using part 1) of Lemma 5.1, it is evident that

$$\frac{d\mathbb{E}[j_{\bar{X}|\bar{S}}(X, D|S)]}{dP_{\bar{X}\bar{S}}(ab)}\Big|_{P_{\bar{X}\bar{S}}=P_{XS}} = 0. \tag{5.59}$$

This completes the proof of the lemma. □

Remark 5.3. The result in Lemma 5.6 is Surprising. The differentiation is w.r.t. $dP_{\bar{X}\bar{S}}(ab)$, not $dP_{\bar{X}|\bar{S}}(a|b)$. According to [55, Theorem 2.2], the D-tilted information density for the source coding without side information is given by

$$j_X(a, D) = R'(P_X(a), D) - \log e = \frac{dR(P_{\bar{X}}, D)}{dP_{\bar{X}}(a)}\Big|_{P_{\bar{X}}=P_X} - \log e. \quad (5.60)$$

This is because

$$\frac{dR(P_{\bar{X}}, D)}{dP_{\bar{X}}(a)}\Big|_{P_{\bar{X}}=P_X} = \frac{d\mathbb{E}[j_{\bar{X}}(\bar{X}, D)]}{dP_{\bar{X}}(a)}\Big|_{P_{\bar{X}}=P_X} \quad (5.61)$$

$$= j_{\bar{X}}(a, D) + \frac{d\mathbb{E}[j_{\bar{X}}(X, D)]}{dP_{\bar{X}}(a)}\Big|_{P_{\bar{X}}=P_X}, \quad (5.62)$$

and in this case we have

$$\frac{d\mathbb{E}[j_{\bar{X}}(X, D)]}{dP_{\bar{X}}(a)}\Big|_{P_{\bar{X}}=P_X} = -\log e. \quad (5.63)$$

Observe that the term $-\log e$ is present in the no-side information setting (5.60) but not in the side information setting (5.54). This is due to (5.63).

As a consequence of Lemma 5.6, the variance of the conditional D-tilted information V, defined in (5.37)–(5.38), can be alternatively expressed as the variance of the gradient $\nabla R(P_{XS})$ with respect to P_{XS}, i.e.,

$$V = \mathsf{var}(\nabla R(P_{XS})) \quad (5.64)$$

$$= \sum_{a\in\mathcal{X}}\sum_{b\in\mathcal{S}} P_{XS}(ab)[R'(P_{X|S}(a|b), D|P_S(b))]^2 - \left[\sum_{a\in\mathcal{X}}\sum_{b\in\mathcal{S}} P_{XS}(ab)R'(P_{X|S}(a|b), D|P_S(b))\right]^2. \quad (5.65)$$

■ 5.5 Gaussian memoryless source with i.i.d. side information

In this section, we consider the i.i.d. Gaussian source. More specifically,

$$X_i \sim \mathcal{N}(0, \sigma_X^2). \tag{5.66}$$

The side information is given by

$$S_i = X_i + Z_i \tag{5.67}$$

where $i = 1, 2, ..., n$,

$$Z_i \sim \mathcal{N}(0, \sigma_Z^2) \tag{5.68}$$

and Z_i is independent of X_i. We consider the squared-error distortion function, i.e.,

$$d(x^n, y^n) \triangleq \sum_{i=1}^{n} (x_i - y_i)^2. \tag{5.69}$$

Define the conditional variance as

$$\sigma_{X|S}^2 \triangleq \frac{\sigma_X^2 \sigma_Z^2}{\sigma_X^2 + \sigma_Z^2} \tag{5.70}$$

The case where $D \geq \sigma_{X|S}^2$ is trivial as $R(X; D|S) = 0$. It is assumed that $0 < D < \sigma_{X|S}^2$. In this case, it is well-known that [36] the conditional rate-distortion function is given by

$$R(X; D|S) = \frac{1}{2} \log \frac{\sigma_{X|S}^2}{D}. \tag{5.71}$$

The second-order rate-distortion function in this case is given by the following theorem.

Theorem 5.2. *The second-order rate-distortion function* $L^*(\epsilon, D, R(X;D|S))$ *for Gaussian source coding with side information is given by*

$$L^*(\epsilon, D, R(X;D|S)) = \sqrt{\frac{1}{2}} Q^{-1}(\epsilon) \log e. \tag{5.72}$$

This theorem is proved in subsection 5.8.3.

5.5.1 Remarks concerning Theorem 5.2

1. From Theorem 5.2, we observe that the dispersion for Gaussian source coding with side information is 1/2 nats squared per source symbol. In other words, the second-order rate-distortion function for Gaussian source coding with side information is the same as that for Gaussian source coding without side information [56] even though the rate-distortion functions for both coding problems are different in general. The presence of side information at both the encoder and the decoder does not affect the second-order coding rate. Intuitively, given the side information s^n, the encoder and the decoder can adapt to it and design a second-order optimal sub-code for each source-encoding sub-test channel (indexed by s^n). The second-order coding rate for each sub-test channel is basically the same as that for the source coding system without side information. The second-order rate-distortion function for Gaussian source coding with side information is the average of all second-order coding rates for sub-test channels, when the average is taken with respect to the side information random variable. Thus, this explains the observation. This observation might not hold if the

side information sequence is not a Gaussian random process.

2. The proof of Theorem 5.1 is also applicable to that of Theorem 5.2. However, due to the special setting in this section, we present an alternative achievability scheme in the proof. This achievability scheme helps us visualize the structure of a code that is optimal in the second-order sense.

3. It would be interesting to investigate if the statement mentioned in the previous item still holds when the side information is available at either only the decoder or only the encoder. Of course, the rate-distortion functions for the cases where the side information is known at both terminals and at the decoder only are identical in the Gaussian case [22, Chapter 11]. Thus one wonders whether the dispersion remains at $1/2$ nats^2 per source symbol for the Gaussian Wyner-Ziv problem [121].

