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Transverse vibrations of clamped and simply-supported circular plates with two dimensional thickness variation

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Abstract. Two dimensional boundary characteristic orthonormal polynomials are used in the Ritz method for the vibration analysis of clamped and simply-supported circular plates of varying thickness. The thickness variation in the radial direction is linear whereas in the circumferential direction the thickness varies according to $\cos\ k\theta$, where k is an integer. In order to verify the validity, convergence and accuracy of the results, comparison studies are made against existing results for the special case of linearly tapered thickness plates. Variations in frequencies for the first six normal modes of vibration and mode shapes for various taper parameters are presented.

1. Introduction

Plates of various shapes and of non-uniform thickness are widely used in engineering structures. Many papers have been written on the vibration of plates covering various shapes, thickness variations according to different power laws and different boundary conditions. Leissa [1] have discussed an excellent series of survey papers in his book: vibration of plates. Focusing attention on plates vibration with varying thickness, Laura and Gutierrez [3] have analyzed the vibration of circular plates with non-uniform thickness by means of the differential quadrature method. Laura and his co-workers [4] have studied the problem of free vibration of a solid circular plate of linearly varying thickness resting on a Winkler foundation. In another paper Bambill and Laura [5] have solved axisymmetric vibration of circular plates with double linear variable thickness. Chakraverty et al. [6] have surveyed the research on vibration of plates using boundary characteristic polynomials in the Rayleigh-Ritz method. Forced axisymmetric and asymmetric responses of linearly tapered circular plates have been studied by Gupta and Goyal [7,8]. Kim and Dickinson [9] have investigated the lateral vibrations of thin annular and circular composite plates subjected to certain complicating effects. Chen [10] has analyzed axisymmetric vibrations of circular and annular plates with arbitrarily varying thickness. Wang [11] has studied classical circular plates with generalized variable thickness and presented power series solutions in the form of recursive relations whereas exact vibration results for stepped

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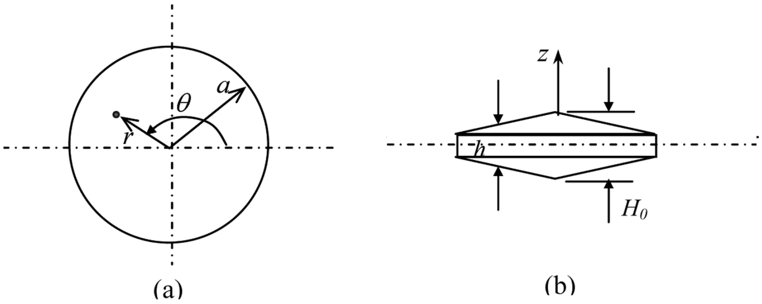

Fig. 1. Plate geometry: (a) plan view and (b) side view.

circular plates with free edges has been obtained by Hang et al. [12]. These vibration results are used for the hydro elastic analysis of circular large floating structures. Duan et al. [13] have treated an interesting problem of modifying the fundamental vibration modes of circular plates with free edges. In recent years, Bhardwaj and his co-workers [14–20] have analyzed different structural orthotropic tapered plate problems resting on Winkler elastic foundations using the Rayleigh-Ritz method. Ansari and Gupta [21] have solved the problem of forced vibration of axisymmetric polar orthotropic plates resting on elastic foundation. Civalek and Ozturk [22] have analyzed free vibration of tapered rectangular plates using the discrete singular convolution method. The same method has used by Civalek [23] to determine the fundamental frequency and to study the effect of mode number of isotropic and orthotropic rectangular plates with linearly varying thickness.

Hitherto, no study has been made on the vibrations of circular plates whose thickness varies linearly in the radial direction and according to $\cos\ k\theta$ in the circumferential (or θ) direction, where k is an integer. Such plate thickness variations find applications in body vibrating /massage machines and can be potentially used as rotating disk in devices to generate vortex flow in fluid or to generate optical vortices. The objective of this study is to investigate the vibration frequencies and the corresponding mode shapes for such varying thickness circular plates. For analysis, boundary characteristic orthonormal polynomials are used as admissible functions in the Ritz method. Four types of thickness variations are considered in θ-direction by adjusting the integer k (i.e., k = 1, 2, 3 and 4). Frequencies for the first six normal modes of vibration for various values of taper parameters for clamped circular and simply-supported plates are computed and these results are presented in graphical forms. Contour plots of mode shapes are also presented. In order to verify the correctness of the converged results, comparison studies are made with particular cases that are available in the literature and shown in tabular form.

2. Problem formulation and analysis

Let us consider an isotropic, thin circular plate of radius a, and varying thickness $h\,(r,\theta)$ as shown in Fig. 1. The formulation is presented in cylindrical coordinates with the axis of the plate along the z-axis and the middle plane of the plate in the $r-\theta$ plane.

The energy functional $J\,(W)$ is obtained by subtracting the maximum kinetic energy from the maximum strain energy, i.e.

$$J\,(W) = \frac{E\,H_0^3}{24\,(1-\nu^2)} \int_0^1 \int_0^{2\pi} \left[F^3\,(R,\theta) \left\{ W_{,RR}^2 + 2\nu\, W_{,RR} \left(\frac{1}{R^2} W_{,\theta\theta} + \frac{1}{R} W_{,R} \right) + \left(\frac{1}{R^2} W_{,\theta\theta} + \frac{1}{R} W_{,R} \right)^2 + 4\,(1-\nu^2) \left(\frac{1}{R^2} W_{,\theta} - \frac{1}{R}\, W_{,R\theta} \right)^2 \right\} - \Omega^2\, F\ W^2 \right] R\, d\theta\, dR\,, \quad \ldots \qquad (1)$$

where $H = h/a\,,\ \ R\ = r/a\,, w\,(r,\theta,t) = a\,W\,(R,\theta)\,\cos\,\omega\,T, T\,=\,(t/a)\,\sqrt{E/\rho}$, $\Omega^2\,=\,12\,(1-\nu^2)\,\omega^2/H_0^2$, H_0 is the thickness at the centre of plate, ρ the mass density, $w\,(r,\theta,t)$ the transverse deflection and ω the natural

