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REGULARITY CONDITIONS AND BERNOULLI PROPERTIES OF EQUILIBRIUM STATES AND g-MEASURES

PETER WALTERS

Abstract

When $T:X\longrightarrow X$ is a one-sided topologically mixing subshift of finite type and $\varphi:X\longrightarrow R$ is a continuous function, one can define the Ruelle operator $\mathcal{L}_\varphi:C(X)\longrightarrow C(X)$ on the space $C(X)$ of real-valued continuous functions on X. The dual operator $\mathcal{L}_\varphi^*$ always has a probability measure ν as an eigenvector corresponding to a positive eigenvalue ($\mathcal{L}_\varphi^*\nu=\lambda\nu$ with $\lambda>0$). Necessary and sufficient conditions on such an eigenmeasure ν are obtained for φ to belong to two important spaces of functions, $W(X,T)$ and $\mathrm{Bow}(X,T)$. For example, $\varphi\in\mathrm{Bow}(X,T)$ if and only if ν is a measure with a certain approximate product structure. This is used to apply results of Bradley to show that the natural extension of the unique equilibrium state μ_φ of $\varphi\in\mathrm{Bow}(X,T)$ has the weak Bernoulli property and hence is measure-theoretically isomorphic to a Bernoulli shift. It is also shown that the unique equilibrium state of a two-sided Bowen function has the weak Bernoulli property. The characterizations mentioned above are used in the case of g-measures to obtain results on the 'reverse' of a g-measure.

Introduction

We consider subshifts of finite type with a finite number of symbols. Let $k\geqslant 2$, and let $\Gamma=\{1,2,\ldots,k\}$ be the set of symbols. Let $A=(a_{ij})$ be a $k\times k$ matrix with each entry $a_{ij}\in\{0,1\}$ and with no zero row and no zero column.

Let

$$X_A=\left\{x=(x_n)_{n=0}^{\infty}\in\prod_0^\infty\Gamma\mid a_{x_n x_{n+1}}=1\ \forall n\geqslant 0\right\}$$

and

$$\hat{X}_A=\left\{x=(x_n)_{n=-\infty}^{\infty}\in\prod_{-\infty}^{\infty}\Gamma\mid a_{x_n x_{n+1}}=1\ \forall n\in Z\right\}.$$

Both are compact sets under the product topologies on $\prod_0^\infty\Gamma$ and $\prod_{-\infty}^\infty\Gamma$ when Γ is equipped with the discrete topology. The one-sided subshift of finite type determined by A is the continuous surjection $T:X_A\longrightarrow X_A$ defined by $T((x_0,x_1,\ldots))=(x_1,x_2,\ldots)$. The two-sided subshift of finite type determined by A is the homeomorphism $S:\hat{X}_A\longrightarrow\hat{X}_A$ defined by

$$S((\ldots x_{-1}\overset{*}{x}_0 x_1\ldots))=(\ldots x_{-1}x_0\overset{*}{x}_1 x_2\ldots),$$

where the symbol $*$ is over the 0th position. Both shifts are called topologically mixing if there exists $M\geqslant 1$ with the product matrix $A^M>0$, that is, every entry of A^M is non-zero. We use M for such a number throughout the paper. This is equivalent to T being topologically mixing (that is, for all non-empty open sets U,V, there exists $M\geqslant 1$ with $U\cap T^{-n}V\neq\varnothing$ for all $n\geqslant M$), and to

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S being topologically mixing. If $p \leqslant q$ and $b_p, \ldots, b_q \in \Gamma$, then ${}_p[b_p, \ldots, b_q] = {}_p[b_p, \ldots, b_q]_q = [b_p, \ldots, b_q]_q = \{x \in X_A \mid x_i = b_i \text{ for } p \leqslant i \leqslant q\}$ when $0 \leqslant p$, and also ${}_p[b_p, .., b_q] = {}_p[b_p, .., b_q]_q = [b_p, \ldots, b_q]_q = \{x \in \hat{X}_A \mid x_i = b_i \text{ for } p \leqslant i \leqslant q\}$ when $p \in Z$. For $n, p \geqslant 1$, we have $T^p({}_0[x_0, \ldots, x_{p+n-1}]) = {}_0[x_p, \ldots, x_{p+n-1}]$ if ${}_0[x_0, \ldots, x_{p+n-1}] \neq \varnothing$, and $T^{-p}({}_0[x_0, \ldots, x_{n-1}]) = {}_p[x_0, \ldots, x_{n-1}]$. An allowable block in X_A or $\hat{X}_A$ is a string $b_0, \ldots, b_{n-1}$ of symbols with $a_{b_0 b_1} a_{b_1 b_2} \ldots a_{b_{n-2} b_{n-1}} = 1$. If $b_0, \ldots, b_{n-1}$ is an allowable block, then we call ${}_p[b_0, \ldots, b_{n-1}]$ an allowable cylinder. If $b, c \in \Gamma$ and $p, q \in N$, then $b^p c^q$ denotes the block $b_0, \ldots, b_{p+q-1}$, where $b_i = b$ for $0 \leqslant i \leqslant p-1$ and $b_i = c$ for $p \leqslant i \leqslant p+q-1$. If $x_0, \ldots, x_{n-1}$ is an allowable block in X_A and $z = (z_0, z_1, \ldots) \in X_A$ with $a_{x_{n-1} z_0} = 1$, then $(x_0, \ldots, x_{n-1}, z)$ denotes the member $y = (y_i)$ of X_A with $y_i = x_i$ for $0 \leqslant i \leqslant n-1$ and $y_{i+n} = z_i$ for $i \geqslant 0$.

We often write X and $\hat{X}$ for X_A and $\hat{X}_A$. Consider a subshift of finite type (SFT) $T: X \longrightarrow X$. We use $C(X)$ to denote the space of all real-valued continuous functions on X, equipped with the supremum norm. We let $M(X)$ denote the space of all probability measures on the Borel subsets of X, equipped with the weak*-topology, and let $M(X, T)$ denote the non-empty subset of T-invariant members of $M(X)$. We say that $\tau \in M(X)$ has support X if $\tau(U) > 0$ for every non-empty open set U. If $\varphi \in C(X)$, we let $P(T, \varphi)$ denote the pressure of T at φ [**13**], and let $T_n \varphi$ be the function $\sum_{i=0}^{n-1} \varphi \circ T^i$. Similar notation applies to $S: \hat{X} \longrightarrow \hat{X}$. When T is a one-sided subshift of finite type, the Ruelle operator of $\varphi \in C(X)$ is denoted by $\mathcal{L}_\varphi : C(X) \longrightarrow C(X)$, so that $(\mathcal{L}_\varphi f)(x) = \sum e^{\varphi(y)} f(y)$, where the sum is over all $y \in T^{-1}x$. The dual operator $\mathcal{L}_\varphi^*$ always has an eigenmeasure in $M(X)$, that is, there exists $\nu \in M(X)$ and $\lambda > 0$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$. For $\varphi \in C(X)$ and T a one-sided subshift of finite type, we define $v_n(\varphi)$, for $n \geqslant 1$, as $v_n(\varphi) = \sup\{\varphi(x) - \varphi(x') \mid x, x' \in X \text{ and } x_i = x_i', 0 \leqslant i \leqslant n-1\}$. We define the space $\mathrm{Bow}(X, T)$ to be $\{\varphi \in C(X) \mid \sup_{n \geqslant 1} v_n(T_n \varphi) < \infty\}$ and the space $W(X, T)$ to be $\{\varphi \in C(X) \mid \sup_{n \geqslant 1} v_{n+p}(T_n \varphi) \to 0 \text{ as } p \to \infty\}$ [**14**, **15**]. We have $W(X, T) \subset \mathrm{Bow}(X, T)$, because $\varphi \in \mathrm{Bow}(X, T)$ if and only if there is some $p \geqslant 0$ with $\sup_{n \geqslant 1} v_{n+p}(T_n \varphi) < \infty$. If $\varphi \in C(X)$ has summable variations (that is, $\sum_{n=1}^{\infty} v_n(\varphi) < \infty$) [**11**], then $\varphi \in W(X, T)$.

In [**12**], the author showed that for a topologically mixing subshift of finite type (TMSFT), if $\varphi \in W(X, T)$, then the Ruelle operator theorem holds (that is, there exists $\lambda > 0, \nu \in M(X)$, and $h \in C(X)$ with $h > 0$ and $\int h \, d\nu = 1$ such that $\mathcal{L}_\varphi h = \lambda h$, $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and for all $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \rightrightarrows h(x) \int f \, d\nu,$$

where $\rightrightarrows$ denotes uniform convergence on X), φ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension. Here $\mu_\varphi = h\nu$, and μ_φ is the unique g-measure for the g-function $g(x) = e^{\varphi(x)} h(x) / \lambda h(Tx)$. In [**14**], the author considered these questions for $\varphi \in \mathrm{Bow}(X, T)$ and proved a weakened version of the Ruelle operator theorem. Each $\varphi \in \mathrm{Bow}(X, T)$ has a unique equilibrium state μ_φ. We show in this paper that (T, μ_φ) has a Bernoulli natural extension. We obtain necessary and sufficient conditions on an eigenmeasure ν for $\varphi \in C(X)$ to ensure that $\varphi \in \mathrm{Bow}(X, T)$ and to ensure that $\varphi \in W(X, T)$. These give characterizations of when $\varphi \in \mathrm{Bow}(X, T)$ in terms of μ_φ. When this is applied to the cases $\varphi = \log g$ and g is a g-function (see Section 3), we obtain results about the 'reverse' of a g-measure.