4. Scarlett [94] showed that the dispersion for dirty paper coding (Gaussian Gel'fand-Pinsker) is the same as that when there is no interference. Furthermore, he showed that the same holds true even if the interference is not Gaussian but satisfies some mild concentration conditions. It would be interesting to investigate if the same is true in the lossy compression with (encoder and decoder) side information scenario.

■ 5.6 Markov source with Markov side information

So far, we have considered only memoryless sources. In this section, we consider the system in which the source and side information jointly forms an irreducible, ergodic and time-homogeneous Markov chain, i.e.,

$$X_1S_1 \to X_2S_2 \to \ldots \to X_nS_n. \tag{5.73}$$

We further assume that the source alphabet $\mathcal{X}$ and the side information alphabet $\mathcal{S}$ are both finite. Denote the stationary distribution of this Markov chain as π_{XS}. Assume that this Markov chain starts from the stationary distribution, i.e.,

$$P_{X_1S_1} = \pi_{XS}. \tag{5.74}$$

Under the assumption in (5.74), all the marginals $P_{X_iS_i}$ for $i \geq 1$ are equal to π_{XS}.

First, we define a few relevant quantities.

Definition 5.11. Define

$$\mu \triangleq R(X; D|S)\big|_{P_{XS}=\pi_{XS}}, \tag{5.75}$$

$$V_n \triangleq \mathsf{var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i)\right). \tag{5.76}$$

We have the following important lemma.

Lemma 5.7. *For the Markov chains considered above, the following limit exists*

$$\lim_{n\to\infty} V_n \tag{5.77}$$

and is equal to

$$\begin{aligned} V_\infty \triangleq\ & \mathsf{var}[j_{X|S}(X, D|S)]\big|_{P_{XS}=\pi_{XS}} \\ & + 2\sum_{i=1}^{\infty} \mathsf{cov}[j_{X_1|S_1}(X_1, D|S_1), j_{X_{1+i}|S_{1+i}}(X_{1+i}, D|S_{1+i})]. \end{aligned} \tag{5.78}$$

Proof. The lemma follows from the fact that

$$V_n = \frac{1}{n}\mathsf{var}\left(\sum_{i=1}^{n} \jmath_{X_i|S_i}(X_i, D|S_i)\right) \tag{5.79}$$

$$= \frac{1}{n}\sum_{k,l=1}^{n} \mathsf{cov}\left[\jmath_{X_k|S_k}(X_k, D|S_k), \jmath_{X_l|S_l}(X_l, D|S_l)\right] \tag{5.80}$$

$$= \mathsf{var}[\jmath(X, D|S)]\big|_{P_{XS}=\pi_{XS}}$$
$$+ \frac{2}{n}\sum_{j=1}^{n}(n-j)\mathsf{cov}\left[\jmath_{X_1|S_1}(X_1, D|S_1), \jmath_{X_{1+j}|S_{1+j}}(X_{1+j}, D|S_{1+j})\right]. \tag{5.81}$$

The equality in (5.81) follows from the time-homogeneity of the chain and simple rearrangements. Now, since the covariance

$$\left|\mathsf{cov}\left(\jmath_{X_1|S_1}(X_1, D|S_1), \jmath_{X_{1+j}|S_{1+j}}(X_{1+j}, D|S_{1+j})\right)\right|$$

decays exponentially fast in the lag j for this class of Markov chains,

$$\lim_{n\to\infty}\sum_{j=1}^{n} j\cdot\mathsf{cov}\left[\jmath_{X_1|S_1}(X_1, D|S_1), \jmath_{X_{1+j}|S_{1+j}}(X_{1+j}, D|S_{1+j})\right] = 0, \tag{5.82}$$

and thus

$$\lim_{n\to\infty} V_n = \mathsf{var}[\jmath(X, D|S)]\big|_{P_{XS}=\pi_{XS}}$$
$$+ 2\sum_{j=1}^{\infty}\mathsf{cov}\left[\jmath_{X_1|S_1}(X_1, D|S_1), \jmath_{X_{1+j}|S_{1+j}}(X_{1+j}, D|S_{1+j})\right]. \tag{5.83}$$

The right-hand-side is exactly V_∞ as desired. □

The second-order rate-distortion function for the Markov sequence is given by the following theorem.

Theorem 5.3. *The second-order rate-distortion function $L^*(\epsilon, D, \mu)$ for the Markov source with side information is given by*

$$L^*(\epsilon, D, \mu) = \sqrt{V_\infty}Q^{-1}(\epsilon). \tag{5.84}$$

This theorem is proved in subsection 5.8.4 and it uses a Markov generalization of the Berry-Esséen theorem due to Tikhomirov [109].

5.6.1 Remarks concerning Theorem 5.3

1. Notice that the second-order coding rate for the Markov case consists of two parts:

$$A \triangleq \mathsf{var}[j_{X|S}(X, D|S)]\big|_{P_{XS}=\pi_{XS}},$$

$$B \triangleq \sum_{i=1}^{\infty} \mathsf{cov}[j_{X_1|S_1}(X_1, D|S_1), j_{X_{1+i}|S_{1+i}}(X_{1+i}, D|S_{1+i})]. \tag{5.85}$$

When the sequence of random variables $\{X_i S_i\}_{i=1}^{\infty}$ is independent and identically distributed, the second part B in (5.85) vanishes and we recover the result in section 5.4. Thus, the infinite sum in the definition of V_∞ in (5.78) quantifies the effect that the mixing of the Markov chain $\{X_i S_i\}_{i=1}^{\infty}$ has on rate of convergence the finite blocklength rate-distortion function to the Shannon limit. The faster the mixing is, the faster the convergence to the Shannon limit is.

2. Denote Ξ as transitional matrix of the Markov chain $X_1 S_1 \to X_2 S_2 \to \ldots \to X_n S_n$. If Ξ is diagonalizable, we can compute V_∞ using the following lemma.