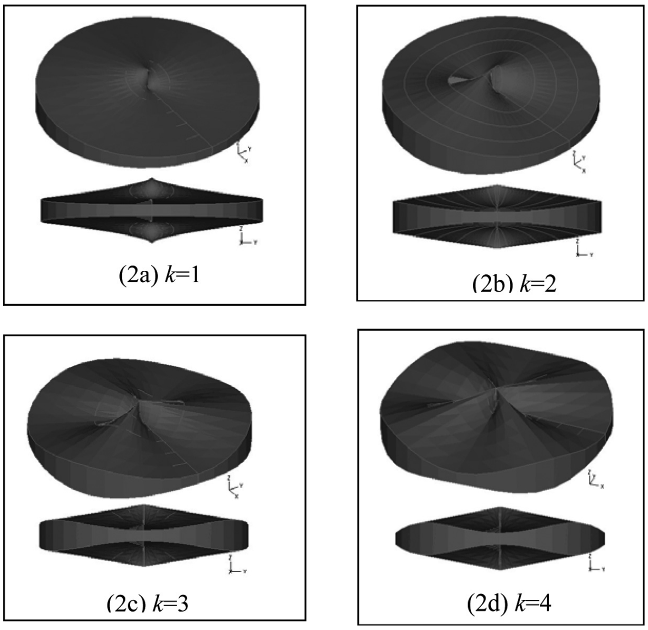

Fig. 2. Three dimensional view of plate geometry when $k = 1$ to 4.

circular frequency of the plate. A comma followed by a suffixed variable denotes differentiation with respect to that variable.

The considered two dimensional variation in thickness is taken as

$$H = H_0\, F\,(R,\theta) \text{ ,where } F\,(R,\theta) = (1 - \alpha\, R)\;\; (1 - \beta\,\cos\;\, k\theta)\,, \ldots \tag{2}$$

where k is an integer and α,β are the taper constants with assumption that $\beta = 0$ at $r = 0$. The plate thickness variations considered here are shown in Figs 2(a) to 2(d) where the four cases show the changes in plate geometry as the parameter k, varies from 1 to 4. In all cases of plate geometry, both taper parameters α and β are assigned positive values. The assignment of negative α (with $\beta = 0$) will lead to plate tapering in decreasing thickness along radial direction towards edge of plate. On the other hand, the assignment of negative β (with $\alpha = 0$) will lead to changes in pattern of thickness variation in circumferential direction from ridge to valley and vice-versa in comparison to the case of positive β.

According to the Ritz method, the functional $J\,(W)$ is to be minimized with respect to W. The deflection W may be approximated by

$$W\,(R,\theta) = \sum_{j=1}^{N} c_j\, \Phi_j\,(R,\theta)\;\;, \quad \ldots \tag{3}$$

where $\Phi_j\,(R,\theta)$ are the two dimensional boundary characteristic orthonormal polynomials and they are generated by the Gram-Schmidt process as follows:

$$\Phi_j\; = \phi_j \Big/ \sqrt{< \phi_j, \phi_j >}, \quad \phi_1 = \left(1 - R^2\right)^p \quad \ldots \tag{4}$$

$$\phi_i = \;\phi_1 f_i\,(R,\theta) - \sum_{j=1}^{i-1} \frac{< \phi_1 f_i, \phi_j >}{< \phi_j, \phi_j >}\, \phi_j\,, \qquad i = \,2\;(1)\;N, \quad \ldots \tag{5}$$

$$< f, g > = \iint_A F\,(R,\theta)\, f\,(R,\theta)\; g\,(R,\theta)\;\; R\, d\theta\;\, dR, \quad \ldots \tag{6}$$

where A is the area of the plate.

The functions $f_i\,(R,\theta)$ are taken as simple polynomials of $R\cos\theta$ and $R\sin\theta$ of the form $R^{m_i+n_i}\cos^{m_i}\theta\ \sin^{n_i}\theta$, where m_i and n_i are non-negative integers taken as:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
m_i	0	1	0	2	1	0	3	2	1	0	4	3	2	1	0	...
n_i	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	...

The substitution of $W\,(R,\theta)$ from Eq. (3) into the energy functional [Eq. (1)] and then minimizing $J\,(W)$ as a function of the coefficients c_j leads to the standard eigenvalue problem:

$$\sum_{j=1}^{N}\left(a_{ij}-\Omega^2\,\delta_{ij}\right)c_j=0\ ,\qquad i=1\ (1)\ N\ ,\quad \ldots \tag{7}$$

where

$$\delta_{ij}=\begin{cases}1 \textit{ when } i=j\\ 0 \textit{ when } i\neq j\end{cases}$$

and

$$\begin{aligned}a_{ij}=\int_0^1\int_0^{2\pi}\Bigg[F^3\,(R,\theta)\Bigg\{&\Phi_{i,RR}\Phi_{j,RR}+\nu\,\Phi_{i,RR}\left(\frac{1}{R^2}\Phi_{j,\theta\theta}+\frac{1}{R}\Phi_{j,R}\right)+\nu\,\Phi_{j,RR}\left(\frac{1}{R^2}\Phi_{i,\theta\theta}+\frac{1}{R}\Phi_{i,R}\right)\\&+\left(\frac{1}{R^2}\Phi_{i,\theta\theta}+\frac{1}{R}\Phi_{i,R}\right)\left(\frac{1}{R^2}\Phi_{j,\theta\theta}+\frac{1}{R}\Phi_{j,R}\right)+4\,(1-\nu^2)\left(\frac{1}{R^2}\Phi_{i,R\theta}-\frac{1}{R}\Phi_{i,\theta}\right)\\&\left(\frac{1}{R}\Phi_{j,R\theta}-\frac{1}{R^2}\Phi_{j,\theta}\right)\Bigg\}+\Phi_i\Phi_j\Bigg]\ R\,d\theta\ dR\end{aligned}\tag{8}$$

