Revista de Humanidades de Valparaíso
Año 7, 2019, 1er semestre, No 13, págs. 34-67
eISSN 0719-4242

Humanities Journal of Valparaiso
No 13 (2019): 34-67

Hybrid Logic as Extension of Modal and Temporal Logics

La lógica híbrida como extensión de las lógicas modal y temporal

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DOI: https://doi.org/10.22370/rhv2019iss13pp34-67
Recibido: 16/07/2019. Aceptado: 14/08/2019

Abstract

Developed by Arthur Prior, Temporal Logic allows to represent temporal information on a logical system using modal (temporal) operators such as P, F, H or G, whose intuitive meaning is "it was sometime in the *P*ast...", "it will be sometime in the *F*uture...", "it *H*as always been in the past..." and "it will always *G*oing to be in the future..." respectively. Valuation of formulae built from these operators are carried out on Kripke semantics, so Modal Logic and Temporal Logic are consequently related. In fact, Temporal Logic is an extension of Modal one. Even when both logics mechanisms are able to formalize modal-temporal information with some accuracy, they suffer from a lack of expressiveness which Hybrid Logic can solve. Indeed, one of the problems of Modal Logic consists in its incapacity of *naming* specific points inside a model. As Temporal Logic is based on it, it cannot make such a thing neither. But First-Order Logic does can by means of constants and equality relation. Hybrid Logic, which results from combining Modal Logic and First-Order Logic, may solve this shortcoming. The main aim of this paper is to explain how Hybrid Logic emanates from Modal and Temporal ones in order to show what it adds to both logics with regard to information representation, why it is more expressive than them and what relation it maintains with the First-Order Correspondence Language.

Keywords: Arthur Prior, First-Order Correspondence Language, Temporal Representation, Nominals, Translations.

Resumen

La lógica temporal fue creada por Arthur Prior para representar información temporal en un sistema lógico mediante operadores modales-temporales como P, F, H o G. Intuitivamente tales operadores pueden entenderse respectivamente como "fue alguna vez en el pasado...", "será alguna vez en el futuro...", "ha sido siempre en el pasado..." y "será siempre en el futuro...". La evaluación de las fórmulas construidas a partir de ellos se lleva a cabo en semánticas kripkeanas y, de este modo, la lógica modal y la temporal están relacionadas. Sin embargo, aunque sus mecanismos permiten formalizar la información modal-temporal con cierta precisión, ambas lógicas adolecen de un problema de expresividad que la lógica híbrida es capaz de solventar. En efecto, uno de los problemas de la lógica modal reside en su incapacidad para *nombrar* puntos concretos dentro de un modelo. La lógica temporal, al basarse en ella, tampoco puede hacerlo. Pero la lógica de primer orden sí es capaz gracias a las constantes y a la relación de identidad. La lógica híbrida, que resulta de combinar la lógica modal con la lógica de primer orden, sería una solución a este problema. El principal objetivo de este artículo consiste en explicar el origen de la lógica híbrida a partir de la modal-temporal para mostrar qué añade a ambos sistemas en la representación de información, porqué es más expresiva que ellos y qué relación guarda con el lenguaje de correspondencia de la lógica de primer orden.

Palabras clave: Arthur Prior, lenguaje de correspondencia de primer orden, representación temporal, nominales, traducciones.

1. Introduction

Arthur Prior (1957) (1967) (2010) developed one of the most classical and fundamental approaches of Temporal Logic (TL), which can be understood in general terms as the representation of propositions with temporal information on logical frames.

In Classical and Modal Logic we are used to deal with formulae of the kind $\neg p$, $p \rightarrow q$, $\Box p$ or $\Diamond p$, among others, where p and q are variables which stand for whatever propositions, $\rightarrow$ is the material conditional and $\Box$ and $\Diamond$ are modal operators standing for necessity and possibility respectively.

Modal Logic syntax is not much wider than Classical one. We just have to add the rest of connectives. Its semantics, on the other hand, is based on the notion of *possible worlds* to valuate formulae such as $\Box\varphi$ or $\Diamond\varphi$: $\Box\varphi$ is satisfied at a world w of a model $\mathfrak{M}$ if and only if is satisfied at every world w' accessible in $\mathfrak{M}$ from w, whereas $\Diamond\varphi$ is satisfied at a world w of a model $\mathfrak{M}$ if and only if there exists a world w' in $\mathfrak{M}$ accessible from w where φ is satisfied.

One of the most relevant features of this possible worlds semantics lies in it provides an *internal* perspective of Kripke models. As we state that a sentence such as $\Diamond\varphi$ is satisfied in $\mathfrak{M}$ at w if and only if there exists a w′ in $\mathfrak{M}$ which is accessible from it and where φ is satisfied, what we are doing is to valuate that formula inside a model and regarding one (or some) possible world. That is why Modal Logic semantics is internal, for we valuate formulae at a specific point of a certain model.

Following Patrick Blackburn´s metaphor (2006, 332), we may conceive a modal formula like a creature located inside a model at a certain point which is compelled to change its position on the basis of some "transition rules" established by the accessibility relation. And that means that, in some sense, modal formulae truth values are context-dependents, for they depend on the possible worlds in which they are valuated.

First-Order Logic (FOL), on the contrary, does not provide an internal perspective of models but an *external* one. Indeed, in FOL, formulae valuation does not answer to some points inside a model, that is, their truth values do not depend on any kind of contextual information. Formulae are plainly true or false in a structure.

This distinction between internal/external perspective is thus closely related to intensional/extensional one. Classical Logic is extensional because formulae are true or false. But Modal Logic is intensional because formulae are true or false *according to a set of possible worlds of a model*. Formally speaking, in Modal Logic we say that a formula φ is satisfied at a world w of a model $\mathfrak{M}$, in symbols $\mathfrak{M}, w \vDash \varphi$, whereas in First-Order Logic we say that a formula φ is true in a structure or, *mutatis mutandis*, in a model $\mathfrak{M}$: $\mathfrak{M} \vDash \varphi$.

Due to this difference between perspectives Prior opted by Modal Logic instead of First-Order one to build Temporal Logic. His main idea is that our language and our thought possess an internal perspective of time by which it is represented by fixing a past moment and a future one in relation to a changing *now*. A logic of time must observe such internal perspective.

As Modal Logic (ML) is the best option for reflecting it (due to the reasons we have already seen), Prior´s proposal consists in extending ML vocabulary with new temporal operators which allow to represent sentences such as «It will be p sometime in the future» or «It has always been p in the past». Formulae resulting from these operators are valuated like modal ones, that is, by means of possible worlds semantic. Although adapted to moments of time now, and by doing so we can express things as «I will be rich sometime in the future» (1), which will be true at this moment (t_0) if and only if there exists at least a future moment t_1 (whenever this may be) when I be rich.

The problem of this approach is however clear: even though thanks to TL it is possible to formalize and valuate formulae referred to temporal events, the expressivity of such formulae is restricted for they cannot allude, e.g., to specific instants. We are not able to express something as «I will be rich on 15th May» (2) or «I am currently rich» (3). To do so we have to appeal to FOL mechanisms.

Quantification, the use of non-logical constants (individual constants, functors and predicates) and the equality relation are some of the greatest advantages which FOL possesses against ML or TL, as they allow to make reference to particular points inside a structure (model). The result of combining the internal perspective of Modal Logic with the external perspective of First-Order one is Hybrid Logic (HL), which as well as TL was developed by Prior.

HL novelty lies in it extends ML language (and consequently TL one too) with new operators and with a new sort of propositional symbol, namely, nominals, which can be merged with other formulae to compose more complex ones and that allow to *name* a specific point inside a model stating the formula they bind is true at that point, and only at that.

This paper aims to expose how HL came up as an ML and TL extension, and to explain why Hybrid Logic is very useful to, among other things, represent temporal information. In order to do so we shall follow the very same path we have followed in this introduction: in section 2 we will introduce the basic features of Modal Logic and the minimal temporal logic developed by Prior; in section 3 we will set the difference between these two systems and First-Order Logic in terms of internal/external perspectives; in section 4 we will explain how Hybrid Logic is developed from combining both perspectives and what its novelty in relation with ML and TL is; in section 5 we will state the conclusions of this paper and all the work which remains to be done in the field of Hybrid Logic; section 6 will be composed by an appendix where we will define the language we are going to use in this paper as well as by an abbreviation index; and finally in section 7 references are specified.

2. Modal Logic and Temporal Logic

Hybrid Logic is largely an extension of Modal Logic and it also comes up due to Arthur Prior and his studies in Temporal Logic. That is why it is essential to know the most relevant aspects of these two systems to fully understand what Hybrid Logic constitutes. This is going to be the aim of this section. We will split it in two parts: in the first one we will briefly explain Modal Logic syntax and semantics; in the second one we will present the minimal temporal logic devised by Prior from a contemporary point of view.

2.1. Basic Modal Logic

Classical Propositional Logic alphabet is composed by $\mathcal{L}$:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi,$$

where $p \in$ PROP, $\varphi, \psi \in$ WFF (see appendix 6.1) and the rest of connectives ($\vee$ and $\equiv$) can be defined from $\neg$ and $\wedge$ (or from $\neg$ and $\rightarrow$).

On the other hand, its semantics is based on the notion of *interpretation*, which is a function V assigning a truth value True (1) or False (0) to every formula. V is recursively defined from f, which is a function assigning a truth value to every propositional variable, i.e., f: PROP → {1, 0}.

V fulfills the following conditions, for every p ∈ PROP and every $\varphi, \psi \in$ WFF:

1. $\mathrm{V}(\mathrm{p}) = f(\mathrm{p})$,
2. $\mathrm{V}(\neg\varphi) = 1$ iff $\mathrm{V}(\varphi) = 0$,
3. $\mathrm{V}(\varphi \wedge \psi) = 1$ iff $\mathrm{V}(\varphi) = 1$ and $\mathrm{V}(\psi) = 1$,
4. $\mathrm{V}(\varphi \rightarrow \psi) = 1$ iff $\mathrm{V}(\varphi) = 0$ or $\mathrm{V}(\psi) = 1$.