For a two-sided topologically mixing subshift of finite type, $S:\hat{X}\longrightarrow\hat{X}$ and $\hat{\varphi}\in C(\hat{X})$, we let $\mathrm{var}_n(\hat{\varphi})$, for $n\geqslant 1$, denote $\mathrm{var}_n(\hat{\varphi})=\sup\{\hat{\varphi}(x)-\hat{\varphi}(x')\mid x,\,x'\in\hat{X}$ and $x_i=x_i'$ for $-(n-1)\leqslant i\leqslant n-1)\}$. Then let $\mathrm{Bow}(\hat{X}_A,S)=\{\hat{\varphi}\in C(\hat{X})\mid \sup_{n\geqslant 1}\mathrm{var}_n(S_n\hat{\varphi})<\infty\}$, where $S_n\hat{\varphi}=\sum_{i=0}^{n-1}\hat{\varphi}\circ S^i$. Bowen showed that each $\hat{\varphi}\in\mathrm{Bow}(\hat{X}_A,S)$ has a unique equilibrium state $\hat{\mu}_{\hat{\varphi}}$ [**2**]. We show that $(S,\hat{\mu}_{\varphi})$ is isomorphic to a Bernoulli shift.

1. *Eigenmeasures of the Ruelle operator*

LEMMA 1.1. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type, let $\varphi\in C(X)$, and let $\nu\in M(X)$ and $\lambda>0$ satisfy $\mathcal{L}_{\varphi}^{*}\nu=\lambda\nu$. Then for all $n\geqslant 1$, $p\geqslant 1$ and $x\in X$, we have*

$$\nu({}_0[x_0,\ldots,x_{n+p-1}])=\frac{1}{\lambda^p}\int\limits_{z\in{}_0[x_p,\ldots,\,x_{n+p-1}]} e^{(T_p\varphi)(x_0,\ldots,\,x_{p-1},\,z)}\,d\nu(z).$$

Proof.

$$\begin{aligned}\nu({}_0[x_0,\ldots,x_{n+p-1}])&=\frac{1}{\lambda^p}\int\limits_X \mathcal{L}_{\varphi}^{p}\chi_{{}_0[x_0,\ldots,\,x_{n+p-1}]}(z)\,d\nu(z)\\&=\frac{1}{\lambda^p}\int\limits_{{}_0[x_p,\ldots,\,x_{n+p-1}]} e^{(T_p\varphi)(x_0,\ldots,\,x_{p-1},\,z)}\,d\nu(z).\end{aligned}$$ □

In part (i) of the next result, the number k is the number of symbols in the topologically mixing subshift of finite type and M is a natural number for which the Mth power A^M of the transition matrix A has every entry non-zero.

THEOREM 1.2. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi\in C(X)$. Let $\nu\in M(X)$ and $\lambda>0$ satisfy $\mathcal{L}_{\varphi}^{*}\nu=\lambda\nu$. Then $\lambda=e^{P(T,\varphi)}$, ν is a supported measure, and we have the following.*

(i) *For all $p\geqslant 1$ and for all $x\in X$,*

$$\big(e^{v_p(T_p\varphi)}\lambda^M e^{M\|\varphi\|}\big)^{-1}\leqslant\frac{\nu({}_0[x_0,\ldots,x_{p-1}])\lambda^p}{e^{(T_p\varphi)(x)}}\leqslant k^M e^{v_p(T_p\varphi)}\frac{e^{M\|\varphi\|}}{\lambda^M}.$$

(ii) *For all $n\geqslant 1$, $p\geqslant 1$ and for all $x\in X$,*

$$e^{-v_{n+p}(T_p\varphi)}\leqslant\frac{\nu({}_0[x_0,\ldots,x_{n+p-1}])}{\nu({}_0[x_p,\ldots,x_{n+p-1}])}\frac{\lambda^p}{e^{(T_p\varphi)(x)}}\leqslant e^{v_{n+p}(T_p\varphi)}.$$

(iii) *The measures ν and $\nu\circ T^{-p}$ are equivalent and*

$$\frac{d\nu\circ T^{-p}}{d\nu}=\frac{\mathcal{L}_{\varphi}^{p}1}{\lambda^p}.$$

(iv) *For each symbol $i\in\{1,2,\ldots,k\}$, $T\,|_{{}_0[i]}$ is injective, so $\nu\circ T\,|_{{}_0[i]}$ is a measure on ${}_0[i]$ and*

$$\frac{d\big(\nu\circ T\,|_{{}_0[i]}\big)}{d\big(\nu\,|_{{}_0[i]}\big)}=\lambda e^{-\varphi}$$

on ${}_0[i]$.

(v) *For all* $n \geqslant 1$, $p \geqslant 1$ *and for all* $w \in X$,

$$e^{-v_n(\log \mathcal{L}_\varphi^p 1)} \leqslant \frac{\nu({}_p[w_0,\ldots,w_{n-1}])}{\nu({}_0[w_0,\ldots,w_{n-1}])}\frac{\lambda^p}{(\mathcal{L}_\varphi^p 1)(w)} \leqslant e^{v_n(\log \mathcal{L}_\varphi^p 1)}.$$

Proof. (i)

$$\begin{aligned}\nu({}_0[x_0,\ldots,x_{p-1}]) &= \int_X \chi_{0[x_0,\ldots,x_{p-1}]}(z)\,d\nu(z)\\ &= \frac{1}{\lambda^{p+M}}\int_X \mathcal{L}_\varphi^{p+M}\chi_{0[x_0,\ldots,x_{p-1}]}(z)\,d\nu(z).\end{aligned}$$

Now

$$\mathcal{L}_\varphi^{p+M}\chi_{0[x_0,\ldots,x_{p-1}]}(z) = \sum e^{(T_{p+M}\varphi)(x_0,\ldots,x_{p-1},b_0,\ldots,b_{M-1},z)},$$

where the sum is over all $b_0,\ldots,b_{M-1}$ with the product $a_{x_{p-1}b_0}a_{b_0b_1}\ldots a_{b_{M-1}z_0} > 0$. For any x_{p-1}, z_0, such $(b_0,\ldots,b_{M-1})$ exist by topological mixing, and there are at most k^M choices. For each such admissable $(b_0,\ldots,b_{M-1})$, we have $|(T_{p+M}\varphi)(x_0,\ldots,x_{p-1},b_0,\ldots,b_{M-1},z) - (T_p\varphi)(x)| \leqslant v_p(T_p\varphi) + M\|\varphi\|$, so we get the inequalities in (i).

The left-hand inequality in (i) shows that ν is supported. To see that $\lambda = e^{P(T,\varphi)}$, we use the fact that $(1/p)\log(\mathcal{L}_\varphi^p 1)(x) \rightrightarrows P(T,\varphi)$ [**14**, Theorem 1.3]. If $\varepsilon > 0$, then there exists N_ε with $e^{pP(T,\varphi)-p\varepsilon} \leqslant (\mathcal{L}_\varphi^p 1)(x) \leqslant e^{pP(T,\varphi)+p\varepsilon}$ for all $x \in X$ and for all $p \geqslant N_\varepsilon$. Integrating with respect to ν gives $e^{p(P(T,\varphi)-\varepsilon)} \leqslant \lambda^p \leqslant e^{p(P(T,\varphi)+\varepsilon)}$ for all $p \geqslant N_\varepsilon$. Hence $e^{P(T,\varphi)-\varepsilon} \leqslant \lambda \leqslant e^{P(T,\varphi)+\varepsilon}$, and since this holds for every $\varepsilon > 0$, we have $\lambda = e^{P(T,\varphi)}$.

(ii) Statement (ii) follows from Lemma 1.1 and the inequality $|(T_p\varphi)(x_0,\ldots, x_{p-1}z) - (T_p\varphi)(x)| \leqslant v_{n+p}(T_p\varphi)$, when $z \in {}_0[x_p,\ldots,x_{n+p-1}]$.

(iii) Fix a cylinder ${}_0[w_0,\ldots,w_{p-1}]$ in X. Then

$$T^{-p}({}_0[w_0,\ldots,w_{n-1}]) = \bigcup {}_0[y_0,\ldots,y_{p-1},w_0,\ldots,w_{n-1}],$$

where the union is over all $(y_0,\ldots,y_{p-1})$ with $a_{y_0y_1}\ldots a_{y_{p-1}w_0} = 1$, and this is a disjoint union. Apply Lemma 1.1 to each such admissable ${}_0[y_0,\ldots,y_{p-1}, w_0,\ldots,w_{n-1}]$ to get

$$\nu({}_0[y_0,\ldots,y_{p-1},w_0,\ldots,w_{n-1}]) = \frac{1}{\lambda^p}\int_{z\in{}_0[w_0,\ldots,w_{n-1}]} e^{(T_p\varphi)(y_0,\ldots,y_{p-1},z)}\,d\nu(z).$$

Now sum over all $(y_0,\ldots,y_{p-1})$ to get

$$(\nu\circ T^{-p})({}_0[w_0,\ldots,w_{n-1}]) = \int_{z\in{}_0[w_0,\ldots,w_{n-1}]} \frac{(\mathcal{L}_\varphi^p 1)(z)}{\lambda^p}\,dv(z).$$

(iv) We have to show that for each cylinder ${}_0[i,i_1,\ldots,i_{p-1}]$, we have

$$\nu(T\,{}_0[i,i_1,\ldots,i_{p-1}]) = \lambda\int_{x\in{}_0[i,i_1,\ldots,i_{p-1}]} e^{-\varphi(x)}\,d\nu(x).$$