The integral evaluated in Eqs (6) and (8) are given by the formula

$$\begin{aligned}&\int_0^1\int_0^{2\pi}(1-R^2)^i\ R^{j+l}\cos^{j}\theta\,\sin^{l}\theta\ R\,d\theta\ dR\\&=\left.\Gamma\!\left(\frac{i+1}{2}\right)\Gamma\!\left(\frac{j+l+2}{2}\right)\Gamma\!\left(\frac{j+1}{2}\right)\Gamma\!\left(\frac{l+1}{2}\right)\middle/\Gamma\!\left(\frac{2i+j+l+4}{2}\right)\Gamma\!\left(\frac{j+l+2}{2}\right)\right.\ \text{when } j \text{ and } l \text{ are even},\\&=0,\ \textit{otherwise},\ldots\end{aligned}\tag{9}$$

where $i,j,l>-1$.

The eigenvalues (Ω) and the eigenvectors (c_j) are computed by the Jacobi method. The mode shapes are computed from Eq. (3) and their corresponding contours are plotted. C++ coding is employed for all the computations.

3. Results and discussion

There are six parameters (i.e., α, β, k, p, N and ν) used in the analysis of the plate. The value of ν is taken as 0.3 for all calculations except for calculating the results given in Table 3 where ν is taken as 0.33 for the sake of the comparison with known results. The value of parameter p is taken as 2 or 1 for clamped edge or simply-supported edge. The taper parameters α and β vary from –0.9 to 0.9 and –0.7 to 0.7, respectively, with a step size of 0.2. Four cases of thickness variations corresponding to $k = 1$, 2, 3 and 4 are considered. Each case is discussed separately.

In Table 1, the convergence for first six normal modes of vibrations at least up to five significant figures for the clamped (**C-plate**) and simply-supported plates (**S-plate**) are investigated for $\alpha = \beta = 0.5$. It is observed that a maximum of 19 terms are required to obtain converged results in all four cases, i.e., $k = 1, 2, 3, 4$. It is observed that the frequencies of **C-plates** are more than that of **S-plates** as expected due to the higher stiffness of the **C-plates** in comparison to **S-plates** for all the four cases of circumferential thickness variations. Moreover, there is no specific

Table 1
Convergence of Ω for C and S - plates when $\alpha = \beta = 0.5$, $e_r = 1.0$, $g_r = 0.384$

Case-I	**C-plate**					**S-plate**				
$\rightarrow$ N $\downarrow \Omega$	15	16	17	18	19	15	16	17	18	19
Ω_1	5.1345	5.1282	5.1282	5.1280	5.1280	2.8402	2.8265	2.8263	2.8261	2.8261
Ω_2	11.424	11.424	11.423	11.423	11.423	8.1840	8.1840	8.0850	8.0850	8.0850
Ω_3	12.203	12.200	12.200	12.149	12.149	9.1957	8.8380	8.8380	8.7684	8.7684
Ω_4	18.991	18.991	18.741	18.741	18.571	15.587	15.488	15.350	15.350	15.350
Ω_5	19.288	19.098	19.098	18.930	18.930	15.620	15.587	15.488	15.487	15.487
Ω_6	23.463	23.110	23.110	23.110	23.110	21.095	20.784	20.780	20.321	20.321
Case-II			**C-plate**					**S-plate**		
Ω_1	5.8316	5.8316	5.8316	5.8316	5.8316	3.4113	3.4113	3.4113	3.4113	3.4113
Ω_2	9.8361	9.8352	9.8352	9.7849	9.7849	9.2911	8.9958	8.8186	8.8186	8.8186
Ω_3	15.497	15.497	15.494	15.494	15.286	11.376	11.376	11.270	11.270	11.270
Ω_4	19.681	19.681	19.681	19.681	19.681	18.308	18.308	18.308	18.308	18.308
Ω_5	19.807	19.807	19.807	19.807	19.807	18.718	18.718	18.718	18.718	18.718
Ω_6	29.471	28.534	28.534	26.136	26.136	24.291	24.291	24.291	24.291	24.291
Case-III			**C-plate**					**S-plate**		
Ω_1	6.3833	6.3515	6.3515	6.3251	6.3251	3.7980	3.7320	3.7008	3.7008	3.7008
Ω_2	12.464	12.434	12.427	12.322	12.322	10.345	10.331	10.258	10.258	10.258
Ω_3	23.848	23.848	23.772	22.347	22.347	10.364	10.346	10.331	10.310	10.310
Ω_4	30.230	29.637	29.637	29.310	29.310	20.249	20.211	19.592	19.592	19.397
Ω_5	32.585	32.585	32.520	32.520	32.520	20.249	20.249	20.210	19.935	19.592
Ω_6	37.003	36.998	36.996	36.994	36.994	24.243	24.230	24.228	24.226	24.226
Case-IV			**C-plate**					**S-plate**		
Ω_1	6.6497	6.6497	6.6497	6.6497	6.6497	3.8259	3.8259	3.8259	3.8259	3.8259
Ω_2	13.773	13.369	13.414	13.414	13.162	9.7779	9.6897	9.3353	9.3353	9.2001
Ω_3	13.773	13.773	13.769	13.630	13.630	9.7779	9.6897	9.6897	9.6326	9.6326
Ω_4	18.162	18.162	18.162	18.162	18.162	13.486	13.486	13.486	13.486	13.486
Ω_5	27.267	27.267	27.267	27.267	27.267	20.248	20.248	20.248	20.248	20.248
Ω_6	30.753	30.753	30.753	30.753	30.753	24.245	24.245	24.245	24.245	24.245

Table 2
Comparison of Ω for isotropic C and S- plates when $\alpha = \beta = 0$ and $e_r = 1$

Edge Condition	Ref.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
S	[9]	4.93515	13.8982	25.6133	29.720	39.954	48.4796	56.8425
	Present	4.93515	13.896	25.6129	29.720	39.954	48.477	56.841
C	[9]	10.2158	21.260	34.877	39.771	51.030	60.829	69.674
	Present	10.216	21.258	34.872	39.771	51.025	60.827	69.666

pattern of variation of the frequencies is found for both **C**- and **S-plates** except that the first mode shows an increase in frequencies with an increase in k (from 1 to 4).

