BOTTOM OF SPECTRA AND COVERINGS OF ORBIFOLDS

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Dedicated to Shiing-shen Chern, a great mathematician and a great man

Abstract. We discuss the behaviour of the bottom of the spectrum of scalar Schrödinger operators under Riemannian coverings of orbifolds. We apply our results to geometrically finite and to conformally compact orbifolds.

Contents

1. Introduction

Spectral invariants of orbifolds are a classical issue in number theory, but have not yet attracted so much attention in Riemannian geometry. The prime examples in number theory are closed or finite volume quotients of Riemannian symmetric spaces of non-compact type with the modular surface as the most classical one. Our investigations were motivated by our article [5], in which the bottom of the spectrum of an orbifold quotient occurred in one of the applications and where the value of that number was in question (answered by Theorem A.1 below). Most of our results are known in the case of manifolds, the point of the present article is their extension to orbifolds.

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These extensions do not come for free, and even some of the fundamentals have to be prepared appropriately.

We consider a covering $p\colon O_1 \to O_0$ of Riemannian orbifolds, where O_0 is connected, a scalar Schrödinger operator $S_0 = \Delta + V_0$ on O_0 and its lift S_1 to O_1. We assume throughout that S_0 is bounded from below (on $C_c^\infty(O_0)$). The orbifold fundamental group $\Gamma_0 = \pi_1^{\mathrm{orb}}(O_0)$ of O_0 acts on the fibers of p, and we say that the covering is *amenable* if the action on fibers over regular points of O_0 is amenable. It is noteworthy that a normal covering between connected orbifolds is amenable if and only if its deck transformation group is amenable. The reader not familiar with these notions is referred to the body of the text below, where we discuss them in some detail.

For a Lipschitz function $f \neq 0$ with compact support on an orbifold O and a scalar Schrödinger operator $S = \Delta + V$ on O,

$$R_S(f) = \frac{\int_O (|\nabla f|^2 + V f^2)}{\int_O f^2} \tag{1.1}$$

is called the *Rayleigh quotient* of f (with respect to S). We then call

$$\lambda_0(S, O) = \inf R_S(f) \tag{1.2}$$

the *bottom of the spectrum of* S, where the infimum is taken over all non-zero $f \in C_c^\infty(O)$, or, equivalently, over all non-zero Lipschitz functions f on O with compact support. In the case of the Laplacian, we also write $R(f)$ and $\lambda_0(O)$ instead of $R_\Delta(f)$ and $\lambda_0(\Delta, O)$, respectively.

If S is bounded from below, that is, $\lambda_0(S, O) > -\infty$, then the Friedrichs extension $\bar{S}$ of S in $L^2(O)$ is defined and $\lambda_0(S, O)$ is equal to the bottom of the spectrum $\sigma(S, O)$ of $\bar{S}$. To state our main result, we will also need the notion of the *bottom* $\inf \sigma_{\mathrm{ess}}(S, O)$ *of the essential spectrum of* S. It is given by

$$\lambda_{\mathrm{ess}}(S, O) = \sup \lambda_0(S, O \setminus K), \tag{1.3}$$

where the supremum is taken over all compact subsets K of O.

In the case of coverings as above, we always have that

$$\lambda_0(S_1, O_1) \geq \lambda_0(S_0, O_0), \tag{1.4}$$

as we will see in Section 7. This result is known in many cases; see our survey article [4] for references.

By (1.4), S_1 is also bounded from below so that both, $\lambda_0(S_1, O_1)$ and $\lambda_0(S_0, O_0)$, realize the bottom of the spectrum of their respective Friedrichs extension.

Theorem A. *Let $p\colon O_1 \to O_0$ be a Riemannian covering of orbifolds, where O_0 is connected, and S_0 and S_1 be compatible scalar Schrödinger operators on O_0 and O_1, respectively, where S_0 is bounded from below. Then we have:*

1. *(1) if p is amenable, then $\lambda_0(S_1, O_1) = \lambda_0(S_0, O_0)$;*
2. *(2) if $\lambda_0(S_1, O_1) = \lambda_0(S_0, O_0) < \lambda_{\mathrm{ess}}(S_0, O_0)$, then p is amenable.*

The study of the behaviour of λ_0 under Riemannian coverings was initiated by Brooks [10, 11]. For connected Riemannian manifolds, Theorem A.1 and A.2 are [2, Theorem 1.2] and [21, Theorem 4.1]. (The latter is also [22, Theorem 1.2].) For further comments and references, we refer to [4].

Note that we do not assume that O_1 is connected. This is important in our proof, where the case that O_1 is–possibly–not connected occurs at an intermediate stage, but it also seems important in some applications. We do assume, however, that the orbifolds considered here are second countable so that they have at most countably many connected components.

Corollary B. *If O_0 contains a compact domain K such that the fundamental groups of the connected components of the complement $O_0 \setminus K$ are amenable, then there are the following two cases:*

(1) *if $\lambda_0(S_0, O_0) < \lambda_{\mathrm{ess}}(S_0, O_0)$, then $\lambda_0(S_1, O_1) = \lambda_0(S_0, O_0)$ if and only if p is amenable;*
(2) *if $\lambda_0(S_0, O_0) = \lambda_{\mathrm{ess}}(S_0, O_0)$, then $\lambda_0(S_1, O_1) = \lambda_0(S_0, O_0)$.*

For manifolds, Corollary B is [4, Corollary D]. Using Theorem A, the proof there extends to orbifolds.

Let O be a complete and connected Riemannian orbifold with sectional curvature $-b^2 \le K_O \le -a^2$. Then O is a quotient $\Gamma\backslash X$, where X is a simply connected Riemannian manifold and Γ a properly discontinuous group of isometries of X. For such an O, let Ω be the complement of the limit set of Γ in the sphere at infinity of X. Following Bowditch [8], we say that O and Γ are *geometrically finite* if $\Gamma\backslash(X \cup \Omega)$ has finitely many ends and each of them is parabolic (in the sense of [8, Section 5.1]).

We say that a Riemannian orbifold O is *hyperbolic* if it can be written as a quotient $\Gamma\backslash X$, where X is one of the hyperbolic spaces H^n_F with $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, endowed with its canonical Riemannian metric, which is unique up to scale. We normalize it so that $\max K_X = -1$. Then the number

$$h_X = m + \dim_{\mathbb{R}} F - 2, \tag{1.5}$$

where $m = \dim X = n \dim_{\mathbb{R}} F$, is equal to the asymptotic volume growth of X. For any hyperbolic space X and non-compact, geometrically finite orbifold $O = \Gamma\backslash X$, we have

$$\lambda_{\mathrm{ess}}(O) = \lambda_0(X) = h_X^2/4. \tag{1.6}$$

For manifolds, the lower estimate $\lambda_{\mathrm{ess}}(O) \ge h_X^2/4$ is contained in Hamenstädt [17, p. 282: Corollary], the equality is explained in J. Li [19, Remark 1.2]. It is likely that their arguments extend to the orbifold case. However, the orbifold case is also contained in [6, Theorem B], which contains corresponding estimates for more general kinds of operators over orbifolds, including the Hodge-Laplacian on differential forms.

Corollary C. *Let $p\colon O_1 \to O_0$ be a Riemannian covering of complete and connected Riemannian orbifolds. Assume that there is a non-compact and geometrically finite hyperbolic orbifold $O_0' = \Gamma\backslash X$ such that $O_0 \setminus K$ is isometric to $O_0' \setminus K'$ for some compact domains $K \subseteq O_0$ and $K' \subseteq O_0'$. Then there are the following two cases:*

(1) *if $\lambda_0(O_0) < h_X^2/4$, then $\lambda_0(O_1) = \lambda_0(O_0)$ if and only if p is amenable;*
(2) *if $\lambda_0(O_0) = h_X^2/4$, then $\lambda_0(O_1) = \lambda_0(O_0)$.*

For manifolds, Corollary C is contained in [4]. Using the characterization of $\lambda_{\mathrm{ess}}(O_0)$ in (1.3), Corollary C is an immediate consequence of Theorem A and (1.6).

Consider now a compact and connected Riemannian orbifold P with boundary $\partial P \neq \emptyset$ and a smooth non-negative function ρ on N defining ∂N, that is,

$$\partial P = \{\rho = 0\} \text{ and } \partial_\nu \rho > 0 \tag{1.7}$$

along ∂P, where ν denotes the inner unit normal of P along ∂P with respect to the given metric h on P. Let O be the interior of P, endowed with the conformally equivalent metric

$$g = \rho^{-2} h \tag{1.8}$$

on O. The metric g is complete since any point in O has infinite distance to ∂P with respect to g. Metrics of this kind were introduced by Mazzeo, who called them *conformally compact*. In [20, Theorem 1.3] he determines the essential spectrum of the Hodge-Laplacian of conformally compact manifolds. In particular, for functions he obtains that the essential spectrum of g is $[a^2(m-1)^2/4, \infty)$, where $a = \min \partial_\nu \rho > 0$ and $m = \dim O$. It seems that his arguments also go through for orbifolds. However, by an easy argument, we will obtain

$$\lambda_{\text{ess}}(\tilde{O}) \leq \lambda_{\text{ess}}(O) = (m-1)^2 a^2/4, \tag{1.9}$$

for any connected and conformally compact Riemannian orbifold of dimension m, where $a = \min \partial_\nu \rho$ as above and $\tilde{O}$ denotes the universal covering space of O. Note the converse monotonicity in (1.9) in comparison to (1.4).

Corollary D. *Let $p\colon O_1 \to O_0$ be a Riemannian covering of orbifolds of dimension m. Assume that O_0 is conformally compact with $a = \min \partial_\nu \rho$ as above. Then we have:*

(1) if $\lambda_0(O_0) < (m-1)^2 a^2/4$, then $\lambda_0(O_1) = \lambda_0(O_0)$ if and only if p is amenable;
(2) if $\lambda_0(O_0) = (m-1)^2 a^2/4$, then $\lambda_0(O_1) = \lambda_0(O_0)$.

For manifolds, Corollary D is [3, Theorem 1.10]. Corollary D is an immediate consequence of Theorems A and (1.9), using that $\lambda_0(O_1) \leq \lambda_0(\tilde{O}) \leq \lambda_{\text{ess}}(\tilde{O})$ for the second assertion.

1.1. **Structure of the paper.** In Sections 2–6, we discuss the structure of orbifolds; in particular, what we need about the geometry of Riemannian orbifolds, the analysis of Schrödinger operators on orbifolds, and coverings of orbifolds. In Section 3.3, we discuss also the Bishop-Gromov volume and Cheng eigenvalue comparison theorems. In the short Section 7, we show the monotonicity (1.4) of the bottom of the spectrum under coverings. In Sections 8–10, we present the proof of Theorem A. The proof of (1.9) is contained in Section 11.

2. Preliminaries on orbifolds

Our exposition of orbifolds follows mostly [26, Chapter 13]; further good references are [9, Section G], [16], [18, Chapter 6], [23, Chapter 13], and [25, §2].

An *(m-dimensional orbifold) chart* of a Hausdorff space O consists of an open subset U of O, a connected m-dimensional manifold $\hat{U}$ with boundary (possibly empty), a finite subgroup G of the diffeomorphism group of $\hat{U}$, and a continuous map $\pi\colon \hat{U} \to U$ such that π induces a homeomorphism $G\backslash\hat{U} \to U$. The information about such a chart is captured by the diagram

$$\begin{array}{ccc} \hat{U} & & \\ {\scriptstyle\pi}\big\downarrow & \searrow & \\ U & \xleftarrow[\cong]{} & G\backslash\hat{U} \end{array} \tag{2.1}$$

Note that $G\backslash\hat{U}$ and U are connected since $\hat{U}$ is connected.

For a chart $a = (U, \hat{U}, G, \pi)$ of O, we call U the *domain,* $\hat{U}$ the *codomain*, G the group, and π the *projection of* a.

Since G is finite, $\hat{U}$ carries a G-invariant Riemannian metric. Hence, given $x \in \hat{U}$, an element $g \in G$ is determined by its value gx and derivative $dg|_x$.

Two m-dimensional charts $(U, \hat{U}, G, \pi)$ and $(U', \hat{U}', G', \pi')$ of O are said to be *compatible* if, for any $x \in \hat{U}$ and $x' \in \hat{U}'$ with $\pi(x) = \pi'(x')$, there is a local diffeomorphism f from $\hat{U}$ to $\hat{U}'$ (a diffeomorphism between open subsets of $\hat{U}$ and $\hat{U}'$), called a *change of charts*, with $f(x) = x'$ and $\pi' f = \pi$. If the domain of f is (chosen to be) connected, then f is unique up to composition with a $g \in G'$ [9, p. 588: Exercise 1.5.1].