We have

$$\lambda \int\limits_{{}_0[i,i_1,\ldots,i_{p-1}]} e^{-\varphi(x)}\, d\nu(x) = \lambda \int\limits_X e^{-\varphi} \chi_{{}_0[i,i_1,\ldots,i_{p-1}]}\, d\nu$$
$$= \int\limits_X \mathcal{L}_\varphi \left(e^{-\varphi} \chi_{{}_0[i,i_1,\ldots,i_{p-1}]}\right) d\nu$$
$$= \int\limits_X \chi_{{}_0[i,i_1,\ldots,i_{p-1}]}(iz)\, d\nu(z)$$
$$= \nu(T_0[i,i_1,\ldots,i_{p-1}]).$$

(v) By (iii),

$$\nu({}_p[w_0,\ldots,w_{n-1}]) = \int\limits_{z\in{}_0[w_0,\ldots,w_{n-1}]} \frac{\mathcal{L}_\varphi^p 1}{\lambda^p}(z)\, d\nu(z).$$

If $w \in X$ and $z \in {}_0[w_0,\ldots,w_{n-1}]$, then

$$e^{-v_n\left(\log \mathcal{L}_\varphi^p 1\right)} \leqslant \frac{\left(\mathcal{L}_\varphi^p 1\right)(z)}{\left(\mathcal{L}_\varphi^p 1\right)(w)} \leqslant e^{v_n\left(\log \mathcal{L}_\varphi^p 1\right)}$$

so (v) holds. □

Recall that we use the symbol $\rightrightarrows$ to denote uniform convergence on X.

COROLLARY 1.3. *Let $T: X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type, and let $\varphi \in C(X)$. Let $\nu \in M(X)$ and let $\lambda > 0$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda\nu$. For each fixed $p \geqslant 1$,*

$$\frac{\nu({}_0[x_0,\ldots,x_{n+p-1}])}{\nu({}_0[x_p,\ldots,x_{n+p-1}])} \rightrightarrows \frac{e^{(T_p\varphi)(x)}}{\lambda^p} \quad \text{as } n \to \infty,$$

and

$$\frac{\nu({}_p[w_0,\ldots,w_{n-1}])}{\nu({}_0[w_0,\ldots,w_{n-1}])} \rightrightarrows \frac{\left(\mathcal{L}_\varphi^p 1\right)(w)}{\lambda^p} \quad \text{as } n \to \infty.$$

Proof. The first statement is by Theorem 1.2(ii), since, for p fixed, $v_{n+p}(T_p\varphi) \to 0$ as $n \to \infty$, and the second statement is by Theorem 1.2(v), since, for fixed p, $v_n(\log \mathcal{L}_\varphi^p 1) \to 0$ as $n \to \infty$. □

In fact, the case $p = 1$ in the first statement of Corollary 1.3 gives

$$\frac{\nu({}_0[x_0,\ldots,x_n])}{\nu({}_0[x_1,\ldots,x_n])} \rightrightarrows \frac{e^{\varphi(x)}}{\lambda} \quad \text{as } n \to \infty,$$

and the case of general p follows from this, as does the second conclusion of the corollary.

Note that we can write the first conclusion of Corollary 1.3 as

$$\frac{\nu({}_0[x_0,\ldots,x_{n+p-1}])}{\nu(T^p{}_0[x_0,\ldots,x_{n+p-1}])} \rightrightarrows \frac{e^{(T_p\varphi)(x)}}{\lambda^p} \quad \text{as } n \to \infty,$$

and the second conclusion as

$$\frac{\nu(T^{-p}{}_0[w_0,\ldots,w_{n-1}])}{\nu({}_0[w_0,\ldots,w_{n-1}])} \rightrightarrows \frac{(\mathcal{L}_\varphi^p 1)(w)}{\lambda^p} \quad \text{as } n\to\infty.$$

The following result gives a condition for a probability measure to be an eigenmeasure of a Ruelle operator.

Corollary 1.4. *Let $T: X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\nu \in M(X)$. Then $\mathcal{L}_\varphi^*\nu = \lambda\nu$ for some $\varphi \in C(X)$ and some $\lambda > 0$ if and only if ν is supported and $\nu({}_0[x_0,\ldots,x_n])/\nu({}_0[x_1,\ldots,x_n])$ converges uniformly on X to a function $f: X \longrightarrow (0,\infty)$.*

Proof. If $\mathcal{L}_\varphi^*\nu = \lambda\nu$ for some $\nu\in M(X)$ and some $\lambda>0$, then

$$\frac{\nu({}_0[x_0,\ldots,x_n])}{\nu({}_0[x_1,\ldots,x_n])} \rightrightarrows \frac{e^{\varphi(x)}}{\lambda}$$

by Corollary 1.3.

If $\nu({}_0[x_0,\ldots,x_n])/\nu({}_0[x_1,\ldots,x_n])$ converges uniformly on X to a positive function, then denote the limit by $e^{\varphi(x)}$. We show that $\mathcal{L}_\varphi^*\nu = \nu$. For any cylinder ${}_0[b_0,\ldots,b_{t-1}]$,

$$\begin{aligned}
&(\mathcal{L}_\varphi^*\nu)({}_0[b_0,\ldots,b_{t-1}]) \\
&\quad = \int \mathcal{L}_\varphi \chi_{{}_0[b_0,\ldots,b_{t-1}]}\,d\nu = \int\limits_{x\in{}_0[b_1,\ldots,b_{t-1}]} e^{\varphi(b_0x)}\,d\nu(x) \\
&\quad = \lim_{n\to\infty} \int\limits_{x\in{}_0[b_1,\ldots,b_{t-1}]} \frac{\nu({}_0[b_0,b_1,\ldots,b_{t-1},x_{t-1},x_t,\ldots,x_{t+n}])}{\nu({}_0[b_1,\ldots,b_{t-1},\ x_{t-1},\ldots,x_{t+n}])}\,d\nu(x) \\
&\quad = \lim_{n\to\infty} \sum_{x_{t-1},\ldots,x_{t+n}} \nu({}_0[b_0,b_1,\ldots,b_{t-1},x_{t-1},\ldots,x_{t+n}]) \\
&\quad = \nu({}_0[b_0,\ldots,b_{t-1}]).
\end{aligned}$$

□

With Corollary 1.3 in mind, we can characterize when $\varphi\in C(X)$ is a member of $W(X,T)$ in terms of an eigenmeasure ν. If $\mathbb{N}$ is the set of natural numbers, then $\mathrm{BC}(\mathbb{N}\times X)$ denotes the space of bounded continuous real-valued functions on $\mathbb{N}\times X$, equipped with the supremum norm.

Theorem 1.5. *Let $T: X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi\in C(X)$. Let $\lambda = e^{P(T,\varphi)}$. The following statements are pairwise equivalent.*

(i) $\varphi \in W(X,T)$.

(ii) *There exists $\tau\in M(X)$ with support X satisfying*

$$\frac{\tau({}_0[x_0,\ldots,x_{p+n-1}])}{\tau({}_0[x_p,\ldots,x_{p+n-1}])}\frac{\lambda^p}{e^{(T_p\varphi)(x)}} \rightrightarrows 1 \quad \text{as } n\to\infty,$$

where the convergence is uniform in both $x\in X$ and $p\in\mathbb{N}$.

(iii) *There exists* $\nu \in M(X)$ *with* $\mathcal{L}_\varphi^* \nu = \lambda\nu$ *such that in* $\mathrm{BC}(\mathbb{N}\times X)$, *the sequence* (ψ_n), *given by*

$$\psi_n(p,x) = \log\left(\frac{\nu({}_0[x_0,\dots,x_{p+n-1}])}{\nu({}_0[x_p,\dots,x_{p+n-1}])}\right),$$

is convergent.

(iv) *There exists* $\mu \in M(X,T)$ *with support* X *and* $\ell \in C(X)$ *satisfying*

$$\log\left(\frac{\mu([x_0,\dots,x_{n+p-1}])}{\mu([x_p,\dots,x_{n+p-1}])}\right) \rightrightarrows (T_p\varphi)(x) + \ell(x) - \ell(T^p x) - p\log\lambda \quad \text{as } n\to\infty,$$

where the convergence is uniform in both $x\in X$ *and* $p \geqslant 1$.

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from Theorem 1.2(ii). To show that (ii) $\Rightarrow$ (i), suppose that $\tau\in M(X)$ satisfies (ii). Then for all $\varepsilon>0$, there exists N_ε such that $n\geqslant N_\varepsilon$ implies that

$$\frac{\tau({}_0[x_0,\dots,x_{n+p-1}])}{\tau({}_0[x_p,\dots,x_{n+p-1}])}\lambda^p e^{-\varepsilon} < e^{(T_p\varphi)(x)} < \frac{\tau({}_0[x_0,\dots,x_{p+n-1}])}{\tau({}_0[x_p,\dots,x_{p+n-1}])}\lambda^p e^{\varepsilon} \quad \forall\, p\geqslant 1,\ \forall\, x\in X.$$

If $x,z\in X$ have $x_i=z_i$ for $0\leqslant i\leqslant p+n-1$, then when $n\geqslant N_\varepsilon$, $e^{-2\varepsilon} < e^{(T_p\varphi)(x)-(T_p\varphi)(z)} < e^{2\varepsilon}$. Hence $n\geqslant N_\varepsilon \Rightarrow \sup_{p\geqslant 1}\nu_{p+n}(T_p\varphi)\leqslant 2\varepsilon$. Hence $\varphi\in W(X,T)$.