A comparison of the frequency parameters Ω with Kim and Dickinson [9] results for isotropic simply-supported and clamped plates of uniform thickness ($\alpha = \beta = 0$) is shown in Table 2 with $\nu = 0.3$. In Table 3 comparison of Ω is shown with the results of Gelos [24], Luisoni [25], Wah [26], Gontkevich [27], Avolos [28], Leissa and Narita [2] and Leissa [1] for isotropic **S-plates** of uniform thickness when $e_r = 1.0$ and $\nu = 0.3$.

Comparison of Ω with Leissa and Narita [2], Singh and Saxena [29] and Laura and his co-worker's [4] for isotropic **S-plates** of linearly varying (L.V.) thickness (α varying and $\beta = 0$) when $e_r = 1.0$ and $\nu = 0.3$ is shown in Table 4 and for the same parameters as **C-plates** with the results of Laura and his co-worker's [4] given in Table 5. In Tables 2 to 5, the maximum quantitative difference in frequencies relative to the existing results is found $\pm$ 0.84%.

The variations of the first six frequencies with respect to the taper parameters α and β for k = 1, 2, 3 and 4 are shown in Figs 3 and 4. Figure 3 shows the variation of Ω for α ranging from -0.7 to 0.7 while β is kept constant (i.e. $\beta = 0.7$). It can be seen that all frequencies decrease with increasing taper parameter α. This is expected since varying α from negative to positive means that the mass of the plates (in all the four cases k = 1, 2, 3 and 4) will be shifted from around the supporting edges towards the plate centre due to the larger thickness at the center, resulting

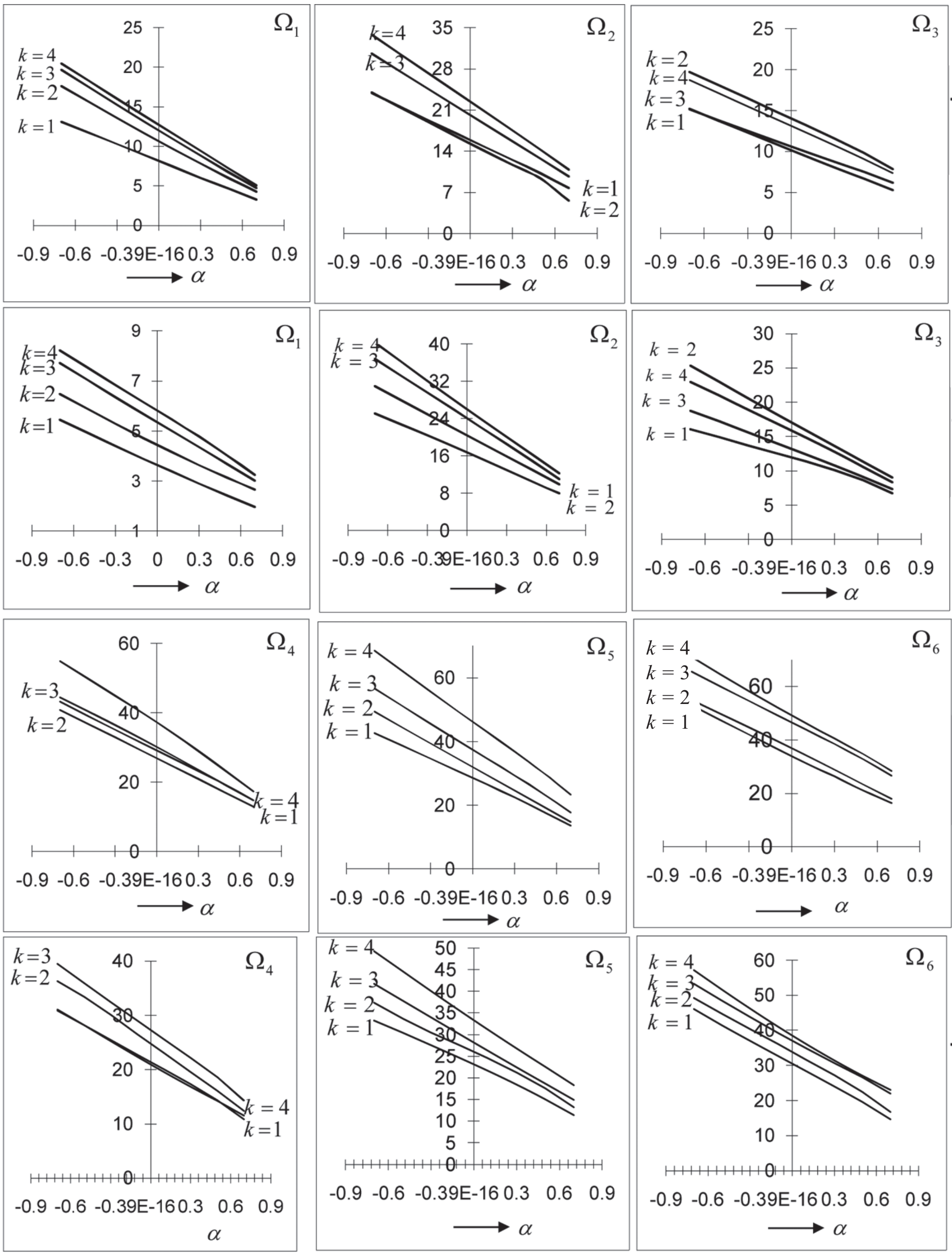

Fig. 3. Variation in Ω with α for corresponding C and S- plates when $\beta = 0.7$ and $k = 1,2,3$ and 4.

