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Positive-measure self-similar sets without interior

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Abstract. We recall the problem posed by Peres and Solomyak in Problems on self-similar and self-affine sets; an update. *Progr. Prob.* **46** (2000), 95–106: *can one find examples of self-similar sets with positive Lebesgue measure, but with no interior*? The method in Properties of measures supported on fat Sierpinski carpets, this issue, leads to families of examples of such sets.

In [**1**] the existence of self-affine sets with positive Lebesgue measure but empty interior was shown. In this appendix we demonstrate that an adjustment of this argument can show the existence of self-similar sets with positive Lebesgue measure but empty interior answering a question from [**2**].

The construction. Let $\underline{t} = (t_1, t_2) \in [0, 1]^2 \subset \mathbb{R}^2$. We consider ten similarities (with the same contraction rate $\frac{1}{3}$) given by

$$\begin{aligned}
T_0(x, y) &= (\tfrac{1}{3}x, \tfrac{1}{3}y), & T_5(x, y) &= (\tfrac{1}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y + 1)\\
T_1(x, y) &= (\tfrac{1}{3}x, \tfrac{1}{3}y + t_1), & T_6(x, y) &= (\tfrac{2}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y)\\
T_2(x, y) &= (\tfrac{1}{3}x, \tfrac{1}{3}y + t_2), & T_7(x, y) &= (\tfrac{2}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y + t_1)\\
T_3(x, y) &= (\tfrac{1}{3}x, \tfrac{1}{3}y + 1), & T_8(x, y) &= (\tfrac{2}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y + t_2)\\
T_4(x, y) &= (\tfrac{1}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y), & T_9(x, y) &= (\tfrac{2}{3} + \tfrac{1}{3}x, \tfrac{1}{3}y + 1).
\end{aligned}$$

This construction is similar in spirit to those in [**3**]. To see that the associated limit set $\Lambda_{\underline{t}}$ has empty interior, observe that the intersection of $\Lambda_{\underline{t}}$ with each of vertical lines $\{(k + \frac{1}{2})3^{-n}\} \times \mathbb{R}$, with $n \geq 0$ and $0 \leq k \leq 3^n - 1$ has zero measure. It remains to show that typically $\Lambda_{\underline{t}}$ has positive measure.

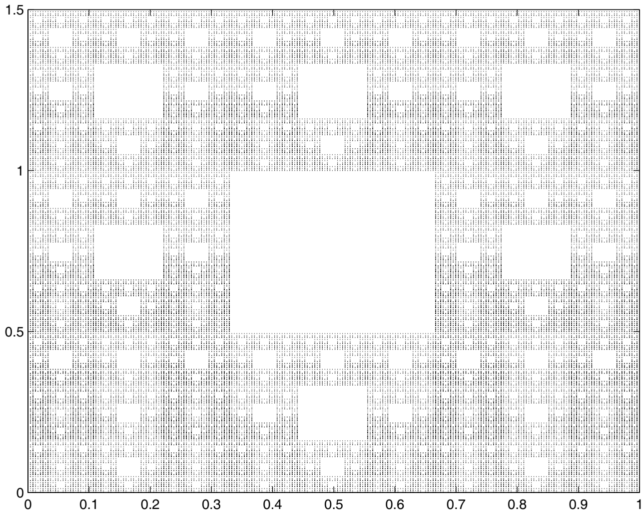

FIGURE 1. A typical limit set $\Lambda_{\underline{t}}$.

Let $\Sigma_{10} = \{0, 1, \ldots, 9\}^{\mathbb{Z}^+}$ denote the full shift on 10 symbols and let $\Pi_{\underline{t}} : \Sigma_{10} \to \Lambda_{\underline{t}}$ be the usual projection map. Let $\mu = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})^{\mathbb{Z}^+}$ be a Bernoulli measure on Σ_{10}. To show that $\Lambda_{\underline{t}}$ has non-zero Lebesgue measure it suffices to show that $\nu := \Pi_{\underline{t}}\mu$ is absolutely continuous. By construction, ν projects to Lebesgue measure on the unit interval in the x-axis; thus it suffices to show the conditional measure $\nu_{\underline{t},x}$ on Lebesgue almost every vertical line $\{x\} \times \mathbb{R}$ is absolutely continuous.