The implication that (i) $\Rightarrow$ (iii) follows from Theorem 1.2(ii). If (iii) holds, then the limit is $(T_p\varphi)(n) - p\log\lambda$ by Corollary 1.3, so (ii) holds.

We now show that (i) $\Rightarrow$ (iv). When $\varphi\in W(X,T)$, the unique $\nu\in M(X)$ with $\mathcal{L}_\varphi^*\nu=\lambda\nu$ satisfies (ii), as we saw above. By [**11**], there exists $h\in C(X)$, $h>0$, with $\mathcal{L}_\varphi h = \lambda h$ and $\int h\,d\nu = 1$. The measure $\mu = h\nu\in M(X,T)$, and

$$e^{-v_t(\log h)} \leqslant \frac{\mu([x_0,\dots,x_{t-1}])}{\nu({}_0[x_0,\dots,x_{t-1}])h(x)} \leqslant e^{v_t(\log h)} \qquad \forall\ t\geqslant 1,\ x\in X.$$

Hence

$$\frac{\mu([x_0,\dots,x_{n+p-1}])}{\mu([x_p,\dots,x_{n+p-1}])}\frac{\lambda^p}{e^{(T_p\varphi)(x)}}\frac{h(T^p x)}{h(x)} \rightrightarrows 1 \quad \text{as } n\to\infty$$

uniformly in $p\geqslant 1$ and $x\in X$. Statement (iv) follows with $\ell = \log h$.

If (iv) holds, then, for $\varepsilon>0$, there exists N_ε such that $n\geqslant N_\varepsilon$ implies that

$$e^{-\varepsilon} < \frac{\mu([x_0,\dots,x_{n+p-1}])}{\mu([x_p,\dots,x_{n+p-1}])}\frac{\lambda^p}{e^{(T_p\varphi)(x)}}\frac{h(T^p x)}{h(x)} < e^{\varepsilon} \qquad \forall\, p\geqslant 1,\ \forall\, x\in X,$$

where $h=e^\ell$. If $(x_0,\dots,x_{n+p-1}) = (z_0,\dots,z_{n+p-1})$, then, for $n\geqslant N_\varepsilon$,

$$\frac{h(T^p x)}{h(T^p z)}\frac{h(z)}{h(x)}e^{-2\varepsilon} < e^{(T_p\varphi)(x)-(T_p\varphi)(z)} < \frac{h(T^p x)}{h(T^p z)}\frac{h(z)}{h(x)}e^{2\varepsilon},$$

so

$$v_{n+p}(T_p\varphi) \leqslant v_n(\log h) + v_{n+p}(\log h) + 2\varepsilon \leqslant 2v_n(\log h) + 2\varepsilon \qquad \forall\, p\geqslant 1,\ \forall\, n\geqslant 1.$$

Hence $\varphi\in W(X,T)$. □

Note that when $\varphi\in W(X,T)$, then the eigenmeasure ν has the property given in (ii). The property in (ii) can be written as

$$\log\left(\frac{\tau({}_0[x_0,\dots,x_{n+p-1}])}{\tau({}_0[x_p,\dots,x_{n+p-1}])}\right) \rightrightarrows (T_p\varphi)(x) - p\log\lambda \quad \text{as } n\to\infty,$$

where the convergence is uniform in both $x\in X$ and $p\geqslant 1$.

We now characterize, in terms of eigenmeasure, those $\varphi \in C(X)$ which belong to $\mathrm{Bow}(X,T)$.

Definition 1.6. Let X be a one-sided topologically mixing subshift of finite type. Let $\tau \in M(X)$. We say that τ is *approximately multiplicative at coordinate zero* if it has support X and there exists $C > 1$ with

$$C^{-1} \leqslant \frac{\tau({}_0[x_0,\dots,x_{n+p-1}])}{\tau({}_0[x_0,\dots,x_{p-1}])\tau({}_0[x_p,\dots,x_{n+p-1}])} \leqslant C \qquad \forall n \geqslant 1,\ p \geqslant 1,\ x \in X.$$

If we let $f^{(n)}(x) = \nu({}_0[x_0,\dots,x_{n-1}])$, then the condition becomes

$$C^{-1} \leqslant \frac{f^{(p+n)}(x)}{f^{(p)}(x) f^{(n)}(T^p x)} \leqslant C.$$

Theorem 1.7. *Let $T : X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\lambda = e^{P(T,\varphi)}$. The following statements are pairwise equivalent.*

(i) $\varphi \in \mathrm{Bow}(X,T)$.

(ii) *There exists $\tau \in M(X)$ with support X and there exists $D > 1$ with*

$$D^{-1} \leqslant \tau({}_0[x_0,\dots,x_{p-1}]) \frac{\lambda^p}{e^{(T_p\varphi)(x)}} \leqslant D \qquad \forall p \geqslant 1,\ \forall x \in X.$$

(iii) *There exists $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and ν is approximately multiplicative at coordinate zero.*

Proof. The implication (i) ⇒ (ii) follows from Theorem 1.2(i). If (ii) holds and $(x_0,\dots,x_{p-1}) = (z_0,\dots,z_{p-1})$, then

$$D^{-2} \leqslant e^{(T_p\varphi)(x) - (T_p\varphi)(z)} \leqslant D^2,$$

so

$$v_p(T_p\varphi) \leqslant 2\log D \quad \text{and} \quad \varphi \in \mathrm{Bow}\,(X,T).$$

To show (i) ⇒ (iii), we use Theorem 1.2(i) and (ii) to get

$$\frac{\lambda^M}{\lambda^M e^{M\|\varphi\|} e^{v_{n+p}(T_p\varphi)+v_p(T_p\varphi)}} \leqslant \frac{\nu({}_0[x_0,\dots,x_{n+p-1}])}{\nu({}_0[x_0,\dots,x_{p-1}])\nu({}_0[x_p,\dots,x_{n+p-1}])} \leqslant \lambda^M e^{M\|\varphi\|} e^{v_{n+p}(T_p\varphi)+v_p(T_p\varphi)}.$$

Since $v_{p+n}(T_p\varphi) \leqslant \mathrm{var}_p(T_p\varphi)$, we have (i) ⇒ (iii). To see that (iii) ⇒ (ii), if C is the constant in the definition of ν being approximately multiplicative at coordinate zero, then from Theorem 1.2(ii) we have

$$C^{-1} e^{-v_{p+n}(T_p\varphi)} \leqslant \nu({}_0[x_0,\dots,x_{p-1}]) \frac{\lambda^p}{e^{(T_p\varphi)(x)}} \leqslant C e^{v_{p+n}(T_p\varphi)} \qquad \text{for all } n \geqslant 1.$$

Let $n \to \infty$ to give (ii). □

The equivalence of (i) and (ii) is well known. Other equivalent conditions for $\varphi \in \mathrm{Bow}(X,T)$ can be found in [**14**].

As we shall see in Section 3, Theorem 1.7 gives a nice characterization of which g-measures correspond to a g with $\log g \in \mathrm{Bow}(X,T)$.

COROLLARY 1.8. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi\in \mathrm{Bow}(X,T)$. The unique equilibrium state μ_φ of φ is approximately multiplicative at coordinate zero.*

Proof. When $\varphi\in\mathrm{Bow}(X,T)$, there is a unique $\nu\in M(X)$ with $\mathcal{L}_\varphi^*\nu=\lambda\nu$ [**14**], and ν is approximately multiplicative at coordinate zero by Theorem 1.7. Also, $\mu_\varphi=h\nu$ for some measurable $h:X\longrightarrow[a,b]$ with $0<a<b$, and $\mathcal{L}_\varphi h=\lambda h$ and $\int h\,d\nu=1$ [**14**]. Hence

$$a\nu({}_p[x_0,\dots,x_{n-1}])\leqslant\mu_\varphi({}_p[x_0,\dots,x_{n-1}])\leqslant b\nu({}_p[x_0,\dots,x_{n-1}])\qquad\forall\, p\geqslant 1,\ n\geqslant 1,\ x\in X,$$

so that μ_φ is approximately multiplicative at coordinate zero. □

Since μ_φ is T-invariant, we can write the approximately multiplicative at coordinate zero condition for μ_φ as follows. There exists $D>1$ with

$$D^{-1}\leqslant\frac{\mu_\varphi[{}_0(x_0,\dots,x_{n+p-1})]}{\mu_\varphi({}_0[x_0,\dots,x_{p-1}])\mu_\varphi({}_p[x_p,\dots,x_{n+p-1}])}\leqslant D\qquad\forall\, p\geqslant 1,\ n\geqslant 1,\ x\in X.$$

We use this condition in the next definition.

DEFINITION 1.9. Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\tau\in M(X)$. We say that τ has *approximate product structure* if it has support X and if there exists $C>1$ with

$$C^{-1}\leqslant\frac{\tau({}_0[x_0,\dots,x_{n+p-1}])}{\tau({}_0[x_0,\dots,x_{p-1}])\tau({}_p[x_p,\dots,x_{n+p-1}])}\leqslant C\qquad\forall\, n\geqslant 1,\ p\geqslant 1,\ x\in X.$$

If $\tau\in M(X,T)$, then clearly τ is approximately multplicative at coordinate zero if and only if τ has approximate product structure. To investigate the relationship between the two conditions when τ is an eigenmeasure ν for $\mathcal{L}_\varphi$, we can use the following deduction from Theorem 1.2(iii).