Consequence relation is thus defined in relation with V, so a formula φ is a consequence of a set Γ of formulae if and only if there is no interpretation where every element of Γ is true and φ is not. We will symbolize it as

$$\Gamma \vDash \varphi,$$

where ⊨ stands for the semantic consequence relation.

In the case of Modal Logic this syntax is extended with new operators. Let $\mathcal{L}_M$ be ML language. $\mathcal{L}_M$ is composed by (Blackburn and van Benthem, 2007, 3):

- The PROP set.
- The MOD set of modal operators, where MOD = {m, m', m'', ...} and there is an accessibility relation for each m.

The distinct elements of MOD can represent any modal operator. We are usually used to □ and ◇ (both related to the same m ∈ MOD). But they can also represent to K and B, which are the epistemic operators for knowledge and belief respectively; to P, H, F and G, which as we have seen are the temporal operators used in TL; or to many others. However, in this part they will only stand for □ and ◇.

$\mathcal{L}_M$ is thus formally defined as:

$$\mathrm{p} \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Diamond\varphi \mid \Box\varphi.$$

Any combination of these formulae constitutes a well-formed formula of $\mathcal{L}_M$. It belongs to the set WFF of well-formed formulae of $\mathcal{L}_M$. The proposition

$$\Diamond\,(\mathrm{r} \wedge \mathrm{p}) \wedge \Diamond\,(\mathrm{r} \wedge \mathrm{q}) \rightarrow \Diamond\,(\mathrm{p} \wedge \mathrm{q}) \qquad (4)$$

would be an instance such formulae. Besides, □ and ◇ are interdefinable: □ ≡ ¬◇¬ and ◇ ≡ ¬□¬.

The alphabet of basic Modal Logic is therefore just the same as Classical Logic but with the addition of $\Box$ and $\Diamond$, so from a syntactic point of view they are not very different. The main difference is on semantics.

As we have pointed out Modal Logic is based on possible worlds semantics, which responds to the so-called *Kripke frames (or models)*. A Kripke frame is a tuple

$$\mathfrak{F} = \langle W, R \rangle,$$

where $W \neq \emptyset$ is the set of possible worlds and $R \subseteq W \times W$ is the accessibility relation between them.

From $\mathfrak{F}$ we can build a model by adding a valuation function. Let V_M be such function. What V_M makes is to assign subsets of W to propositional variables in such a way that the set of possible worlds assigned to a variable e.g. p is the set of worlds where that variable is true. Formally speaking, V_M: PROP $\rightarrow \wp(W)$. The result of adding V_M to $\mathfrak{F}$ is the model $\langle W, R, V_M \rangle$, denoted by $\mathfrak{M}_K$.

In order to express "φ is satisfied at a world w of model $\mathfrak{M}_K$" we write $\mathfrak{M}_K, w \vDash \varphi$. The satisfiability conditions of the rest of formulae are defined as follows, for any $\varphi, \psi \in$ WFF; w, w' $\in$ W and p $\in$ PROP:

1. $\mathfrak{M}_K, w \vDash p$ iff $w \in V_M(p)$,
2. $\mathfrak{M}_K, w \vDash \neg\varphi$ iff $\mathfrak{M}_K, w \nvDash \varphi$,
3. $\mathfrak{M}_K, w \vDash \varphi \wedge \psi$ iff $\mathfrak{M}_K, w \vDash \varphi$ and $\mathfrak{M}_K, w \vDash \psi$,
4. $\mathfrak{M}_K, w \vDash \varphi \rightarrow \psi$ iff $\mathfrak{M}_K, w \nvDash \varphi$ or $\mathfrak{M}_K, w \vDash \psi$,
5. $\mathfrak{M}_K, w \vDash \Diamond\varphi$ iff $\exists w' \in W$ such that wRw' and $\mathfrak{M}_K, w' \vDash \varphi$,
6. $\mathfrak{M}_K, w \vDash \Box\varphi$ iff $\forall w' \in W$ if wRw' then $\mathfrak{M}_K, w' \vDash \varphi$.

A formula is globally satisfied in a model $\mathfrak{M}_K$ if it is satisfied at every world of that model. We will symbolize it as $\mathfrak{M}_K \vDash \varphi$. A formula is valid if it is globally satisfied in every model. $\vDash \varphi$ symbolizes such thing. Finally, a formula φ is consequence of a set Γ of formulae if for every model $\mathfrak{M}_K$, every world w of $\mathfrak{M}_K$ and every $\gamma \in \Gamma$, if $\mathfrak{M}_K, w \vDash \gamma$ then $\mathfrak{M}_K, w \vDash \varphi$. And as well as in Modal Logic $\Gamma \vDash \varphi$ denotes such thing.

This system we have introduced is the most basic one of ML, and it is said it is minimal because it does not make any particular requirement from accessibility relation, i.e., it does not impose any property (reflexivity, transitivity, symmetry, etc.) when dealing with possible worlds. We will call this system K. Hence we have denoted Modal Logic models by $\mathfrak{M}_K$.

The valuation of a formula such as (4) in K would carry out as follows:

$$\mathfrak{M}_K, w \models \Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)$$

holds if and only if it is verified that if

$$\mathfrak{M}_K, w \models \Diamond(r \wedge p) \qquad (4.1)$$

and

$$\mathfrak{M}_K, w \models \Diamond(r \wedge q) \qquad (4.2)$$

then

$$\mathfrak{M}_K, w \models \Diamond(p \wedge q). \qquad (4.3)$$

(4.1) holds if there exists at least a possible world w' accessible from w where $(r \wedge p)$ holds. (4.2) holds if there exists at least a possible world w'' accessible from w where $(r \wedge q)$ holds. And (4.3) holds if there exists a possible world accessible from w where $(p \wedge q)$ holds. As p is true at world w' and q does it at world w'' there is no guarantee of p and q being true at the same world accessible from w, so (4.3) is false. That then means that (4) is false too, for its antecedent is true but its consequent it is not. Therefore, formally,

$$\mathfrak{M}_K, w' \models r \wedge p,$$
$$\mathfrak{M}_K, w'' \models r \wedge q$$

and then

$$\mathfrak{M}_K, w \not\models \Diamond(p \wedge q).$$

In consequence, in a model where $R = \{\langle w, w' \rangle, \langle w, w'' \rangle\}$

$$\mathfrak{M}_K, w \not\models \Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q).$$

In other words, given the model $\mathfrak{M}_K$ of K where R has any particular property, if $R = \{\langle w, w' \rangle, \langle w, w'' \rangle\}$ and $w'' \notin V_M(p)$, $w' \notin V_M(q)$ and $w', w'' \in V_M(r)$ then, if $\mathfrak{M}_K, w' \models r \wedge p$ and $\mathfrak{M}_K, w'' \models r \wedge q$, $\mathfrak{M}_K, w \not\models \Diamond(p \wedge q)$ and thus $\mathfrak{M}_K, w \not\models \Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)$.

Let us look at the valuation conditions of V both in Classical Propositional Logic and in Modal Logic. As we can see, in the second one V_M depends on the model and on the set of possible worlds of that model. In the first one this does not happen, however, and it shows that whereas ML formulae are intensionally interpreted Classical Propositional Logic ones are extensionally interpreted.

It is this intensionality the one which highlights the internal perspective of Modal Logic that we spoke about at the Introduction and that we shall closely discuss in Section 3. By now it is enough to know Prior´s temporal logic is one of the systems of Modal Logic that better captures that perspective for to him, as we exist in time and we make use of it in a context-dependent way by means of our utterances, a logic which tries to reflect temporal information may respect such dependence. Hence he appealed Modal Logic to do so.

2.2. Prior´s Temporal Logic

Prior´s main motivation for developing Temporal Logic was eminently philosophical. In 1941 John Findlay published "Time: A Treatment of Some Puzzles", an article where he exposes some problems derived from our conception of time and change. From then on Prior set out to build a formal system similar to Modal Logic that allows to mirror the internal perspective of time that, according to him, our language and out thought possess. His aim was anything but to formally solve those problems, specially determinism.

Prior mainly developed TL in three papers: the article "Diodoran Modalities" (1955) and the books *Time and Modality* (1962) and *Past, Present and Future* (1967). It is on them in which we shall base this section on. Although we will highlight (1967) for it is on it where the minimal system of Temporal Logic is definitely presented (Prior 1967, 176). Our way of exposing it will nevertheless differ from his as Prior´s notation is Polish one.

If K is the basic system of Modal Logic we have presented above, let K_T be the system (also basic) of Temporal Logic we are going to present hereunder.

K_T language, $\mathcal{L}_{KT}$, is composed by $\mathcal{L}$ plus the temporal operators F, P, G and H. F can be read as "it will be sometime in the *F*uture that...", P can be read as "it was sometime in the *P*ast that...", G can be read as "it will always *G*oing to be in the future..." and H can be read as "it *H*as always been in the past that...". $\mathcal{L}_{KT}$ is consequently defined as:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid F\varphi \mid P\varphi \mid G\varphi \mid H\varphi.$$

And as before, any combination of such kind of formulae constitutes a well-formed formula of $\mathcal{L}_{KT}$. That is, it belongs to the set WFF of well-formed formulae of $\mathcal{L}_{KT}$. A proposition such as (4) (page 38) may be now arose in terms of

$$F(r \wedge p) \wedge F(r \wedge q) \rightarrow F(p \wedge q) \quad (5)$$

or of

$$P(r \wedge p) \wedge P(r \wedge q) \rightarrow P(p \wedge q) \quad (6)$$

As we are going to see in a moment at the conditions of the valuation function, F and P are similar to Modal Logic possibility operator, whereas G and H are with regard to necessity one. And just like □ and ◇ are interdefinable, F and G and P and H do are too: $F \equiv \neg P \neg$, $P \equiv \neg F \neg$, $G \equiv \neg H \neg$ and $H \equiv \neg G \neg$.

Regarding to its semantics Temporal Logic, being modal, is also based on possible worlds Kripke semantics. However, instead of ⟨W, R⟩ its frames will consist on pairs such as ⟨T, <⟩ where T is the nonempty set of temporal instants and < is the accessibility relation between them but understood as an ulteriority relation. So, T = {t, t', t'', ...}[1] and $< \subseteq T \times T$.