Assume from now on that O is endowed with an *(m-dimensional orbifold) atlas of* O, that is, a collection $\mathcal{A}$ of m-dimensional charts of O such that the domains of the charts from $\mathcal{A}$ cover O and all the charts of $\mathcal{A}$ are compatible with each other. Since any two charts, which are compatible with each chart from $\mathcal{A}$, are also compatible with each other, $\mathcal{A}$ determines a unique maximal atlas $\mathcal{S}$, a so-called *orbifold structure*, namely the atlas consisting of all charts compatible with all charts from $\mathcal{A}$. The pair $(O, \mathcal{S})$ is called an *(m-dimensional) orbifold with boundary* (possibly empty). Since $\mathcal{A}$ determines $\mathcal{S}$ uniquely, we usually view O together with $\mathcal{A}$ also as an orbifold.

Convention: We assume throughout that orbifolds with boundary are second countable, although paracompactness would be sufficient in most places. Moreover, we follow the corresponding tradition for manifolds and speak of orbifolds when assuming that their boundaries are empty.

Let O be an orbifold with boundary. Then for any $y \in O$, chart a of O as above with $y \in U$, and point $x \in \hat{U}$ with $\pi(x) = y$, the order $|G_x|$ of the stabilizer G_x of x in G does not depend on x and a and is called the *order of* y, denoted by $|y|$. A point z in O is called *regular* if it has order one, otherwise it is called *singular*. The *regular set*, that is, the set $\mathcal{R} = \mathcal{R}_O$ of regular points, is open and dense in O. The complement $\mathcal{S} = \mathcal{S}_O$ is the *singular set.*

The orbifold structure of a manifold M with boundary, given by the connected components of M, where the corresponding groups G are trivial and the maps π the identity, will be called the *trivial (orbifold) structure (of M).* In the context of orbifolds, manifolds are always endowed with the trivial structure, unless specified otherwise.

A map $f\colon O' \to O$ between orbifolds with boundary is called *smooth* if, for all charts $(U', \hat{U}', G', \pi')$ of O', $(U, \hat{U}, G, \pi)$ of O, and points $x' \in \hat{U}'$ and $x \in \hat{U}$ with $f(\pi'(x')) = \pi(x)$, there is a smooth map φ from a neighborhood of x' in $\hat{U}'$ to $\hat{U}$ such that $\pi\varphi = f\pi'$. A map $f\colon M \to O$ between a manifold M with boundary and an orbifold O with boundary is smooth if it is smooth as a map between orbifolds with boundary, where M is endowed with its trivial orbifold structure. It is clear by the definition of smoothness that the rank of a smooth map f is well defined. In particular, we can speak of regular points and regular values of f and can apply the implicit function and Sard theorems.

Using charts in an analogous way, we define tensor fields on orbifolds with boundary. For example, a *Riemannian metric* on an orbifold O with boundary consists of a family of Riemannian metrics on the codomains of the charts from an atlas of O such that the actions of the corresponding groups G on the codomains and the changes of charts are isometric. An orbifold with boundary together with a Riemannian metric is called a *Riemannian orbifold with boundary.*

2.1. **Domains in orbifolds.** A connected subset D of an orbifold O is called a *domain with smooth boundary* if the topological boundary ∂D of D admits a covering by *adapted charts* $a = (U, \hat{U}, G, \pi)$ of O, that is, $\hat{U} = (-\varepsilon, \varepsilon) \times \hat{V}$ with G acting trivially on $(-\varepsilon, \varepsilon)$,

$$\pi^{-1}(D \cap U) = (-\varepsilon, 0] \times \hat{V}, \quad \text{and} \quad \pi^{-1}(\partial D \cap U) = \{0\} \times \hat{V}. \tag{2.2}$$

The vector field $\partial/\partial r$, where r denotes the variable in $(-\varepsilon, \varepsilon)$, is invariant under G and vanishes nowhere in $\hat{U}$. This restricts the nature of points which can be boundary points of domains with smooth boundary. Clearly, any domain in O with smooth boundary is the sublevel set of a regular value of a smooth function on O. Note also that $\pi_W^{-1}(\partial D \cap W)$ is a submanifold of $\hat{W}$, for any chart $b = (W, \hat{W}, H, \pi_W)$ of O.

The adapted charts turn ∂D into an orbifold, and regular and singular set of ∂D are equal to $\mathcal{R}_{\partial D} = \partial D \cap \mathcal{R}_O$ and $\mathcal{S}_{\partial D} = \partial D \cap \mathcal{S}_O$, respectively.

3. Riemannian orbifolds

As defined above, an orbifold O with boundary together with a Riemannian metric on O is called a Riemannian orbifold with boundary.

3.1. **Distance and completeness.** Given a Riemannian orbifold O with boundary, the length of a piecewise smooth curve is defined to be the sum of the lengths of the local lifts of the curve to codomains of charts. The distance $d(x, y)$ of two points $x, y \in O$ is defined to be the infimum of the lengths of piecewise smooth curves in O joining x to y. Since O is a Hausdorff space, the distance function d is a metric on O in the standard sense if O is connected. It is then easy to see that O is an *interior metric space*, that is, $d(x, y)$ is the infimum of the lengths of rectifiable curves in O joining x to y.

In what follows, let O be a Riemannian orbifold with boundary.

Lemma 3.1. *Let $(U, \hat{U}, G, \pi)$ be a chart for O and $c\colon [a, b] \to U$ be a rectifiable curve. Then there is a lift $\sigma\colon [a, b] \to \hat{U}$ of c such that $L(\sigma|_{[s,t]}) = L(c|_{[s,t]})$, for all $a \le s, t \le b$.*

Proof. We can assume without loss of generality that c has unit speed.

For any $\varepsilon > 0$, there are a partition $a = t_0 < \cdots < t_k = b$ of $[a, b]$, points $p_i \in \hat{U}$ above $c(t_i)$, and minimizing geodesics $\sigma_i : [t_{i-1}, t_i] \to \hat{U}$ from p_{i-1} to q_i such that $\pi(q_i) = c(t_i)$ and

$$\sum L(\sigma_i) = L(c) \pm \varepsilon.$$

By rearranging the points p_i, we can assume that $q_i = p_i$. Then

$$\sigma = \sigma_1 * \cdots * \sigma_k$$

is a continuous curve in $\hat{U}$. Since the σ_i are minimizing, we have

$$L(\sigma_i) \le L(c|_{[t_{i-1}, t_i]}) \le t_i - t_{i-1}.$$

It is not hard to see that we may apply the Arzela-Ascoli theorem to a sequence of such curves, where $\varepsilon = \varepsilon_n \to 0$, to get a lift σ of c as asserted. □

The following result is an immediate consequence of Cohn-Vossen's generalization of the Hopf-Rinow Theorem [14] ([1, Section I.2]).

Theorem 3.2. *If O is connected, then the following are equivalent:*

(1) O is complete as a metric space;
(2) any minimizing geodesic $c\colon [0, 1) \to O$ can be extended to $[0, 1]$;
(3) for some $x \in O$, any minimizing geodesic $c\colon [0, 1) \to O$ with $c(0) = x$ can be extended to $[0, 1]$;
(4) bounded subsets of O are relatively compact.

Moreover, each of these properties implies that, for any pair $x, y \in O$, there is a minimizing geodesic from x to y.

The proof of the following result is close to the proof of the corresponding result for manifolds.

Proposition 3.3. *Any orbifold with boundary admits a complete Riemannian metric.*

3.2. **Riemannian measure.** Let O be a Riemannian orbifold of dimension m. Then the volume element dv of the Riemannian metric of O is well defined on the manifold $\mathcal{R}$ of regular points of O, and we define the *(m-dimensional) measure* of a Borel set B of O by

$$|B| = |B|_m = \int_{\mathcal{R}} \mathrm{dv}(z) \chi_B(z), \tag{3.4}$$

where χ_B denotes the characteristic function of B. To justify this definition, we observe that O can be covered by the domains of countably many charts of O and that, in the codomain of each chart of O, the set of singular points has measure zero. With (3.4), we obtain a positive measure dv on the σ-algebra of Borel sets of O. Furthermore, if $B \subseteq O$ is a Borel set which is contained in the domain U of a chart $(U, \hat{U}, G, \pi)$, then

$$|G||B| = \int_{\hat{U}} \mathrm{dv}(z) \chi_B(\pi(z)) = |\pi^{-1}(B)|, \tag{3.5}$$

where the right hand side denotes the Riemannian volume of $\pi^{-1}(B)$.

Suppose that D is a compact domain with smooth boundary ∂D in O as in Section 2.1. Then adapted charts turn ∂D into an orbifold, endowed with the induced Riemannian metric. Hence by (3.4), applied to ∂D, we obtain an (($m-1$)-dimensional) measure on the σ-algebra of Borel sets in ∂D,

$$|B| = |B|_{m-1} = \int_{\partial D\cap\mathcal{R}} \mathrm{dv}_{m-1}(z)\chi_B(z), \tag{3.6}$$

where dv_{m-1} denotes the induced volume element of the submanifold $\partial D\cap\mathcal{R}$ in the Riemannian manifold $\mathcal{R}$. Here we recall that $\partial D\cap\mathcal{R} = \mathcal{R}_{\partial D}$.

Let now $b = (V, \hat V, H, \pi_V)$ be an adapted chart for ∂D, $a = (U, \hat U, G, \pi_U)$ an arbitrary chart of O, $z \in U\cap V$ a singular point, $x \in \pi_U^{-1}(z)$, and $y \in \pi_V^{-1}(z)$. Then there is a local diffeomorphism φ respecting the projections from a neighborhood of x in $\hat U$ to a neighborhood of y in $\hat V$. In particular, $\varphi(x) = y$. Moreover, φ respects the order of points so that φ maps regular and singular points in $\hat U$ to respective points in $\hat V$. In particular, since the set of singular points in $\pi_V^{-1}(\partial D)$ has $(m-1)$-dimensional measure zero in $\hat V$, the same is true for the set of singular points in the submanifold $\pi_U^{-1}(\partial D)$ in $\hat U$. Therefore, if $B \subseteq \partial D$ is a Borel set which is contained in U, then

$$|G||B|_{m-1} = \int_{\pi_U^{-1}(\partial D)} \mathrm{dv}_{m-1}(z)\chi_B(\pi(z)) = |\pi_U^{-1}(B)|_{m-1}, \tag{3.7}$$

where the right hand side denotes the Riemannian volume of $\pi_U^{-1}(B)$ in the manifold $\pi_U^{-1}(\partial D)$ with the induced Riemannian metric.

Lemma 3.8 (Coarea formula)**.** *For $f \in C^\infty(O)$ and $\varphi \in C^0_c(O)$*

$$\int_O |\nabla f|\,\mathrm{dv}_m\,\varphi = \int_{\mathbb{R}}\int_{\{f=t\}} \mathrm{dt}\,\mathrm{dv}_{m-1}\,\varphi.$$

Proof. Only the regular points of f matter. In the neighborhood of such a point x, we may introduce coordinates by using the local flow (F_s) of $X = \nabla f/|\nabla f|^2$, that is, $(s,y)\sim F_s(y)$ with $y\in\{f(y)=t\}$, where $f(x)=t$. In such coordinates, we have $|\nabla f|\,\mathrm{dv}_m = \mathrm{dt}\,\mathrm{dv}_{m-1}$. □

3.3. **Cut locus and two comparison results.** Let O be a connected Riemannian orbifold (that is, orbifold with empty boundary). It is then easy to see that a curve $c\colon I\to O$, that is minimizing locally, has local lifts to codomains of charts over it and that, up to parametrization, these local lifts are geodesics in the usual sense.

Proposition 3.9 (Proposition 15 in [7])**.** *If $c\colon [a,b]\to O$ is a minimal geodesic and $c(t)\in\mathcal{S}$ for some $t\in(a,b)$, then c is contained in $\mathcal{S}$.*

Assume for the rest of this subsection that O is complete. Let $x\in O$ and c be a (non-constant) geodesic starting at x. Then the first point on c behind which c is not a minimal connection to x anymore is called the *cut point* of x along c. The set $C(x)$ of all cut points of x along geodesics from x is called the *cut locus of* x.

Proposition 3.10. *We have:*

(1) for any direction v at x, the time $t_0(v)>0$, at which the unit speed geodesic from x in the direction of v stops being minimizing, depends continuously on v;

(2) for any $y \in O \setminus C(x)$, there is a unique minimal geodesic from x to y;
(3) $C(x)$ is closed and $|C(x)|_m = 0$, where $m = \dim O$;
(4) if $x \in \mathcal{R}$, then $\mathcal{S} \subseteq C(x)$.