PROPOSITION 1.10. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi\in C(X)$. Let $\nu\in M(X)$ satisfy $\mathcal{L}_\varphi^*\nu=\lambda\nu$ with $\lambda=e^{P(T,\varphi)}$. Let $D\geqslant 1$, and let $p\in\mathbb{N}$. Then*

$$D^{-1}\leqslant\frac{\left(\mathcal{L}_\varphi^p 1(x)\right)}{\lambda^p}\leqslant D\quad\forall\, x\in X$$

if and only if

$$D^{-1}\leqslant\frac{\nu({}_p[z_0,\dots,z_{n-1}])}{\nu({}_0[z_0,\dots,z_{n-1}])}\leqslant D\qquad\forall\, n\geqslant 1,\ z\in X.$$

Proof. By Theorem 1.2(iii),

$$\nu({}_p[z_0,\dots,z_{n-1}])=\int\limits_{w\in{}_0[z_0,\dots,z_{n-1}]}\frac{\left(\mathcal{L}_\varphi^p 1\right)}{\lambda^p}(w)\,d\nu(w).\tag{$*$}$$

Clearly the first statement of the proposition implies the second statement. Now suppose that the second statement holds and let

$$U=\left\{x\in X|\frac{\left(\mathcal{L}_\varphi^p 1\right)(x)}{\lambda^p}(x)>D\right\}.$$

Then U is open. Suppose that $U \neq \varnothing$. For $z \in U$, choose some n with ${}_0[z_0, \dots, z_{n-1}] \subset U$. Then, by $(*)$, $\nu({}_p[z_0, \dots, z_{n-1}]) > D\nu({}_0([z_0, \dots, z_{n-1}])$, which contradicts the assumption. Hence $U = \varnothing$, so $(\mathcal{L}_\varphi^p 1)(x)/\lambda^p \leqslant D$ for all $x \in X$. Similarly, we get $D^{-1} \leqslant (\mathcal{L}_\varphi^p 1)(x)/\lambda^p$ for all $x \in X$. □

Corollary 1.11. *Let $T : X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\nu \in M(X)$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda\nu$ with $\lambda = e^{P(T,\varphi)}$. Suppose that there exists $D \geqslant 1$ with*

$$D^{-1} \leqslant \frac{(\mathcal{L}_\varphi^p 1)(x)}{\lambda^p} \leqslant D \qquad \forall\, p \geqslant 1,\ \forall\, x \in X.$$

Then ν is approximately multiplicative at coordinate zero if and only if ν has approximate product structure.

Corollary 1.12. *Let $T : X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\lambda = e^{P(T,\varphi)}$. Then $\varphi \in \mathrm{Bow}(X,T)$ if and only if both of the following statements hold.*

(i) *There exists $B > 1$ with $B^{-1} \leqslant (\mathcal{L}_\varphi^p 1)(x)/\lambda^p \leqslant B$ for all $p \geqslant 1$, $x \in X$.*

(ii) *There exists $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda\nu$ and ν has approximate product structure.*

Proof. If $\varphi \in \mathrm{Bow}(X,T)$, then (i) holds by [**14**, p. 337], and (ii) holds by Theorem 1.7 and Corollary 1.11. If (i) and (ii) hold, then, by Corollary 1.11, ν is approximately multiplicative at coordinate zero and hence $\varphi \in \mathrm{Bow}(X,T)$ by Theorem 1.7. □

The following lemma is well known.

Lemma 1.13. *Let $T : X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$ and $\lambda = e^{P(T,\varphi)}$. Suppose that $\nu \in M(X)$ satisfies $\mathcal{L}_\varphi^* \nu = \lambda\nu$, and suppose that there is a measurable $h : X \longrightarrow [a,b]$ with $0 < a \leqslant b$ and $\mathcal{L}_\varphi h = \lambda h$. Suppose that h is normalized so that $\int h\, d\nu = 1$. Then $\mu = h\nu$ is an equilibrium state for φ.*

Proof. We have $\mu \in M(X,T)$ since, for $f \in C(X)$,

$$\int f \circ T\, d\mu = \int f \circ T h\, d\nu = \lambda^{-1} \int \mathcal{L}_\varphi(f \circ T h)\, d\nu = \lambda^{-1} \int f \mathcal{L}_\varphi(h)\, d\nu = \int f h\, d\nu = \int f\, d\mu.$$

To see that μ is an equilibrium state, we use Theorem 1.2(i) to get

$$\begin{aligned} -\frac{1}{p}\log(\lambda^M e^{M\|\varphi\|}) - \frac{v_p(T_p\varphi)}{p} &\leqslant -\frac{1}{p}\log \nu({}_0[x_0, \dots, x_{p-1}]) + \frac{1}{p}(T_p\varphi)(x) - \log\lambda \\ &\leqslant \frac{1}{p}\log\left(\frac{k^M e^{M\|\varphi\|}}{\lambda^M}\right) + \frac{v_p(T_p\varphi)}{p}. \end{aligned}$$

Since $a\nu({}_0[x_0, \dots, x_{p-1}]) \leqslant \mu({}_0[x_0, \dots, x_{p-1}]) \leqslant b\nu({}_0[x_0, \dots, x_{p-1}])$ and

$$v_p(T_p\varphi) \leqslant \sum_{i=1}^{p} v_i(\varphi),$$

we have

$$-\frac{1}{p}\log\mu({}_0[x_0,\ldots,x_{p-1}])+\frac{1}{p}(T_p\varphi)(x)\Rightarrow\log\lambda=P(T,\varphi).$$

Integrating via μ gives

$$\frac{1}{p}H\left(\bigvee_{i=0}^{p-1}T^{-i}\xi\right)+\int\varphi\,d\mu\longrightarrow P(T,\varphi),$$

where ξ is the partition into states at coordinate zero. Since ξ is a one-sided generator, we have $h_\mu(T)+\int\varphi\,d\mu=P(T,\varphi)$. □

There is another characterization of $\varphi\in\mathrm{Bow}(X,T)$.

THEOREM 1.14. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi\in C(X)$ and $\lambda=e^{P(T,\varphi)}$. Then $\varphi\in\mathrm{Bow}(X,T)$ if and only if all of the following three statements hold.*
(i) *φ has a unique equilibrium state μ_φ.*
(ii) *μ_φ has approximate product structure.*
(iii) *There exists $B>1$ with $B^{-1}\leqslant(\mathcal{L}_\varphi^p1)(x)/\lambda^p\leqslant B$ for all $p\geqslant1$, $x\in X$.*

Proof. If $\varphi\in\mathrm{Bow}(X,T)$, (i) and (iii) hold by [**14**], and (ii) holds by Corollary 1.8. Now assume that the three conditions hold. Let $\nu\in M(X)$ satisfy $\mathcal{L}_\varphi^*\nu=\lambda\nu$. By (iii), there is a measurable $h:X\longrightarrow[a,b]$ with $0<a<b$, $\int h\,d\nu=1$, and $\mathcal{L}_\varphi h=\lambda h$ [**14**, p. 341]. Then, by Lemma 1.13, $\mu=h\nu$ is T-invariant and is an equilibrium state for φ. By (i), $\mu_\varphi=h\nu$. By (ii), μ_φ is approximately multiplicative at coordinate zero, so ν is also. Hence $\varphi\in\mathrm{Bow}(X,T)$ by Theorem 1.7. □

2. *The weak Bernoulli property*

In this section, we show that if $\varphi\in\mathrm{Bow}(X,T)$, then the natural extension of T with respect to the unique equilibrium state μ_φ of φ is a Bernoulli shift. We shall also show that when $S:\hat{X}\longrightarrow\hat{X}$ is a two-sided topologically mixing subshift of finite type and $\hat{\varphi}\in\mathrm{Bow}(\hat{X},S)$, then $(S,\hat{\mu}_{\hat{\varphi}})$ is isomorphic to a Bernoulli shift where $\hat{\mu}_{\hat{\varphi}}$ is the unique equilibrium state of $\hat{\varphi}$.

When $\hat{\varphi}\in\mathrm{Bow}(\hat{X},S)$, then Bowen [**2**] showed that there exists $C>1$ with

$$C^{-1}\leqslant\frac{\hat{\mu}_{\hat{\varphi}}([x_0,\ldots,x_{n-1}])}{e^{(S_n\hat{\varphi})(x)-nP(S,\hat{\varphi})}}\leqslant C\qquad\forall\,x\in\hat{X},\ \forall\,n\geqslant1.$$

It readily follows that $\hat{\mu}_{\hat{\varphi}}$ has approximate product structure (or rather that $\hat{\mu}\circ\pi^{-1}$ does, where $\pi:\hat{X}\longrightarrow X$ is the natural projection). The definition of approximate product structure also makes sense for a measure on the two-sided shift space $\hat{X}$.

Whereas every $\hat{\varphi}\in W(\hat{X},S)$ is cohomologous in $C(\hat{X})$ to some $\varphi\circ\pi$ with $\varphi\in W(X,T)$ [**1**], there are examples of $\hat{\varphi}\in\mathrm{Bow}(\hat{X},S)$ which are not cohomologous in $C(\hat{X})$ to a one-sided function [**10**].

For every $\mu\in M(X,T)$, there is a unique $\hat{\mu}\in M(\hat{X},S)$ with $\hat{\mu}\circ\pi^{-1}=\mu$. We have $\hat{\mu}({}_p[b_0,\ldots,b_{n-1}])=\mu({}_q[b_0,\ldots,b_{n-1}])$, for all $p\in Z$, $q\geqslant0$. This gives a bijection between $M(X,T)$ and $M(\hat{X},S)$.