From ⟨T, <⟩ a model, $\mathfrak{M}_{KT}$, is defined as a pair $\langle T, <, V_T \rangle$ where V_T is evidently the interpretation function. Like V_M, V_T aims to assign a subset of temporal instants of T to each formula: the set where that formula is true. In formal terms, V_T: WFF → $\wp$(T).

As usual, we shall denote the satisfaction of a formula φ in the model $\mathfrak{M}_{KT}$ and at instant t as $\mathfrak{M}_{KT}, t \vDash \varphi$. Satisfaction conditions, for whatever $\varphi, \psi \in$ WFF, t, t' ∈ T and propositional variable p, are:

1. $\mathfrak{M}_{KT}, t \vDash p$ iff $t \in V_T(p)$,
2. $\mathfrak{M}_{KT}, t \vDash \neg\varphi$ iff $\mathfrak{M}_{KT}, t \nvDash \varphi$,
3. $\mathfrak{M}_{KT}, t \vDash \varphi \wedge \psi$ iff $\mathfrak{M}_{KT}, t \vDash \varphi$ and $\mathfrak{M}_{KT}, t \vDash \psi$,
4. $\mathfrak{M}_{KT}, t \vDash \varphi \rightarrow \psi$ iff $\mathfrak{M}_{KT}, t \nvDash \varphi$ or $\mathfrak{M}_{KT}, t \vDash \psi$,
5. $\mathfrak{M}_{KT}, t \vDash F\varphi$ iff $\exists t' \in T$ such that $t < t'$ and $\mathfrak{M}_{KT}, t' \vDash \varphi$,
6. $\mathfrak{M}_{KT}, t \vDash P\varphi$ iff $\exists t' \in T$ such that $t' < t$ and $\mathfrak{M}_{KT}, t' \vDash \varphi$,
7. $\mathfrak{M}_{KT}, t \vDash G\varphi$ iff $\forall t' \in T$ if $t < t'$ then $\mathfrak{M}_{KT}, t' \vDash \varphi$,
8. $\mathfrak{M}_{KT}, t \vDash H\varphi$ iff $\forall t' \in T$ if $t' < t$ then $\mathfrak{M}_{KT}, t' \vDash \varphi$.

1. stands for propositional variables truth conditions, which are true at instant t of model $\mathfrak{M}_{KT}$ if and only if t belongs to the set of temporal instants where that variable is true. 2. represents the principle of bivalence according to which if a formula is true then its negation must be false, and vice versa. 3. and 4. reflect the interpretation of conjunction and conditional. 5. shows the truth conditions of a formula of the kind Fφ, which is satisfied at instant t of model $\mathfrak{M}_{KT}$ if and only if there exists an instant t' after it where φ is satisfied. 6. shows the same but from a formula as Pφ, which is satisfied at instant t of model $\mathfrak{M}_{KT}$ if and only if there exists an instant t' before it where φ is satisfied. 7., on the other hand, sets that a formula of the kind Gφ is satisfied at instant t of model $\mathfrak{M}_{KT}$ if and only if φ is satisfied at every instant after t of that model. Finally, 8. states that a formula as Hφ is

[1]In Temporal Logic, instants are usually represented by means of variable t along with integer numbers as subscripts. The current moment, where the valuation is carried out, is always symbolized as t_0. For further moments, the t subscript value is a positive integer, and for previous moments its value is a negative one.

satisfied at instant t of model $\mathfrak{M}_{KT}$ if and only if φ is satisfied at every instant before t of that model.

Global satisfaction, validity and semantic consequence relation are defined just as in ML, although in this case adapted to the model $\langle T, <, V_T \rangle$. Besides, $\mathfrak{M}_{KT}$ does not make any demand to < either.

Let us see how a formula like (4) would be valuated in K_T, but on its (5) and (6) versions. For (5) being true at instant t of model $\mathfrak{M}_{KT}$, that is,

$$\mathfrak{M}_{KT}, t \vDash F(r \land p) \land F(r \land q) \rightarrow F(p \land q),$$

there must be held that if $F(r \land p)$ and $F(r \land q)$ are true then $F(p \land q)$ has to be so. If $F(r \land p)$ is true at t means there exists another instant t' after it where $r \land p$ is true. The same applies to $F(r \land q)$: there must be an instant t'' after t where $r \land q$ is true. As in (4), that does not necessarily mean that there exists an instant later than t where $p \land q$ is true and therefore $F(p \land q)$ is false at t in some models.

What follows from this is that (5) is false, for its antecedent is true but its consequent is not, and then

$$\mathfrak{M}_{KT}, t \nvDash F(r \land p) \land F(r \land q) \rightarrow F(p \land q).$$

That is to say, if $< = \{\langle t, t' \rangle, \langle t, t'' \rangle\}$ and $t' \notin V_T(q)$ or $t'' \notin V_T(p)$ then it happens that

$$\mathfrak{M}_{KT}, t' \vDash r \land p,$$
$$\mathfrak{M}_{KT}, t'' \vDash r \land q.$$

But

$$\mathfrak{M}_{KT}, t \nvDash F(p \land q)$$

and so

$$\mathfrak{M}_{KT}, t \nvDash F(r \land p) \land F(r \land q) \rightarrow F(p \land q).$$

In the case of (6) exactly the same takes place, but regarding to instants t' and t'' prior to t: if $< = \{\{\langle t', t \rangle, \langle t'', t \rangle\}$ and $t' \in V_T(p)$, $t'' \in V_T(q)$ and $t', t'' \in V_T(r)$ then, if $\mathfrak{M}_{KT}, t' \vDash r \land p$ and $\mathfrak{M}_{KT}, t'' \vDash r \land q$, $\mathfrak{M}_{KT}, t \nvDash P(r \land p) \land P(r \land q) \rightarrow P(p \land q)$ holds.

Nonetheless, natural language does not work like that. For instance, let us suppose we are talking to a friend about what we are going to do this summer and we are telling him/her we are going to go to a water park where we will sunbath while we drink beer at its slow river–fun guaranteed. If we were to speak logically, we may say something as: «If I will go sometime in the future (this simmer) to a water park to enjoy sunbathing and I will go sometime in the future to a water park to drink beer at its slow river then I will go sometime in the future to sunbath and drink beer at a slow river».

If variable r stands for "going to a water park", variable p stands for "sunbath" and variable q stands for "drinking a beer at a slow river" then $F(r \wedge p) \wedge F(r \wedge q) \rightarrow F(p \wedge q)$, which is (5), stands for the aforementioned proposition. A proposition that is true today (t) as long as it is true that this summer (t′) we will go to a water park where we will sunbath and drink a beer at its slow river. But it would be false if we do not set that the moment where p and q are true is the same.

In any case, what this example shows is that thanks to Modal Logic mechanisms Temporal Logic is able to reflect both time flows and that internal perspective of temporality we talked about at the beginning and that, according to Prior, our language possesses. Indeed, whenever we allude to some fact, we do so by taking as point of reference the present moment and by setting that such fact either happened at some earlier moment or it is happening now or it will happen at a later moment. That context-dependence is the main characteristic of Modal Logic, as we have said, and hence TL constitutes a nice example about how ML deals with models from inside.

However, if we look at conditions 5 and 6 of V_M in ML and at conditions 5-8 of V_T in TL we shall see that we may use First Order Logic to capture at least a part of that internal perspective of modal representation. ML (and thus TL) then is weaker than FOL, and in fact by means of what is known as *Standard Translation* Modal Logic can be represented by First Order Correspondence Language mechanisms. In particular, by using free variables to contextualize FOL formulae valuations based on certain individuals (counterparts of possible worlds).

Therefore, even although First Order Logic provides an external point of view of models it can be employed to reflect Modal Logic internal perspective, and that is what we are going to present in the next section.

3. Standard Translation and Modal Logic Problems

At the beginning of section 2 we said that Hybrid Logic maintains a close relation with Modal Logic. But it is also closely related to First Order Logic. Specifically, to a particular part of FOL called *First Order Correspondence Language* (FOCL)[2]. ML and FOCL in turn possess a very interesting connection, which is necessary to know to understand what the true innovation of Hybrid Logic is and what it gains with respect to ML, TL and FOCL.

The aim of this section is thus to show that connection between Modal Logic and First Order Correspondence Language as well as the problems of the former in representing certain kind of propositions. To do so the section will be divided in two parts: on the first

[2]From now on any allusion to FOL must be understood as referred to this language and not to First Order Logic entirely.

one we will address the relation between ML and FOCL, and on the second one we will deal with ML problems.

3.1. Relation between Modal Logic and First Order Correspondence Language

Modal Logic formulae may be expressed by First Order Correspondence Language formulae with at least one free variable. Nevertheless, this does not happen conversely: First Order Correspondence Language formulae cannot be expressed by Modal Logic ones for the latter is weaker than the former. There are FOCL formulae which are not expressible in ML. That is why it is possible to conceive ML as a part of its corresponding first order language.

Modal Logic Kripke models are no (Blackburn and van Benthem 2007, 10) more than relational structures composed by a domain over which quantification is carrying out (W in the case of ML and T in the case of TL), a range of binary relations over that domain (R in the case of ML and < in the case of TL) and another range of unary relations which are applied to formulae of that structure (V_M in the case of ML and V_T in the case of TL).

The problem —one may think— is, if so, then we do not actually have to resort to Modal Logic to talk about possible worlds. It is enough to First Order Logic to do so. Indeed, if we add to FOL language a binary relator R applied to every element of MOD, i.e., a relation R^m for every m ∈ MOD, and a unary relation P for every element of PROP, i.e., a relation P for every p ∈ PROP, then we can set a correspondence between ML (or TL) formulae and FOL formulae by means of the Standard Translation. That is what FOCL consists on.