Proof. Mutatis mutandis, (1)–(3) follow easily from corresponding arguments in the case of Riemannian manifolds. Proposition 3.9 implies (4). □

The Bishop-Gromov volume and Cheng eigenvalue comparison theorems extend to orbifolds. Denote by $B_r(k)$ the ball of radius r in M_k^m, the model space of dimension $m = \dim O$ and constant sectional curvature k.

Theorem 3.11 (Bishop-Gromov volume comparison)**.** *Assume that* Ric $\geq (m-1)k$ *on* $B_s(x)$*. Then*

$$\frac{|B_s(x)|}{\beta(s)} \leq \frac{|B_r(x)|}{\beta(r)} \xrightarrow[r \to 0]{} \frac{1}{|x|}$$

for all $0 < r < s$*, where* $\beta(r) = |B_r(k)|$ *. Moreover, equality on the left holds for some* $0 < r < s$ *if and only if* $B_s(x)$ *is isometric to* $G \backslash B_s(k)$*, where* G *is a finite group of isometries of* M_k^m *fixing the center of* $B_s(k)$*.*

Except for the equality discussion, Theorem 3.11 is [7, Proposition 20].

Theorem 3.12 (Cheng eigenvalue comparison)**.** *Assume that* Ric $\geq (m-1)k$ *on* $B_r(x)$*. Then*

$$\lambda_0(B_r(x)) \leq \lambda_0(B_r(k)).$$

Moreover, equality holds if and only if $B_r(x)$ *is isometric to* $G \backslash B_r(k)$*, where* G *is a finite group of isometries of* M_k^m *fixing the center of* $B_r(k)$*.*

Sketch of proofs of Theorems 3.11 and 3.12. Recall that the standard proofs of Theorems 3.11 and 3.12 in the manifold case are obtained via integrating associated inequalities along radial geodesics; see [13, Theorem 1.1] for Theorem 3.12. Using Proposition 3.10, the same procedure leads to the above assertions for orbifolds. □

3.4. **Cheeger constants.** The *Cheeger constant* of a Riemannian orbifold O of dimension m is defined to be

$$h(O) = \inf_A \frac{|\partial A|_{m-1}}{|A|_m}, \tag{3.13}$$

where the infimum is taken over all compact domains A of O with smooth boundary, and where $|.|_k$ indicates k-dimensional Riemannian volume. The Cheeger constant is related with the bottom of the spectrum of the Laplacian on O via the *Cheeger inequality*,

$$\lambda_0(O) \geq \frac{1}{4} h(O)^2. \tag{3.14}$$

For the convenience of the reader, we present a short proof of (3.14). Namely, for the Rayleigh quotient $R(f)$ of a non-vanishing function $f \in C_c^\infty(O)$,

$$\begin{aligned} \int_O |\nabla f^2| &= \int_O 2|f||\nabla f| \\ &\leq 2 \left(\int_O f^2 \right)^{1/2} \left(\int_O |\nabla f|^2 \right)^{1/2} \leq 2R(f)^{1/2} \int_O f^2, \end{aligned}$$

by the Schwarz inequality. On the other hand, by Lemma 3.8, the definition of $h(O)$, and Cavalieri's principle, respectively,

$$\begin{aligned}\int_O |\nabla f^2| &= \int_0^\infty |\{f^2 = t\}|_{m-1}\,\mathrm{dt} \\ &\geq h(O)\int_0^\infty |\{f^2 \geq t\}|_m\,\mathrm{dt} = h(O)\int_O f^2.\end{aligned}$$

We conclude that $h(O)^2 \leq 4R(f)^2$. Now $\lambda_0(O)$ is the infimum of Rayleigh quotients $R(f)$ over non-vanishing $f \in C_c^\infty(O)$, hence (3.14) follows.

In analogy with (1.3), we define the *asymptotic Cheeger constant of* O by

$$h_{\mathrm{ess}}(O) = \sup h(O \setminus K), \tag{3.15}$$

where the supremum is taken over all compact subsets K of O. In (4.9), we obtain an analogue of (3.14), relating the asymptotic Cheeger constant to the bottom of the essential spectrum of O.

Recall that, for a relatively compact open domain D in a Riemannian manifold M, the Cheeger constant of D with respect to Neumann boundary condition is defined to be

$$h^N(D) = \inf \frac{|\partial A \cap D|}{|A|}, \tag{3.16}$$

where the infimum is taken over all domains $A \subseteq D$ with $|A| \leq |D|/2$ and smooth intersection $\partial A \cap D$.

Lemma 3.17 (Lemma 5.1 of Buser [12])**.** *If M is of dimension m and complete with Ricci curvature bounded from below by $1-m$, $D \subseteq M$ is starlike with respect to a point $x \in M$, and $B_r(x) \subseteq D \subseteq B_R(x)$, then*

$$h^N(D) \geq C_m^{1+R}\frac{r^{m-1}}{R^m}.$$

4. Analysis on orbifolds

Throughout this section, let O be a Riemannian orbifold. Then the Laplace operator Δ is well defined on $C^\infty(O)$.

Lemma 4.1. *For functions $f, g \in C^\infty(O)$ and a compact domain $D \subseteq O$ with smooth boundary ∂D, we have*

$$\int_D f\Delta g = \int_D \langle \nabla f, \nabla g\rangle - \int_{\partial D} f\partial_\nu g.$$

Proof. Using a smooth partition of unity on O, the proof reduces to the two cases where the support of f is contained in a coordinate domain in the interior of D or in one for the boundary of D. We only discuss the less trivial second case, where the codomain $\hat U = \hat V \times [0,\varepsilon)$ with the corresponding finite group G acting trivially on the factor $[0,\varepsilon)$. We obtain

$$\begin{aligned}|G|\int_D f\Delta g &= \int_{\hat U} \hat f\Delta\hat g \\ &= \int_{\hat U}\langle\nabla\hat f, \nabla\hat g\rangle - \int_{\hat V}\hat f\partial_\nu\hat g \\ &= |G|\int_D\langle\nabla f, \nabla g\rangle - |G|\int_{\partial D} f\partial_\nu g,\end{aligned}$$

where $\hat{f} = f \circ \pi$, $\hat{g} = g \circ \pi$ and ν denotes the exterior normal vector field of $\hat{U}$ and D, respectively. $\square$

We consider a Schrödinger operator $S = \Delta + V$ on O with smooth potential V. In view of Lemma 4.1, the operator

$$S\colon D(S) \subseteq L^2(O) \to L^2(O)$$

with domain $D(S) = C_c^\infty(O)$ is symmetric. It is evident that S is bounded from below if V is bounded from below.

Assume from now on that S is a Schrödinger operator on O that is bounded from below, and choose $\beta \in \mathbb{R}$ such that

$$\langle Sf, f\rangle_{L^2} \geq \beta \|f\|_{L^2}^2$$

for any $f \in C_c^\infty(O)$. Denote by $H_S \subseteq L^2(O)$ the completion of $C_c^\infty(O)$ with respect to the inner product

$$\langle f, h\rangle_{H_S} = \langle f, h\rangle_{L^2} + \langle (S - \beta)f, h\rangle_{L^2}.$$

Since $\|f\|_{H_S} \geq \|f\|_{L^2}$, we view $H_S \subseteq L^2(O)$. The Friedrichs extension of S is the self-adjoint operator

$$\bar{S}\colon D(\bar{S}) \subseteq L^2(O) \to L^2(O)$$

with $\bar{S} = S^*$ on its domain $D(\bar{S}) = H_S \cap D(S^*)$, where S^* denotes the adjoint of S.

We denote by $\lambda_0(S)$ the bottom of the spectrum of $\bar{S}$. The Rayleigh quotient of a non-zero $f \in H_S$ with respect to S is defined by

$$R_S(f) = \frac{\|f\|_{H_S}^2}{\|f\|_{L^2}^2} + \beta - 1. \tag{4.2}$$

It is well known from functional analysis that

$$\lambda_0(S) = \inf R_S(f), \tag{4.3}$$

where the infimum is taken over all non-zero $f \in C_c^\infty(O)$ or over all non-zero $f \in H_S$.

Lemma 4.4. *Any compactly supported Lipschitz function on O belongs to H_S.*

Proof. It suffices to prove the assertion for any compactly supported Lipschitz function f such that there exists a coordinate chart $(U, \hat{U}, G, \pi)$ with U precompact and $\operatorname{supp} f \subseteq U$. Setting $\hat{f} := f \circ \pi$, we compute

$$|\hat{f}(x) - \hat{f}(y)| = |f(\pi(x)) - f(\pi(y))| \leq Cd(\pi(x), \pi(y)) \leq Cd(x, y),$$

where $x, y \in \hat{U}$ and C is the Lipschitz constant of f. This yields that $\hat{f}$ is a Lipschitz function, and, therefore, there exists a sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\hat{U})$ with $\hat{f}_n \to \hat{f}$ in $H_0^1(\hat{U})$. It is evident that the functions

$$\hat{h}_n := \frac{1}{|G|} \sum_{g\in G} \hat{f}_n \circ g$$

descend to functions $h_n \in C_c^\infty(O)$. Using that $\hat f$ is G-invariant, we have that $\hat h_n \to \hat f$ in $H_0^1(\hat U)$ and, hence, that $h_n \to f$ in $H_0^1(U)$. Since the h_n are smooth,

$$\|h_n - h_m\|_{H_S}^2 \le (1 - \beta + C_V)\|h_n - h_m\|_{L^2}^2 + \|\nabla h_n - \nabla h_m\|_{L^2}^2,$$

where C_V is the supremum of $|V|$ on U. Therefore $h_n \to f$ in H_S, as we wished. ☐

As an immediate consequence of Lemma 4.4, we obtain the following assertion from the introduction (see (1.2)).

Corollary 4.5. *The bottom $\lambda_0(S)$ of the spectrum of $\bar S$ is given by*

$$\lambda_0(S) = \inf R_S(f),$$

where the infimum is taken over all non-zero $f \in \mathrm{Lip}_c(O)$.

Lemma 4.6. *Suppose that O is connected and that $\lambda_0(S)$ is an eigenvalue of $\bar S$. Then any eigenfunction φ of $\bar S$ corresponding to $\lambda_0(S)$ is smooth and nowhere vanishing. In particular, the eigenspace corresponding to $\lambda_0(S)$ is one-dimensional.*

Proof. We know from elliptic regularity that any eigenfunction φ is smooth. Since $\varphi \in H_S$, there exists a sequence $(f_n)_{n\in\mathbb N}$ in $C_c^\infty(O)$ with $f_n \to \varphi$ in H_S. Then the sequence $(|f_n|)_{n\in\mathbb N}$ belongs to H_S and is bounded in H_S. Hence it has a weakly convergent subsequence in H_S. On the other hand, $|f_n| \to |\varphi|$ in $L^2(O)$. Therefore, $|\varphi| \in H_S$ and $R_S(|\varphi|) = \lambda_0(S)$, which yields that $|\varphi|$ is an eigenfunction of $\bar S$ corresponding to $\lambda_0(S)$, and, in particular, that $|\varphi|$ is smooth. From the maximum principle it follows that $|\varphi|$ is positive, that is, φ is nowhere vanishing. The second assertion is a consequence of the first since the first implies that there are no L^2-perpendicular eigenfunctions corresponding to $\lambda_0(S)$. ☐

Lemma 4.7. *Suppose that O is connected and that $\lambda_0(S) < \lambda_{\mathrm{ess}}(S)$. Let $(f_n)_{n\in\mathbb N}$ be a sequence in $\mathrm{Lip}_c(O)$ with $\|f_n\|_{L^2} \to 1$ and $R_S(f_n) \to \lambda_0(S)$. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb N}$ such that $f_{n_k} \to \varphi$ in $L^2(O)$ for some eigenfunction φ of $\bar S$ corresponding to $\lambda_0(S)$.*

Proof. We know from Lemma 4.4 that any compactly supported Lipschitz function can be approximated by compactly supported smooth functions in H_S. Hence, it suffices to prove the assertion for the case where the $f_n \in C_c^\infty(O)$. Since $\lambda_0(S) < \lambda_{\mathrm{ess}}(S)$, we have that $\lambda_0(S)$ is an isolated eigenvalue of $\bar S$ of finite multiplicity. Let E be the corresponding eigenspace, and write $f_n = g_n + h_n$, where $g_n \in E$ and h_n is L^2-perpendicular to E. From our assumption, it follows that $R_S(h_n) \ge \lambda_0(S) + c$ for some $c > 0$. Since E is one-dimensional, after passing to a subsequence if necessary, we may assume that $g_n \to \varphi$ in $L^2(O)$ for some $\varphi \in E$.