Lemma 5.8. *Assume* $\Xi = U \mathsf{diag}(1, \lambda_2, \ldots, \lambda_{|\mathcal{X}||\mathcal{S}|}) U^\dagger$. *We have*

$$V_\infty = \mathsf{cov}[j_{X|S}(X, D|S), j_{X'|S'}(X', D|S')]\big|_{P_{XS,X'S'}=\pi_{XS} P_{X'S'|XS}} \tag{5.86}$$

where

$$P_{X'S'|XS}(x's'|xs) = \left[U \mathsf{diag}\left(1, \frac{1+\lambda_2}{1-\lambda_2}, \ldots, \frac{1+\lambda_{|\mathcal{X}||\mathcal{S}|}}{1-\lambda_{|\mathcal{X}||\mathcal{S}|}}\right) U^\dagger \right]_{x's'xs}. \tag{5.87}$$

This lemma can be proved using techniques presented by Tomamichel and Tan in [111, Appendix A]. Briefly, we make use of the fact that the Markov chain $\{X_i S_i\}_{i=1}^{\infty}$ is time-homogeneous and starts from the stationary distribution. Secondly, in the diagonalization of the transition matrix Ξ, except for eigenvalue $\lambda_1 \triangleq 1$, the rest of the eigenvalues satisfy $|\lambda_i| < 1$. Thus, we have $\sum_{k=1}^{\infty} \lambda_i^k = \frac{\lambda_i}{1-\lambda_i}$ for all but the leading eigenvalue.

■ 5.7 Conclusion

In this chapter, the second-order coding rates for the source coding problem with side information available at both the encoder and the decoder are characterized for three different kinds of sources: discrete memoryless sources, Gaussian memoryless sources and Markov sources. The conditional D-tilted information density is found to play a key role in our second-order analysis.

■ 5.8 Appendix

■ 5.8.1 Proof of Lemma 5.3

Lemma 5.3 is a corollary of Lemma 5.2. From Lemma 5.2, we can show the existence of an (M_n, D, n, ϵ_n)-code such that

$$\epsilon_n \leq \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n))|S^n)^M]\} \tag{5.88}$$

$$= \sum_{s^n} P_{S^n}(s^n)\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n))|S^n = s^n)^M]. \tag{5.89}$$

Using techniques from [55, Corollary 2.20], we can show that for every s^n,

$$\begin{aligned}
&\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n))|S^n = s^n)^M] \\
&\leq \Pr[j_{X^n|S^n}(X^n, D|S^n) > \log \gamma_n - \log \beta_n - \lambda_n^* \delta_n | S^n = s^n] \\
&\quad + \mathbb{E}[|1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n]|^+ | S^n = s^n] \\
&\quad + e^{-\frac{M}{\gamma_n}} \mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n)))|S^n = s^n], \qquad (5.90)
\end{aligned}$$

for any γ_n, β_n, and δ_n.

Taking the average of both sides of inequality (5.90) over all sequences s^n completes the proof of this lemma.

5.8.2 Proof of Theorem 5.1

5.8.2.1 Achievability proof of Theorem 5.1

In this part, we prove that, for any $\delta > 0$, $\sqrt{V}Q^{-1}(\epsilon) + \delta$ is second-order (ϵ, D, κ)-achievable when $\kappa = R(X; D|S)$.

We apply Lemma 5.3 to construct a sequence of (M_n, D, n, ϵ_n)-codes as follows. Choose $\delta_n = \frac{\mathsf{const}}{n}$. Similar to the proof in [56, Lemma 4], it can be proved that

$$\Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D | X^n = x^n, S^n = s^n] \geq \frac{C}{\sqrt{n}}, \qquad (5.91)$$

when n is sufficiently large, for some constant C.

Choose $\beta_n = \frac{\sqrt{n}}{C}$. We have

$$\mathbb{E}[\mathbb{E}[|1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D|X^n]|^+|S^n]] = 0, \tag{5.92}$$

when n is sufficiently large.

Choose $\gamma_n = \frac{M}{\sqrt{n}}$. We have

$$e^{-\frac{M}{\gamma_n}}\mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n)))|S^n]\}$$

$$= e^{-\sqrt{n}}\mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n)))|S^n]\} \tag{5.93}$$

$$\leq e^{-\sqrt{n}}\mathbb{E}\{\mathbb{E}[1|S^n]\} \tag{5.94}$$

$$= e^{-\sqrt{n}}. \tag{5.95}$$

Choose

$$\log M_n = nR(X; D|S) + \sqrt{nV}Q^{-1}(\hat{\epsilon}_n) + \log\sqrt{n} + \lambda_n^* \frac{D}{100} + \log\frac{\sqrt{n}}{C}, \tag{5.96}$$

where

$$\hat{\epsilon}_n \triangleq \epsilon - \frac{B_n}{\sqrt{n}} - e^{-\sqrt{n}} \tag{5.97}$$

$$B_n \triangleq 6\frac{T_n}{V^{3/2}} \tag{5.98}$$

$$T_n \triangleq \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[|j_{X|S}(X, D|S) - R(X; D|S)|^3]. \tag{5.99}$$

Applying Lemma 5.3, for n sufficiently large, we have

$$\epsilon_n \leq \Pr\left[j_{X^n|S^n}(X^n, D|S^n) > nR(X; D|S) + \sqrt{nV}Q^{-1}(\hat{\epsilon}_n)\right] + e^{-\sqrt{n}} \tag{5.100}$$

$$\leq \Pr\left[\sum_{i=1}^{n} j_{X|S}(X_i, D|S_i) > nR(X; D|S) + \sqrt{nV}Q^{-1}(\hat{\epsilon}_n)\right] + e^{-\sqrt{n}} \tag{5.101}$$

$$\leq \epsilon \tag{5.102}$$

where equation (5.102) follows from Theorem 2.10.