Let ST_x be a function assigning to every modal formula its corresponding first order formula. ST_x consists on the following, for every propositional variable p, whatever φ, ψ ∈ WFF and whatever [m], ⟨m⟩ ∈ MOD (where [m] represents any modal operator similar to □, that is, whose valuation depends on every possible world accessible from the actual one, and ⟨m⟩ represents any modal operator similar to ◇, that is, whose valuation depends on some possible world accessible from the actual one) (Blackburn 2006, 334):

1. $ST_x(p) = P(x)$,
2. $ST_x(\neg\varphi) = \neg ST_x(\varphi)$,
3. $ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$,
4. $ST_x(\varphi \rightarrow \psi) = ST_x(\varphi) \rightarrow ST_x(\psi)$,
5. $ST_x(\langle m\rangle\varphi) = \exists y\ (R^m\langle x, y\rangle \wedge ST_y(\varphi))$,
6. $ST_x([m]\varphi) = \forall y\ (R^m\langle x, y\rangle \rightarrow ST_y(\varphi))$.

1-6 are very similar to 1-6 V_M conditions we saw at part 2.1. But what they set is a correspondence between Modal Logic and First Order Logic by resorting to free variable x, whose importance we shall see hereunder.

1. shows that a ML propositional variable may be translated via FOCL as a predicate P whose argument is x. In Modal Logic we saw that a proposition such as p is satisfied at world w of model $\mathfrak{M}_K$ if and only if w belongs to the set of possible worlds where p is true. That is what $\mathfrak{M}_K, w \vDash p$ iff $w \in V_M(p)$ meant.

As in First Order Logic formulae valuations do not depend on Kripke semantics we cannot assert the same while we are talking about p truth. To do so we use free variables (x in this case), which are what reflect the internal perspective of Modal Logic through Classical Logic. By assigning a value to x what we are doing is something similar to what happens in ML when we set that p is true at w, namely, we indicate that the formula concerned by x is true (or false) in x. $P(x)$ then means that every propositional symbol p is true in x. $\mathfrak{M}_K, w \vDash p$ iff $w \in V_M(p)$ and $P(x)$ are therefore analogous, for assigning a value to x is equivalent to valuate a modal formula at some world of a model.

Conditions 3. and 4. work similarly so let us focus on 5. and 6. ones. As we have a formula as $\Diamond\varphi$ in Modal Logic we say it is satisfied at world w of model $\mathfrak{M}_K$ if and only if there exists some world w' accessible from w where φ is true. $\mathfrak{M}_K, w \vDash \Diamond\varphi$ iff $\exists w' \in W$ such that wRw' and $\mathfrak{M}_K, w' \vDash \varphi$ sets such thing. In the case of Classical Logic the valuation of $\Diamond\varphi$ does not differ too much. The only remarkable difference is naturally that the truth of φ does not depend on any possible world of $\mathfrak{M}_K$ but on free variable x and bound variable y.

So, what condition 5. shows is that a formula as $\langle m\rangle\varphi$ is true in Classical Logic, on the basis of x, if and only if there exists at least one y such that x and y are related and φ is true in y. Something similar happens to formulae of the kind $[m]\varphi$, which are true in Classical Logic on the basis of x if and only if, for every variable y, if x and y are related then φ is true in y.

Consequently,

$$\mathfrak{M}_K, w \vDash \Diamond\varphi \text{ iff } \exists w' \in W \text{ such that } wRw' \text{ and } \mathfrak{M}_K, w' \vDash \varphi$$

and

$$ST_x(\langle m\rangle\varphi) = \exists y\, (R^m\langle x, y\rangle \wedge ST_y(\varphi)),$$

are equivalent. And

$$\mathfrak{M}_K, w \vDash \Box\varphi \text{ iff } \forall w' \in W \text{ if } wRw' \text{ then } \mathfrak{M}_K, w' \vDash \varphi$$

and

$$ST_x([m]\varphi) = \forall y\ (R^m\langle x, y\rangle \rightarrow ST_y(\varphi)),$$

are too. This evinces that modal formulae and their first order translation express the same.

Standard Translation may be also applied to Temporal Logic. There would just have to modify conditions 5. and 6. to be adapted to F, P, G and H operators. In order to do so we need to turn the order of P and H R^m pairs around to reflect conditions 6. and 8. of V_T:

5*. $ST_x(F\varphi) = \exists y\ (R^m\langle x, y\rangle \wedge ST_y(\varphi))$,

6*. $ST_x(P\varphi) = \exists y\ (R^m\langle y, x\rangle \wedge ST_y(\varphi))$,

7*. $ST_x(G\varphi) = \forall y\ (R^m\langle x, y\rangle \rightarrow ST_y(\varphi))$,

8*. $ST_x(H\varphi) = \forall y\ (R^m\langle y, x\rangle \rightarrow ST_y(\varphi))$.

An example of how Standard Translation would work is the following one, where R is a unary relation for every r ∈ PROP and Q is a unary relation for every q ∈ PROP; and P keeps the previous meaning:

$$\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q) =$$
$$ST_x(\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)) =$$
$$\exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge P(y))) \wedge \exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge Q(y))) \rightarrow$$
$$\exists y\ (R^m\langle x, y\rangle \wedge (P(y) \wedge Q(y))).$$

The formula we have recursively translated into Classical Logic is our example (4), whose Standard Translation is $ST_x(\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q))$. By condition 4. of ST_x, $ST_x(\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q))$ is equal to $ST_x(\Diamond(r \wedge p) \wedge \Diamond(r \wedge q)) \rightarrow ST_x(\Diamond(p \wedge q))$. And by the recursive application of conditions 1., 3. and 5. we obtain $\exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge P(y))) \wedge \exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge Q(y))) \rightarrow \exists y\ (R^m\langle x, y\rangle \wedge (P(y) \wedge Q(y)))$, which is the full Standard Translation of (4).

(5) and (6) translation would be similar, but by following conditions 5*.-8*., instead of 5. and 6., this time. (5) would then be:

$$F(r \wedge p) \wedge F(r \wedge q) \rightarrow F(p \wedge q) =$$
$$ST_x(F(r \wedge p) \wedge F(r \wedge q) \rightarrow F(p \wedge q)) =$$

$$\exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge P(y))) \wedge \exists y\ (R^m\langle x, y\rangle \wedge (R(y) \wedge Q(y))) \rightarrow$$
$$\exists y\ (R^m\langle x, y\rangle \wedge (P(y) \wedge Q(y))).$$

(6), on its part, would be alike but with R^m applied to $\langle y, x\rangle$.

The most interesting thing about Standard Translation is it preserves satisfiability, i.e., both

$$\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q) \text{ and } ST_x(\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q))$$

and

$$F(r \land p) \land F(r \land q) \rightarrow F(p \land q) \text{ and } ST_x(F(r \land p) \land F(r \land q) \rightarrow F(p \land q))$$

(along with their analogous with (6)) are equisatisfiable, what means that if one of them are satisfiable in First Order Logic then its corresponding one will be so. We shall denote such property by means of symbol ≈. Hence, we picture that

$$\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q)$$

and

$$ST_x(\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q))$$

are equisatisfiable as

$$\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q) \approx ST_x(\Diamond(r \land p) \land \Diamond(r \land q) \rightarrow \Diamond(p \land q)).$$

Such result may be expressed by the following proposition (Blackburn and van Benthem 2007, 11; Blackburn 2006, 335):

> **Definition 3.1.1. ML-FOCL Equisatisfiability (ML ≈ FOCL)** For any basic modal formula φ, any model $\mathfrak{M}_K$ and any possible world w in $\mathfrak{M}_K$, $\mathfrak{M}_K$, w $\vDash \varphi$ if and only if $\mathfrak{M}_K \vDash ST_x(\varphi)$ [$x \leftarrow$ w].

[$x \leftarrow$ w] means that x takes world w value, that is, w is assigned to free variable x. The proof of **ML ≈ FOCL** directly arises from conditions 1.-6. of Standard Translation. Every modal formula thus can be transformed into its corresponding First Order Logic one but, as we remarked at the beginning of this section, not every FOCL formula may be transformed into a ML one. $\neg(x\mathrm{R}x)$ or $\neg(x < x)$ would be two examples. Indeed, whereas in FOL we can express properties such as irreflexivity by means of $\neg(x\mathrm{R}x)$ or $\neg(x < x)$, in ML we cannot. The reason lies in any modal formula is satisfied at every irreflexive points of a model. To explain why it is necessary to introduce the notion of *bisimulation*.

Let $\mathfrak{M}_K$ and $\mathfrak{M}_K$* be two whatever models of ML and let B be a binary relation between these two models. B is a bisimulation between $\mathfrak{M}_K$ and $\mathfrak{M}_K$* if the following holds:

- If B is applied to two worlds w and w' of $\mathfrak{M}_K$ and $\mathfrak{M}_K$* respectively then, for every propositional variable p, if w is on the set of possible worlds of $\mathfrak{M}_K$ where p is true, w' has also to be on the set of possible worlds of $\mathfrak{M}_K$* where p is true.
- If w and w' are related through B and w is related in $\mathfrak{M}_K$ through R to another possible world v, then w' must be related in $\mathfrak{M}_K$* through R' to another possible world v' too and, besides, v is related to v' through B.
- If w and w' are related through B and w' is related in $\mathfrak{M}_K$* through R' to v' then w is also related to v through R in $\mathfrak{M}_K$ and, moreover, v and v' are related through B.

Formally speaking:

Definition 3.1.2. ML Bisimulation If $\mathfrak{M}_K$ and $\mathfrak{M}_K$* are two models of basic Modal Logic such that $\mathfrak{M}_K = \langle W, R, V_M \rangle$ and $\mathfrak{M}_K{}^* = \langle W', R', V'_M \rangle$ then the relation $B \subseteq W \times W'$ is a bisimulation between $\mathfrak{M}_K$ and $\mathfrak{M}_K$* if the following conditions are satisfied:

1. If wBw' then, for any p ∈ PROP, $w \in V_M(p)$ if and only if $w' \in V'_M(p)$.
2. If wBw' and wRv then there exists a point v' in $\mathfrak{M}_K$* such that w'R'v' and vBv'.
3. If wBw' and w'R'v' then wRv and vBv'.

We say that a possible world w in $\mathfrak{M}_K$ is bisimilar to another possible world w' in $\mathfrak{M}_K$* if wBw'.

