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EXOTIC KNOTTINGS OF SURFACES IN THE 4-SPHERE

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1. The main result.

THEOREM. *There exists an infinite series* $S_1, S_2, \ldots$ *of smooth submanifolds of* S^4 *such that*:

(1) *for any* i, j *the pairs* (S^4, S_i), (S^4, S_j) *are homeomorphic,*

(2) *for any* $i \neq j$ *the pairs* (S^4, S_i), (S^4, S_j) *are not diffeomorphic,*

(3) *each* S_n *is homeomorphic to the connected sum* $\#_{10}\mathbf{R}P^2$ *of* 10 *copies of the projective plane,*

(4) $\pi_1(S^4 \backslash S_n) = \mathbf{Z}_2$,

(5) *the normal Euler number* (*with local coefficients*) *of* S_n *in* S^4 *is* 16.

Actually we show instead of (1) a slightly stronger result, namely, that there are smooth isomorphisms φ_i of tubular neighborhoods of S_i and S_1 which can be extended to homeomorphisms of the exterior. But according to (2) $\varphi_j^{-1} \circ \varphi_i$ cannot be extended to a diffeomorphism of the exteriors. This is surprising, as there is no analogous result in other dimensions. Let N be a closed smooth submanifold of a closed manifold M of dimension $\neq 4$. Let U be a smooth tubular neighborhood. Then there are only finitely many diffeomorphism types rel. boundary of smooth manifolds X with $\partial X = \partial U$ and X homeomorphic to $M - \overset{\circ}{U}$. If $\dim M = 3$ the number of diffeomorphism types is 1 and if $\dim M \geq 5$ the number of smoothings rel. boundary (which is an upper bound for the number of diffeomorphism types) is finite by [**KS**].

In fact we describe an infinite family $F_1, F_2, \ldots$ of smooth submanifolds of S^4 satisfying conditions (2)–(5) of the Theorem, and we prove that there are only finitely many homeomorphism types of (S^4, F_n) in the sense described above.

The F_n's are obtained from a fixed smooth submanifold $F \subset S^4$ by a family of new knotting constructions. F is the obvious simplest submanifold satisfying the conditions (3), (4), and (5): the pair (S^4, F) is the connected sum of the standard pair $(S^4, \mathbf{R}P^2)$ (with normal Euler number -2) and nine copies of it with the orientation of S^4 reversed.

Our knotting constructions can be applied to "smaller" submanifolds, e.g. the Klein bottle with normal Euler number 0 and the torus, which are standardly embedded in S^4. The only thing we fail to prove in these situations is the nonexistence of diffeomorphisms.

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2. Outline of the proof. The construction of F_n is motivated by the recent work of S. Donaldson [**D**$_1$, **D**$_2$] C. Okonek and A. van de Ven [**OV**] resp. R. Friedman and J. Morgan [**FM**]. They considered the Dolgachev surfaces [**Do**], which are complex elliptic surfaces organized in families $D_{p,q}$ with $p, q \in \mathbf{N}$, $(p,q) = 1$. The $D_{p,q}$-surfaces are 1-connected and permit an elliptic fibration over the sphere $\mathbf{C}P^1$ with two multiple fibres of multiplicity p and q. Any $D_{p,q}$-surface can be obtained from some rational elliptic surface diffeomorphic to $\mathbf{C}P^2 \# 9\overline{\mathbf{C}P}^2$ (blow up 9 points of $\mathbf{C}P^2$) by logarithmic transformations [**BPV**] of multiplicity p and q along two nonsingular fibres. The rational elliptic surfaces themselves are included in this system as $D_{1,1}$. By Freedman's classification of 1-connected closed 4-manifolds [**F**$_1$], all Dolgachev surfaces are homeomorphic to $\mathbf{C}P^2 \# 9\overline{\mathbf{C}P}^2$. But Donaldson [**D**$_1$, **D**$_2$] proved that no $D_{2,3}$-surface is diffeomorphic to $\mathbf{C}P^2 \# 9\overline{\mathbf{C}P}^2$, and this was extended in [**OV**, **FM**], showing that no $D_{2,q}$-surface is diffeomorphic to a $D_{1,1}$-surface or a $D_{2,r}$-surface with odd $r \neq q$.

PROPOSITION 1. *For any p, q there exists a $D_{p,q}$-surface M which admits an antiholomorphic involution c with M/c diffeomorphic to S^4.*

In the case of $D_{1,1}$ such an involution can easily be constructed via the usual complex conjugation $c\colon \mathbf{C}P^2 \to \mathbf{C}P^2\colon (z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$. The orbit space $\mathbf{C}P^2/c$ is diffeomorphic to S^4 [**K**, **M**]. The fixed point set $\mathbf{R}P^2$ is standardly embedded in it with normal Euler number -2. For any $D_{1,1}$-surface M obtained by blowing up 9 *real* points (i.e. points in $\mathbf{R}P^2$) of $\mathbf{C}P^2$, we can extend this involution to an antiholomorphic involution c on M with M/c diffeomorphic to $S^4 (= \#_{10} S^4)$. The fixed point set is $F = \mathbf{R}P^2 \# 9\mathbf{R}P^2 = \#_{10}\mathbf{R}P^2 \hookrightarrow S^4$ with normal Euler number $-2 + 9 \cdot 2 = 16$, the standardly embedded $\#_{10}\mathbf{R}P^2$ with this normal Euler number. Proposition 1 is easily deduced from this and so is the following lemma, closely related to surgery of two-fold branched coverings considered by O. Viro [**V**] and J. M. Montesinos [**Mo**].