Therefore, we have constructed a sequence of (M_n, D, n, ϵ_n)-codes satisfying

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_n - nR(X;D|S)) = \sqrt{V}Q^{-1}(\epsilon) \tag{5.103}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon. \tag{5.104}$$

5.8.2.2 Converse proof of Theorem 5.1

Let L be a second-order $(\epsilon, D, R(X;D|S))$-achievable. We want to show

$$Q^{-1}(\epsilon)\sqrt{V} \leq L + \delta,$$

for any $\delta > 0$.

Since L is second-order $(\epsilon, D, R(X;D|S))$-achievable, by definition, there exists a sequence of (M_n, n, D, ϵ_n)-codes satisfying

$$\log M_n \leq nR(X;D|S) + \sqrt{n}(L+\delta), \tag{5.105}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon, \tag{5.106}$$

when n is sufficiently large.

Using Lemma 5.4 for M_n satisfying equation (5.105) and $\gamma = \log\sqrt{n}$, we have

$$\epsilon_n \geq \Pr[j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \log\sqrt{n}] - \frac{1}{\sqrt{n}} \tag{5.107}$$

$$= \Pr\left[\sum_{i=1}^{n} j_{X|S}(X_i, D|S_i) \geq \log M_n + \log\sqrt{n}\right] - \frac{1}{\sqrt{n}} \tag{5.108}$$

$$\geq \Pr\left[\sum_{i=1}^{n} \jmath_{X|S}(X_i, D|S_i) \geq nR(X;D|S) + \sqrt{n}(L+\delta) + \log\sqrt{n}\right] - \frac{1}{\sqrt{n}} \tag{5.109}$$

$$\geq \Pr\left[\sum_{i=1}^{n} \jmath_{X|S}(X_i, D|S_i) - nR(X;D|S) \geq \sqrt{nV}\left(\frac{L+\delta}{\sqrt{V}} + \frac{\log\sqrt{n}}{\sqrt{nV}}\right)\right] - \frac{1}{\sqrt{n}} \tag{5.110}$$

$$\geq Q\left(\frac{L+\delta}{\sqrt{V}} + \frac{\log\sqrt{n}}{\sqrt{nV}}\right) - \frac{B_n}{\sqrt{n}} - \frac{1}{\sqrt{n}} \tag{5.111}$$

$$= Q\left(\frac{L+\delta}{\sqrt{V}}\right) + O\left(\frac{\log\sqrt{n}}{\sqrt{n}}\right) - \frac{B_n+1}{\sqrt{n}} \tag{5.112}$$

where equation (5.111) follows from Theorem 2.10 and in this equation B_n is defined in (5.98), and (5.112) follows from the continuity of $Q(\cdot)$ and Taylor expansion.

Combining (5.112) and (5.106), we have

$$\epsilon \geq \limsup_{n\to\infty} \epsilon_n \tag{5.113}$$

$$= Q\left(\frac{L+\delta}{\sqrt{V}}\right). \tag{5.114}$$

Thus, all second-order achievable rates L must satisfy $L \geq Q^{-1}(\epsilon)\sqrt{V} - \delta$. Taking $\delta \downarrow 0$, we complete the proof of the converse.

■ 5.8.3 Proof of Theorem 5.2

Define the correlation coefficient ρ between X_i and S_i, for $i = 1, 2, \ldots, n$ as

$$\rho \triangleq \frac{\mathbb{E}[XS]}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[S^2]}} = \frac{\sigma_X}{\sqrt{\sigma_Z^2 + \sigma_X^2}}. \tag{5.115}$$

Next, we define the conditional mean of X given $S = s$ as

$$\mu(s) \triangleq \rho \cdot \frac{\sigma_X}{\sigma_S} \cdot s = \rho^2 \cdot s = \frac{\sigma_X^2}{\sigma_Z^2 + \sigma_X^2} \cdot s. \tag{5.116}$$

This is simply the minimum mean squared estimate of X given $S = s$.

5.8.3.1 Achievability proof of Theorem 5.2

In this part, we prove that, for any $\delta > 0$, $\sqrt{\frac{1}{2}}Q^{-1}(\epsilon)\log(e) + \delta$ is second-order $(\epsilon, D, \frac{1}{2}\log\frac{\sigma^2_{X|S}}{D})$-achievable. We apply Lemma 5.2 to construct a sequence of (M_n, D, n, ϵ_n)-codes as follows. For each s^n, choose the distribution $P_{\bar{Y}^n|S^n}(\cdot|S^n = s^n)$ in equation (5.22) as the uniform distribution on the surface of the n-dimensional sphere, with radius $r_0 \triangleq \sqrt{n(\sigma^2_{X|S} - D)}$ and centre at

$$\mu(s^n) \triangleq (\mu(s_1), \mu(s_2), \ldots, \mu(s_n)). \tag{5.117}$$

Observe that $P_{\bar{Y}^n|S^n}(B_D(x^n))|S^n = s^n) = 0$ if

$$|x^n - \mu(s^n)| < \sqrt{n(\sigma^2_{X|S} - D)} - \sqrt{nD} \triangleq r_1 \tag{5.118}$$

or

$$|x^n - \mu(s^n)| > \sqrt{n(\sigma^2_{X|S} - D)} + \sqrt{nD} \triangleq r_2. \tag{5.119}$$

Therefore, we have a sequence of (M_n, D, n, ϵ_n)-codes that satisfies

$$\epsilon_n \leq \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n))|S^n)^{M_n}]\} \tag{5.120}$$

$$\begin{aligned}
&\leq \mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)))^{M_n}.\Pr(r_1 \leq |x^n - \mu(S^n)| \leq r_2)|S^n]\} \\
&\quad + \mathbb{E}\{\mathbb{E}[\Pr(r_2 < |X^n - \mu(S^n)|)|S^n]\} \\
&\quad + \mathbb{E}\{\mathbb{E}[\Pr(r_1 > |X^n - \mu(S^n)|)|S^n]\}.
\end{aligned} \tag{5.121}$$

By the weak law of large numbers, we observe that the second term and the third term become vanishingly small as $n \to \infty$. Now, we analyze the first term.