Given $\varepsilon > 0$, we have that $R_S(f_n) < \lambda_0(S) + \varepsilon$ for sufficiently large $n \in \mathbb N$. Using that $\langle \bar S h_n, g_n\rangle = 0$, we compute

$$\begin{aligned}(\lambda_0(S) + c)\|h_n\|_{L^2}^2 &\le \langle \bar S h_n, h_n\rangle_{L^2} = \langle \bar S f_n, f_n\rangle_{L^2} - \langle \bar S g_n, g_n\rangle_{L^2} \\ &\le (\lambda_0(S) + \varepsilon)\|f_n\|_{L^2}^2 - \lambda_0(S)\|g_n\|_{L^2}^2 \\ &\le \lambda_0(S)\|h_n\|_{L^2}^2 + \varepsilon\|f_n\|_{L^2}^2\end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. This shows that $h_n \to 0$ and hence that $f_n \to \varphi$ in $L^2(O)$. □

Proposition 4.8. *The bottom of the essential spectrum of S is given by*

$$\lambda_{\mathrm{ess}}(S, O) = \sup \lambda_0(S, O \setminus K),$$

where the supremum is taken over all compact subsets $K \subseteq O$.

Proof. By Weyl's criterion, $\lambda \in \mathbb{R}$ belongs to the essential spectrum of $\bar{S}$ if and only if there exists a *Weyl sequence* $(f_n)_{n\in\mathbb{N}}$ for λ in the domain $D(\bar{S})$, which means that $\|f_n\|_{L^2} \to 1$, $f_n \rightharpoonup 0$, and $(\bar{S} - \lambda)f_n \to 0$ in $L^2(O)$.

Given such a sequence and a compact subset K of O, we want to show that there exists a cut off function $\chi \in C_c^\infty(O)$ such that, after passing to a subsequence if necessary, $((1 - \chi)f_n)_{n\in\mathbb{N}}$ is a Weyl sequence for λ with supports disjoint from K.

It is evident that it suffices to prove this for any compact domain K contained in a coordinate region U of a chart $(U, \hat{U}, G, \pi)$ of O. Fix $\chi \in C_c^\infty(O)$ with $\chi = 1$ in a neighborhood of K and $\operatorname{supp} \chi \subseteq U$. From elliptic estimates (on $\hat{U}$), it is not hard to see that $(\hat{\chi}\hat{f}_n)_{n\in\mathbb{N}}$ is bounded in $H^2(\hat{U})$ and that $\hat{\chi}\hat{f}_n \rightharpoonup 0$ in $L^2(\hat{U})$, where $\hat{\chi} = \chi \circ \pi$ and $\hat{f}_n = f_n \circ \pi$. After passing to a subsequence if necessary, this yields that $\hat{\chi}\hat{f}_n \rightharpoonup 0$ in $H^2(\hat{U})$. Since the supports of these functions are contained in a compact subset of $\hat{U}$, it follows from Rellich's lemma that $\hat{\chi}\hat{f}_n \to 0$ in $H^1(\hat{U})$. Therefore $\chi f_n \to 0$ and $(\bar{S} - \lambda)(\chi f_n) \to 0$ in $L^2(O)$. We conclude that $((1-\chi)f_n)_{n\in\mathbb{N}}$ is a Weyl sequence for λ with supports disjoint from K. In particular,

$$\lambda_{\mathrm{ess}}(S, O) \geq \lambda_0(S, O \setminus K).$$

To finish the proof, we may assume that $\lambda_\infty = \sup \lambda_0(S, O\setminus K)$ is finite. Now for any $\varepsilon > 0$, we may choose an exhaustion of O by compact subsets K_n and a sequence of functions $f_n \in C_c^\infty(O\setminus K_n)$ with pairwise disjoint supports such that $\|f_n\|_{L^2} = 1$ and $R_S(f_n) \leq \lambda_n + \varepsilon$, where $\lambda_n = \lambda_0(S, O \setminus K_n)$. We see that the space of functions $f \in C_c^\infty(O)$ with Rayleigh quotient at most $\lambda_\infty + 2\varepsilon$ is infinite dimensional. Hence $\lambda_{\mathrm{ess}}(S, O) \leq \lambda_\infty$. □

The asymptotic Cheeger inequality

$$\lambda_{\mathrm{ess}}(O) \geq \frac{1}{4} h_{\mathrm{ess}}(O)^2 \tag{4.9}$$

is an immediate consequence of (3.14) and Proposition 4.8.

Consider a positive $\varphi \in C^\infty(O)$ satisfying $S\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{R}$. Denote by $L^2_\varphi(O)$ the L^2-space of O with respect to the measure $\varphi^2\,\mathrm{dv}$, where dv is the measure induced from the Riemannian metric of O. It is easy to see that the isometric isomorphism $m_\varphi \colon L^2_\varphi(O) \to L^2(O)$, defined by $m_\varphi f = f\varphi$, intertwines $S - \lambda$ with the diffusion operator

$$L = m_\varphi^{-1}(S - \lambda)m_\varphi = \Delta - 2\nabla \ln \varphi.$$

The operator L is called *renormalization of S with respect to φ*. The Rayleigh quotient of a non-zero $f \in C_c^\infty(O)$ is defined by

$$R_L(f) = \frac{\langle Lf, f\rangle_{L^2_\varphi}}{\|f\|^2_{L^2_\varphi}} = \frac{\int_O |\nabla f|^2\varphi^2}{\int_O f^2\varphi^2}.$$

Lemma 4.10. *For any non-zero $f \in C_c^\infty(O)$, we have $R_L(f) = R_S(f\varphi) - \lambda$.*

5. Coverings of orbifolds

A map $p\colon O' \to O$ between orbifolds with boundary is called a *covering (of orbifolds)* if, for each chart $a = (U, \hat{U}, G, \pi)$ of O with simply connected domain U, the preimage $p^{-1}(U)$ is the disjoint union of connected open subsets U' of O', which belong to charts of O' of the form $a' = (U', \hat{U}, G', \pi')$, where $G' \subseteq G$ and G' may depend on a', such that the diagram

$$\begin{array}{ccccc} \hat{U} & \xrightarrow{\pi'} & U' & \xrightarrow{\cong} & G'\backslash\hat{U} \\ \downarrow q & & \downarrow p & & \downarrow \\ \hat{U} & \xrightarrow{\pi} & U & \xrightarrow{\cong} & G\backslash\hat{U} \end{array} \tag{5.1}$$

commutes, where q denotes the identification of the codomain $\hat{U}$ of a' with the codomain $\hat{U}$ of a and the right vertical arrow and each of the compositions of horizontal arrows denote the natural projections.

In contrast to standard coverings, the restrictions $p\colon U' \to U$ are, in general, not homeomorphisms, but correspond to the projections $G'\backslash\hat{U} \to G\backslash\hat{U}$. Nevertheless, charts a and the respective open subset U of O will be called *evenly covered* by the charts a' and respective open sets U' as above. Conversely, a' will be called a *lift of a* or said *to be above a*, and similarly for U' and U. We will also say that U' is a *local leaf of p over U*.

Note that coverings are smooth. An orbifold is said to be *good* if it admits an orbifold covering by a manifold.

Example 5.2. If $\Gamma \curvearrowright M$ is a properly discontinuous action of a countable group Γ on a manifold M via diffeomorphisms, then the space $O = \Gamma\backslash M$ of orbits is a good orbifold with M as a covering space.

A covering $\tilde{p}\colon \tilde{O} \to O$ of connected orbifolds with boundary is called *universal* if, for any covering $p\colon O' \to O$ with O' connected, and any points $x \in O$, $x' \in O'$, and $\tilde{x} \in \tilde{O}$ with $p(x') = \tilde{p}(\tilde{x}) = x$, there is a covering $p'\colon \tilde{O} \to O'$ such that $\tilde{p} = p \circ p'$ and $p'(\tilde{x}) = x'$.

Theorem 5.3 (Thurston, Proposition 13.2.4 in [26])**.** *A connected orbifold with boundary has a universal cover, and any such cover is unique up to isomorphism.*

For a covering $p\colon O' \to O$ of orbifolds with boundary, we say that a diffeomorphism τ of O' is a *deck transformation* if $p\tau = p$. We say that p is *normal*, if its group of deck transformations is transitive on the fibers of p. By definition, universal coverings of O are normal. Up to natural isomorphism, the groups of deck transformations does not depend on the universal covering and is called the *fundamental group* of O, denoted by $\pi_1^{\mathrm{orb}}(O)$.

5.1. **Riemannian coverings.**

A covering $p\colon O' \to O$ of Riemannian orbifolds with boundary is called *Riemannian* if p is a local isometry.

Lemma 5.4. *Let $p\colon O' \to O$ be a Riemannian covering of Riemannian orbifolds with boundary, and assume that O' is complete. Let $c\colon I \to O$ be a minimizing geodesic, where $I = [a, b]$ or $I = [a, b)$. Let $x' \in O'$ be a point with $p(x') = c(a)$. Then there is a lift $c'\colon I \to O'$ with $c'(a) = x'$ such that $L(c'|_{[s,t]}) = L(c|_{[s,t]})$ for all $s \le t$ in I. Any such lift c' of c is a minimizing geodesic in O'.*

Proof. We consider the case $I = [a, b]$ first. We can assume that $a = 0$ and that c has unit speed. Let A be the set of $a \in [0, b]$ such that a corresponding lift exists for $c|_{[0,a]}$. By the assumption on the length of subcurves, any such lift is of unit speed and minimizing since p does not increase distances and c is minimizing. Furthermore, A is not empty since $0 \in A$.

We show now that A is closed. Let (a_n) be an increasing sequence in A with limit a and $c_n\colon [0, a_n] \to O'$ be lifts of c as asserted. Then for $m \ge n$, c_m is also such a lift on $[0, a_n]$. Since O' is complete, we may apply the Arzela-Ascoli theorem and get a subsequence of the $c_m|_{[0,a_n]}$ which converges to a corresponding lift of c on $[0, a_n]$. We apply this argument again, but now to the subsequence and for $a_m > a_n$. We get a subsequence of the subsequence of the $c_k|_{[0,a_m]}$ which converges, and the limit will coincide with the previous limit on $[0, a_n]$. Iterating this argument, we get a lift of c on $[0, a)$. By the completeness of O' it can be extended to $[0, a]$, and hence $a \in A$. Therefore A is closed.

Suppose now that $a \in A$ with $a < b$, and let $c'\colon [0, a] \to O'$ be a lift of $c|_{[0,a]}$. Let $(U, \hat{U}, G, \pi)$ be an evenly covered chart of O with $c(a) \in U$ and $(U', \hat{U}, G', \pi')$ be the associated chart of O' with $c'(a) \in U'$. By Lemma 3.1, there is an $\varepsilon > 0$ and a lift σ to $\hat{U}$ of $c|_{[a,a+\varepsilon]}$ such that $L(\sigma) = L(c|_{[a,a+\varepsilon]})$. Evidently, there exists $g \in G$ with $(\pi' \circ g \circ \sigma)(a) = c'(a)$. Extending c' to $[a, a + \varepsilon]$ by $\pi' \circ g \circ \sigma$, we obtain a lift c' of c on $[0, a + \varepsilon]$ such that $L(c') = L(c|_{[0,a+\varepsilon]})$. As we observed above, c' is minimizing since p does not increase distances and c is minimizing.

The assertion in the case where $I = [a, b)$ is now by reduction to the case $I = [a, b_n]$, where the sequence of b_n increases to b, and applying the Arzela-Ascoli theorem as above. □

The next result is an immediate consequence of Theorem 3.2.2.

Proposition 5.5. *For a covering $p\colon O' \to O$ of connected orbifolds with boundary, a Riemannian metric on O is complete if and only if the lifted Riemannian metric on O' is complete.*

Proof. Suppose first that O' is complete, and let $c\colon [0, 1) \to O$ be a minimizing geodesic. Let $c'\colon [0, 1) \to O'$ as in Lemma 5.4. Then Theorem 3.2.2 implies that c' can be extended to $[0, 1]$, hence the composition of the extension with p is an extension of c to $[0, 1]$. Hence O is complete.

Suppose now that O is complete and let $c'\colon [0, 1) \to O'$ be a minimizing geodesic. Then $c = p \circ c'$ can be extended to $[0, 1]$, by the completeness of O. Then using an evenly covered chart about $c(1)$ and the leaf above it containing the image of $c'|_{(1-\varepsilon,1)}$ for some $\varepsilon > 0$, we see that c' can be extended to $[0, 1]$. Hence O' is complete. □

Proposition 5.6. *Let $p\colon O' \to O$ be a Riemannian covering of complete and connected Riemannian orbifolds with boundary. Let $x_0, x_1 \in O$ and $y_0 \in p^{-1}(x_0)$. Then there is a point $y_1 \in p^{-1}(x_1)$ such that $d(y_0, y_1) = d(x_0, x_1)$.*

Note that $d(y_0, y_1) \geq d(x_0, x_1)$ for all $y_1 \in p^{-1}(x_1)$ since p does not increase distances.