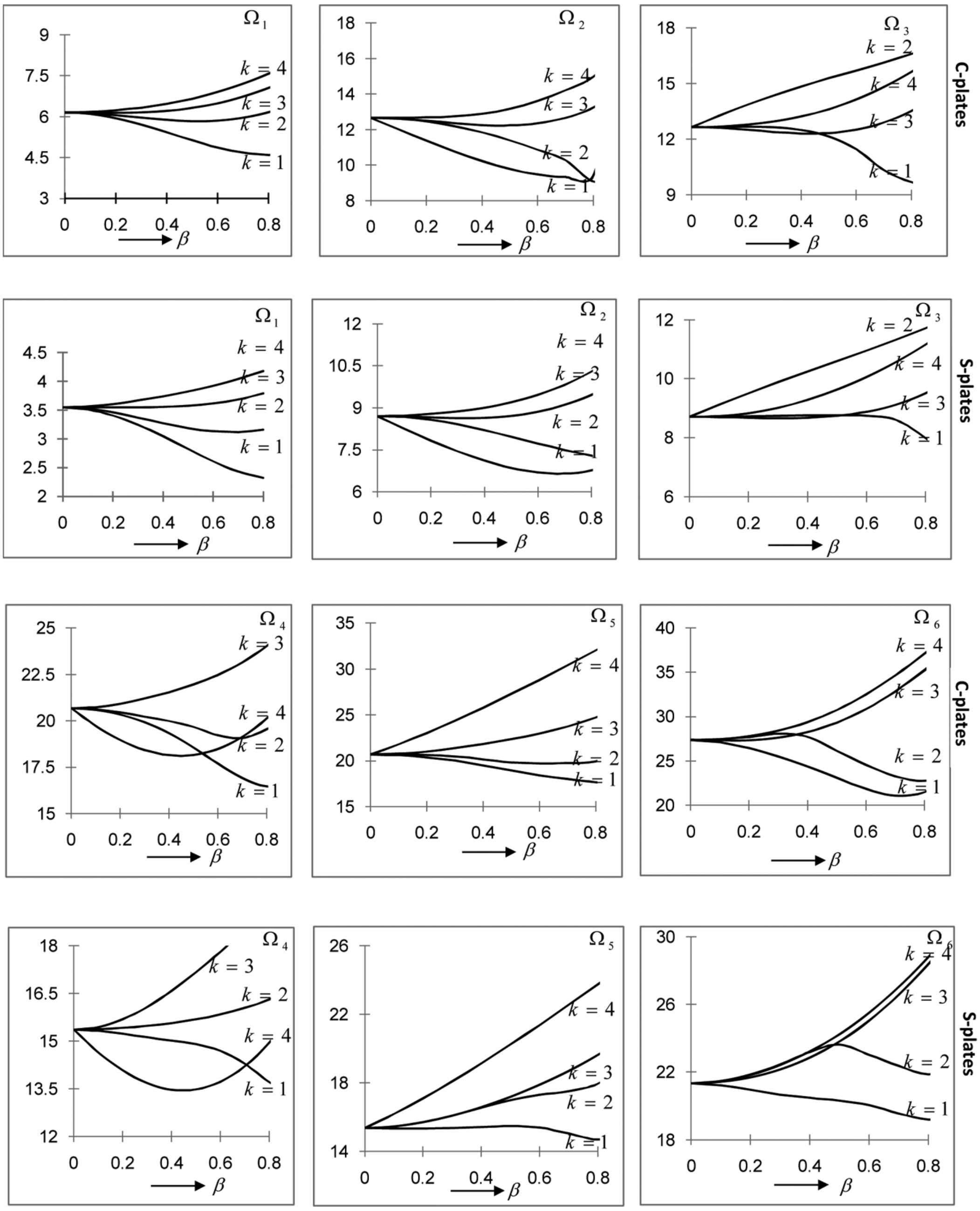

Fig. 4. Variation in Ω with β for corresponding C and S- plates when $\alpha = 0.5$ and $k = 1,2,3$ and 4.