Theorem 2.1. *Let $T:X\longrightarrow T$ be a one-sided topologically mixing subshift of finite type with transition matrix A. Let $\mu\in M(X,T)$ have approximate product structure. If $A^M>0$, then there exists $D>1$ such that for all $n\geqslant 2M$, all $p,q\geqslant 1$, and all allowable cylinders ${}_0[x_0,\dots,x_{p-1}]$, ${}_0[y_0,\dots,y_{q-1}]$, we have*

$$D^{-1}\leqslant\frac{\mu({}_0[x_0,\dots,x_{p-1}]\cap T^{-(p+n)}{}_0[y_0,\dots,y_{q-1}])}{\mu([x_0,\dots,x_{p-1}])\mu([y_0,\dots,y_{q-1}])}\leqslant D.$$

Proof. Let C be the constant in the approximate product structure condition for μ. Let $[x_0,\dots,x_{p-1}]$ and $[y_0,\dots,y_{q-1}]$ be given. Let $n\geqslant 2M$. Since $A^M>0$, there is a point $w\in X$ of the form $w=(x_0,\dots,x_{p-1},w_p,\dots,w_{p+n-1},y_0,y_1,\dots)$. Moreover, for any allowable choice of the cylinder $[w_{p+M-1},\dots,w_{p+n-M}]$, there is such a point. Since

$$C^{-4}\leqslant\frac{\mu[x_0,\dots,x_{p-1},w_p,\dots,w_{p+n-1},y_0,\dots,y_{q-1}]}{\mu[x_0,\dots,x_{p-1}]\mu[w_p,\dots,w_{p+M-2}]\mu[w_{p+M-1},\dots,w_{p+n-M}]\mu[w_{p+n-M+1},\dots,w_{p+n-1}]\mu[y_0,\dots,y_{q-1}]}\leqslant C^4,$$

we have

$$C^{-4}d^2\leqslant\frac{\mu\left({}_0[x_0,\dots,x_{p-1}]\cap T^{-(p+n)}\,{}_0[y_0,\dots,y_{q-1}]\right)}{\mu({}_0[x_0,\dots,x_{p-1}])\mu({}_0[y_0,\dots,y_{q-1}])}\leqslant C^4,$$

where d is the minimal μ-measure of a cylinder of length $M-1$.

Therefore, put $D=C^4d^{-2}$. □

If we consider the corresponding two-sided measure $\hat\mu$, then the conclusion of Theorem 2.1 can be written as

$$D^{-1}\leqslant\frac{\hat\mu({}_{-p}[x_{-p},\dots,x_{-1}]\cap S^{-n}{}_0[y_0,\dots,y_{q-1}])}{\hat\mu([x_{-p},\dots,x_{-1}])\hat\mu([y_0,\dots,y_{q-1}])}\leqslant D$$

for all $n\geqslant 2M$ and all cylinders ${}_{-p}[x_{-p},\dots,x_{-1}]$, ${}_0[y_0,\dots,y_{q-1}]$ with $p,q\geqslant 1$.

By the usual approximation arguments, one can readily get

$$D^{-1}\leqslant\frac{\hat\mu(B_1\cap S^{-n}B_2)}{\hat\mu(B_1)\hat\mu(B_2)}\leqslant D \tag{†}$$

whenever $n\geqslant 2M$, $B_1\in\mathcal{B}_{-\infty}^{-1}$, $B_2\in\mathcal{B}_0^{\infty}$ and $\hat\mu(B_1)\hat\mu(B_2)>0$. Here $\mathcal{B}_{-\infty}^{-1}$ is the σ-algebra generated by all cylinders ${}_{-p}[x_{-p},\dots,x_{-1}]_{-1}$ with $p\geqslant 1$, and $\mathcal{B}_0^{\infty}$ is the σ-algebra generated by all cylinders ${}_0[y_0,\dots,y_{q-1}]_{q-1}$ with $q\geqslant 1$.

We shall use the following result.

Corollary 2.2. *Let $S:\hat X\longrightarrow\hat X$ be a one-sided topologically mixing subshift of finite type and let $\hat\mu\in M(\hat X,S)$ be of approximate product type. Then S is strongly mixing with respect to $\hat\mu$.*

Proof. This follows from a result of Ornstein [**9**] if we can show that S^i is ergodic for each $i\geqslant 1$ and there is a constant d with

$$\limsup_{n\to\infty}\hat\mu(B_1\cap S^{-n}B_2)\leqslant d\hat\mu(B_1)\hat\mu(B_2)\qquad\forall\,B_1,B_2\in\mathcal{B}.$$

From (†), we get

$$\limsup_{n\to\infty}\hat\mu(B_1\cap S^{-n}B_2)\leqslant D\hat\mu(B_1)\hat\mu(B_2)$$

whenever $B_1 \in \mathcal{B}_{-\infty}^{\ell}$ and $B_2 \in \mathcal{B}_{-j}^{\infty}$, $\ell, j > 0$, so an approximation argument gives the same inequality whenever $B_1, B_2 \in \mathcal{B}$.

Similarly,

$$\liminf_{n\to\infty} \hat{\mu}(B_1 \cap S^{-n}B_2) \geqslant D^{-1}\hat{\mu}(B_1)\hat{\mu}(B_2)$$

whenever $B_1, B_2 \in \mathcal{B}$. This latter inequality gives the total ergodicity required for Ornstein's result, since, if $S^i B_2 = B_2$, then

$$0 = \hat{\mu}((\hat{X} \setminus B_2) \cap B_2) \geqslant D^{-1}\hat{\mu}(\hat{X} \setminus B_2)\hat{\mu}(B_2),$$

$$\text{so either } \hat{\mu}(B_2) \text{ or } \hat{\mu}(\hat{X} \setminus B_2) \text{ is } 0.$$

We now use results of Bradley to show that having approximate product structure implies the weak Bernoulli property. □

THEOREM 2.3. *Let $S : \hat{X} \longrightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type, and let $\hat{\mu} \in M(\hat{X}, S)$ have approximate product structure. Then for all $\varepsilon > 0$, there exists N so that $n \geqslant N$ implies that*

$$e^{-\varepsilon}\hat{\mu}(B_1)\hat{\mu}(B_2) \leqslant \hat{\mu}(B_1 \cap S^{-n}B_2) \leqslant e^{\varepsilon}\hat{\mu}(B_1)\hat{\mu}(B_2)$$

whenever $B_1 \in \mathcal{B}_{-\infty}^{-1}$ and $B_2 \in \mathcal{B}_0^{\infty}$. Hence the natural partition into states at coordinate zero is a weak Bernoulli partition for S, so $(S, \hat{\mu})$ is isomorphic to a Bernoulli shift.

Proof. From Theorem 2.1, the inequalities (†) hold. Since, by Corollary 2.2, $(S, \hat{\mu})$ is strongly mixing, then, by a result of Bradley [**3**], if

$$\psi_n^* = \sup\left\{ \frac{\hat{\mu}(B_1 \cap S^{-n}B_2)}{\hat{\mu}(B_1)\hat{\mu}(B_2)} \middle| B_1 \in \mathcal{B}_{-\infty}^{-1},\ B_2 \in \mathcal{B}_0^{\infty},\ \hat{\mu}(B_1)\hat{\mu}(B_2) > 0 \right\}$$
$$\psi_n' = \inf\left\{ \frac{\hat{\mu}(B_1 \cap S^{-n}B_2)}{\hat{\mu}(B_1)\hat{\mu}(B_2)} \middle| B_1 \in \mathcal{B}_{-\infty}^{-1},\ B_2 \in \mathcal{B}_0^{\infty},\ \hat{\mu}(B_1)\hat{\mu}(B_2) > 0 \right\},$$

then (i) either $\psi_n^* \to 1$ as $n \to \infty$ or $\psi_n^* = \infty$ for all n, (ii) either $\psi_n' \to 1$ as $n \to \infty$ or $\psi_n' = 0$ for all n. Since inequalities (†) hold, we have $\psi_n^* \leqslant D$ and $D^{-1} \leqslant \psi_n'$, so $\psi_n^* \to 1$ and $\psi_n' \to 1$. This gives the condition in the statement of the theorem. The isomorphism result is due to Friedman and Ornstein [**5**]. □

We state the following results.

THEOREM 2.4. *Let $T : X \longrightarrow X$ be a one-sided, and let $S : \hat{X} \longrightarrow \hat{X}$ be the corresponding two-sided topologically mixing subshift of finite type.*

(i) *If $\varphi \in \mathrm{Bow}(X, T)$ and μ_φ is its unique equlibrium state, then the natural extension of (T, μ_φ) is isomorphic to a Bernoulli shift.*

(ii) *If $\hat{\varphi} \in \mathrm{Bow}(\hat{X}, S)$ and $\hat{\mu}_{\hat{\varphi}}$ is its unique equilibrium state, then $(S, \hat{\mu}_\varphi)$ is isomorphic to a Bernoulli shift.*

Proof. (i) By Corollary 1.8, μ_φ has approximate product structure, so $\hat{\mu}_\varphi$ has approximate product structure. Now Theorem 2.3 gives the result.

