LINEAR PROGRAMMING BASED MODELS

FOR

RESILIENT SUPPLY SYSTEMS DESIGN

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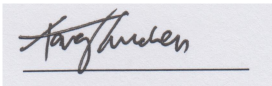
Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.



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Contents

Summary

This thesis develops a framework to quantitatively model the concept of resilience in general supply systems design problems that can be cast as linear programming models. First, we build a general notion of resilience measures from the decision-making perspective. Specifically, we use a penalty cost function to measure the supply system's service quality by comparing it with a given tolerance level. We then propose the formal definition of resilience measure as a mapping from a space of uncertain penalty positions to a range of nonnegative numbers. An axiomatic framework is used to describe the salient characteristics of resilience measures that reflect the decision-makers risk attitude and at the same time preserve computational tractability. After giving some examples of such resilience measures, we propose two different approaches of constructing new resilience measures from a decision-making and computational perspective respectively. We next focus on two important applications to illustrate the efficiency of the resilience measure based framework. In the energy supply system application, no distributional information is available at the planning stage, and we adopt the idea of adjustable uncertainty sets for the construction of resilience index. We then develop efficient algorithms for evaluation of the resilience index defined by various types of adjustable uncertainty sets, and extend it to a design optimization problem which aims to maximize the resilience of the energy supply system. In the telecommunication network application, historical data are available at the planning stage. We model uncertain demands as ambiguous random variables with known first- and second-order moments. We then propose a design model based on two-stage robust optimization modelling for maximizing the resilience of the resulting telecommunication network and present a class of tractable approximations to solve it. All the computational results suggest the superiority of our proposed resilience measure based approach over traditional methods in terms of supply system's service quality.

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Nomenclature and Abbreviations

$\mathbf{x}$	Lower-case bold face letters denote common vectors
$\mathbf{A}$	Upper-case bold face letters denote common matrices
$\emptyset$	Empty set
$\Re^n$	Set of all real valued vectors with n entries
$\Re^n_+$	Set of all real valued vectors with n nonnegative entries
$\mathbb{Z}$	Set of all integers
$\mathbb{Z}_+$	Set of all nonnegative integers
$\mathbb{R}$	Set of all rational numbers
$\mathbb{R}_+$	Set of all nonnegative rational numbers
$\mathbb{S}^n$	Set of symmetric matrices with dimension $n \times n$
$\mathbb{M}(\mathcal{D})$	Set of finite Borel measures supported by set $\mathcal{D}$
$\lfloor \cdot \rfloor$	Floor of a scalar, i.e., $\lfloor x \rfloor \triangleq \max\{t \in \mathbb{Z} : t \leq x\} : \forall x \in \Re$
$\lceil \cdot \rceil$	Ceiling of a scalar, i.e., $\lceil x \rceil \triangleq \min\{t \in \mathbb{Z} : t \geq x\} : \forall x \in \Re$
$\lvert \cdot \rvert$	Absolute value of a scalar, or the determinant of a matrix, or the number of elements in a set

$\|\cdot\|_p$	Polynomial norm of a vector, i.e., $\|\mathbf{x}\|_p \triangleq \sum_{i=1}^{n} (\|x_i\|^p)^{1/p} : \forall p > 0$. By convention, we denote $\|\mathbf{x}\|_\infty \triangleq \lim\limits_{p\to\infty} \|\mathbf{x}\|_p$
$(\cdot)^+$	Positive part of a scalar, i.e., $x^+ \triangleq \max\{x, 0\} : \forall x \in \Re$
$\mathbf{x}^+$	Positive part of a vector, i.e., $\mathbf{x}^+ \triangleq (x_1^+; \cdots; x_n^+) : \mathbf{x} \in \Re^n$
$\lfloor \mathbf{x} \rfloor$	Integer part of a vector, i.e., $\lfloor \mathbf{x} \rfloor \triangleq (\lfloor x_1 \rfloor; \cdots; \lfloor x_n \rfloor) : \mathbf{x} \in \Re^n$
diag$(\cdot)$	Transformed diagonal matrix of a vector, i.e., diag$(\mathbf{x}) : \mathbf{x} \in \Re^n$ returns a diagonal matrix with its main diagonal coincides with $\mathbf{x}$
$\mathcal{CH}(\mathcal{D})$	Convex hull of the set $\mathcal{D}$
cl$\{\mathcal{D}\}$	Closure of the set $\mathcal{D}$
int$\{\mathcal{D}\}$	The interior of the set $\mathcal{D}$
$\mathbb{E}(\cdot)$	Expected value of a random quantity
$\mathbf{x} \geq \mathbf{y}$	Component-wise dominance for two vectors $\mathbf{x}, \mathbf{y} \in \Re^n$, i.e., $x_i \geq y_i : i = 1, \cdots, n$
$\mathbf{A} \succ \mathbf{B}$	The symmetric matrix $(\mathbf{A} - \mathbf{B})$ is positive definite
$\mathbf{A} \succcurlyeq \mathbf{B}$	The symmetric matrix $(\mathbf{A} - \mathbf{B})$ is positive semi-definite
$\circ$	A customized operator that $\mathbf{x} \circ \mathbf{y} \triangleq$ diag$(\mathbf{x})\mathbf{y} = (x_1 y_1; \cdots, x_n y_n) : \mathbf{x}, \mathbf{y} \in \Re^n$
VaR	Abbreviation of the popular term "value at risk". Mathematically, under distribution $\mathbb{P}$, VaR$_{\mathbb{P}}(\tilde{v}, \gamma) \triangleq \inf\{t : \mathbb{P}(\tilde{v} > t) \leq 1 - \gamma\}$
CVaR	Abbreviation of the popular term "conditional value at risk", which is also known as expected shortfall, expected tail loss and average value at risk. Mathematically, under distribution $\mathbb{P}$, CVaR$_{\mathbb{P}}(\tilde{v}, \gamma) \triangleq \inf\limits_{\nu} \left\{ \nu + \frac{\mathbb{E}_{\mathbb{P}}(\tilde{v} - \nu)^+}{1 - \gamma} \right\}$

Chapter 1

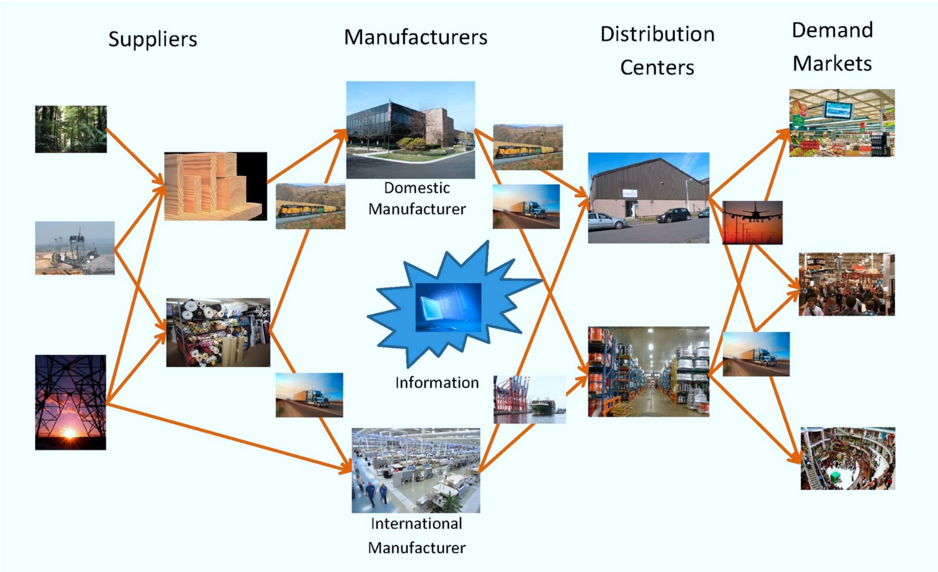
Introduction

1.1 Background

We consider "Supply System" as a system that delivers one or multiple commodities from supply sides to demand sides. In real world applications, various forms of supply systems appear in various disciplines as diverse as logistics (supply chain), telecommunication, transportation, power generation, natural gas routing, potable water distribution, to name a few. These supply systems differ from each other in a thousand ways, but share a similar network structure. More precisely, the whole system consists of separated nodes (categorized either by their physical locations or by their logical relationships) and arcs linking different nodes. Among all these nodes, some of which are sources at which the commodity is injected into the system. Correspondingly, some nodes are sinks at which consumption takes place. Normally, either affected by physical restriction or the intrinsic demand pattern, some pairs of sources and sinks locate in a long distance with each other such that directly routing the commodity from source to sink is expensive. If that is the case, some intermediate nodes are necessary for transition, storage and processing purposes. Let's take the supply chain system as an example (Figure 1.1)[1]. The sources are the suppliers that provide the necessary raw materials. The raw materials are

[1]Source: http://annanagurney.blogspot.sg/2012/01/how-us-can-compete-and-win-in-global.html

Figure 1.1: An illustrative supply chain system



then transported to the manufacturers to produce the final products. Then the qualified products are transported to different distribution centers (DCs), which can be regarded as intermediate nodes. Finally, products are transported to demand markets upon placed orders. In supply chain system, we move raw materials or the final products through common transportation vehicles such as ship, railway, or airline. Therefore, these different means of transportation play as arcs linking different nodes in the system.

Efficient decision making is necessary to guarantee the sustainability of the supply system's service in fulfilling customers' demands. Typically, two types of critical decisions are involved during the planing and operating of the supply system. At the planning stage, managers have to make strategic decisions determining the topology structure of supply system. Let's take the supply chain system as an example again. At the very beginning, the managers have to properly choose the suppliers to form a long-term agreement on raw material supply. Meanwhile, they have to choose the location and capacity of the manufacturing sites and distribution centers. After all these decisions being made, the topology of the supply chain network is fixed and no more temporary changes are allowed during the operational stage. At the operational stage, managers have to make the operational decisions to fulfill the demands at different sinks. For supply chain network, the operational decisions consist of determining the order quantity from suppliers and the shipment quantities from manufacturers to distribution centers and that from distribution centers to the final markets. Generally, these two stages of decisions conflict to each to some extent. In the planning stage the managers tend to investment less, which would in return affect the quality of service at the operational stage. To overcome this obstacle, many researchers consider these two types of decisions jointly to build integrated mathematical optimization models for decision supporting. This mathematical optimization based approach has been extensively studied in different disciplines of supply system and we refer it as the classical approach.

In real world applications, it is well accepted that data are always subject to uncertainty when critical decisions need to be made. In the planning stage of the supply chain system, the actual demand amount at the final demand markets and the actual supply amount of raw material suppliers can not be precisely known. In natural gas supply network, supply disruptions can occur due to unforeseen circumstances such as natural disasters, geopolitical crises and terrorist attacks, etc. These kind of uncertainties are inevitable in nature and failing to plan for them in an appropriate way can lead to severe degradation in service quality. Therefore, questions of how to measure the supply system's ability of mitigating the impact of uncertainties and how to design a supply system which is resilient against these uncertainties have received significant interest in supply security. This is especially true for supply systems of critical resources such that its quality of service is our first concern. The ultimate goal of this thesis is to develop a mathematical optimization based approach to provide efficient decision support in designing resilient supply systems. To achieve this, a fundamental step is to develop a unifying framework for quantitative and rigorous modeling of supply system resilience. Indeed, our framework of modeling supply system resilience is motivated by the the definition of resilience proposed in the UK Energy Research Centre (UKERC) white paper (Chaudry et al., 2011): "Resilience is the capacity of an energy system to tolerate disturbance and to continue to deliver affordable energy services to consumers. A resilient energy system can speedily recover from shocks and can provide alternative means of satisfying energy service needs in the event of changed external circumstances.”

In summary, this thesis aims to address three fundamental issues: (i) how to model the uncertainty arising from noisy and incomplete data, (ii) how to develop a general approach to quantify the supply system’s resilience, i.e., how well the supply system serves the demands under uncertainty, (iii) and how to develop practically efficient methods to compute and optimize supply system resilience, which would be conducive to solving the design problem. Theoretically, we consider an optimization problem to be tractable if it can be solved in polynomial time (e.g., via interior point method or ellipsoid method). To my knowledge, this is the case if the optimization problem can be formulated into standard convex optimization problems, such as Linear Program (LP), Second Order Cone Program (SOCP), and Linear Matrix Inequality (LMI), that can be solved by off-the-shelf solvers. Here “practically efficient” means that we can solve the corresponding problem by limited number of tractable optimization problems such that the total computational time would not be too large for moderate sized problem. In the following we will explain how we address the proposed three issues in details, followed by which the outline of the thesis will be presented.

1.2 Research Objectives

While uncertainty has been addressed in different disciplines of supply systems from different perspectives (e.g. Carvalho et al., 2014 for natural gas supply system; Georgiadis et al., 2011 for supply chain system; Qadrdan et al., 2014 for electricity supply system), their approaches are far from perfect in modeling uncertainty and identifying the supply system’s resilience. As stated, our fundamental goal is to develop a unifying framework of modeling system resilience which can provide efficient decision support in designing resilient supply systems. More specifically, this thesis is to:

1. Build an explicit model of uncertainty which will reflect some real world concerns. In the literature, people tend to apply the two stage Stochastic Programming (SP) approach of modeling uncertainty as random variables with given distribution (Dantzig, 1955). However, the exact distribution of the uncertainty is rarely known in reality. This has rekindled recent interests in the Robust Optimization (RO) paradigm as an alternative approach of addressing data uncertainty. Instead of assuming the exact distribution of uncertainty, RO decides an uncertainty set defining all the realizations for which the decision maker is willing to be prepared. A critical criticism on RO is its conservatism because it totally ignores the distributional knowledge of uncertainty, which could be partially available in some cases. In this thesis, we use an ambiguous model of uncertainty as a compromise of the stochastic model of uncertainty in SP and the uncertainty set based model of uncertainty in RO. Specifically, we model the primitive uncertain variables as ambiguous random variables that their joint distribution is known to belong to a family of distributions specified by some distributional information such as mean, support, and covariance matrix. Our proposed ambiguous model of uncertainty is consistent with the *distributionally robust optimization* paradigm which aims to bridge the gap between SP and RO. It is worth pointing out that when only support information $\mathcal{W}$ of the random vector $\tilde{\mathbf{z}}$ is available, i.e., the specified distributional family $\mathcal{P} = \left\{\mathbb{P} \in \mathbb{M}(\mathcal{W}) : \mathbb{E}_{\mathbb{P}}\left(1_{\{\tilde{\mathbf{z}} \in \mathcal{W}\}}\right) = 1\right\}$, our proposed ambiguous model of uncertainty recovers to the classical RO approach and thenceforth constitutes a generalization of the classical RO approach.

2. Build a quantitative and rigorous approach of measuring supply system resilience, which will reflect the supply system's ability of continuing its service under uncertainty whilst at the same time capture the decision maker's attitude towards uncertainty. To do this, we consider to map the supply system's service quality to a scalar value and give the formal definition of resilience measure as a family of these mappings. Instead of defining a unique resilience measure, we propose an axiomatic framework for describing the salient characteristics of resilience measures. Specifically, we measure the supply

system resilience by evaluating the random penalty cost function which uniquely depends on the random demand loss vector. We then construct the axiomatic framework by several axioms which are stimulated from some basic and reasonable intuitions. This proposed axiomatic framework enables us to choose specific resilience measure according to specific concerns.

3. Develop practically efficient algorithms for the computation and optimization of resilience measures. Specifically, we apply our resilience measure based framework to two specific applications. In the first application, we consider energy supply system in which supply disruptions are the major source of uncertainty. Since rare distributional knowledge of the supply realizations is available in this case, we adopt the RO approach of addressing uncertainty. More precisely, we define the resilience index as the size of largest *adjustable* uncertainty set containing no unacceptable event of supply realizations and develop efficient solution algorithms to compute the resilience index for cardinality-constrained adjustable uncertainty sets. Based on this, we also study the design problem maximizing the energy supply system's resilience with limited investment budget. In the next application we consider telecommunication network resilience under demand uncertainty, where vast amount of historical demand data are available at the planning stage. In this case we model the demands as ambiguous random variables with known support and moments (first- and second-order moments). To this end, we use the distributionally-ambiguous shortfall awareness measure to address both the distributional ambiguity and computational tractability. Since the telecommunication network design problem is generally difficult to solve, we propose decision rule based approximation of the proposed resilient telecommunication network design model.

The remainder of this thesis is organized as follows. In Chapter 2 the related literature is reviewed. In Chapter 3 we build the fundamental framework of defining supply system resilience measure. Chapter 4 investigates the construction of specific resilience measure. In particular, we propose two approaches of building resilience measures: from subjectively specified reference measures or simply from adjustable uncertainty sets. Chapter

5 considers the resilience related issues of energy supply system. Chapter 6 deals with the resilience of telecommunication network. Chapter 7 concludes the whole thesis and gives some lights on future research.

Chapter 2

Literature Review

2.1 Optimization under Uncertainty

In classical mathematical optimization, we seek to minimize (or maximize) an objective function subject to a set of constraints as follows:

$$\begin{aligned} \min \quad & f_0(\mathbf{x}, \mathbf{a}_0) \\ \text{s.t.} \quad & f_i(\mathbf{x}, \mathbf{a}_i) \geq 0 : \forall i \in \mathcal{I}, \end{aligned} \tag{2.1}$$

where $\mathbf{x}$ is the vector of decision variables and $\mathbf{a}_i : i \in \mathcal{I} \cup \{0\}$ are the parameters which are primarily assumed to be deterministic.

When parameters in the objective function are uncertain, we are unlike to obtain the "real" optimal solution. When parameters in the constraints differ from the assumed nominal values, the computed "optimal solution" might not even satisfy all the constraints. The stochastic programming approach might be the first attempt of addressing parameter uncertainties in optimization problems. Specifically, stochastic programming assumes that the joint distribution of the parameters $\tilde{\mathbf{a}} = (\tilde{\mathbf{a}}_i)_{i \in \mathcal{I} \cup \{0\}}$, denoted as $\mathbb{Q}$, is precisely given. Based on this, stochastic programming adopts the two-stage based modeling framework which is an important building block in the literature of optimization under uncertainty. In particular, we classify the decision variables $\mathbf{x}$ into two categories:

the *here and now* decisions $\mathbf{x}_1$ which should be made prior to the realization of the $\tilde{\mathbf{a}}$, and the *wait and see* decisions $\mathbf{x}_2$ which can be adjusted after realizing the exact values of $\tilde{\mathbf{a}}$. Generally, the first stage cost is independent of the realized uncertainty and we thenceforth can decompose the original objective as $f_0(\mathbf{x}, \mathbf{a}_0) = \hat{f}_0(\mathbf{x}_1) + \check{f}_0(\mathbf{x}_2, \mathbf{a}_0)$. The stochastic programming version of the original optimization problem (2.1) is them given as:

$$\min \hat{f}_0(\mathbf{x}_1) + \mathbb{E}_{\mathbb{Q}}\left[\mathcal{Q}(\mathbf{x}_1, \tilde{\mathbf{a}})\right], \tag{2.2}$$

where $\tilde{\mathbf{a}} = (\tilde{\mathbf{a}}_i)_{i \in \mathcal{I} \cup \{0\}}$ now becomes a random vector and the expectation is taken with respect to its joint distribution $\mathbb{Q}$. For each realization of $\tilde{\mathbf{a}}$, $\mathcal{Q}(\mathbf{x}_1, \tilde{\mathbf{a}})$ is the optimal objective value of the following optimization problem:

$$\begin{aligned} \mathcal{Q}(\mathbf{x}_1, \tilde{\mathbf{a}}) \triangleq \min \quad & \check{f}_0(\mathbf{x}_2, \tilde{\mathbf{a}}_0) \\ \text{s.t.} \quad & f_i(\mathbf{x}_1, \mathbf{x}_2, \tilde{\mathbf{a}}_i) \geq 0 : \forall i \in \mathcal{I}. \end{aligned}$$

Comprehensive introduction to stochastic programming can be found in Birge and Louveaux (2011); Shapiro and Ruszczyński (2003).

The two stage SP model (2.2) is expressively rich and has been widely applied in the supply chain literature. However, there are some fundamental drawbacks. In practice, we can rarely obtain the actual distribution of the uncertainties. Besides, even if the precise distribution is known, the corresponding SP model is computationally challenging to solve. In addition, in the absence of some structural property of the SP model, some popular solution methods of SP model such as Sampling Average Approximation (SAA) may yield meaningless first stage solutions $\mathbf{x}_1^*$ that its corresponding recourse problem $\mathcal{Q}(\mathbf{x}_1^*, \tilde{\mathbf{a}})$ is infeasible for some realizations of $\tilde{\mathbf{a}}$. To overcome these shortcomings, Soyster (1973) establishes the robust optimization scheme as an alternative approach of addressing data uncertainty by replacing probability distributions with specified uncertainty set. Attracted by the tractable representation of the robust counterpart by either Lagrange duality (e.g., Ben-Tal et al., 2009) or Fenchel duality (Ben-Tal et al., 2012), robust optimization has witnessed an explosive growth in recent years (See Ben-Tal and Nemirovski, 1998, 1999, 2000; Bertsimas and Sim, 2003, 2004; El Ghaoui and Lebret, 1997; El Ghaoui et al., 1998 for classical robust counterpart and Ben-Tal et al., 2004 for adjustable robust counterpart). Extensive review about RO is given in Ben-Tal et al. (2009); Bertsimas et al. (2011). However, it has been observed that when some level of distributional knowledge of the uncertainty is available, RO may yield overly conservative solutions.

To overcome both the over specificity of SP and the conservatism of RO, a new variant of optimization scheme, which is named as *distributionally robust optimization approach* has been promoted. In distributionally robust optimization, we assume the probability distribution $\mathbb{Q}$ itself be uncertain. In particular, the distributionally robust optimization version of the two stage SP model (2.2) is given as:

$$\min \hat{f}_0(\mathbf{x}_1) + \sup_{\mathbb{Q}\in\mathbb{F}} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{Q}(\mathbf{x}_1, \tilde{\mathbf{a}})\right], \tag{2.3}$$

where $\mathbb{F}$ is the ambiguity set of distributions. Unlike classical SP or RO, the distributionally robust optimization approach address the decision maker's attitudes towards both risk (exposure to uncertain outcomes whose probability distribution is known) and ambiguity (exposure to uncertainty about the probability distribution of the outcomes). Recently, the duality results on moment problems (Bertsimas and Popescu, 2005; Isii, 1962; Popescu, 2005) have motivated a growing body of research on distributionally robust optimization with the belief that the distributionally robust version of the classical optimization model (2.1) is tractable for specific ambiguity set $\mathbb{F}$ and constraints $\{f_i\}_{i\in\mathcal{I}}$. Ghaoui et al. (2003) study a portfolio optimization problem of minimizing the worst-case Value at Risk under the assumption that only bounds on the mean and covariance matrix of the returns are known. Calafiore and El Ghaoui (2006) extend this result to linear optimization problem with ambiguous chance constraints in which the underlying distribution is only known to belong to a family of distributions specified by mean, covariance matrix and support. Erdoğan and Iyengar (2006) propose a robust sampling approach to approximate the joint ambiguous chance constraint problem in which the underlying distribution family $\mathbb{F}$ is defined by Prohorov metric. These works on single stage distributionally robust optimization has also stimulated some new thoughts on the complicated two stage problem (2.3). Bertsimas et al. (2010) study two stage distributionally robust linear programs with given mean and covariance. They show that this problem is NP-Hard if uncertainty impacts the right hand side of the recourse problem and build semidefinite programs to solve it for the special case that uncertainty only affects the objective function. Delage and Ye (2010) provide an ellipsoid method based polynomial time algorithm of the two stage distributionally robust optimization program with the ambiguity set specified by uncertain mean and covariance matrix. As a side result, then also give a data-driven approach of constructing the ambiguity set of distributions. Tractable approximations by linear decision rules or its advanced versions are proposed in Chen et al. (2007); Goh and Sim (2010); Kuhn et al. (2011), in which the ambiguity sets are specified by mean, support, covariance matrix and/or directional deviations.

2.2 Supply Chain Management under Supply Uncertainty

The literature of supply chain management is rich. In this section we review some of the related works that consider supply uncertainty. Typically, supply uncertainty is modeled as *yield uncertainty* or *capacity uncertainty* where the supply quantity various stochastically, or *lead-time uncertainty* where stochasticity presents in order lead time, or *supply chain disruption* under which some suppliers in the supply chain system stop functioning during the entire period of disruption. These forms of supply uncertainty are not distinct to each other since we can regard the supply chain disruption as the extreme case of yield uncertainty with a Bernoulli random yield. We refer the readers to Grosfeld-Nir and Gerchak (2004) for a review on yield uncertainty and to Zipkin (2000) for a textbook on inventory management under lead-time uncertainty. Indeed, the abstract concept of supply chain disruption has been addressed in a wide range of supply chain management related applications. Basically, these applications can be divided into three major categories: inventory management models, facility location models and the joint location-inventory models.

The studies by Parlar and Berkin (1991) and Berk and Arreola-Risa (1994) are regarded as early works that incorporate supply disruption in the EOQ model, although there might be earlier ones. Snyder (2006) propose a convex approximation of Berk and Arreola-Risa's model. His approximation behaves similarly to the simple EOQ cost function and thus a close-form solution of the optimal inventory policy can be derived. Based on this, Qi et al. (2009) further build an extended model by considering supply disruptions at both supplier and retailer with proper approximation of the cost function. Beside these EOQ model based studies, supply disruptions are also investigated from some other perspectives. For instance, supply disruption is embedded in different inventory systems. The impact of supply disruptions in (Q, r) inventory system is studied in Parlar (1997) and Mohebbi (2003). Dada et al. (2007) extend the newsvendor model by considering multiple unreliable suppliers which might be subject to supply disruptions. In addition, simulation methods are also proposed to study the effect of supply disruption (Schmitt and Singh, 2009, 2012; Snyder and Shen, 2006).

Supply disruption is also considered in the facility location problem. Snyder and Daskin (2005) consider two reliability based extensions of the the classical fixed-charge location problem and the p-median problem, in which the facilities are subject to failure with the same disruption probability. Their models assume that the customers can be reassigned to alternate distribution centers (DC) in the case that their originally assigned DC is disrupted. In their model, a weighted sum of the nominal cost when no disruption occurs and the expected additional transportation cost caused by reassignment in presence of disruptions is minimized. One typical critique of Snyder and Daskin's approach is on its uniform disruption probability assumption, which is hardly suitable for many real cases. Several approaches have been proposed to relax the uniform disruption probability assumption, such as enumerating all the possible disruption scenarios (Snyder et al., 2006), using nonlinear mixed integer formulations (Cui et al., 2010; Berman et al., 2007) and applying the continuum approximation by assuming that customers are uniformly spread (Cui et al., 2010; Li and Ouyang, 2010).

Motivated by the intuition that jointly considering the location and inventory decisions can save total cost, integrated supply chain design problems have been studied recently. Qi et al. (2010) consider a joint location-inventory model in which supply disruptions can occur both at suppliers and retailers. Basically, their approach of modeling supply disruption is the same as that in Qi et al. (2009) and its impact is reflected in the working inventory cost. Different from Qi et al. (2010) in which the customers are not allowed to be temporarily reassigned to other non-disrupted retailer when its pre-assigned DC is disrupted, Mak and Shen (2012) studies this problem by allowing customers to be reassigned after disruptions occur. Snyder et al. (2010) gives a comprehensive review.

2.3 Robust Network Design

With the beautiful interplay between the constraints and the geometry of the uncertainty set, RO has provided computationally scalable antidotes for various difficult problems. This is especially true for the network design problem. Atamtürk and Zhang (2007) develop a two-stage modeling framework of robust network flow and design problem, in which part of the network flows along with the arc capacities are determined in the first stage, and the rest of the flows are determined in the second stage. Extensions to multi-commodity are also presented. However, the proposed two stage robust network flow formulation is generally NP-Hard and does not inherit an efficient exact method to solve. Instead, the authors propose an approximation method.

To our knowledge, Atamtürk and Zhang (2007) is the only work dealing with general network flow problem. The rest of the literature mainly focus on the specific application on telecommunication network, in which the capacities of the arcs should be determined to fulfill multiple pairwise demands. Specifically, we choose to minimize the capital expenditures such that there exists a capacity-feasible, multi-commodity flow routing that can accommodate all realized demands. When the demands are precisely known, i.e., deterministic, this problem and its related multi-commodity network model extension have been well studied in the literature (see Atamtürk, 2002; Bienstock et al., 1998; Bienstock and Günlük, 1996; Dahl and Stoer, 1998; Frangioni and Gendron, 2009; Günlük, 1999; Raack et al., 2011, and the references therein).

To address demand uncertainty, which constitutes the major source of uncertainty in telecommunication network design, the *robust telecommunication network design problem* aims to find a capacity installation with minimum possible capital expenditures such that the resulting network is operational for any demand values residing in a prescribed *uncertainty set*. In the robust telecommunication network design literature, polyhedral uncertainty (Lemaréchal et al., 2010) sets are commonly used due to its simplicity. Chekuri et al. (2007); Gupta et al. (2001) report that the design problem is NP-Hard for general polyhedral uncertainty sets. Ben-Ameur and Kerivin (2005) propose an vertices enumeration based algorithm to this NP-Hard problem.

There have been various efforts to overcome the computational difficulties by limiting the flexibility of routing solutions, and using simple uncertainty sets such as the hose model and the $\Gamma-$model. For example, the *oblivious* (or static) routing policy restricts the routing of a commodity at each arc to be a linear function of the realized demand value of this commodity. As a consequence, for each demand commodity, a set of routes is designated to carry a fixed proportion of the realized demand, and the value of this proportion is independent of the any realized demand values. To address demand uncertainty, the *hose model,* which assumes that only the upper bounds of the sum of incoming and outgoing traffic is known, is popularly used in virtual private network (VPN) design problems due to its ease of specification and the resulting model simplicity. Altın et al. (2007) develop a mixed-integer programming (MIP) model for

VPN design and propose a branch-and-price-and-cut algorithm to solve it. By applying a decomposition property obtained from projecting out flow variables, Altın et al. (2011) simplify the resulting polyhedral analysis with metric inequalities, based on which they simplify the branch-and-price-and-cut algorithm. Koster et al. (2013) investigate the computational aspects of the robust network design problem with the more complicated $\Gamma-$model of uncertainty (Bertsimas and Sim, 2003, 2004), which is effective in addressing demand uncertainty when only the ranges of the demand values of each commodity are given. Lee et al. (2012) also use the $\Gamma-$model for the robust network design problem with discrete capacity installation and unsplittable flows. Other than the conservative oblivious routing policy, more flexible routing policies is also studied (see Ben-Ameur, 2007; Scutellà, 2009). We also note that there has been some progress in investigating the relationship between oblivious routing policy and the fully flexible dynamic routing (Mudchanatongsuk et al., 2007; Poss and Raack, 2012).

Chapter 3

Axiomatic Framework of Resilience Measure

In this chapter we give a formal definition of resilience measure following the abstract approach of target based satisficing measure. In particular, we develop an axiomatic definition of resilience measures to guarantee the consistency with intuitions on the target oriented decision trend.

First, we give our implicit model of supply system uncertainty to motivate the definition of resilience measure. Let $\tilde{\mathbf{z}}$ be a vector of primitive uncertain variables constituting the source of uncertainty involved in the design and operations supply systems. Throughout this thesis, we interchangeably use the notation with tilde sign (e.g., $\tilde{v}$) or a function of $\tilde{\mathbf{z}}$ (e.g., $v(\tilde{\mathbf{z}})$) to denote a random scalar for simplicity of representation. Suppose that $\tilde{\mathbf{z}}$ is an ambiguous random vector defined on a measurable space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is the corresponding $\sigma-$algebra of events and $\mathbb{P}$ corresponds to the actual probability distribution defined on $(\Omega, \mathcal{F})$. Without loss of generality, we assume that the sample space Ω is convex and compact (If not, we can replace it with its convex hull and the results still go through). To address distributional ambiguity, we also assume that $\mathbb{P}$ is not precisely known.

As previously stated, resilience corresponds to the system's ability of continuing its service under uncertainty. To measure the service quality of a given supply system $\mathcal{G}$, we introduce a penalty function denoting the total penalty cost associated with unfulfilled demands, which is of course uncertain and we denote it as $\phi_{\mathcal{G}}(\tilde{\mathbf{z}})$. In the context of this thesis, we specifically call the penalty cost function $\phi_{\mathcal{G}}(\tilde{\mathbf{z}})$ as penalty position and abbreviate it by $\phi(\tilde{\mathbf{z}})$ when it is not necessary to distinguish different supply systems. Our fundamental philosophy of quantifying supply system resilience is to relate every penalty position $\phi(\tilde{\mathbf{z}})$ to a real number and use that real number to quantify its resilience. A typical critique might be that measuring supply system resilience by a single number may lead to a loss of information about the stochastic system performance $\phi(\tilde{\mathbf{z}})$. However, many actual decisions about supply systems in practice fundamentally involve ranking the resilience of different systems. Therefore, we can benefit by the simplicity of this single number and use it as the decision criterion when designing the supply system.

To this end, we carry our definition of resilience measure as a consistent mapping from a space of penalty position $\mathcal{V}$ to an interval $[0, \bar{\rho}]$. In this thesis, we originally specify $\mathcal{V}$ as the set of penalty positions defined as piece-wise affine functions as:

$$\mathcal{V} \triangleq \left\{ \tilde{v} \,\middle|\, \exists \mathbf{v}_i \geq \mathbf{0}, v_{i0} : i \in \mathcal{I} \text{ such that } \tilde{v} = \max_{i \in \mathcal{I}} \left\{ v_{i0} + \mathbf{v}_i' \tilde{\mathbf{z}} \right\} \right\}. \tag{3.1}$$

The space $\mathcal{V}$ is expressive enough and possesses some nice properties that can make our analysis intact:

1. For every $\tilde{v} \in \mathcal{V}$ and a scalar $a \in \Re$, we have $\tilde{v} + a \in \mathcal{V}$.

2. For every $\tilde{v} \in \mathcal{V}$ and $\theta \geq 0$, we have $\theta \tilde{v} \in \mathcal{V}$.

3. For every $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$, we have $\tilde{v}_1 + \tilde{v}_2 \in \mathcal{V}$ and $\max\{\tilde{v}_1, \tilde{v}_2\} \in \mathcal{V}$.

In real world decision making, it is prevalent that decision maker is primarily concerned about attaining the target other than maximizing his outcome. Simon built the well-known bounded rationality model in Simon (1955) to interpret this target oriented decision trend and coins the term *satisficing* to describe this behavior in Simon (1959). This satisficing behavior enjoys ample empirical justifications (see Mao, 1970; Payne et al., 1980, 1981) and has stimulated new decision criteria named as satisficing and aspiration measures which admit preference diversification (Brown and Sim, 2009; Brown et al., 2012). To embody the target oriented decision trend, we denote by τ_0 as the threshold of maximum acceptable level of demand loss penalty (or tolerance level). Therefore, we consider the performance of the system as acceptable whenever $\phi(\tilde{\mathbf{z}}) \leq \tau_0$ and not acceptable otherwise (For rest exposition in this chapter, if not specified, we can assume that $\tau_0 = 0$ by embedding τ_0 within the penalty position $\phi(\tilde{\mathbf{z}})$).

Perhaps the most common candidate that admits the satisficing trend is the success probability $\mathbb{P}(\phi(\tilde{\mathbf{z}}) \leq 0)$. However, the success probability is not appropriate in our context because firstly, its evaluation requires precise distributional information of the uncertainty, which is not available when critical decisions need to be made. Even if the distributional information is precisely known, its evaluation and optimization is generally intractable (Nemirovski and Shapiro, 2006 show that evaluating the success probability of a linear constraint on uniformly and independently distributed random variables is NP-Hard). Brown and Sim (2009) propose a general notion of "probability-like" measures which are natural to specify. They also pointed out that when such measures are quasi-concave, optimization of which can be approached by computationally tractable methods. Chen and Sim (2009) propose a goal-driven model that encompasses the success probability and the expected level of under-performance. This goal driven objective, also named as shortfall awareness measure, is closely related to the quasi-concave conditional value-at-risk (CVaR) satisficing measure so that computational tractability can be approached. However, adopting the shortfall aspiration criterion in our case of measuring system resilience also poses some issues as follows. First, as in the case of success probability, the evaluation of the shortfall aspiration criterion also requires distributional information on the uncertainty. Even though tractable robust approximations can be generated using partial distributional information, for complex problems where second stage actions are involved (such as in our case, where second stage actions refer to uncertainty recovery and mitigation actions), further approximations such as affine decision rules (or enhanced version) are required. These approximations would question the accuracy of the resilience measure, which is primarily adopted to rank resilience of different systems. In light of these, we return to consider the basic qualifying characteristics of such probability-like performance measures using an axiomatic definition below in order to develop resilience measures suitable for our purpose, instead of restricting our attention to a specific one such as the shortfall awareness measure.

Definition 3.1. A mapping $\rho : \mathcal{V} \longmapsto [0, \bar{\rho}]$, where $\bar{\rho} \in (0, +\infty)$, is a resilience measure if the following axioms hold for every $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$:

1. Satisfaction: $\rho(0) = \bar{\rho}$ and $\rho(1) = 0$.
2. Monotonicity: For $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$, if $v_1(\tilde{\mathbf{z}}) \leq v_2(\tilde{\mathbf{z}}) : \forall \tilde{\mathbf{z}} \in \Omega$, we have $\rho(\tilde{v}_1) \geq \rho(\tilde{v}_2)$.
3. Left continuity: For $\tilde{v} \in \mathcal{V}$, we have $\lim_{a \downarrow 0} \rho(\tilde{v} - a) = \rho(\tilde{v})$.
4. Scale invariance: If $\alpha > 0, \tilde{v} \in \mathcal{V}$, then $\rho(\alpha\tilde{v}) = \rho(\tilde{v})$.
5. Quasi-concavity: For $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$ and $0 \leq \theta \leq 1$, we have $\rho(\theta\tilde{v}_1 + (1 - \theta)\tilde{v}_2) \geq \min\{\rho(\tilde{v}_1), \rho(\tilde{v}_2)\}$.

In addition, we say that $\rho(\cdot)$ is *computationally proper* if the following holds:

- For all $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}$, we have $\rho(\max\{\tilde{v}_1, \tilde{v}_2\}) = \min\{\rho(\tilde{v}_1), \rho(\tilde{v}_2)\}$.

The first three axioms are naturally adapted from characteristics of the probability-like measures in Brown and Sim (2009) termed also as *satisficing measures*. In particular, the *satisfaction* condition in Axiom 1 states that all the penalty positions that are always nonpositive should be mostly preferred, while the positions that are always positive should be least preferred. The monotonicity property in Axiom 2 states the intuitive reasoning that when one system "A" *always* outperforms the other system "B", then "A" should never be less preferred as "B". More importantly, it is worth stressing that the monotonicity is also consistent with the first and second order stochastic dominance, which are widely used in decision theory as an alternative of the risk neutral decision criterion. The left continuity property in Axiom 3 means if we reduce the demand loss penalty by a small amount, the gain in terms of supply system resilience should approach zero at the limit. The scale invariance property in Axiom 4 suggests that, when comparing systems, mere differences in absolute units and scales should not influence its resilience evaluation. This axiom is much based on the intuition that re-scaling the penalty position, by changing the cost unit, or by adding several completely correlated penalty positions, would not change our attitude towards the system resilience.

The quasi-concavity in Axiom 5, and computational properness are carried out to rule out some measures like success probability, which we have pointed out to be computationally prohibited. Indeed, these two properties admit the potentials of simplifying the optimization and the evaluation of resilience measures. For instance, consider the problem of selecting the most "resilient" supply system from a convex space $\mathcal{C}$ using $\rho(\mathcal{G})$ as:

$$\rho^* \triangleq \max\{\rho(\mathcal{G}) : \mathcal{G} \in \mathcal{C}\}.$$

It is commonly known that developing a computationally efficient method to find a feasible solution that lies in a convex set is relatively easier than non-convex sets. When quasi-concavity holds, the set

$$\mathcal{S}(\gamma) = \{\mathcal{G} : \rho(\mathcal{G}) \geq \gamma\}$$

is convex, and hence, we assume that there exists a computationally tractable oracle of finding a feasible solution in the convex set $\mathcal{S}(\gamma)$. We then can solve ρ^* as a sequence of tractable problems with certain accuracy by bisection on ρ^*, provided that $0 \leq \rho^* \leq \bar{\rho}$. The quasi-concavity property admits the tractability of optimizing resilience measure, while the computationally properness simplifies its evaluation. By the monotonicity of resilience measure $\rho(\cdot)$ (Axiom 3), we deduce that $\rho(v(\tilde{\mathbf{z}})) \leq \min_{i\in\mathcal{I}}\{\rho(v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}})\}$ for all $v(\tilde{\mathbf{z}}) = \max_{i\in\mathcal{I}}\{v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}}\} : \mathbf{v}_i \geq \mathbf{0}$. When $\rho(\cdot)$ is computationally proper, the equality holds so that the resilience measure of the complex penalty position $v(\tilde{\mathbf{z}})$ can be simplified to the simple affine penalty positions $\{v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}}\}_{i\in\mathcal{I}}$.

Indeed, our general notion of resilience measure does not single out any specific resilience measure. (While our proposed resilience measure is actually a coherent satisficing measure by replacing $\tilde{v}$ with $-\tilde{v}$, we specifically call it resilience measure in the context of this thesis.) We now provide a few important examples which are commonly used in the literature. It is worth stressing that, most of them (except the distributionally-ambiguous forms of resilience measure) depend uniquely on the actual distribution of the penalty position $\tilde{v}$, which is also named as *law-invariant*. The law-invariant resilience measures are generally difficult to compute because multi-dimensional integration is computational intractable. Coincidently, they also do not possess the additional computational properness property.