LEMMA (ON REAL LOGARITHMIC TRANSFORMATION). *Let $E \to B$ be an elliptic fibration commuting with antiholomorphic involutions $c\colon E \to E$ and $\sigma\colon B \to B$. Let F be a fibre with $c(F) = F$ and $F \cap \operatorname{fix}(c) \neq \varnothing$. Then there exists a logarithmic transform E' of E along F of any given multiplicity which admits an antiholomorphic involution extending $c|_{E \setminus F}$ with orbit space diffeomorphic to E/c.*

For any involution c of a $D_{p,q}$-surface M with M/c diffeomorphic to S^4, the topology of the fixed point set and its normal Euler number are determined by the topology of M, and thus $\operatorname{fix}(c)$ is again $\#_{10}\mathbf{R}P^2$ embedded in S^4 with normal Euler number 16. Under appropriate conditions we can also control the fundamental group of the complement of the fixed point set in S^4:

PROPOSITION 2. *For any odd q there exists a $D_{2,q}$-surface M and an involution c as in Proposition* 1 *with abelian $\pi_1((M/c) - \operatorname{fix}(c))$ (implying $\pi_1((M/c) - \operatorname{fix}(c)) = \mathbf{Z}_2$).*

We will indicate the proof of this proposition in the next section. For $p = 2$ and $q = 2n + 1$ let us take such M and c and denote by F_n the image of $\mathrm{fix}(c)$ under some diffeomorphism $M/c \to S^4$. Since we can get M back from F_n as a 2-fold covering of S^4 branched along F_n, the results of [**OV**, **FM**] imply that for $n \neq m$ pairs (S^4, F_n), (S^4, F_m) are not diffeomorphic. This result together with the following proposition implies our theorem.

PROPOSITION 3. *Let S be a fixed nonorientable 2-manifold and k a fixed integer. Consider pairs (S^4, S), S a smooth submanifold of S^4 with normal Euler number k and $\pi_1(S^4 \backslash S) = \mathbf{Z}_2$. Choose for ech pair a smooth isomorphism of a tubular neighborhood of S with a fixed 2-dimensional disk bundle over S with normal Euler number k and identify all boundaries of these tubular neighborhoods by them.*

Then the number of homeomorphism types rel. boundary of the complements of the tubular neighborhoods is finite.

We hope to prove that all (S^4, F_n) are homeomorphic, extending a result of T. Lawson [**L**], who showed that if $S = \mathbf{R}P^2$ there is a unique homeomorphism type of such knottings. At present, in our proof of Proposition 3, which uses the surgery method of [**Kr**] (applicable in dimension 4 by Freedman's results [$\mathbf{F}_2$]), there occur several obstructions sitting in nontrivial finite groups. We don't see an obvious reason for them to be trivial.

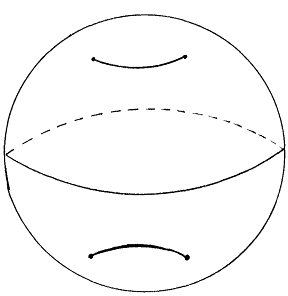

FIGURE 1

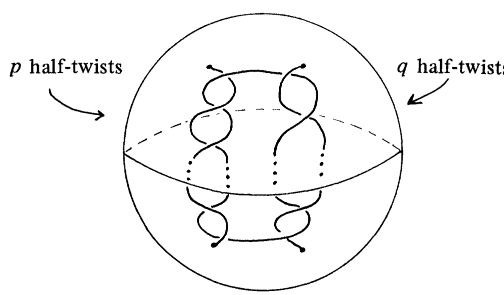

FIGURE 2

3. Knotting constructions. Let X be a smooth 4-manifold and F a smooth closed 2-submanifold of X. Let $\mathfrak{M} \subset X$ be a membrane homeomorphic to $S^1 \times I$ with $\partial\mathfrak{M} = \mathfrak{M} \cap F$ and let $\mathfrak{M}$ have index 0 or, equivalently, there exists a diffeomorphism of a regular neighborhood N of $\mathfrak{M}$ in X, $\varphi\colon N \to S^1 \times D^3$, mapping $N \cap F$ onto $S^1 \times (I \amalg I)$, and such that the segments $I \amalg I$ are embedded unknotted and unlinked into D^3 as in Figure 1. ($\amalg$ means the disjoint sum operation.) For arbitrary relatively prime p, q denoted by $K_{p,q}(F, \mathfrak{M}, \varphi)$ a new smooth submanifold of X obtained from F by replacing the embedded segments $I \amalg I \hookrightarrow D^3$ drawn in Figure 1 by the two embedded segments in Figure 2.

PROPOSITION 4. *If $F \backslash \partial\mathfrak{M}$ is connected and $\pi_1(X \backslash (F \cup \mathfrak{M}))$ is abelian, then $\pi_1(X \backslash K_{2,q}(F, \mathfrak{M}, \varphi))$ is abelian for any odd q.*

PROPOSITION 5. *Let T be a nonsingular fibre of a real (i.e., equivariant with respect to the standard complex conjugations) elliptic fibration $M \to \mathbf{C}P^1$, M a $D_{1,1}$-surface. Let T be invariant under c and intersect the fixed point set F of c in two disjoint circles. Then the 2-fold covering of $S^4 = M/c$ branched over $K_{p,q}(F, T/c, \varphi)$ for some φ is equivariantly diffeomorphic to a $D_{p,q}$-surface.*

In the situation of Proposition 5 the conditions of Proposition 4 are satisfied as can be shown by the method of [**Fi**]. Thus Propositions 4 and 5 imply Proposition 2.

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