Proof of Proposition 5.6. Let $c\colon [0,1] \to O_0$ be a minimizing geodesic from x_0 to x_1 and c' be a lift of c to O' with $c'(0) = y_0$. Then c' is minimizing and $p(c'(1)) = x_1$. Hence $y_1 = c'(1) \in p^{-1}(x_1)$ and $d(y_0, y_1) = d(x_0, x_1)$. $\square$

5.2. **Dirichlet domains.** Consider now a Riemannian covering $p\colon O' \to O$ of complete Riemannian orbifolds without boundary, where O is connected.

Lemma 5.7. *If $N \subseteq O$ has measure zero, then also $p^{-1}(N) \subseteq O'$.*

Fix $x \in O$. For $y \in p^{-1}(x)$, the *Dirichlet domain* of p centered at y is defined to be

$$D_y = \{z \in O' \mid d(z, y) \leq d(z, y') \text{ for any } y' \in p^{-1}(x)\}. \tag{5.8}$$

By Proposition 5.6, $d(y, z) = d(x, p(z))$ for any $z \in D_y$.

Proposition 5.9. *If $x \in \mathcal{R}_O$ and $y \in p^{-1}(x)$, then*

(1) $y \in \mathcal{R}_{O'}$;
(2) $\partial D_y = \{z \in O' \mid d(z, y) = d(z, y') \text{ for some } y' \neq y \text{ in } p^{-1}(x)\}$;
(3) $\partial D_y \subseteq p^{-1}(C(x))$ and $\mathrm{int}(D_y) \subseteq \mathcal{R}_{O'}$;
(4) $|D_y \cap p^{-1}(C(x))|_m = 0$;
(5) $p\colon D_y \setminus p^{-1}(C(x)) \to O \setminus C(x)$ is an isometry;
(6) for any integrable function f on O, $f \circ p$ is integrable on D_y and

$$\int_{D_y} f \circ p = \int_O f.$$

Proof. (1) is clear from the definition of coverings of orbifolds.

(2) Let $z \in D_y$ be such that there is a point $y' \neq y$ in $p^{-1}(x)$ with $d(z, y') = d(z, y)$. Now there is a minimal geodesic from y' to z, by the completeness of O_1. Hence any neighborhood of z contains points which are strictly closer to y' than to y. This shows that the given set belongs to ∂D_y. The converse direction is obvious.

(3) Let $z \in \partial D_y$ and $y' \in p^{-1}(x)$ be a point with $d(z, y') = d(z, y)$. If $p(z)$ is singular, then we know from Proposition 3.10 that $p(z) \in C(x)$. If $p(z)$ is regular, consider minimizing geodesics from y to z and y' to z. Since their velocity vectors at z are different, their projections to O are two different geodesics from x to $p(z)$. By Proposition 5.6, both are minimal, and hence $p(z) \in C(x)$. The second assertion follows immediately from Proposition 3.9 since $y \in \mathcal{R}_{O'}$.

(4) is clear from Proposition 3.10 and Lemma 5.7.

(5) We only need to check that $p\colon D_y \setminus p^{-1}(C(x)) \to O \setminus C(x)$ is bijective. Let $z \neq z'$ be points in D_y, and suppose that $p(z) = p(z')$. Then p maps minimal geodesics from y to z and z' to different geodesics from x to $u = p(z) = p(z')$. By Proposition 5.6, they are minimal, and hence $u \in C(x)$. Therefore p is injective on $D_y \setminus p^{-1}(C(x))$.

Given $u \in O$, let $c\colon [0,1] \to O$ be a minimizing geodesic from x to u. Then the lift c' of c starting at y is a minimizing geodesic. Since p does not increase distances, it follows that $d(y, c'(1)) = d(x,u) \leq d(y', c'(1))$ for any $y' \in p^{-1}(x)$, which means that $c'(1) \in D_y$. We conclude that $p\colon D_y \to O$ is surjective, and so is $p\colon D_y \setminus p^{-1}(C(x)) \to O \setminus C(x)$.

(6) is clear from (4) and (5). $\square$

5.3. **Action of the fundamental group on the fiber.** Let $p\colon O_1 \to O_0$ be a Riemannian covering of orbifolds with boundary, where O_0 is connected. Consider a (connected) component O_1' of O_1. Then the universal covering $p_0\colon \tilde{O} \to O_0$ of O_0 factors through O_1',

$$\begin{array}{ccc} \tilde{O} & \xrightarrow{p_1'} & O_1' \\ & \searrow^{p_0} & \downarrow p \\ & & O_0 \end{array} \tag{5.10}$$

and p_1' is the universal covering of O_1'. In general, the covering p_1' is not unique, but we fix a choice for each component O_1' of O_1.

Recall that the fundamental group $\Gamma_0 = \pi_1^{\mathrm{orb}}(O_0)$ of O_0 is defined to be the group of deck transformation of p_0 and that Γ_0 is transitive on the fibers of p_0 and simply transitive on the fibers over regular points of O_0. The corresponding statements hold for p_1', where we denote the group of deck transformations of p_1' by Γ_1'. Since $p_0 = p \circ p_1'$, we have $\Gamma_1' \subseteq \Gamma_0$.

For $x_0 \in O_0$, choose $\tilde{x} \in \tilde{O}$ with $p_0(\tilde{x}) = x_0$ and, for each connected component O_1' of O_1, set $x_1' = p_1'(\tilde{x}) \in O_1'$. Then $p(x_1') = x_0$ and the part of the fiber of p over x_0 in O_1' is given by

$$p^{-1}(x_0) \cap O_1' = p_1'(\Gamma_0 \tilde{x}). \tag{5.11}$$

If x_0 is regular, we obtain a right action of Γ_0 on $p^{-1}(x_0)$ by setting

$$yg = p_1'(hg\tilde{x}) \quad \text{for} \quad y = p_1'(h\tilde{x}). \tag{5.12}$$

Identifying $p_1'(h\tilde{x})$ with $\Gamma_1' h$, this action corresponds to the right action of Γ_0 on $\Gamma_1' \backslash \Gamma_0$. Clearly, the action on the fiber depends on the choice of the point $\tilde{x}$ over x_0.

We fix a complete background metric on O_0 and its lift to O_1 and consider distances and geodesics with respect to these metrics.

Fix $x_0 \in \mathcal{R}_0$ and choose $\tilde{x} \in \tilde{O}$ and x_1' in the components O_1' of O_1 as above. For $r > 0$, set

$$G_r = \{ g \in \Gamma_0 \mid d(g\tilde{x}, \tilde{x}) < r \} \quad \text{and} \quad N(r) = |G_r|. \tag{5.13}$$

Note that $G_r = G_r^{-1}$. For any $y_1, y_2 \in p^{-1}(x_0)$,

$$d(y_1, y_2) < r \iff y_2 = y_1 g \text{ for some } g \in G_r, \tag{5.14}$$

by Proposition 5.6. For any $\tilde{y} \in \tilde{O}$, we have

$$|\{ g \in \Gamma_0 \mid d(g\tilde{x}, \tilde{y}) < r \}| \leq N(2r), \tag{5.15}$$

by the triangle inequality.

Lemma 5.16. *For any $r > 0$ and $y_1 \in O_1$, we have*

$$|p^{-1}(x_0) \cap B_r(y_1)| \le N(2r).$$

Proof. Suppose that $y_1 \in B_r(x_1' g_i)$, $1 \le i \le n$, where the $[g_i]$ are pairwise different in $\Gamma_1 \backslash \Gamma_0$. Then there exist $h_i \in \Gamma_1$ such that

$$d(h_i g_i \tilde{x}, \tilde{y}) = d(x_1' g_i, y_1) < r,$$

by Proposition 5.6, where $\tilde{y}$ is any point in $p_1^{-1}(y_1)$. Since the $h_i g_i$ are pairwise different, we get $n \le N(2r)$, by (5.15). □

6. Amenability of actions

Consider a right action of a countable group Γ on a countable set X. The action is called *amenable* if there exists an invariant mean on $\ell^\infty(X)$; that is, a linear map $\mu\colon \ell^\infty(X) \to \mathbb{R}$ such that

$$\inf f \le \mu(f) \le \sup f \text{ and } \mu(g^* f) = \mu(f)$$

for any $f \in \ell^\infty(X)$ and any $g \in \Gamma$. The group Γ is called *amenable* if the right action of Γ on itself is amenable.

Clearly, any (right) action of Γ on any finite set is amenable. Furthermore, an action of Γ on a countable set X is amenable if its restriction to a non-empty invariant subset of X is amenable.

Amenability refers to some kind of asymptotic smallness of X with respect to the action of Γ. This is made precise by the following characterization, due to Følner in the case of groups [15, Main Theorem and Remark] and then extended to actions by Rosenblatt [24, Theorems 4.4 and 4.9]. Given a finite $G \subseteq \Gamma$ and an $\varepsilon > 0$, a Følner set F for G and ε is a non-empty, finite subset of X satisfying $|Fg \setminus F| < \varepsilon |F|$ for any $g \in G$.

Theorem 6.1. *The action of Γ on X is amenable if and only if, for any finite $G \subseteq \Gamma$ and $\varepsilon > 0$, there exists a Følner set for G and ε.*

In particular, it follows that the action of Γ on X is amenable if and only if the restriction to any finitely generated subgroup of Γ is amenable.

Let $p\colon O_1 \to O_0$ be a covering of orbifolds with boundary, where O_0 is connected, but O_1 possibly not. As explained in Section 5.3, for any connected component O_1' of O_0, we have for the fundamental groups that $\Gamma_1' \subseteq \Gamma_0$. The covering p is called *amenable* if the right action of Γ_0 on the disjoint union of the $\Gamma_1' \backslash \Gamma_0$ is amenable, where union is taken over all connected components O_1' of O_1. After fixing a regular $x_0 \in O_0$, an $\tilde{x} \in \tilde{O}$ over x_0, and a covering $p_1'\colon \tilde{O} \to O_1'$ for any connected component O_1' of O_1, the aforementioned action coincides with the action of Γ_0 on $p^{-1}(x_0)$, defined in (5.12). Therefore, the covering p is amenable if and only if the latter action is amenable.

Example 6.2. For any covering $p\colon O_1 \to O_0$ (where O_0 is connected),

$$p \sqcup \mathrm{id}\colon O_1 \sqcup O_0 \to O_0$$

is an amenable covering.

Recall that Følner's condition allows us to characterize amenability of an action of a group Γ in terms of the restriction of the action to finitely generated subgroups of Γ. In the context of coverings, this is reflected by the following characterization of amenability.

Proposition 6.3. *Let $p\colon O_1 \to O_0$ be a covering with O_0 connected, and $K_1 \subseteq K_2 \subseteq \cdots$ be an exhaustion of O_0 by compact domains with smooth boundary. Then p is amenable if and only if the restrictions $p\colon p^{-1}(K_n) \to K_n$ of p are amenable.*

Proof. Endow O_0 with a complete Riemannian metric and fix a regular point x_0 in the interior of K_1. Denote by $p_0\colon \tilde{O} \to O_0$ and $p_n\colon \tilde{K}_n \to K_n$ the universal coverings, and choose $\tilde{x} \in p_0^{-1}(x_0)$ and $x_n \in p_n^{-1}(x_0)$. Given $r > 0$, consider the finite set G_r defined in (5.13) and the finite sets

$$G_{r,n} := \{g \in \pi_1^{\mathrm{orb}}(K_n) \mid d(gx_n, x_n) < r\}.$$

Assume first that p is amenable and let $n \in \mathbb{N}$. Given $\varepsilon > 0$ and a finite subset G' of $\pi_1^{\mathrm{orb}}(K_n)$, there exists $r > 0$ such that $G \subseteq G_{r,n}$. Let F be a Følner set for G_r and $\varepsilon/|G_r|$. It follows from (5.14) that $d_{p^{-1}(K_n)}(y, yg') < r$ for any $y \in p^{-1}(x_0)$ and $g' \in G_{r,n}$. In particular, $d_{O_1}(y, yg') < r$ and (5.14) yields that there exists $g \in G_r$ with $yg' = yg$. Therefore, for any $g' \in G_{r,n}$, we have that Fg' is contained in the union of Fg with $g \in G_r$, and in particular,

$$|Fg' \setminus F| \leq \sum_{g \in G_r} |Fg \setminus F| < \varepsilon,$$

which yields that the covering $p\colon p^{-1}(K_n) \to K_n$ is amenable.