Example 1. (Shortfall awareness measure): The shortfall awareness measure in Chen and Sim (2009) is motivated by a convex approximation of the success probability $\mathbb{P}(\tilde{v} \leq 0)$. Specifically, for every $t > 0$, we have the following inequality:

$$
\begin{aligned}
\mathbb{P}(\tilde{v} \leq 0) &= \mathbb{E}_{\mathbb{P}}\left(1_{(-\infty,0]}(\tilde{v})\right) \\
&\leq \mathbb{E}_{\mathbb{P}}\left(1 - (\tilde{v}/t + 1)^{+}\right) \\
&= 1 - \mathbb{E}_{\mathbb{P}}(\tilde{v}/t + 1)^{+}.
\end{aligned}
$$

The value of t plays as the degree of shortfall for the upside risk $\tilde{v} > 0$. By selecting the infimum of this bound for t in $(0, +\infty)$, Chen and Sim propose the shortfall awareness measure as:

$$\rho_{\mathrm{SAM}}(\tilde{v}) = 1 - \inf_{t>0} \mathbb{E}_{\mathbb{P}}(\tilde{v}/t + 1)^{+}.$$

It is not difficult to prove that the shortfall awareness measure is indeed a resilience measure by checking all of these five axioms. In addition, they point out that the shortfall aspiration awareness measure is connected to the CVaR measure as

$$\rho_{\mathrm{SAM}}(\tilde{v}) = \sup\left\{\gamma \in (0,1) : \mathrm{CVaR}_{\mathbb{P}}(\tilde{v},\gamma) \leq 0\right\},$$

where

$$\mathrm{CVaR}_{\mathbb{P}}(\tilde{v},\gamma) = \inf_{\nu\in\Re}\left\{\nu + \frac{\mathbb{E}_{\mathbb{P}}(\tilde{v}-\nu)^{+}}{1-\gamma}\right\}.$$

Example 2. (Distributionally-ambiguous shortfall awareness measure): Instead of assuming precise knowledge of the distribution, Zhu and Fukushima (2009) consider the distributionally-ambiguous case that $\mathbb{P}$ is only known to belong to a family of distribution $\mathbb{F}$. Therefore, we give the definition of distributionally-ambiguous shortfall awareness measure as

$$\rho_{\mathrm{DSAM}}(\tilde{v},\mathbb{F}) = \sup\left\{\gamma \in (0,1) : \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{CVaR}_{\mathbb{P}}(\tilde{v},\gamma) \leq 0\right\}.$$

In fact, for any law-invariant resilience measure, the corresponding distributionally-ambiguous version of resilience measure specified by a closed convex family of probability measure $\mathbb{F}$ remains a resilience measure. This is because all the five axioms are preserved when the supremum operator is placed over a closed convex distribution family. Therefore, the distributionally-ambiguous shortfall awareness measure is also a resilience measure when $\mathbb{F}$ is closed and convex.

Example 3. (Bernstein approximation based measure): Nemirovski and Shapiro (2006) state that for every $t > 0$, if $\ln\left(\mathbb{E}_{\mathbb{P}}[\exp\{\tilde{v}/t\}]\right) \leq t\ln(1-\gamma)$, we then have the following probability bound

$$\begin{aligned}
\mathbb{P}(\tilde{v} \leq 0) &= \mathbb{P}(\exp\{\tilde{v}\} \leq 1) \\
&= 1 - \mathbb{P}(\exp\{\tilde{v}\} > 1) \\
&\geq 1 - \mathbb{E}_{\mathbb{P}}[\exp\{\tilde{v}/t\}] \quad \text{Markov's Inequality} \\
&\geq 1 - (1-\gamma) \\
&= \gamma.
\end{aligned}$$

Since the infimum operator on $t \in (0, +\infty)$ would not affect the result, we have the following implication on the Bernstein approximation of success probability:

$$\inf_{t>0} \{t \ln (\mathbb{E}_{\mathbb{P}}[\exp\{\tilde{v}/t\}]) - t \ln(1-\gamma)\} \leq 0 \text{ implies } \mathbb{P}(\tilde{v} \leq 0) \geq \gamma.$$

Motivated by this, the Bernstein approximation based measure defined as

$$\rho_{\text{BAM}}(\tilde{v}) = \sup \left\{ \gamma \in (0,1) : \inf_{t>0} \{t \ln (\mathbb{E}_{\mathbb{P}}[\exp\{\tilde{v}/t\}]) - t \ln(1-\gamma)\} \leq 0 \right\}$$

also satisfies the axioms of resilience measure. Similarly, we can also construct a distributionally-ambiguous Bernstein approximation based measure $\rho_{\text{DBAM}}(\cdot)$ by specifying a family of probability measure $\mathbb{F}$.

Since CVaR is the tightest law-invariant convex approximation of VaR, it is worth stressing that the following inequality holds:

$$\rho_{\text{BAM}}(\tilde{v}) \leq \rho_{\text{SAM}}(\tilde{v}) \leq \mathbb{P}(\tilde{v} \leq 0).$$

Actually the result that CVaR is tighter than Bernstein approximation can also be derived from the simple inequality $x^+ \leq t \exp(x/t - 1) : \forall t > 0, x \in \Re$. Based on this, we have

$$\text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma) = \inf_{\nu \in \Re} \left\{ \nu + \frac{\mathbb{E}_{\mathbb{P}}(\tilde{v} - \nu)^+}{1-\gamma} \right\}$$

$$
\begin{aligned}
&\leq \inf_{\nu\in\Re}\inf_{t>0}\left\{\nu+\frac{1}{1-\gamma}t\mathbb{E}_{\mathbb{P}}\left[\exp((\tilde{v}-\nu)/t-1)\right]\right\}\\
&=\inf_{t>0}\inf_{\nu\in\Re}\left\{\nu+\frac{1}{1-\gamma}t\mathbb{E}_{\mathbb{P}}\left[\exp((\tilde{v}-\nu)/t-1)\right]\right\}\\
&=\inf_{t>0}\left\{t\ln\left(\mathbb{E}_{\mathbb{P}}[\exp\{\tilde{v}/t\}]\right)-t\ln(1-\gamma)\right\}.
\end{aligned}
$$

Example 4. (One side moment based measure): Choi et al. (2011) define an important law-invariant coherent risk measure

$$
\mu_{\text{OSM}}(\tilde{v},\gamma)=\mathbb{E}_{\mathbb{P}}(\tilde{v})+\gamma\left[\mathbb{E}_{\mathbb{P}}\left((\tilde{v}-\mathbb{E}_{\mathbb{P}}(\tilde{v}))^{+}\right)^{p}\right]^{1/p},
$$

where $\gamma\in[0,1],p\geq 1$. The one side moment based measure is a generalization of the mean variance risk measure (when $p=2$, it is exactly the mean variance measure). This risk measure is also special case of mean-risk functions which are usually expressed as a risk neutral term plus a weighted factor of variability. The value of γ reflects the degree of risk aversion and the p-th semideviation is used to measure the variability of the outcome.

Since, the one side moment based measure is a coherent risk measure for $\gamma\in[0,1],p\geq 1$, its dual presentation

$$
\rho_{\text{OSM}}=\sup\left\{\gamma\in[0,1]:\mu_{\text{OSM}}(\tilde{v},\gamma)\leq 0\right\}
$$

is a resilience measure. Similarly, the distributionally-ambiguous one side moment based measure $\rho_{\text{DOSM}}(\cdot)$ is also a resilience measure.

Chapter 4

Constructing Resilience Measure

The wide range of resilience measures allows us to choose one according to our specific considerations. In this chapter, we illustrate two different approaches of constructing resilience measures, either by subjective references or by adjustable uncertainty sets.

4.1 Resilience Measures and Subjective References

In practice, decision-makers may possess specific opinions on penalty positions that reflect their risk attitude and preferences. We now propose a general approach to incorporate such preferences to synthesize resilience measures that are consistent with the axioms presented in Definition 3.1. We assume that the decision-maker is behaviorally modeled with a reference function $\Psi : \mathcal{X} \longmapsto [0, \bar{\rho}]$, where $\mathcal{X}$ is defined as a "reference space" of penalty positions. The reference function $\Psi(\cdot)$ may be solicited by inquiring the decision-maker's scores on the real interval $[0, \bar{\rho}]$ for a set of penalty positions. A score of 0 means 'worst possible' resilience and $\bar{\rho}$ means 'best possible' resilience. To construct a new resilience measure that satisfies all the axioms, it might be inappropriate to directly use the reference function $\Psi(\cdot)$ because the decision maker might not be aware of consistently following the axioms when they are about to give the scores of the reference penalty positions. Therefore, the problem of how to synthesis a resilience measure that satisfies the axioms and preserve as much information of the reference function $\Psi(\cdot)$ as possible is crucial. We now focus, in the rest of this section, the important problem of extending $\Psi(\cdot)$ to a quasi-concave resilience measure $\rho_\Psi(\cdot)$ that is defined on a larger (extended) space $\mathcal{V}$. We assume $\mathcal{X}$ and $\mathcal{V}$ are sub-spaces of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the family of bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We also assume that $\mathcal{X}$ is closed under of constant addition, i.e., $\tilde{x} \in \mathcal{X} \Rightarrow \tilde{x} + a \in \mathcal{X} : \forall a \in \Re$. A basic artifact of interest in the synthesis of a resilience measure is that of a "support" of a given penalty position $\tilde{v} \in \mathcal{V}$, defined in the following.

Definition 4.1. *Supports of a penalty position:* Suppose that we have a family $\mathcal{X}$ of bounded random variables which is closed by under constant translation. We consider a family, indexed by the elements of $\mathcal{X}$, of nonnegative real numbers $\chi = \chi(\tilde{x}) : \tilde{x} \in \mathcal{X}$, all of them but a finite number being positive. We state that the pair (χ, a), where a is a real number, "supports" the random penalty position $\tilde{v} \in \mathcal{V}$ if

$$\tilde{v} \leq \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x})\tilde{x} - a.$$

In addition, we denote by $S_\mathcal{X}(\tilde{v})$ as the set of all pairs of (χ, a) that support $\tilde{v}$ on $\mathcal{X}$.

Intuitively, the definition of a support allows us to construct upper approximations of penalty positions in the space $\mathcal{V}$ using non-negative affine combinations of finite referenced penalty positions in $\mathcal{X}$. Specifically, χ is an infinite dimension vector that the elements in it are the coefficients of a nonnegative combination of the elements in space $\mathcal{X}$. By adding a constant shifting amount a, we finally construct an upper approximation of each $\tilde{v} \in \mathcal{V}$ as $\sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x})\tilde{x} - a$. Our definition of "support" is adapted from Artzner et al. (1999), who use nonnegative combination of "standard risk" to bound from below other risk positions. Inspired by the similarity between the penalty positions in our case and the risk positions in finance, we adjust our concept of "support" accordingly by bounding the penalty position from above by a positive combination of several reference penalty positions. Motivated by the above definition, we will show in Theorem 4.1 that when the reference measure $\Psi(\cdot)$ fulfills the *consistency condition* defined in the following, we can extend it to a resilience measure $\rho_\Psi(\cdot)$ on $\mathcal{V}$.

Definition 4.2. Consistency condition: Given a mapping $\Psi : \mathcal{X} \longmapsto [0, \bar{\rho}]$, we say that $\Psi(\cdot)$ fulfills the consistency condition if for every $\gamma > 0$, we have

$$\inf_{(\chi,a)\in S_{\mathcal{X}}(1)} \left\{ \sum_{\tilde{x}\in\mathcal{X}} \chi(\tilde{x})\mu(\tilde{x},\gamma) - a \right\} > 0,$$

where $\mu(\tilde{x},\gamma) := \inf\{a : \Psi(\tilde{x} - a) \geq \gamma\}$, for every $\tilde{x} \in \mathcal{X}$ and $\gamma \in [0, \bar{\rho}]$. We define $\inf \emptyset = \infty$ and $0 \cdot \inf \emptyset = 0$ as a convention.

Remark 4.1. Here $\mu(\cdot,\gamma)$ describes a dual characterization of $\Psi(\cdot)$. This format of dual characterization has been widely used to connect two different ways of measuring risk in finance. For example, when $\Psi(\cdot)$ is the success probability of $\mathbb{P}(\tilde{x} \leq 0)$, the corresponding $\mu(\cdot,\gamma)$ gives the classical risk of *value-at-risk*. More generally, when Ψ gives a *satisficing measure*, $\mu(\cdot,\gamma)$ gives a family of risk measures which is nondecreasing in γ.

Remark 4.2. Actually, the term "1" appears in the above equation can be replaced by any positive constant penalty positions because the left hand side in the equation is positive homogeneous. This condition is mainly specified to guarantee the axiom of satisfaction in resilience measure. The detailed explanation can be induced in the proof of Theorem 4.1.

We now give an important lemma on left continuity, which is useful for the proof of Theorem (4.1), as follows:

Lemma 4.1. *Given a mapping* $\mu : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \times [0, \bar{\rho}] \longmapsto \Re$ *such that:*

1. $\mu(\tilde{v},\gamma)$ *is nondecreasing in* γ*: for every* $\tilde{v} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$*,* $\gamma_1, \gamma_2 \in [0, \bar{\rho}], \gamma_1 < \gamma_2$*, we have* $\mu(\tilde{v},\gamma_1) \leq \mu(\tilde{v},\gamma_2)$*.*

2. $\mu(\tilde{v},\gamma)$ *is semi-nondecreasing in* $\tilde{v}$*: for every* $\gamma \in [0, \bar{\rho}]$*,* $a > 0$*, we have* $\mu(\tilde{v} - a,\gamma) \leq \mu(\tilde{v},\gamma)$*.*

3. $\mu(\tilde{v},\gamma)$ *is uniformly left continuous: for every* $\tilde{v} \in L^\infty(\Omega,\mathcal{F},\mathbb{P})$ *and* $\delta > 0$*, there exists* $\hat{\epsilon} > 0$ *such that* $|\mu(\tilde{v}-a,\gamma) - \mu(\tilde{v},\gamma)| < \delta$ *for every* $\gamma \in [0,\bar{\rho}], a \in [0,\hat{\epsilon}]$.

Then the mapping $\rho : L^\infty(\Omega,\mathcal{F},\mathbb{P}) \longmapsto [0,\bar{\rho}]$ *defined by*

$$\rho(\tilde{v}) = \sup\{\gamma : \mu(\tilde{v},\gamma) \leq 0\}$$

follows the left continuity property.

Proof. Observe that $\mu(\tilde{v},\gamma)$ is semi-nondecreasing in $\tilde{v}$, easily we have $\rho(\tilde{v}-a) \geq \rho(\tilde{v})$ indicating that $\lim_{a\downarrow 0}\rho(\tilde{v}-a) \geq \rho(\tilde{v})$. Hence, it suffices to show that for every sequence $\{a_k\}_{k=1,2,\cdots} \downarrow 0$ and positive number ϵ, there exists a positive integer N such that

$$\rho(\tilde{v}-a_k) \leq \rho(\tilde{v}) + \epsilon \ : \forall k \geq N \tag{4.1}$$

By the definition of $\rho(\cdot)$, we have $\mu(\tilde{v},\gamma) > 0$ for every $\gamma > \rho(\tilde{v})$. Regarding to the fact that $\mu(\tilde{v},\gamma)$ is nondecreasing in γ, we surely can find $\delta > 0$ for every positive ϵ such that the implication

$$\mu(\tilde{v},\gamma) \leq \delta \Longrightarrow \gamma \leq \rho(\tilde{v}) + \epsilon$$

holds. Note that $\mu(\tilde{v},\gamma)$ is uniformly left continuous, there exists $\hat{\epsilon} > 0$ such that $|\mu(\tilde{v}-a,\gamma) - \mu(\tilde{v},\gamma)| < \delta : \forall a \in [0,\hat{\epsilon}]$ and $\gamma \in [0,\bar{\rho}]$. Hence, for every $a \in [0,\hat{\epsilon}]$, if $\mu(\tilde{v}-a,\gamma) \leq 0$, we have

$$\mu(\tilde{v},\gamma) \leq \mu(\tilde{v}-a,\gamma) + \delta \leq \delta$$

indicating $\gamma \leq \rho(\tilde{v}) + \epsilon$. The arbitrariness of γ suggests that $\rho(\tilde{v}-a) \leq \rho(\tilde{v}) + \epsilon : \forall a \in [0,\hat{\epsilon}]$. Recall that $a_k \downarrow 0$, it is obvious that (4.1) holds. □

Theorem 4.1. *Given a family of bounded random variables $\mathcal{X} \subseteq L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ which is closed by any constant translation, and a mapping $\Psi : \mathcal{X} \longmapsto [0, \bar{\rho}]$. Define $\mu(\tilde{x}, \gamma) := \inf\{a : \Psi(\tilde{x} - a) \geq \gamma\}$ for every $\tilde{x} \in \mathcal{X}$ and $\gamma \in [0, \bar{\rho}]$. The following equation:*

$$\rho_{\Psi}(\tilde{v}) = \sup \left\{ \gamma : \inf_{(\chi, a) \in S_{\mathcal{X}}(\tilde{v})} \left\{ \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \mu(\tilde{x}, \gamma) - a \right\} \leq 0 \right\} \tag{4.2}$$

defines a resilience measure $\rho_{\Psi}(\cdot)$ if and only if $\Psi(\cdot)$ fulfills the consistency condition in Definition 4.2. Moreover, ρ_{Ψ} is the smallest quasi-concave resilience measure $\rho(\cdot)$ on $\mathcal{V}$ such that $\rho \geq \Psi$ on $\mathcal{X}$.

Proof. 1. The necessity of consistency condition follows straightforwardly from $\rho_{\Psi}(1) = 0$. We next show the sufficiency in 2-6, i.e., $\rho_{\Psi}(\cdot)$ is a resilience measure if the consistency condition given in Definition 4.2 holds.

2. Since $(0, 0)$ is a support of 0 on $\mathcal{X}$, we have $\rho_{\Psi}(0) = \bar{\rho}$ because $0 \cdot \mu(\tilde{x}, \gamma) = 0$. Meanwhile, the consistency condition ensures $\rho_{\Psi}(1) = 0$. Consequently, the satisfaction axiom holds.

3. Suppose $\tilde{v}_1 \leq \tilde{v}_2$, then we have $S_{\mathcal{X}}(\tilde{v}_1) \supseteq S_{\mathcal{X}}(\tilde{v}_2)$, which implies the following inequality for every $\gamma \in [0, \bar{\rho}]$:

$$\inf_{(\chi, a) \in S_{\mathcal{X}}(\tilde{v}_1)} \left\{ \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \mu(\tilde{x}, \gamma) - a \right\} \leq \inf_{(\chi, a) \in S_{\mathcal{X}}(\tilde{v}_2)} \left\{ \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \mu(\tilde{x}, \gamma) - a \right\}.$$

Consequently, we can verify that $\rho_{\Psi}(\tilde{v}_1) \geq \rho_{\Psi}(\tilde{v}_2)$, i.e, $\rho_{\Psi}(\cdot)$ satisfies the monotonicity.

4. Define $\hat{u}(\tilde{v}, \gamma) := \inf_{(\chi, a) \in S_{\mathcal{X}}(\tilde{v})} \left\{ \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \mu(\tilde{x}, \gamma) - a \right\}$. Since $\hat{\mu}(\tilde{v} - t, \gamma) = \hat{\mu}(\tilde{v}, \gamma) - t$, it is not hard to verify that $\hat{\mu}(\tilde{v}, \gamma)$ fulfills the required properties of $\mu(\tilde{v}, \gamma)$ in Lemma 4.1. Recall that $\rho_{\Psi}(\tilde{v}) = \sup\{\gamma : \hat{\mu}(\tilde{v}, \gamma) \leq 0\}$, the left continuity holds according to Lemma 4.1.

5. Since $(\chi, a) \in S_{\mathcal{X}}(\tilde{v})$ if and only if $(\alpha\chi, \alpha a) \in S_{\mathcal{X}}(\alpha\tilde{v})$, $\rho_{\Psi}(\cdot)$ is scale invariant.

6. To show the quasi-concavity, it suffices to show $\rho_{\Psi}(\tilde{v}_1 + \tilde{v}_2) \geq \min\{\rho_{\Psi}(\tilde{v}_1), \rho_{\Psi}(\tilde{v}_2)\}$ due to the fact that $\rho_{\Psi}(\cdot)$ is scale invariant. Recall that if $(\chi_1, a_1) \in S_{\mathcal{X}}(\tilde{v}_1)$ and $(\chi_2, a_2) \in$

$S_{\mathcal{X}}(\tilde{v}_2)$, there must be $(\chi_1 + \chi_2, a_1 + a_2) \in S_{\mathcal{X}}(\tilde{v}_1 + \tilde{v}_2)$. We then can deduce that $\hat{\mu}(\tilde{v}_1 + \tilde{v}_2, \gamma) \leq \hat{\mu}(\tilde{v}_1, \gamma) + \hat{\mu}(\tilde{v}_2, \gamma)$, for every $\gamma \in [0, \bar{\rho}]$. Hence, if $\hat{\mu}(\tilde{v}_1, \gamma) \leq 0$ and $\hat{\mu}(\tilde{v}_2, \gamma) \leq 0$ we surely have $\hat{\mu}(\tilde{v}_1 + \tilde{v}_2, \gamma) \leq 0$ implying $\rho_\Psi(\tilde{v}_1 + \tilde{v}_2) \geq \gamma$. By letting γ converge to $\min\{\rho_\Psi(\tilde{v}_1), \rho_\Psi(\tilde{v}_2)\}$, we have $\rho_\Psi(\tilde{v}_1 + \tilde{v}_2) \geq \min\{\rho_\Psi(\tilde{v}_1), \rho_\Psi(\tilde{v}_2)\}$.

7. We now show the rest part. Given a resilience measure $\rho(\cdot)$ such that $\rho(\cdot) \geq \Psi(\cdot)$ on $\mathcal{X}$. Since $\rho(\cdot)$ follows left continuity, to prove $\rho_\Psi \leq \rho$ on $\mathcal{V}$, it suffices to show that $\rho_\Psi(\tilde{v}) \leq \rho(\tilde{v} - \epsilon)$ for every $\tilde{v} \in \mathcal{V}$ and $\epsilon > 0$. Observe that for every $\gamma \in [0, \bar{\rho}]$ such that $\hat{\mu}(\tilde{v}, \gamma) \leq 0$, certainly we can find $(\chi, a) \in S_{\mathcal{X}}(\tilde{v})$ such that the following inequality holds

$$\sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \mu(\tilde{x}, \gamma) - a < \epsilon_0$$

for arbitrarily small $\epsilon_0 > 0$. By the definition of $\mu(\tilde{x}, \gamma) : \tilde{x} \in \mathcal{X}$, the arbitrariness of ϵ_0 ensures that we can find a family of positive numbers $\epsilon(\tilde{x}) > 0$ such that

$$\Psi(\tilde{x} - \mu(\tilde{x}, \gamma) - \epsilon(x)) \geq \gamma : \tilde{x} \in \mathcal{X}; \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \epsilon(\tilde{x}) + \epsilon_0 < \epsilon$$

Hence, by monotonicity of $\rho(\cdot)$ we have

$$\begin{aligned}
\rho(\tilde{v} - \epsilon) \quad & \geq \rho(\tilde{v} - \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \epsilon(\tilde{x}) - \epsilon_0) \quad \text{Because } \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \epsilon(\tilde{x}) + \epsilon_0 < \epsilon \\
& \geq \rho\left(\sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \tilde{x} - \sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \epsilon(\tilde{x}) - (a + \epsilon_0) \right) \quad : (\chi, a) \in S_{\mathcal{X}}(\tilde{v}) \\
& \geq \rho\left(\sum_{\tilde{x} \in \mathcal{X}} \chi(\tilde{x}) \left[\tilde{x} - \mu(\tilde{x}, \gamma) - \epsilon(\tilde{x}) \right] \right) \\
& \geq \min_{\chi(\tilde{x}) > 0} \left\{ \rho(\tilde{x} - \mu(\tilde{x}, \gamma) - \epsilon(\tilde{x})) \right\} \quad \text{Quasi-concavity of } \rho(\cdot) \\
& \geq \min_{\chi(\tilde{x}) > 0} \left\{ \Psi(\tilde{x} - \mu(\tilde{x}, \gamma) - \epsilon(\tilde{x})) \right\} \geq \gamma
\end{aligned}$$

Hence we have $\rho_\Psi(\tilde{v}) = \sup\{\gamma : \hat{\mu}(\tilde{v}, \gamma) \leq 0\} \leq \rho(\tilde{v} - \epsilon)$ completing the proof of $\rho(\tilde{v}) \geq \rho_\Psi(\tilde{v})$. □

The intuition of relating the extended resilience measure $\rho_\Psi(\cdot)$ in (4.2) to the reference function $\Psi(\cdot)$ can be viewed by the concept of "support". In fact, the dual representation $\mu(\tilde{x},\gamma)$ can be interpreted as the minimum constant shifting amount to make the original position $\tilde{x}$ be γ resilient in terms of the reference measure $\Psi(\cdot)$. Similarly, from the definition of "support", the constant a can be interpreted as the possible constant shifting amount of the nonnegative combination $\sum_{\tilde{x}\in\mathcal{X}} \chi(\tilde{x})\tilde{x}$ to bound $\tilde{v}$ from above. Therefore, if we disregard the internal infimum operation in equation (4.2), the inequality $\chi(\tilde{x})\mu(\tilde{x},\gamma) - a \leq 0$ means that the weighted summation of the "standard" shift amount by $\Psi(\cdot)$ should not be greater than the maximum possible shifting amount a. More precisely, let us assume that the infimum operations of defining $\rho_\Psi(\cdot)$ and $\mu(\cdot,\gamma)$ can be achieved exactly for sake of clarity (which of course is not always true in real cases). In other words, we assume that $\Psi(\tilde{x} - \mu(\tilde{x},\gamma)) = \gamma : \forall\gamma$ and, for $\gamma^* = \rho_\Psi(\tilde{v})$, there $\exists(\chi, a) \in S_{\mathcal{X}}(\tilde{v})$ such that $\sum_{\tilde{x}\in\mathcal{X}} \chi(\tilde{x})\mu(\tilde{x},\gamma^*) - a \leq 0$. It is not hard to see that $\tilde{v} \leq \sum_{\tilde{x}\in\mathcal{X}} \chi(\tilde{x})\tilde{x} - a \leq \sum_{\tilde{x}\in\mathcal{X}} \chi(\tilde{x})(\tilde{x} - \mu(\tilde{x},\gamma^*))$. Note that $\Psi(\tilde{x} - \mu(\tilde{x},\gamma^*)) = \gamma^*$, our principle of constructing $\rho_\Psi(\cdot)$ is that we think $\tilde{v}$ is γ-resilient if and only if we can bound it from above by a nonnegative linear combination of γ-resilient (by $\Psi(\cdot)$) penalty positions in the reference space $\mathcal{X}$.

Remark 4.3. Indeed, the intuitive interpretation of the consistency condition may not be straightforward, beyond the fact that it is a necessary condition for the *satisfaction* property of $\rho_\Psi(\cdot)$ in (4.2). At the same time, the consistency condition is weaker than the axioms in resilience measure (necessary but not sufficient). Therefore, one important implication of the extension is that we can build resilience measures with assistance of reference measures which are less restrictive. Moreover, if we apply Theorem 4.1 to $(\mathcal{X},\Psi) = (\mathcal{V},\rho)$, we have the following corollary:

Corollary 4.1 $\rho : \mathcal{V} \longmapsto [0,\bar{\rho}]$ is a resilience measure if and only if it is of the form $\rho_\Psi(\cdot)$ defined as (4.2) for a mapping $\Psi : \mathcal{V} \longmapsto [0,\bar{\rho}]$ that fulfills the consistency condition given in Definition 4.2.

4.2 Resilience Measures and Adjustable Uncertainty Sets

Though equation (4.2) gives precisely the extended resilience measure by subjective references, a practical issue arises that the computation of $\rho(\cdot)$ is generally difficult because we need to search the overall set of support functions $(\chi, a) \in S_{\mathcal{X}}(\tilde{v})$. In the following, we will propose an adjustable uncertainty sets based representation of resilience measure. This result is particularly important because we can then concentrate on uncertainty sets for choosing and studying resilience measures, and leverage on existing robust optimization technology for computational purposes. Furthermore, the robust optimization constructs require very little distributional assumptions on the uncertainties, which are quite suitable for the modeling of uncertainties in our context.

The intuition of relating resilience measure to uncertainty sets is quite straightforward. By considering $\mathcal{U}_{\tilde{v}}$ as the set of $\tilde{\mathbf{z}}$ such that $v(\tilde{\mathbf{z}}) \leq 0$, we can then think of relating the resilience measure of $\tilde{v}$ to the "size" of this uncertainty set $\mathcal{U}_{\tilde{v}}$. In the following we will show that this indeed yields resilience measures consistent with our axiomatic description in Definition 3.1. Before that we first give an auxiliary lemma on regular cones in conic optimization.

Lemma 4.2. *Suppose that $\rho : \mathcal{V} \longmapsto [0, \bar{\rho}]$, where $\mathcal{V}$ is defined in (3.1), is a resilience measure, then for all $\gamma \in [0, \bar{\rho}]$, the set*

$$\mathcal{K}(\gamma) = \left\{(\mathbf{v}, t) : \mathbf{v} \geq \mathbf{0}, \rho(\mathbf{v}'\tilde{\mathbf{z}} - t) \geq \gamma\right\}$$

is a regular cone which is closed, convex, pointed and has a non-empty interior.

Proof. The positive homogeneity of $\mathcal{K}(\gamma)$ comes straightforwardly from the scale invariance property of $\rho(\cdot)$. It is also not hard to see that $\mathcal{K}(\gamma)$ is closed and pointed cone by its definition. For any $(\mathbf{v}_1, t_1), (\mathbf{v}_2, t_2) \in \mathcal{K}(\gamma)$ and $\theta \in [0,1]$, we observe that

$$\begin{aligned}
&\rho\left((\theta\mathbf{v}_1 + (1-\theta)\mathbf{v}_2)'\tilde{\mathbf{z}} - (\theta t_1 + (1-\theta)t_2)\right) \\
&= \rho(\theta(\mathbf{v}_1'\tilde{\mathbf{z}} - t_1) + (1-\theta)(\mathbf{v}_2'\tilde{\mathbf{z}} - t_2)) \\
&\geq \min\left\{\rho(\mathbf{v}_1'\tilde{\mathbf{z}} - t_1), \rho(\mathbf{v}_2'\tilde{\mathbf{z}} - t_2)\right\} \\
&\geq \gamma.
\end{aligned}$$

where the first inequality is because $\rho(\cdot)$ is quasi-concave. Hence $(\theta\mathbf{v}_1 + (1-\theta)\mathbf{v}_2, \theta t_1 + (1-\theta)t_2) \in \mathcal{K}(\gamma)$, which verifies the convexity.

Recall that the sample space Ω is compact, which means $\sup_{\mathbf{z}\in\Omega} \|\mathbf{z}\|_\infty$ is bounded above by a finite number $\hbar$, where $\|\cdot\|_\infty$ is the commonly known Maximum norm. For every $\mathbf{v} \geq \mathbf{0}$ and $t \geq \hbar\|\mathbf{v}\|_1$, we observe that

$$\max_{\tilde{\mathbf{z}}\in\Omega}\{\mathbf{v}'\tilde{\mathbf{z}} - t\} \leq \hbar\|\mathbf{v}\|_1 - t \leq 0,$$

where $\|\cdot\|$ is the commonly known polynomial norm with $p = 1$. Hence, we have $\rho(\mathbf{v}'\tilde{\mathbf{z}} - t) \geq \rho(0) \geq \gamma$ and therefore the cone $\{(\mathbf{v}, t) : \mathbf{v} \geq \mathbf{0}, t \geq \hbar\|\mathbf{v}\|_1\}$ is a subset of $\mathcal{K}(\gamma)$. As a consequence, $\mathcal{K}(\gamma)$ has non-empty interior. □

The following Theorem 4.2 suggests that every computationally proper resilience measure has an equivalent adjustable uncertainty set based representation. Moreover, if we further restrict the family of nondecreasing uncertainty sets $\{\Omega(\gamma)\}_{\gamma\in[0,\bar{\rho}]}$ to be subsets of the support set Ω, we then can construct a computationally proper resilience measure in the reverse way.

Theorem 4.2. *For every computationally proper resilience measure $\rho : \mathcal{V} \longmapsto [0, \bar{\rho}]$, we have*

$$\rho(\tilde{v}) = \begin{cases} 0 & \text{if } \min_{\tilde{\mathbf{z}}\in\Omega} v(\tilde{\mathbf{z}}) > 0 \\ \sup\left\{\gamma : \sup\left\{v(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in \Omega(\gamma)\right\} \leq 0\right\} & \text{otherwise,} \end{cases} \tag{4.3}$$

where

$$\Omega(\gamma) = \left\{ \mathbf{z} : \sup_{\mathbf{v} \geq \mathbf{0}} \left\{ \mathbf{v}'\mathbf{z} : \rho(\mathbf{v}'\tilde{\mathbf{z}} - 1) \geq \gamma \right\} \leq 1 \right\} \tag{4.4}$$

is nondecreasing in γ and convex. Moreover, for every family of nondeceasing sets $\{\Omega(\gamma)\}_{\gamma \in [0,\bar{\rho}]}$ such that $\Omega(\gamma) \subseteq \Omega : \gamma \in [0, \bar{\rho}]$ (here, nondecreasing means $\Omega(\gamma_1) \subseteq \Omega(\gamma_2) : \forall \gamma_1 \leq \gamma_2$), the corresponding mapping $\rho(\cdot)$ defined in (4.3) is a computationally proper resilience measure on $\mathcal{V}$.

Proof. We first show that all computationally proper resilience measure $\rho(\cdot)$ can be expressed as (4.3) with the specified adjustable uncertainty set $\Omega(\gamma)$ defined as (4.4). By Lemma 4.2 we know that the set $\mathcal{K}(\gamma) = \{(\mathbf{v}, t) : \mathbf{v} \geq \mathbf{0}, \rho(\mathbf{v}'\tilde{\mathbf{z}} - t) \geq \gamma\}$ is a regular cone for every $\gamma \in (0, \bar{\rho}]$. Therefore, its dual cone

$$\mathcal{K}^*(\gamma) = \{(\mathbf{z}, s) \,|\, \langle(\mathbf{z}, s), (\mathbf{v}, t)\rangle \geq 0 : \forall (\mathbf{v}, t) \in \mathcal{K}(\gamma)\}$$

is also a regular cone and thus we can make use of the strong conic duality theorem to yield our desired result. Suppose that the penalty position $\tilde{v}$ can be expressed as the piece-wise linear form as $v(\tilde{\mathbf{z}}) = \max_{i \in \mathcal{I}} \{v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}}\}$, where $\mathbf{v}_i \geq \mathbf{0}$. We then have

$$\begin{aligned}
\rho(v(\tilde{\mathbf{z}})) &= \sup\{\gamma : \rho(v(\tilde{\mathbf{z}})) \geq \gamma\} \\
&= \sup\{\gamma : \inf\{t : \rho(v(\tilde{\mathbf{z}}) - t) \geq \gamma\} \leq 0\} \quad \text{left continuity} \\
&= \sup\{\gamma : \{\inf\{t : \rho(\mathbf{v}_i'\tilde{\mathbf{z}} - (t - v_{i0})) \geq \gamma\} \leq 0 : i \in \mathcal{I}\}\} \ \text{computationally proper} \\
&= \sup\{\gamma : \{\inf\{t : (\mathbf{v}_i, t - v_{i0}) \in \mathcal{K}(\gamma)\} \leq 0 : i \in \mathcal{I}\}\} \\
&= \sup\left\{\gamma : \left\{\inf\{t : (\mathbf{0}, 1)t \succeq_{\mathcal{K}(\gamma)} (-\mathbf{v}_i, v_{i0})\} \leq 0 : i \in \mathcal{I}\right\}\right\} \\
&= \sup\{\gamma : \{\sup\{\mathbf{v}_i'\mathbf{z} + s v_{i0} : (-\mathbf{z}, s) \in \mathcal{K}^*(\gamma), s = 1\} \leq 0 : i \in \mathcal{I}\}\} \ \text{strong conic duality} \\
&= \sup\left\{\gamma : \max_{i \in \mathcal{I}} \{\sup\{\mathbf{v}_i'\mathbf{z} + v_{i0} : (-\mathbf{z}, 1) \in \mathcal{K}^*(\gamma)\}\} \leq 0\right\} \\
&= \sup\left\{\gamma : \sup\left\{\max_{i \in \mathcal{I}}\{\mathbf{v}_i'\mathbf{z} + v_{i0}\} : (-\mathbf{z}, 1) \in \mathcal{K}^*(\gamma)\right\} \leq 0\right\} \\
&= \sup\{\gamma : \sup\{v(\tilde{\mathbf{z}}) : (-\mathbf{z}, 1) \in \mathcal{K}^*(\gamma)\} \leq 0\} \\
&= \sup\left\{\gamma : \sup_{\mathbf{z} \in \Omega(\gamma)} \{v(\tilde{\mathbf{z}})\} \leq 0\right\},
\end{aligned}$$

where $\Omega(\gamma)$ is the uncertainty set defined as $\Omega(\gamma) = \{\mathbf{z} : (-\mathbf{z}, 1) \in \mathcal{K}^*(\gamma)\}$.

Hence, it suffices to show that $\Omega(\gamma)$ is exactly the same as that in (4.4), which is verified as follows:

$$
\begin{aligned}
\Omega(\gamma) &= \{\mathbf{z} : (-\mathbf{z}, 1) \in \mathcal{K}^*(\gamma)\} \\
&= \{\mathbf{z} : \{\langle(-\mathbf{z}, 1), (\mathbf{v}, t)\rangle \geq 0 : \forall(\mathbf{v}, t) \in \mathcal{K}(\gamma)\}\} \\
&= \{\mathbf{z} : \{t \geq \mathbf{v}'\mathbf{z} : \forall \rho(\mathbf{v}'\tilde{\mathbf{z}} - t) \geq \gamma, \mathbf{v} \geq \mathbf{0}\}\} \\
&= \{\mathbf{z} : \{1 \geq \mathbf{v}'\mathbf{z} : \forall \rho(\mathbf{v}'\tilde{\mathbf{z}} - 1) \geq \gamma, \mathbf{v} \geq \mathbf{0}\}\} \\
&= \left\{\mathbf{z} : \max_{\mathbf{v} \geq \mathbf{0}} \{\mathbf{v}'\mathbf{z} : \rho(\mathbf{v}'\tilde{\mathbf{z}} - 1) \geq \gamma\} \leq 1\right\}.
\end{aligned}
$$

It is not hard to verify the monotonicity of $\Omega(\gamma) : \gamma \in [0, \bar{\rho}]$. The convexity of $\Omega(\gamma)$ can be verified by making use of the quasiconcavity of $\rho(\cdot)$.

We now show the reverse direction.

1. Satisfaction: Clear.

2. Monotonicity: Suppose $\tilde{v}_1 \leq \tilde{v}_2 : \forall \tilde{\mathbf{z}} \in \Omega$. Let γ_k be an increasing sequence that converge to $\rho(\tilde{v}_2)$. Note that $\Omega(\gamma_k)$ is a subset of Ω, it follows that

$$
\sup\{v_1(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in \Omega(\gamma_k)\} \leq \sup\{v_2(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in \Omega(\gamma_k)\} \leq 0,
$$

which implies $\rho(\tilde{v}_1) \geq \gamma_k : \forall k$. Hence, $\rho(\tilde{v}_1) \geq \rho(\tilde{v}_2)$ by letting the right hand side of the inequality γ_k approach to $\rho(\tilde{v}_2)$.

3. Left continuity: Observe that $\mu(\tilde{v}, \gamma) = \max\{v(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in \Omega(\gamma)\}$ is translation invariant for $\tilde{v}$, we can conclude that $\mu(\tilde{v}, \gamma)$ is semi-nondecreasing and uniformly left continuous. Meanwhile, $\mu(\tilde{v}, \gamma)$ is also nondecreasing in γ. Thus we can verify its left continuity of $\rho(\cdot)$ according to Lemma 4.1.

4. Scale invariance: Clear.

5. Quasi-concavity: Observe that for every $\tilde{v}_1, \tilde{v}_2 \in \mathcal{V}, \theta \in [0,1], \gamma \in [0, \bar{\rho}]$, the following implication holds:

$$\sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \{\theta v_1(\tilde{\mathbf{z}}) + (1-\theta) v_2(\tilde{\mathbf{z}})\} \leq \max\left\{ \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \{v_1(\tilde{\mathbf{z}})\}, \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \{v_2(\tilde{\mathbf{z}})\} \right\}.$$

Based on this, we can verify the quasi-concavity via arguments similar to those in the proof of the monotoncity.

6. Finally, we need to show the computational properness of $\rho(\cdot)$. That is, for every $\tilde{v} = \max_{i \in \mathcal{I}} \tilde{v}_i$,

$$\rho(\tilde{v}) = \min_{i \in \mathcal{I}} \{\rho(\tilde{v}_i)\}. \tag{4.5}$$

Observe that

$$\begin{aligned} \rho(\tilde{v}) &= \sup\left\{ \gamma : \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \left\{ \max_{i \in \mathcal{I}} \{\tilde{v}_i\} \right\} \leq 0 \right\} \\ &= \sup\left\{ \gamma : \max_{i \in \mathcal{I}} \left\{ \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \{\tilde{v}_i\} \right\} \leq 0 \right\} \\ &= \sup\left\{ \gamma : \left\{ \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \{\tilde{v}_i\} \leq 0 : i \in \mathcal{I} \right\} \right\} \\ &\leq \sup\left\{ \gamma : \{\gamma \leq \rho(\tilde{v}_i) : i \in \mathcal{I}\} \right\} \\ &= \min_{i \in \mathcal{I}} \{\rho(\tilde{v}_i)\}. \end{aligned}$$

In addition, let us construct an increasing sequence $\{\gamma_k\}_{k=1,2,\cdots}$ that converges to $\min_{i \in \mathcal{I}} \{\rho(\tilde{v}_i)\}$. Observe that

$$\sup_{\tilde{\mathbf{z}} \in \Omega(\gamma_k)} \{\tilde{v}\} = \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma_k)} \left\{ \max_{i \in \mathcal{I}} \{\tilde{v}_i\} \right\} = \max_{i \in \mathcal{I}} \left\{ \sup_{\tilde{\mathbf{z}} \in \Omega(\gamma_k)} \{\tilde{v}_i\} \right\} \leq 0,$$

we finally have $\rho(\tilde{v}) \geq \min_{i \in \mathcal{I}} \{\rho(\tilde{v}_i)\}$ completing the proof of (4.5). □

From equation (4.3) we can see that, if we regard the case $v(\tilde{\mathbf{z}}) \leq 0$ as acceptable and $v(\tilde{\mathbf{z}}) > 0$ as unacceptable in a target oriented manner, we can express every computationally proper resilience measure as the largest size of adjustable uncertainty set $\Omega(\gamma)$ that contains no unacceptable uncertainty realizations. In addition, the inner part of this equation $\sup\{v(\tilde{\mathbf{z}}) : \tilde{\mathbf{z}} \in \Omega(\gamma)\}$ is indeed a robust optimization representation, which motivates the possibility of leveraging robust optimization techniques for the computation of resilience measure. To do this, we need to obtain an explicit representation of the adjustable uncertainty set $\{\Omega(\gamma)\}_{\gamma\in[0,\bar{\rho}]}$ via (4.4). In fact, we can achieve this by imposing some structural assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the underlying space $\mathcal{V}$. To demonstrate this, we now introduce an important lemma.