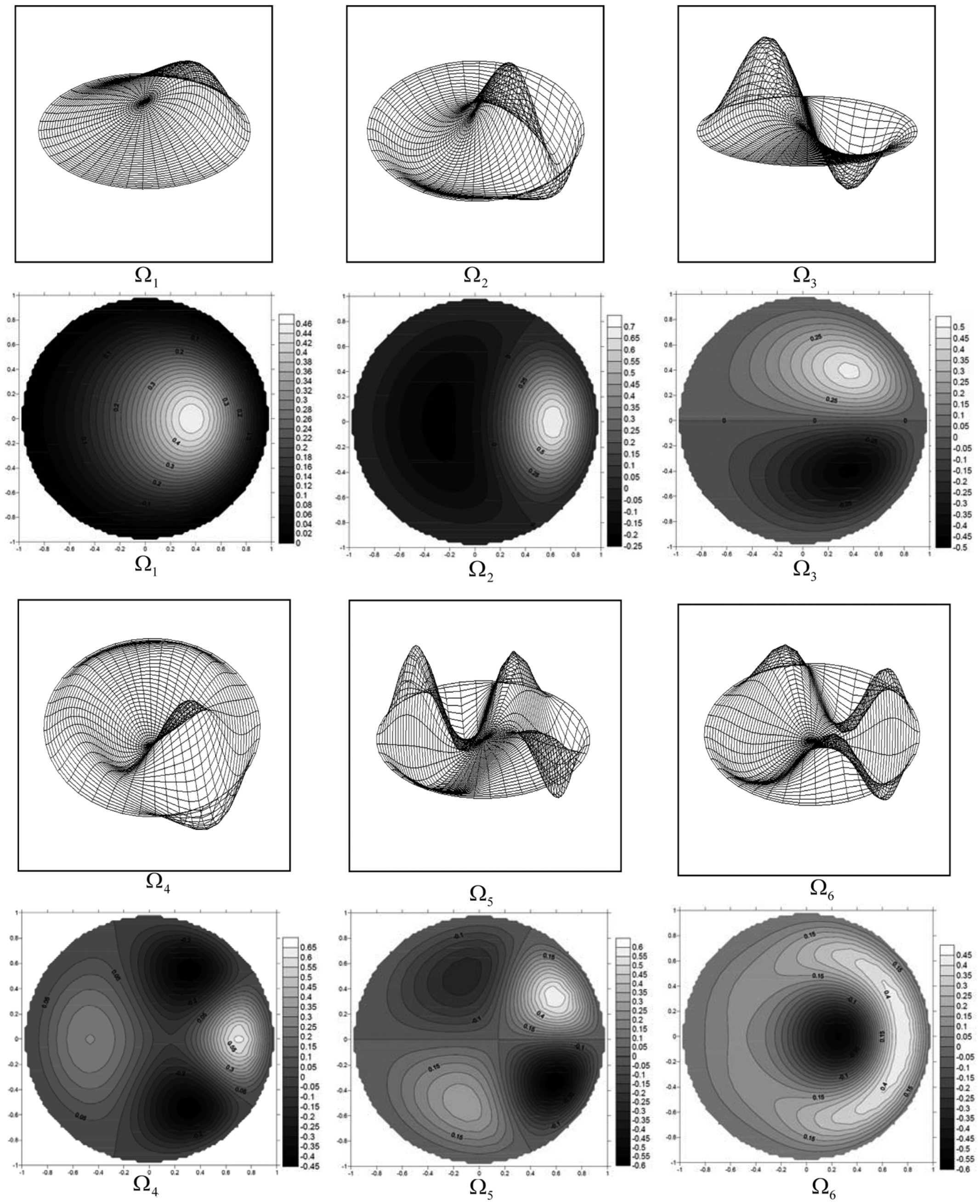

Fig. 5. First six normal modes and their contours for C-plates when $\alpha = \beta = 0.5$ and $k = 1$.

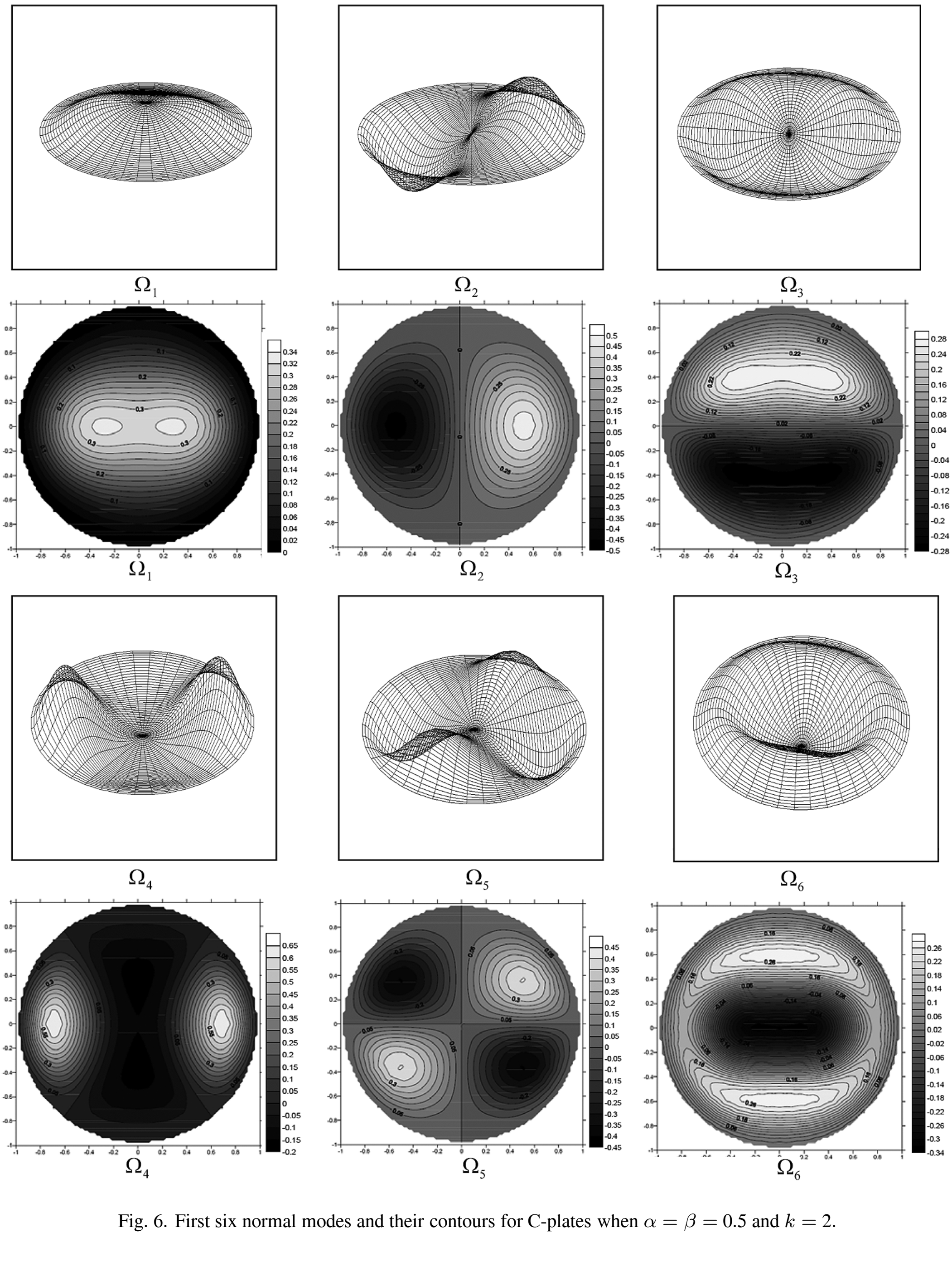

Fig. 6. First six normal modes and their contours for C-plates when $\alpha = \beta = 0.5$ and $k = 2$.