(ii) Bowen [**2**] showed that $\hat{\mu}_{\hat{\varphi}}$ has the property that there exists $C > 1$ with

$$C^{-1} \leqslant \frac{\hat{\mu}_{\hat{\varphi}}[x_0, \ldots, x_{n-1}])\varphi}{e^{(S_n\hat{\varphi})(x) - nP(S,\hat{\varphi})}} \leqslant C \qquad \forall n \geqslant 1,\ x \in \hat{X}.$$

Hence $\hat{\mu}_{\hat{\varphi}}$ has approximate product structure, so the result follows by Theorem 2.3. □

3. *g-measures*

We interpret the results in Sections 1 and 2 for the case of g-measures. If $T: X \longrightarrow X$ is a one-sided topologically mixing subshift of finite type, let $\mathcal{G}(X,T)$, or $\mathcal{G}$, denote the set

$$\left\{ g \in C(X) | g(x) > 0 \ \forall x \in X \text{ and } \sum_{y \in T^{-1}x} g(y) = 1 \ \forall x \in X \right\}.$$

If $g \in \mathcal{G}$, we can consider $\mathcal{L}_{\log g}$, and $\mu \in M(X)$ is called a g-measure if $\mathcal{L}^*_{\log g}\mu = \mu$ [**7**]. Such a measure always belongs to $M(X,T)$. The condition can be formulated in several ways. For example, one can show that $\mu \in M(X,T)$ is a g-measure if and only if μ is an equilibrium state of $\log g$ ([**8**], see also [**11**]). Since $P(T, \log g) = 0$ for $g \in \mathcal{G}$, this condition becomes $h_\mu(T) + \int \log g \, d\mu = 0$. Let $\mathcal{M}(X,T) = \{\mu \in M(X,T) \mid \mu \text{ is a } g\text{-measure for some } g \in \mathcal{G}(X,T)\}$.

The following results are special cases of the results in Sections 1 and 2, obtained by considering φ of the form $\log g$ with $g \in \mathcal{G}(X,T)$. Again, k is the number of symbols used for the subshift of finite type, and M is a natural number with $A^M > 0$.

Theorem 3.1. *Let $T: X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g \in \mathcal{G}$. Let μ be a g-measure. Then μ has support X and each of the following holds.*

(i) *For all $p \geqslant 1$, $x \in X$,*

$$(\inf g)^M e^{-v_p(T_p \log g)} \leqslant \frac{\mu[x_0, \ldots, x_{p-1}]}{g(x)g(Tx)\ldots g(T^{p-1}x)} \leqslant k^M e^{v_p(T_p \log g)}.$$

(ii) *For all $n, p \leqslant 1$, $x \in X$,*

$$e^{-v_{n+p}(T_p \log g)} \leqslant \frac{\mu([x_0, \ldots, x_{p+n-1}])}{\mu([x_p, \ldots, x_{p+n}])g(x)g(Tx)\ldots g(T^{p-1}x)} \leqslant e^{v_{n+p}(T_p \log g)}.$$

Corollary 3.2. *For $T: X \longrightarrow X$, $g \in \mathcal{G}$ and $\mu \in M(X,T)$ as in Theorem 3.1, we have for each fixed $p \geqslant 1$,*

$$\frac{\mu([x_0, \ldots, x_{p+n-1}])}{\mu([x_p, \ldots, x_{p+n-1}])} \rightrightarrows g(x)g(Tx)\ldots g(T^{p-1}x) \quad \text{as } n \to \infty.$$

Corollary 3.3. *Let $T: X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\mu \in M(X,T)$. Then μ is a g-measure for some $g \in \mathcal{G}$ if and only if μ has support X and $\mu([x_0, \ldots, x_{n-1}])/\mu([x_1, \ldots, x_{n-1}])$ converges uniformly on X as $n \to \infty$ to a function $f: X \longrightarrow (0, \infty)$.*

With Corollary 3.2 in mind, we can characterize those g with $\log g \in W(X,T)$ as follows.

Theorem 3.4. *Let $T: X \longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g \in \mathcal{G}$. The following statements are pairwise equivalent.*

(i) $\log g \in W(X,T)$.

(ii) *There exists $\mu \in M(X,T)$, with support X, satisfying*

$$\log\left(\frac{\mu([x_0,\ldots,x_{p+n-1}])}{\mu([x_p,\ldots,x_{p+n-1}])}\right) \rightrightarrows (T_p(\log g))(x) \quad \text{as } n\to\infty,$$

where the convergence is uniform in both $x\in X$ and $p\geqslant 1$.

(iii) *There exists a g-measure μ such that in* $\mathrm{BC}(\mathbb{N}\times X)$, *the sequence (ψ_n), given by*

$$\psi_n(p,x) = \log\left(\frac{\mu([x_0,\ldots,x_{p+n-1}])}{\mu([x_p,\ldots,x_{p+n-1}])}\right),$$

is convergent.

Note that the unique g-measure μ, when $\log g\in W(X,T)$, satisfies the condition in (ii).

We shall use Theorem 3.4 later in an application. The following result characterizes those g with $\log g\in \mathrm{Bow}(X,T)$.

THEOREM 3.5. *Let $T:X\longrightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g\in\mathcal{G}(X,T)$. Then $\log g\in\mathrm{Bow}(X,T)$ if and only if there is a g-measure which has approximate product structure. When $\log g\in B(X,T)$, there is a unique g-measure μ and the coordinate zero partition is weak Bernoulli for μ so that the natural extension of (T,μ) is isomorphic to a Bernoulli shift.*

Proof. Since a g-measure is exactly an eigenmeasure for $\mathcal{L}^*_{\log g}$ and is T-invariant, the first statement follows from Theorem 1.7. Let $\log g\in\mathrm{Bow}(X,T)$. By [**14**, Theorem 3.2], there is a unique g-measure, and the Bernoulli properties follow from Theorem 2.4. □

Notice that when μ is g-measure, then μ has approximate product structure if and only if $\log g\in\mathrm{Bow}(X,T)$.

We now consider the question of whether the 'reverse' of a g-measure is also a g-measure. Let $T:X\longrightarrow X$, where $X=X_A$, be a one-sided topologically mixing subshift of finite type, and let $S:\hat{X}\longrightarrow\hat{X}$ be the corresponding two-sided topologically mixing subshift of finite type. Let $\pi:\hat{X}\longrightarrow X$ be the natural projection given by $\pi\{x_n\}_{-\infty}^{\infty}=\{x_n\}_0^{\infty}$. Then $\pi S=T\pi$, and there is a natural bijection $M(\hat{X},S)\longrightarrow M(X,T)$ given by $\hat{\mu}\longrightarrow\hat{\mu}\circ\pi^{-1}$. We denote $\hat{\mu}\circ\pi^{-1}$ by $\hat{\mu}_+$.

The other one-sided space $X_-=\{\{x_n\}_{-\infty}^0 \mid \exists x_i \text{ for } i\geqslant 1 \text{ with } \{x_n\}_{-\infty}^{\infty}\in\hat{X}\}$ together with the shift $T_-:X_-\longrightarrow X_-$, given by $T_-((\ldots,x_{-2},x_{-1},x_0))=(\ldots,x_{-2}, x_{-1})$, can be considered as the one-sided shift on the space $X_{A^{\mathrm{t}}}$, where A^{t} is the transpose of the matrix A. Let $\pi:\hat{X}\longrightarrow X_-$ be given by $\pi_-\{x_n\}_{-\infty}^{\infty}=\{x_n\}_{-\infty}^0$ and then $\pi S^{-1}=T_-\pi_-$. Since $M(\hat{X},S^{-1})=M(\hat{X},S)$, we have a natural bijection $M(\hat{X},S)\longrightarrow M(X_-,T_-)$ given by $\hat{\mu}\longrightarrow\hat{\mu}\circ\pi_-^{-1}\equiv\hat{\mu}_-$, so that $\hat{\mu}_+\longrightarrow\hat{\mu}_-$ gives a natural bijection $M(X,T)\longrightarrow M(X_-,T_-)$, For an allowed cylinder $[i_1,\ldots,i_r]$ in $\hat{X}$, we have $\hat{\mu}_+({}_s[i_1,\ldots,i_r])=\mu_-([i_1,\ldots,i_r]_t)$ for all $s\geqslant 0$, $t\leqslant 0$. Clearly $\hat{\mu}_+$ has support X if and only if $\hat{\mu}_-$ has support X_-. We can define what it means for $\hat{\mu}_-$ to have approximate product structure by considering the natural conjugacy $X_-\longrightarrow X_{A^{\mathrm{t}}}$, given by $(\ldots,x_{-2},x_{-1},x_0)\longrightarrow(x_0,x_{-1},x_{-2},\ldots)$, of T_- to the topologically mixing subshift of finite type on $X_{A^{\mathrm{t}}}$. Then $\hat{\mu}_-$ has approximate product structure if and only if $\hat{\mu}_+$ has approximate product structure if and only if $\hat{\mu}$ has approximate product structure.

Let $\mathcal{G}_+$ denote $\mathcal{G}(X,T)$, the space of all positive g-functions for T, and let $\mathcal{G}_-$ denote $\mathcal{G}(X_-,T_-)$, the space of all positive g-functions for T_-. Hence

$$\mathcal{G}_- = \left\{ g \in C(X_-) | g(z) > 0 \;\forall\, z \in X_- \text{ and } \sum_{w \in T_-^{-1} z} g(w) = 1 \;\forall\, z \in X_- \right\}.$$

If $\mathcal{M}_+$ denotes $\mathcal{M}(X,T)$ and $\mathcal{M}_-$ denotes $\mathcal{M}(X_-,T_-)$, then the map $\hat{\mu}_+ \longrightarrow \hat{\mu}_-$ need not map $\mathcal{M}_+$ into $\mathcal{M}_-$. Kalikow constructed examples to show this when X is the full shift space on two symbols [**6**]. One can construct a family of such examples, inspired by Kalikow, as follows.