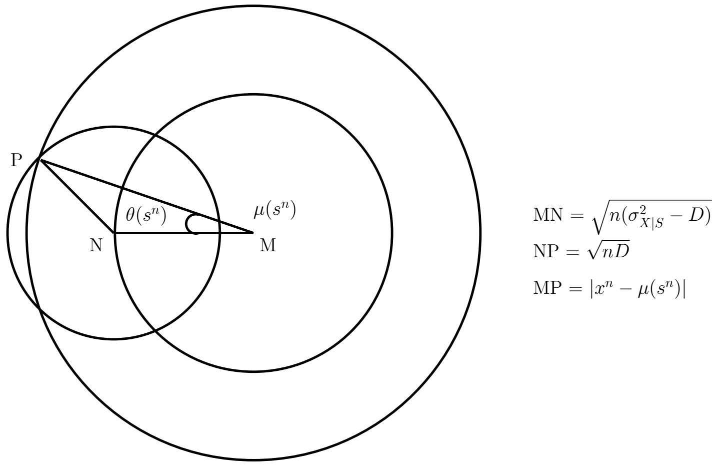

Figure 5.2. Encoding for Gaussian source

Note that $\frac{|X^n - \mu(s^n)|^2}{\sigma^2_{X|S}}$ has a central χ^2_n distribution. Denote

$$A_n(r_0) \triangleq \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r_0^{n-1}$$

as the surface area of an n-dimensional sphere of radius r_0. Denote $A_n(r_0, \theta(s^n))$ as the surface area of n-dimensional polar cap of radius r_0 and angle $\theta(s^n)$ (see Figure 5.2), where the angle $0 < \theta(s^n) < \pi$ is given by

$$\theta(s^n) \triangleq \cos^{-1}\left(\frac{|x^n - \mu(s^n)|^2 + r_0^2 - nD}{2|x^n - \mu(s^n)|r_0}\right). \tag{5.122}$$

We have

$$\mathbb{E}\{\mathbb{E}[(1 - P_{\bar{Y}^n|S^n}(B_D(X^n)))^{M_n}. \Pr(r_1 \leq |x^n - \mu(S^n)| \leq r_2)|S^n]\} \tag{5.123}$$

$$= \mathbb{E}\left\{\mathbb{E}\left[\left(1 - \frac{A_n(r_0)}{A_n(r_0, \theta(s^n))}\right)^{M_n}. \Pr(r_1 \leq |x^n - \mu(S^n)| \leq r_2)\Big|S^n\right]\right\} \tag{5.124}$$

$$\leq \mathbb{E}\Bigg\{\mathbb{E}\Bigg[\left(1 - \frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}n\Gamma(\frac{n-1}{2}+1)}(\sin(\theta(s^n)))^{n-1}\right)^{M_n}$$

$$. \Pr(r_1 \leq |x^n - \mu(S^n)| \leq r_2)\Big|S^n\Bigg]\Bigg\} \tag{5.125}$$

$$\leq \mathbb{E}\left\{\left[n\int_0^\infty (1 - f(n,z))^{M_n} 1\{r_1 \leq z \leq r_2\} P_{\chi^2_n}(nz)dz\Big|S^n\right]\right\} \tag{5.126}$$

$$= n\int_0^\infty (1 - f(n,z))^{M_n} 1\{r_1 \leq z \leq r_2\} P_{\chi^2_n}(nz)dz \tag{5.127}$$

where

- (5.124) comes from geometry,
- (5.125) comes from a lower bound on $A_n(r_0, \theta(s^n))$ [91], and
- in (5.126), the function $f(n,z)$ is defined as

$$f(n,z) \triangleq \frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}n\Gamma(\frac{n-1}{2}+1)} \left(1 - \frac{\left(1+z-2\frac{D}{\sigma^2_{X|S}}\right)^2}{4\left(1-\frac{D}{\sigma^2_{X|S}}\right)z}\right)^{\frac{n-1}{2}} \tag{5.128}$$

and $P_{\chi^2_n}$ is the central χ^2_n probability density function.

Next, we choose the sequence M_n such that

$$\frac{\log M_n}{n} = \frac{1}{2}\log\frac{\sigma^2_{X|S}}{D} + \sqrt{\frac{1}{2n}}Q^{-1}(\epsilon)\log e + \frac{\log n}{2n} + \frac{\log\log n}{n} + O\left(\frac{1}{n}\right). \tag{5.129}$$

We can check that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}\left(\log M_n - \frac{n}{2}\log\frac{\sigma^2_{X|S}}{D}\right) = \sqrt{\frac{1}{2}}Q^{-1}(\epsilon)\log e. \tag{5.130}$$

Using similar techniques as in [56, Appendix K], we can show that the bound in (5.127) can be analyzed using the Gaussian approximation to yield

$$\limsup_{n\to\infty} \epsilon_n \le \epsilon. \tag{5.131}$$

■ 5.8.3.2 Converse proof of Theorem 5.2

The conditional D-tilted information in the jointly Gaussian case is

$$\jmath_{X^n|S^n}(x^n, D|s^n) = \frac{n}{2}\log\frac{\sigma^2_{X|S}}{D} + \frac{\left|x^n - \frac{\sigma^2_X}{\sigma^2_X+\sigma^2_Z}s^n\right|^2}{2\sigma^2_{X|S}}\log e - \frac{n}{2}\log e. \tag{5.132}$$