Lemma 4.3. *For every convex and closed set $\mathcal{Z} \in \Re^n$, we have the following result:*

$$\left\{\mathbf{z} : \sup_{\mathbf{v}\geq\mathbf{0}} \left\{\mathbf{z}'\mathbf{v} : \max_{\mathbf{z}\in\mathcal{Z}} \mathbf{v}'\mathbf{z} \leq 1\right\} \leq 1\right\} = \left\{\mathbf{z} : \exists \check{\mathbf{z}} \in \check{\mathcal{Z}}, \mathbf{z} \leq \check{\mathbf{z}}\right\},$$

where $\check{\mathcal{Z}} = \mathcal{CH}\{\mathcal{Z} \cup \{\mathbf{0}\}\}$.

Proof. Denote by $\mathcal{Z}_L$ and $\mathcal{Z}_R$ as the set on the left side and right side, respectively. Observe that for every $\mathbf{v} \geq \mathbf{0}$, we have

$$\begin{aligned}\max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} &\leq \max_{\mathbf{z}\in\check{\mathcal{Z}}} \mathbf{v}'\mathbf{z} \\ &= \max_{\mathbf{z}\in\mathcal{Z}} \max_{\lambda\in[0,1]} \mathbf{v}'(\lambda\mathbf{z} + (1-\lambda)\mathbf{0}) \\ &\leq \max_{\mathbf{z}\in\mathcal{Z}} \mathbf{v}'\mathbf{z}.\end{aligned}$$

In addition, we also have

$$\max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} \geq \max_{\mathbf{z}\in\mathcal{Z}} \mathbf{v}'\mathbf{z}$$

because $\mathcal{Z} \subseteq \check{\mathcal{Z}} \subseteq \mathcal{Z}_R$. Hence, $\max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} = \max_{\mathbf{z}\in\mathcal{Z}} \mathbf{v}'\mathbf{z} : \forall \mathbf{v} \geq \mathbf{0}$. Consequently, we deduce that

$$\mathcal{Z}_L = \left\{\mathbf{z} : \sup_{\mathbf{v}} \left\{\mathbf{z}'\mathbf{v} : \max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} \leq 1\right\} \leq 1\right\},$$

which implies that $\mathcal{Z}_R \subseteq \mathcal{Z}_L$.

To prove the reverse side $\mathcal{Z}_L \subseteq \mathcal{Z}_R$, we assume by contradiction that $\exists \mathbf{z}^* \in \mathcal{Z}_L$ such that $\mathbf{z}^* \notin \mathcal{Z}_R$. Because $\mathcal{Z}_R$ is closed, it follows from the separating hyperplane theorem that

$$\exists \mathbf{v}^* \in \Re^n \text{such that } \mathbf{z}^{*\prime}\mathbf{v}^* > \max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}^{*\prime}\mathbf{z}.$$

Hence, $\mathbf{v}^* \geq \mathbf{0}$ (Otherwise the right hand side is infinity). Because $\mathbf{0} \in \mathcal{Z}_R$, certainly one of the following two cases hold.

a) $\max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}^{*\prime}\mathbf{z} = 0$. Because $\mathbf{z}^{*\prime}\mathbf{v}^* > 0$ and $\max_{\mathbf{z}\in\mathcal{Z}_R} (\theta\mathbf{v}^*)'\mathbf{z} = 0 : \forall\theta \geq 0$, we have

$$\sup_{\mathbf{v}} \left\{ \mathbf{z}^{*\prime}\mathbf{v} : \max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} \leq 1 \right\} \geq \sup_{\theta\geq 0} \left\{ (\theta\mathbf{z}^*)'\mathbf{v}^* \right\} = +\infty$$

indicating a contradiction that $\mathbf{z}^* \notin \mathcal{Z}_L$.

b) $\max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}^{*\prime}\mathbf{z} > 0$. There exists $\theta > 0$ such that $\max_{\mathbf{z}\in\mathcal{Z}_R} (\theta\mathbf{v}^*)'\mathbf{z} = 1$. Thus,

$$\sup_{\mathbf{v}} \left\{ \mathbf{z}^{*\prime}\mathbf{v} : \max_{\mathbf{z}\in\mathcal{Z}_R} \mathbf{v}'\mathbf{z} \leq 1 \right\} \geq \mathbf{z}^{*\prime}(\theta\mathbf{v}^*) > \max_{\mathbf{z}\in\mathcal{Z}_R} (\theta\mathbf{v}^*)'\mathbf{z} = 1,$$

which indicates a contradiction that $\mathbf{z}^* \notin \mathcal{Z}_L$. □

Proposition 4.1. *Suppose that random vector $\tilde{\mathbf{z}}$ follows a discrete distribution on sample space $\Omega = \{\mathbf{z}_i\}_{i=1,\cdots,S}$ with uncertain probability $\mathbb{P}(\tilde{\mathbf{z}} = \mathbf{z}_i) = \alpha_i$ which is only known to reside in a convex closed set $\mathbb{F}_\alpha$. In addition, let the penalty position space reduce to the set of affine functions, i.e., $\mathcal{V} = \{\tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}} : \mathbf{v} \geq \mathbf{0}, v_0 \in \Re\}$. Then we can express the distributionally-ambiguous shortfall awareness measure $\rho_{\mathrm{DSAM}}(\tilde{v})$ defined on $\mathcal{V}$ in the form of (4.3) with adjustable uncertainty sets*

$$\Omega(\gamma) = \left\{ \mathbf{z} : \begin{array}{l} \exists \mathbf{u} \in \Re^S, (\boldsymbol{\alpha}, \lambda) \in \Pi_\alpha \\ \mathbf{z} \leq \sum_{i=1}^S u_i \mathbf{z}_i \\ \sum_{i=1}^S u_i = \lambda \\ \mathbf{0} \leq \mathbf{u} \leq \frac{1}{1-\gamma}\boldsymbol{\alpha} \end{array} \right\},$$

where

$$\Pi_\alpha = \{(\boldsymbol{\alpha}, \lambda) : \boldsymbol{\alpha}/\lambda \in \mathbb{F}_\alpha, 0 < \lambda \leq 1\} \cup \{(\boldsymbol{\alpha}, 0) : \boldsymbol{\alpha} \in \mathbb{F}_\alpha\}.$$

Similarly, the distributionally-ambiguous one side moment based measure $\rho_{\rm DOSM}(\tilde{v})$ *defined on* $\mathcal{V}$ *can also be represented as the form of (4.3) with adjustable uncertainty sets*

$$\Omega(\gamma) = \left\{ \mathbf{z} : \begin{array}{c} \exists \hat{\mathbf{z}} \in \Re^n, \mathbf{u} \in \Re^S_+, \boldsymbol{\alpha} \in \mathbb{F}_\alpha, \lambda \in [0,1] \\ \mathbf{z} \leq \left(\lambda - \gamma \sum_{i=1}^S u_i\right) \hat{\mathbf{z}} + \gamma \sum\limits_{i=1}^S u_i \mathbf{z}_i \\ \hat{\mathbf{z}} = \sum\limits_{i=1}^S \alpha_i \mathbf{z}_i, \|\boldsymbol{\alpha}^{-1/p} \circ \mathbf{u}\|_q \leq \lambda \end{array} \right\},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Because $\rho_{\rm DSAM}(\cdot)$ and $\rho_{\rm DOSM}(\cdot)$ satisfy all the properties except the computational properness, representation (4.3) in Theorem 4.2 also holds if we restrict the space of penalty positions $\mathcal{V} = \{\tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}} : \mathbf{v} \geq \mathbf{0}, v_0 \in \Re\}$.

1. Observe that

$$\begin{array}{l} \rho_{\rm DSAM}(\mathbf{v}'\tilde{\mathbf{z}} - 1) \geq \gamma \\ \Leftrightarrow \max\limits_{\boldsymbol{\alpha}} \min\limits_{\nu} \left\{ \nu + \dfrac{1}{1-\gamma} \sum\limits_{i=1}^S \alpha_i y_i : y_i + \nu \geq \mathbf{v}'\mathbf{z}_i, y_i \geq 0 \right\} \leq 1 \\ \Leftrightarrow \max\limits_{\boldsymbol{\alpha}, \mathbf{u}} \left\{ \mathbf{v}'(\sum\limits_{i=1}^S u_i \mathbf{z}_i) : \sum\limits_{i=1}^S u_i = 1, \mathbf{0} \leq \mathbf{u} \leq \frac{1}{1-\gamma}\boldsymbol{\alpha}, \boldsymbol{\alpha} \in \mathbb{F}_\alpha \right\} \leq 1. \end{array}$$

Let us define a set

$$\mathcal{Z} = \{\sum_{i=1}^S u_i \mathbf{z}_i : \sum_{i=1}^S u_i = 1, \mathbf{0} \leq \mathbf{u} \leq \frac{1}{1-\gamma}\boldsymbol{\alpha}, \boldsymbol{\alpha} \in \mathbb{F}_\alpha\}.$$

By the definition of convex hull, we can deduce that for every element $\mathbf{z} \in \mathcal{CH}\{\mathcal{Z} \cup \{\mathbf{0}\}\}$, $\exists \lambda \in [0,1], \check{\mathbf{z}} \in \mathcal{Z}$ such that

$$\mathbf{z} = \lambda \check{\mathbf{z}} + (1-\lambda)\mathbf{0}.$$

Therefore, if $\lambda > 0$, we surely can find $\mathbf{u} \in \Re^S_+$ such that

$$\mathbf{z} = \sum_{i=1}^S u_i \mathbf{z}_i, \sum_{i=1}^S u_i = \lambda, \mathbf{0} \leq \mathbf{u} \leq \tfrac{1}{1-\gamma}\boldsymbol{\alpha}, \boldsymbol{\alpha}/\lambda \in \mathbb{F}_\alpha.$$

Consequently, we have

$$\mathcal{CH}\{\mathcal{Z}\cup\{\mathbf{0}\}\} = \left\{ \mathbf{z} : \begin{array}{l} \exists \mathbf{u}\in\Re^S, (\boldsymbol{\alpha},\lambda)\in\Pi_\alpha \\ \mathbf{z} = \sum_{i=1}^S u_i\mathbf{z}_i \\ \sum_{i=1}^S u_i = \lambda \\ \mathbf{0}\le\mathbf{u}\le\frac{1}{1-\gamma}\boldsymbol{\alpha} \end{array} \right\}.$$

The result for the distributionally-ambiguous shortfall awareness measure follows directly from (4.4) by applying Lemma 4.3 for $\mathcal{Z}$.

2. For fixed $\boldsymbol{\alpha}$, let $\hat{\mathbf{z}} = \sum_{i=1}^{S} \alpha_i\mathbf{z}_i$, we have

$$\begin{aligned}
&\rho_{\rm OSM}(\mathbf{v}'\tilde{\mathbf{z}}-1)\ge\gamma\\
\Leftrightarrow\ &\mu_{\rm OSM}(\mathbf{v}'\tilde{\mathbf{z}}-1,\gamma)\le 0\\
\Leftrightarrow\ &\min_{\mathbf{y}\ge\mathbf{0}}\{\mathbf{v}'\hat{\mathbf{z}}+\gamma\|\boldsymbol{\alpha}^{1/p}\circ\mathbf{y}\|_p : y_i\ge\mathbf{v}'(\hat{\mathbf{z}}-\mathbf{z}_i)\}\le 1\\
\Leftrightarrow\ &\min_{\mathbf{y}\ge\mathbf{0}}\left\{\max_{\boldsymbol{\delta}\ge\mathbf{0},\|\boldsymbol{\delta}\|_q\le 1}\left\{\mathbf{v}'\hat{\mathbf{z}}+\gamma(\boldsymbol{\alpha}^{1/p}\circ\boldsymbol{\delta})'\mathbf{y}\right\} : y_i\ge\mathbf{v}'(\hat{\mathbf{z}}-\mathbf{z}_i)\right\}\le 1\\
\Leftrightarrow\ &\max_{\boldsymbol{\delta}\ge\mathbf{0},\|\boldsymbol{\delta}\|_q\le 1}\left\{\min_{\mathbf{y}\ge\mathbf{0}}\left\{\mathbf{v}'\hat{\mathbf{z}}+\gamma(\boldsymbol{\alpha}^{1/p}\circ\boldsymbol{\delta})'\mathbf{y} : y_i\ge\mathbf{v}'(\hat{\mathbf{z}}-\mathbf{z}_i)\right\}\right\}\le 1\\
\Leftrightarrow\ &\max_{\boldsymbol{\delta}\ge\mathbf{0},\|\boldsymbol{\delta}\|_q\le 1}\ \max_{\mathbf{u}\ge\mathbf{0}}\left\{\mathbf{v}'\hat{\mathbf{z}}+\gamma\textstyle\sum_{i=1}^S u_i(\mathbf{v}'\mathbf{z}_i-\mathbf{v}'\hat{\mathbf{z}})\right\}\le 1 \quad \text{(LP duality)}\\
\Leftrightarrow\ &\max\left\{\mathbf{v}'\left(\left(1-\gamma\textstyle\sum_{i=1}^S u_i\right)\hat{\mathbf{z}}+\gamma\textstyle\sum_{i=1}^S u_i\mathbf{z}_i\right) : \mathbf{u}\ge\mathbf{0}, \|\boldsymbol{\alpha}^{-1/p}\mathbf{u}\|_q\le 1\right\}\le 1,
\end{aligned}$$

where the exchange of “min” and “max” follows from Sion’s minimax theorem (Sion, 1957). The result is similar for the distributionally-ambiguous one side moment measure $\rho_{\rm DOSM}(\cdot)$ by letting the probability vector $\boldsymbol{\alpha}$ be uncertain. According to the equation of the budget uncertainty sets (4.4), the result follows directly by letting the set $\mathcal{Z}$ in Lemma 4.3 be

$$\mathcal{Z} = \left\{\left(1-\gamma\sum_{i=1}^S u_i\right)\hat{\mathbf{z}}+\gamma\sum_{i=1}^S u_i\mathbf{z}_i : \hat{\mathbf{z}} = \sum_{i=1}^S\alpha_i\mathbf{z}_i, \mathbf{u}\ge\mathbf{0}, \|\boldsymbol{\alpha}^{-1/p}\mathbf{u}\|_q\le 1, \boldsymbol{\alpha}\in\mathbb{F}_\alpha\right\}.$$

□

It seems from the above result that we can construct the corresponding adjustable uncertainty sets for every specific resilience measure and use these uncertainty sets for the computation and optimization of the original resilience measure. Unfortunately, this approach seems a mirage due to various reasons. First of all, the specification of these adjustable uncertainty sets depends explicitly on the structure of the probability space. Thus, we can not say much of the resulting uncertainty sets without any specific assumptions on it. Even if the assumptions on the probability space are acceptable, two technical difficulties remain. One issue arises from the fact that the evaluation of these measures for the piece-wise linear penalty positions in $\mathcal{V}$ in (3.1) is generally difficult. To achieve this, one typical approach is to approximate them by linear positions (e.g. affine decision rule approximations) but the quality of the approximation is unknown. Besides, even if we restrict our attention to linear penalty positions $\mathcal{V} = \{\tilde{v} = v_0 + \mathbf{v}'\tilde{\mathbf{z}} : \mathbf{v} \geq \mathbf{0}, v_0 \in \Re\}$ only, the computational difficulty remains due to the complex structure of the resulting uncertainty sets given in Proposition 4.1.

An exciting news is that the opposite direction of Theorem 4.2 suggests us a new way of constructing new resilience measures by specifying adjustable uncertainty sets. Even though the proposed approach requires $\Omega(\gamma)$ to be subset of the support Ω, this restriction would not lose much generality from the intuition that we do not need to protect against realizations of uncertainty $\tilde{\mathbf{z}}$ that do not belong to the support Ω. To make the evaluation of the constructed resilience measure tractable, a nature choice is to use the intersection of a closed convex body $\Upsilon(\gamma)$ and the support space Ω. Mathematically,

$$\Omega(\gamma) = \underbrace{\{\mathbf{z} : \|\mathbf{z} - \mathbf{c}\|_\ell \leq \kappa(\gamma)\}}_{\Upsilon(\gamma)} \cap \Omega, \tag{4.6}$$

where $\|\cdot\|_\ell$, indexed by ℓ, is a general norm defined in $\Re^n$ such that $\Upsilon(\gamma)$ is a compact set, $\kappa(\cdot)$ is a nonnegative valued increasing function on $[0, \bar{\rho}]$ and $\mathbf{c}$ is the central point of $\Upsilon(\gamma)$. For technical reasons, we further assume that the cone

$$\Pi \triangleq \text{cl}\{(\mathbf{z}, t) : t \geq \|\mathbf{z}\|_\ell\} \tag{4.7}$$

is a regular cone which is closed, convex, pointed with a non-empty interior. The notion of relating popular uncertainty sets to general norms here is much in the spirit of Bertsimas et al. (2004); Bertsimas and Sim (2006); Chen et al. (2007). Here we give examples of some commonly used norms along with two fundamental principles of constructing new norms based on current ones, by which we can express most uncertainty sets in the literature as the form of $\Upsilon(\gamma)$ in (4.6):

1. The most commonly used norm is the polynomial norm $\|\mathbf{z}\|_p : p \in \Re_+ \cup \{+\infty\}$.

2. An invertible linear mapping $\mathbf{Q}$ can define a new norm from the current one as $\|\mathbf{z}\|_{\ell'} := \|\mathbf{Q}\mathbf{z}\|_\ell$.

- The box uncertainty set $\underline{\mathbf{z}} \leq \mathbf{z} \leq \bar{\mathbf{z}}$ (e.g. Ang et al., 2012; Sy et al., 2012). Let $\mathbf{Q} := 2\kappa(\gamma)(\text{diag}(\bar{\mathbf{z}} - \underline{\mathbf{z}}))^{-1}, \mathbf{c} = \underline{\mathbf{z}} + \bar{\mathbf{z}}$, then the norm based representation $\|\mathbf{z}\|_{Q-\infty} := \|\mathbf{Q}\mathbf{z}\|_\infty$ gives the box uncertainty set in the form of $\Upsilon(\gamma)$.

- The ellipsoid uncertainty set $\|\mathbf{Q}(\mathbf{z} - \mathbf{c})\|_2 \leq 1$. Let the ellipsoidal norm $\|\mathbf{z}\|_{Q-2} := \kappa(\gamma)\|\mathbf{Q}\mathbf{z}\|_2$. Then it can be expressed as the form of $\Upsilon(\gamma)$.

3. The intersection of multiple given norms $\|\mathbf{z}\|_{\cap_i \ell_i} = \max_i\{\|\mathbf{z}\|_{\ell_i}\}$ defines a new norm.

- In Ben-Tal and Nemirovski (1999), they consider the intersection of multiple ellipsoidal norms $\|\mathbf{z}\|_\ell := \max_i\{\|\mathbf{z}\|_{Q_i-2}\}$ to model the uncertainty set of the linear programming parameters.

- In Ben-Tal and Nemirovski (2000) they consider the intersection of $\|\cdot\|_2$ and $\|\cdot\|_\infty$ as the generalized norm $\|\mathbf{z}\|_\ell := \max\{\|\mathbf{z}\|_2, \Gamma\|\mathbf{z}\|_\infty\}$.

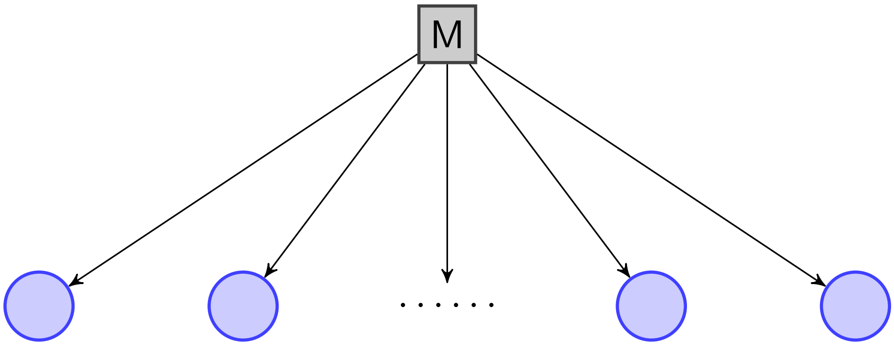


Figure 4.1: An example of a supply system

- In Bertsimas and Sim (2003) and Bertsimas and Sim (2004) they consider the intersection of $\|\cdot\|_1$ and $\|\cdot\|_\infty$ as the generalized norm $\|\mathbf{z}\|_\ell := \max\{\frac{1}{\Gamma}\|\mathbf{z}\|_1, \|\mathbf{z}\|_\infty\}$ to bound the uncertain parameters.

By choosing the norm based uncertainty set (4.6), we can quickly calculate the resilience measure of some simple supply systems. Let's consider a simplified one period inventory rationing system, which consists of a manufacturer and n customers depicted in Figure 4.1. In this system, the manufacturer has to fulfill its customers as much as possible with limited stock s. When the stock amount is not enough to satisfy all its customers, a rationing solution is necessary. Specifically, if we denote d_i as the demand amount of customer i and τ_i as the corresponding shortage cost. Without loss of generality, we assume that $\tau_1 \geq \tau_2, \cdots, \geq \tau_n$. Then, the linear programming model of the optimal rationing policy minimizing total penalties is:

$$
\begin{aligned}
\min \quad & \boldsymbol{\tau}'\mathbf{e} \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq s \\
& x_i + e_i \geq d_i : i = 1, \cdots, n \\
& \mathbf{x} \geq \mathbf{0} \\
& \mathbf{e} \geq \mathbf{0}.
\end{aligned}
$$

Let the total shortage cost in the above LP be the penalty function we considered. Suppose that uncertainty in this inventory rationing system only affects the stock level by $s = b_0 - \mathbf{b}_1'\tilde{\mathbf{z}}$. If we measure the system resilience $\varrho(\mathcal{G})$ by the resilience measure defined by adjustable uncertain sets (4.6) with $\Omega = \Re^n$, the supply system resilience is then given by:

$$\varrho(\mathcal{G}) = \kappa^{-1}\left(\frac{b_0 + \wp(\tau_0) - \mathbf{b}_1'\mathbf{c} - \sum_{i=1}^{n} d_i}{\|\mathbf{b}_1\|_{\ell^*}}\right),$$

where $\kappa^{-1}(\cdot)$ is the inverse function of $\kappa(\cdot)$, $\mathbf{c}$ is the central point of $\Upsilon(\gamma)$ in (4.6), τ_0 is the tolerance level of penalty,

$$\|\mathbf{h}\|_{\ell^*} \triangleq \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \mathbf{h}'\mathbf{z} : \forall \mathbf{h} \in \Re^n,$$

and

$$\wp(\tau_0) = \begin{cases} \tau_0/\tau_1 & \text{if } \tau_0 \le \tau_1 d_1 \\ \sum_{i=1}^{j-1} d_i + (\tau_0 - \sum_{i=1}^{j-1} \tau_i d_i)/d_j & \text{if } \sum_{i=1}^{j-1} \tau_i d_i < \tau_0 \le \sum_{i=1}^{j} \tau_i d_i : j = 2, \cdots, n \\ \sum_{i=1}^{n} d_i & \text{Otherwise.} \end{cases}$$

Chapter 5

Energy Supply System Resilience under Supply Disruptions

The sharp increase in energy consumption has motivated the surge in energy supply security and resilience against supply disruptions, which usually result from some unexpected crises such as natural disasters, geopolitical crises and terrorist, etc. To my knowledge, most proposed methods on energy system resilience remain at a fairly qualitative level based on intuition and hindsight developed from past case studies or scenario based analysis. There is thus strong impetus to develop more quantitative and rigorous modeling of energy supply resilience concepts suitable for effective decision support. In this chapter we apply the previously proposed resilience measure based approach to investigate energy supply system resilience under supply disruptions. In particular, Section 5.1 gives the definition of resilience index by relating energy supply system resilience to adjustable uncertainty sets based resilience measure of the uncertain penalty function. Section 5.2 discusses the computation of the proposed resilience index, and develop efficient algorithms for evaluating resilience index defined by various types of adjustable uncertainty sets. In Section 5.3, we investigate the design problem by optimizing the resilience index with limited investment budget. In Section 5.4, we present numerical experiments on natural gas supply system. Section 5.5 presents some concluding remarks.

5.1 Modeling Energy Supply System Resilience

We consider an energy supply system of interest that can be modeled as a generalized linear network with n nodes and m arcs. Let $\mathbf{A}_s \in \Re^{n_1 \times m}$, $\mathbf{A}_d \in \Re^{n_2 \times m}$ be the arc-incidence matrices describing the supply network structure, $\mathbf{s} \in \Re^{n_1}$ be the supply levels on n_1 supply nodes, $\mathbf{d} \in \Re^{n_2}$ be the set of demand levels on n_2 demand nodes, and $\mathbf{x} \in \Re^m$ be the flow levels in the supply network. The following linear programming (LP) model of the energy supply problem is of interest in this chapter:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{e}} \quad & \boldsymbol{\tau}'\mathbf{e} \\ \text{s.t.} \quad & \mathbf{A}_s\mathbf{x} + \mathbf{w} = \mathbf{s} \\ & \mathbf{A}_d\mathbf{x} + \mathbf{e} = \mathbf{d} \\ & \mathbf{G}\mathbf{x} \geq \mathbf{g} \\ & \mathbf{x}, \mathbf{e}, \mathbf{w} \geq \mathbf{0}, \end{aligned} \tag{5.1}$$

where $\mathbf{w}$ and $\mathbf{e}$ are defined as supply surplus and unfulfilled demands respectively, and $\boldsymbol{\tau}'\mathbf{e}$ is the penalty cost associated with unfulfilled $\mathbf{e}$. In addition, parameters $\mathbf{G}$ and $\mathbf{g}$ are used to model other system constraints (e.g., flow balance at transfer nodes, flow capacities restrictions, gas pipeline pressure constraints) depending on the specific application. Hence, given a set of supply $\mathbf{s} \in \Re^{n_1}$ and demands $\mathbf{d} \in \Re^{n_2}$, model (5.1) is solved to achieve a load-flow plan that minimizes total penalty cost on demand shortages. From system resilience point of view, the solution in the above can also be seen as a mitigating response to minimize the impact of supply disruptions in $\mathbf{s} \in \Re^{n_1}$. Model (5.1) can also be extended by embedding multiple time periods or multiple commodities, etc., using simple formulation techniques applied to classical network flow models. We refer the readers to Ahuja et al. (1993) for a comprehensive description of these extensions. The focus of our work in this chapter are problems that can be reasonably modeled in the format in (5.1), since such models are the basis of many existing legacy decision-support tools for supply planning problems.

Remark 5.1. In this chapter we only consider energy supply systems in which the routing operations can be formulated or approximately solved by LP models of the form (5.1). This restriction originates from the difficulty of evaluating the subsequently proposed resilience index. Indeed, the extension of the framework of resilience measure from this LP model to more general and complicated optimization models (e.g., nonlinear or even nonconvex problems) is not clear and this might be one of the future research directions.

For simplicity of representation, we re-write the network flow model (5.1) as

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \boldsymbol{\tau}'\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{G}\mathbf{x} \geq \mathbf{g} \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0} \end{aligned} \tag{5.2}$$

with $\mathbf{y} = (\mathbf{w}; \mathbf{e}), \mathbf{b} = (\mathbf{s}; \mathbf{d})$ (here $\boldsymbol{\tau} = (\mathbf{0}, \boldsymbol{\tau})$, we repeat this notation for notational convenience).

Energy supply disruptions can arise from events such as extreme weather conditions, over extraction, failure of critical infrastructure, or political situations, etc., which are often difficult to predict in advance. In reality, the precise probability specifications of their occurrences are also often not available (or not even appropriate). Here we model the uncertain supply disruption levels using a set of primitive uncertain variables $\tilde{\mathbf{z}}$, which are ambiguous random variables defined on measure space $(\Omega, \mathcal{F}, \mathbb{P})$ as described in Chapter 3. Since energy supply disruptions rarely happen, it is nearly impossible for us to obtain any distributional information on it and thus we assume that $\mathbb{P}$ is totally unknown. In addition, we assume that supply disruptions can only affect supply-demand parameters $\mathbf{b}$ (denoted as $\mathbf{b}(\tilde{\mathbf{z}})$) in model (5.2). Without loss of generality, we further make the following assumption:

Assumption 5.1. *For all $\tilde{\mathbf{z}} \in \Omega$, the corresponding network flow model (5.2) is feasible and bounded.*

The supply losses at various supply nodes may also be correlated. To address this, define a *load matrix* $\mathbf{B}$ such that we can express the uncertain supply-demand vector as $\mathbf{b}(\tilde{\mathbf{z}}) = \overline{\mathbf{b}} - \mathbf{B}\tilde{\mathbf{z}}$, where $\overline{\mathbf{b}}$ is the supply-demand vector under nominal condition (no disruption occurs). Without loss of generality, we normalize the support set Ω as $[0,1]^n$, where n is the dimension of $\tilde{\mathbf{z}}$. Because supply disruptions are undesirable events that tend to reduce the supply levels, we further assume that the elements in the first n_1 rows of $\mathbf{B}$ are nonnegative. The demand vector $\mathbf{d}$ corresponds to the threshold of minimum required demand quantity, which is assumed to be deterministic throughout this chapter. Therefore, the elements in row n_1 to row n_1+n_2 are all zero. We denote the energy supply system of interest as $\mathcal{G}$, distinguished by its system model parameters $(\mathbf{A}, \mathbf{G}, \mathbf{B}, \overline{\mathbf{b}}, \mathbf{g})$. The demand loss penalty function $\phi_{\mathcal{G}}(\tilde{\mathbf{z}})$ under supply scenario $\tilde{\mathbf{z}}$ is then given as

$$\begin{aligned} \phi_{\mathcal{G}}(\tilde{\mathbf{z}}) = \min_{\mathbf{x},\mathbf{y}} \quad & \boldsymbol{\tau}'\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \overline{\mathbf{b}} - \mathbf{B}\tilde{\mathbf{z}} \\ & \mathbf{G}\mathbf{x} \geq \mathbf{g} \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{5.3}$$

By Assumption 5.1, (5.3) is bounded and feasible for all $\tilde{\mathbf{z}} \in \Omega$. Then strong duality holds indicating that we can compute $\phi(\tilde{\mathbf{z}})$ exactly by the following dual problem:

$$\begin{aligned} \phi(\tilde{\mathbf{z}}) = \max_{\boldsymbol{\pi},\boldsymbol{\mu}} \quad & \overline{\mathbf{b}}'\boldsymbol{\pi} - \boldsymbol{\pi}'\mathbf{B}\tilde{\mathbf{z}} + \mathbf{g}'\boldsymbol{\mu} \\ \text{s.t.} \quad & \mathbf{A}'\boldsymbol{\pi} + \mathbf{G}'\boldsymbol{\mu} \leq \mathbf{0} \\ & \boldsymbol{\pi} \leq \boldsymbol{\tau} \\ & \boldsymbol{\mu} \geq \mathbf{0}. \end{aligned} \tag{5.4}$$

Observe that the optimality of (5.4) can be reached at one extreme point of its the feasible region. Let us define $\mathcal{Y}$ as the feasible region of (5.4) and the corresponding set of extreme points as $\bar{\mathcal{Y}} = \{(\boldsymbol{\pi}_i, \boldsymbol{\mu}_i)\}_{i=1,\cdots,I}$. Then we can deduce that

$$\phi(\tilde{\mathbf{z}}) = \max_{i=1,\cdots,I} \left\{ \overline{\mathbf{b}}'\boldsymbol{\pi}_i + \mathbf{g}'\boldsymbol{\mu}_i - (\mathbf{B}'\boldsymbol{\pi}_i)'\tilde{\mathbf{z}} \right\},$$

and hence, $\phi(\tilde{\mathbf{z}})$ is a piece-wise linear convex function in $\tilde{\mathbf{z}}$. In other words, we can express it as $\phi(\tilde{\mathbf{z}}) = \max_{i\in\mathcal{I}} \{v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}}\}$, by denoting $v_{i0} = \overline{\mathbf{b}}'\boldsymbol{\pi}_i + \mathbf{g}'\boldsymbol{\mu}_i, \mathbf{v}_i = -\mathbf{B}'\boldsymbol{\pi}_i, \mathcal{I} = \{1, \cdots, I\}$. Note that the first n_1 elements of $\boldsymbol{\tau}$ equal to zero. Hence we have $\mathbf{v}_i = -\mathbf{B}'\boldsymbol{\pi}_i \geq -\mathbf{B}'\boldsymbol{\tau} = \mathbf{0}$ implying that $\phi(\tilde{\mathbf{z}})$ is exactly an element of the specific penalty position space $\mathcal{V}$ defined in (3.1). Therefore, we can measure the resilience of energy supply system $\varrho(\mathcal{G})$ by a resilience measure $\rho(\cdot)$ defined on penalty position space $\mathcal{V}$ in (3.1). Specifically, if we define $\rho(\cdot)$ as (4.3) with adjustable uncertainty sets $\{\Omega(\gamma)\}_{\gamma\in[0,\bar{\rho}]}$, we give the following definition of resilience index.

Definition 5.1. Suppose that we have a family of convex and closed adjustable uncertainty sets $\{\Omega(\gamma) \subseteq \Omega\}_{\gamma\in[0,\bar{\rho}]} : \bar{\rho} \in (0, +\infty)$, which is nondecreasing in γ. Then the following mapping $\varrho : \mathcal{G} \longmapsto [0, \bar{\rho}]$, defines a resilience index of energy supply system $\mathcal{G}$:

$$\varrho(\mathcal{G}) = \begin{cases} 0 & \text{if } \min\limits_{\tilde{\mathbf{z}}} \phi_{\mathcal{G}}(\tilde{\mathbf{z}}) > \tau_0 \\ \sup\{\gamma : \psi(\gamma) \leq \tau_0\} & \text{otherwise,} \end{cases} \tag{5.5}$$

where

$$\psi(\gamma) = \sup_{\tilde{\mathbf{z}}\in\Omega(\gamma)} \phi_{\mathcal{G}}(\tilde{\mathbf{z}}).$$

5.2 Evaluating Resilience Index

In the following subsections we consider the computation of $\varrho(\mathcal{G})$ defined by different families of adjustable uncertainty sets $\{\Omega(\gamma)\}_{\gamma\in[0,\bar{\rho}]}$. Specifically, Subsection 5.2.1 considers general norm based uncertainty sets (4.6). In Subsection 5.2.2 we investigate two specific families of adjustable uncertainty sets: the box uncertainty set and the cardinality uncertainty set.

5.2.1 General Method by Extreme Points Enumeration

Note that $\varrho(\mathcal{G}) = \sup\{\gamma : \psi(\gamma) \leq 0\}$ and $\psi(\gamma)$ is non-decreasing in γ, we can evaluate the resilience index $\varrho(\mathcal{G})$ using a bisection procedure on γ within ϵ accuracy, provided that an effective routine of asserting whether $\psi(\gamma) \leq 0$ or not is available. Thus, the evaluation of $\psi(\gamma)$, which corresponds to the maximum (worst-case) penalty function value realization over all supply realizations $\tilde{\mathbf{z}} \in \Omega(\gamma)$, is of key importance. It can be observed from the dual representation of $\phi(\tilde{\mathbf{z}})$ in (5.4) that

$$\begin{aligned} \psi(\gamma) = -\tau_0 + \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \max_{\boldsymbol{\pi}, \boldsymbol{\mu}} \quad & \overline{\mathbf{b}}' \boldsymbol{\pi} - \boldsymbol{\pi}' \mathbf{B} \tilde{\mathbf{z}} + \mathbf{g}' \boldsymbol{\mu} \\ \text{s.t.} \quad & \mathbf{A}' \boldsymbol{\pi} + \mathbf{G}' \boldsymbol{\mu} \leq \mathbf{0} \\ & \boldsymbol{\pi} \leq \boldsymbol{\tau} \\ & \boldsymbol{\mu} \geq \mathbf{0}. \end{aligned} \tag{5.6}$$

When $\mathbf{0} \in \text{int}\{\Upsilon(\gamma)\}$, the dual norm $\|\cdot\|_{\ell^*}$ defined as

$$\|\mathbf{h}\|_{\ell^*} \triangleq \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \mathbf{h}'\mathbf{z} : \forall \mathbf{h} \in \Re^n$$

is a norm and the worst case penalty function $\psi(\gamma)$ can be expressed as:

$$\psi(\gamma) = \max \left\{ \overline{\mathbf{b}}' \boldsymbol{\pi} + \mathbf{g}' \boldsymbol{\mu} + \| -\mathbf{B}' \boldsymbol{\pi} \|_{\ell^*} : (\boldsymbol{\pi}, \boldsymbol{\mu}) \in \mathcal{Y} \right\}.$$

It is well-known that the general norm maximization over a bounded polyhedron is NP-Hard (see Bodlaender et al., 1990; Mangasarian and Shiau, 1986). Thus, we believe that the general problem of computing $\psi(\gamma)$ is also hard because a norm type objective function is involved. Note that the dual problem (5.4) is bounded and feasible for all $\tilde{\mathbf{z}} \in \Omega$ (by Assumption 5.1), which indicates that, for every $\tilde{\mathbf{z}} \in \Omega$, there exists an extreme point reaching optimality of (5.4). Hence, in the case that all the extreme points are explicitly known to us (Assumption 5.2 below), the problem can be slightly simplified.

Assumption 5.2. *The extreme points set* $\overline{\mathcal{Y}} = \{(\boldsymbol{\pi}_1, \boldsymbol{\mu}_1), \cdots, (\boldsymbol{\pi}_I, \boldsymbol{\mu}_I)\}$ *is explicitly known.*

Remark 5.2. Although the problem of generating all extreme points of a polyhedron is also NP-Hard (see Dyer, 1983; Khachiyan et al., 2008), many practical algorithms can efficiently solve it. Among those are pivot-based algorithm in Avis and Fukuda (1992), improved reverse search algorithm in Avis (2000), primal-dual methods in Bremner et al. (1998) and others. Therefore, the abovementioned assumption is reasonable if we can leverage on the existing techniques to enumerate all the extreme points.

The next result provides a conic optimization reformulation of the sub-problem $\{\psi(\gamma) \leq 0\}$ as follows:

Theorem 5.1. *Under Assumption 5.2, the sub-problem of asserting* $\{\psi(\gamma) \leq 0\}$ *or not is equivalent to the following feasibility problem:*

$$\left\{ \begin{array}{ll} \exists \mathbf{p}_i, \mathbf{q}_i \in \Re^n, t_i \in \Re \text{ such that} & \\ \overline{\mathbf{b}}'\boldsymbol{\pi}_i + \mathbf{g}'\boldsymbol{\mu}_i + \mathbf{c}'\mathbf{p}_i + \mathbf{1}'\mathbf{q}_i + \kappa(\gamma)t_i \leq \tau_0 & \\ \mathbf{p}_i + \mathbf{q}_i + \mathbf{B}'\boldsymbol{\pi}_i \geq 0 & : i = 1, \cdots, I \\ (-\mathbf{p}_i, t_i) \in \Pi^*, \mathbf{q}_i \geq \mathbf{0} & \end{array} \right\}, \tag{5.7}$$

where Π^* *is the dual cone of* Π *defined in (4.7).*

Proof. Since cone Π is a regular cone which is closed, convex, pointed and has a non-empty interior, we can make use of strong conic duality.