Let $X = \prod_0^\infty \{0,1\}$ be the space of all sequences $(x_0, x_1, \ldots)$ with each $x_n \in \{0,1\}$. Let $\{d_n\}_{n=0}^\infty$ be such that $d_n \in [0,1)$ for all $n \geqslant 0$, $d_n \to 0$ as $n \to \infty$, and $\sum_{n=0}^\infty (d_n/(1+d_n)) = \infty$. Such a sequence is given by $d_n = 1/(n+1)$. Define $g: X \longrightarrow (0,1)$ as follows. For $k \geqslant 0$, $\ell \geqslant 0$, put

$$\begin{aligned} g(000^k 1^\ell 101\ldots) &= \tfrac{1}{2}(1 - d_{k+\ell}), \\ g(100^k 1^\ell 101\ldots) &= \tfrac{1}{2}(1 + d_{k+\ell}), \\ g(000^k 1^\ell 100\ldots) &= \tfrac{1}{2}(1 + d_{k+\ell}), \\ g(100^k 1^\ell 100\ldots) &= \tfrac{1}{2}(1 - d_{k+\ell}), \end{aligned}$$

and $g(x) = \frac{1}{2}$ at all other points. The value of g depends on the first occurrence of cylinder [10] in $(x_2, x_3, \ldots)$ and on whether this occurrence of [10] is followed by a 0 or a 1. Then $g \in \mathcal{G}_+$. Suppose that $\hat{\mu}_+$ is a g-measure and let $\hat{\mu}_-$ correspond to it under the natural bijection $M(X,T) \longrightarrow M(X_-,T_-)$. Here X_- is the space $\prod_{-\infty}^0 \{0,1\}$. One can easily show that $\hat{\mu}_+([1^m 00])/\hat{\mu}_+([1^m 0])$ does not depends on m for $m \geqslant 2$ so has a constant value $c \in (0,1)$. Hence

$$\frac{\hat{\mu}_-([1^m 00])}{\hat{\mu}_-([1^m 0])} = \frac{\hat{\mu}_+([1^m 00])}{\hat{\mu}_+([1^m 0])} = c.$$

Suppose that $\hat{\mu}_-$ is a g_--measure for some $g_- \in \mathcal{G}_-$. Then $g_-(1^\infty 00) = \hat{\mu}_-[1^m 00]/\hat{\mu}_-[1^m 0]$ for all $m \geqslant 2$, so $g_-(1^\infty 00) = c$. However, one can use the properties of $\{d_n\}$ to show that, for each fixed $m \geqslant 1$, $\hat{\mu}_+([0^n 1^m 0])/\hat{\mu}_+([0^n 1^m 00]) \to 0$ as $n \to \infty$. Hence $\hat{\mu}_-([0^n 1^m 00])/\hat{\mu}_-([0^n 1^m 0]) \to 1$ as $n \to \infty$, so $g_-(0^\infty 1^m 00) = 1$ for all $m \geqslant 1$. Therefore g_- cannot be continuous, because $\lim_{m\to\infty} g_-(0^\infty 1^m 00) = 1 \neq c = g_m(1^\infty 00)$. ∎

We now show that if $g_+ \in \mathcal{G}_+$ and $\log g_+ \in W(X,T)$ and $\hat{\mu}_+$ is the unique g_+-measure, then $\hat{\mu}_-$ is the unique g_--measure for some $g_- \in \mathcal{G}_-$ with $\log g_- \in W(X_-,T_-)$.

THEOREM 3.6. *Let $S: \hat{X} \longrightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type and let $T: X \longrightarrow X$, $T_-: X_- \longrightarrow X_-$ be the corresponding one-sided topologically mixing subshifts of finite type. Let $\hat{\mu}_+ \longrightarrow \hat{\mu}_-$ be the natural bijection from $M(X,T)$ to $M(X_-,T_-)$ described above. Let $g_+ \in \mathcal{G}_+$ and let $\hat{\mu}_+$ be a g_+-measure. If $\log g_+ \in W(X,T)$, then $\hat{\mu}_-$ is a g_--measure for some $g_- \in \mathcal{G}_-$ and $\log g_- \in W(X_-,T_-)$. We have*

$$\left| \log\left(\frac{\hat{\mu}_-([x_{-n}, \ldots, x_0])}{\hat{\mu}_-([x_{-n}, \ldots, x_{-1}])} \right) - \log g_-(x) \right| \leqslant 2 \liminf_{j\to\infty} v_{n+j}(T_j \log g_+) \quad \forall\, n \geqslant 1,\; x \in X_-.$$

The functions $\log g_+ \circ \pi$ and $\log g_- \circ \pi_-$ are cohomologous in $C(\hat{X})$.

Proof. For $n \geqslant 1$, define $b_n : X_- \longrightarrow (0,1)$ by

$$b_n(x) = \frac{\hat{\mu}_-([x_{-n}, \ldots, x_0])}{\hat{\mu}_-([x_{-n}, \ldots, x_{-1}])} = \frac{\hat{\mu}_+([x_{-n}, \ldots, x_0])}{\hat{\mu}_+([x_{-n}, \ldots, x_{-1}])}.$$

We show that $(\log b_n)$ is a Cauchy sequence in (C, X_-). Since

$$\frac{b_n(x)}{b_{n+j}(x)} = \frac{\hat{\mu}_+([x_{-n}, \ldots, x_0])}{\hat{\mu}_+([x_{-n-j}, \ldots, x_0])} \frac{\hat{\mu}_+([x_{-n-j}, \ldots, x_{-1}])}{\hat{\mu}_+([x_{-n}, \ldots, x_{-1}])},$$

we can use Theorem 3.1 to get

$$e^{-v_{n+j+1}(T_j \log g_+) - v_{n+j}(T_j \log g_+)} \leqslant \frac{b_n(x)}{b_{n+j}(x)} \leqslant e^{v_{n+j+1}(T_j \log g_+) + v_{n+j}(T_j \log g_+)}.$$

Therefore $|\log b_n(x) - \log b_{n+j}(x)| \leqslant 2v_{n+j}(T_j \log g_+)$, and since $\log g_+ \in W(X,T)$, we have $(\log b_n)$ is a Cauchy sequence in $C(X_-)$. Hence $\log b_n(x) \rightrightarrows \psi(x)$ for some $\psi \in C(X_-)$. Since $\sum_{z \in T_-^{-1}(x)} b_n(z) = 1$ for all $n \geqslant 1$, we have $\sum_{z \in T_-^{-1} x} e^{\psi(z)} = 1$. Let $g_- = e^{\psi}$. Then g_- is a g-function for $T_- : X_- \longrightarrow X_-$ and $\hat{\mu}_-$ is a g_--measure by Corollories 3.2 and 3.3.

We get

$$|\log b_n(x) - \log g_-(x)| \leqslant 2 \liminf_{j \to \infty} v_{n+j}(T_j \log g_+).$$

To see that $\log g_- \in W(X_-, T_-)$, we use Theorem 3.4. Since $\log g_+ \in W(X,T)$, we have that

$$(p,x) \longrightarrow \log \left(\frac{\hat{\mu}_+([x_0, \ldots, x_{p+n-1}])}{\hat{\mu}_+([x_p, \ldots, \, x_{p+n-1}])} \right)$$

is a Cauchy sequence in $\mathrm{BC}(\mathbb{N} \times X)$. This is the equivalent to

$$(p,z) \longrightarrow \log \left(\frac{\hat{\mu}_-([z_{-(n+p-1)}, \ldots, \, z_0])}{\hat{\mu}_-([z_{-(n+p-1)}, \ldots, \, z_{-p}])} \right)$$

being a Cauchy sequence in $\mathrm{BC}(\mathbb{N} \times X_-)$, and hence $\log g_- \in W(X_-, T_-)$.

We use [**15**, Theorem 1.4] to see that $\log g_+ \circ \pi$ is cohomologous to $\log g_- \circ \pi_-$ in $C(\hat{X})$. Since $\log g_+ \in W(X,T), \log g_+ \circ \pi$ is cohomologous in $C(\hat{X})$ to $\varphi_- \circ \pi_-$ for some $\varphi_- \in C(X_-)$. By of [**15**, Lemma 13], $\varphi_- \in W(X_-, T_-)$. Hence φ_- is cohomologous in $C(X_-)$ to $\log g_1$ for some g-function $g_1 : X_- \longrightarrow (0,1)$ [**12**]. Since μ_- is a g-measure for g_1 and g_-, we have $g_1 = g_-$. Hence $\log g_+ \circ \pi$ is cohomologous to $\log g_- \circ \pi_-$ in $C(\hat{X})$. □

We do not know if the corresponding result holds when $W(X,T)$ is replaced by $\mathrm{Bow}(X,T)$, but we do have the following.

THEOREM 3.7. *Let $S : \hat{X} \longrightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type, and let $T : X \longrightarrow X$, $T : X_- \longrightarrow X_-$ be the corresponding one-sided topologically mixing subshifts of finite type. Let $\hat{\mu}_+ \longrightarrow \hat{\mu}_-$ be the natural bijection from $M(X,T)$ to $M(X_-, T_-)$ described above. Let $\hat{\mu}_+$ be a g_+-measure and $\hat{\mu}_-$ be a g_--measure for some $g_+ \in \mathcal{G}_+$ and some $g_- \in \mathcal{G}_-$. Then* $\log g_+ \in \mathrm{Bow}(X,T)$ *if and only if* $\log g_- \in \mathrm{Bow}(X_-, T_-)$.

Proof. From Theorem 3.5, we know that $\log g_+ \in \mathrm{Bow}(X,T)$ if and only if $\hat{\mu}_+$ has approximate product structure, and $\log g_- \in \mathrm{Bow}(X_-, T_-)$ if and only if $\hat{\mu}_-$ has approximate product structure. □

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