Let $\Sigma_3 = \{0, 1, 2\}^{\mathbb{Z}^+}$ be the full shift on 3 symbols coding the horizontal coordinate. In particular, there is a natural semi-conjugacy $p : \Sigma_{10} \to \Sigma_3$ given by $p(\omega)_n = i(\omega_n)$ where $i|_{\{0,1,2,3\}} \equiv 0$; $i|_{\{4,5\}} \equiv 1$; and $i|_{\{6,7,8,9\}} \equiv 2$. Then $p\mu = \overline{\mu} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathbb{Z}^+}$ is the Bernoulli measure on Σ_3. Given $\xi \in \Sigma_3$ let μ_ξ denote the induced measure on $p^{-1}(\xi)$. Clearly, if $\Pi_{\underline{t},\xi} : p^{-1}(\xi) \to \{x\} \times \mathbb{R}$ is the restriction of $\Pi_{\underline{t}}$, then by construction $\Pi_{\underline{t},\xi}\mu_\xi = \nu_{\underline{t},x}$. We also let $\pi : \Sigma_3 \to [0, 1]$ be the natural projection from Σ_3 to the x-axis given by $\pi(\xi) = \sum_{n=0}^{\infty} \xi_n (\frac{1}{3})^{n+1}$. The analogue of transversality is that there exists $C > 0$ such that

$$\Delta_\xi(r; \omega, \tau) := \text{leb}\{\underline{t} \in [0, 1]^2 : |\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| \le r\} \le C3^{|\omega\wedge\tau|}r, \quad \text{for } r > 0. \quad (1)$$

This can be seen by a simple adjustment of the arguments in [**3**, Lemma 3.1] and [**4**, Proposition 3.1 Part (i)]. However, in our special case it can be easily verified directly, as follows.

Let $\omega, \tau \in p^{-1}(\xi)$ with $|\omega \wedge \tau| = n$; that is, $\tau_i = \omega_i$ for $i < n$ and $\tau_n \neq \omega_n$. Since $\omega, \tau \in p^{-1}(\xi)$ we have $i(\omega_n) = i(\tau_n)$ for all n, and $\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau) = (0, \phi_{\underline{t},\xi}(\omega, \tau))$, where

$$\phi_{\underline{t},\xi}(\omega, \tau) = 3^{-n}\bigg((t_{j(\omega_n)} - t_{j(\tau_n)}) + \sum_{k=1}^{\infty} 3^{-k}(t_{j(\omega_{k+n})} - t_{j(\tau_{k+n})})\bigg).$$

Here we let $j|_{\{0,4,6\}} \equiv 0$, $j|_{\{1,7\}} \equiv 1$, $j|_{\{2,8\}} \equiv 2$, $j|_{\{3,5,9\}} \equiv 3$, and $t_0 = 0, t_3 = 1$ for convenience. If $\{j(\omega_n), j(\tau_n)\} = \{0, 3\}$, then

$$|\phi_{\underline{t},\xi}(\omega, \tau)| \geq 3^{-n}\left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

in view of $t_j \in [0, 1]$ for all j, and (1) follows. Otherwise, let $j \in \{j(\omega_n), j(\tau_n)\} \cap \{1, 2\}$. Then

$$\left|\frac{\partial \phi_{\underline{t},\xi}(\omega, \tau)}{\partial t_j}\right| \geq 3^{-n}\left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

which also implies (1).