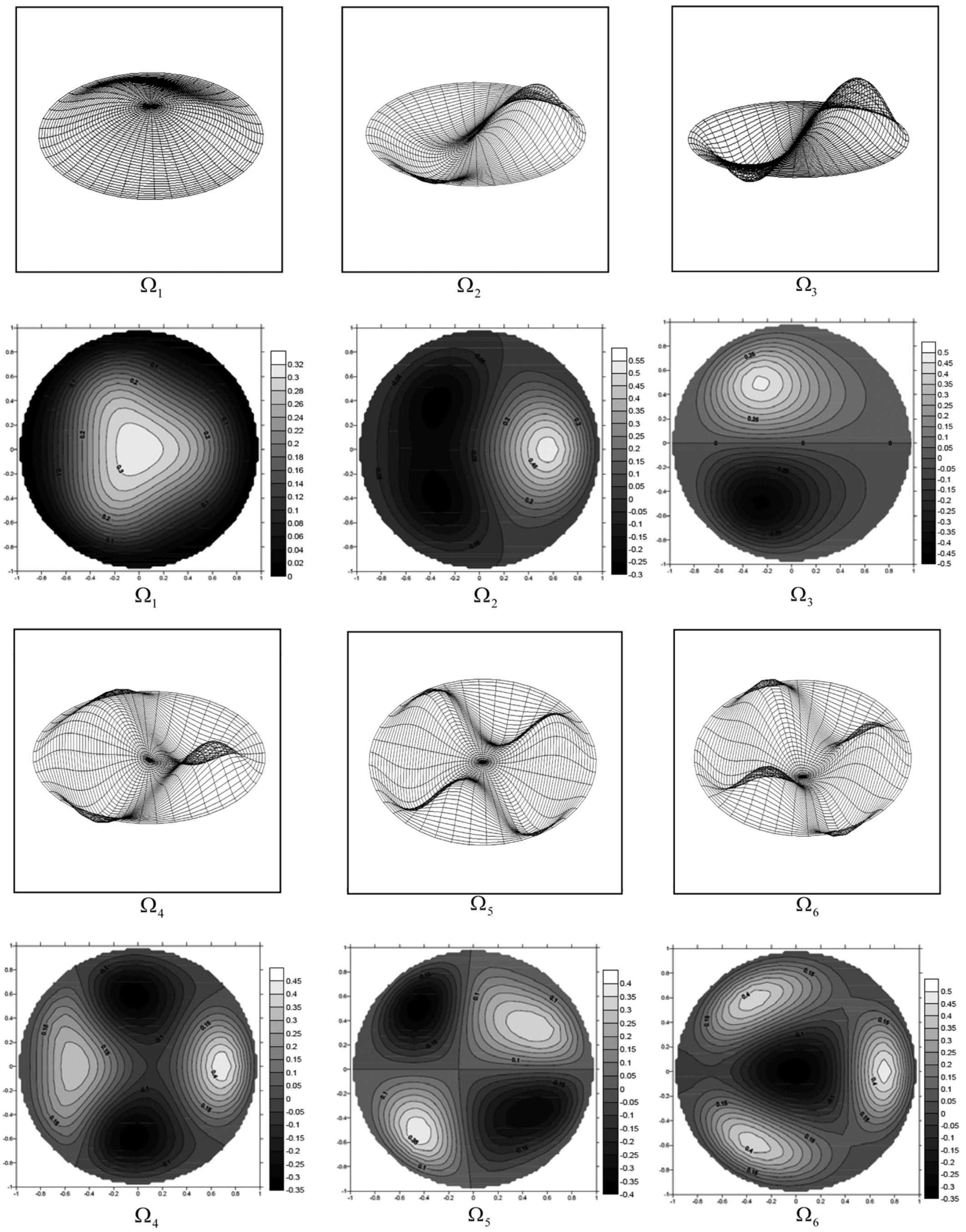

Fig. 7. First six normal modes and their contours for C-plates when $\alpha = \beta = 0.5$ and $k = 3$.