For each $i \in \{1, 2, \ldots, n\}$, we have

$$\mathbb{E}[j_{X_i|S_i}(X_i, D|S_i)] = \frac{1}{2}\log\frac{\sigma^2_{X|S}}{D}, \tag{5.133}$$

and

$$\begin{aligned}
\mathsf{var}[j_{X_i|S_i}(X_i, D|S_i)] &= \mathbb{E}\left[\frac{|X_i - \mu(S_i)|^2}{2\sigma^2_{X|S}}\log e - \frac{\log e}{2}\right]^2 &(5.134)\\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{|X_i - \mu(S_i)|^2}{2\sigma^2_{X|S}}\log e - \frac{\log e}{2}\right)^2 \middle| S_i\right]\right] &(5.135)\\
&= (\log e)^2\mathbb{E}\left[\mathbb{E}\left[\left(\frac{|X_i - \mu(S_i)|^4}{4\sigma^4_{X|S}} - \frac{|X_i - \mu(S_i)|^2}{2\sigma^2_{X|S}} + \frac{1}{4}\right) \middle| S_i\right]\right] &(5.136)\\
&= \frac{1}{2}(\log e)^2. &(5.137)
\end{aligned}$$

Let L be second-order $(\epsilon, D, \frac{1}{2}\log\frac{\sigma^2_{X|S}}{D})$-achievable. We want to show that

$$Q^{-1}(\epsilon)\sqrt{\frac{1}{2}}\log e \leq L + \delta$$

for any $\delta > 0$. Since L is second-order $(\epsilon, D, \frac{1}{2}\log\frac{\sigma^2_{X|S}}{D})$-achievable, there exists a sequence of (M_n, n, D, ϵ_n)-codes satisfying

$$\log M_n \leq \frac{n}{2}\log\frac{\sigma^2_{X|S}}{D} + \sqrt{n}(L + \delta), \tag{5.138}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon, \tag{5.139}$$

where (5.138) holds for all n sufficiently large.

Using Lemma 5.4 for M_n satisfying equation (5.138) and $\gamma = \log\sqrt{n}$, we have

$$\epsilon_n \geq \Pr[j_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \log\sqrt{n}] - \frac{1}{\sqrt{n}} \tag{5.140}$$

$$\geq Q\left(\frac{L+\delta}{\sqrt{\frac{1}{2}\log e}} + \frac{\log\sqrt{n}}{\sqrt{\frac{1}{2}n\log e}}\right) - \frac{B_n}{\sqrt{n}} - \frac{1}{\sqrt{n}} \tag{5.141}$$

$$= Q\left(\frac{L+\delta}{\sqrt{\frac{1}{2}\log e}}\right) + O\left(\frac{\log\sqrt{n}}{\sqrt{n}}\right) - \frac{B_n+1}{\sqrt{n}} \tag{5.142}$$

where equation (5.141) follows from Theorem 2.10 and in this equation B_n is the constant in Theorem 2.10, and (5.142) follows from the continuous differentiability of $Q(\cdot)$ and Taylor expansion.

Combining (5.142) and (5.139), we have

$$\epsilon \geq \limsup_{n\to\infty} \epsilon_n \tag{5.143}$$

$$= Q\left(\frac{L+\delta}{\sqrt{\frac{1}{2}\log e}}\right). \tag{5.144}$$

This completes the proof of the converse upon taking $\delta \downarrow 0$.

5.8.4 Proof of Theorem 5.3

To prove Theorem 5.3, we use a variant of Berry-Esséen Theorem [109] to deal with a sequence of random variables that forms a Markov chain. This theorem is stated as follows.

Theorem 5.4. *Consider a stationary process $\{X_k : k \geq 1\}$, with $\mathbb{E}X_1 = 0$ and finite variance. Define the strong mixing coefficient $\alpha(n)$ as*

$$\alpha(n) \triangleq \sup\{|\Pr(A\cap B) - \Pr(A)\Pr(B)| : A \in \mathcal{F}_{-\infty}^{k}, B \in \mathcal{F}_{k+n}^{\infty}, k \in \mathbb{Z}\}, \tag{5.145}$$

where $\mathcal{F}_a^b = \sigma \langle X_i : i \in [a,b] \cap \mathbb{Z} \rangle$ *is the* σ*-field generated by* $\{X_i : i \in [a,b] \cap \mathbb{Z}\}$, $-\infty \leq a \leq b \leq \infty$. *Denote*

$$\sigma_n^2 \triangleq \mathbb{E}\left[\left(\sum_{j=1}^{n} X_j\right)^2\right]. \tag{5.146}$$

Assume that the strong mixing coefficient is exponentially decaying, i.e., $\alpha(n) \leq Ke^{-\kappa_1 n}$ *for some* K *and* κ_1 *and all* $n \geq 1$. *Assume* $\mathbb{E}[|X_1^{2+\gamma}] < \infty$ *for some* γ, $1 \geq \gamma > 0$. *Then, there is a constant* $B(K, \kappa, \gamma) > 0$ *such that, for all* $n \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}} \left| \Pr\left[\frac{1}{\sigma_n} \sum_{k=1}^{n} X_k \leq \lambda\right] - \Phi(\lambda) \right| \leq \frac{B(K, \kappa_1, \gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}}. \tag{5.147}$$

Note that the strong mixing coefficient of a time-homogeneous, irreducible and ergodic Markov chain decays to zero and, in fact, vanishes exponentially fast[10, Theorem 3.1].

In this proof, we make use of the following lemma.

Lemma 5.9. *If the sequence* $X_1S_1 \to X_2S_2 \to X_3S_3 \to \ldots$ *forms a Markov chain, then the sequence of D-tilted information densities* $\{j_{X_i|S_i}(X_i, D|S_i)\}_{i=1}^{\infty}$ *also forms a Markov chain.*

This lemma is proved in section 5.8.5

■ 5.8.4.1 Achievability proof of Theorem 5.3

In this part, we prove that, for any $\delta > 0$, $\sqrt{V_\infty}Q^{-1}(\epsilon) + \delta$ is second-order (ϵ, D, μ)-achievable.

We apply Lemma 5.3 to construct a sequence of (M_n, D, n, ϵ_n)-codes as follows. Choose $\delta_n = \frac{\mathsf{const}}{n}$. Similar to the proof in [56, Lemma 4], it can be proved that

$$\Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D | X^n = x^n, S^n = s^n] \geq \frac{C}{\sqrt{n}}, \tag{5.148}$$

when n is sufficiently large, for some constant C.