Condition 1. states that two bisimilar worlds satisfy the same propositions, i.e., if p is true at w in $\mathfrak{M}_K$ it will also be true at w' in $\mathfrak{M}_K$*. 2. and 3., on their part, set that if there are two bisimilar worlds and one of them access to another in $\mathfrak{M}_K$ or $\mathfrak{M}_K$* then that accessibility relation will also hold in the other model and these two accessed worlds will be bisimilar too. Therefore, two worlds are bisimilar if they satisfy the same formulae and access to the same worlds.

What follows from this is that two bisimilar worlds also satisfy the same modal formulae, for their satisfaction only requires the access to another world(s) and that in that world(s) such formula is satisfied. In consequence, two modal formulae are not able to distinguish between two bisimilar worlds.

The clearest example is the one we have previously highlighted: $\neg(x\mathrm{R}x)$ or $\neg(x < x)$. Let us suppose $\mathfrak{M}_K$ is a model where every propositional variable is false at every world and $\mathfrak{M}_K$* is a model where there is only a reflexive possible world where every propositional variable is false too. In this case B relates every $\mathfrak{M}_K$ world to the only $\mathfrak{M}_K$* world (w'). By the result derived from **ML Bisimulation** every $\mathfrak{M}_K$ and $\mathfrak{M}_K$* world satisfies the same modal formulae, but that means there cannot be a modal formula true at every irreflexive worlds for, if it existed, it would be true at every $\mathfrak{M}_K$ worlds while false at w' of $\mathfrak{M}_K$*. As this is impossible for there is a bisimulation between $\mathfrak{M}_K$ worlds and w′ it follows that no modal formula satisfies something as $\neg(x\mathrm{R}x)$ or $\neg(x < x)$.

Modal Logic expressive capacity is thus less than First Order Logic one. That is the reason why every ML formula can be translated into a FOCL one but not conversely. There is not a modal formula equisatisfiable to the first order formula $\neg(x\mathrm{R}x)$. Modal Logic may consequently be conceived as a proper fragment of First Order Logic. In particular, as the fragment composed by those formulae closed under bisimulation, that is, Modal Logic is that part of First Order Logic holding every formula invariant for bisimulation.

It is said a FOL formula φ with one free variable is invariant for bisimulation if its valuation at bisimilar worlds is always the same. The fact that ML is a proper fragment of FOCL follows from this definition and the so-called van Benthem´s *Characterisation Theorem* (van Eijck 2006, 15; Blackburn 2006, 337-338; Blackburn and van Benthem 2007, 21), according to which if a first order formula φ with one free variable is invariant for bisimulation then φ is equivalent to the standard translation of its corresponding modal formula.

Why thus making use of Modal Logic instead of First Order one if the former is only a small part of the latter? For two reasons, mainly:

I. The first one is because of decidability: Modal Logic is PSPACE-decidable whereas First Order Logic is not.

II. The second one is due to *perspectivism*: Modal Logic provides an internal perspective of models whereas First Order Logic one is external.

These two reasons, along with language simplicity, justify the importance of Modal Logic. Although ML has still great problems to represent certain kind of propositions, as we will see down below.

3.2. Problems of Modal Logic

In the previous section we have seen that, despite Modal Logic is properly included in First Order Correspondence Language, it is still useful as the preceding I. and II. conditions show. Specially thanks to the second one, which was Prior´s reason for choosing ML to build Temporal Logic.

However, Modal Logic main problem lies in its expressivity. As mentioned previously, there is no modal formula which satisfies an irreflexive proposition, and that constitutes a very remarkable expressive limitation. But now we are going to focus on another limitation, namely, its inability to name points inside a model.

The main characteristic of ML and TL languages is they are composed by modal expressions ($\Box\varphi$, $\Diamond\varphi$, Fφ, Pφ, Gφ or Hφ) which allow to relativize φ truth to a range of worlds or instants (or one in the case of $\Diamond$, F and P). But they do not allow for instance to set that φ is true at exactly *this* world (instant). First Order Logic can do it. Indeed, by means of constants and the identity relation we are able to state that certain individual possesses such and such property, or that two individuals with x property are equal. But ML and TL do not count with these mechanisms.

What directly follows from that is Temporal Logic, being based on Modal Logic, is not able to accurately represent the natural conception of time. As we allude to facts which have happened or will happen at any time different from current one we do not usually make do with claiming that such facts have happened sometime in the past or will happen sometime in the future. We generally aim to state when they happened or will happen, and Temporal Logic cannot formally represent.

In Classical Modal Logic, either purely modal or temporal, epistemic, etc., formulae are valuated regarding a point of reference which tends to be a possible world. That point of reference (or possible world) can change and so formulae truth value can change too. If for instance we say «I am going to make tea later» it is evident that such claim shall have different truth values depending on the possible world (or the moment and history in the case of Temporal Logic) where it is valuated. Nonetheless, sometimes we want to be more precise and set the exact moment where such fact happened, happens or will happen. Following our example, maybe we want to specify that we are going to make tea at five o´clock this afternoon.

The first proposition, «I am going to make tea later», can be formalized in terms of TL as F(1)p, where F is the possibly future operator, value 1 indicates p will take place one time unit (let us put eight hours) later than the utterance moment[3] and p stands for "making tea later". But the second proposition, «I am going to make tea at five o´clock this afternoon», cannot be formalized.

It cannot be so for, to do it, it is required to introduce nominals, i.e., symbols which exactly determine when a proposition is valuated. Therefore, the point of reference which will allow to set the truth value of propositions bound with nominals is just the point such nominal designates, and just that one. In our case, it may be the 4^{th} of July at 17:00.

Following Hans Reichenbach´s (1947) distinction between point of utterance (S), point of event (E) and point of reference (R) we can see that basic Temporal Logic only counts with the first and the second ones but not with the third one. As we valuate a formula such as F(1)p at t_0, S, which is the point of valuation, is precisely t_0 whereas E, which is the point where the fact the proposition talks about takes place, is some future moment t_1 where p is true. The same applies to P, G and H.

[3]We are dealing with Metric Temporal Logic (Prior 2010, 159-170; Øhrstrøm and Hasle 1995, 231-240).

This structure allows us to formalize things as «I am going to make tea later» or «I made tea before», but it does not allow us to formalize «I had made tea», for instance, since for this claim be valuated it is necessary to count on a point of reference such that succeeds S and precedes E. Graphically:

$$E \leftarrow R \leftarrow S.$$

By stating that at t_0 we had made tea what we are asserting is there is a previous moment t_{-1}, corresponding to R, where it is true that at some previous moment t_{-2} we made tea. That moment corresponds to E.

In the same way, if we said «I will have made tea» at t_0 what we were asserting is that it is true at some future moment t_2 (R) that in some past moment t_1 (E) it is true that we will have made tea. (Although the structure of this tense is not always this way (Reichenbach, 1947).) Graphically

$$S \leftarrow E \leftarrow R.$$

Structures like these ones are what basic Temporal Logic is not able to represent. Its syntax and semantics do not allow to allude to specific instants. That is why it is narrowed. And also that is why Prior saw the necessity of embracing some FOL mechanisms to build a more expressive system than TL. That system is Hybrid Logic.

We thus have seen that Modal Logic may be subsumed under First Order Logic by means of Standard Translation. That makes the internal perspective of the former can be expressed through the external perspective of the latter and that we are able to call Modal Logic usefulness into question. Its utility lies, however, in its capacity for talking about models from inside them. In the case of Temporal Logic that internal perspective is crucial for developing a logical system which aims to represent time natural conception. The problem, nevertheless, is TL does not achieve it for it is based on Modal Logic, whose expressive ability is limited. A solution may be to merge ML/TL and FOL mechanisms in order to increase that expressivity. The result of this merging is Hybrid Logic, to which we shall devote the following section.

4. Hybrid Logic

At the beginning of this paper we said Prior´s motivation for building both Temporal Logic and Hybrid Logic was mainly philosophical. According to him any issue related to time (and almost anything) may be logically solved and that is why, as we will see, when the time comes he has to resort to Hybrid Logic to solve one of the greatest problems of TL. Problem linked to TL expressivity, which we have set out at the previous section.

At this one what we are going to do is, on the one hand, to explain Hybrid Logic motivation to, on the other hand, introduce its basic system and extend it in accordance with Prior´s approach.

4.1. Hybrid Logic Motivation

In (1908) John McTaggart raises two ways of conceiving time, i.e., two ways of understanding the arrangement of facts in time: A series and B series. The first one consists in sorting events in terms of past, present and future. The second one consists in sorting them on the basis of ulteriority relation, that is, on the basis of the before/after relation. A series accurate reflections are F, P, G and H operators whereas B series one is the $<$ relation.

By sorting facts according to whether they take place at present, at past or at future we embrace an internal conception of time, namely, we place ourselves inside it and we reflect the chain of events from the future to the past, through the present. Temporal Logic, by arising from Modal one, picks up this internal view of time by means of F, P, G and H.

In contrast, by sorting facts according to whether they take place before or after each other we embrace an external view of time, for we place ourselves from outside it and we just reflect its course through the mere succession of events. The $<$ relation, characteristic of First Order Logic, embodies this idea very well.

That Prior has developed to a larger extent Temporal Logic and that he has conferred it a greater importance on his writings proves that, for him, A series takes priority over B series. In fact, in his opinion, the former presupposes the latter. But not only that. Besides, B series has two important problems: firstly, it does not actually represent the way we experience time for our experience is not external but internal; secondly, it entails accepting the existence of instants. Indeed, in TL quantification is carrying out over instants (as we have seen in V_T 5-8 conditions) and by claiming that "There exists an instant t such that..." or "For every instant t..." what we are doing is to commit ourselves to their existence, which is very doubtful.

This is why Prior prioritizes A series over B one, although he must prove it is possible to subsume it under A, and why he resorts to Hybrid Logic.

In "Tense Logic and the Logic of Earlier and Later" (2010) Prior avers there are four types (grades) of logical-temporal entailment:

In the first one he presents TL as some kind of abbreviation of what he calls *U-calculus*, which is no more than his version of Standard Translation. What he advocates is that formulae as Fφ or Pφ are shorthand of «There exists an instant t' later than t where φ is true» and «There exists an instant t' earlier than t where φ is true», respectively, and so temporal operators may be understood as ways of synthesizing $<$ properties. In other words: Prior appeals to Standard Translation to show that Temporal Logic may be conceived as a Temporal First Order Logic. A series is then reduced to B one.