Observe that for every $\mathbf{h} \in \Re^n$, we have

$$\begin{aligned} & \max_{\mathbf{z} \in \Upsilon(\gamma)} \mathbf{h}'\mathbf{z} \\ & = \max \{\mathbf{h}'\mathbf{z} : (\mathbf{z} - \mathbf{c}, \kappa(\gamma)) \in \Pi\} \\ & = \max \{\mathbf{h}'\mathbf{z} : (\mathbf{z}, 0) - (\mathbf{c}, -\kappa(\gamma)) \in \Pi\} \\ & \underbrace{=}_{(a)} \min\{\langle(-\mathbf{c}, \kappa(\gamma)), (\boldsymbol{v}, t)\rangle : \boldsymbol{v} = -\mathbf{h}, (\boldsymbol{v}, t) \in \Pi^*\} \\ & = \min\{\mathbf{c}'\mathbf{h} + \kappa(\gamma)t : (-\mathbf{h}, t) \in \Pi^*\}, \end{aligned}$$

where the equality (a) is due to the strong conic duality of regular cones Π and Π^*. It follows that

$$\max_{\mathbf{z} \in \Omega(\gamma)} \mathbf{h}'\mathbf{z}$$

$$
\begin{aligned}
&= \max\left\{\mathbf{h}'\mathbf{z} : \mathbf{z} = \mathbf{z}_1 = \mathbf{z}_2, \mathbf{z}_1 \in \Upsilon(\gamma), \mathbf{z}_2 \in \Omega\right\} \\
&= \max_{\mathbf{z},\mathbf{z}_1,\mathbf{z}_2}\left\{\min_{\mathbf{p},\mathbf{q}}\left\{\mathbf{h}'\mathbf{z} + \mathbf{p}'(\mathbf{z}_1 - \mathbf{z}) + \mathbf{q}'(\mathbf{z}_2 - \mathbf{z})\right\} : \mathbf{z}_1 \in \Upsilon(\gamma), \mathbf{z}_2 \in \Omega\right\} \\
&\underbrace{=}_{(b)} \min_{\mathbf{p},\mathbf{q}}\left\{\max_{\mathbf{z},\mathbf{z}_1,\mathbf{z}_2}\left\{(\mathbf{h} - \mathbf{p} - \mathbf{q})'\mathbf{z} + \mathbf{p}'\mathbf{z}_1 + \mathbf{q}'\mathbf{z}_2 : \mathbf{z}_1 \in \Upsilon(\gamma), \mathbf{z}_2 \in \Omega\right\}\right\} \\
&= \min_{\mathbf{p}+\mathbf{q}=\mathbf{h}}\left\{\max_{\mathbf{z}_1,\mathbf{z}_2}\left\{\mathbf{p}'\mathbf{z}_1 + \mathbf{q}'\mathbf{z}_2 : \mathbf{z}_1 \in \Upsilon(\gamma), \mathbf{z}_2 \in \Omega\right\}\right\} \\
&= \min_{\mathbf{p}+\mathbf{q}=\mathbf{h}}\left\{\max_{\mathbf{z}_1}\left\{\mathbf{p}'\mathbf{z}_1 + \mathbf{1}'\mathbf{q}^+ : \mathbf{z}_1 \in \Upsilon(\gamma)\right\}\right\} \\
&= \min_{\mathbf{p}+\mathbf{q}=\mathbf{h}}\left\{\min_{t}\left\{\mathbf{c}'\mathbf{p} + \kappa(\gamma)t + \mathbf{1}'\mathbf{q}^+ : (-\mathbf{p}, t) \in \Pi^*\right\}\right\} \\
&= \min_{\mathbf{p},\mathbf{q},t}\left\{\mathbf{c}'\mathbf{p} + \kappa(\gamma)t + \mathbf{1}'\mathbf{q} : (-\mathbf{p}, t) \in \Pi^*, \mathbf{p} + \mathbf{q} \geq \mathbf{h}, \mathbf{q} \geq \mathbf{0}\right\},
\end{aligned}
$$

where the interchange of "max" and "min" in (b) is due to the Sion's minimax theorem (Sion, 1957). Specifically, we observe that the continuous objective $(\mathbf{h}-\mathbf{p}-\mathbf{q})'\mathbf{z}+\mathbf{p}'\mathbf{z}_1+\mathbf{q}'\mathbf{z}_2$ is quasiconvex (actually bi-linear) in $\mathbf{p}, \mathbf{q}$ and quasiconcave (actually bi-linear) in $\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2$. In addition, the set $\Upsilon(\gamma)$ and Ω are convex and compact and thus we can deduce (b) according to Sion's minimax theorem. Recall that the counterpart $\{\psi(\gamma) \leq \tau_0\}$ is equivalent to

$$
\overline{\mathbf{b}}'\boldsymbol{\pi}_i + \mathbf{g}'\boldsymbol{\mu}_i + \max_{\mathbf{z}\in\Omega(\gamma)} \mathbf{h}'\tilde{\mathbf{z}} \leq \tau_0 : i = 1, \cdots, I.
$$

So, we can easily show the equivalence by augmenting $\boldsymbol{\pi}_i, \mathbf{p}_i, \mathbf{q}_i$ for $i = 1, \cdots, I$. □

Remark 5.3. When the actual norm $\|\cdot\|_\ell$ is constructed by transformations introduced in Section 4.2 (invertible linear mapping or intersection) from rational polynomial norms $\|\cdot\|_p : p \in \mathbb{R}_+ \cup +\infty$, we can reformulate the conic feasible constraint $(-\mathbf{p}_i, t_i) \in \Pi^*$ by several second order cone constraints. Therefore, (5.7) can be reformulated into a second order cone program which can be solved efficiently by interior point methods. For the modeling details, we refer the readers to Alizadeh and Goldfarb (2003) on second order cone formulation of general rational norms $\|\cdot\|_p$ and to Ben-Tal and Nemirovski (1999) on second order cone reformulation of new norms defined by above mentioned transformations (invertible linear mapping and intersection).

5.2.2 Box Uncertainty Set and Cardinality Uncertainty Set

When the number of extreme points is extremely large, the conic formulation (5.7) becomes computationally hard due to huge number of constraints. To overcome this, we sacrifice some level of modeling generality by considering two specific types of adjustable uncertainty sets: the box uncertainty set and the cardinality uncertainty set. The simplicity of these two types of uncertainty sets leads us practically efficient methods of computing the proposed supply resilience index.

With a subjective guess that $\mathbb{P}(\tilde{z}_i \leq z_i(\gamma)) = \gamma : \gamma \in [0,1]$, we define a family of nondecreasing mappings $z_i(\gamma) : [0,1] \longmapsto [0,1]$ such that $z_i(0) = 0, z_i(1) = 1$ for $i = 1, \cdots, n$. Based on this, we define the box uncertainty set as:

$$\Omega(\gamma) = \{\mathbf{z} \in \Re^n \, | 0 \leq z_i \leq z_i(\gamma) : i = 1, \cdots, n\}. \tag{5.8}$$

Notice that, for the box uncertainty set (5.8), the worst-case demand loss penalty occurs when $\tilde{z}_i$ reaches the upper bound, i.e., $\tilde{\mathbf{z}} = \mathbf{z}(\gamma)$. The evaluation of $\psi(\gamma)$ with $\Omega(\gamma)$ defined as (5.8) reduces to the following single linear programming problem:

$$\begin{aligned} \psi(\gamma) = \min_{\mathbf{x},\mathbf{y}} \quad & \boldsymbol{\tau}'\mathbf{y} - \tau_0 \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \overline{\mathbf{b}} - \mathbf{B}\mathbf{z}(\gamma) \\ & \mathbf{G}\mathbf{x} \geq \mathbf{g} \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

We now turn to the following *cardinality uncertainty set*, which is an adaption of the "budget of uncertainty" model in Bertsimas and Sim (2003, 2004):

$$\Omega(\gamma) = \left\{ \mathbf{z} \in \Re^n \, \left| \sum_{i=1}^{n} z_i \leq \gamma, z_i \in [0,1] : i = 1, \cdots, n \right. \right\}. \tag{5.9}$$

Here γ takes value from $[0, n]$. When $\gamma = 0$, we are actually considering the nominal supply case. The $\gamma = n$ case corresponds to the worst supply case. Indeed, if n is total the number of supply nodes, the resilience index has the intuitive interpretation as the maximum number of supply nodes that can be completely destroyed before compromising the tolerance target τ_0. In Proposition 5.1, we will illustrate how the computation of $\psi(\gamma)$ defined in (5.6) can be reformulated into mixed-integer programs (MIP), for integer valued γ and non-integer valued γ.

Proposition 5.1. *When the adjustable uncertainty set $\Omega(\gamma)$ is given by (5.9), we can reformulate $\psi(\gamma)$ defined in (5.6) into mixed-integer programs as follows:*

1. *When γ is an integer, we can compute $\psi(\gamma)$ by*

$$\begin{aligned} \psi(\gamma) = -\tau_0 + \max_{\boldsymbol{\pi},\boldsymbol{\mu},\mathbf{h},\mathbf{z},\boldsymbol{\varpi}} \quad & \mathbf{1}'\boldsymbol{\varpi} + \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} \\ & \mathbf{A}'\boldsymbol{\pi} + \mathbf{G}'\boldsymbol{\mu} \leq \mathbf{0} \\ & \mathbf{h} + \mathbf{B}'\boldsymbol{\pi} = 0 \\ & \mathbf{1}'\mathbf{z} = \gamma \\ & \boldsymbol{\varpi} \leq \mathbf{h} \\ & \boldsymbol{\varpi} \leq M\mathbf{z} \\ & \boldsymbol{\pi} \leq \boldsymbol{\tau} \\ & \boldsymbol{\mu} \geq \mathbf{0} \\ & \mathbf{z} \in \{0,1\}^n, \end{aligned} \tag{5.10}$$

where M is a sufficiently large constant. Moreover, the optimal solution of (5.10) gives the optimal solution of (5.6) as $(\boldsymbol{\pi}, \boldsymbol{\mu}, \mathbf{z})$.

2. *When γ is not integer, then we can compute $\psi(\gamma)$ by*

$$\begin{aligned} \psi(\gamma) = -\tau_0 + \max_{\boldsymbol{\pi},\boldsymbol{\mu},\mathbf{h},\mathbf{z},\mathbf{z}^\dagger,\boldsymbol{\varpi},\boldsymbol{\varpi}^\dagger} \quad & \mathbf{1}'(\boldsymbol{\varpi} + \boldsymbol{\varpi}^\dagger) + \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} \\ & \mathbf{A}'\boldsymbol{\pi} + \mathbf{G}'\boldsymbol{\mu} \leq \mathbf{0} \\ & \mathbf{h} + \mathbf{B}'\boldsymbol{\pi} = 0 \\ & \mathbf{1}'\mathbf{z} = \lfloor \gamma \rfloor \\ & \mathbf{1}'\mathbf{z}\dagger = 1 \end{aligned}$$

$$
\begin{aligned}
&\boldsymbol{\varpi} \leq \mathbf{h} \\
&\boldsymbol{\varpi} \leq M\mathbf{z} \\
&\boldsymbol{\varpi}^{\dagger} \leq (\gamma - \lfloor\gamma\rfloor)\mathbf{h} \\
&\boldsymbol{\varpi}^{\dagger} \leq M\mathbf{z}^{\dagger} \\
&\mathbf{z} + \mathbf{z}^{\dagger} \leq \mathbf{1} \\
&\boldsymbol{\pi} \leq \boldsymbol{\tau} \\
&\boldsymbol{\mu} \geq \mathbf{0} \\
&\mathbf{z}, \mathbf{z}^{\dagger} \in \{0,1\}^n,
\end{aligned} \tag{5.11}
$$

where M is a sufficiently large constant. Moreover, the optimal solution of (5.11) gives the optimal solution of (5.6) as $(\boldsymbol{\pi}, \boldsymbol{\mu}, \mathbf{z} + (\gamma - \lfloor\gamma\rfloor)\mathbf{z}^{\dagger})$.

Proof. We present the proof of (b) here only because the integer case is analogous and simpler. Because the dual problem (5.4) is always feasible and bounded, the optimal value of (5.6) remains the same by restricting $\tilde{\mathbf{z}}$ to be one extreme point of $\Omega(\gamma)$. For every extreme point $\tilde{\mathbf{z}}$ and a feasible solution $(\boldsymbol{\pi}, \boldsymbol{\mu}) \in \mathcal{Y}$, we next show that we can construct a feasible solution of (5.11) such that their corresponding objective value equals. In particular, we can define $(\mathbf{z}, \mathbf{z}^{\dagger}, \boldsymbol{\varpi}, \boldsymbol{\varpi}^{\dagger}, \mathbf{h})$ as:

$$
\begin{aligned}
&\mathbf{z} = \lfloor\tilde{\mathbf{z}}\rfloor \\
&\mathbf{z}^{\dagger} = \frac{1}{\gamma - \lfloor\gamma\rfloor}(\tilde{\mathbf{z}} - \mathbf{z}) \\
&\mathbf{h} = -\mathbf{B}'\boldsymbol{\pi} \\
&\boldsymbol{\varpi} = \mathrm{diag}(\mathbf{z})\mathbf{h} \\
&\boldsymbol{\varpi}^{\dagger} = \mathrm{diag}(\tilde{\mathbf{z}} - \mathbf{z})\mathbf{h}.
\end{aligned}
$$

Conversely, we can also construct a feasible solution of (5.6) based on a given feasible solution of (5.11). Therefore, these two problems are equivalent. □

Although commercial software such as CPLEX and MOSEK enable us to quickly solve single mixed-integer program in moderate size, solving the above MIPs repeatedly in a binary search scheme can still be extremely time-consuming when high accuracy of $\varrho(\mathcal{G})$ is required. Note that formulation (5.10) is relatively simpler than (5.11) because less integer variables are involved, we consider the following modified binary search scheme of computing the integer part of $\varrho(\mathcal{G})$ using (5.10) only.

Algorithm 1 Binary search algorithm

Input: A routine that solves Model (5.6) optimally for any integer $\gamma \in [1, n]$.
Output: An integer $k = \lfloor \varrho(\mathcal{G}) \rfloor$.

1. Set $\gamma_L := 0$, $\gamma_U := n, l = 0$
2. Check the value $\gamma_U - \gamma_L$.
   If $\gamma_U - \gamma_L = 1$, output $\lfloor \varrho(\mathcal{G}) \rfloor := \gamma_L$, terminate the algorithm.
   Else Go to Step 3.
3. Set $\gamma = \lfloor \frac{\gamma_L + \gamma_U}{2} \rfloor$ and compute $\psi(\gamma)$ by (5.10).
   If $\psi(\gamma) = 0$, update $\gamma_L := \gamma$.
   Else, update $\gamma_U = \gamma$.
   Update $l := l + 1$ and go back to Step 2.

Our next result reports the computational performance of Algorithm 1.

Proposition 5.2. *Suppose that the worst case demand loss penalty $\psi(\gamma)$ given in (5.6) is defined by cardinality uncertainty set (5.9) and $\varrho(\mathcal{G}) \in [0, n)$, then Algorithm 1 can compute the exact value of $\lfloor \varrho(\mathcal{G}) \rfloor$ by solving at most $\lceil \log_2(n) \rceil$ sub-problems of the form (5.6).*

Proof. Obviously γ_L and γ_U are always integers when performing the algorithm. In addition, it is not difficult to verify that $\varrho(\mathcal{G}) \in [\gamma_L, \gamma_U)$ and $\gamma_U - \gamma_L \leq 2^{\lceil \log_2(n) \rceil - l}$ at the l th iteration by induction. As a consequence, the algorithm terminates after at most $\lceil \log_2(n) \rceil$ iterations and γ_L gives the value of $\lfloor \varrho(\mathcal{G}) \rfloor$ upon the termination criterion. □

Algorithm 1 gives the integer part of $\varrho(\mathcal{G})$. To compute the exact value, we first study some geometry properties of function $\psi(\cdot)$.

Proposition 5.3. *Define* $\Omega(\gamma)$ *as (5.9), function* $\psi(\cdot)$ *given by (5.6) is nondecreasing and convex in the interval* $[k, k+1]$ *for any integer* $k \in [0, n)$.

Proof. The nondecreasing property is clear because $\{\Omega(\gamma)\}_{\gamma\in[0,n]}$ is nondecreasing.

To prove the convexity, it suffices to show that for any $\gamma = \theta\gamma_1 + (1-\theta)\gamma_2 : \gamma_1, \gamma_2 \in [k, k+1], \theta \in (0, 1)$, we have

$$\psi(\gamma) \leq \theta\psi(\gamma_1) + (1-\theta)\psi(\gamma_2). \tag{5.12}$$

As discussed, we can express $\psi(\gamma)$ as the form $\max_{i\in\mathcal{I}}\{v_{i0} + \mathbf{v}_i'\tilde{\mathbf{z}}\}$. Suppose that $\mathbf{z}$ is the extreme point that $\psi(\gamma) = \phi(\mathbf{z})$. Let the index $i^* = \arg\max_i(v_{i0} + \mathbf{v}_i'\mathbf{z})$. (If multiple indices attain the maximum, we arbitrarily select one of them). Let us define $\mathbf{z}_1, \mathbf{z}_2$ as:

$$\mathbf{z}_1 = \lfloor\mathbf{z}\rfloor + \frac{\gamma_1 - \lfloor\gamma\rfloor}{\gamma - \lfloor\gamma\rfloor}(\mathbf{z} - \lfloor\mathbf{z}\rfloor)$$
$$\mathbf{z}_2 = \lfloor\mathbf{z}\rfloor + \frac{\gamma_2 - \lfloor\gamma\rfloor}{\gamma - \lfloor\gamma\rfloor}(\mathbf{z} - \lfloor\mathbf{z}\rfloor).$$

It is not hard to see that $\mathbf{z}_1 \in \Omega(\gamma_1), \mathbf{z}_2 \in \Omega(\gamma_2)$ and $\mathbf{z} = \theta\mathbf{z}_1 + (1-\theta)\mathbf{z}_2$. Thus, we can conclude that

$$\begin{aligned}
\psi(\gamma) &= v_{i^*0} + \mathbf{v}_{i^*}'\mathbf{z} \\
&= \theta(v_{i^*0} + \mathbf{v}_{i^*}'\mathbf{z}_1) + (1-\theta)(v_{i^*0} + \mathbf{v}_{i^*}'\mathbf{z}_2) \\
&\leq \theta\phi(\mathbf{z}_1) + (1-\theta)\phi(\mathbf{z}_2) \\
&\leq \theta\psi(\gamma_1) + (1-\theta)\psi(\gamma_2),
\end{aligned}$$

which verifies the convexity of $\psi(\cdot)$ in $[k, k+1]$. □

It is well-known that for any nondecreasing and differentiable function $\psi(\cdot)$, we can apply the Newton-Raphson method to find the $\gamma^\dagger$ such that $\psi(\gamma^\dagger) = 0$ iteratively when its gradient is given (here $\psi(\cdot)$ is a one dimension function, the gradient is also the first order derivative). Furthermore, this method converges when $\psi(\cdot)$ is convex. The convexity of $\psi(\cdot)$ encourages us to apply gradient-based method to calculate $\varrho(\mathcal{G})$. Since $\psi(\cdot)$ is continuous but not necessarily differentiable, we modify the Newton-Raphson method by using sub-gradients. The adapted "gradient-based" search algorithm is given as follows:

Algorithm 2 Gradient based search algorithm

Input: An integer k such that $\varrho(\mathcal{G}) \in [k, k+1)$, and a routine that solves model (5.6) with cardinality uncertainty set (5.9) for every $\gamma \in [k, k+1]$.
Output: An real number $\gamma^\dagger = \varrho(\mathcal{G})$.

1. Set $l := 0, \gamma_l = k+1$.
2. Solve $\psi(\gamma_l)$ in (5.6) by the routine, and extract the corresponding optimal solution $(\boldsymbol{\pi}, \boldsymbol{\mu})$.
3. **If** $\psi(\gamma_l) = 0$, output $\gamma^\dagger := \gamma_l$, terminate the algorithm.
4. **Otherwise**, compute the nonnegative vector $\mathbf{h} = -\mathbf{B}'\boldsymbol{\pi}$, and sort the elements of $\mathbf{h}$ in a noninceasing order as $h_{i_1} \geq h_{i_2} \cdots h_{i_k} \geq \cdots \geq h_{n_2}$. Let $\gamma := \gamma_l - \dfrac{\psi(\gamma_l)}{h_{i_{k+1}}}$. Update $l := l+1$, $\gamma_l = \gamma$. Go to Step 2.

Actually, $h_{i_{k+1}}$ is always positive whenever we find that $\psi(\gamma_k) > 0$ (this claim is a concomitant result of Theorem 5.2, which follows directly from $(-\mathbf{B}'\boldsymbol{\pi})_{k+1} > 0$). Figure 5.1 gives a graphical illustration of the iterative procedure of Algorithm 2. In each iteration, we compute the value of $\psi(\gamma_l)$ using and its corresponding sub-gradient of $\psi(\cdot)$ in $[k, k+1]$ by $h_{i_{k+1}}$ (see Lemma 5.1 below). With this sub-gradient, we then update the iteration by the linear approximation of $\psi(\gamma)$ (The dash-dotted line). Fortunately, our proposed "gradient-based" search algorithm can give the exact value of $\varrho(\mathcal{G})$ within a finite number of iterations, which is suggested in Theorem 5.2.

Lemma 5.1. *For every iteration (the lth iteration) of performing Algorithm 2 such that $\psi(\gamma_l) > 0$, the corresponding optimal solution $(\boldsymbol{\pi}, \boldsymbol{\mu})$ gives a sub-gradient of $\psi(\cdot)$ in $[k, k+1]$*

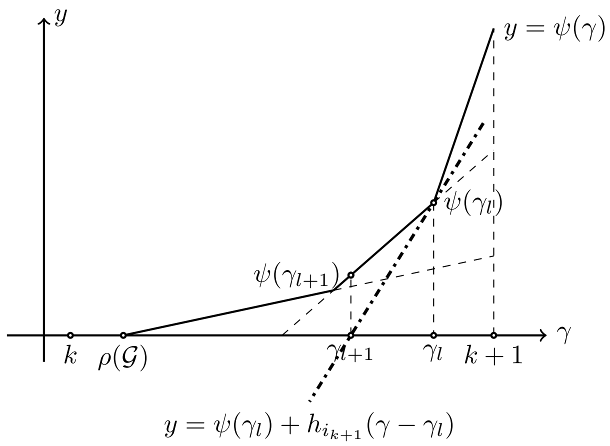


Figure 5.1: A graphical illustration of the "gradient-based" search algorithm. The solid line is $\psi(\gamma)$, the dashed dotted line gives a lower linear approximation based on the computed sub-gradient $h_{i_{k+1}}$.

by

$$\psi(\gamma) \geq \psi(\gamma_l) + h_{i_{k+1}}(\gamma - \gamma_l) : \forall \gamma \in [k, k+1],$$

where $h_{i_{k+1}}$ is the same as what we defined in Algorithm 2.

Proof. According to the definition of $\psi(\gamma) : \gamma \in [k, k+1]$, we have

$$\begin{aligned} \psi(\gamma) &\geq \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} + (-\mathbf{B}'\boldsymbol{\pi})'\tilde{\mathbf{z}} \\ &= \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} + \mathbf{h}'\tilde{\mathbf{z}} \\ &= \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} + \sum_{t=1}^{k} h_{i_t} + (\gamma - k) h_{i_{k+1}} \\ &= \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} + \sum_{t=1}^{k} h_{i_t} + (\gamma_l - k) h_{i_{k+1}} + (\gamma - \gamma_l) h_{i_{k+1}} \\ &= \psi(\gamma_l) + h_{i_{k+1}}(\gamma - \gamma_l), \end{aligned}$$

where the first inequality is because $(\boldsymbol{\pi}, \boldsymbol{\mu})$ is a feasible solution of (5.6). This finishes the proof of the lemma. $\square$

We now present an elementary result on the existence of sharper gradient upon computed optimal solution $(\boldsymbol{\pi}, \boldsymbol{\mu})$, which is a fundamental auxiliary lemma for a rigorous proof of the convergence of the gradient based search algorithm. For ease of exposition, we denote by $(\mathbf{h})_\gamma$ as the $\lceil\gamma\rceil$th largest entry of $\mathbf{h}$ for every vector $\mathbf{h} \in \Re^n$ (If there exist multiple entries that attain the $\lceil\gamma\rceil$th largest, we break ties by arbitrarily select one of them), and define two auxiliary functions

$$\begin{aligned}\beta(\mathbf{h},\gamma) &= \max_{\tilde{\mathbf{z}}\in\Omega(\gamma)} \mathbf{h}'\tilde{\mathbf{z}} \\ \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma) &= \overline{\mathbf{b}}'\boldsymbol{\pi} + \mathbf{g}'\boldsymbol{\mu} + \beta(-\mathbf{B}'\boldsymbol{\pi},\gamma),\end{aligned}$$

where $\Omega(\gamma)$ is the cardinality uncertainty set defined in (5.9).

It follows directly from the definition that

$$\beta(\mathbf{h},\gamma) = \sum_{i=1}^{\lceil\gamma\rceil-1} (\mathbf{h})_i + (\gamma - \lfloor\gamma\rfloor)(\mathbf{h})_\gamma : \mathbf{h} \in \Re^n, \gamma \in \Re \tag{5.13}$$

and

$$\begin{aligned}\beta(\alpha\mathbf{h},\gamma) &= \alpha\beta(\mathbf{h},\gamma) : \alpha \geq 0, \mathbf{h} \in \Re^n, \gamma \in [0,n] \\ \beta(\mathbf{h}_1+\mathbf{h}_2,\gamma) &\leq \beta(\mathbf{h}_1,\gamma) + \beta(\mathbf{h}_2,\gamma) : \mathbf{h}_1,\mathbf{h}_2 \in \Re^n, \gamma \in [0,n].\end{aligned} \tag{5.14}$$

Lemma 5.2. *For any $\gamma \in (k, k+1)$, where k is an integer in $[0, n-1]$. Suppose that $(\boldsymbol{\pi},\boldsymbol{\mu})$ is an optimal solution of (5.6) such that $\psi(\gamma) = \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma)$, then there exists $(\boldsymbol{\pi}^\dagger,\boldsymbol{\mu}^\dagger) \in \bar{\mathcal{Y}}$ such that*

$$\begin{aligned}(-\mathbf{B}'\boldsymbol{\pi}^\dagger)_{k+1} &\geq (-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\ \eta(\boldsymbol{\pi}\dagger,\boldsymbol{\mu}^\dagger,\gamma) &= \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma).\end{aligned}$$

Proof. We begin with an elementary result on the geometry of polyhedron that every element in a polyhedron can be expressed as a convex combination of its extreme points plus a nonnegative combination of its extreme rays. Based on this, we can express the optimal solution $(\boldsymbol{\pi},\boldsymbol{\mu})$ as

$$(\boldsymbol{\pi},\boldsymbol{\mu}) = \sum_{j\in J_1\cup J_2} \lambda_j(\boldsymbol{\pi}_j,\boldsymbol{\mu}_j) : \lambda_j > 0, \forall j \in J_1\cup J_2; \sum_{j\in J_1}\lambda_j = 1,$$

where J_1 and J_2 are index sets such that $(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j)$ corresponds to extreme point for $\forall j \in J_1$ and extreme ray for $\forall j \in J_2$. By applying (5.14), we have the following inequality:

$$\beta\left(\sum_{j \in J} \theta_j \mathbf{h}_j, \gamma\right) \leq \sum_{j \in J} \theta_j \beta\left(\mathbf{h}_j, \gamma\right). \tag{5.15}$$

for every $\theta_j \geq 0 : j \in J$. Recall that, for every fixed $\tilde{\mathbf{z}} \in \Omega$, LP (5.6) is bounded under Assumption 5.1. Thus, the objective gain along any extreme ray is nonpositive. It follows that for every $j \in J_2$,

$$\eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, n) = \overline{\mathbf{b}}'\boldsymbol{\pi}_j + \mathbf{g}'\boldsymbol{\mu}_j - (\mathbf{B}'\boldsymbol{\pi})'\mathbf{1} \leq 0.$$

As assumed, $(\boldsymbol{\pi}, \boldsymbol{\mu})$ gives the optimal value $\psi(\gamma)$. Hence,

$$\begin{aligned}
&\psi(\gamma) \\
&= \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma) \\
&= \overline{\mathbf{b}}'\left(\sum_{j \in J_1 \cup J_2} \lambda_j \boldsymbol{\pi}_j\right) + \mathbf{g}'\left(\sum_{j \in J_1 \cup J_2} \lambda_j \boldsymbol{\mu}_j\right) + \beta\left(-\sum_{j \in J_1 \cup J_2} \lambda_j \mathbf{B}'\boldsymbol{\pi}_j, \gamma\right) \\
&\leq \sum_{j \in J_1 \cup J_2} \lambda_j \left(\overline{\mathbf{b}}'\boldsymbol{\pi}_j + \mathbf{g}'\boldsymbol{\mu}_j + \beta(-\mathbf{B}'\boldsymbol{\pi}_j, \gamma)\right) \\
&\leq \sum_{j \in J_1} \lambda_j \eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, \gamma) + \sum_{j \in J_2} \lambda_j \eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, n) \\
&\leq \sum_{j \in J_1} \lambda_j \eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, \gamma) \\
&\leq \sum_{j \in J_1} \lambda_j \psi(\gamma) \\
&= \psi(\gamma),
\end{aligned}$$

where the first inequality is because (5.15), the second inequality follows from $\mathbf{B}'\boldsymbol{\pi}_j \leq \mathbf{0} : \forall j \in J_2$ and the third inequality follows from $\eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, n) \leq 0 : \forall j \in J_2$. Because the left side coincides with the right side, all the involved inequalities should be tight and we further have

$$\eta(\boldsymbol{\pi}_j, \boldsymbol{\mu}_j, \gamma) = \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma) : \forall j \in J_1 \tag{5.16}$$

$$\beta(-\mathbf{B}'\boldsymbol{\pi},\gamma) = \sum_{j \in J_1 \cup J_2} \lambda_j \beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma) \tag{5.17}$$

$$\beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma) = \beta(-\mathbf{B}'\boldsymbol{\pi}_j,n) : \forall j \in J_2. \tag{5.18}$$

For $t = 1, \cdots, k+1$, let i_t be the index that $-\mathbf{B}'_{i_t}\boldsymbol{\pi} = (-\mathbf{B}'\boldsymbol{\pi})_t$, where $\mathbf{B}_{i_t}$ denotes the i_tth column of the matrix $\mathbf{B}$ (We break ties by arbitrarily select one of them if multiple columns attain the same value). According to (5.13) and (5.17), we have

$$\begin{aligned}
&\beta(-\mathbf{B}'\boldsymbol{\pi},\gamma) \\
&= \sum_{t=1}^{k} (-\mathbf{B}'\boldsymbol{\pi})_t + (\gamma - k)(-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= \sum_{t=1}^{k} (-\mathbf{B}'_{i_t}\boldsymbol{\pi}) + (\gamma - k)(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}) \\
&= \sum_{j \in J_1 \cup J_2} \lambda_j \left[\sum_{t=1}^{k} (-\mathbf{B}'_{i_t}\boldsymbol{\pi}_j) + (\gamma - k)(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j)\right] \\
&\leq \sum_{j \in J_1 \cup J_2} \lambda_j \beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma) \\
&= \beta(-\mathbf{B}'\boldsymbol{\pi},\gamma).
\end{aligned}$$

Therefore, the involved inequality should be tight and we further can deduce that

$$\beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma) = \sum_{t=1}^{k} (-\mathbf{B}'_{i_t}\boldsymbol{\pi}_j) + (\gamma - k)(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j) : \forall j \in J_1 \cup J_2.$$

Note that $k < \gamma < k+1$ indicating that the right hand side would be smaller than $\beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma)$ if $(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j) > \min_{t=1,\cdots,k}\{(-\mathbf{B}'_{i_t}\boldsymbol{\pi}_j)\}$. Thus, certainly we have

$$(-\mathbf{B}'\boldsymbol{\pi}_j)_{k+1} \geq (-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j) : \forall j \in J_1 \cup J_2.$$

Furthermore, note from (5.18) that $\beta(-\mathbf{B}'\boldsymbol{\pi}_j,\gamma) = \beta(-\mathbf{B}'\boldsymbol{\pi}_j,n)$ for every $j \in J_2$, we have

$$-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j = 0 : \forall j \in J_2.$$

Let $j^* \in J_1$ be the index such that $-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_{j^*} = \max_{j \in J_1}\{-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j\}$. Therefore, we have

$$\begin{aligned}
&(-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= -\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi} \\
&= \sum_{j \in J_1 \cup J_2} \lambda_j(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j) \\
&= \sum_{j \in J_1} \lambda_j(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_j) \\
&\leq \sum_{j \in J_1} \lambda_j(-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_{j^*}) \\
&= (-\mathbf{B}'_{i_{k+1}}\boldsymbol{\pi}_{j^*}) \\
&\leq (-\mathbf{B}'\boldsymbol{\pi}_{j^*})_{k+1}.
\end{aligned}$$

Along with (5.16), we can deduce that $(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger) := (\boldsymbol{\pi}_{j^*}, \boldsymbol{\mu}_{j^*})$ gives the exact extreme point we attempt to find. □

Theorem 5.2. *Suppose that the resilience index $\varrho(\cdot)$ is defined by cardinality uncertainty set (5.9) and $\varrho(\mathcal{G}) \in [k, k+1)$, where k is an integer given as the input of Algorithm 2. Then Algorithm 2 would terminate in at most I steps with its output $\gamma^\dagger = \varrho(\mathcal{G})$, where I is the number of extreme points of $\mathcal{Y}$.*

Proof. Let $\{\gamma_l\}_{l=0,1,\cdots}$ be the sequence of γ generated by the "gradient-based" search algorithm (If the algorithm stops at the l^*th iteration, we let $\gamma_l = \gamma_{l^*}$ for $l \geq l^*$).

It is not difficult to see that the function $\eta(\cdot)$ possesses the following "first-order" type equation

$$\eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma - \delta) = \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma) - \delta \cdot (-\mathbf{B}'\boldsymbol{\pi})_\gamma$$

for every $0 \leq \delta \leq \gamma - \lfloor \gamma \rfloor$. Hence, $\psi(\cdot)$ is left continuous which indicates that $\psi(\varrho(\mathcal{G})) = 0$.

Observe that (5.4) is always bounded and feasible for all $\tilde{\mathbf{z}} \in \Omega$, the optimality can be reached at one extreme point of the feasible region. Thus, we can replace the feasible region of (5.4) $\mathcal{Y}$ with the convex hull of the extreme points set $\bar{\mathcal{Y}}$, denoted as $\mathcal{CH}(\bar{\mathcal{Y}})$, without affecting optimality. Thus, we have

$$\psi(\gamma) = \max_{(\boldsymbol{\pi},\boldsymbol{\mu})\in\mathcal{CH}(\bar{\mathcal{Y}})} \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma) - \tau_0.$$

We then complete the proof by showing the following two claims:

Claim (a): For all $l \in \mathbb{Z}$ we have $\gamma_l \geq \varrho(\mathcal{G})$.

Claim (b): There exists $l^* < \infty, l^* \in \mathbb{Z}$ such that $\psi(\gamma_{l^*}) = 0$.

We first prove (a) by induction.

Suppose that it holds for l and $\gamma_l > \varrho(\mathcal{G})$, or equivalently, $\psi(\gamma_l) > 0$ (Otherwise we have $\gamma_{l+1} = \gamma_l \geq \varrho(\mathcal{G})$ completing the induction step). For simplicity of representation we denote by $(\boldsymbol{\pi},\boldsymbol{\mu})$ as the computed optimal solution of (5.6) for $\gamma = \gamma_l$, i.e., $\psi(\gamma_l) = \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma_l) - \tau_0$.

Observe that if $(-\mathbf{B}'\boldsymbol{\pi})_{k+1} \leq 0$, we have

$$\begin{aligned}
&\psi(k)\\
&\geq \eta(\boldsymbol{\pi},\boldsymbol{\mu},k) - \tau_0\\
&= \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma_l) - (\gamma_l - k)(-\mathbf{B}'\boldsymbol{\pi})_{k+1} - \tau_0\\
&\geq \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma_l) - \tau_0\\
&= \psi(\gamma_l) > 0
\end{aligned}$$

contradicting the fact that $k \leq \varrho(\mathcal{G})$.

Hence, we have $(-\mathbf{B}'\boldsymbol{\pi})_{k+1} > 0$. In addition, we observe that, for any $0 < \epsilon \leq \gamma_l - \gamma_{l+1}$,

$$\begin{aligned}
&\psi(\gamma_{l+1} + \epsilon)\\
&= \max_{(\boldsymbol{\pi},\boldsymbol{\mu})\in\mathcal{CH}(\bar{\mathcal{Y}})} \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma_{l+1}+\epsilon) - \tau_0\\
&\geq \eta(\boldsymbol{\pi},\boldsymbol{\mu},\gamma_{l+1}+\epsilon) - \tau_0
\end{aligned}$$

$$
\begin{aligned}
&= \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma_l) - (\gamma_l - \gamma_{l+1} - \epsilon) \cdot (-\mathbf{B}'\boldsymbol{\pi})_{k+1} - \tau_0 \\
&= \psi(\gamma_l) - (\gamma_l - \gamma_{l+1} - \epsilon) \cdot (-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= \psi(\gamma_l) - \left(\frac{\psi(\gamma_l)}{(-\mathbf{B}'\boldsymbol{\pi})_{k+1}} - \epsilon \right) \cdot (-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= \epsilon \cdot \left(-\mathbf{B}'\boldsymbol{\pi}\right)_{k+1} \\
&> 0.
\end{aligned}
$$

The arbitrariness of ϵ indicates that $\gamma_{l+1} \geq \varrho(\mathcal{G})$, which completes the induction step.

We now show (b), we assume by the way of contradiction that the claim does not hold, or equivalently, $\psi(\gamma_l) > 0 : \forall l \in \mathbb{Z}_+$. To give a contradiction, it suffices to show the following claim because $\bar{\mathcal{Y}}$ is actually a finite set.

Claim (c): There exists a sequence of sets $\{\mathcal{Y}_l \subseteq \bar{\mathcal{Y}}\}_{l=1,\cdots}$, such that

$$
\begin{aligned}
&|\mathcal{Y}_l| = l - 1 \\
&\eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma) - \tau_0 \leq 0 : \forall (\boldsymbol{\pi}, \boldsymbol{\mu}) \in \mathcal{Y}_l, \gamma \leq \gamma_l.
\end{aligned}
$$

Obviously (c) holds for $l = 1$ by letting $\mathcal{Y}_1 = \emptyset$. We next prove it by induction. Assume that (c) holds for l. For notational simplicity, we denote by $(\boldsymbol{\pi}, \boldsymbol{\mu})$ as the corresponding computed optimal solution of the case $\gamma = \gamma_l$. By Lemma 5.2, we can find $(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger) \in \bar{\mathcal{Y}}$ such that

$$
\begin{aligned}
&\eta(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger, \gamma_l) = \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma_l) > \tau_0; \\
&(-\mathbf{B}'\boldsymbol{\pi}^\dagger)_{k+1} \geq (-\mathbf{B}'\boldsymbol{\pi})_{k+1}.
\end{aligned}
$$

By the induced assumption that (c) holds for l, we can deduce that $(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger) \notin \mathcal{Y}_l$. Defining $\mathcal{Y}_{l+1} = \mathcal{Y}_l \cup \{(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger)\}$, it is obvious that $\mathcal{Y}_{l+1} \subseteq \bar{\mathcal{Y}}$ and $|\mathcal{Y}_{l+1}| = l + 1$. We also observe that,

$$
\eta(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger, \gamma_{l+1})
$$

$$
\begin{aligned}
&= \eta(\boldsymbol{\pi}^\dagger, \boldsymbol{\mu}^\dagger, \gamma_l) - (\gamma - \gamma_{l+1}) \cdot (-\mathbf{B}'\boldsymbol{\pi}^\dagger)_{k+1} \\
&= \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma_l) - (\gamma - \gamma_{l+1}) \cdot (-\mathbf{B}'\boldsymbol{\pi}^\dagger)_{k+1} \\
&\leq \eta(\boldsymbol{\pi}, \boldsymbol{\mu}, \gamma_l) - (\gamma - \gamma_{l+1}) \cdot (-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= \psi(\gamma_{l+1}) + \tau_0 - (\gamma - \gamma_{l+1}) \cdot (-\mathbf{B}'\boldsymbol{\pi})_{k+1} \\
&= \tau_0,
\end{aligned}
$$

which means that $\mathcal{Y}_{l+1}$ is exactly the set we are trying to find. Hence, claim (c) holds for $l+1$ completing the induction step. □

Remark 5.4. Note that I can be exponentially large, which means that Theorem 5.2 fails to guarantee the computational efficiency of Algorithm 2. However, in our tested experiments, only one or two additional iterations are necessary for algorithm termination. Thus, we can compute $\varrho(\mathcal{G})$ by solving roughly $\lceil \log_2(n) \rceil$ MIPs. In practice, the value of n is not very large, this further suggests that our method is practically acceptable for moderate sized industrial problems.

5.3 Designing Resilient Energy Supply System

A practical problem of interest is improving the resilience of a given energy supply system. Let us denote by $\mathbf{u}$ as the investment actions, and $\mathcal{G}(\mathbf{u})$ as the corresponding energy supply system under $\mathbf{u}$. We now consider the problem of maximizing the supply resilience index of $\mathcal{G}(\mathbf{u})$ over the investment decision space $\mathcal{U}$, the mathematical formulation of which is given as:

$$
\begin{aligned}
\max_{\mathbf{u}} \quad & \rho(\mathcal{G}(\mathbf{u})) \\
\text{s.t.} \quad & Q(\mathbf{u}) \leq B \\
& \mathbf{u} \in \mathcal{U},
\end{aligned}
\tag{5.19}
$$

where $Q(\mathbf{u})$ is the total cost associated with investment decision $\mathbf{u}$ and B is the available investment budget at the planning stage.

Remark 5.5. In (5.19), we maximize the energy supply system resilience within a given investment budget. This modeling approach is different from those in the energy literature dealing with design and procurement problems, where typically, a sum of the investment cost and the expected operational cost is minimized. Such approaches however have several drawbacks in our context, in contrast to our proposed model. First, minimizing a total cost objective may make trade-offs that are difficult to justify when dealing with losses associated with severe disruptions, since a consensus on the monetary values attached with social costs needs to be established, which can be challenging. Furthermore, minimizing the expected total costs requires a prior specification of the uncertainties, either the probability distributions or the supports of the uncertainty sets. This can be difficult to solicit from stakeholders in practice. Also, the investment budget is a one-time cost incurred during the investment stage, and the operational cost is realized over a long time study period, so their accounting is of rather different in nature. Finally, as we argued, the supply disruptions are rare events that can not be predicted precisely, thus the operating cost under supply disruptions is uncertain and can not be explicitly evaluated. Consequently, a small deviation on the estimated probability of supply disruptions might drive the computed investment decision far from optimal.

Applying a binary search on γ, problem (5.19) can be decomposed into a sequence of the following sub-problems:

$$\left\{ \text{Find } \mathbf{u} \in \mathcal{U} : \begin{array}{c} \rho(\mathcal{G}(\mathbf{u})) \geq \gamma \\ Q(\mathbf{u}) \leq B \end{array} \right\}. \tag{5.20}$$

Generally, problem (5.20) is still intractable due to the inherent dependency between supply network parameters $(\mathbf{A}, \mathbf{G}, \mathbf{B}, \overline{\mathbf{b}}, \mathbf{g})$ and the investment decision $\mathbf{u}$. To simplify it, we make two implicit assumptions.