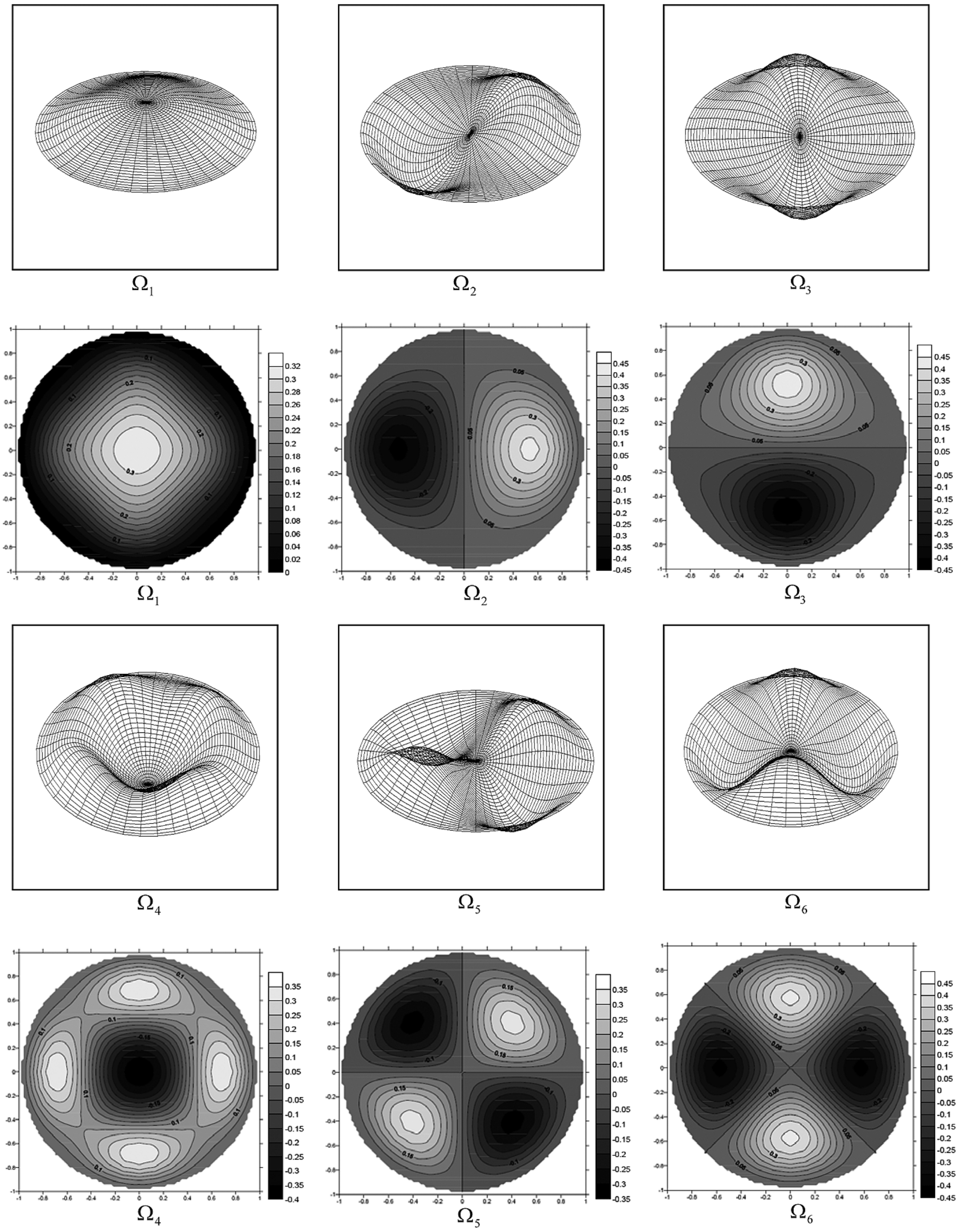

Fig. 8. First six normal modes and their contours for C-plates when $\alpha = \beta = 0.5$ and $k = 4$.

Table 3
Comparison of Ω for isotropic circular plates of uniform thickness with simply supported edge when $\alpha = \beta = 0$ **and** $e_r = 1$

Ref.	Ω_1	Ω_2	Ω_3	Ω_4
[24]	4.93	–	–	–
[25]	4.935	–	–	–
[26]	4.94	29.72	–	–
[27]	4.977	29.72	–	–
[28]	4.935	29.720	74.24	–
[2]	4.9352	29.720	74.156	–
[1]	4.977	29.76	74.20	138.34
Present	4.9351	29.720	74.156	138.32

Table 4
Comparison of Ω for isotropic circular plates of linearly varying thickness with simply supported

$\rightarrow \Omega$ $\downarrow \alpha$	Ω_1				Ω_2				Ω_3		
	[2]	[29]	[4]	**Present**	[2]	[29]	[4]	**Present**	[2]	[29]	**Present**
0.2	–	–	4.44	4.4345	–	–	26.54	26.462	–	–	–
0.1	4.667	4.6637	4.71	4.7076	28.080	28.077	28.16	28.118	70.225	70.213	70.213
0.0	–	4.9351	4.98	4.9351	–	29.720	29.77	29.720	–	74.156	74.156
−0.1	5.209	5.2061	5.25	5.2495	31.349	31.346	31.38	31.374	78.015	78.032	78.030
−0.2	–	–	5.52	5.5198	–	–	32.98	32.980	–	–	–

Edge when $\beta = 0$, $e_r = 1.0$, $g_r = 0.384$, $\nu_r = 0.3$ and $K_f = 0$.

Table 5
Comparison of Ω for C-plate when $\beta = 0$

$\rightarrow \Omega$ $\downarrow \alpha$	Ω_1		Ω_2	
	[4]	**Present**	[4]	**Present**
0.2	8.63	8.6020	35.10	34.973
0.1	9.44	9.4085	37.47	37.438
0.0	10.25	10.215	39.81	39.921
−0.1	11.05	11.024	42.15	42.432
−0.2	11.86	11.835	44.49	44.977

in a decrease in the stiffness of the plate. Also, when α is positive (i.e. larger mass concentrating at the center) makes the plate vibrates for a longer period or a lower frequency.

The variations of Ω for β varying from –0.7 to 0.7 while α is kept constant (i.e. $\alpha = 0.5$) are shown in Fig. 4. It can be seen that in all the four cases ($k = 1$, 2, 3 and 4), when circumferential thickness $(1 - \beta \cos k\theta)$ varies only and the radial thickness $(1 - \alpha R)$ is kept constant, there is no fix pattern in frequencies. For the fundamental frequency Ω_1 of both **C**- and **S-plates**, it can be seen that the values remain almost constant for different k when β is small ($\beta \leqslant 0.15$ for **C-plate** and $\beta \leqslant 0.1$ for **S-plate**). Moreover, it is seen that the rate of increase of fundamental frequency with respect to k is faster for **S-plates** in comparison with **C-plates** for all k.

Three dimensional mode shapes and their corresponding contours of **C-plates** for the first six normal modes of vibrations corresponding to $k = 1$, 2, 3, 4 are shown in Figs 5 to 8 when $\alpha = \beta = 0.5$, $e_r = 1.0$ and $\nu = 0.3$. From the modes shown for a particular k, it is found that the boundary condition does not have significant influence on the mode shape when changes from clamped to simply-supported.

4. Conclusion

This paper deals with the free vibration of a class of plate having thickness variation in both radial and circumferential directions. Two dimensional boundary characteristic orthonormal polynomials are used in the Ritz method for the formulation of the eigenvalue problem. The accuracy and rate of convergence are shown. Comparison with results of special cases already reported in the literature shows good agreement for all the cases. It is found that the taper parameter for thickness variation in radial direction has more uniform influence on the changes in the magnitude of the frequencies in comparison with that in circumferential direction. Investigation of mode shapes shows that the clamped and simply-supported boundary condition has no significant influence.

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