In the second grade temporal operators are no longer subsumed under < relation. They are now at the same level. The key to understand such thing lies in how atomic propositions are treated. As we saw at 2.2 section K_T semantics demands for referring to instants in order to valuate formulae. That means a proposition as p cannot be valuated, by itself, in K_T because we do not count with the reference to an instant to carry it out. To do so we would need a moment t and stating that p is true in it, i.e., to say that $\mathfrak{M}_{KT}, t \vDash p$. Thus, p on its own is an uncompleted formula.

Nevertheless, what Prior claims is that p actually constitutes what Nicholas Rescher and Alasdair Urquhart have called *chronologically definite propositions* (Rescher and Urquhart 1971; Prior 2010, 118; Øhrstrøm and Hasle 1995, 218). A chronologically definite proposition is that one which implicitly alludes to an instant though it does not explicitly allude to it. This means that a formula as p is equivalent to $\mathfrak{M}_{KT}, t \vDash p$, that is, p is a way of stating "p is true at instant t". And what results from this is A and B series are at the same conceptual level. None of them are constrained by the other one but both are related.

In the third grade this connection between both series becomes more evident thanks to the introduction of what Prior calls *world-variables*, which are just nominals. According to Prior nominals depict the (possible) world just as it is at the very moment they allude to. That implies their nature is two-fold: on the one hand, they are indexes naming a certain moment; on the other hand, they are propositions depicting the world just as it is at that very moment.

If so, that is, if nominals are not only terms which refer to time instants but they are also propositions, then one of the greatest problems of B series, namely its commitment with instants existence, is solved. By being propositions, instants are no longer fictitious entities (Prior 1967, 188-189), and as claiming that «φ is true just at instant i» is the same that claiming «It is necessary that if i then φ», it is possible to entirely derive Temporal Logic from A series plus the necessity operator, i.e., from F, P, G, H and $\Box$.

Finally, the fourth grade is no more than an attempt to define the necessity operator in terms of temporal ones. In it Prior suggests what is known as *universal modality* and besides he reduces the entire Temporal Logic (including $\Box$) to F and P operators.

Third and fourth grades allow to subsume B series under A one, and Hybrid Logic arises due to them. The system which reflects the third one is more general than the one which reflects the fourth grade, which is more specific, but in any case both are origin of Hybrid Logic. A logic that, as Prior later acknowledges in articles such as "Quasi-Propositions and Quasi-Individuals" (Prior 2010, 213-221) or "Egocentric Logic" (Prior 2010, 223-240), may be extended to any domain and allows to consider as propositions not only predicates alluding to instants but basically any predicate. Description Logic, posed by him in the second aforementioned article, results from this way of conceiving HL.

Hybrid Logic arising is then due to Prior´s pretence of proving that A series truly reflects our conception of time and that B series may be reduced to it. And it means that we can reduce one part of First Order Logic to a purely temporal logic. A reduction which just may be carried out by Hybrid Logic mechanisms. In the following section we shall see what they are.

4.2. Two Systems of Hybrid Logic

Hybrid Logic relevance both for Temporal Logic and for Modal Logic has been evinced in the previous section. In this one what we are going to do is to formally present two systems of HL: the first one is its basic system whereas the second one is the strongest system presented by Prior himself.

HL alphabet is based on ML one but with two main differences: it adds a new sort of propositional symbols and a new set of modal operators.

Apart from PROP set of propositional variables Hybrid Logic adds a second sort of symbols to represent nominals: NOM. Those symbols are *i, j, k, l*, etc.: NOM = {*i, j, k, l*, ...}. One of the most interesting and relevant characteristics of nominals is they constitute terms, that is, they are not only variables indicating a specific moment but those variables are, by themselves, propositions. Hence, if we say «*i*» what we are asserting is, on the one hand, there is an instant called "*i*" and, on the other hand, that formula is true at the instant whose name is "*i*".

Nominals can be combined with other formulae to build more complex formulae, and such formulae are true just at the instant named by the nominal. Their function is thus to name a point inside a model and to set that the proposition bound by them is true at exactly that point. Thanks to that we are able to solve the Modal Logic expressive limitation.

Recall proposition (4):

$$\Diamond(\mathrm{r} \wedge \mathrm{p}) \wedge \Diamond(\mathrm{r} \wedge \mathrm{q}) \rightarrow \Diamond(\mathrm{p} \wedge \mathrm{q}).$$

(4) is not valid in K, but it does be in Hybrid Logic as we substitute r by *i*. The resulting formula would be:

$$\Diamond(i \wedge \mathrm{p}) \wedge \Diamond(i \wedge \mathrm{q}) \rightarrow \Diamond(\mathrm{p} \wedge \mathrm{q}). \qquad (4^*)$$

(4*) is always true for what it states is the world where p is true and the world where q is true are the same, namely, *i*. And hence it is true that there exists a world w' (*i*) where p and q are simultaneously true.

Regarding (5) and (6) the same happens. If these propositions are transformed in

$$\mathrm{F}(i \wedge \mathrm{p}) \wedge \mathrm{F}(i \wedge \mathrm{q}) \rightarrow \mathrm{F}(\mathrm{p} \wedge \mathrm{q}) \qquad (5^*)$$

and

$$P(i \land p) \land P(i \land q) \rightarrow P(p \land q) \qquad (6^*)$$

by substituting r by *i* then the moments where p and q are true overlap, and thus the consequent of these conditionals is always true.

But nominals may also formally represent propositions which Modal/Temporal logics are not able to do such as «I had made tea», which we discussed at section 3.2. In Hybrid Logic the formula representing such proposition would be:

$$P(i \land Pp) \qquad (7)$$

which sets there is a moment *i* earlier than t_0 and a moment t_{-1} earlier than *i* such that p is true at it. In other words, the moment where that fact takes place is earlier than the point of reference, which in turn predates the moment where (7) is uttered. (7) structure is, as we may see,

$$E \leftarrow R \leftarrow S,$$

where *i* is R, corresponding to what Reichenbach poses in (1947) and we remarked at point 3.2[4].

So nominals increase and enhance Modal/Temporal logics expressivity by allowing to allude to specific points and they constitute the first step in Hybrid Logic building. The second one lies in the satisfaction operator @. Its function consists in binding both a nominal and a propositional variable to determine the very moment that variable is true. A formula such as $@_i\varphi$, read as "At *i*, φ", sets that φ is satisfied only and exclusively at *i*. @ is therefore a modal operator whose range is just a possible world or moment, and alike □ and ◇ it fulfills the following properties:

- Distributive:

$$@_i(\varphi \rightarrow \psi) \rightarrow (@_i\varphi \rightarrow @_i\psi).$$

- Generalization:

$$\text{If } \vdash\varphi \text{ then } \vdash @_i\varphi,$$

for any *i*.

- Self-duality:

$$@_i\varphi \equiv \neg @_i \neg\varphi.$$

[4]It is worthy to highlight that, notwithstanding the tremendous link between Reichenbach´s approach and Prior´s one, the latter never thought there was any. In fact, Prior thought Reichenbach´s proposal was too messy because of his distinction between R and E (Prior 1967, 13); however using nominals implies such distinction.

One of the main advantages of satisfaction operator is it provides a modal interpretation of identity relation. Indeed, thanks to @ we may represent identity in Hybrid Logic by the formula

$$@_i j,$$

which states that, at point *i*, *j* is satisfied. As *i* and *j* are nominals that means point *i* is identical to point *j* and consequently we may express = by means of $@_i j$. @ is also able to express other properties of identity for which it is necessary to resort to = and First Order Logic in other systems, namely, reflexivity, symmetry, transitivity and substitution (Blackburn 2006, 343):

- Reflexivity:

  $\forall x\,(x = x)$

  is represented in HL as

  $@_i i.$

- Symmetry:

  $\forall x \forall y\,(x = y \rightarrow y = x)$

  is represented in HL as

  $@_i j \rightarrow @_j i.$

- Transitivity:

  $\forall x \forall y \forall z\,(x = y \wedge y = z \rightarrow x = z)$

  is represented in HL as

  $@_i j \wedge @_j k \rightarrow @_i k.$

- Substitution:

  $\forall x \forall y\,(x = y \wedge \Psi(x) \rightarrow \Psi(y))$

  is represented in HL as

  $@_i \varphi \wedge @_i j \rightarrow @_j \varphi,$

  where Ψ symbolizes any property.

Hybrid Logic, due to @, is thus able to modally translate FOL formulae. This is why (among other things) its expressive capacity is bigger than other systems as ML and TL one.

Basic Hybrid Logic alphabet, called L_H, consists of:

$$i \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid @_i\varphi,$$

where $i \in$ NOM, p $\in$ PROP, $\varphi, \psi \in$ WFF and $\Box$, $\Diamond$ and @ $\in$ MOD (all of them under the same accessibility relation). (4*), (5*) and (6*) would be examples of HL well-formed formulae.

As HL is no more than ML enriched with NOM and @ it addresses possible worlds semantics too. So, from the frame $\mathfrak{F} = \langle W, R\rangle$ we build the model $\mathfrak{M}_H = \langle W, R, V_H\rangle$, where W is a nonempty set of possible worlds (or states or points), R is the accessibility relation between them such that $R \subseteq W \times W$ and V_H is the valuation function assigning subsets of W both to propositional variables and to nominals, i.e., V_H: PROP $\cup$ NOM $\rightarrow \wp(W)$. Hence, V_H domain is PROP $\cup$ NOM and its range is $\wp(W)$.

Regarding nominals V_H is always a singleton subset of W, that is, for any $i \in$ NOM and any $w \in W$, $w \in V_H(i) \equiv w = i$. In other words: every nominal has only and the same value at W. $V_H(i)$ single world is the denotation of i.