Conversely, consider $\varepsilon > 0$ and a finite subset G of $\pi_1^{\mathrm{orb}}(O_0)$. Then there exists $r > 0$ such that $G \subseteq G_r$ and $n \in \mathbb{N}$ such that $B(x_0, r) \subseteq K_n$. Since $p\colon p^{-1}(K_n) \to K_n$ is amenable, there exists a Følner set F for $G_{r,n}$ and $\varepsilon/|G_{r,n}|$. For $y \in F$ and $g \in G_r$, we obtain from (5.14) that $d_{O_1}(yg, y) < r$. Since $B(x_0, r) \subseteq K_n$, it follows that any minimizing geodesic from y to yg lies in $p^{-1}(K_n)$, and hence, $d_{p^{-1}(K_n)}(yg, y) < r$. In view of (5.14), we obtain that there exists $g' \in G_{r,n}$ such that $yg = yg'$, which means that Fg is contained in the union of Fg' with $g' \in G_{r,n}$. We conclude that F is a Følner set for G and ε, which yields that p is amenable. ☐

Proposition 6.3 illustrates the importance of considering non-connected covering spaces. Namely, the preimage $p^{-1}(K)$ of a compact domain K in O_0 with smooth boundary may not be connected even if O_1 is.

7. Monotonicity of λ_0

In our standard setup of a Riemannian covering $p\colon O_1 \to O_0$ of Riemannian orbifolds with compatible Schrödinger operators $S_1 = \Delta + V_1$ and $S_0 = \Delta + V_0$, respectively, let $f \in C_c^\infty(O_1)$, and define a function $f_0 \geq 0$ on O_0 by

$$f_0^2(x) = \sum_{y \in p^{-1}(x)} \frac{|x|}{|y|} f^2(y) \tag{7.1}$$

on O_0. We call f_0 the *pushdown* of f.

Lemma 7.2. *Given a non-zero $f \in C_c^\infty(O_1)$, its pushdown f_0 is a Lipschitz function on O_0 with compact support such that f_0^2 is smooth and*

$$R_{S_0}(f_0) \le R_{S_1}(f).$$

Since $\lambda_0(S_0, O_0)$ and $\lambda_0(S_1, O_1)$ are the infimum of Rayleigh quotients of non-vanishing compactly supported Lipschitz or smooth functions on O_0 and O_1 (either way), (1.4) is an immediate consequence of Lemma 7.2.

Proof of Lemma 7.2. Let $U = \hat{U}/G$ be an evenly covered domain of O_0. Denote by $V_j \cong \hat{U}/G_j$, $j \in J$, the connected components of $p^{-1}(U)$, by p_j the restriction $p|_{V_j}$, and by $\pi\colon \hat{U} \to U$ and $\pi_j\colon \hat{U} \to V_j$ the projections. Given $x \in U$ and $u \in \pi^{-1}(x)$,

$$\begin{aligned}
(f_0^2 \circ \pi)(u) = f_0^2(x) &= \sum_{j\in J} \sum_{y \in p_j^{-1}(x)} \frac{|x|}{|y|} f^2(y) \\
&= \sum_{j\in J} \sum_{y \in p_j^{-1}(x)} \frac{|x|}{|y|} \frac{1}{|\pi_j^{-1}(y)|} \sum_{v \in \pi_j^{-1}(y)} (f^2 \circ \pi_j)(v) \\
&= \sum_{j\in J} \frac{|x|}{|G_j|} \sum_{v \in \pi^{-1}(x)} (f^2 \circ \pi_j)(v) \\
&= \sum_{j\in J} \frac{1}{|G_j|} \sum_{g\in G} (f^2 \circ \pi_j)(gu),
\end{aligned}$$

where we use that $p_j \pi_j = \pi$ and $|y||\pi_j^{-1}(y)| = |G_j|$ for the penultimate equality and that $|x||\pi^{-1}(x)| = |G|$ for the last. It follows that $f_0^2 \in C_c^\infty(O_0)$.

For $h \in C^\infty(O_0)$, let $h_1 = h \circ \pi$ be its lift to O_1. Using that the set $\mathcal{R}_0$ of regular points of O_0 and its preimage in O_1 are of full measure and that the preimage is contained in the set $\mathcal{R}_1$ of regular points of O_1, we get

$$\int_{O_0} h f_0^2 = \int_{\mathcal{R}_0} h f_0^2 = \int_{\mathcal{R}_0} \Sigma_{y \in p^{-1}(x)} h_1(y) f^2(y) = \int_{O_1} h_1 f^2.$$

Therefore

$$\|f_0\|_{L^2(O_0)} = \|f\|_{L^2(O_1)} \quad \text{and} \quad \int_{O_0} V_0 f_0^2 = \int_{O_1} V_1 f^2.$$

Moreover

$$f_0^2(x) = \sum_{y \in p^{-1}(x)} f^2(y)$$

on $\mathcal{R}_0$, hence

$$|\nabla f_0(x)|^2 \le \sum_{y \in p^{-1}(x)} |\nabla f(y)|^2$$

on $\mathcal{R}_0 \cap \{f_0^2 \ne 0\}$. This shows that f_0 is non-negative on O_0, and smooth with bounded gradient on $\{f_0^2 \ne 0\}$, and thus, f_0 is Lipschitz. Furthermore, it follows that

$$\int_{O_0} |\nabla f_0|^2 \le \int_{O_1} |\nabla f|^2.$$

In conclusion, $R_{S_0}(f_0) \leq R_{S_1}(f)$. □

8. Stability of λ_0 for amenable coverings

The aim of the section is the proof of Theorem A.1. To that end, fix a complete background metric on O_0 and consider its lift to O_1. In what follows, distances and geodesics are taken with respect to the given complete background metrics. However, gradients, Laplace operators, volumes, and integrals are taken with respect to the original metrics, since the main issue of our discussion are Rayleigh quotients with respect to the original metrics. Fix a regular point $x_0 \in O_0$, a point in the universal covering space above it, and recall the definition of $G_r \subseteq \Gamma_0$ and $N(r)$ from (5.13).

For $r > 0$ and $y \in p^{-1}(x_0)$, consider the function ψ_y on O_1 defined by

$$\psi_y(z) = \begin{cases} 1 & \text{if } d(z,y) \leq r, \\ r+1-d(z,y) & \text{if } r \leq d(z,y) \leq r+1, \\ 0 & \text{if } d(z,y) \geq r+1, \end{cases}$$

a Lipschitz function with Lipschitz constant 1. For any $z \in O_1$, there are at most $N(2r+3)$ points $y \in p_0^{-1}(y_0)$ with $z \in \operatorname{supp}\psi_y$, by Lemma 5.16. Hence the function

$$\psi_1 = \max\left\{0, 1 - \sum_{y \in p^{-1}(x)} \psi_y\right\}$$

on O_1 is well-defined and admits $N(2r+3)$ as a Lipschitz constant. Thus we obtain a partition of unity on O_1 consisting of

$$\varphi_1 = \frac{\psi_1}{\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y} \quad \text{and the} \quad \varphi_y = \frac{\psi_y}{\psi_1 + \sum_{y \in p^{-1}(x)} \psi_y} \tag{8.1}$$

with $y \in p^{-1}(x)$, the *partition of unity corresponding to* $r > 0$. Clearly $\operatorname{supp}\varphi_y = \operatorname{supp}\psi_y$ and $\sum_{y \in p^{-1}(x)} \varphi_y = 1$ in $B_r(y)$ for any $y \in p^{-1}(x)$.

Lemma 8.2. *The functions φ_y admit Lipschitz constant $3N(2r+3)$, the function φ_1 admits Lipschitz constant $9N(2r+3)^2$.*

Proof. The numerator of φ_y in (8.1) takes values in $[0,1]$ and admits 1 as a Lipschitz constant, the denominator takes values in $[1, N(2r+3)]$ and admits $2N(2r+3)$ as a Lipschitz constant. The first assertion follows now from an easy calculation. Since $\varphi_1 = 1 - \sum \varphi_y$, the second is an immediate consequence. □

For a finite subset $P \subseteq p^{-1}(x_0)$ consider the non-negative function

$$\chi = \sum_{y \in P} \varphi_y \tag{8.3}$$

and the sets

$$\begin{aligned} Q_+ &= \{y \in p^{-1}(x) \mid \chi = 1 \text{ in } B_r(y)\}, \\ Q_- &= \{y \in p^{-1}(x) \mid 0 < \chi(z) < 1 \text{ for some } z \in B_r(y)\}, \end{aligned} \tag{8.4}$$

In virtue of Lemma 8.2, we obtain that χ admits $3N(2r+3)^2$ as a Lipschitz constant.

Proposition 8.5. *Suppose that $p\colon O_1 \to O_0$ is amenable, and let $\varepsilon > 0$. Then there exists a finite subset $P \subseteq p^{-1}(x_0)$ such that $|Q_-| < \varepsilon|Q_+|$.*

Proof. Since the right action of Γ_0 on $p^{-1}(x_0)$ is amenable, there exists a finite subset $P \subseteq p^{-1}(x_0)$ such that $|Pg \setminus P| < \varepsilon|P|$ for all $g \in G_{2r+2}$.

Let $y \in Q_-$ and $z \in B_r(y)$ such that $0 < \chi(z) < 1$. Since

$$\sum_{y\in p^{-1}(x_0)} \varphi_y(z) = 1,$$

it follows that there is $y_0 \in P$ and $y_1 \in p^{-1}(x_0) \setminus P$, such that $\varphi_{y_i}(z) > 0$, $i = 0, 1$. This yields that $d(y_i, z) < r+1$, and, in particular, that $d(y_0, y_1) < 2r + 2$. In view of (5.14), we obtain that there exists $g \in G_{2r+2}$ such that $y_1 = y_0g$, which shows that $y_1 \in Pg \setminus P$, for some $g \in G_{2r+2}$. Hence, there exist at most $\varepsilon N(2r + 2)|P|$ such y_1. Since $d(y, y_1) < 2r + 1$, it follows from Lemma 5.16 that, for any such y_1, there exist at most $N(4r + 2)$ such y. Therefore, we obtain that

$$|Q_-| \le \varepsilon N(4r + 2)N(2r + 2)|P| \le \varepsilon N(4r + 2)N(2r + 2)|Q_- \cup Q_+|,$$

where we use that $P \subseteq Q_- \cup Q_+$. Proposition 8.5 is an immediate consequence of this inequality. □

Proof of Theorem A.1. Let $f \in C_c^\infty(O_0)$, $f \neq 0$, and $f_1 = f \circ p$ be the lift of f to O_1. Choose $x_0 \in \mathcal{R}_0$ and $r > 0$ such that $\operatorname{supp} f \subseteq B_r(x_0)$. Then f_1 has support in the neighborhood $U_r(p^{-1}(x_0))$ of radius r about $p^{-1}(x_0)$.

Consider the partition of unity on O_1 corresponding to r as in (8.1). Given $\varepsilon > 0$, choose the finite set $P \subseteq p^{-1}(x_0)$ according to Proposition 8.5 with χ as in (8.3) and Q_- and Q_+ as in (8.4). Then χ has compact support contained in the closed neighborhood $N_{r+1}(P)$ of radius $r + 1$ about P, and admits $L_\chi = 3N(2r + 3)^2$ as a Lipschitz constant.

Since $\operatorname{supp} f \subseteq B_r(x_0)$, it is easy to see that $\operatorname{supp} f_1 \cap D_y \subseteq B_r(y)$ for any $y \in p^{-1}(x_0)$. This yields that $\operatorname{supp}(\chi f)$ is contained in the union of D_y with $y \in Q = Q_+ \cup Q_-$.