Choose $\beta_n = \frac{\sqrt{n}}{C}$. We have

$$\mathbb{E}[\mathbb{E}[|1 - \beta_n \Pr[D - \delta_n \leq d(X^n, Y^{n*}) \leq D | X^n]|^+ | S^n]] = 0, \tag{5.149}$$

when n is sufficiently large.

Choose $\gamma_n = \frac{M}{\sqrt{n}}$. We have

$$e^{-\frac{M}{\gamma_n}} \mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n))) | S^n]\}$$

$$= e^{-\sqrt{n}} \mathbb{E}\{\mathbb{E}[\min(1, \gamma_n \exp(-j_{X^n|S^n}(X^n, D, S^n))) | S^n]\} \tag{5.150}$$

$$\leq e^{-\sqrt{n}} \mathbb{E}\{\mathbb{E}[1 | S^n]\} \tag{5.151}$$

$$= e^{-\sqrt{n}}. \tag{5.152}$$

Choose

$$\log M_n = n\mu + \sqrt{nV_n} Q^{-1}(\hat{\epsilon}_n) + \log \sqrt{n} + \lambda_n^* \frac{D}{100} + \log \frac{\sqrt{n}}{C}, \tag{5.153}$$

where

$$\hat{\epsilon}_n \triangleq \epsilon - \frac{B(K, \kappa_1, \gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}} - e^{-\sqrt{n}} \tag{5.154}$$

and $B(K, \kappa_1, \gamma)$ is found in Theorem 5.4.

Applying Lemma 5.3, for n sufficiently large, we have

$$\epsilon_n \leq \Pr\left[j_{X^n|S^n}(X^n, D|S^n) > n\mu + \sqrt{nV_n}Q^{-1}(\hat{\epsilon}_n)\right] + e^{-\sqrt{n}} \tag{5.155}$$

$$\leq \Pr\left[\sum_{i=1}^{n} j_{X_i|S_i}(X_i, D|S_i) > n\mu + \sqrt{nV_n}Q^{-1}(\hat{\epsilon}_n)\right] + e^{-\sqrt{n}} \tag{5.156}$$

$$\leq \epsilon \tag{5.157}$$

where equation (5.157) follows from Theorem 5.4.

Therefore, we have constructed a sequence of (M_n, D, n, ϵ_n)-codes satisfying

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_n - n\mu) = \sqrt{V_\infty}Q^{-1}(\epsilon) \tag{5.158}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon. \tag{5.159}$$

5.8.4.2 Converse proof of Theorem 5.3

Let L be second-order (ϵ, D, μ)-achievable. In this part, we want to show that $Q^{-1}(\epsilon)\sqrt{V_\infty} \leq L + \delta$, for any $\delta > 0$.

Since L is (ϵ, D, μ)-second-order achievable, by definition there exists a sequence of (M_n, n, D, ϵ_n)-codes satisfying

$$\log M_n \leq n\mu + \sqrt{n}(L + \delta), \tag{5.160}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon, \tag{5.161}$$

when n is sufficiently large.

Using Lemma 5.4 for M_n satisfying equation (5.160) and $\gamma = \log \sqrt{n}$, we have

$$\begin{aligned}
\epsilon_n &\geq \Pr[\jmath_{X^n|S^n}(X^n, D|S^n) \geq \log M_n + \log \sqrt{n}] - \frac{1}{\sqrt{n}} && (5.162)\\
&= \Pr\left[\sum_{i=1}^{n} \jmath_{X_i|S_i}(X_i, D|S_i) \geq \log M_n + \log \sqrt{n}\right] - \frac{1}{\sqrt{n}} && (5.163)\\
&\geq \Pr\left[\sum_{i=1}^{n} \jmath_{X_i|S_i}(X_i, D|S_i) \geq n\mu + \sqrt{n}(L+\delta) + \log \sqrt{n}\right] - \frac{1}{\sqrt{n}} && (5.164)\\
&\geq \Pr\left[\sum_{i=1}^{n} \jmath_{X_i|S_i}(X_i, D|S_i) - n\mu \geq \sqrt{nV_n}\left(\frac{L+\delta}{\sqrt{V_n}} + \frac{\log \sqrt{n}}{\sqrt{nV_n}}\right)\right] - \frac{1}{\sqrt{n}} && (5.165)\\
&\geq Q\left(\frac{L+\delta}{\sqrt{V_n}} + \frac{\log \sqrt{n}}{\sqrt{nV_n}}\right) - \frac{B(K,\kappa_1,\gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}} - \frac{1}{\sqrt{n}} && (5.166)\\
&= Q\left(\frac{L+\delta}{\sqrt{V_n}}\right) + O\left(\frac{\log \sqrt{n}}{\sqrt{n}}\right) - \frac{B(K,\kappa_1,\gamma)(\log n)^{1+\frac{\gamma}{2}}}{n^{\frac{\gamma}{2}}} - \frac{1}{\sqrt{n}} && (5.167)
\end{aligned}$$

where equation (5.166) follows from Theorem 5.4 and in this equation $B(K, \kappa_1, \gamma)$ is defined in Theorem 5.4, and (5.167) follows from the continuity of $Q(\cdot)$ and Taylor expansion.

Combining (5.167) and (5.161), we have

$$\begin{aligned}
\epsilon &\geq \limsup_{n\to\infty} \epsilon_n && (5.168)\\
&= Q\left(\frac{L+\delta}{\sqrt{V_\infty}}\right) && (5.169)
\end{aligned}$$

where in (5.169), we use the fact that $V_n \to V_\infty$.

■ 5.8.5 Proof of Lemma 5.9

In the proof of this lemma, we make use of the following lemma.