As well as in Modal Logic we shall symbolize that a formula φ is satisfied at world w of model $\mathfrak{M}_H$ as $\mathfrak{M}_H, w \vDash \varphi$. HL formulae are satisfied if the following holds, for any $\varphi, \psi \in$ WFF; w, w' $\in$ W; $i \in$ NOM and propositional variable p (Blackburn, 2000, 347; Areces and ten Cate, 2007, 825):

1. $\mathfrak{M}_H, w \vDash p$ iff $w \in V_H(p)$,
2. $\mathfrak{M}_H, w \vDash i$ iff $w \in V_H(i)$,
3. $\mathfrak{M}_H, w \vDash \neg\varphi$ iff $\mathfrak{M}_H, w \nvDash \varphi$,
4. $\mathfrak{M}_H, w \vDash \varphi \wedge \psi$ iff $\mathfrak{M}_H, w \vDash \varphi$ and $\mathfrak{M}_H, w \vDash \psi$,
5. $\mathfrak{M}_H, w \vDash \varphi \rightarrow \psi$ iff $\mathfrak{M}_H, w \nvDash \varphi$ or $\mathfrak{M}_H, w \vDash \psi$,
6. $\mathfrak{M}_H, w \vDash \Box\varphi$ iff $\forall w' \in W$ if wRw' then $\mathfrak{M}_H, w' \vDash \varphi$,
7. $\mathfrak{M}_H, w \vDash \Diamond\varphi$ iff $\exists w' \in W$ such that wRw' and $\mathfrak{M}_H, w' \vDash \varphi$,
8. $\mathfrak{M}_H, w \vDash @_i\varphi$ iff $\mathfrak{M}_H, w' \vDash \varphi$, where $w' \in V_H(i)$.

Conditions 1., 3., 4., 5., 6., and 7. are akin to K and K_T ones. The only two different are 2. and 8. What 2. states is a nominal i is satisfied at world w if and only if i is the name of w, that is, if w and i are equal. 8., on its part, states that $@_i\varphi$-like formulae are satisfied at some world if and only if φ is satisfied at world denoted by the nominal.

If φ is satisfied at every possible worlds of every model $\mathfrak{M}_H$ based on $\mathfrak{F}$ then we say φ is valid in $\mathfrak{F}$. Logically speaking: $\mathfrak{F} \Vdash \varphi$. If φ is valid in every frame $\mathfrak{F}$ we say it is just valid, i.e., $\Vdash \varphi$.

Basic Hybrid Logic system basically consists in this. It possesses three main characteristics: firstly, as well as in Modal Logic, it is decidable in PSPACE; secondly, it can be translated to First Order Logic if we extend the Standard Translation to nominals and satisfaction operators; and thirdly, it constitutes the FOL invariant for bisimulation fragment of formulae with constants and identity relation.

Let us focus on the second and the third characteristics. In order to express by means of First Order Correspondence Language Hybrid Logic formulae we just have to extend the Standard Translation conditions (see page 45) to represent nominals and satisfaction operators. These two conditions may be (Blackburn 2006, 344):

7. $ST_x(i) = (x = i)$,
8. $ST_x(@_i\varphi) = ST_i(\varphi)$.

In FOCL nominal *i* is symbolized by constant *i* whereas the satisfaction operator is represented by substituting the free variable *x* by the constant *i*.

(4*) Standard Translation would then be:

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q) =$$
$$ST_x(\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)) =$$
$$\exists y\, (R^m\langle x, y\rangle \wedge (y = i \wedge P(y))) \wedge \exists y\, (R^m\langle x, y\rangle \wedge (y = i \wedge Q(y))) \rightarrow$$
$$\exists y\, (R^m\langle x, y\rangle \wedge (P(y) \wedge Q(y))).$$

As in ML and TL, Standard Translation also preserves satisfiability so that:

> **Definition 4.2.1. HL-FOCL Equisatisfiablity (HL ≈ FOCL)** For any basic hybrid formula φ, any model $\mathfrak{M}_H$ and any possible world w in $\mathfrak{M}_H$, $\mathfrak{M}_H$, w $\vDash \varphi$ if and only if $\mathfrak{M}_H \vDash ST_x(\varphi)\ [x \leftarrow w]$.

Therefore, from **HL ≈ FOCL** is deduced that

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

and

$$ST_x(\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q))$$

are equisatisfiables.

If so, then HL is a FOL fragment, as we have declared. In particular, it is the fragment of any invariant for bisimulation formula composed by constants and identity relation. The following proposition reflects that:

Definition 4.2.2. HL Bisimulation If $\mathfrak{M}_H$ and $\mathfrak{M}_H$* are two basic Hyrbid Logic models such that $\mathfrak{M}_H = \langle W, R, V_H \rangle$ and $\mathfrak{M}_H^* = \langle W', R', V'_H \rangle$ then the relation $B \subseteq W \times W'$ is a bisimulation between $\mathfrak{M}_H$ and $\mathfrak{M}_H$* if the following conditions are satisfied:

1. If wBw' then, for any p ∈ PROP, $w \in V_H(p)$ if and only if $w' \in V'_H(p)$.
2. If wBw' and wRv then w'R'v' and vBv'.
3. If wBw' and w'R'v' then wRv and vBv'.
4. If, for any w ∈ W and w' ∈ W', w and w' are denoted by the same nominal then wBw'.

It is said a FOL formula φ with one free variable is invariant for bisimulation if its valuation at bisimilar worlds is always the same. That HL is a proper fragment of FOL is derived from this definition and from the *Characterization Theorem* (Blackburn 2006, 345; Areces and ten Cate 2007, 838-839), stating that if a first order formula φ with one free variable is invariant for bisimulation then φ is equivalent to its corresponding standard translation formula in HL.

The above system is basic for it just adds NOM and @ to ML language. However, Prior´s one is stronger. Despite us, who have drawn from Modal Logic to build Hybrid Logic, Prior draws from Temporal Logic and extends it by adding three elements: nominals, the Universal Modality and the ∀ and ∃ quantifiers.

Regarding nominals, it cannot be said any more but he adds a new operator, Q, as an alternative to them. Hence, a formula as $Q\varphi$ is true at any point of the model if and only if there is just one point on it where φ is true. That means that, through $Q\varphi$, φ is transformed into a nominal, for this is equivalent to say that φ is true at w if and only if w belongs to the set of points satisfying φ. As there is just one, φ and w are identical.

The Universal Modality is a tool which lies in most of the contemporary Hybrid Logic systems and which possesses two forms: one equivalent to □, symbolized as A, and another equivalent to ◇, symbolized as E. $A\varphi$ means that φ is true at every world of the model, whereas $E\varphi$ means that φ is true at some world in the model.

As it is easily seen, $E(i \wedge \varphi)$ truth condition is equivalent to $@_i\varphi$ one and thus $@_i\varphi$ may be defined in terms of E as:

$$@_i\varphi \overset{\text{def}}{=} E(i \wedge \varphi),$$

which states there exists some point in the model where i and φ are simultaneously true. But $@_i\varphi$ may be also defined in terms of A as follows:

$$@_i\varphi \overset{\text{def}}{=} A(i \rightarrow \varphi),$$

which sets that at every world of the model where i is true φ is true too.

Finally, Prior adds the quantifiers ∀ and ∃, notwithstanding the novelty that, besides their regular use, they can bind nominals. In this way, formulae such as

$$\exists x @_x \varphi$$

or

$$\forall x @_x \mathrm{F}\varphi$$

could also be included in the strong Hybrid Logic language (HL^+), named $\mathcal{L}_{HL+}$.

$\mathcal{L}_{HL+}$ is composed, apart for PROP, NOM and MOD, by a new sort of state variables SVAR = {x, y, z, ...} representing nominals. The difference between NOM and SVAR lies in NOM elements are constants, they always denote the same world, whereas SVAR ones are variables. Hence, $\mathcal{L}_{HL+}$ is defined as (Blackburn 2006, 351):

$$x \mid i \mid \mathrm{p} \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid @_i\varphi \mid \mathrm{A}\varphi \mid \mathrm{E}\varphi \mid \forall x\varphi \mid \exists x\varphi.$$

Many of these operators may be interdefined (the rest of logical constants and $\Box\varphi \equiv \neg\Diamond\neg\varphi$, $@_i\varphi \equiv \mathrm{E}(i \wedge \varphi)$, $\mathrm{A}\varphi \equiv \neg\mathrm{E}\neg\varphi$, $\forall x\varphi \equiv \neg\exists x\neg\varphi$, and vice versa), but we have chosen for adding them all in order to fully present HL^+ alphabet.

With respect to its semantics, frames and models still are Kripkean, i.e., $\mathfrak{M}_{H+} = \langle \mathrm{W}, \mathrm{R}, \mathrm{V}_{H+}\rangle$. However, as now there are free and bound variables it is necessary to assign truth values in accordance with SVAR and consequently to relativize formulae valuation to variables assignments.

Let g be a function assigning values in $\mathfrak{M}_{H+}$ to variables. g is a function from SVAR to W, that is, g: SVAR → W and moreover, if g and g' are functions assigning values to variables in $\mathfrak{M}_{H+}$ but g possibly differs from g' in x value, we say that g' is a x-variant of g, and we denote it as $g' \sim^x g$. We also represent that any formula φ is satisfied at world w of model $\mathfrak{M}_{H+}$ under the assignment g as $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \varphi$.

Satisfiability conditions are defined as follows:

1. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash x$ iff $\mathrm{w} = g(x)$,
2. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash i$ iff $\mathrm{w} \in \mathrm{V}_{H+}(i)$,
3. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \mathrm{p}$ iff $\mathrm{w} \in \mathrm{V}_{H+}(\mathrm{p})$,
4. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \neg\varphi$ iff $\mathfrak{M}_{H+}, g, \mathrm{w} \nvDash \varphi$,
5. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \varphi \wedge \psi$ iff $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \varphi$ and $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \psi$,
6. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \varphi \rightarrow \psi$ iff $\mathfrak{M}_{H+}, g, \mathrm{w} \nvDash \varphi$ or $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \psi$,
7. $\mathfrak{M}_{H+}, g, \mathrm{w} \vDash \Box\varphi$ iff $\forall \mathrm{w}' \in \mathrm{W}$ if wRw' then $\mathfrak{M}_{H+}, g, \mathrm{w}' \vDash \varphi$,

8. $\mathfrak{M}_{H+}, g, w \vDash \Diamond\varphi$ iff $\exists w' \in W$ such that wRw' and $\mathfrak{M}_{H+}, g, w' \vDash \varphi$,
9. $\mathfrak{M}_{H+}, g, w \vDash @_i\varphi$ iff $\mathfrak{M}_{H+}, g, w' \vDash \varphi$, where $w' \in V_H(i)$,
10. $\mathfrak{M}_{H+}, g, w \vDash A\varphi$ iff $\forall w' \in W\ \mathfrak{M}_{H+}, g, w' \vDash \varphi$,
11. $\mathfrak{M}_{H+}, g, w \vDash E\varphi$ iff $\exists w' \in W\ \mathfrak{M}_{H+}, g, w' \vDash \varphi$,
12. $\mathfrak{M}_{H+}, g, w \vDash \forall x\varphi$ iff, for any $g' \sim^{x} g$, $\mathfrak{M}_{H+}, g', w \vDash \varphi$,
13. $\mathfrak{M}_{H+}, g, w \vDash \exists x\varphi$ iff, for some $g' \sim^{x} g$, $\mathfrak{M}_{H+}, g', w \vDash \varphi$.