Now we use (1) to prove that $\nu_{\underline{t},x}$ is absolutely continuous for almost every x. For a sequence $\xi \in \Sigma_3$ we define $n_i(\xi)$ to be the number of i's in the first n terms of ξ. By the strong law of large numbers, given $\epsilon, \delta > 0$ we can use Egorov's theorem to choose a set $X \subset [0, 1]$ of measure $\mathrm{leb}(X) > 1 - \epsilon$ (equivalently $\overline{\mu}(\pi^{-1}X) > 1 - \epsilon$) such that there exists $N \in \mathbb{N}$ where for $n \geq N$, $n_i(\xi) \geq (\frac{1}{3} - \delta)n$, for $i = 0, 1, 2$. We can bound

$$\begin{aligned}
&\int_{[0,1]^2} \int_X \left(\int_{\{x\}\times\mathbb{R}} \underline{D}(\nu_{\underline{t},x})(y)\, d\nu_{\underline{t},x}(y)\right) d(\mathrm{leb})(x)\, d\underline{t} \\
&\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\pi^{-1}X} \left(\int_{p^{-1}(\xi)} \int_{p^{-1}(\xi)} \Delta_\xi(r; \omega, \tau)\, d\mu_\xi(\omega)\, d\mu_\xi(\tau)\right) d\overline{\mu}(\xi) \\
&\leq C \int_{\pi^{-1}X} \left(\sum_{n=0}^{\infty} \sum_{\tau_0,\ldots,\tau_{n-1}} \mu_\xi[\tau_0, \ldots, \tau_{n-1}]^2 3^n\right) d\overline{\mu}(\xi) \\
&\leq C \int_{\pi^{-1}X} \left(\sum_{n=0}^{\infty} 4^{-n_0(\xi)} 2^{-n_1(\xi)} 4^{-n_2(\xi)} 3^n\right) d\overline{\mu}(\xi) \\
&\leq CC_1 + C \sum_{n=N}^{\infty} \left(4^{-(2/3-2\delta)} 2^{-(1/3-\delta)} 3\right)^n,
\end{aligned}$$

for some $C_1 > 0$ bounding the first N terms of the series, and observe that the series is finite for δ sufficiently small. This implies the desired (typical) absolute continuity, as in the main article.

We have proved the following.

THEOREM A. *For almost every $\underline{t} \in [0, 1]^2$ the limit set $\Lambda_{\underline{t}}$ has positive Lebesgue measure and empty interior.*

Remark. We can also construct examples with fewer similarities using different contraction rates. Let $0 < \lambda < \frac{1}{2}$ and $\underline{t} = (t_1, t_2, t_3) \in [0, 1]^3$. Consider the six similarities of $\mathbb{R}^2$ defined by

$$\begin{aligned}
T_0(x, y) &= (\lambda x, \lambda y), & T_3(x, y) &= (\lambda + \lambda x, \lambda y) \\
T_1(x, y) &= (\lambda x, \lambda y + t_1), & T_4(x, y) &= (\lambda + \lambda x, \lambda y + t_1) \\
T_2(x, y) &= (\lambda x, \lambda y + t_2), & T_5(x, y) &= (2\lambda + (1 - 2\lambda)x, (1 - 2\lambda)y + t_3).
\end{aligned}$$

Let $\Lambda_{\underline{t}}$ again denote the self-similar set. Let $\mu = (\frac{\lambda}{3}, \frac{\lambda}{3}, \frac{\lambda}{3}, \frac{\lambda}{2}, \frac{\lambda}{2}, (1-2\lambda))^{\mathbb{Z}^+}$ be the Bernoulli measure on Σ_6. Let $\overline{\mu} = (\lambda, \lambda, (1-2\lambda))^{\mathbb{Z}^+}$ denote the induced measure on Σ_3. The proof of Theorem A can be adapted to this setting provided

$$-(h(\mu) - h(\overline{\mu})) = -\lambda \log 2 - \lambda \log 3 \leq 2\lambda \log \lambda + (1-2\lambda)\log(1-2\lambda),$$

which is true provided λ is sufficiently close to $\frac{1}{2}$. More precisely, we have the following result.

THEOREM B. *If* $\lambda \in (0.4759, \frac{1}{2})$ *then for almost every* $\underline{t} \in [0,1]^3$ *the limit set* $\Lambda_{\underline{t}}$ *has positive Lebesgue measure and empty interior.*

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