Assumption 5.3. *There exist investment independent matrix* $\mathbf{F}$ *and vector* $\mathbf{f}$ *such that the total demand loss penalty of energy supply system* $\mathcal{G}(\mathbf{u})$ *under disruption realization* $\tilde{\mathbf{z}}$ *can be computed by*

$$\begin{aligned} \phi_{\mathcal{G}(\mathbf{u})}(\tilde{\mathbf{z}}) = \min_{\mathbf{x},\mathbf{y}} \quad & \boldsymbol{\tau}'\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \overline{\mathbf{b}} - \mathbf{B}\tilde{\mathbf{z}} \\ & \mathbf{G}\mathbf{x} \geq \mathbf{g} \\ & \mathbf{F}\mathbf{x} \leq \mathbf{U}\mathbf{u} + \mathbf{f} \\ & \mathbf{x}, \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Assumption 5.4. *We can specify an* investment independent worst case scenario set $\mathcal{Z}(\gamma) = \{\mathbf{z}_l\}_{l=1,\cdots,L}$, *which is independent of the investment decision* $\mathbf{u} \in \mathcal{U}$, *such that*

$$\max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \phi_{\mathcal{G}(\mathbf{u})}(\tilde{\mathbf{z}}) = \max_{\tilde{\mathbf{z}} \in \mathcal{Z}(\gamma)} \phi_{\mathcal{G}(\mathbf{u})}(\tilde{\mathbf{z}}).$$

Remark 5.6. In Section 5.4, we will illustrate how this auxiliary linear program model in Assumption 5.3 can be constructed. The proposed modeling tricks can be extended to general energy supply system problems, which indicates this assumption is not quite practically limiting. Assumption 5.4 is motivated by the computational benefits of the box uncertainty set and the cardinality uncertainty set. For the box uncertainty set, the worst case scenario set reduces to the singleton $\{\mathbf{z}(\gamma)\}$. For the cardinality uncertainty set or a general polytope, the set of its extreme points gives the investment independent worst case scenario set. For complicated uncertainty sets such as ellipsoid or intersection of multiple ellipsoids that we can not specify an investment decision independent worst case set of finite cardinality, we may approximate it by an inner polytope to give an upper bound and a circumscribed polytope to give a lower bound of the corresponding resilience index, respectively.

We next show that under these two assumptions, the sub-problem (5.20) can be solved via a deterministic mixed-integer program.

Proposition 5.4. *Under Assumption 5.3 and Assumption 5.4 with specified* $(\mathbf{F}, \mathbf{f})$ *and* $\mathcal{Z}(\gamma) = \{\mathbf{z}_l\}_{l=1,\cdots,L}$*, (5.20) is equivalent to*

$$\left\{ \text{Find } \mathbf{u} \in \mathcal{U} : \begin{array}{l} \boldsymbol{\tau}'\mathbf{y}_l \leq \tau_0 : l = 1, \cdots, L \\ \mathbf{A}\mathbf{x}_l + \mathbf{y}_l = \overline{\mathbf{b}} - \mathbf{B}\mathbf{z}_l : l = 1, \cdots, L \\ \mathbf{G}\mathbf{x}_l \geq \mathbf{g} : l = 1, \cdots, L \\ \mathbf{F}\mathbf{x}_l \leq \mathbf{U}\mathbf{u} + \mathbf{f} : l = 1, \cdots, L \\ \mathbf{x}_l, \mathbf{y}_l \geq \mathbf{0} : l = 1, \cdots, L \\ Q(\mathbf{u}) \leq B \end{array} \right\}$$

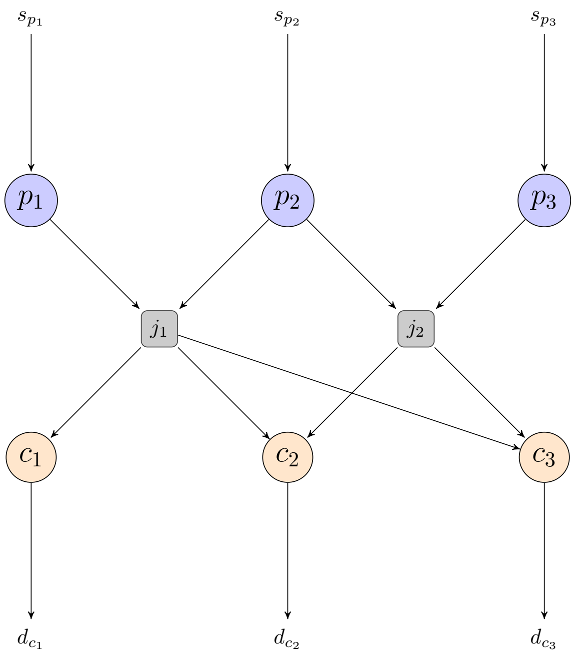
Proof. Because $\rho(\mathcal{G}(\mathbf{u})) \leq \gamma \iff \max_{\tilde{\mathbf{z}} \in \Omega(\gamma)} \phi_{\mathcal{G}(\mathbf{u})}(\tilde{\mathbf{z}})$, the modeling equivalence follows straightforwardly from Assumption 5.3 and 5.4. □

5.4 Application and Computational Experiments

The design and operations of natural gas networks is a very important industrial problem that has been extensively studied from different perspectives (e.g., Babonneau et al., 2012; De Wolf and Smeers, 1996, 2000; Martin et al., 2006). However, few of the studies in the literature consider the impact of gas supply disruptions. Indeed, gas supply disruptions can cause severe economic and social losses so that the resilience of natural gas network should be highlighted (see chap. 6 of Chaudry et al., 2011, for instances of natural gas supply disruptions). In this section, we apply our proposed supply resilience index to natural gas supply networks. The results illustrate how the resilience index can reveal and improve the system performance under natural gas supply disruptions.

In the following computational studies, all related optimization models are implemented in a MATLAB 2012 platform which calls the commercial software CPLEX 12.3 to solve. In addition, all the programs run on a PC with Windows 7 operating system, 8 GB RAM and Intel Dual Core i5-2500 CPU with 3.30GHz.

Figure 5.2: Illustration of the natural gas network



5.4.1 Linear Programming Model of Natural Gas Transmission Network

We consider a natural gas supply network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ with node set $\mathcal{N}$ and arc set $\mathcal{A}$. Node is an abstract object referring to a collection of physical infrastructures in a city or a specific region, arc denotes a pipeline linking a pair of nodes. Typically, there are three types of nodes in a natural gas supply network: gas supply nodes, junction nodes and gas demand nodes. Gas supply nodes are gas fields or import terminals in which the natural gas is injected into the system. Before flowing to the final market, the natural gas is transported to the junction nodes. In reality, the gas from different supply nodes has different chemical compositions, the junction nodes also play a role of mixing natural gas with different quality into a homogeneous one. For simplicity of modeling, we treat the natural gas flows through the system as a homogeneous one. The gas demand nodes denote the cities or regions where natural gas is consumed or traded. Figure 5.2 gives a schematic view of a natural gas network.

When modeling the natural gas operations, the fundamental flow balance condition should hold. In particular, we write it as:

$$\sum_{a\in\delta^+(\mathcal{G},i)} x(a) - \sum_{a\in\delta^-(\mathcal{G},i)} x(a) \begin{cases} \geq -s_i(\tilde{\mathbf{z}}) : i \in \mathcal{P} \\ = 0 : i \in \mathcal{J} \\ \geq d_i - e_i : i \in \mathcal{D} \end{cases} \tag{5.21}$$

where e_i is a nonnegative auxiliary variable denoting the demand loss at $i \in \mathcal{D}$, $\delta^+(\mathcal{G}, i)$ and $\delta^-(\mathcal{G}, i)$ denote the set of arcs into and out of node $i \in \mathcal{N}$, respectively, $\mathcal{P}$, $\mathcal{J}$ and $\mathcal{D}$ denote the set of gas supply nodes, junction nodes and gas demand nodes, respectively. Moreover, we denote by $s_i(\tilde{\mathbf{z}})$ as the uncertain gas supply at supply node $i \in \mathcal{P}$ and denote by d_i as the gas demand at demand node $i \in \mathcal{D}$. To address supply disruptions, we model the uncertain gas supply as:

$$s_i(\tilde{\mathbf{z}}) = \bar{s}_i - \hat{s}_i\tilde{z}_i : i \in \mathcal{P},$$

where $\tilde{z}_i$ is a primitive random variable with support $[0, 1]$.

Aside from the flow balance constraints, the upper and lower bound of pressure level at each node should also be enforced.

$$\underline{r}_i \leq r_i \leq \bar{r}_i \quad : i \in \mathcal{N} \tag{5.22}$$

Assumption 5.5. *In the natural gas supply network, we have $\bar{r}_i < +\infty : \forall i \in \mathcal{N}$. In addition, for any pair of nodes $i \in \mathcal{N}, j \in \delta^-(\mathcal{G}, i)$, we have $\bar{r}_i \geq \bar{r}_j$ and $\underline{r}_i \geq \underline{r}_j$. In addition, $\bigcap_{i\in\mathcal{N}} [\underline{r}_i, \bar{r}_i] \neq \emptyset$.*

Remark 5.7. Assumption 5.5 is necessary for us to specify the value of parameters $\bar{r}_i$ and $\underline{r}_i$ for all $i \in \mathcal{N}$, which is helpful for the simplification of the problem. For instance, suppose there exist a supply node “A” and a junction node “B” which are linked by a pipeline. The gas pressure at “A” actually should be lower than a certain upper bound $\bar{r}_A$

due to the limitation of the compressor at field "A". On the contrary, there is no specific requirement for the pressure level at "B", which means $\bar{r}_B = +\infty$. For any feasible flow solution with certain pressure value r_A and r_B, r_B should not be greater than r_A (otherwise the gas would goes from "B" to "A"). Therefore, we would not clear out any feasible flow solution if we set $\bar{r}_B = \bar{R}_A$ instead of $\bar{r}_B = +\infty$. In reality, the gas pressure at all production fields has a finite upper bound $\bar{r}_i < +\infty$ due to physical limitations, so we can assume all the $\bar{r}_i$ in the network is finite. The argument on the pressure lower bound $\underline{r}_i : i \in \mathcal{N}$ is similar.

In addition, the Weymouth equation that expresses the physical relationship between the gas flow amount along a pipeline and pressure levels at both sides of that pipeline should also be included. Here we use the most commonly used form of Weymouth equation as follows:

$$x_{ij} = C_{ij}\sqrt{r_i^2 - r_j^2} \triangleq W_{ij}(r_i, r_j) \quad : (i,j) \in \mathcal{A},$$

where C_{ij} is a constant depending on the physical condition of the pipeline such as temperature, pipeline length and diameter, etc; r_i and r_j denote the pressure level at node i and j, respectively. The Weymouth equation states that the natural gas would only go from one node with higher pressure to the one with lower pressure, and the corresponding flow amount is proportion to the square root of the pressure square difference. To overcome the non-linearity caused by the Weymouth equation, we approximate it by its first order Taylor expansion around several breakpoints $(r_i = RI, r_o = RO)$.

$$\begin{aligned}
W_{ij} &\simeq W_{ij}(RI, RO) + \frac{\eth W}{\eth r_i}(r_i - RI) + \frac{\eth W}{\eth r_j}(r_j - RO) \\
&= C_{ij}\sqrt{RI^2 - RO^2} + \frac{RI \times C_{ij}}{\sqrt{RI^2 - RO^2}}(r_i - RI) - \frac{RO \times C_{ij}}{\sqrt{RI^2 - RO^2}}(r_j - RO) \\
&= C_{ij}\sqrt{RI^2 - RO^2} - C_{ij}\frac{RI^2 - RO^2}{\sqrt{RI^2 - RO^2}} + C_{ij}\left(\frac{RI}{\sqrt{RI^2 - RO^2}}r_i - \frac{RO}{\sqrt{RI^2 - RO^2}}r_j\right)
\end{aligned}$$

$$= C_{ij}\left(\frac{RI}{\sqrt{RI^2 - RO^2}}r_i - \frac{RO}{\sqrt{RI^2 - RO^2}}r_j\right).$$

Combining the above Taylor approximation at different pairs of (RI, RO), we can approximate the Weymouth equation with several linear constraints. For convenience of representation, we express them as

$$x_{(i,j)} - g_{ijl}r_i + h_{ijl}r_j \leq 0 \quad : (i,j) \in \mathcal{A}, l \in \mathcal{L}, \tag{5.23}$$

where $\mathcal{L}$ is the set of breakpoints.

Remark 5.8. The coefficients in (5.23) are determined by the selected break points (RI_{ijl}, RO_{ijl}), the selection of which would of course affect the approximation accuracy of the Weymouth equation. The most straightforward way is letting the break points uniformly divide the feasible pressure interval $[\underline{r}_i, \overline{r}_i]$ and $\left[\underline{r}_j, \overline{r}_j\right]$ by setting $RI_{ijl} = \underline{r}_i + \frac{l-1}{|\mathcal{L}|-1}\left(\overline{r}_i - \underline{r}_i\right), RO_{ijl} = \underline{r}_j + \frac{l-1}{|\mathcal{L}|-1}\left(\overline{r}_j - \underline{r}_j\right) \quad l \in \mathcal{L}$.

In our tested experiments, we let the penalty function be the total demand loss and its corresponding target level is set to be zero. In summary, the minimum demand loss model for a given natural gas transmission network $\mathcal{G}$ is given as the following LP:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{D}} e_i \\ \text{s.t.} \quad & (5.21), (5.22), (5.23) \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{e} \geq \mathbf{0}. \end{aligned}$$

5.4.2 Resilient Natural Gas Network Design

When considering the design problem, we denote by A as the set of potentially installed pipelines and by N as the set of potentially installed nodes. We define design vector $\mathbf{u} \in \{0,1\}^{|A|+|N|}$, where u_i denotes the investment decision of building the corresponding pipeline for $i \in A$ and the corresponding node for $i \in N$. For modeling purpose, we denote an auxiliary network $\overline{\mathcal{G}}$ as the one with investment decision $\mathbf{u} = \mathbf{1}$. Thus, the minimum possible total demand loss of for network $\mathcal{G}(\mathbf{u})$ under gas supply realization $\mathbf{s}(\tilde{\mathbf{z}})$, denoted as $\mathcal{Q}(\mathbf{u}, \tilde{\mathbf{z}})$, can be computed by the following a deterministic optimization problem:

$$
\begin{aligned}
\mathcal{Q}(\mathbf{u}, \tilde{\mathbf{z}}) \triangleq \min \quad & \sum_{i \in \mathcal{D}} e_i \\
\text{s.t.} \quad & (5.21), (5.22), (5.23) \\
& \sum_{a \in \delta^-(\overline{\mathcal{G}}, i)} x_a \leq \overline{s}_i u_i : i \in \mathcal{P} \cap N \\
& \sum_{a \in \delta^+(\overline{\mathcal{G}}, i)} x_a \leq s u_i : i \in \mathcal{J} \cap N \\
& x_a \leq s u_a : a \in \mathcal{A} \cap A \\
& \mathbf{x} \geq \mathbf{0} \\
& \mathbf{e} \geq \mathbf{0},
\end{aligned}
\tag{5.24}
$$

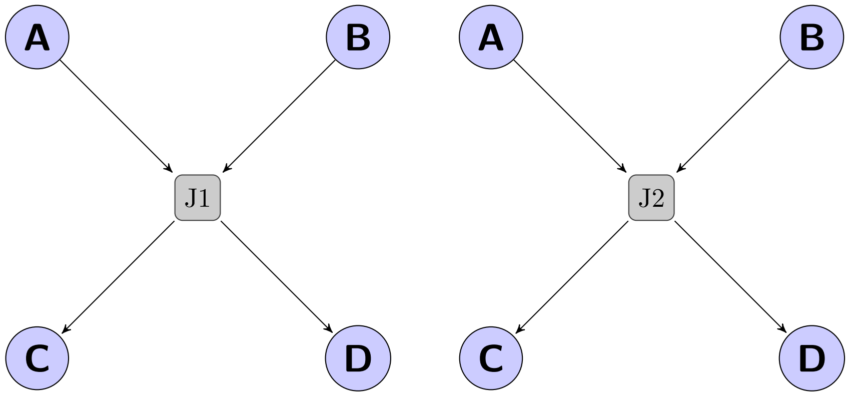
where $s = \sum_{i \in \mathcal{P}} \overline{s}_i$. By abbreviating the three added constraints into vector form as $\mathbf{Fx} \leq \mathbf{Uu}$, formulation (5.24) gives exactly an investment decision independent linear formulation as what we described in Assumption 5.3. If the selected resilience index is built by adjustable uncertainty set (5.8) or (5.9), we therefore can resolve the design problem with respect to their investment decision independent worst case sets.

5.4.3 Experiment Settings and Numerical Results

Resilience index & system performance

We now investigate the relevance of our proposed supply resilience index and the energy supply system's stochastic system performance under supply disruptions. For illustration, we give a gas network with two supply nodes A and B, two demand nodes C and D. The gas supply at node A is $s_A = 75 - 30\tilde{z}_1$, and supply at node B is $s_B = 60 - 20\tilde{z}_2$, where $\tilde{z}_1, \tilde{z}_2$ are random variables fluctuate in $[0, 1]$. The gas demand at node C and D is $d_C = d_D = 50$. Two network designs (a) and (b), given in Figure 5.3, are considered. In design (a) all the supply nodes and demand nodes are linked by a junction node J1, while

Figure 5.3: Example of a gas network consisting of two supply nodes and two demand nodes



(a) (b)

design (b) uses a different junction node J2. The corresponding experiment parameters are presented in Table 5.1. The difference of these two designs in the mathematical model is only reflected by the corresponding Weymouth constants. As we can see in the table, the Weymouth constants from node A and B to J1 are higher than that from node A and B to J2. Differently, the constants of the pipelines linking J1 to node C and node D is lower than that of J2. The weymouth constant is affected by the pipeline diameter, distance, and some other factors (like the external temperature). The differences in the weymouth is usually caused by every aspect of the transmission network, such as the junction node location, and the investment effort on every pipeline. A simple physical interpretation of the differences of these two designs might be that the junction node J1 is closer to the supply nodes A and B whereas node J2 is closer to the demand side. Due to the limitation that we can not enumerate all the possible instances, we use this case for illustration only (we also test some other parameter settings, the results are consistent with this illustrative example).

Even though the size of the network is small, comparing their system performances against supply disruptions is not a trivial task. In this experiment, 100 pairs of pressure

Table 5.1: Design parameters: "IP" denotes inlet pressure level, "OP" denotes outlet pressure level and C_a denotes the corresponding Weymouth Constant.

	Max IP	Min IP	Max OP	Min OP	C_a
A to J1	200	180	160	140	0.62
B to J1	180	160	160	140	0.52
J1 to C	160	140	120	100	0.41
J1 to D	160	140	120	100	0.41
A to J2	200	180	160	140	0.46
B to J2	180	160	160	140	0.54
J2 to C	160	140	120	100	0.45
J2 to D	160	140	120	100	0.45

break points at each arc are used for accurate approximation. Based on this, the computed cardinality uncertainty set based resilience indexes of design (a) and design (b) are 0.73 and 0.97, respectively. Therefore, design (b) is preferred with consideration of resilience related issues.

The stochastic system performance suggests this resilience index based comparison. To evaluate the system's stochastic performance, we draw the system performance profile using Monte Carlo simulation via a sample of 10,000 supply scenarios $\tilde{\mathbf{z}}^i$. For all tested experiments, if not specified, two sample distributions are tested to investigate the stochastic system performance: uniform distribution $U(0,1)$ and normal distribution $N(0.5, 0.2^2)$ (the standard deviation 0.2 ensures that $\tilde{z}$ lies in $[0,1]$). We list the mean, standard deviation (StDev), success probability (Suc Prob), 95% VaR, 95% CVaR, 90% VaR, 90% CVaR and 95% CVaR of the total demand loss in Table 5.2. Let TDL be the total demand loss, then "Suc Prob" gives $\mathbb{P}(\mathrm{TDL} \leq 0)$, α-VaR gives $\inf\{t : \mathbb{P}(\mathrm{TDL} > t) \leq 1-\alpha\}$, and α-CVaR gives $\inf_t \left\{ t + \frac{\mathbb{E}\left(\mathrm{TDL} - t\right)^+}{1-\alpha} \right\}$ for every $\alpha \in (0,1)$. As we can see, the system performance of design (b) dominates design (a) for all of these statistics, especially for the success probability (more than 10% higher than (a)). The results are consistent for other tested distributions and experiment parameters, the difference in stochastic system performance is significant when the difference in resilience index is larger than 0.15.

Table 5.2: Summary statistics for design (a) and design (b)

	Normal data		Uniform data	
	Design (a)	Design (b)	Design (a)	Design (b)
Mean	0.4757	0.2776	1.3618	0.9377
StDev	1.3627	1.2018	2.5706	2.4591
Suc Prob	78.95%	91.31 %	65.43%	80.34%
90% VaR	1.7104	0.0000	4.8991	4.0911
95% VaR	3.2187	1.9941	7.1273	7.1233
90% CVaR	3.9108	2.7728	7.7167	7.6123
95% CVaR	5.4025	4.8304	9.7166	9.7114

Computational performance of "gradient-based" search method

We now study the practical performance of the "gradient-based" search algorithm. For each individual experiment setting, we test the computational time of sub-problem (5.6) and the number of additional iterations for algorithm termination, provided with $\lfloor \varrho(\mathcal{G}) \rfloor$. Specifically, we consider a natural gas network with m supply nodes, one junction node and m demand nodes, where m varies from 5 to 15. For each m, we randomly generate the problem parameters. More precisely, we generate $\overline{s}_i$ using a uniform distribution on the interval $[60, 80]$, $\hat{s}_i$ on $[0.3\overline{s}_i, 0.5\overline{s}_i]$. Similarly, we randomly generate d_i on interval $[45, 60]$, the Weymouth constant on $[0.55, 0.75]$, pressure bounds on $[180, 200]$ [bar], $[140, 160]$ and $[100, 120]$ for gas supply nodes, junction nodes and gas demand nodes, respectively. The linear model is built using 25 pairs of equally broken pressure breakpoints for the Weymouth equation. With these settings, we can ensure that the nominal operational problem (the gas supply takes $\overline{s}_i$ at each supply node $i \in \mathcal{P}$) for each randomly generated experiments is feasible.

To compute the supply resilience index, we need to solve a sequence of sub-problems (5.6), by either the MIP model (5.10) or (5.11). Intuitively, (5.6) is the most difficult when $\gamma = \lfloor \frac{m}{2} \rfloor + \frac{1}{2}$. Actually, it takes less than 10 seconds to solve an instance of (5.6) for $m = 15, \gamma = 7.5$. Therefore, we can roughly give an upper bound of the total time required to compute the resilience index when $m = 15$ as $(\lceil \log_2 15 \rceil + \delta) * 10$ seconds, where δ is the number of additional iterations required after $\lfloor \varrho(\mathcal{G}) \rfloor$ is given. If δ is not

Table 5.3: Running time and the number of iterations required when applying Algorithm 2, 50 instances are tested for the evaluation of mean and standard deviation.

	No. of iterations		Computational Time (sec)	
m	Mean	StDev	Mean	StDev
5	1.14	0.3505	0.1262	0.0510
7	1.22	0.4647	0.1827	0.1146
9	1.10	0.3030	0.2980	0.1238
11	1.32	0.5127	0.7592	0.4466
13	1.44	0.7329	1.7377	1.1118
15	1.46	0.6131	6.3008	3.0944

too large, our proposed method of evaluating the resilience index can easily handle moderate problems in practice (The natural gas network example in De Wolf and Smeers (2000) has 20 nodes and 24 arcs, which has a smaller problem size than our example of $m = 15$).

We next investigate the value of δ for various m. For example, we randomly conduct 15 instances of experiment for the case that $m = 10$. Among all these 15 instances, $\delta = 1$ for 3 instances, $\delta = 1$ for 10 instances and $\delta = 0$ for 2 instances (the zero value takes when the network is fully resilient for the worst case). Table 5.3 presents the statistics of the running time and δ estimated by 50 replications (here the running time only includes the computational time of applying the "gradient-based" algorithm after $\lfloor \varrho(\mathcal{G}) \rfloor$ is given). The result shows that there is no significant increasing trend in the number of iterations when m increases, i.e., we can believe that δ tends to be bounded by some constant, which suggests the practical efficiency of our proposed method of computing $\varrho(\mathcal{G})$ defined by cardinality uncertainty set (5.9).

Example of a design problem

We now compare design optimization problem using our proposed supply resilience index with two benchmark approaches: the nominal value based approach and the stochastic programming approach. The nominal value approach is based on the deterministic design and procurement problem in literature which minimizes the total cost (investment cost plus the procurement cost) under the assumption that the gas supply equals its nominal value $\overline{s}_i : \forall i \in \mathcal{P}$. In other words, we solve the following problem:

$$
\begin{aligned}
&\min \sum_{i\in\mathcal{P}} c_i(\overline{s}_i - \hat{e}_i) + \mathbf{q}'\mathbf{u} \\
&\text{s.t.} (5.22), (5.23) \\
&\quad \mathbf{F}\mathbf{x} \le \mathbf{U}\mathbf{u} \\
&\quad \mathbf{q}'\mathbf{u} \le B \\
&\quad \sum_{a\in\delta^+(\overline{\mathcal{G}},i)} x(a) - \sum_{a\in\delta^-(\overline{\mathcal{G}},i)} x(a) \begin{cases} = -\overline{s}_i + \hat{e}_i : i \in \mathcal{P} \\ = 0 : i \in \mathcal{J} \\ \ge d_i : i \in \mathcal{D} \end{cases} \\
&\quad \mathbf{x} \ge \mathbf{0}, \hat{\mathbf{e}} \ge \mathbf{0}.
\end{aligned}
$$

In the above model, c_i denotes the unit gas price (or drilling cost) at potential gas supply node $i \in \mathcal{P}$, B denotes the total available amount of investment budget and $\mathbf{q}'\mathbf{u}$ gives the total investment cost associated with investment decision $\mathbf{u}$. We carry out the nominal value approach for comparison because it is commonly used in the natural gas literature. To benchmark the performance of our proposed resilience optimization design model, we also use a stochastic programming approach to maximize the success probability of achieving zero penalty positions via sample average approximations (SAA). More precisely, when K generalized samples of uncertainty $\tilde{\mathbf{z}}^j : j = 1, \cdots, N$ is given, the SAA approach solves the following optimization problem:

$$
\begin{aligned}
\max_{\mathbf{u}, p_j} \quad & \frac{1}{N} p_j \\
\text{s.t.} \quad & \mathcal{Q}(\mathbf{u}, \tilde{\mathbf{z}}^j) \le M(1 - p_j) : j = 1, \cdots, N \\
& \mathbf{q}'\mathbf{u} \le B,
\end{aligned}
$$

where M is a constant which is sufficiently large so that the constraint $\mathcal{Q}(\mathbf{u}, \tilde{\mathbf{z}}^j) \leq M(1 - p_j)$ would force $p_j = 0$ when $\mathcal{Q}(\mathbf{u}, \tilde{\mathbf{z}}^j) > 0$. Thus, the proposed SAA method aims to find a design maximizing the estimated success probability.

We now turn to the problem of designing a natural gas network with 5 potential supply nodes, 2 potential junction nodes, and 5 given gas demand nodes. The experiment data is randomly generated to avoid uniqueness. In particular, $\overline{s}_i$ is independently and randomly generated by uniform distribution on $[60, 100]$, $\hat{s}_i$ on $[0.25\overline{s}_i, 0.35\overline{s}_i]$, d_i on $[45, 55]$, C_{ij} on $[0.4, 0.8]$, pressure bounds $\underline{r}_i$ and $\overline{r}_i$ on $[180, 200]$[bar], $[140, 160]$[bar] and $[100, 120]$[bar] for gas supply nodes, junction nodes and gas demand nodes, respectively. Installation cost is randomly generated on $[35, 45]$, $[20, 30]$ and $[10, 15]$ for gas supply nodes junction nodes and pipeline, respectively. Finally, to apply the nominal value design approach, we generate the unit gas price c_i on $[0.2, 0.6]$ by the same randomness rule. Moreover, we test the experiment on various investment budget B in $[\underline{B}, \overline{B}]$. The lower bound $\underline{B}$ denotes the minimum budget required to make the design and procurement problem feasible, which means the comparison among different approaches is unnecessary when the available budget $B < \underline{B}$ because lacking of budget. The upper bound $\bar{B}$ denotes the minimum investment budget required when the gas supply reaches its lowest possible value $\overline{s}_i - \hat{s}_i : i \in \mathcal{P}$, which means any comparison when $B \geq \overline{B}$ is unnecessary because we can design a system that is fully resilient against the worst supply case.

For our resilience index approach, we test it for both the box uncertainty set (5.8) (where $\tilde{z}_i(\gamma) := \gamma$) and cardinality uncertainty set (5.9). Generally, the stochastic programming approach is expected to outperform the resilience index optimization when the number of samples is sufficiently large because it uses more distributional information. However, due to the curse of dimensionality, the computational time of the SAA model would exponentially increase with respect to the used sample size. To make the comparison fair, we test the SAA with 50 samples under exact distribution so that both the SAA model and our proposed resilience index model can solve a design instance within several minutes.

Table 5.4 and 5.5 summarize the statistics of the stochastic system performances of these methods, for one specific randomly generated test experiment. All these statistics are estimated from a randomly generalized sample of 10,000 realizations. In these two tables, “SRI-B” denotes the supply resilience index using the box uncertainty set, “SRI-C” denotes the supply resilience index using the cardinality uncertainty set, “SAA-U” denotes the SAA method using uniform randomized data, and “NV” is short for the nominal value approach of the design problem. It can be observed that the nominal value approach performs worst among all these methods, and the improvement of system performance by increasing the investment budget is negligible. Conversely, both the supply resilience index approach and the SAA approach can attain significant system performance improvement upon additional investment budget. Not surprisingly, the SAA method also performs well except the extreme case of $B = \underline{B}$. More precisely, its average performance is quite close to the resilience index optimization by box uncertainty set in terms of these statistics. This indicates that our proposed resilience index based approach is as comparable as the SAA method under exact distribution. However, in the case that the exact distribution is unknown, it is not difficult to imagine that our approach would be a better choice because the performance of the SAA method highly depends on the quality of generated samples. The computational result in Chen and Sim (2009) shows that the advantage of SAA method would disappear when a wrong distribution is assumed.

It is worth noting that, for low investment budget $(B - \underline{B})/(\overline{B} - \underline{B}) < 0.6$, the box uncertainty set based resilience index lead us to solutions providing better system performance than the cardinality uncertainty based one. The opposite is observed for large investment budget $(B - \underline{B})/(\overline{B} - \underline{B}) > 0.6$. In this experiment, the overall performance of the box uncertainty set based resilience index is generally better than the cardinality set based one. A possible explanation is that the exact support information is known and the two tested distributions are quite consistent with the assumption of the box uncertainty set. In the case that only limited data is available to specify the support set and the underlying distribution of uncertainty is different, the cardinality uncertainty might be a better choice.

We also observe sudden changes in system performance of the "SRI-B" and "SRI-C" when we increase the investment budget (the break point is 0.4 for "SRI-B" and 0.6 for "SRI-C"). This phenomenon can be explained through the discrete system cost structure. In particular, the investment cost would suddenly change if we build an additional new node or pipeline. When the total available amount of investment budget is below some break point, we are not capable of installing all the critical nodes and arcs. As a consequence, the system performance would be poor due to lack of gas supply or pipeline capacity. In contrast, when the budget exceeds the breakpoint, all the critical installations can be made and thus the system would be capable of most uncertainty realizations, which results in a sudden change in system performance. Indeed, the main change in the investment decisions around the breakpoints is the installation of a new gas supply node, which would significantly improve the gas supply amount.

This result is also suggested in Table 5.6, in which the real value of supply resilience index of design solutions obtained by different methods (both the resilience index-cardinality and resilience index-box are computed). As we can see, a sudden change in system resilience index for both method is also observed at break points 0.4 and 0.6, respectively. In the table, the ratio $r = (B - \underline{B})/(\overline{B} - \underline{B})$ is the normalized budget level. "SAA-U" and "SAA-N" correspond to SAA method using uniformly and normally distributed data, respectively. "SRI-B" and "SRI-C" denote the resilience index optimization defined by box uncertainty set and the cardinality uncertainty set, respectively. Not surprisingly, the resilience optimization model that optimizes the correct resilience index dominates the others for every B. When the budget amount B goes close to $\bar{B}$, the specific choice of resilience index does not affect the result significantly. For example, if we choose to maximize the resilience index by box uncertainty set, the derived optimal solution also has a high valued cardinality uncertainty set based resilience index (close to the optimal one) when B is close to $\bar{B}$. In this specific case, $\overline{B} = 1.286\underline{B}$, which means the system can reach full resilience by only additional more 28.65% of the least required budget $\underline{B}$. If we look back to the stochastic system performance, approximately 15% additional investment ($(B-\underline{B})/(\overline{B}-\underline{B})=0.6$) can help us find a design that has close stochastic system performance as the fully resilient one. Apparently, the trend in supply resilience index with respect to the investment budget varies from case to case. For each individual case, by querying the curve of resilience index on investment budget, the decision maker can decide the amount of the investment budget B accordingly.

5.5 Conclusion

By relating to resilience measures of the uncertain demand loss penalty function, we give a formal definition of supply resilience index to measure energy supply system's resilience against supply disruptions in this chapter. Based on that, a general algorithm of computing resilience index defined by general adjustable uncertainty sets is proposed. Besides, to circumvent the computational difficulty resulting from exponentially large number of extreme points, we consider two special families of adjustable uncertainty sets and develop specific algorithms to evaluate the corresponding resilience indexes. We also investigate the problem of design a resilient supply system, in which the investment decisions must be made before the realization of any uncertainty. As the general design problem is computationally difficult, we propose a solution procedure by making two implicit assumptions.

The computational experiments on the natural gas transmission network illustrate the advantage of the resilience index. In the first computational experiment on a simple natural gas network, we explore that the resilience index is highly relevant to the stochastic system performances. Besides, the second experiment illustrates that our proposed "gradient-based" search method is practically efficient for moderate sized problems, despite the NP-Hardness of the general problem. The last experiment shows the superiority of our proposed method to the traditional approach.

Table 5.4: Summary statistics system performance under normal distributed data.

Statistic	Method	$(B-\underline{B})/(\overline{B}-\underline{B})$										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Mean	NV	33.5495	29.4358	29.3132	29.1293	29.4875	29.5264	29.4915	29.4426	29.3616	29.3532	29.4613
	SAA-N	246.3670	20.6772	27.9762	19.4044	0.1210	0.1230	1.0753	0.0025	0.1334	0.0220	0.0177
	SRI-B	33.5495	20.6099	20.6288	19.3854	0.1195	0.1239	0.0779	0.0071	0.0078	0.0021	0.0000
	SRI-C	33.5495	20.6099	27.9783	19.3854	19.5162	19.6776	0.0779	0.0025	0.0051	0.0000	0.0000
Suc Prob	NV	0.0000	0.0036	0.0043	0.0040	0.0040	0.0043	0.0041	0.0054	0.0043	0.0044	0.0038
	SAA-N	0.0000	0.0313	0.0142	0.0784	0.9146	0.9165	0.8104	0.9991	0.9140	0.9951	0.9959
	SRI-B	0.0000	0.0447	0.0411	0.0784	0.9151	0.9166	0.9714	0.9972	0.9981	0.9994	1.0000
	SRI-C	0.0000	0.0447	0.0140	0.0784	0.0753	0.0769	0.9714	0.9991	0.9992	0.9999	1.0000
VaR0.9	NV	47.6228	44.5042	44.7276	44.1668	44.5105	44.7584	44.7457	44.6152	44.5839	44.5580	44.6472
	SAA-N	246.3670	36.4350	44.4638	35.8105	0.0000	0.0000	4.1685	0.0000	0.0000	0.0000	0.0000
	SRI-B	47.6228	36.4641	36.5341	35.7807	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	SRI-C	47.6228	36.4641	44.5128	35.7807	36.2120	36.4180	0.0000	0.0000	0.0000	0.0000	0.0000
VaR0.95	NV	51.9562	49.3159	49.4508	48.8272	49.1199	49.6695	49.0993	49.2031	49.0127	49.2239	49.1253
	SAA-N	246.3670	41.1274	49.4041	40.6376	0.6962	0.7080	8.0333	0.0000	0.7827	0.0000	0.0000
	SRI-B	51.9562	41.1362	41.4477	40.6376	0.6820	0.6859	0.0000	0.0000	0.0000	0.0000	0.0000
	SRI-C	51.9562	41.1362	49.4314	40.6376	40.8279	40.9187	0.0000	0.0000	0.0000	0.0000	0.0000
CVaR0.9	NV	53.3267	50.4733	50.7424	50.3686	50.5170	50.8671	50.4694	50.6903	50.5925	50.5111	50.4604
	SAA-N	246.3670	42.4525	50.6373	42.3459	1.2092	1.2289	9.0304	0.0245	1.3330	0.2201	0.1768
	SRI-B	53.3267	42.4608	42.8999	42.3214	1.1942	1.2380	0.7778	0.0713	0.0774	0.0210	0.0000
	SRI-C	53.3267	42.4608	50.6609	42.3214	42.2932	42.3511	0.7778	0.0245	0.0513	0.0006	0.0000
CVaR0.95	NV	57.0485	54.2218	54.6219	54.4012	54.4204	54.6741	54.1173	54.6890	54.5360	54.3437	54.2799
	SAA-N	246.3670	46.2893	54.5753	46.6300	2.1938	2.2456	12.2035	0.0490	2.4152	0.4397	0.3532
	SRI-B	57.0485	46.2875	47.0527	46.6132	2.1721	2.2662	1.5540	0.1424	0.1547	0.0419	0.0000
	SRI-C	57.0485	46.2875	54.5930	46.6132	46.3629	46.1243	1.5540	0.0490	0.1025	0.0012	0.0000

Table 5.5: Summary statistics system performance under uniform distributed data.

Statistic	Method	$(B-\underline{B})/(\overline{B}-\underline{B})$										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Mean	NV	34.9533	30.9619	31.0632	30.9235	31.1094	30.9806	31.0932	30.8483	30.7303	31.1234	31.0341
	SAA-U	246.3670	23.1788	28.4715	20.5469	0.6080	0.6452	0.5109	0.0750	0.0644	0.0338	0.2498
	SRI-B	34.9533	23.1561	22.9908	20.5348	0.6097	0.6449	0.5059	0.1023	0.0487	0.0515	0.0000
	SRI-C	34.9533	23.1561	28.4764	20.5348	20.6379	20.4900	0.5059	0.0751	0.0324	0.0319	0.0000
Suc Prob	NV	0.0009	0.0267	0.0241	0.0250	0.0268	0.0283	0.0282	0.0261	0.0264	0.0277	0.0287
	SAA-U	0.0000	0.1044	0.0631	0.1716	0.7432	0.7453	0.8867	0.9857	0.9850	0.9934	0.9578
	SRI-B	0.0009	0.1045	0.0974	0.1720	0.7439	0.7448	0.8885	0.9728	0.9849	0.9856	1.0000
	SRI-C	0.0009	0.1045	0.0653	0.1720	0.1625	0.1741	0.8885	0.9859	0.9926	0.9936	1.0000
VaR0.9	NV	54.7014	52.6400	52.6513	52.4521	52.7008	53.1560	52.8347	52.3468	52.3360	52.5402	52.4996
	SAA-U	246.3670	45.1970	51.9934	44.4202	2.1401	2.1893	0.6013	0.0000	0.0000	0.0000	0.0000
	SRI-B	54.7014	45.1527	45.1571	44.3299	2.1519	2.1951	0.5366	0.0000	0.0000	0.0000	0.0000
	SRI-C	54.7014	45.1527	52.0097	44.3299	43.9422	44.2735	0.5366	0.0000	0.0000	0.0000	0.0000
VaR0.95	NV	60.4958	58.7557	58.2639	58.3863	58.3405	58.8227	58.8229	57.7281	57.6602	58.3489	58.2003
	SAA-U	246.3670	51.1368	58.2159	50.7031	3.7806	4.1504	3.6712	0.0000	0.0000	0.0000	0.0000
	SRI-B	60.4958	51.0442	51.2535	50.6122	3.8025	4.1655	3.6651	0.0000	0.0000	0.0000	0.0000
	SRI-C	60.4958	51.0442	58.2639	50.6122	50.8417	50.8855	3.6651	0.0000	0.0000	0.0000	0.0000
CVaR0.9	NV	61.7134	60.1001	59.9109	59.8113	59.7097	60.2287	60.2118	59.6461	59.2969	59.7401	59.8439
	SAA-U	246.3670	52.8495	59.7318	52.3173	4.8243	5.1472	5.0627	0.7490	0.6436	0.3381	2.4954
	SRI-B	61.7134	52.7957	53.0031	52.2360	4.8413	5.1412	5.0220	1.0217	0.4868	0.5144	0.0000
	SRI-C	61.7134	52.7957	59.7923	52.2360	52.1326	52.2887	5.0220	0.7507	0.3233	0.3182	0.0000
CVaR0.95	NV	66.1629	64.8282	64.6227	64.5672	64.1441	64.6302	64.8467	64.4760	63.7945	64.2536	64.6153
	SAA-U	246.3670	57.7968	64.5556	57.2741	6.7823	7.2369	8.1212	1.4965	1.2859	0.6756	4.9858
	SRI-B	66.1629	57.7470	57.9689	57.1967	6.8065	7.2249	8.0920	2.0413	0.9725	1.0277	0.0000
	SRI-C	66.1629	57.7470	64.6227	57.1967	57.2051	57.2474	8.0920	1.4999	0.6460	0.6358	0.0000

Table 5.6: Supply resilience index under different B.

r	Resilience index (Box)				
	NV	SAA-N	SAA-U	SRI-B	SRI-C
0.0	0.0278	0.0005	0.0005	0.0278	0.0239
0.1	0.1685	0.3042	0.3091	0.3091	0.3081
0.2	0.1685	0.2476	0.2476	0.3081	0.2476
0.3	0.1685	0.3354	0.3354	0.3354	0.3354
0.4	0.1685	0.7319	0.7319	0.7280	0.3354
0.5	0.1685	0.7319	0.7319	0.7280	0.3354
0.6	0.1685	0.6353	0.7798	0.7798	0.7798
0.7	0.1685	0.8286	0.8286	0.8286	0.8286
0.8	0.1685	0.7319	0.8286	0.8374	0.8384
0.9	0.1685	0.7749	0.8384	0.8374	0.8384
1.0	0.1685	0.7749	0.7749	1.0000	1.0000
r	Resilience index (Cardinality)				
	NV	SAA-N	SAA-U	SRI-B	SRI-C
0.0	0.0235	0.0003	0.0003	0.0235	0.0244
0.1	0.3140	0.2994	0.4886	0.4886	0.4932
0.2	0.3140	0.6772	0.4898	0.4886	0.6787
0.3	0.3140	0.9616	0.9616	0.9616	0.9619
0.4	0.3140	0.8731	0.8731	0.8731	0.9619
0.5	0.3140	0.8731	0.8731	0.8731	0.9619
0.6	0.3140	0.9647	1.5610	1.5576	1.5610
0.7	0.3140	2.6181	2.6181	1.8625	2.6221
0.8	0.3140	0.8731	2.6181	2.5449	3.2666
0.9	0.3140	2.3721	3.2681	2.5449	3.2666
1.0	0.3140	2.7640	2.3721	5.0000	5.0000

Chapter 6

Telecommunication Network Resilience under Ambiguous Demands

As we reviewed in Chapter 2, the telecommunication network design literature tends to address demand uncertainty via robust models. A drawback of the robust model is its conservatism because it totally ignores the distributional knowledge of uncertainty, which can be obtained from substantial historical data. In addition, the robust models in the cited works implicitly do not allow demand shortages to happen. In practice however, since large demand peaks can occur with non-zero probability, some demand shortfalls are expected (and do happen), and may be tolerable if below a certain level. In this chapter, we apply our proposed framework of resilience measure as a quantitative way of measuring telecommunication network's service quality to address these issues. To achieve this, we introduce a penalty function that captures the demand shortage penalties, and relate the network resilience to the uncertain penalty functions using the popular *distributionally-ambiguous shortfall awareness measure*. To this end, we propose a capacity design model that optimizes the telecommunication network resilience, with respect to an investment budget. An important novelty of the model is the use of a family of routing decision rules we term as the *improved affine decision rules* that generalizes those in the recent literature. We develop tractable approximations of this problem based on linear matrix inequalities, and show rigorously how the proposed approximation is less conservative than existing solutions. finally, computational studies are performed to demonstrate the effectiveness of our proposed model compared to some existing approaches.