We want to extimate the Rayleigh quotient $R_{S_1}(\chi f_1)$. Since the intersection of different D_y's is of measure zero, we compute

$$\begin{aligned} R_{S_1}(\chi f_1) = \frac{\int_{O_1} \chi f_1 S_1(\chi f_1)}{\int_{O_1}(\chi f_1)^2} &= \frac{\int_{O_1}\{|\nabla(\chi f_1)|^2 + \chi f_1 V_1 \chi f_1\}}{\int_{O_1} \chi^2 f_1^2} \\ &= \frac{\sum_{y\in Q}\int_{D_y}\{|\nabla(\chi f_1)|^2 + V_1\chi^2 f_1^2\}}{\sum_{y\in Q}\int_{D_y}\chi^2 f_1^2} \\ &\le \frac{\sum_{y\in Q}\int_{D_y}\{|\nabla(\chi f_1)|^2 + V_1\chi^2 f_1^2\}}{\sum_{y\in Q_+}\int_{D_y}\chi^2 f_1^2}. \end{aligned}$$

We now estimate the terms arising from the right hand side. Since $\chi = 1$ on $B_y(y)$ for any $y \in Q_+$, we have that $\chi f_1 = f_1$ in a neighborhood of $\operatorname{supp}(\chi f_1) \cap D_y$. This, together with Proposition 5.9.6, yields that

$$\int_{D_y} \chi^2 f_1^2 = \int_{O_0} f^2 \text{ and } \int_{D_y}\{|\nabla(\chi f_1)|^2 + V_1\chi^2 f_1^2\} = \int_{O_0}\{|\nabla f|^2 + V f^2\}$$

for any $y \in Q_+$. Denote by L_χ and L_f the respective Lipschitz constants of χ and f and by C_f and C_V the respective maximum of $|f|$ and of $|V|$ on $\operatorname{supp} f$. It is easy to see that (at any point of O_1)

$$\begin{aligned}|\nabla(\chi f_1)|^2 + V_1\chi^2 f_1^2 &\leq 2\chi_1^2|\nabla f_1|^2 + 2f_1^2|\nabla\chi|^2 + |V_1|\chi^2 f_1^2\\ &\leq 2L_f^2 + 2C_f^2 L_\chi^2 + C_V C_f^2 =: C,\end{aligned}$$

where we use that $0 \leq \chi \leq 1$. Therefore, using again Proposition 5.9.6, we obtain that

$$\int_{D_y} \{|\nabla(\chi f_1)|^2 + V_1\chi^2 f_1^2\} \leq C|\operatorname{supp} f|$$

for any $y \in Q_-$. From the above estimates, we conclude that

$$R_{S_1}(\chi f_1) \leq R_{S_0}(f) + \frac{C|\operatorname{supp} f|}{\int_{O_0} f^2}\frac{|Q_-|}{|Q_+|} < R_{S_0}(f) + \frac{C|\operatorname{supp} f|}{\int_{O_0} f^2}\varepsilon.$$

This shows that, for any $\delta > 0$, we have $R_{S_1}(\chi f_1) = R_{S_0}(f) + \delta$ if ε is chosen sufficiently small. The proof is completed by (1.2). □

9. Stability implies amenability: the case of a closed base

Let $p\colon O_1 \to O_0$ be a Riemannian covering with O_0 closed (that is, compact and without boundary) and connected, and O_1 possibly non-connected. The aim of this section is to prove the following:

Theorem 9.1. *If $\lambda_0(O_1) = 0$, then p is amenable.*

We have the right action of $\pi_1^{\mathrm{orb}}(O_0)$ on the right cosets of the fundamental group of any connected component of O_1 in $\pi_1^{\mathrm{orb}}(O_0)$, and, by definition, amenability of p means that this action on the disjoint union of these cosets is amenable. Recall that this action is equivalent to the action on the fiber of p, presented in Section 5.3. After fixing a regular point $x_0 \in O_0$ and a point in the universal covering space above it as in Section 5.3, for any $y_1, y_2 \in p^{-1}(x_0)$, we have that $d(y_1, y_2) < r$ if and only if $y_2 = y_1 g$ for some $g \in G_r$.

Cover O_0 with finitely many evenly covered, coordinate systems $\pi_i\colon \hat{U}_i \to U_i = \hat{U}_i/G_i$, which are extensible; that is, each π_i can be extended to an evenly covered, coordinate system $\pi_i\colon \hat{V}_i \to V_i$, where $\hat{V}_i \subseteq \mathbb{R}^n$ is bounded and the closure of $\hat{U}_i$ contained in $\hat{V}_i$. Since O_0 is compact, there exists $r_0 > 0$ such that for any $x \in O_0$, we have that $B_{3r_0}(x)$ is contained in some U_i. It should be noticed that for any $y \in O_1$ there exists a lifted coordinate system $\pi_{ij}\colon \hat{U}_i \to V_{ij} = \hat{U}_i/G_{ij}$ such that $B_{3r_0}(y) \subseteq V_{ij}$.

Since O_0 is closed, it has Ricci curvature bounded from below, by $1 - m$, say, and so does O_1.

Lemma 9.2. *There exists a constant $C > 0$ such that, for any i, $x \in U_i$ and $\hat{x} \in \pi_i^{-1}(x)$, we have that*

$$h^N(B_{r_0}(\hat{x})) \geq C \text{ and } h^N(B_{2r_0}(\hat{x})) \geq C.$$

Proof. We may extend the Riemannian metrics on the $\hat{U}_i$ to complete Riemannian metrics on $\mathbb{R}^n$ such that there Ricci curvature is bounded from below. Then we can apply Lemma 3.17 to arrive at Lemma 9.2. □

Recall that a subset X of an orbifold O is called a complete $2r$-separated subset if X is a maximal subset with the property $d(x,y) \geq 2r$ for any $x \neq y$ in X. It is clear that if X is a complete $2r$-separated subset of O, then the balls $B_{2r}(x)$ with $x \in X$ cover O.

Corollary 9.3. *There exists $C(r_0) > 0$ such that for any complete $2r_0$-package X of O_1, we have that any $x \in O_1$ belongs to at most $C(r_0)$ of the balls $B_{2r_0}(y)$ with $y \in X$.*

Proof. Given a complete $2r_0$-package X and $x \in O_1$, set $E_x := \{y \in X : x \in B_{2r_0}(y)\}$. It is evident that the disjoint balls $B_{r_0}(y)$ with $y \in E_x$, are contained in $B_{3r_0}(x)$. It should be observed that $|B_{r_0}(y)| \geq |B_{r_0}(p(y))| \geq c > 0$, since O_0 is closed. We conclude from Proposition 3.11 that

$$c|E_x| \leq \sum_{y \in E_x} |B_{r_0}(y)| \leq |B_{3r_0}(x)| \leq \frac{1}{|\Gamma_x|}\beta(3r_0) \leq \beta(3r_0),$$

as we wished. □

We are ready to prove an analogue of a special version of Buser's [12, Lemma 7.2].

Proposition 9.4. *If $h(O_1) = 0$, then for any $\varepsilon, r > 0$, there exists open bounded $A \subseteq O_1$ such that $|U_r(\partial A)| < \varepsilon |A|$.*

Proof. In view of the volume comparison theorem, it suffices to prove the assertion for any $\varepsilon > 0$ and a fixed $r > 0$. Set $r = r_0$ from the beginning of this section, and

$$C_0 := \max_i |G_i|,$$

where G_i are the groups corresponding to the coordinate systems in the beginning of this section. Since $h(O_1) = 0$, we have that for any $\varepsilon > 0$, there exists a smoothly bounded, compact domain $A \subseteq O_1$ with

$$\frac{|\partial A|}{|A|} < \delta := \min\left\{\frac{C\beta(r)\varepsilon}{2C_0^2\beta(4r)}, \frac{C\beta(r)\varepsilon}{2C(r)C_0\beta(2r)}\right\} \tag{9.5}$$

We partition O_1 into the sets

$$A_+ = \{x \in O_1 \mid |A \cap B_r(x)| > \frac{1}{2C_0}|B_r(x)|\}, \tag{9.6}$$

$$A_0 = \{x \in O_1 \mid |A \cap B_r(x)| = \frac{1}{2C_0}|B_r(x)|\}, \tag{9.7}$$

$$A_- = \{x \in O_1 \mid |A \cap B_r(x)| < \frac{1}{2C_0}|B_r(x)|\}. \tag{9.8}$$

Clearly, $|A \cap B_r(x)| \neq 0$ for all $x \in A_+ \cup A_0$. Since $|B_r(x)|$ and $|A \cap B_r(x)|$ depend continuously on x, a path from A_- to A_+ will pass through A_0. Since A is bounded, A_+ and A_0 are bounded. Moreover, $\partial A_+ \subseteq A_0$, A_+ and A_- are open, and A_0 is closed, hence compact. We will show that A_+ satisfies the asserted inequality. By passing from A to A_+, we get rid of a possibly "hairy structure" along the "outer part" of A. We pay by possibly losing regularity of the boundary.

We now choose a $2r$-separated subset X of O_1 as follows. We start with a $2r$-separated subset $X_0 \subseteq A_0$ such that A_0 is contained in the union of the balls $B_{2r}(x)$ with $x \in X_0$. (If $A_0 = \emptyset$, then $X_0 = \emptyset$.) We extend X_0 to a $2r$-separated subset $X_0 \cup X_+$ of $A_0 \cup A_+$ such that $A_0 \cup A_+$ is contained in the union of the balls $B_{2r}(x)$ with $x \in X_0 \cup X_+$. (If $A_+ = \emptyset$, then $X_+ = \emptyset$.) We finally extend $X_0 \cup X_+$ to a complete $2r$-separated subset $X = X_0 \cup X_+ \cup X_-$ of O_1. (If $A_- = \emptyset$, then $X_- = \emptyset$.) By definition, $X_+ \subseteq A_+$ and $X_- \subseteq A_-$. Since A is bounded and $|A \cap B_r(x)| \neq 0$ for all $x \in X_0 \cup X_+$, the sets X_0 and X_+ are finite. By the same reason, the set Y of $x \in X_-$ with $|A \cap B_{2r}(x)| \neq 0$ is finite.

The neighborhood $U_{2r}(A_0)$ is covered by the balls $B_{4r}(x)$ with $x \in X_0$. Using Proposition 3.11 and (9.7), we therefore get

$$\begin{aligned} |U_{2r}(A_0)| &\leq \sum_{x \in X_0} |B_{4r}(x)| \\ &\leq \frac{\beta(4r)}{\beta(r)} \sum_{x \in X_0} |B_r(x)| \\ &= \frac{2C_0\beta(4r)}{\beta(r)} \sum_{x \in X_0} |A \cap B_r(x)|. \end{aligned}$$

For any $x \in X_0$ there exists a (lifted) coordinate system $\pi_{ij} \colon \hat{U}_i \to V_{ij} = \hat{U}_i / G_{ij}$ with $B_{3r}(x) \subseteq V_{ij}$. Fixing $\hat{x} \in \pi_{ij}^{-1}(x)$, we compute

$$|\pi_{ij}^{-1}(A) \cap B_r(\hat{x})| \leq |\pi_{ij}^{-1}(A \cap B_r(x))| = |G_{ij}||A \cap B_r(x)| = \frac{|G_{ij}|}{2C_0}|B_r(x)|.$$

It is easily checked that

$$|B_r(x)| = \frac{1}{|G_{ij}|}|\pi_{ij}^{-1}(B_r(x))| \leq \frac{1}{|G_{ij}|} \sum_{z \in \pi_{ij}^{-1}(x)} |B_r(z)| \leq |B_r(\hat{x})|,$$

which shows that

$$|\pi_{ij}^{-1}(A) \cap B_r(\hat{x})| \leq \frac{1}{2}|B_r(\hat{x})|. \tag{9.9}$$

From Lemmas 9.2 and 3.7 we derive that

$$\frac{|\partial A \cap B_r(x)|}{|A \cap B_r(x)|} \geq \frac{1}{|G_{ij}|} \frac{|\pi_{ij}^{-1}(\partial A) \cap B_r(\hat{x})|}{|\pi_{ij}^{-1}(A) \cap B_r(\hat{x})|} \geq \frac{C}{C_0} \tag{9.10}$$

for any $x \in X_0$, where we used that G_{ij} is a subgroup of G_i. Hence

$$\begin{aligned} |U_{2r}(A_0)| &\leq \frac{2C_0^2\beta(4r)}{C\beta(r)} \sum_{x \in X_0} |\partial A \cap B_r(x)| \\ &\leq \frac{2C_0^2\beta(4r)}{C\beta(r)} |\partial A| \\ &\leq \frac{2C_0^2\beta(4r)}{C\beta(r)} \delta |A| \leq \varepsilon |A| \end{aligned} \tag{9.11}$$

where we use that A satisfies (9.5).

Since any curve from A_+ to A_- passes through A_0, A_+ has distance at least $2r$ to $A_- \setminus U_{2r}(A_0)$. Hence $A_- \setminus U_{2r}(A_0)$ is covered by the open balls $B_{2r}(x)$ with $x \in X_-$.