2.-9. conditions are similar to the ones we have already seen. The new ones are 1., 10., 11., 12. and 13. 1. sets that if variable *x* is true at w according to *g* it is because the value assigned to *x* by *g* is w. On their part, 10.-13. are no more than the formal representation of what we have pointed out above.

$\mathcal{L}_{HL+}$ and $\mathfrak{M}_{H+}$ may be extended to Temporal Logic and to accept propositions composed by temporal operators whose valuation is carried out at instants and not at worlds. And $\mathcal{L}_{HL+}$ can also be extended by means of other operators such as ↓, **Until** and **Since** (Areces and ten Cate 2007, 822-823) to increase its expressive power.

Anyway, the most relevant thing about strong Hybrid Logic is that, apart from it is still decidable despite its huge expressive capacity, it is also powerful enough to translate FOCL to its own language. Until now we have claimed that FOCL is able to translate ML, TL and basic HL formulae by means of Standard Translation but HL^{+} is, in turn, able to translate FOCL formulae thanks to what it is called *Hybrid Translation*.

Let HT be a function assigning to each FOCL formula its corresponding HL^{+} one. HT consists in the following, for any variable $x, y, z \in SVAR$, any propositional symbol $p \in PROP$ and any $\varphi, \psi \in WFF$ (Blackburn 2006, 352):

1. $HT(R^{m}\langle x, y\rangle) = @_x\langle m\rangle y$,
2. $HT(P(x)) = @_x p$,
3. $HT(x = y) = @_x y$,
4. $HT(\neg\varphi) = \neg HT(\varphi)$,
5. $HT(\varphi \wedge \psi) = HT(\varphi) \wedge HT(\psi)$,
6. $HT(\varphi \rightarrow \psi) = HT(\varphi) \rightarrow HT(\psi)$,
7. $HT(\exists z\varphi) = \exists z HT(\varphi)$,
8. $HT(\forall z\varphi) = \forall z HT(\varphi)$.

Similarly for the rest of formulae.

As it may be observed the main key of Hybrid Translation is @. Thanks to it we can reduce FOCL to HL^+, so without @ Hybrid Logic would not be capable enough of carrying out this translation.

Hybrid Logic (both basic and strong) basically consists in this, and its main advantage over the rest logics we have presented here is Hybrid Translation.

5. Conclusion

As we asserted at the Introduction this paper aims to show how Hybrid Logic was developed by Arthur Prior in order to be an extent of Temporal Logic to solve its problems (and thus Modal Logic ones) and why it is stronger than both of them.

With the aim of fulfilling this goal in section 2 we exposed ML and TL to see, in the first place, how Temporal Logic is built by Prior from Modal Logic. Compared with ML, TL constitutes a huge advance at formally representing propositions suffused with temporal information, but it suffers from the same Modal Logic disadvantages.

In section 3 we entered such disadvantages. They are mainly two: on the one hand, both ML and TL may be subsumed under the First Order Correspondence Language. In the first subsection, we saw that by means of the Standard Translation FOCL may translate any ML/TL formula into its own language. We explained why: for Modal Logic is the FOCL fragment composed by every invariant for bisimulation formulae. On the other hand, ML and TL are incapable of naming specific points inside a model, that is, they have no mechanism allowing to point that such and such propositions is true at some particular point. First Order Logic does have, however.

That means FOL is stronger than ML and TL. Nevertheless, ML and TL main advantage is they provide an internal perspective of models in contrast to FOL, whose perspective is external. That is precisely why at the end of the second subsection of section 3 we claimed that both are better to propose a logical system which does its best to reflect the natural concept of time. It is just necessary to combine ML/TL mechanisms with FOL ones to achieve it. The result of such combination is Hybrid Logic.

Section 4 has been devoted entirely to it. In the first part, we explained its motivation. The need to pose a more powerful system than Temporal Logic lied in the Priorean approach regarding Time B series may be reduced to A ones. As TL language is not expressive enough to allow such thing, by means of the four grades of tense-logical involvement we saw how Prior formulates a system which does allow to reduce First Order Logic to a purely temporal logic. A reduction which can only be carried out by HL mechanisms.

In the second part of this section we exposed such mechanisms. The use of a new set of symbols representing nominals and of satisfaction operators constitute the Hybrid Logic main achievements. Thanks to them, and to operators such as A, E, ∀ and ∃, we are able to develop an even stronger system than HL to which it may now be translated the First

Order Correspondence Language. At the end of this part it has been indicated how to do that by using Hybrid Translation.

Hybrid Translation is the most relevant result proposed in this paper. By means of it, it is proved that HL is an incredibly powerful system due to its huge expressivity. An expressiveness which, as pointed, allows to translate FOCL to HL language. And since Standard Translation allows to do the same with ML/TL languages then both logics may also be translated to Hybrid one.

In formal terms, we could say:

$$\text{ML/TL} \subseteq \text{FOCL} \subseteq \text{HL},$$

that is, Modal Logic and Temporal Logic are properly included in the First Order Correspondence Language, which in turn it is included (but not properly) in Hybrid Logic. So, both FOCL and HL may translate each other; a thing that ML and TL are not able to do with respect to FOCL. Hence Hybrid Translation relevance.

Throughout the entire exposition we have always delivered examples with the aim of clarifying the questions at stake. Therefore, it only remains to conclude by pointing out that Hybrid Logic may be applied not only to what we have mentioned but "hybridization" may be applied to a great range of contexts.

As we said in section 4.1., Prior realized that HL mechanisms allow to consider as propositions not only predicates alluding to instants but just any predicate. In this paper, we have exclusively focused on Kripke semantics, but Hybrid Logic can also be extended to topological and algebraic semantics. HL tools may be added both to basic Modal Logic and to First Order and Second Order Modal Logics (intuitionism-based even), allowing to obtain very interesting results. An example may be found on Carlos Areces, Patrick Blackburn, Antonia Huertas and María Manzano´s (2011) paper where a Hybrid Types Theory is developed. Or on Areces, Blackburn, Huertas and Manzano´s (2014) paper where a Completeness proof, via Henkin, of this Hybrid Types Theory is provided.

Some of these applications have already been investigated, but there are many others which remain unstudied. In our case, we think it would be so interesting due to its pragmatic involvements to combine Hybrid Logic with Epistemic Logic and Fuzzy one in order to build models reflecting more accurately the way we, human beings, reason and argue.

When we utter «I think φ» or «I know φ» this believe or knowledge is usually context-dependent, i.e., we can believe or know φ in some particular moment, but not to believe it or know it in another one (for our set of knowledges has changed, for instance). And even more. In may times we do not totally believe or know φ. We may have doubts about its truth (or falsehood) and not to believe it *too much*, or not to know it *very well*.

Classical Epistemic Logic allows to formalize propositions such as «I think φ» or «I know φ» and to valuate them according to the epistemic states of some model. By combining it with Fuzzy Logic we obtain a many-valued system allowing to increase the number of truth values we may assign to formulae to represent values such as *too much* or *not too much*, *very well* or not *very well*, *a little*, *enough*, etc. And if we add all this to Hybrid Logic we could build a system in which, moreover, it could be possible to reflect the fact that certain propositions possess just a truth value at some specific point. For instance, we could formally reflect and valuate a statement as (9) «I am not quite sure in this moment of φ».

Utterances as (9) are quite common in our daily communicative exchange. Our believes and knowledges usually tend to be fuzzy and context-dependents, and that is why we ought to appeal to mechanisms from these three systems (epistemic, fuzzy and hybrid) if we wish to develop models which depict such characteristics. Models whose application may go from Knowledge and Reasoning Representation in Artificial Intelligence to Argumentation.

6. Appendix

6.1. Notation

The syntax of the logical language $\mathcal{L}$ used in this paper is composed by:

A. Primitive Symbols

1. Propositional variables: p, q, r, ... That is, lowercase letters of the alphabet beginning from p.
2. Connectives:
    i. Negation: $\neg$,
    ii. Conjunction: $\wedge$,
    iii. Disyunction: $\vee$,
    iv. Material conditional: $\rightarrow$,
    v. Logical equivalence: $\equiv$.
3. Quantifiers:
    i. Universal: $\forall$,
    ii. Existential: $\exists$.

B. Metavariables

Representing either propositional variables or well-formed formulae: φ, ψ, ...

C. Well-Formed Formulae (WFF)

Which are any formulae fulfilling the following conditions:

1. Both propositional variables and metavariables are wff.
2. If p is a wff, then $\neg$p is a wff too.
3. If φ and ψ are wff, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \equiv \psi)$, $\forall\varphi$ and $\exists\varphi$ are wff too.
4. The outermost parentheses of any well-formed formula may be omitted.
5. Anything not followed from the recursive application of rules 1.-4. is a wff.

We shall call PROP to the set of propositional variables and WFF to the set of well-formed formulae. Besides, apart from these symbols over the course of our paper we have introduced new ones inasmuch as we have needed it.

6.2. Abbreviation Index

We have used the following abbreviations throughout the paper:

- FOCL for *First Order Correspondence Language*.
- FOL for *First Order Logic*.
- HL for *Hybrid Logic*.
- HL^{+} for *Strong Hybrid Logic*.
- ML for *Modal Logic*.
- TL for *Temporal Logic*.

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