The rest of this chapter is organized as follows. Section 6.1 introduces the path-flow based formulation of the routing problem along with the second stage problem of minimizing total penalty cost, which is introduced in Babonneau et al. (2009); Ouorou and Vial (2007) as an alternative of the network flow formulation. In Section 6.2, we motivate the resilience index via distributionally-ambiguous shortfall awareness measure and formulate the resilient telecommunication network design model. In Section 6.3, we build tractable approximations of the proposed model via decision rules. Section 6.4 describes an alternative reformulation of the resilience index based on which the resilient telecommunication network design model can be approximately solved by a single optimization model. Section 6.5 reports the computational results and Section 6.6 concludes this chapter.

6.1 Path Flow Formulation of Telecommunication Network

We consider a telecommunication network described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ in which pairwise demand occurs. In other words, each demand commodity $k \in \mathcal{K}$ is associated with a pair of nodes: source and destination nodes. In most practical situations, the demand vector $\mathbf{d} = (d_k)_{k\in\mathcal{K}}$ changes over time and is thus uncertain (denoted as $\tilde{\mathbf{d}}$). To avoid the conservatism of the robust optimization approach and address distributional ambiguity at the same time, we assume that $\tilde{\mathbf{d}}$ fluctuates in its support $\mathcal{D}$ and its underlying distribution $\mathbb{P}$ belongs to the distribution family

$$\mathbb{F} = \left\{ \mathbb{P} \in \mathbb{M}(\mathcal{D}) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}} \left(\tilde{\mathbf{d}} \tilde{\mathbf{d}}' \right) = \boldsymbol{\mu} \boldsymbol{\mu}' + \boldsymbol{\Sigma} \right\}, \tag{6.1}$$

where $\mathbb{M}(\mathcal{D})$ is the set of finite Borel measures supported by $\mathcal{D}$. Let $K = |\mathcal{K}|$ and $A = |\mathcal{A}|$. For notational convenience, we sometimes replace $\mathcal{K}$ or $\mathcal{A}$ by $[K]$ or $[A]$ when they are referred as index sets of vector or matrix components. Without loss of much generality, we further assume that $\mathcal{D} \subseteq \Re_+^K$ is closed and bounded. This assumption is based on a reasonable guess that the real demand amount $\tilde{d}_k$ of commodity k should be bounded below some finite value.

Remark 6.1. The mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ of the uncertain demand vector $\tilde{\mathbf{d}}$ are assumed to be known here. This assumption is consistent with the real cases because we can estimate them in high accuracy by large amount of historical data.

Remark 6.2. Indeed, the general problem of determining whether there exists a distribution $\mathbb{P}$ that is consistent with the support set $\mathcal{D}$ along with the first- and second-order moments $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is difficult. However, this problem is beyond the scope of this study and we only simply assume that $\mathbb{F}$ is not empty and the provided moment information $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ do not further restrict the support set $\mathcal{D}$. In other words, for every $\mathbf{d} \in \mathcal{D}$, we can find $\mathbb{P} \in \mathbb{F}$ such that $\mathbb{P}(\tilde{\mathbf{d}} = \mathbf{d}) > 0$. By this assumption, we can express the distributionally ambiguous inequality $\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{P}(w_0 + \mathbf{w}'\tilde{\mathbf{d}} \geq 0) = 1$ as a robust form $w_0 + \mathbf{w}'\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}$.

For simplicity of representation, we formulate the telecommunication network design problem in a path flow based pattern. Specifically, we assume that, for each commodity $k \in \mathcal{K}$, the set of possible paths going from source node to destination node is explicitly known as $\mathcal{P}(k)$. Let N be the cardinality of all these commodity-path pairs. We term a *path flow* routing $\mathbf{z} \in \Re_+^N$ of the graph $\mathcal{G}$ as a routing solution sending z_{kp} amount of commodity $k \in \mathcal{K}$ along the path $p \in \mathcal{P}_k$. Indeed, two sets of important decisions are involved in telecommunication network design. At the very beginning of the planning stage, we have to determine the capacity of every arc $a \in \mathcal{A}$. Here we consider continuous capacity installation for model simplicity. We denote by $\mathbf{x} \in \Re_+^A$ as the capacity installation. Let $\mathcal{Z}_R(\mathbf{x})$ be the set of path flows respecting the capacity installation $\mathbf{x}$ such that the corresponding total flow amount passing through each arc a does not exceed its capacity x_a. Specifically, $\mathcal{Z}_R(\mathbf{x})$ can be expressed as:

$$\mathcal{Z}_R(\mathbf{x}) = \left\{ \mathbf{z} \in \Re_+^N \,\middle|\, \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}_k} \delta_{pa} z_{kp} \le x_a : a \in \mathcal{A} \right\},$$

where δ_{pa} equals 1 if arc a is involved in path p and 0 otherwise. Once the capacity installation decision is made, it is not allowed to be changed during the operation. After the uncertain demand vector $\tilde{\mathbf{d}}$ is realized, we have to make the recourse decisions of routing the flows to match the demand. Let $\mathcal{Z}_S(\tilde{\mathbf{d}})$ be the set of path flows supporting demand realization $\tilde{\mathbf{d}}$, which can also be expressed as:

$$\mathcal{Z}_S(\tilde{\mathbf{d}}) = \left\{ \mathbf{z} \in \Re_+^N \,\middle|\, \sum_{p \in \mathcal{P}_k} z_{kp} \ge \tilde{d}_k : k \in \mathcal{K} \right\}.$$

Therefore, the routing decision is actually finding a feasible path flow $\mathbf{z} \in \mathcal{Z}_R(\mathbf{x}) \cap \mathcal{Z}_S(\tilde{\mathbf{d}})$.

Remark 6.3. The justification on using the path-flow formulation instead of the standard multi-commodity network flow formulation is as follows. First, the path-flow formulation is more adaptable to complicated decision rules such as affine decision rule and improved affine decision rule (see Section 6.3). On the other hand, if we extend the multi-commodity network flow model with affine decision rule, the resulting counterpart of applying affine decision rule becomes much more complicated due to the large number of involved robust constraints. Secondly, actual telecommunication networks in practice are quite sparse and its dimension is of reasonable size (with tens of nodes and tens of arcs). Thus, the specification of the path set $\mathcal{P}_k$ can be enumerated within reasonable time because the number of available paths from each source node to its destination node is small.

It is worth stressing that, for some $\mathbf{x}$ and $\tilde{\mathbf{d}} \in \mathcal{D}$, we cannot find a feasible capacity solution $\mathbf{z}$ simultaneously respects $\mathbf{x}$ and supports $\tilde{\mathbf{d}}$. This is especially true when the original investment budget is not enough and/or realized demand amount is very high. In these cases, demand shortage occurs. Failing to accounting for the demand shortage in the proposed model would affect the its validation.

In the literature, researchers choose to skip this issue by considering smaller uncertainty sets, which are also named as budget uncertainty sets $\mathcal{D}(\gamma)$. However, their robust approaches face some crucial drawbacks. Firstly, the robust design is sensitive to the selected uncertainty sets $\mathcal{D}(\gamma)$ but there is no universal guideline of specifying them (The $\Gamma-$model requires the uncertainty be symmetrically distributed, which might be inconsistent with real situations). Secondly, in telecommunication network design problem, vast historical data are available to support the design decisions, which means that it is possible for the decision maker to access certain distributional properties of the uncertain demands. Thus, the robust approach may give overly conservative design solutions because it ignores the stochastic nature of the uncertainty. Also, as previously stated, they do not accept any demand shortages.

To address these issues, we relax the supporting constraint by introducing shortage variable s_k. Also, we use $p_k(s_k)$ to denote the penalty cost of s_k demand loss of commodity k. Here $p_k(\cdot) : k \in \mathcal{K}$ is a continuous function possessing the following properties:

1. $s \leq 0 \Rightarrow p_k(s) = 0 : k \in \mathcal{K}$.

2. $p_k(\cdot) : k \in \mathcal{K}$ is nondecreasing and convex in $[0, \infty)$.

A natural choice is to let the penalty functions be nondecreasing piecewise-affine functions, which arise naturally under practical situations. Due to the fact that the piecewise-affine functions are most commonly used to approximate general convex functions, we can also treat them as approximations of the penalty functions. Specifically, we express them as:

$$p_k(s) = \max_{m \in [M]} \{p_{km} s + q_{km}\},$$

where $[M]$ denote the index set $\{1, \cdots, M\}$ of the function pieces, and $\max_{m \in [M]}\{q_{km}\} \geq 0$. We further assume that $0 < p_{k1} < \cdots < p_{kM}$ without loss of generality.

Consequently, we can formulate the recourse problem of finding the optimal path flow that minimizes the total penalty cost as follows:

$$\begin{aligned} \mathcal{Q}\left(\mathbf{x}, \tilde{\mathbf{d}}\right) \triangleq \ & \min \sum_{k \in \mathcal{K}} p_k(s_k) \\ \text{s.t.} \ & \sum_{p \in \mathcal{P}_k} z_{kp} + s_k \geq \tilde{d}_k : k \in \mathcal{K} \\ & \mathbf{z} \in \mathcal{Z}_R(\mathbf{x}), \mathbf{s} \in \Re^K_+. \end{aligned} \tag{6.2}$$

It is worth stressing that the flow routing cost is not involved in the cost function. This is because firstly, our main interest relies on modeling the impact of demand shortage so that only the demand shortage cost should be involved. Secondly, due to the nature of telecommunication network, the overall routing cost is insensitive to the routing solutions and we thenceforth can regard it as a constant.

6.2 Telecommunication Network Resilience by Distributionally-ambiguous Shortfall Awareness Measure

The distributionally-ambiguous shortfall awareness measure $\rho_{\text{DSAM}}(\cdot, \mathbb{F})$ given in Chapter 3 is favored because it simultaneously addresses distributional ambiguity and tail risk. Here we revisit its definition for convenience of reading. Given every $\tilde{v} \in L^\infty(\mathcal{D}, \mathcal{F}, \mathbb{P})$, $\rho_{\text{DSAM}}(\cdot, \mathbb{F})$ is defined as:

$$\rho_{\text{DSAM}}(\tilde{v}, \mathbb{F}) = \sup\left\{\gamma \in (0,1) : \sup_{\mathbb{P} \in \mathbb{F}} \text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma) \leq 0\right\},$$

where

$$\mathrm{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma) = \inf_{\nu} \left\{ \nu + \frac{\mathbb{E}_{\mathbb{P}}(\tilde{v} - \nu)^+}{1 - \gamma} \right\}.$$

Therefore, similar to the energy supply system, we define the *resilience index* of a telecommunication network with capacity installation $\mathbf{x}$ by relating it to the distributionally-ambiguous shortfall awareness measure as

$$\varrho(\mathcal{G}, \mathbf{x}) \triangleq \rho_{\mathrm{DSAM}} \left(\mathcal{Q} \left(\mathbf{x}, \tilde{\mathbf{d}} \right) - \tau, \mathbb{F} \right).$$

Based on this, we formulate the resilient telecommunication network design model as:

$$\begin{aligned} Z_0^* = \max \quad & \varrho(\mathcal{G}, \mathbf{x}) \\ \text{s.t.} \quad & \mathbf{c}'\mathbf{x} \le B \\ & \mathbf{x} \ge \mathbf{0} \end{aligned} \tag{6.3}$$

In model (6.3), $\mathbf{c}$ is the cost vector so that $\mathbf{c}'\mathbf{x}$ gives the total investment cost associated with capacity installation decision $\mathbf{x}$. Through this model we attempt to find an optimal capacity installation decision $\mathbf{x}$ within investment budget limit B such that the resilience index of the resulting telecommunication network is maximized. Unfortunately, model (6.3) is difficult to solve in general due to the complex relationship between its resilience index $\varrho(\mathcal{G}, \mathbf{x})$ and the capacity installation $\mathbf{x}$. To do this, we can decompose it into a sequence of the following sub-problems by performing a binary search on $\gamma \in (0, 1)$ upon checking whether its optimal objective value exceeds B or not:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}(\gamma). \end{aligned}$$

Here $\mathcal{X}(\gamma)$ denotes the set of γ-admissible capacity installation such that the resulting telecommunication network is at least γ-resilient. A mathematical expression is given as

$$\mathcal{X}(\gamma) \triangleq \left\{ \mathbf{x} \in \Re_+^A : \Psi_{\mathbb{F}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) \le \tau \right\}, \tag{6.4}$$

where $\Psi_{\mathbb{F}}(\cdot,\gamma)$ denotes the distributionally-ambiguous conditional value of risk (CVaR). In other words, for every $\tilde{v}$ such that $\tilde{v} \in L^{\infty}(\mathcal{D},\mathcal{F},\mathbb{P})$ for every $\mathbb{P} \in \mathbb{F}$, we have

$$\Psi_{\mathbb{F}}(\tilde{v},\gamma) = \sup_{\mathbb{P}\in\mathbb{F}} \mathrm{CVaR}_{\mathbb{P}}(\tilde{v},\gamma). \tag{6.5}$$

Still, computational difficulty remains because a compact reformulation of the set $\mathcal{X}(\gamma)$ is not clear. Therefore, proper approximations of the recourse problem objective $\mathcal{Q}(\mathbf{x},\tilde{\mathbf{d}})$ are necessary to obtain tractable approximations of $\mathcal{X}(\gamma)$, which are exactly what we are going to discuss in the subsequent section.

6.3 Decision Rule Based Approximations of $\mathcal{X}(\gamma)$

Consider an extreme case that the penalty functions $p_k(s) : k \in \mathcal{K}$ equals 0 if $s \leq 0$ and approaches to $+\infty$ otherwise. Then for every $\gamma \in (0,1)$, $\mathcal{X}(\gamma)$ denotes the set of capacity installations s such that $\mathcal{Z}_R(\mathbf{x}) \cap \mathcal{Z}_S(\tilde{\mathbf{d}})$ is always not empty for every $\tilde{\mathbf{d}} \in \mathcal{D}$. However, it is known that the problem of determining a given capacity installation $\mathbf{x}$ belongs to $\mathcal{X}(\gamma)$ or not is NP-Complete (Chekuri et al., 2007). Therefore, unless coNP=P, it is impossible to construct a compact formulation of $\mathcal{X}(\gamma)$. To overcome this computational difficulty, one possible way is to add proper restrictions on the path flow solutions $\mathbf{z}$ to approximate the complex function $\mathcal{Q}(\mathbf{x},\tilde{\mathbf{d}})$.

For representation simplicity, we first give several important notations on decision rules. For every integer $n \in \mathbb{Z}_+$, we define $\mathcal{M}(n)$ as the set of functions as:

$$\mathcal{M}(n) \triangleq \{f : \mathcal{D} \longmapsto \Re^n\}.$$

Similarly, we define the parameterized family of functions $S(k) : k \in \mathcal{K}$ as:

$$\mathcal{S}(k) \triangleq \{f : \mathcal{D} \longmapsto \Re : \exists w \in \Re \text{ such that } f(\mathbf{d}) = w d_k\},$$

and the family of affine functions $\mathcal{L}(n)$ as:

$$\mathcal{L}(n) \triangleq \left\{f : \mathcal{D} \longmapsto \Re^n : \exists(\mathbf{w}_0, \mathbf{W}) \in \Re^n \times \Re^{n\times K} \text{ such that } f(\mathbf{d}) = \mathbf{w}_0 + \mathbf{W}\mathbf{d}\right\}.$$

From now on, we concentrate on decision rule based approximation of the set $\mathcal{X}(\gamma)$. To do this, we give the following proposition of conducting an equivalent representation of the distributionally-ambiguous CVaR measure $\Psi_{\mathbb{F}}(\cdot, \gamma)$ defined in (6.5).

Proposition 6.1. *For every capacity installation decision $\mathbf{x} \in \Re^A_+$ and $\gamma \in (0,1)$, the distributionally-ambiguous CVaR of $\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})$ with respect to distribution family $\mathbb{F}$ can also be equivalently expressed as*

$$\Psi(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) = \inf_{\nu\in\Re} \sup_{\mathbb{P}\in\mathbb{F}} \left\{\nu + \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+\right\}.$$

Proof. Observe that $\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \in [0, p^*]$, where $p^* = \max\left\{\sum_{k\in\mathcal{K}} p_k(\tilde{\mathbf{d}}) : \tilde{\mathbf{d}} \in \mathcal{D}\right\} \in [0, +\infty)$ due to the fact that the support set $\mathcal{D}$ is bounded. Therefore the set of minima $\nu^* \in [0, p^*]$. It can be seen that

$$\text{CVaR}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) = \min_{\nu\in[0,p^*]} \left\{\nu + \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+\right\}.$$

Therefore, it follows immediately from the stochastic saddle point theorem (Natarajan et al., 2009; Shapiro and Kleywegt, 2002) that

$$\begin{aligned}
\Psi(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) &= \sup_{\mathbb{P}\in\mathbb{F}} \min_{\nu\in[0,p^*]} \left\{\nu + \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+\right\} \\
&= \min_{\nu\in[0,p^*]} \sup_{\mathbb{P}\in\mathbb{F}} \left\{\nu + \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+\right\} \\
&\geq \min_{\nu\in\Re} \sup_{\mathbb{P}\in\mathbb{F}} \left\{\nu + \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+\right\}.
\end{aligned}$$

On the other hand, the reverse direction of this inequality can be induced from the well known min-max inequality.

Specifically, we observe that

$$\begin{aligned}
&\Psi(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) \\
&= \sup_{\mathbb{P}\in\mathbb{F}} \min_{\nu\in\Re} \left\{ \nu + \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^{+} \right\} \\
&\leq \min_{\nu\in\Re} \sup_{\mathbb{P}\in\mathbb{F}} \left\{ \nu + \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^{+} \right\}.
\end{aligned}$$

Consequently, the result can be verified by these two inequalities. □

Proposition 6.1 gives a relatively tractable formulation of the distributionally-ambiguous CVaR. Before defining the decision rule based representation of $\mathcal{X}(\gamma)$, we first give two auxiliary index sets $\hat{\Phi} \triangleq \{(k, p) : k \in \mathcal{K}, p \in \mathcal{P}_k\}, \check{\Phi} \triangleq \{(k, m) : k \in \mathcal{K}, m \in [M]\}$. By introducing auxiliary decision variables $(\mathbf{t}, \nu, \theta)$, $\mathcal{X}(\gamma)$ can be equivalently expressed as

$$\mathcal{X}(\gamma) = \left\{ \mathbf{x} \in \Re^{A}_{+} : \begin{array}{l}
\exists (\mathbf{z}, \mathbf{s}, \mathbf{t}, \theta) \in \mathcal{M}(N + 2K + 1), \nu \in \Re \\
\text{s.t. } \nu + \dfrac{1}{1-\gamma} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}[\theta(\tilde{\mathbf{d}})] \leq \tau \\
\displaystyle\sum_{k\in\mathcal{K}} t_k(\tilde{\mathbf{d}}) - \nu \leq \theta(\tilde{\mathbf{d}}) \\
p_{km} s_k(\tilde{\mathbf{d}}) + q_{km} \leq t_k(\tilde{\mathbf{d}}) : (k, m) \in \check{\Phi} \\
\displaystyle\sum_{p\in\mathcal{P}_k} z_{kp}(\tilde{\mathbf{d}}) + s_k(\tilde{\mathbf{d}}) \geq \tilde{d}_k : k \in \mathcal{K} \\
\displaystyle\sum_{(k,p)\in\hat{\Phi}} \delta_{pa} z_{kp}(\tilde{\mathbf{d}}) \leq x_a : a \in \mathcal{A} \\
z_{kp}(\tilde{\mathbf{d}}) \geq 0 : (k, p) \in \hat{\Phi} \\
s_k(\tilde{\mathbf{d}}) \geq 0 : k \in \mathcal{K} \\
\theta(\tilde{\mathbf{d}}) \geq 0
\end{array} \right\} \quad (6.6)$$

The freedom of choosing the decision variables $(\mathbf{z}, \mathbf{s}, \mathbf{t}, \theta)$ in the decision space $\mathcal{M}(N + 2K + 1)$ constitutes the computational difficulties. If we narrow down the decision space $\mathcal{M}(1)$ by simple decision rules such as $\mathcal{S}(k)$ or $\mathcal{L}(1)$, we can obtain tractable approximations.

6.3.1 Oblivious Routing Policy and Affine Routing Policy

Many works in the literature (e.g. Altın et al., 2007, 2011; Koster et al., 2013; Mudchanatongsuk et al., 2007; Lee et al., 2012) consider a simple version of the recourse problem $\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})$ by restricting the commodity-flow routing solution to be a linear function of the demand $\tilde{d}_k$, which is known as oblivious routing policy or static routing policy. In the context of our flow path based formulation, the static routing policy forces the decision variables $z_{kp}(\tilde{\mathbf{d}})$ and $s_k(\tilde{\mathbf{d}})$ to belong to $\mathcal{S}(k)$ instead of $\mathcal{M}(1)$. By further restricting $t_k(\tilde{\mathbf{d}}) = t_k^0 + t_k^1 \tilde{d}_k$ and $\theta(\tilde{\mathbf{d}}) \in \mathcal{L}(1)$, we obtain an oblivious routing policy based approximation of $\mathcal{X}(\gamma)$ as

$$\mathcal{X}_{ORP}(\gamma) = \left\{ \mathbf{x} \in \Re_+^A : \begin{array}{l} \exists \nu, \{\theta^0, \boldsymbol{\theta}\}, \{z_{kp}^0\}_{(k,p)\in\hat{\Phi}}, \{s_k, t_k^0, t_k^1\}_{k\in\mathcal{K}} \\ \text{s.t. } \nu + \dfrac{1}{1-\gamma}(\theta^0 + \boldsymbol{\theta}'\boldsymbol{\mu}) \le \tau \\ \displaystyle\sum_{k\in\mathcal{K}} t_k^0 - \nu - \theta^0 + (\mathbf{t}^1 - \boldsymbol{\theta})\tilde{\mathbf{d}} \le 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D} \\ q_{km} - t_k^0 + (p_{km}s_k - t_k^1)\tilde{d}_k \le 0 : \forall \tilde{d}_k \in \mathcal{D}_k, (k,m) \in \check{\Phi} \\ \displaystyle\sum_{p\in\mathcal{P}_k} z_{kp}^0 + s_k \ge 1 : k \in \mathcal{K} \\ \displaystyle\sum_{(k,p)\in\hat{\Phi}} \delta_{pa} z_{kp}^0 \tilde{d}_k \le x_a : \forall \tilde{\mathbf{d}} \in \mathcal{D}, a \in \mathcal{A} \\ \theta^0 + \boldsymbol{\theta}'\tilde{\mathbf{d}} \ge 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, \end{array} \right\} \quad (6.7)$$

where we define $\mathbf{t}^1 \triangleq (t_k^1)_{k\in\mathcal{K}}$, $\mathcal{D}_k \triangleq \{d : \exists \tilde{\mathbf{d}} \in \mathcal{D}, d = \tilde{d}_k\}$.

The above formulation of $\mathcal{X}_{ORP}(\gamma)$ is a standard robust counterpart based formulation, which can be solved using Lagrange duality (see Ben-Tal et al., 2009) for a wide range of uncertainty sets $\mathcal{D}$. Specifically, if $\mathcal{D}$ is a polyhedron, $\mathcal{X}_{ORP}(\gamma)$ can be expressed by linear constraints and thenceforth $\min\{\mathbf{c}'\mathbf{x} : \mathbf{x} \in \mathcal{X}_{ORP}(\gamma)\}$ reduces to a simple linear optimization problem. However, a critical issue is the conservatism. To address this, a less conservative approximation named affine routing policy is introduced in (Lemaréchal et al., 2010; Ouorou, 2011; Poss and Raack, 2012). Instead of restricting the path flow solution z_{kp} be a portion of the realized demand component $\tilde{d}_k$, affine decision rules allows it to be an affine function of all the components of the realized demand $\tilde{\mathbf{d}}$. More precisely, we let the variables $(\mathbf{z}, \mathbf{s}, \mathbf{t}, \theta)$ in (6.6) be affine functions of the demand vector $\tilde{\mathbf{d}}$, i.e., $(\mathbf{z}, \mathbf{s}, \mathbf{t}, \theta) \in \mathcal{L}(N + 2K + 1)$.

Hence, we construct the affine routing policy based approximation of $\mathcal{X}(\gamma)$ as

$$\mathcal{X}_{ARP}(\gamma) = \left\{ \mathbf{x} \in \Re^A_+ : \begin{array}{l} \exists \nu, \{\theta^0, \boldsymbol{\theta}\}, \{z^0_{kp}, \mathbf{z}_{kp}\}_{(k,p)\in\hat{\Phi}}, \{s^0_k, \mathbf{s}_k, t^0_k, \mathbf{t}_k\}_{k\in\mathcal{K}} \\ \text{s.t. } \nu + \dfrac{1}{1-\gamma}(\theta^0 + \boldsymbol{\theta}'\boldsymbol{\mu}) \leq \tau \\ \displaystyle\sum_{k\in\mathcal{K}} t^0_k - \nu - \theta^0 + (\sum_{k\in\mathcal{K}} \mathbf{t}_k - \boldsymbol{\theta})'\tilde{\mathbf{d}} \leq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D} \\ p_{km} s^0_k + q_{km} - t^0_k + (p_{km}\mathbf{s}_k - \mathbf{t}_k)'\tilde{\mathbf{d}} \leq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, (k,m) \in \check{\Phi} \\ \displaystyle\sum_{p\in\mathcal{P}_k} z^0_{kp} + s^0_k + (\mathbf{s}_k + \sum_{p\in\mathcal{P}_k} \mathbf{z}_{kp})'\tilde{\mathbf{d}} \geq \tilde{d}_k : \forall \tilde{\mathbf{d}} \in \mathcal{D}, k \in \mathcal{K} \\ \displaystyle\sum_{(k,p)\in\hat{\Phi}} \delta_{pa} z^0_{kp} + (\sum_{(k,p)\in\hat{\Phi}} \delta_{pa}\mathbf{z}_{kp})'\tilde{\mathbf{d}} \leq x_a : \forall \tilde{\mathbf{d}} \in \mathcal{D}, a \in \mathcal{A} \\ z^0_{kp} + \mathbf{z}'_{kp}\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, (k,p) \in \hat{\Phi} \\ s^0_k + \mathbf{s}'_k\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, k \in \mathcal{K} \\ \theta^0 + \boldsymbol{\theta}'\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}. \end{array} \right\} \tag{6.8}$$

Note the fact that $\mathcal{S}(k) \subseteq \mathcal{L}(1) \subseteq \mathcal{M}(1)$, we have the following elementary result about these two approximations.

Proposition 6.2. *The tractable sets $\mathcal{X}_{ORP}(\gamma)$ and $\mathcal{X}_{ARP}(\gamma)$ give two conservative approximation of $\mathcal{X}(\gamma)$, and we have $\mathcal{X}_{ORP}(\gamma) \subseteq \mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}(\gamma)$.*

Proof. $\mathcal{X}_{ORP}(\gamma) \subseteq \mathcal{X}_{ARP}(\gamma)$ follows straightforwardly from the fact that oblivious routing solution is a specific instance of the affine routing solution. To complete the proof, it suffices to show that $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}(\gamma)$. For any feasible solution of the feasibility formulation (6.8), it can be seen that

$$
\begin{aligned}
& z_{kp}^*(\tilde{\mathbf{d}}) = z_{kp}^0(\tilde{\mathbf{d}}) + \mathbf{z}_{kp}'\tilde{\mathbf{d}} : (k,p) \in \hat{\Phi} \\
& s_k^*(\tilde{\mathbf{d}}) = s_k^0 + \mathbf{s}_k'\tilde{\mathbf{d}} : k \in \mathcal{K} \\
& t_k^*(\tilde{\mathbf{d}}) = t_k^0 + \mathbf{t}_k'\tilde{\mathbf{d}} : k \in \mathcal{K} \\
& \theta^*(\tilde{\mathbf{d}}) = \theta^0 + \boldsymbol{\theta}'\tilde{\mathbf{d}} \\
& \nu^* = \nu
\end{aligned}
$$

is a feasible solution of (6.6). Hence, $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}(\gamma)$ completing the proof. □

6.3.2 Improved Affine Decision Rule Based Approximation

Both the oblivious routing policy and affine routing policy give rise to tractable approximations of $\mathcal{X}(\gamma)$. It can be seen that only the first moment information $\boldsymbol{\mu}$ appears in the resulting formulation, which means that these approximations are as tight as the ones for a larger distributional family $\mathbb{F}^\dagger = \left\{\mathbb{P} \in \mathbb{M}(\mathcal{D}) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu}\right\}$. To make use of the covariance matrix $\boldsymbol{\Sigma}$, we now provide an improved affine decision rule based approximation as

$$
\mathcal{X}_{IARP}(\gamma) = \left\{ \mathbf{x} \in \Re_+^A : \begin{array}{l}
\exists \nu \in \Re; \{z_{kp}(\tilde{\mathbf{d}})\}_{(k,p)\in\hat{\Phi}} \in \mathcal{L}(N) \\
\{h_{km}(\tilde{\mathbf{d}})\}_{(k,m)\in\check{\Phi}} \in \mathcal{L}(|\check{\Phi}|) \\
\{s_k(\tilde{\mathbf{d}}), t_k(\tilde{\mathbf{d}}), r_k(\tilde{\mathbf{d}})\}_{k\in\mathcal{K}} \in \mathcal{L}(K) \\
\text{s.t. } \nu + \dfrac{1}{1-\gamma} g\left(\nu, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r}\right) \le \tau \\
\displaystyle\sum_{p\in\mathcal{P}_k} z_{kp}(\tilde{\mathbf{d}}) + s_k(\tilde{\mathbf{d}}) \ge \tilde{d}_k : \forall \tilde{\mathbf{d}} \in \mathcal{D}, k \in \mathcal{K} \\
\displaystyle\sum_{(k,p)\in\hat{\Phi}} \delta_{pa} z_{kp}(\tilde{\mathbf{d}}) \le x_a : \forall \tilde{\mathbf{d}} \in \mathcal{D}, a \in \mathcal{A} \\
z_{kp}(\tilde{\mathbf{d}}) \ge 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, (k,p) \in \hat{\Phi}
\end{array} \right\}, \quad (6.9)
$$

where we define the auxiliary function g as

$$
\begin{aligned}
g(\nu,\mathbf{z},\mathbf{s},\mathbf{t},\mathbf{h},\mathbf{r}) \triangleq & \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\sum_{k\in\mathcal{K}} t_k(\tilde{\mathbf{d}}) + \sum_{(k,m)\in\check{\Phi}} h_{km}(\tilde{\mathbf{d}}) + \sum_{k\in\mathcal{K}} p_{kM} r_k(\tilde{\mathbf{d}}) - \nu \right)^+ \\
& + \sum_{(k,m)\in\check{\Phi}} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(p_{km} s_k(\tilde{\mathbf{d}}) + q_{km} - t_k(\tilde{\mathbf{d}}) - h_{km}(\tilde{\mathbf{d}}) \right)^+ \\
& + \sum_{(k,m)\in\check{\Phi}} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-h_{km}(\tilde{\mathbf{d}}) \right)^+ + \sum_{k\in\mathcal{K}} p_{kM} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-r_k(\tilde{\mathbf{d}}) \right)^+ \\
& + \sum_{k\in\mathcal{K}} p_{kM} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-s_k(\tilde{\mathbf{d}}) - r_k(\tilde{\mathbf{d}}) \right)^+ .
\end{aligned}
\tag{6.10}
$$

To explore the connection between $\mathcal{X}_{IARP}(\gamma)$ and $\mathcal{X}(\gamma)$, we first present two important lemmas.

Lemma 6.1. *(Chen et al., 2010; Goh and Hall, 2013) For every scalar-valued convex piece-wise linear function* $V(\tilde{\mathbf{d}}) = \max_{m\in[M]} \left\{ p_m s(\tilde{\mathbf{d}}) + q_m \right\}$*, where* $s(\tilde{\mathbf{d}}) \in \mathcal{M}(1)$*. We can equivalently express* $V(\tilde{\mathbf{d}})$ *as*

$$
V(\tilde{\mathbf{d}}) = \min \left\{ t(\tilde{\mathbf{d}}) + \sum_{m\in[M]} \left(p_m s(\tilde{\mathbf{d}}) + q_m - t(\tilde{\mathbf{d}}) \right)^+ : t(\tilde{\mathbf{d}}) \in \mathcal{M}(1) \right\}.
$$

Lemma 6.2. *For every* $y(\tilde{\mathbf{d}}) \in \mathcal{M}(1)$ *and* $\{x_i(\tilde{\mathbf{d}})\}_{i\in[I]} \in \mathcal{M}(I)$*, we have*

$$
\begin{aligned}
\left(y(\tilde{\mathbf{d}}) + \sum_{i\in[I]} \left(x_i(\tilde{\mathbf{d}}) \right)^+ \right)^+ = \min \Bigg\{ & \sum_{i\in[I]} \left(\left(-h_i(\tilde{\mathbf{d}}) \right)^+ + \left(x_i(\tilde{\mathbf{d}}) - h_i(\tilde{\mathbf{d}}) \right)^+ \right) \\
& + \left(y(\tilde{\mathbf{d}}) + \sum_{i\in[I]} h_i(\tilde{\mathbf{d}}) \right)^+ : \{h_i(\tilde{\mathbf{d}})\}_{i\in[I]} \in \mathcal{M}(I) \Bigg\}
\end{aligned}
$$

Proof. According to the proof of Theorem 4 in See and Sim (2010), the inequality

$$
\left(y + \sum_{i\in[I]} (x_i)^+ \right)^+ \leq \sum_{i\in[I]} \left((-h_i)^+ + (x_i - h_i)^+ \right) + \left(y + \sum_{i\in[I]} h_i \right)^+
$$

holds for every $y, \{x_i, h_i\}_{i\in[I]} \in \Re$. Hence, the left hand side of the proposed equality is less than or equal to the right hand side. In addition, by letting $h_i^*(\tilde{\mathbf{d}}) = \left(x_i^*(\tilde{\mathbf{d}})\right)^+$, we have

$$\left(y(\tilde{\mathbf{d}}) + \sum_{i\in[I]} \left(x_i(\tilde{\mathbf{d}})\right)^+\right)^+ = \sum_{i\in[I]} \left(\left(-h_i^*(\tilde{\mathbf{d}})\right)^+ + \left(x_i(\tilde{\mathbf{d}}) - h_i^*(\tilde{\mathbf{d}})\right)^+\right) + \left(y(\tilde{\mathbf{d}}) + \sum_{i\in[I]} h_i^*(\tilde{\mathbf{d}})\right)^+$$

indicating that the left hand side is greater than or equal to the right hand side of the proposed equality. Hence, the equality holds. □

Proposition 6.3. *For every $\gamma \in (0,1)$, $\mathcal{X}_{IARP}(\gamma)$ constitutes a conservative approximation of $\mathcal{X}(\gamma)$ and is tighter than $\mathcal{X}_{ARP}(\gamma)$. More precisely, we have $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}_{IARP}(\gamma) \subseteq \mathcal{X}(\gamma)$.*

Proof. Note the fact that for every $\mathbf{x} \in \mathcal{X}_{ARP}(\gamma)$, we define the following variables from its associated feasible variables $\nu, \{\theta^0, \boldsymbol{\theta}\}, \{z_{kp}^0, \mathbf{z}_{kp}\}_{(k,p)\in\hat{\Phi}}, \{s_k^0, \mathbf{s}_k, t_k^0, \mathbf{t}_k\}_{k\in\mathcal{K}}$ in (6.8) as

$$\begin{aligned}
&z_{kp}^*(\tilde{\mathbf{d}}) = z_{kp}^0(\tilde{\mathbf{d}}) + \mathbf{z}_{kp}'\tilde{\mathbf{d}} : (k,p) \in \hat{\Phi}\\
&s_k^*(\tilde{\mathbf{d}}) = s_k^0 + \mathbf{s}_k'\tilde{\mathbf{d}} : k \in \mathcal{K}\\
&t_k^*(\tilde{\mathbf{d}}) = t_k^0 + \mathbf{t}_k'\tilde{\mathbf{d}} : k \in \mathcal{K}\\
&h_{km}^*(\tilde{\mathbf{d}}) = 0 : (k,m) \in \check{\Phi}\\
&r_k^*(\tilde{\mathbf{d}}) = 0 : k \in \mathcal{K}\\
&\nu^* = \nu.
\end{aligned}$$

It can be seen that all the nonlinear terms from row two to the end of the function $g(\cdot)$ in (6.10) vanish and we therefore have

$$g(\nu^*, \mathbf{z}^*, \mathbf{s}^*, \mathbf{t}^*, \mathbf{h}^*, \mathbf{r}^*) = \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\sum_{k\in\mathcal{K}} t_k^*(\tilde{\mathbf{d}}) - \nu\right)^+$$

$$\leq \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\theta^0 + \boldsymbol{\theta}'\tilde{\mathbf{d}})$$
$$= \theta^0 + \boldsymbol{\theta}'\boldsymbol{\mu}.$$

Hence, we can conclude that $\nu + \frac{1}{1-\gamma} g(\nu^*, \mathbf{z}^*, \mathbf{s}^*, \mathbf{t}^*, \mathbf{r}^*, \mathbf{h}^*) \leq \tau$ indicating that $\mathbf{x} \in \mathcal{X}_{IARP}(\gamma)$. Hence, we can conclude that $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}_{IARP}(\gamma)$.

We next show the rest part that $\mathcal{X}_{IARP}(\gamma) \subseteq \mathcal{X}(\gamma)$.

For every $\mathbf{x} \in \mathcal{X}_{IARP}(\gamma)$ with its associated feasible affine solutions $\nu \in \Re$, $(\mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r}) \in \mathcal{L}$ in (6.9). It is not hard to see that

$$\begin{aligned} & z^*_{kp} = z_{kp}(\tilde{\mathbf{d}}) : \forall (k,p) \in \hat{\Phi} \\ & s^*_k = s_k(\tilde{\mathbf{d}}) + \left(-s_k(\tilde{\mathbf{d}})\right)^+ : \forall k \in \mathcal{K} \end{aligned}$$

is a feasible solution of the recourse problem (6.2) with respect to $\mathbf{x}$. As a consequence, we have $\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \leq \sum_{k\in\mathcal{K}} p_k(s^*_k)$. Therefore, we have

$$\begin{aligned}
& \Psi_{\mathbb{F}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}), \gamma) \\
& = \inf_{\nu\in\Re} \sup_{\mathbb{P}\in\mathbb{F}} \left\{ \nu + \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \nu)^+ \right\} \\
& \leq \inf_{\nu\in\Re} \sup_{\mathbb{P}\in\mathbb{F}} \left\{ \nu + \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}} \left(\sum_{k\in\mathcal{K}} p_k(s^*_k) - \nu \right)^+ \right\} \\
& \leq \nu + \frac{1}{1-\gamma} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\sum_{k\in\mathcal{K}} p_k(s^*_k) - \nu \right)^+ \\
& \leq \nu + \frac{1}{1-\gamma} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-\nu + \sum_{k\in\mathcal{K}} \max_{m\in[M]} \{p_{km} s_k(\tilde{\mathbf{d}}) + q_{km}\} + \sum_{k\in\mathcal{K}} p_{kM} \left(-s_k(\tilde{\mathbf{d}})\right)^+ \right)^+ \\
& \leq \nu + \frac{1}{1-\gamma} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \Bigg(\sum_{k\in\mathcal{K}} t_k(\tilde{\mathbf{d}}) - \nu + \sum_{(k,m)\in\check{\Phi}} \left(p_{km} s_k(\tilde{\mathbf{d}}) + q_{km} - t_k(\tilde{\mathbf{d}})\right)^+ \\
& \quad + \sum_{k\in\mathcal{K}} p_{kM} \left(-s_k(\tilde{\mathbf{d}})\right)^+ \Bigg)^+ \\
& \leq \nu + \frac{1}{1-\gamma} g(\nu, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r}) \\
& \leq \tau.
\end{aligned}$$

The first equality follows directly from Proposition 6.1. The third inequality results from the assumption that $p_{kM} \geq p_{km} : \forall m \in [M]$. The forth inequality is an immediate consequence of applying Lemma 6.1. The second last inequality results from Lemma 6.2; we also make use the fact that the operator $\sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(\cdot)$ satisfies sub-additivity to obtain the second last inequality. Finally, the last inequality is clear because $(\nu, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r})$ is a feasible solution of (6.9) with respect to $\mathbf{x}$. According to the definition of $\mathcal{X}(\gamma)$ in (6.4), we then can conclude that $\mathbf{x} \in \mathcal{X}(\gamma)$ completing the proof that $\mathcal{X}_{IARP}(\gamma) \subseteq \mathcal{X}(\gamma)$. $\square$

Proposition 6.3 states that $\mathcal{X}_{IARP}(\gamma)$ constitutes a less conservative approximation of $\mathcal{X}(\gamma)$ than $\mathcal{X}_{ARP}(\gamma)$. Unfortunately, it is less tractable due to the distributionally-ambiguous expectation over the truncated term $(\cdot)^+$. Let us define $\chi(w_0, \mathbf{w}) \triangleq \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(w_0 + \mathbf{w}'\tilde{\mathbf{d}}\right)^+$. It is known from Murty and Kabadi (1987) that the computation of $\chi(\cdot,\cdot)$ is generally NP-Hard. Therefore, tractable approximation of $\chi(\cdot,\cdot)$ is necessary to build tractable approximation of $\mathcal{X}_{IARP}(\gamma)$.