With Y as above, we let $Z = X_0 \cup Y$. Again, for any $x \in Z$ there exists a (lifted) coordinate system $\pi_{ij} \colon \hat{U}_i \to V_{ij} = \hat{U}_i / G_{ij}$ with $B_{3r}(x) \subseteq V_{ij}$. Arguing as above, using (9.7) and (9.8), we readily see that

$$|\pi_{ij}^{-1}(A) \cap B_r(\hat{x})| \leq \frac{1}{2}|B_r(\hat{x})|$$

for any $\hat{x} \in \pi_{ij}^{-1}(x)$. Letting $A^c = O_1 \setminus A$, we obtain from Proposition 3.11 that

$$\begin{aligned}|\pi_{ij}^{-1}(A)^c \cap B_{2r}(\hat{x})| &\geq |\pi_{ij}^{-1}(A)^c \cap B_r(\hat{x})| \geq \frac{1}{2}|B_r(\hat{x})| \\ &\geq \frac{\beta(r)}{2\beta(2r)}|B_{2r}(\hat{x})| \geq \frac{\beta(r)}{2\beta(2r)}|\pi_{ij}^{-1}(A) \cap B_{2r}(\hat{x})| > 0.\end{aligned}$$

for any $x \in Z$. With the constant C from Lemma 9.2, we therefore get

$$\begin{aligned} C &\leq h^N(B_{2r}(\hat{x})) \\ &\leq \frac{|\pi_{ij}^{-1}(\partial A) \cap B_{2r}(\hat{x})|}{\min\{|\pi_{ij}^{-1}(A) \cap B_{2r}(\hat{x})|, |\pi_{ij}^{-1}(A)^c \cap B_{2r}(\hat{x})|\}} \\ &\leq \frac{2\beta(2r)}{\beta(r)} \frac{|\pi_{ij}^{-1}(\partial A) \cap B_{2r}(\hat{x})|}{|\pi_{ij}^{-1}(A) \cap B_{2r}(\hat{x})|} \\ &\leq \frac{2C_0\beta(2r)}{\beta(r)} \frac{|\partial A \cap B_{2r}(x)|}{|A \cap B_{2r}(x)|} \end{aligned} \tag{9.12}$$

for any $x \in Z$, and the last inequality follows similarly to (9.10). Using Corollary 9.3, (9.12) and (9.5), we conclude that

$$\begin{aligned} |A \cap (A_- \setminus U_{2r}(A_0))| &\leq \sum_{x \in Z} |A \cap B_{2r}(x)| \\ &\leq \frac{2C_0\beta(2r)}{C\beta(r)} \sum_{x \in Z} |\partial A \cap B_{2r}(x)| \\ &\leq \frac{2C(r)C_0\beta(2r)}{C\beta(r)}|\partial A| \\ &< \frac{2C(r)C_0\beta(2r)}{C\beta(r)}\delta|A| \leq \varepsilon|A|, \end{aligned} \tag{9.13}$$

where we use (9.5) in the last step.

Since $A \subseteq A_+ \cup U_{2r}(A_0) \cup (A \cap (A_- \setminus U_{2r}(A_0)))$, we obtain

$$\begin{aligned} |A_+| &\geq |A| - |U_{2r}(A_0)| - |A \cap (A_- \setminus U_{2r}(A_0))| \\ &\geq (1 - 2\varepsilon)|A|. \end{aligned}$$

In particular, A_+ is not empty. Since $\partial A_+ \subseteq A_0$, we conclude that

$$|U_{2r}(\partial A_+)| \leq |U_{2r}(A_0)| \leq \varepsilon|A| \leq \frac{\varepsilon}{1 - 2\varepsilon}|A_+|.$$

In conclusion, A_+ is a bounded open subset of O_1 that satisfies the asserted inequality, albeit with 2ε in place of ε (assuming w.l.o.g. that $\varepsilon < 1/4$). □

We are now ready to prove the main result of the section. As a consequence of the Cheeger inequality, if $\lambda_0(O_1) = 0$, then $h(O_1) = 0$. We know from Lemma 9.4 that for any $\varepsilon > 0$ and $r > 2 \operatorname{diam} O_0$ there exists an open, bounded $A \subseteq O_1$ with $|U_{3r}(A) \setminus A| \leq |U_{3r}(\partial A)| < \varepsilon |A|$.

Fix a regular point $x \in O_0$ and consider the finite set $F := p^{-1}(x) \cap U_r(A)$. Taking into account that

$$\operatorname{diam} D_y \leq 2 \operatorname{diam} O_0 < r,$$

it is immediate to verify that A is contained in the union of D_y, with $y \in F$. Moreover, given $g \in G_r$, it is easy to see that any $y \in Fg \setminus F$ belongs to $U_{2r}(A) \setminus U_r(A)$, which shows that $U_{3r}(A) \setminus A$ contains the corresponding D_y. Using that $|D_y| = |O_0|$ and that the intersection of different D_y's is measure zero, we conclude that

$$|Fg \setminus F||O_0| \leq |U_{3r}(A) \setminus A| < \varepsilon |A| \leq \varepsilon |F||O_0|$$

for any $g \in G_r$. Since any finite subset G of $\pi_1^{\mathrm{orb}}(O_0)$ is contained in G_r for some $r > 2 \operatorname{diam} O_0$, this implies that the covering is amenable.

10. Stability implies amenability: the role of λ_{ess}

Let K_0 be a compact and connected Riemannian orbifold with non-empty boundary. Let $p \colon K_1 \to K_0$ be a Riemannian covering of orbifolds, where we do not assume that K_1 is connected. We assume that $\lambda_0(K_1) = \lambda_0(K_0) = 0$, where we recall the notation $\lambda_0(K) = \lambda(\Delta, K)$, where we use the definition of λ_0 as the infimum of the usual Rayleigh quotients over non-zero functions f in $C_c^\infty(K_1)$ and $C_c^\infty(K_0)$, respectively. Note that we do not require that the functions f vanish on the corresponding boundary.

Change the given Riemannian metric on K_0 in a neighborhood $U \cong [0, \varepsilon) \times \partial K_0$ of ∂K_0 so that the new metric is a product metric $dr^2 + g_0$ on U, where g_0 is a Riemannian metric on ∂K_0, and endow K_1 with the lifted metric. Since K_0 is compact, the old and new Riemannian metrics on K_0 and K_1 are uniformly equivalent, and hence $\lambda_0(K_1) = \lambda_0(K_0) = 0$ with respect to the new metric.

Denote by $2K_0$ and $2K_1$ the doubles of K_0 and K_1. Since the new Riemannian metrics above are product metrics in neighborhoods of the boundaries, they fit together to define Riemannian metrics on $2K_0$ and $2K_1$ so that p extends to a Riemannian covering $2p \colon 2O_1 \to 2O_0$. Since $\lambda_0(K_1) = 0$ with respect to the new metric and test functions in $C_c^\infty(K_1)$ can be doubled to test functions in $\operatorname{Lip}_c(2K_1)$ with the same Rayleigh quotient, we get that $\lambda_0(2K_1) = 0$. Since $2K_0$ is closed, we conclude from Theorem 9.1 that the covering $2p$ is amenable. It follows from Proposition 6.3 that the restriction of $2p$ over K_0, which is the original covering p, is amenable. Hence, we arrive at the following:

Theorem 10.1. *If $\lambda_0(K_1) = 0$, then p is amenable.*

Proof of Theorem A.2. Assume that

$$\lambda_0(S_1, O_1) = \lambda_0(S_0, O_0)$$

and, to arrive at a contradiction, that p is non-amenable. According to (1.2), there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_c^\infty(O_1)$ with L^2-norm one and $R_{S_1}(f_n) \to \lambda_0(S_1, O_1)$. Since the covering is non-amenable, we obtain from Proposition 6.3 that there exists a smoothly bounded, compact domain

$K \subseteq O_0$ such that the covering $p\colon p^{-1}(K) \to K$ is non-amenable. Then Theorem 10.1 implies that $\lambda_0(p^{-1}(K)) > 0$.

We know from Lemma 4.6 that there exists a positive $\varphi_0 \in C^\infty(O_0)$ with L^2-norm one and $S_0\varphi_0 = \lambda_0(S_0, O_0)\varphi_0$. Denote by φ_1 the lift of φ_0 to O_1 and by L the renormalization of S_1 with respect to φ_1. Since φ_1 is positive, we may write $f_n = h_n\varphi_1$, and in view of Lemma 4.10 we have that

$$R_L(h_n) = \frac{\int_{O_1} |\nabla h_n|^2\varphi_1^2}{\int_{O_1} h_n^2\varphi_1^2} = R_{S_1}(f_n) - \lambda_0(S_0, O_0) \to 0.$$

Denoting by $c_1, c_2 > 0$ the minimum and the maximum of φ_0 on K, respectively, we have that

$$\frac{\int_{p^{-1}(K)} |\nabla h_n|^2\varphi_1^2}{\int_{p^{-1}(K)} h_n^2\varphi_1^2} \geq \frac{c_1^2}{c_2^2}\lambda_0(p^{-1}(K)) > 0,$$

which shows that

$$\int_{p^{-1}(K)} h_n^2\varphi_1^2 \to 0 \quad \text{and} \quad \int_{O_1\smallsetminus p^{-1}(K)} h_n^2\varphi_1^2 \to 1.$$

Let K_0 be a compact domain (of positive measure) in the interior of K and consider $\chi_0 \in C_c^\infty(O_1)$ with $\chi_0 = 1$ in a neighborhood of K_0 and $\operatorname{supp}\chi_0 \subseteq K$. Denote by χ_1 the lift of χ_0 to O_1 and set $h_n' = h_n(1-\chi_1)$. It is not difficult to verify that

$$\int_{O_1} (h_n')^2\varphi_1^2 \to 1 \quad \text{and} \quad \int_{O_1} |\nabla h_n'|^2\varphi_1^2 \to 0,$$

and hence $R_L(h_n') \to 0$. Now setting $f_n' = h_n'\varphi_1$, we derive from Lemma 4.10 that $R_{S_1}(f_n') \to \lambda_0(S_0, O_0)$. It should be noticed that $\|f_n'\|_{L^2} \to 1$ and $\operatorname{supp} f_n' \cap p^{-1}(K_0) = \emptyset$.

It is easy to see that the sequence $(g_n)_{n\in\mathbb{N}}$ in $\operatorname{Lip}_c(O_0)$, consisting of the pushdowns of f_n' as defined in (7.1), satisfies $\|g_n\|_{L^2} \to 1$, $\operatorname{supp} g_n \cap K_0 = \emptyset$, and $R_{S_0}(g_n) \to \lambda_0(S_0, O_0)$. Now we see the role of λ_{ess}. Namely, by Lemma 4.7, the assumption that $\lambda_0(S_0, O_0) < \lambda_{\mathrm{ess}}(S_0, O_0)$ yields that $g_n \to \varphi_0$ in $L^2(O_0)$, after passing to a subsequence if necessary. This is a contradiction since φ_0 is positive, whereas K_0 has positive measure and $\operatorname{supp} g_n \cap K_0 = \emptyset$. □

11. Conformally compact orbifolds

Let $O = P \setminus \partial P$, $\partial P = \{\rho = 0\}$, and $g = h/\rho^2$ as in the introduction. Then the normalized gradient field $X = \nabla\rho/|\nabla\rho|$ of ρ with respect to h is well defined in a neighborhood of ∂P, and $V = \rho X$ is the normalized gradient field of ρ with respect to g. The divergence of V with respect to g is given

$$\begin{aligned}
\operatorname{div}_g V &= \frac{m}{2}V(\ln(1/\rho^2)) + \operatorname{div} V \\
&= \frac{m}{2}\frac{\rho}{|\nabla\rho|}d(\ln(1/\rho^2)(\nabla\rho) + \operatorname{div} V \\
&= -m|\nabla\rho| + \operatorname{div} V \\
&= -(m-1)|\nabla\rho| + \rho\operatorname{div} X.
\end{aligned}$$

Proof of (1.9). Since $\operatorname{div} X$ is a smooth function in a neighborhood of ∂P and $\partial P = \{\rho = 0\}$, we conclude that, for any $\varepsilon > 0$, there is a neighborhood U of ∂P such that

$$\operatorname{div}_g V = -(m-1)|\nabla\rho| \pm \varepsilon \geq -a(m-1) - \varepsilon$$

in U. By the divergence formula, we have

$$\min \operatorname{div} V |D|_m \leq \int_D \operatorname{div} V = \int_{\partial D} \langle V, \nu \rangle \leq |\partial D|_{m-1}$$

for any compact domain in U with smooth boundary and hence

$$h_{\mathrm{ess}}(O) \geq (m-1)a.$$

Using the Cheng eigenvalue comparison Theorem 3.12 for orbifolds, the proof of the inequalities $\lambda_{\mathrm{ess}}(\tilde{O}), \lambda_{\mathrm{ess}}(O) \leq a^2(m-1)^2/4$ is the same as that for the corresponding inequalities for λ_0 in the case of manifolds in [3, Theorem 1.10]. The Cheeger inequality (4.9) then implies the asserted equality $\lambda_{\mathrm{ess}}(O) = a^2(m-1)^2/4$. □

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