Lemma 5.10. *Let $\{A_i\}_{i=1}^{\infty}$ be a Markov chain in state space $\mathcal{A}$. Consider the sequence $\{B_i = f(X_i)\}_{i=1}^{\infty}$, where $f : \mathcal{A} \to \mathcal{B}$ is a function from $\mathcal{A}$ to $\mathcal{B}$. Suppose* *that there exists a function* $g : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ *such that*

$$\Pr(B_{i+1} = b | X_i = a) = g(f(a), b) \tag{5.170}$$

for any $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$*. Then the sequence* $\{B_i\}_{i=1}^{\infty}$ *forms a Markov chain.*

The proof of this lemma can be found in [52, Lemma 13]. Note that if f is one-to-one, then it is obvious that the sequence generated by f acting on a Markov chain is also a Markov chain.

Here, $j_{X|S}$ is a composition of several functions log, $\frac{1}{t}$ for $t \neq 0$, exp, summation and $d(.|.)$. So, Lemma 5.9 follows from Lemma 5.10.

Chapter 6

Reflections and Future Works

6.1 Reflections

In this thesis, we have made progress to address the following three questions:

- What is role of noisy feedback in interference networks?
- How does the restriction to operate in the finite-blocklength regime affect the performance of interference networks?
- How do the restriction to operate in the finite-blocklength regime and the presence of side information affect the compression and decompression of an information source?

6.1.1 Role of noisy feedback

- Even though noisy feedback has less information than full feedback, it is found that noisy feedback can still improve the capacity region of an interference channel when the feedback link strength exceeds certain threshold.

This performance gain is due to the fact that noisy feedback can help communication nodes to learn about each other's messages. As a result, communication nodes can cooperate with each other.

- Intuitively, the most important part of a feedback is the information about other transmitters. When the feedback link strength is too small, this important message is submerged in the feedback noise. Therefore, noisy feedback is useless in this case. However, when feedback link strength is sufficiently large, noisy feedback starts to contain information of other transmitters and the capacity region of the interference channel starts to enlarge. When the performance gain due to noisy feedback is large, it is justified economically to build a feedback system for a communication system.

6.1.2 Interference networks in the finite-blocklength regime

In the strictly very strong interference regime, even in the finite-blocklength communication, we still have an interesting observation that receivers can still decode information from the non-intended receivers in such a short duration. As a result, they can remove interference and decode information from the intended receivers.

6.1.3 Combined effect of side information and finite-blocklength communication on source coding

Even in the finite-blocklength communication, it is found that the presence of side information can help the encoder and the decoder to choose the most effective coding strategy up to the second-order terms and to adapt to changes in the environment.

6.2 Future Works

Various avenues for further research follow naturally from the contributions in this thesis. Some possible extensions are mentioned below.

- One possible direction is to further reduce the gap between inner bounds and outer bounds for the symmetric Gaussian IC with noisy feedback. The current constant gap of 4.7 bits/s/Hz can potentially be improved.
- Another interesting work is to obtain the approximate capacity region for the asymmetric Gaussian interference channel with noisy feedback. New techniques might be needed to deal with the asymmetric setting.
- The class of *mixed channels* forms an important class of models for theoretical study as they are the canonical class of non-ergodic channels [39]. The second-order source coding rate region has been considered for the mixed correlated source for the Slepian-Wolf problem in [81]. The corresponding point-to-point channel coding problem was also studied in [86, 111]. It

would be also interesting to find the second-order capacity region for the mixed Gaussian IC in the strictly very strong interference regime. The key difficulty is that characterizing the second-order capacity region for the mixed Gaussian IC appears to involve manipulating the modified information densities and the auxiliary output distributions. Previous works in mixed channels in [39, 81] do not involve auxiliary output distributions. New achievability and converse techniques will be needed to find the second-order capacity region for the mixed Gaussian IC.

- It is also interesting to characterize the second-order capacity region of Gaussian, or discrete memoryless, interference channel in the non-strictly very strong interference regime.

- It is interesting to carry out the second-order analysis for Gaussian source with a quadratic distortion measure.

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Publications

Journals

J1. S.-Q. Le, R. Tandon, M. Motani, and H. V. Poor. Approximate capacity region for the symmetric Gaussian interference channel with noisy feedback. IEEE Transactions on Information Theory, Submitted, Dec 2012.
J2. S.-Q. Le, V. Y. F. Tan, and M. Motani. A case where interference does not affect the channel dispersion. IEEE Transactions on Information Theory, Submitted, Apr. 2014.

Conference

C1. S.-Q. Le, R. Tandon, M. Motani, and H. V. Poor. On the sum-capacity of the linear deterministic interference channels with partial feedback. In Proc. IEEE International Symposium on Information Theory, pages 2102 2106, Cambridge, MA, 2012.
C2. S.-Q. Le, R. Tandon, M. Motani, and H. V. Poor. On linear deterministic interference channels with partial feedback. In Proc. IEEE Information Theory and Application Workshop, pages 131136, San Diego, CA, 2012.
C3. S.-Q. Le, R. Tandon, M. Motani, and H. V. Poor. The capacity region of the symmetric linear deterministic interference channels with partial feedback. In Proc. 50th Annual Allerton Conference on Communication, Control and Computing, Monticello, IL, 2012.
C4. S.-Q. Le, V. Y. F. Tan, and M. Motani. On the dispersions of the discrete memoryless interference channel. In Proc. IEEE International Symposium on Information Theory, pages 1859-1864, Istanbul, Turkey, 2013.
C5. S.-Q. Le, V. Y. F. Tan, and M. Motani. On the discrete memoryless interference channel in the finite blocklength regime. In Proc. IEEE Information Theory and Application Workshop, San Diego, CA, 2013.
C6. S.-Q. Le, V. Y. F. Tan, and M. Motani. Second-order asymptotics for the Gaussian interference channel with strictly very strong interference. In Proc.

IEEE International Symposium on Information Theory, pages 2514-2518, Honolulu, HI, 2014.
C7. S.-Q. Le, V. Y. F. Tan, and M. Motani. Second-order rate-distortion function for source coding with side information. NUS Technical Report, A1001, Aug. 2014.