6.3.3 Tractable Approximation of $\chi(\cdot,\cdot)$

To obtain tractable approximation, we consider the case that $\mathcal{D} = [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$. For more complicated set $\mathcal{D}$, this can be achieved by relaxing $\mathcal{D}$ to $\mathcal{D}_1 \times \cdots \times \mathcal{D}_K : \mathcal{D}_k \triangleq \{d : \exists \tilde{\mathbf{d}} \in \mathcal{D}, d = \tilde{d}_k\}$). In the following we give two alternative approximations. The first one can be represented as a second order cone program (SOCP), which possesses considerably computational efficiency in both theory and practice. The second bound can be represented in terms of linear matrix inequality (LMI).

Lemma 6.3. *Chen and Sim (2009); Goh and Sim (2010) Suppose that the support* $\mathcal{D} = [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$ *and the distribution family* $\mathbb{F} = \left\{\mathbb{P} \in \mathbb{M}(\mathcal{D}) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}\tilde{\mathbf{d}}') = \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\Sigma}\right\}$*, in which* $\boldsymbol{\Sigma} \succ \mathbf{0}$*, then we have*

$$\chi(w_0, \mathbf{w}) \leq \pi(w_0, \mathbf{w}) \triangleq \min \theta_1 + \theta_2 + \theta_3$$

$$\text{s.t. } w_{10} + \mathbf{u}_1'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_1'(\boldsymbol{\mu} - \underline{\mathbf{d}}) \leq \theta_1$$

$$0 \le \theta_1$$

$$\mathbf{u}_2'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_2'(\boldsymbol{\mu} - \underline{\mathbf{d}}) \le \theta_2$$

$$w_{20} \le \theta_2$$

$$\frac{1}{2}\left(w_{30} + \sqrt{w_{30}^2 + \|\boldsymbol{\Sigma}^{1/2}\mathbf{w}_3\|_2^2}\right) \le \theta_3$$

$$\mathbf{u}_1 - \mathbf{v}_1 - \mathbf{u}_2 + \mathbf{v}_2 + \mathbf{w}_3 = \mathbf{w}$$

$$w_{10} + w_{20} + w_{30} = w_0 + \mathbf{w}'\boldsymbol{\mu}, \tag{6.11}$$

Lemma 6.4. *Suppose* $\mathcal{D} = [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$ *and* $\mathbb{F} = \Big\{\mathbb{P} \in \mathbb{M}(\mathcal{D}) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}\tilde{\mathbf{d}}') = \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\Sigma}\Big\}$*, in which* $\boldsymbol{\Sigma} \succ \mathbf{0}$*. We obtain a tractable upper bound of* $\chi(w_0, \mathbf{w})$ *in terms of LMI as*

$$\chi(w_0, \mathbf{w}) \le \eta(w_0, \mathbf{w}) \triangleq \min_{\mathbf{u},\mathbf{v} \ge \mathbf{0}} \left\{\mathbf{u}'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}'(\boldsymbol{\mu} - \underline{\mathbf{d}}) + \eta^\dagger(w_0 - \mathbf{u}'\overline{\mathbf{d}} + \mathbf{v}'\underline{\mathbf{d}}, \mathbf{w} + \mathbf{u} - \mathbf{v})\right\}, \tag{6.12}$$

where $\eta^\dagger(w_0, \mathbf{w})$ *is LMI representable as*

$$\begin{aligned}
\eta^\dagger(w_0, \mathbf{w}) \triangleq \inf_{\mathbf{M} \in \mathbb{S}^{K+1}, \mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2} & \langle \boldsymbol{\Omega}, \mathbf{M} \rangle \\
\text{s.t. } & \mathbf{M} \succcurlyeq \sum_{k=1}^K \beta_{1k}\mathbf{M}_k \\
& \mathbf{M} \succcurlyeq \sum_{k=1}^K \beta_{2k}\mathbf{M}_k + \begin{bmatrix} \mathbf{0} & \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v}) \\ \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v})' & w_0 + \mathbf{u}'\overline{\mathbf{d}} - \mathbf{v}'\underline{\mathbf{d}} \end{bmatrix} \\
& \mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \ge \mathbf{0}.
\end{aligned} \tag{6.13}$$

Here $\{\mathbf{M}_k\}_{k \in [K]} \in \mathbb{S}^{K+1}$ *are symmetric matrices defined as*

$$\mathbf{M}_k(k,k) = -1, \mathbf{M}_k(k,K) = \mathbf{M}_k(K,k) = \frac{1}{2}(\underline{d}_k + \overline{d}_k);$$

$$\mathbf{M}_k(K,K) = -\underline{d}_k\overline{d}_k, \mathbf{M}_k(i,j) = 0 : \forall i,j \in [K] \setminus \{k, K\},$$

and we define $\mathbf{\Omega}$ *as*

$$\mathbf{\Omega} = \begin{bmatrix} \boldsymbol{\mu}\boldsymbol{\mu}' + \mathbf{\Sigma} & \boldsymbol{\mu} \\ \boldsymbol{\mu}' & 1 \end{bmatrix}.$$

Proof. We first prove that $\chi(w_0, \mathbf{w}) \leq \eta^{\dagger}(w_0, \mathbf{w})$. Note $\chi(w_0, \mathbf{w})$ is the worst case expectation of $(w_0 + \mathbf{w}'\tilde{\mathbf{d}})^+$ with respect to distribution family $\mathbb{F}$, which is a classical problem of moments that can be equivalently expressed as an infinite-dimensional linear program with respect to the probability density function $p(\tilde{\boldsymbol{\xi}})$ of the distribution $\mathbb{P}$

$$\begin{aligned} \chi(w_0, \mathbf{w}) = \sup \quad & \int_{[\underline{\mathbf{d}}, \overline{\mathbf{d}}]} \max\{0, w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}}\} p(\tilde{\boldsymbol{\xi}}) \, \mathrm{d}(\tilde{\boldsymbol{\xi}}) \\ \text{s.t.} \quad & \int_{[\underline{\mathbf{d}}, \overline{\mathbf{d}}]} \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix}' \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix} p(\tilde{\boldsymbol{\xi}}) \, \mathrm{d}(\tilde{\boldsymbol{\xi}}) = \mathbf{\Omega} \\ & p(\tilde{\boldsymbol{\xi}}) \geq 0 : \forall \tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]. \end{aligned}$$

Associating dual variables $\mathbf{M} \in \mathbb{S}^{K+1}$ with respect to $\mathbf{\Omega}$ we obtain its equivalent dual formulation

$$\begin{aligned} \chi(w_0, \mathbf{w}) = \inf \quad & \langle \mathbf{\Omega}, \mathbf{M} \rangle \\ \text{s.t.} \quad & \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix}' \geq 0 : \forall \tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}] \\ & \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix}' \geq w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}} : \forall \tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]. \end{aligned} \quad (6.14)$$

Since $\mathbf{\Sigma} \succ \mathbf{0}$, the moment information matrix $\mathbf{\Omega}$ resides in the interior of the space of feasible ones. This slater type condition enforces strong duality (Bertsimas and Popescu, 2005; Popescu, 2005; Zuluaga and Peña, 2005) and we therefore can deduce that its optimal value coincides with $\chi(w_0, \mathbf{w})$.

Note that when $\tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$, we have

$$\begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix} \beta_{1k} \mathbf{M}_k \begin{bmatrix} \tilde{\boldsymbol{\xi}}' & 1 \end{bmatrix}' \geq 0 : \forall k \in [K]$$

for all $\boldsymbol{\beta}_1 = (\beta_{1k})_{k \in [K]} \geq \mathbf{0}$.

Hence, it can be seen that the following constraint

$$\left[\tilde{\boldsymbol{\xi}}'\ 1\right] \mathbf{M} \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' \geq \left[\tilde{\boldsymbol{\xi}}'\ 1\right] \left(\sum_{k \in [K]} \beta_{1k} \mathbf{M}_k \right) \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' : \forall \tilde{\boldsymbol{\xi}} \in \Re^K$$

gives a conservative approximation of the semi-infinite constraint

$$\left[\tilde{\boldsymbol{\xi}}'\ 1\right] \mathbf{M} \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' \geq 0 : \forall \tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}].$$

Similarly, for all $\mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_2 \geq \mathbf{0}$, the following constraint

$$\begin{aligned}\left[\tilde{\boldsymbol{\xi}}'\ 1\right] \mathbf{M} \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' \geq w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}} + \mathbf{u}'(\overline{\mathbf{d}} - \tilde{\boldsymbol{\xi}}) + \mathbf{v}'(\tilde{\boldsymbol{\xi}} - \underline{\mathbf{d}}) + \\ \left[\tilde{\boldsymbol{\xi}}'\ 1\right] \left(\textstyle\sum_{k \in [K]} \beta_{2k} \mathbf{M}_k \right) \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' : \forall \tilde{\boldsymbol{\xi}} \in \Re^K\end{aligned}$$

gives a conservative approximation of the second semi-infinite constraint of (6.14)

$$\left[\tilde{\boldsymbol{\xi}}'\ 1\right] \mathbf{M} \left[\tilde{\boldsymbol{\xi}}'\ 1\right]' \geq w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}} : \forall \tilde{\boldsymbol{\xi}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}].$$

Hence, for all $\mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_2 \geq \mathbf{0}$, a matrix $\mathbf{M} \in \mathbb{S}^{K+1}$ satisfying

$$\mathbf{M} \succcurlyeq \sum_{k=1}^{K} \beta_{1k} \mathbf{M}_k, \mathbf{M} \succcurlyeq \sum_{k=1}^{K} \beta_{2k} \mathbf{M}_k + \begin{bmatrix} \mathbf{0} & \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v}) \\ \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v})' & w_0 + \mathbf{u}'\overline{\mathbf{d}} - \mathbf{v}'\underline{\mathbf{d}} \end{bmatrix}$$

is feasible in (6.14).

The arbitrariness in choosing $\mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \geq \mathbf{0}$ states that $\eta^{\dagger}(\cdot, \cdot)$ is actually a minimization problem with a decision space smaller than that of $\chi(\cdot, \cdot)$, which indicates that

$$\chi(w_0, \mathbf{w}) \leq \eta^{\dagger}(w_0, \mathbf{w}) : \forall (w_0, \mathbf{w}) \in \Re^{K+1}$$

In addition, we observe that when $\tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$

$$\mathbf{u}'(\overline{\mathbf{d}} - \tilde{\mathbf{d}}) \geq 0, \mathbf{v}'(\tilde{\mathbf{d}} - \underline{\mathbf{d}}) \geq 0$$

holds for all $\mathbf{u}, \mathbf{v} \geq \mathbf{0}$. Therefore, it can be seen that

$$\begin{aligned}
&\chi(w_0, \mathbf{w}) \\
&= \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\mathbf{d}})^+ \\
&= \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(w_0 + \mathbf{w}'\tilde{\mathbf{d}} - \mathbf{u}'(\overline{\mathbf{d}} - \tilde{\mathbf{d}}) - \mathbf{v}'(\tilde{\mathbf{d}} - \underline{\mathbf{d}}) + \mathbf{u}'(\overline{\mathbf{d}} - \tilde{\mathbf{d}}) + \mathbf{v}'(\tilde{\mathbf{d}} - \underline{\mathbf{d}})\right)^+ \\
&\leq \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(w_0 - \mathbf{u}'\overline{\mathbf{d}} + \mathbf{v}'\underline{\mathbf{d}} + (\mathbf{w} + \mathbf{u} - \mathbf{v})'\,\tilde{\mathbf{d}}\right)^+ + \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\mathbf{u}'(\overline{\mathbf{d}} - \tilde{\mathbf{d}}) + \mathbf{v}'(\tilde{\mathbf{d}} - \underline{\mathbf{d}})\right) \\
&= \chi\left(w_0 - \mathbf{u}'\overline{\mathbf{d}} + \mathbf{v}'\underline{\mathbf{d}}, \mathbf{w} + \mathbf{u} - \mathbf{v}\right) + \mathbf{u}'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}'(\boldsymbol{\mu} - \underline{\mathbf{d}}) \\
&\leq \mathbf{u}'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}'(\boldsymbol{\mu} - \underline{\mathbf{d}}) + \eta^\dagger(w_0 - \mathbf{u}'\overline{\mathbf{d}} + \mathbf{v}'\underline{\mathbf{d}}, \mathbf{w} + \mathbf{u} - \mathbf{v}).
\end{aligned}$$

The arbitrariness of $\mathbf{u}, \mathbf{v}$ enforces the inequality $\chi(w_0, \mathbf{w}) \leq \eta(w_0, \mathbf{w})$. □

When $K = 1$, the LMI based bound $\eta(\cdot, \cdot)$ is tight because the proposed approximation of the semi-infinite constraints in (6.14) turns to be equivalent due to the famous $\mathcal{S}$-lemma. This indicates that $\eta(\cdot, \cdot)$ is tighter than $\pi(\cdot, \cdot)$ when $K = 1$. In fact, $\eta(\cdot, \cdot)$ is always tighter than $\pi(\cdot, \cdot)$ for every $K \in \mathbb{Z}_+$. In order to prove this tightness, we give the following closed form of a specific LMI.

Lemma 6.5. *For $\boldsymbol{\Sigma} \succ \mathbf{0}, \boldsymbol{\mu} \in \Re^K$ and $(w_0, \mathbf{w}) \in \Re^{K+1}$, we have*

$$\inf \left\{ \langle \Omega, \mathbf{M} \rangle : \mathbf{M} \succcurlyeq \mathbf{0}, \mathbf{M} \succcurlyeq \begin{bmatrix} \mathbf{0} & \frac{1}{2}\mathbf{w} \\ \frac{1}{2}\mathbf{w}' & w_0 \end{bmatrix}, \mathbf{M} \in \mathbb{S}^{K+1} \right\}$$
$$= \frac{1}{2}\left(w_0 + \mathbf{w}'\boldsymbol{\mu} + \sqrt{(w_0 + \mathbf{w}'\boldsymbol{\mu})^2 + \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}\right),$$

where we define $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\Sigma} & \boldsymbol{\mu} \\ \boldsymbol{\mu}' & 1 \end{bmatrix}.$$

Proof. Let $\mathbb{F}_{\boldsymbol{\xi}}$ be the family of distributions of random vector $\tilde{\boldsymbol{\xi}}$ such that the first- and second-order moments are $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, i.e.,

$$\mathbb{F}_{\boldsymbol{\xi}} = \left\{ \mathbb{P} \in \mathbb{M}(\Re^K) : \mathbb{E}_{\mathbb{P}}\left(\left[\tilde{\boldsymbol{\xi}}'\ 1\right]' \left[\tilde{\boldsymbol{\xi}}'\ 1\right] \right) = \boldsymbol{\Omega} \right\}.$$

For $(w_0, \mathbf{w}) \in \Re^{K+1}$, let $\mu = w_0 + \mathbf{w}'\boldsymbol{\mu}$ and $\sigma = \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$. Then clearly the distribution of $w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}}$ belongs to the distribution family of random scalar $\tilde{v}$ defined as

$$\mathbb{F}_v = \left\{\mathbb{P} \in \mathbb{M}(\Re) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{v}) = \mu, \mathbb{E}_{\mathbb{P}}(\tilde{v}^2) = \mu^2 + \sigma^2\right\}.$$

From the general projection property (Theorem 1 of Popescu (2007)), we have

$$\sup_{\mathbb{P}\in\mathbb{F}_{\boldsymbol{\xi}}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}})^+ = \sup_{\mathbb{P}\in\mathbb{F}_v} \mathbb{E}_{\mathbb{P}}(\tilde{v})^+. \tag{6.15}$$

The left side of the above equation can equivalently be expressed as an infinite-dimensional linear program with respect to its probability density function $p(\tilde{\boldsymbol{\xi}})$ as:

$$\begin{aligned}
\sup_{\mathbb{P}\in\mathbb{F}_{\boldsymbol{\xi}}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}})^+ = \sup & \int_{\Re^K} \max\{0, w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}}\} p(\tilde{\boldsymbol{\xi}})\,\mathrm{d}\tilde{\boldsymbol{\xi}} \\
\text{s.t.} & \int_{\Re^K} \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix}' \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix} p(\tilde{\boldsymbol{\xi}})\,\mathrm{d}\tilde{\boldsymbol{\xi}} = \boldsymbol{\Omega} \\
& p(\tilde{\boldsymbol{\xi}}) \geq 0 : \forall \tilde{\boldsymbol{\xi}} \in \Re^K
\end{aligned}$$

By introducing dual variables $\mathbf{M} \in \mathbb{S}^{K+1}$ corresponding to the $(K+1)\times(K+1)$ constraints, we obtain its equivalent dual formulation of this infinite-dimensional linear program by strong duality condition

$$\begin{aligned}
\sup_{\mathbb{P}\in\mathbb{F}_{\boldsymbol{\xi}}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}})^+ = \inf \quad & \langle \boldsymbol{\Omega}, \mathbf{M}\rangle \\
\text{s.t.} \quad & \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix} \mathbf{M} \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix}' \geq 0 : \forall \tilde{\boldsymbol{\xi}} \in \Re^K \\
& \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix} \mathbf{M} \begin{bmatrix}\tilde{\boldsymbol{\xi}}' & 1\end{bmatrix}' \geq w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}} : \forall \tilde{\boldsymbol{\xi}} \in \Re^K.
\end{aligned}$$

We can therefore expressed $\sup_{\mathbb{P}\in\mathbb{F}_{\boldsymbol{\xi}}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}})^+$ as:

$$\sup_{\mathbb{P}\in\mathbb{F}_{\boldsymbol{\xi}}} \mathbb{E}_{\mathbb{P}}(w_0 + \mathbf{w}'\tilde{\boldsymbol{\xi}})^+ = \inf \left\{ \langle \boldsymbol{\Omega}, \mathbf{M}\rangle : \mathbf{M} \succcurlyeq \mathbf{0}, \mathbf{M} \succcurlyeq \begin{bmatrix} \mathbf{0} & \frac{1}{2}\mathbf{w} \\ \frac{1}{2}\mathbf{w}' & w_0 \end{bmatrix} \right\},$$

which is exactly the left side of our claimed equation. Recall equation (6.15), it suffices to show

$$\sup_{\mathbb{P}\in\mathbb{F}_v} \mathbb{E}_{\mathbb{P}}(\tilde{v})^+ = \frac{1}{2}\left(\mu + \sqrt{\mu^2+\sigma^2}\right).$$

Observe that $\tilde{v}^+ \le \frac{1}{4\sqrt{\mu^2+\sigma^2}}\left(\tilde{v} + \sqrt{\mu^2+\sigma^2}\right)^2$ for every $\tilde{v} \in \Re$, we certainly have

$$\sup_{\mathbb{P}\in\mathbb{F}_v} \mathbb{E}_{\mathbb{P}}(\tilde{v})^+ \le \frac{1}{4\sqrt{\mu^2+\sigma^2}} \sup_{\mathbb{P}\in\mathbb{F}_v} \mathbb{E}_{\mathbb{P}}\left(\tilde{v} + \sqrt{\mu^2+\sigma^2}\right)^2 = \frac{1}{2}\left(\mu + \sqrt{\mu^2+\sigma^2}\right).$$

In addition, Let $\tilde{v}$ be a random variable such that

$$\tilde{v} = \begin{cases} \sqrt{\mu^2+\sigma^2} & \text{w.p. } \dfrac{\mu + \sqrt{\mu^2+\sigma^2}}{2\sqrt{\mu^2+\sigma^2}} \\ -\sqrt{\mu^2+\sigma^2} & \text{w.p. } \dfrac{-\mu + \sqrt{\mu^2+\sigma^2}}{2\sqrt{\mu^2+\sigma^2}}. \end{cases}$$

It is no hard to see that $\mathbb{E}(\tilde{v}) = \mu$, $\mathbb{E}(\tilde{v})^2 = \mu^2 + \sigma^2$ and $\mathbb{E}(\tilde{v})^+ = \frac{1}{2}\left(\mu + \sqrt{\mu^2+\sigma^2}\right)$. Hence,

$$\sup_{\mathbb{P}\in\mathbb{F}_v} \mathbb{E}_{\mathbb{P}}(\tilde{v})^+ \ge \frac{1}{2}\left(\mu + \sqrt{\mu^2+\sigma^2}\right);$$

the claim follows. □

Proposition 6.4. *Suppose* $\mathcal{D} = [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$ *and* $\mathbb{F} = \left\{\mathbb{P} \in \mathbb{M}(\mathcal{D}) : \mathbb{E}_{\mathbb{P}}(1) = 1, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}\tilde{\mathbf{d}}') = \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\Sigma}\right\}$*, in which* $\boldsymbol{\Sigma} \succ \mathbf{0}$*. For every* $(w_0, \mathbf{w}) \in \Re^{K+1}$*, we can deduce that*

$$\chi(w_0, \mathbf{w}) \le \eta(w_0, \mathbf{w}) \le \pi(w_0, \mathbf{w}),$$

where $\pi(\cdot,\cdot)$ *and* $\eta(\cdot,\cdot)$ *are given in (6.11) and (6.12), respectively.*

Proof. We first show the following elementary result:

$$\pi(w_0, \mathbf{w}) = \min_{\mathbf{u}_1,\mathbf{v}_1,\mathbf{u}_2,\mathbf{v}_2\ge\mathbf{0}} \left\{\frac{1}{2}\Big(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + \mathbf{u}_2'\overline{\mathbf{d}} - \mathbf{v}_2'\underline{\mathbf{d}}\right.$$

$$+ (\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1 - \mathbf{u}_2 + \mathbf{v}_2)'\boldsymbol{\mu} + \sqrt{\left(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}} + \boldsymbol{\omega}'\boldsymbol{\mu}\right)^2 + \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}\Bigg)\Bigg\}, \tag{6.16}$$

where we define $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = \mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{u}_2 - \mathbf{v}_2$.

For the special case $\boldsymbol{\mu} = \mathbf{0}$, the right hand side of the above equation is exactly the same as the bound of $\chi(\cdot,\cdot)$ provided in Theorem 1(a) of Chen et al. (2008). To keep our proof self-contained, here we prove this result in a different way. If we fix the variables $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2$, the minimum objective function of (6.11) reduces to

$$\min \left\{ (w_{10} + a)^+ + \max\{w_{20}, b\} + \frac{1}{2}(w_{30} + \sqrt{w_{30}^2 + \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}) \right.$$
$$\left. : w_{10} + w_{20} + w_{30} = w_0 + \mathbf{w}'\boldsymbol{\mu} \right\},$$

where $a = \mathbf{u}_1'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_1'(\boldsymbol{\mu} - \underline{\mathbf{d}}), b = \mathbf{u}_2'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_2'(\boldsymbol{\mu} - \underline{\mathbf{d}})$. It can be seen that, when we fix a, b, we obtain the minima at

$$\begin{aligned}
w_{10}^* &= -a \\
w_{20}^* &= b \\
w_{30}^* &= w_0 + \mathbf{w}'\boldsymbol{\mu} + a - b = w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}} + \boldsymbol{\omega}'\boldsymbol{\mu},
\end{aligned}$$

and its corresponding objective value coincides with the right hand side of (6.16) when fixing $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2$. This completes the proof of (6.16).

Consider the LMI representation of (6.13). Note the fact that the variables $(\mathbf{M}, \mathbf{u}, \mathbf{v}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ satisfying

$$\begin{aligned}
&\mathbf{M} \succcurlyeq \mathbf{0} \\
&\mathbf{M} \succcurlyeq \begin{bmatrix} \mathbf{0} & \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v}) \\ \frac{1}{2}(\mathbf{w} - \mathbf{u} + \mathbf{v})' & w_0 + \mathbf{u}'\overline{\mathbf{d}} - \mathbf{v}'\underline{\mathbf{d}} \end{bmatrix} \\
&\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \mathbf{0}
\end{aligned}$$

$$\mathbf{u}, \mathbf{v} \geq \mathbf{0}$$

is feasible for (6.13). Hence, we have

$$\begin{aligned}
&\eta^\dagger(w_0, \mathbf{w}) \\
&\leq \min_{\mathbf{M}\in\mathbb{S}^K, \mathbf{u}_1, \mathbf{v}_1 \geq \mathbf{0}} \left\{ \langle \mathbf{\Omega}, \mathbf{M} \rangle : \mathbf{M} \succcurlyeq \mathbf{0}, \mathbf{M} \succcurlyeq \begin{bmatrix} \mathbf{0} & \frac{1}{2}(\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1) \\ \frac{1}{2}(\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1)' & w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} \end{bmatrix} \right\} \\
&= \min_{\mathbf{u}_1, \mathbf{v}_1 \geq \mathbf{0}} \left\{ \frac{1}{2} \Big(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + (\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1)'\boldsymbol{\mu} + \right. \\
&\quad \left. \sqrt{\left(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + (\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1)'\boldsymbol{\mu}\right)^2 + (\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1)'\mathbf{\Sigma}(\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1)} \Big) \right\}.
\end{aligned}$$

The equality is the consequence of applying Lemma 6.5 by letting $\mathbf{w} := \mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1$ and $w_0 := w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}}$. It follows that

$$\begin{aligned}
&\eta(w_0, \mathbf{w}) \\
&= \min_{\mathbf{u}_2, \mathbf{v}_2 \geq \mathbf{0}} \left\{ \mathbf{u}_2'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_2'(\boldsymbol{\mu} - \mathbf{d}) + \eta^\dagger(w_0 - \mathbf{u}_2\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}}, \mathbf{w} + \mathbf{u}_2 - \mathbf{v}_2) \right\} \\
&\leq \min_{\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 \geq \mathbf{0}} \left\{ \mathbf{u}_2'(\overline{\mathbf{d}} - \boldsymbol{\mu}) + \mathbf{v}_2'(\boldsymbol{\mu} - \mathbf{d}) + \frac{1}{2} \Big(w_0 - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\mathbf{d} + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} \right. \\
&\quad \left. + (\mathbf{w} + \mathbf{u}_2 - \mathbf{v}_2 - \mathbf{u}_1 + \mathbf{v}_1)'\boldsymbol{\mu} + \sqrt{\left(w_0 - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}} + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + \boldsymbol{\omega}'\boldsymbol{\mu}\right)^2 + \boldsymbol{\omega}'\mathbf{\Sigma}\boldsymbol{\omega}} \Big) \right\} \\
&= \min_{\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 \geq \mathbf{0}} \left\{ \frac{1}{2} \Big(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + \mathbf{u}_2'\overline{\mathbf{d}} - \mathbf{v}_2'\underline{\mathbf{d}} + (\mathbf{w} - \mathbf{u}_1 + \mathbf{v}_1 - \mathbf{u}_2 + \mathbf{v}_2)'\boldsymbol{\mu} \right. \\
&\quad \left. + \sqrt{\left(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}} + \boldsymbol{\omega}'\boldsymbol{\mu}\right)^2 + \boldsymbol{\omega}'\mathbf{\Sigma}\boldsymbol{\omega}} \Big) \right\} \\
&= \pi(w_0, \mathbf{w}).
\end{aligned}$$

The last equality above is due to (6.16). The claim follows. □

By using these bounds, we can obtain tractable approximations of $\mathcal{X}_{IARP}(\gamma)$. Specifically, let us denote by $\mathcal{X}_{IARP}^{SOCP}(\gamma)$ and $\mathcal{X}_{IARP}^{LMI}(\gamma)$ as the approximated sets associated with SOCP based bound $\pi(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$, respectively. In fact, we can conclude that both approximations are less conservative than $\mathcal{X}_{ARP}(\gamma)$.

Proposition 6.5. *For every* $\gamma \in (0,1)$*, we have* $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}_{IARP}^{SOCP}(\gamma) \subseteq \mathcal{X}_{IARP}^{LMI}(\gamma) \subseteq \mathcal{X}_{IARP}(\gamma)$.

Proof. It follows directly from Proposition 6.4 that

$$\mathcal{X}_{IARP}^{SOCP}(\gamma) \subseteq \mathcal{X}_{IARP}^{LMI}(\gamma) \subseteq \mathcal{X}_{IARP}(\gamma).$$

To prove the rest part $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}_{IARP}^{SOCP}(\gamma)$, we denote by $\hat{g}(\nu, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r})$ as the corresponding approximation of function $g(\nu, \mathbf{z}, \mathbf{s}, \mathbf{t}, \mathbf{h}, \mathbf{r})$ using bound $\pi(\cdot,\cdot)$. We observe that $\pi(\cdot,\cdot)$ possesses two elementary properties:

1. If $w_0 + \mathbf{w}'\tilde{\mathbf{d}} \leq 0 : \forall \tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$, then $\pi(w_0, \mathbf{w}) = 0$.
2. If $w_0 + \mathbf{w}'\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$, then $\pi(w_0, \mathbf{w}) = w_0 + \mathbf{w}'\boldsymbol{\mu}$.

The proof of these two properties closely follows to Theorem 1(c) in Chen et al. (2008) by considering the equivalent formulation (6.16).

If $w_0 + \mathbf{w}'\tilde{\mathbf{d}} \leq 0 : \forall \tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{d}]$, let $\mathbf{u}_1 = \mathbf{w}^+, \mathbf{v}_1 = \mathbf{w}^-, \mathbf{u}_2 = \mathbf{v}_2 = 0$. It is not hard to see that

$$\begin{aligned} &\mathbf{w} = \mathbf{u}_1 - \mathbf{v}_1 \\ &w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} = \max_{\tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{d}]} \left\{ w_0 + \mathbf{w}'\tilde{\mathbf{d}} \right\} \leq 0. \end{aligned}$$

Hence,

$$0 \leq \pi(w_0, \mathbf{w}) \leq \frac{1}{2}\left(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}} + \sqrt{\left(w_0 + \mathbf{u}_1'\overline{\mathbf{d}} - \mathbf{v}_1'\underline{\mathbf{d}}\right)^2} \right) = 0$$

indicating that $\pi(w_0, \mathbf{w}) = 0$.

Similarly, when $w_0 + \mathbf{w}'\tilde{\mathbf{d}} \geq 0 : \forall \tilde{\mathbf{d}} \in [\underline{\mathbf{d}}, \overline{\mathbf{d}}]$ then let $\mathbf{u}_1 = \mathbf{v}_1 = \mathbf{0}, \mathbf{u}_2 = \mathbf{w}^-, \mathbf{v}_2 = \mathbf{w}^+$. We can also deduce the following inequality, which verifies property 2:

$$w_0 + \mathbf{w}'\boldsymbol{\mu} \leq \pi(w_0, \mathbf{w}) \leq \frac{1}{2}\left(w_0 + \mathbf{u}_2'\overline{\mathbf{d}} - \mathbf{v}_2'\underline{\mathbf{d}} + 2\mathbf{w}'\boldsymbol{\mu} + \sqrt{\left(w_0 - \mathbf{u}_2'\overline{\mathbf{d}} + \mathbf{v}_2'\underline{\mathbf{d}}\right)^2} \right) = w_0 + \mathbf{w}'\boldsymbol{\mu}.$$

Note the fact that for every $\mathbf{x} \in \mathcal{X}_{ARP}(\gamma)$, we define the following variables from its associated feasible variables $\nu, \{\theta^0, \boldsymbol{\theta}\}, \{z_{kp}^0, \mathbf{z}_{kp}\}_{(k,p)\in\hat{\Phi}}, \{s_k^0, \mathbf{s}_k, t_k^0, \mathbf{t}_k\}_{k\in\mathcal{K}}$ in (6.8) as

$$
\begin{aligned}
& z_{kp}^*(\tilde{\mathbf{d}}) = z_{kp}^0(\tilde{\mathbf{d}}) + \mathbf{z}_{kp}'\tilde{\mathbf{d}} : (k,p) \in \hat{\Phi} \\
& s_k^*(\tilde{\mathbf{d}}) = s_k^0 + \mathbf{s}_k'\tilde{\mathbf{d}} : k \in \mathcal{K} \\
& t_k^*(\tilde{\mathbf{d}}) = t_k^0 + \mathbf{t}_k'\tilde{\mathbf{d}} : k \in \mathcal{K} \\
& h_{km}^*(\tilde{\mathbf{d}}) = 0 : (k,m) \in \check{\Phi} \\
& r_k^*(\tilde{\mathbf{d}}) = 0 : k \in \mathcal{K} \\
& \nu^* = \nu.
\end{aligned}
$$

It can be seen that all the nonlinear terms appears in rows two to three of $\hat{g}$ vanishes and we therefore have

$$
\begin{aligned}
& \hat{g}(\nu^*, \mathbf{z}^*, \mathbf{s}^*, \mathbf{t}^*, \mathbf{r}^*, \mathbf{h}^*) \\
& = \pi\left(\sum_{k\in\mathcal{K}} t_k^0 - \nu, \sum_{k\in\mathcal{K}} \mathbf{t}_k\right) \\
& \leq \pi(\theta^0, \boldsymbol{\theta}) + \pi\left(\sum_{k\in\mathcal{K}} t_k^0 - \nu - \theta_0, \sum_{k\in\mathcal{K}} \mathbf{t}_k - \boldsymbol{\theta}\right) \\
& = \pi(\theta^0, \boldsymbol{\theta}) \\
& = \theta_0 + \boldsymbol{\theta}'\boldsymbol{\mu}.
\end{aligned}
$$

The second last equality is due to the first property of $\pi(\cdot,\cdot)$ and the last equality is because its second property. Hence, we can conclude that $\nu + \frac{1}{1-\gamma}\hat{g}(\nu^*, \mathbf{z}^*, \mathbf{s}^*, \mathbf{t}^*, \mathbf{r}^*, \mathbf{h}^*) \leq \tau$ indicating that $\mathbf{x} \in \mathcal{X}_{IARP}^{SOCP}(\gamma)$. Consequently, the claim $\mathcal{X}_{ARP}(\gamma) \subseteq \mathcal{X}_{IARP}^{SOCP}(\gamma)$ follows.

□

6.4 A Single Convex Formulation of the Resilience Index

So far, we manage to build tractable approximations of $\mathcal{X}(\gamma)$ such that the original problem (6.3) can be approximately solved by a sequence of sub-problems

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in \widehat{\mathcal{X}}(\gamma) \end{array}$$

with $\widehat{\mathcal{X}}(\gamma)$ being a tractable approximation of $\mathcal{X}(\gamma)$. Hence, the overall computational time might be high if high accuracy of the optimal objective γ^* is required because we have to solve the sub-problem in many times. In this section we show that under mild conditions, the original problem can be approximately solved by a single convex optimization problem with its computational complexity the same as that of the above sub-problem. The following result gives an equivalent representation of the resilience index.

Proposition 6.6. *Let us define* $\rho_{\text{SAM}}(\tilde{v}, \mathbb{P}) \triangleq \sup\{\gamma : \text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma) \leq 0\}$ *for every* $\tilde{v} \in L^\infty(\mathcal{D}, \mathcal{F}, \mathbb{P})$. *Suppose that for every* $\mathbb{P} \in \mathbb{F}$, $\rho_{\text{SAM}}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{P}\right) \in (0, 1)$. *Then,*

$$\varrho(\mathcal{G}, \mathbf{x}) = \rho_{\text{DSAM}}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{F}\right) = \sup_{\alpha \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min\left\{1, \alpha\left(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})\right)\right\}\right).$$

In addition, the optimal α *value in the right hand side of the above equality resides in* $(0, +\infty)$.

Proof. We first show the following equality:

$$\rho_{\text{DSAM}}(\tilde{v}, \mathbb{F}) = \inf_{\mathbb{P} \in \mathbb{F}} \{\rho_{\text{SAM}}(\tilde{v}, \mathbb{P})\}. \tag{6.17}$$

Denote by γ_L and γ_R as the left side and right side of the above equality, respectively. We complete the proof of $\gamma_L = \gamma_R$ by two separate parts.

a) $\gamma_R \leq \gamma_L$. Suppose by contradiction that $\gamma_R > \gamma_L$, or equivalently, $\exists \epsilon > 0$ such that $\gamma_R > \gamma_L + \epsilon$. By definition we can deduce that $\rho_{\text{SAM}}(\tilde{v}, \mathbb{P}) > \gamma_L + \epsilon$ for every $\mathbb{P} \in \mathbb{F}$. It follows that for every $\mathbb{P} \in \mathbb{F}$, $\text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma_L + \epsilon) \leq 0$ indicating that $\sup_{\mathbb{P}\in\mathbb{F}} \text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma_L + \epsilon) \leq 0$. Consequently, we have $\gamma_L = \rho_{\text{DSAM}}(\tilde{v}, \mathbb{F}) \geq \gamma_L + \epsilon$ which is impossible. In conclusion, we have $\gamma_R \leq \gamma_L$.

b) $\gamma_L \leq \gamma_R$. Similarly, we assume by contradiction that there exists $\epsilon > 0$ such that $\gamma_L > \gamma_R + \epsilon$. It follows from the definition of γ_L that $\sup_{\mathbb{P}\in\mathbb{F}} \text{CVaR}_{\mathbb{P}}(\tilde{v}, \gamma_R + \epsilon) \leq 0$. Hence, for every $\mathbb{P} \in \mathbb{F}$ we have $\rho_{\text{SAM}}(\tilde{v}, \mathbb{P}) \geq \gamma_R + \epsilon$ indicating that $\gamma_R = \inf_{\mathbb{P}\in\mathbb{F}} \{\rho_{\text{SAM}}(\tilde{v}, \mathbb{P}) \geq \gamma_R + \epsilon\}$. This contradiction leads to the claim $\gamma_L \leq \gamma_R$.

From (6.17) we directly deduce that $\varrho(\mathcal{G}, \mathbf{x}) = \inf_{\mathbb{P}\in\mathbb{F}}\{\rho_{\text{SAM}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{P})\}$. For every $\mathbb{P} \in \mathbb{F}$, it follow from $\rho_{\text{SAM}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{P}) \in (0, 1)$ that $\mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) > \tau) > 0$, or equivalently $\exists \epsilon > 0$ such that $\mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon) > 0$.

Observe that for every $\mathbb{P} \in \mathbb{F}$,

$$\begin{aligned}
\rho_{\text{SAM}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{P}) = \quad & \sup_{\gamma} \left\{ \gamma : \text{CVaR}_{\mathbb{P}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \gamma) \leq 0 \right\} \\
= \quad & \sup_{\gamma,\nu} \left\{ \gamma : \nu + \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}} \left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau - \nu \right)^+ \leq 0 \right\} \\
\underbrace{=}_{(a)} \quad & \sup_{\gamma,\nu} \left\{ \gamma : \frac{1}{1-\gamma} \mathbb{E}_{\mathbb{P}} \left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau + \nu \right)^+ \leq \nu, \nu > 0 \right\} \\
= \quad & \sup_{\gamma} \left\{ \gamma : \gamma \leq 1 - \inf_{\nu>0} \mathbb{E}_{\mathbb{P}} \left(\frac{1}{\nu}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau) + 1 \right)^+ \right\} \\
= \quad & \sup_{\nu>0} \mathbb{E}_{\mathbb{P}} \left(\min \left\{ \frac{1}{\nu}(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})), 1 \right\} \right) \\
\underbrace{=}_{(b)} \quad & \sup_{\alpha\geq 0} \mathbb{E}_{\mathbb{P}} \left(\min\{1, \alpha(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}))\} \right).
\end{aligned}$$

Because $\mathbb{P}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) > \tau\right) > 0$, we have $\mathbb{E}_{\mathbb{P}} \left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau\right)^+ > 0$. Therefore, $\nu + \dfrac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}} \left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau - \nu\right)^+ \leq 0$ implying $\nu < 0$. Hence, equality (a) is enforced by substituting ν with $-\nu$. Because $\rho_{\text{SAM}}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau, \mathbb{P}) \in (0, 1)$, the optimal α value α^*

should be positive. Observe that

$$\begin{aligned} 0 &\leq \mathbb{E}_{\mathbb{P}}\left(\min\{1, \alpha^*(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}))\}\right) \\ &\leq \left(1 - \mathbb{P}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon\right)\right) - \mathbb{P}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon\right)\alpha^*\epsilon, \end{aligned}$$

we then have

$$\alpha^* \leq \frac{1}{\epsilon\mathbb{P}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon\right)}\left(1 - \mathbb{P}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon\right)\right).$$

Hence, the additional relaxation from $\alpha > 0, \alpha < +\infty$ to $\alpha \geq 0$ would not affect its maximum value. This implies equality (b). Therefore, we have

$$\begin{aligned} \varrho(\mathcal{G}, \mathbf{x}) &= \inf_{\mathbb{P}\in\mathbb{F}} \sup_{\alpha\geq 0} \mathbb{E}_{\mathbb{P}}\left(\min\{1, \alpha(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}))\}\right) \\ &= \sup_{\alpha\geq 0} \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min\left\{1, \alpha(\tau - \mathcal{Q}\left(\mathbf{x}, \tilde{\mathbf{d}})\right)\right\}\right). \end{aligned}$$

The second equality is due to the stochastic saddle point theorem in Shapiro and Kleywegt (2002). The first claim is verified. Moreover, since the right side of the above equation takes value in $(0, 1)$, we can deduce that $\alpha^* > 0$. Meanwhile, for distribution $\mathbb{P}$ such that $\mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon) > 0$, we have $\alpha^* \leq \left(1 - \mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon)\right) / \left(\mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \geq \tau + \epsilon)\epsilon\right) < +\infty$. All the claims are verified. □

By replacing the objective function $\varrho(\mathcal{G}, \mathbf{x})$ by $\sup_{\alpha\geq 0} \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min\left\{1, \alpha(\tau - \mathcal{Q}\left(\mathbf{x}, \tilde{\mathbf{d}})\right)\right\}\right)$, we can solve the following distributionally robust model instead of (6.3):

$$\begin{aligned} \hat{Z}_0^* = \max_{\mathbf{x},\alpha} \quad & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min\{1, \alpha(\tau - \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}))\}\right) \\ \text{s.t.} \quad & \mathbf{c}'\mathbf{x} \leq B \\ & \mathbf{x} \geq 0, \alpha \geq 0 \end{aligned} \tag{6.18}$$

It can be seen from Proposition 6.6 that Z_0^* and $\hat{Z}_0^*$ coincide except in the degenerate cases (In these degenerate cases, $\varrho(\mathcal{G}, \mathbf{x})$ equals to 0 or 1. The 0 cases can be identified by checking $\min_{\tilde{\mathbf{d}}\in\mathcal{D}} \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) > \tau$, and the 1 cases can be identified by checking $\max_{\tilde{\mathbf{d}}\in\mathcal{D}} \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \leq \tau$). Model (6.18) can be formulated into a single optimization problem via variables substitution from $\mathbf{x} := \alpha\mathbf{x}$. This formulation can avoid the binary search scheme in the computation of the resilience index and thus has the potential of reducing the overall computational effort.

It is also worth pointing out that (6.18) is still intractable because the appearance of nonlinear term $\mathbb{E}_{\mathbb{P}}(\cdot)^+$ in the objective function and the nonlinear penalty cost structure $p_k(s_k) : k \in \mathcal{K}$. Nonetheless, tractable approximation can be obtained by using decision rules. The principles of deriving these approximations are nearly the same as what we have discussed for the convex set $\mathcal{X}(\gamma)$ in Section 6.3. For instance, if we apply the improved affine routing policy, we can approximately solve (6.18) via the following optimization problem:

$$
\begin{aligned}
\hat{Z}^*_{IARP} =& 1 - \\
& \min \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(1 - \alpha\tau + \sum_{k\in\mathcal{K}} t_k(\tilde{\mathbf{d}}) + \sum_{(k,m)\in\check{\Phi}} h_{km}(\tilde{\mathbf{d}}) + \sum_{k\in\mathcal{K}} p_{kM} r_k(\tilde{\mathbf{d}}) \right)^+ \\
& \quad + \sum_{(k,m)\in\check{\Phi}} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(p_{km} s_k(\tilde{\mathbf{d}}) + \alpha q_{km} - t_k(\tilde{\mathbf{d}}) - h_{km}(\tilde{\mathbf{d}}) \right)^+ \\
& \quad + \sum_{(k,m)\in\check{\Phi}} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-h_{km}(\tilde{\mathbf{d}}) \right)^+ + \sum_{k\in\mathcal{K}} p_{kM} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-r_k(\tilde{\mathbf{d}}) \right)^+ \\
& \quad + \sum_{k\in\mathcal{K}} p_{kM} \sup_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(-s_k(\tilde{\mathbf{d}}) - r_k(\tilde{\mathbf{d}}) \right)^+ \\
& \text{s.t.} \{z_{kp}(\tilde{\mathbf{d}})\}_{(k,p)\in\hat{\Phi}} \in \mathcal{L}(N) \\
& \quad \{h_{km}(\tilde{\mathbf{d}})\}_{(k,m)\in\check{\Phi}} \in \mathcal{L}(|\check{\Phi}|) \\
& \quad \{s_k(\tilde{\mathbf{d}}), t_k(\tilde{\mathbf{d}}), r_k(\tilde{\mathbf{d}})\}_{k\in\mathcal{K}} \in \mathcal{L}(K) \\
& \quad \sum_{p\in\mathcal{P}_k} z_{kp}(\tilde{\mathbf{d}}) + s_k(\tilde{\mathbf{d}}) \geq \alpha \tilde{d}_k : \forall \tilde{\mathbf{d}} \in \mathcal{D}, k \in \mathcal{K}
\end{aligned}
$$

$$\sum_{(k,p)\in\hat{\Phi}} \delta_{pa} z_{kp}(\tilde{\mathbf{d}}) \leq x_a : \forall \tilde{\mathbf{d}} \in \mathcal{D}, a \in \mathcal{A}$$

$$z_{kp}(\tilde{\mathbf{d}}) \geq 0 : \forall \tilde{\mathbf{d}} \in \mathcal{D}, (k,p) \in \hat{\Phi}$$

$$\mathbf{c}'\mathbf{x} \leq B\alpha$$

$$\alpha \geq 0$$

$$\mathbf{x} \geq \mathbf{0}.$$

Indeed, we can also obtain the inequality $\hat{Z}^*_{IARP} \leq \hat{Z}^*_0$. The proof is similar to that of Proposition 6.3 and hence omitted.

6.5 Computational Studies

In this section, we conduct some computational experiments to study the following issues of interest. First, we investigate the performance improvement of using additional moment information over the traditional uncertainty set based approach. Second, we investigate if our proposed distributionally-ambiguous shortfall awareness measure can provide reasonable design solution under uncertainty. In the first experiment, we investigate the value of moment information in providing less conservative solutions. The improved affine routing policy is believed to be less sensitive to the specified support set $\mathcal{D}$ because the incorporated second moment information would play a role of correcting the errors in support set specification. The second experiment focuses on the general telecommunication network design problem and compares the performances of three different design strategies.

We consider the general problem of designing a telecommunication network under demand uncertainty, and compare our proposed strategy with two benchmark solution strategies: (1) the robust optimization approach of maximizing the budget of uncertainty (MaBU) and (2) minimizing the expected total cost (MiETC) using a sample average approximation scheme. The robust approach is favored in the literature because it requires less information on the uncertainty and its resulting robust counterpart is relatively easier to solve (see Altın et al., 2007, 2011; Lee et al., 2012; Lemaréchal et al., 2010; Koster et al., 2010, 2011a,b, 2013, and the references therein). In contrast, MiETC is based on a Utopian assumption that the distributional information of the under demand vector $\tilde{\mathbf{d}}$ is precisely known, which is unrealistic in practice.

Maximizing budget of uncertainty With the belief that the uncertain vector $\tilde{\mathbf{d}}$ symmetrically resides in $[\boldsymbol{\mu} - \hat{\mathbf{d}}, \boldsymbol{\mu} + \hat{\mathbf{d}}]$, Bertsimas and Sim (2003, 2004) introduce the budget of uncertainty Γ and define budget uncertainty sets as $\mathcal{D}_\Gamma = \left\{ \mathbf{d} : \sum_{k \in \mathcal{K}} \frac{|d_k - \mu_k|}{\hat{d}_k} \leq \Gamma \right\}$. For fixed Γ value, the robust telecommunication network design problem is formulated as:

$$
\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & \max_{\tilde{\mathbf{d}} \in \mathcal{D}_\Gamma} \sum_{(k,p) \in \hat{\Phi}} \delta_{pa} z^0_{kp} \tilde{d}_k \leq x_a : \forall a \in \mathcal{A} \\
& \sum_{p \in \mathcal{P}_k} z^0_{kp} = 1 : \forall k \in \mathcal{K} \\
& z^0_{kp} \geq 0 : \forall (k,p) \in \hat{\Phi}.
\end{aligned}
$$

The above model aims to find a minimum cost capacity installation $\mathbf{x}$ such that, for every demand vector $\tilde{\mathbf{d}} \in \mathcal{D}_\Gamma$, we can always find a path flow solution that simultaneously supports $\tilde{\mathbf{d}}$ and respects capacity installation $\mathbf{x}$. It is worth noting that oblivious routing policy $z_{kp} = z^0_{kp} \tilde{d}_k$ is adopted for model simplicity. When limited investment budget B is available, we slightly modify the model objective to maximize the budget of uncertainty Γ. A precise formulation is given as:

$$
\begin{aligned}
\Gamma^* = \max \quad & \Gamma \\
\text{s.t.} \quad & \mathbf{c}'\mathbf{x} \leq B \\
& \max_{\tilde{\mathbf{d}} \in \mathcal{D}_\Gamma} \sum_{(k,p) \in \hat{\Phi}} \delta_{pa} z^0_{kp} \tilde{d}_k \leq x_a : \forall a \in \mathcal{A} \\
& \sum_{p \in \mathcal{P}_k} z^0_{kp} = 1 : \forall k \in \mathcal{K} \\
& z^0_{kp} \geq 0 : \forall (k,p) \in \hat{\Phi}.
\end{aligned}
$$

Minimizing expected total cost For fixed capacity installation $\mathbf{x}$, the penalty cost depends critically on the demand vector $\tilde{\mathbf{d}}$ and thus the total cost is uncertain. A typical decision criterion is to find a capacity installation such that the total expected cost is minimized. To make the comparison fair, the investment budget B is also imposed. Therefore, we give the MiETC model as:

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} + \mathbb{E}_{\mathbb{P}}\left[\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})\right] \\ \text{s.t.} & \mathbf{c}'\mathbf{x} \leq B. \end{array}$$

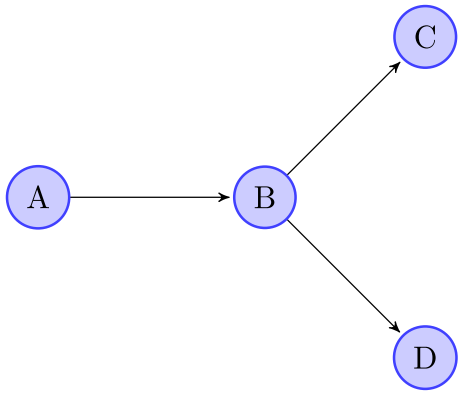
The above problem is intractable because multi-dimensional integration is involved in computing the expectation of nonlinear term $\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}})$, which is believed to be intractable in general (see Nemirovski and Shapiro, 2006). Suppose that the precise distribution of $\tilde{\mathbf{d}}$ is known and we have a random sample of L outcomes $\left\{\mathbf{d}^l\right\}_{l\in[L]}$ from this distribution. We adopt the following sampling average approximation scheme to solve it.

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} + \dfrac{1}{L}\sum\limits_{l\in[L]} \mathcal{Q}(\mathbf{x}, \mathbf{d}^l) \\ \text{s.t.} & \mathbf{c}'\mathbf{x} \leq B. \end{array}$$

It is noted that the success probability $\mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \leq \tau)$ is also commonly used as a decision criterion. However, it is highly intractable because it is non-convex such that the corresponding sampling average approximation scheme leads to MIP formulation. We exclude this criterion in comparison because calculation time would be quite long even for a small sample size. For our proposed distributionally-ambiguous shortfall aspiration awareness measure approach, we solve model (6.3) in a binary search scheme by using the improved affine routing policy, i.e., we approximate $\mathcal{X}(\gamma)$ with $\mathcal{X}_{IARP}(\gamma)$.

All the involved optimization problems are coded in MATLAB 2012 platform by calling the commercial software CPLEX 12.3 as the solver. The program is running on an Intel Dual Core i5-2500 PC with 8 GB RAM and 3.30 GHz CPU. The improved affine routing policy and the SOCP bound $\pi(\cdot,\cdot)$ is modeled using the MATLAB toolbox ROME (Goh and Sim, 2011).

Figure 6.1: A simple telecommunication network with four nodes.



6.5.1 Value of Moment Information

Bertsimas and Sim (2004) report that the Γ-model of uncertainty perform well when the random vector $\tilde{\mathbf{d}}$ is symmetrically distributed in the specified support $\mathcal{D} = [\boldsymbol{\mu} - \hat{\mathbf{d}}, \boldsymbol{\mu} + \hat{\mathbf{d}}]$ with mean $\boldsymbol{\mu}$. However, this is hardly the case under many real situations. Without the exact distribution of uncertainty, the only source of estimating $\mathcal{D}$ is observed data or some descriptive statistics. Hence, errors in the specified $\mathcal{D}$ are inevitable and therefore the quality and robustness of the solution obtained by the robust approach would be affected. Unlike the pure robust optimization approach, our proposed distributionally robust approach exploit moment information on the uncertainty, which is believed to reduce the impact of the errors in support estimation.

To verify this, we consider a telecommunication network with four nodes A, B, C and D (see Figure 6.1). There are two communication demands in this network: pairwise commodities AC and AD. In this experiment, we assume that the demand amount of these two commodities are $\tilde{d}_{AC} = \mu + \tilde{\xi}$ and $\tilde{d}_{AD} = \mu - \tilde{\xi}$, where $\tilde{\xi}$ is a random scalar that follows a two point distribution as:

$$\mathbb{P}\left(\tilde{\xi}=\xi\right)=\begin{cases}\beta & \text{if } \xi=-\dfrac{1}{\beta}\\ 1-\beta & \text{if } \xi=\dfrac{1}{1-\beta}.\end{cases}$$

Furthermore, we also assume that only the first- and second-order moments of the random variable $\tilde{\xi}$ are known to the decision maker. Specifically, $\tilde{\xi}$ is known to be a random scalar with zero mean and a standard deviation of $\sigma=\dfrac{1}{\sqrt{\beta(1-\beta)}}$.

In the absence of data, we implicitly assume that the support of $\tilde{\xi}$ is $[-3\sigma,3\sigma]$ based on a normal approximation. Indeed, this specified support set is corrupted by estimation error resulting from the assumed symmetric normal distribution. When β is smaller than $\frac{1}{2}$, $\tilde{\xi}$ is asymmetrically distributed and the value of $\frac{1}{\beta}$ reflects its level of asymmetry. Now let us consider a simple telecommunication network design problem of minimizing the total expected demand loss as:

$$\begin{aligned}
&\min \mathbb{E}\left[\max\left\{\left(\mu-x_1+\tilde{\xi}\right)^+ + \left(\mu-x_2-\tilde{\xi}\right)^+, (2\mu-x_3)^+\right\}\right]\\
&\text{s.t. } x_1+x_2+x_3\leq B\\
&\qquad \mathbf{x}\geq \mathbf{0},
\end{aligned}$$

where x_1, x_2 and x_3 denote the capacity of arc BC, BD and AB, respectively. We set the mean demand $\mu=10$, installation cost vector $\mathbf{c}=\mathbf{1}$ and the investment budget $B=4\mu+\iota\sigma$ in this experiment. Since the exact distribution of $\tilde{\xi}$ is assumed to be unknown, we adopt the distributionally robust approach by applying Lemma 6.2 with the $\pi(\cdot,\cdot)$ bound as follows:

$$\begin{aligned}
\min\, y &+ \pi\left(-y+h_1^0+h_2^0, h_1^1+h_2^1\right)+\pi\left(-h_1^0,-h_1^1\right)+\pi\left(-h_2^0,-h_2^1\right)\\
&+\pi\left(\mu-x_1-h_1^0, 1-h_1^1\right)+\pi\left(\mu-x_2-h_2^0,-1-h_2^1\right)
\end{aligned}$$

$$\text{s.t. } y \geq 2\mu - x_3$$
$$x_1 + x_2 + x_3 \leq B$$
$$x_1, x_2, x_3, y \geq 0,$$

where the variables $h_1^0, h_1^1, h_2^0, h_2^1$ are used to construct affine decision rules $h_1(\tilde{\xi}) = h_1^0 + h_1^1\tilde{\xi}$ and $h_2(\tilde{\xi}) = h_2^0 + h_2^1$ when applying Lemma 6.2.

We also compute the actual optimal design with exact distributional information as a Utopian benchmark. We term it as the stochastic approach for simplicity. Provided with the exact distributional information of $\tilde{\xi}$, the corresponding optimal design can be solved by the following linear program:

$$\begin{aligned}
\min\ & \beta l_1 + (1-\beta) l_2 \\
\text{s.t. } & l_1 \geq 2\mu - x_3 \\
& l_1 \geq y_1^1 + y_1^2 \\
& y_1^1 \geq \mu - \frac{1}{\beta} - x_1 \\
& y_1^2 \geq \mu + \frac{1}{\beta} - x_2 \\
& l_2 \geq 2\mu - x_3 \\
& l_2 \geq y_2^1 + y_2^2 \\
& y_2^1 \geq \mu + \frac{1}{1-\beta} - x_1 \\
& y_2^2 \geq \mu - \frac{1}{1-\beta} - x_2 \\
& y_1^1, y_1^2, y_2^1, y_2^2, l_1, l_2 \geq 0,
\end{aligned}$$

where l_1 and l_2 denote the total demand loss corresponding to the realized scenarios $\tilde{\xi} = -\frac{1}{\beta}$ and $\tilde{\xi} = \frac{1}{1-\beta}$, respectively.

It can be seen that the MaBU approach is infeasible when $\iota < 0$. When $\iota \geq 0$, the optimal solution is $(\mu + \iota\sigma/3, \mu + \iota\sigma/3, 2\mu + \iota\sigma/3)$. To make the comparison fair, we vary ι from 0.1 to 1 (When ι is large, the total expected demand loss is zero for all three methods). Table 6.1, 6.2 and 6.3 summarize the normal approximation based 95% confidence interval of the expected total demand loss estimated by 10,000 randomly generated instances for each tested parameter setting under each design strategy. In terms of the estimated average total demand loss, we observe that the stochastic approach performs best for most tested instances (The exceptions that $1/\beta = 2, \iota = 0.2, 0.8, 0.9$ is due to the sampling error). This is straightforward because the stochastic approach aims to find the "real" optimal design minimizing the expected total demand loss. The gap between the MaBU approach and the stochastic approach consists of two parts: the optimality gap caused by the conservative worst-case scheme and the one caused by the estimation error of the support set. Not surprisingly, our proposed approach outperforms the MaBU approach for all the tested instances. This suggest our preliminary conjecture that the first- and second-order moment information can reduce the optimality gap caused by the estimation error of the support set. We also normalize the performance ratio by letting the stochastic approach as 1 to make the comparison clear, and plot the expected total demand loss ratio for each design strategies in Figure 6.2. The value of first- and second-order moment information can be viewed as the relative improvement of the proposed approach against the MaBU approach. An apparent increasing trend of the value of moment information is observed when the level of distributional asymmetry of the uncertainty (value of $1/\beta$) increases. Therefore, we can deduce the intuition that the value of the moment information increases when the discrepancy between the actual and estimated support sets increases.

6.5.2 Computation Study of Telecommunication Network Design Problem

We next consider the general telecommunication network design problem. Since the introduction of decision rule increase the problem complexity with many new decision variables, one typical concern is that how much time is required for our proposed method to solve a typical real sized telecommunication network problem? Benefited from the linear capacity setting, our method can solve the network instances dfn-bwin and polska in Orlowski et al. (2010) by less than 10 minutes. However, it is reported in Lee et al. (2012) that their proposed branch-and-price-and-cut method is not able to solve them to optimality in one hour. This means that our proposed method is competitive to existing methods and we do not need to worry too much about its computational performance.

Our main concern is, how would our proposed approach address the stochastic system performance compared with the benchmarks? To make the comparison fair, we test them by various problem instances and report their overall performances. Since the computation of the stochastic system performance requires a lot of replications, our tested samples are of small scale so that the overall simulation time would not be huge. More precisely, we randomly generate the arcs of the telecommunication networks to form a tree structure, which is sparse and the available path for each pairwise demand is unique. In the following we briefly discuss our experiment data generation procedure. We consider 100 instances of telecommunication networks to estimate the average performance of each design strategy. Due to the long computational time required to estimate the stochastic system performance with 10,000 samples, we use a smaller test set of 1,000 demand scenarios for each telecommunication network instance. To form a tree structure, we randomly generate the arcs of the telecommunication network by the following procedure. We design a network with 10 nodes by adding them to the network one by one. For each new added node, we randomly assign an arc linking it to one of the previously added node set. Finally, we obtain a random tree such that the path linking each node pairs is uniquely defined.

For each tree instance, we randomly generate the problem parameters. The capacity installation cost rate of each arc is randomly generated from a uniform distribution between 0 and 2. In addition, we randomly select 15 of the 45 node pairs as the demand pairs. For each pairwise demand, we assume the demand amount $\tilde{d}_k$ fluctuates in

$[\mu_k - 3\sigma_k, \mu_k + 3\sigma_k]$, where the mean demand μ_k is randomly generated from a uniform distribution between 10 and 20, and the corresponding standard deviation is generated from 0 to $\mu_k/3$. For modeling simplicity, the associated penalty function is assumed to be a linear function of the actual demand loss amount, where the penalty rate is also randomly generated from 0.5 to 1.5. In this experiment, we artificially set the investment budget $B = (B_L + B_U)/2$, where B_L and B_U correspond to the minimum investment budget required to support the extreme demand scenarios $\underline{\mathbf{d}}$ and $\overline{\mathbf{d}}$, respectively. To choose the total penalty tolerance level τ, we obtain the optimal design $\mathbf{x}_N^*$ minimizing the total penalty cost under the nominal case that the demand $\tilde{\mathbf{d}} = \boldsymbol{\mu}$, and choose τ as the 40% quantile of the stochastic penalty cost profile with the design solution $\mathbf{x}_N^*$ when the uncertain demands are normally and independently distributed.

For each network instance, we randomly generate 1,000 sets of demand realizations following our assumed distributions. More precisely, we test two subcases: (a) the independent case where the demands are independently distributed (b) the correlated case where the demands follows a multivariate normal distribution such that the correlation matrices are generated by MATLAB command *gallery('randcorr',n)*. For the independent case, we test both the normal distribution and the uniform distribution. We calculate the optimal design solution under three design strategies: the proposed DSAM approach, the MaBU and MiETC approach. With the help of the 1,000 randomized realizations, we estimate the normalized expected penalty value EPV $= \frac{1}{\tau}\mathbb{E}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}))$, success probability SuP $= \mathbb{P}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) \leq \tau)$, normalized expected penalty overrun EPO $= \frac{1}{\tau}\mathbb{E}(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau)^+$ and the normalized conditional expected penalty overrun CEPO $= \frac{1}{\tau}\mathbb{E}\left(\mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) - \tau \,\middle|\, \mathcal{Q}(\mathbf{x}, \tilde{\mathbf{d}}) > \tau\right)$.

Table 6.4 summarizes the 95% confidence interval derived by normal approximation for the independent case. The results indicate that the proposed DSAM approach gives the largest EPV, but provides designs with significantly higher success probability, lower EPO and CEPO than the MaBU approach. It can be seen that the general performance of the MiETC approach is the best among all these three strategies. It is worthwhile to highlight that our DSAM approach performs competitively well with the MiETC approach, which assumes the distributional information on the uncertainty is precisely known. We then compare the results for the correlated case in Table 6.5. As we can see, our proposed DSAM approach gives the best performance among all these three design strategies in terms of all performance measures. In summary, our proposed DSAM approach is as comparable when the demands are independently distributed, and dominates both the MaBU and MiETC design strategies when demands are correlated.

6.6 Conclusion

In this chapter, we consider the resilience of telecommunication network under demand uncertainty. Different from the pure robust approach in the literature, we assume that the stochastic nature of the demand is partially known. More precisely, we assume that the distribution of the demand vector is known to reside in a family of distributions described by known support, first- and second-order moments. Our assumption on demand uncertainty is more reasonable than the purely uncertainty set based robust approach in the literature because vast historical demand data is available at the designing stage.

Attracted by the advantages of the distributionally-ambiguous shortfall aspiration measure in addressing tail risk, distributional ambiguity and computational tractability, we adopt it to describe the telecommunication network resilience against demand fluctuation. Based on this, we build a mathematical model of designing telecommunication network with maximized resilience under given investment budget, and derive tractable approximations to solve it via decision rules. To make use of the second-order moment information, we propose the improved affine routing policy via the approximation of the truncated expectation. Our result gives several important insights on this problem. First, the incorporation of the moment information in the improved affine routing policy provides us less conservative solutions than the simply uncertainty set based robust approach. This is especially true when the support of the uncertainty is not precisely true.

Second, our computational study suggests that our proposed distributionally-ambiguous shortfall aspiration measure based approach gives design solutions with higher stochastic performance than the pure robust approach in the literature. Additionally, when demands are correlated, its performance can be even better than the stochastic approach based on the Utopian assumption of precise distributional information.

6.7 Attached Tables and Figures

Figure 6.2: Performance ratio when ι varies.

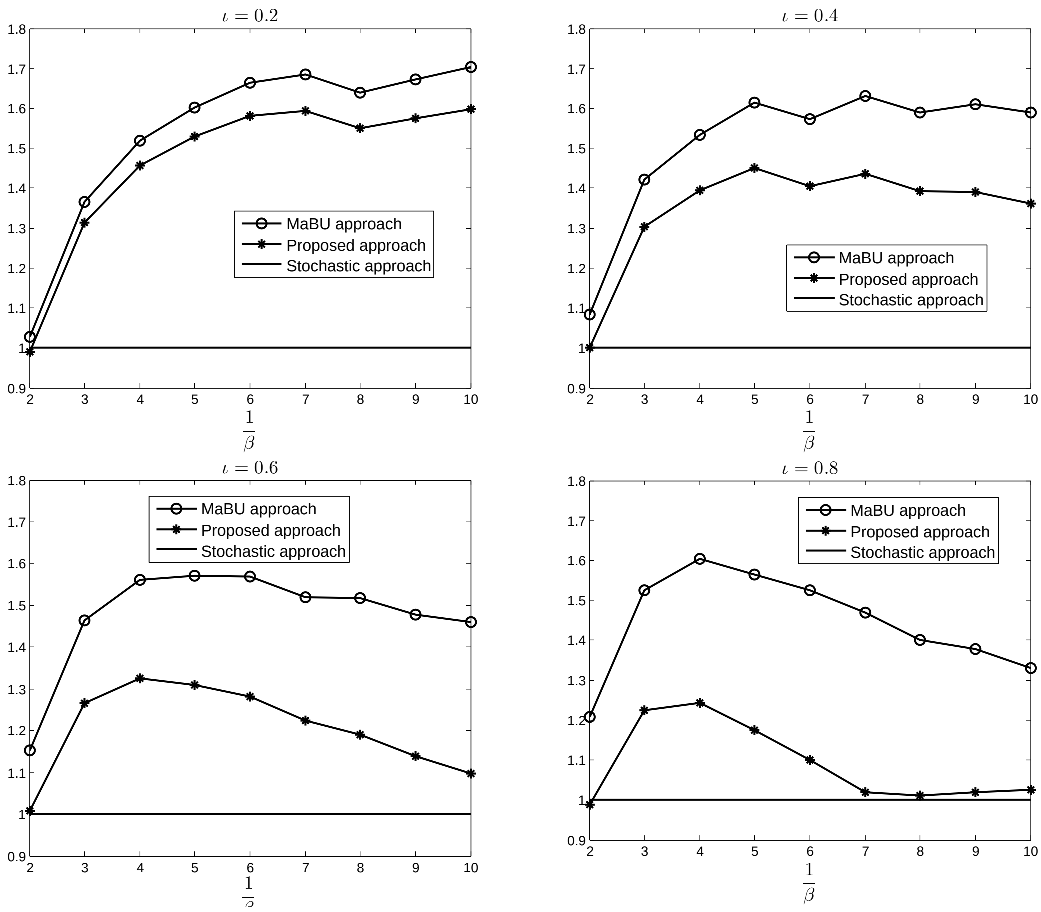


Table 6.1: Expected total demand loss under the MaBU approach.

	ι				
$1/\beta$	0.1	0.2	0.3	0.4	0.5
2	1.9333±3.77E-15	1.8667±2.44E-15	1.8000±6.66E-15	1.7333±3.11E-15	1.6667±4.22E-15
3	1.9331±0.0139	1.8603±0.0139	1.7770±0.0138	1.7117±0.0138	1.6495±0.0139
4	1.9342±0.0228	1.8434±0.0226	1.7771±0.0227	1.6929±0.0226	1.6239±0.0227
5	1.9024±0.0292	1.8266±0.0293	1.7125±0.0288	1.6438±0.0291	1.6115±0.0298
6	1.8910±0.0347	1.8006±0.0347	1.7530±0.0354	1.6784±0.0357	1.5443±0.0349
7	1.9079±0.0401	1.7946±0.0397	1.7034±0.0398	1.6011±0.0397	1.5770±0.0411
8	1.9218±0.0450	1.8444±0.0455	1.7244±0.0450	1.6215±0.0450	1.4940±0.0444
9	1.9183±0.0491	1.8122±0.0491	1.6652±0.0481	1.5670±0.0483	1.4775±0.0487
10	1.8880±0.0523	1.7760±0.0522	1.6907±0.0529	1.5582±0.0523	1.4844±0.0533

	ι				
$1/\beta$	0.6	0.7	0.8	0.9	1.0
2	1.6000±4.88E-15	1.5333±2.44E-15	1.4667±2.00E-15	1.4000±4.66E-15	1.3333±3.11E-15
3	1.5763±0.0139	1.5102±0.0139	1.4398±0.0139	1.3552±0.0138	1.2984±0.0139
4	1.5365±0.0226	1.4470±0.0225	1.3719±0.0225	1.3013±0.0226	1.2182±0.0225
5	1.5079±0.0295	1.4095±0.0293	1.3382±0.0295	1.2305±0.0291	1.1764±0.0295
6	1.4645±0.0351	1.3693±0.0350	1.2856±0.0351	1.1779±0.0348	1.1115±0.0352
7	1.4684±0.0408	1.3241±0.0398	1.2464±0.0402	1.1336±0.0398	1.0582±0.0402
8	1.4021±0.0446	1.3096±0.0448	1.2396±0.0455	1.0757±0.0441	0.9784±0.0441
9	1.3895±0.0491	1.2330±0.0479	1.1222±0.0478	1.0437±0.0485	0.9605±0.0490
10	1.3378±0.0524	1.2507±0.0530	1.0667±0.0511	0.9893±0.0520	0.8667±0.0517

Table 6.2: Expected total demand loss under the proposed distributionally robust approach.

	ι				
$1/\beta$	0.1	0.2	0.3	0.4	0.5
2	1.9000±6.79E-07	1.8000±5.46E-06	1.7000±5.27E-07	1.6000±3.96E-06	1.5000±1.98E-06
3	1.8978±0.0139	1.7897±0.0139	1.6709±0.0138	1.5703±0.0138	1.4727±0.0139
4	1.8958±0.0228	1.7665±0.0226	1.6615±0.0227	1.5387±0.0227	1.4315±0.0227
5	1.8608±0.0292	1.7431±0.0293	1.5871±0.0288	1.4768±0.0291	1.4031±0.0298
6	1.8463±0.0347	1.7110±0.0347	1.6184±0.0354	1.4992±0.0357	1.3207±0.0349
7	1.8602±0.0401	1.6990±0.0397	1.5601±0.0398	1.4104±0.0397	1.3388±0.0411
8	1.8713±0.0450	1.7433±0.0455	1.5731±0.0450	1.4202±0.0450	1.2420±0.0444
9	1.8651±0.0491	1.7059±0.0491	1.5063±0.0481	1.3548±0.0483	1.2124±0.0487
10	1.8323±0.0523	1.6650±0.0522	1.5240±0.0529	1.3361±0.0523	1.2067±0.0533

	ι				
$1/\beta$	0.6	0.7	0.8	0.9	1.0
2	1.4000±6.38E-06	1.3000±3.13E-06	1.2000±2.58E-06	1.1000±2.24E-06	1.0000±2.39E-06
3	1.3641±0.0139	1.2627±0.0139	1.1569±0.0139	1.0368±0.0138	0.9448±0.0139
4	1.3056±0.0226	1.1777±0.0225	1.0640±0.0225	0.9548±0.0226	0.8335±0.0225
5	1.2577±0.0295	1.1175±0.0293	1.0048±0.0295	0.8556±0.0291	0.7598±0.0295
6	1.1960±0.0351	1.0560±0.0350	0.9279±0.0351	0.7817±0.0347	0.7821±0.0341
7	1.1826±0.0408	0.9912±0.0398	0.8654±0.0402	0.8074±0.0390	0.8062±0.0384
8	1.1001±0.0446	0.9569±0.0448	0.8943±0.0450	0.8133±0.0427	0.7981±0.0418
9	1.0711±0.0491	0.8618±0.0479	0.8299±0.0469	0.8393±0.0466	0.8432±0.0461
10	1.0043±0.0524	0.9116±0.0527	0.8233±0.0498	0.8398±0.0497	0.8125±0.0485

Table 6.3: Expected total demand loss under the stochastic approach.

	ι				
$1/\beta$	0.1	0.2	0.3	0.4	0.5
2	1.8863±0.0372	1.8162±0.0353	1.6796±0.0333	1.5990±0.0314	1.4961±0.0294
3	1.4403±0.0397	1.3633±0.0377	1.2599±0.0355	1.2039±0.0336	1.1536±0.0318
4	1.2970±0.0436	1.2130±0.0413	1.1740±0.0395	1.1037±0.0374	1.0584±0.0356
5	1.1772±0.0467	1.1397±0.0449	1.0450±0.0423	1.0180±0.0407	1.0375±0.0397
6	1.1271±0.0501	1.0821±0.0482	1.0942±0.0472	1.0673±0.0456	0.9660±0.0426
7	1.1301±0.0541	1.0656±0.0517	1.0306±0.0499	0.9819±0.0477	1.0241±0.0474
8	1.1342±0.0580	1.1245±0.0566	1.0616±0.0541	1.0202±0.0521	0.9516±0.0494
9	1.1199±0.0611	1.0836±0.0592	0.9996±0.0560	0.9737±0.0543	0.9567±0.0528
10	1.0767±0.0634	1.0424±0.0614	1.0384±0.0602	0.9807±0.0576	0.9869±0.0566

	ι				
$1/\beta$	0.6	0.7	0.8	0.9	1.0
2	1.3888±0.0274	1.2940±0.0255	1.2144±0.0235	1.1059±0.0216	0.9866±0.0196
3	1.0769±0.0298	1.0155±0.0279	0.9446±0.0260	0.8490±0.0238	0.8016±0.0220
4	0.9846±0.0335	0.9095±0.0313	0.8554±0.0294	0.8066±0.0275	0.7424±0.0255
5	0.9600±0.0374	0.8915±0.0352	0.8555±0.0334	0.7792±0.0311	0.7598±0.0295
6	0.9330±0.0409	0.8818±0.0388	0.8434±0.0369	0.7804±0.0347	0.7584±0.0331
7	0.9659±0.0451	0.8713±0.0421	0.8486±0.0405	0.7905±0.0382	0.7682±0.0366
8	0.9234±0.0477	0.8937±0.0459	0.8855±0.0446	0.7866±0.0413	0.7527±0.0394
9	0.9399±0.0513	0.8529±0.0481	0.8140±0.0460	0.8053±0.0447	0.7901±0.0432
10	0.9157±0.0537	0.9059±0.0523	0.8022±0.0485	0.8014±0.0474	0.7583±0.0452

Table 6.4: Performances of various design strategies with independent demand data.

		Design strategy		
Distribution	Statistics	MaBU	MiETC	DSAM
Uniform	EPV%	54.2463±2.8737	51.0159±3.0521	60.5178±3.4795
	SuP%	81.8236±1.9984	93.1618±2.4597	92.0389±2.1190
	EPO%	12.4588±2.5853	2.7757±0.9894	2.1246±0.8508
	CEPO%	65.8669±7.8372	36.9605±3.0086	25.0758±2.9870
Normal	EPV%	50.1014±3.3775	48.1088±3.2631	61.9610±4.0301
	SuP%	85.8830±2.1626	94.5669±2.6797	94.8313±2.6551
	EPO%	8.3534±1.8404	2.5547±0.5531	3.0787±0.8542
	CEPO%	63.0541±7.5636	48.5796±4.5150	57.7021±6.1889

Table 6.5: Performances of various design strategies with correlated demand data.

Statistics	MaBU	MiETC	DSAM
EPV%	52.6688±3.1413	50.1989±3.7035	58.1434±4.2612
SuP%	84.3762±1.8985	88.7171±2.2926	96.8481±1.2366
EPO%	12.3472±1.8469	6.8683±2.5688	0.6123±0.4558
CEPO%	76.7690±10.2357	84.5170±13.0536	19.3711± 4.3178

Chapter 7

Conclusion and Future Research

The fundamental goal of this thesis is to build a quantitative framework of modeling supply systems resilience. In particular, our analytic framework relies on a network flow based LP model and thus our framework is restricted to specific supply systems with a network structure and the operational actions of mitigating the impact of uncertainty can be formulated as an LP. The specific contributions are summarized as follows.

In Chapter 3, an axiomatic framework of defining resilience measures is proposed by analyzing the uncertain demand loss penalty function. More precisely, these five axioms are carried to address both the target oriented decision trend and computational tractability. The target oriented decision trend, which is also termed as “satisficing” trend, possesses strong justification in decision theory and has recently received a surge interest in the area of decision under uncertainty. In addition, the proposed five axioms themselves do not single out any specific type of resilience measure so that we can adjust it accordingly.

Chapter 4 explores two generic approaches of constructing new resilience measures. We build an explicit approach of extending resilience measure from certain reference measures by making use of the concept of “supporting” in finance. This approach enables us to incorporate the decision maker’s specific risk attitudes towards penalty positions in the synthesized resilience measures. We then show the fundamental result that, if the reference measure fulfills the consistency condition, the corresponding extended mapping on the extended space is indeed a resilience measure which admits preference for diversification. We then proposed an adjustable uncertainty set based representation for resilience measures in Theorem 4.2. More precisely, every computationally proper resilience measure has an explicitly robust optimization representation and conversely, this robust optimization format of expression uniquely defines a computationally proper resilience measure with properly specified adjustable uncertainty sets. This result is of crucial importance of connecting resilience measure to robust optimization. Even though we cannot say much about the structure of the corresponding adjustable uncertainty set without assuming a structural form of the resilience measure and the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the reverse direction of Theorem 4.2 gives us a generic way of constructing resilience measures by simply specifying adjustable uncertainty sets. This result takes a major step towards the computation of resilience because we can take advantage of the robust optimization paradigm.

In chapter 5, we apply our resilience measure based framework to study energy supply system resilience against supply disruptions. In particular, we give the definition of *resilience index* by relating the energy supply system resilience to a resilience measure on the uncertain penalty position towards a certain tolerance level. Since supply disruptions are really hard to predict, we consider resilience measure defined by adjustable uncertainty sets. To this end, we investigate algorithms of computing the resilience index. By making use of vertex enumeration, a conic optimization based algorithm was firstly proposed to compute resilience measure generated by general norm based adjustable uncertainty sets. After that, to circumvent the computational difficulty of the general algorithm, we considered two special families of adjustable uncertainty sets: the box uncertainty set and the cardinality uncertainty set. The box uncertainty set is favored due to its simplicity. When some degree of basic information on the uncertainty is available, carnality uncertainty set is preferred from the modeling perspective. The complex structure of the carnality uncertainty set challenged the efficiency of pure binary search scheme.

We then develop practically efficient solution algorithms for cardinality-constrained uncertainty set case, and extend our framework to design optimization problem of maximizing the supply system resilience index over the whole decision space. Finally, we conduct numerical studies using a natural gas supply network application to verify the effectiveness of the proposed resilience index approach. The first experiment shows that the proposed resilience measure offers an interesting relevance to the stochastic system performances. Besides, even though the computation of resilience measure is generally NP-Hard, the second experiment shows that gradient-based" search method can handle moderate sized problems. The third experiment demonstrates the superiority of our proposed method to the traditional approach when natural gas supply disruptions present. Moreover, its stochastic performance is also as comparable as the stochastic method with concise distributional information of the uncertainty.

Chapter 6 applies our proposed framework of resilience measure to telecommunication network design problem. The primary goal is to provide a more flexible alternative of the traditional robust telecommunication network design problem which allows no demand shortage and might lead to overly conservative solutions due to the ignorance of the distributional knowledge. Similar to the energy supply system application, we also propose the same concept of resilience index to measure the telecommunication network's service quality with fluctuated demands. In this application, our major concern is on the design of telecommunication network. To avoid the conservatism suffered by the robust telecommunication network design approach due to lack of distributional information, we model the demands as ambiguous random variables with known first- and second-order moments and support. To address this ambiguous demand model and achieve computationally tractability at the same time, we adopt the distributionally-ambiguous shortfall awareness measure to build the resilience index. We then build a design model for maximizing the resilience index of the resulting telecommunication network with continuous capacity installation. To achieve tight and tractable approximations of this problem, we then propose a class of improved affine decision rule models.

Our computational experiments illustrate that our proposed resilience measure based approach can provide solution with better stochastic performance than the conventional robust network design method.

These thesis can be regarded as an opening work of adopting mathematical optimization methods (especially optimization under uncertainty paradigm) to build an analytic framework of modeling supply system resilience against unpredictable uncertainties, which is important but not well studied in the literature. On the whole a number of contributions have been achieved in this thesis. Nevertheless, some further research is necessary to extend our work. Here I list some possible topics for future extension.

In this thesis we restrict our attention to supply system with a network structure and its corresponding operational problem of delivering the service can be formulated or approximately solved by LP models. The network structure restriction is necessary for explicit quantitative analysis. In contrast, the LP model assumption is purely made from the computational concerns. Hence, the extension from this LP model to more general and complicated optimization models (e.g., nonlinear or even nonconvex problems) could be one of the future research directions. Of course there is no free lunch in the world, the computation and optimization of the resilience index would definitely be more difficult. Therefore, new analytic or approximation techniques are necessary for this extension.

In the two separate applications, uncertainties are adopted to address supply disruptions or demand fluctuations. That means, the right hand side of the linear programming model is affected by uncertainty. Indeed, it is necessary to extend our work to cases in which the supply system is subject to unknown attacks (such as terrorist attack) such that the intrinsic structure of the supply system is subject to (removing some critical arcs of the network). This extension has significant importance, especially for network planning of critical resources such as ambulances or blood banks. In fact, some specific instances of this problem has been studied in the literature, such as the robust maximum flow problem proposed in Bertsimas et al. (2013b) and the network interdiction problem considered in Bertsimas et al. (2013a). These two problems are known to be difficult to solve, which means that the new extended problem would also be difficult because it is more general. Therefore, it would be interesting to find some useful approximations of this new extended problem.

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