Solving System of Fractional Order Differential Equations Using Legendre Operational Matrix of Derivatives

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Abstract: This paper presents approximate solutions of linear system of fractional differential equations (FDEs) by extending the approach of shifted Legendre operational matrix of derivatives together with spectral method. The results obtained in solving differential systems of linear FDE shows that the proposed method is factual. Fractional differential equations (FDEs) have been a useful tool for computing and modelling of computer components in telecommunication companies, mobile companies and for industrial practitioners and also plays a vital role in science and engineering.

1. Introduction

Consider the system of FDEs:

$$\begin{gathered} D^{\alpha_1}y_1(x) = f_1(x, y_1, y_{2,\cdots}y_n) \\ D^{\alpha_2}y_2(x) = f_2(x, y_1, y_{2,\cdots}y_n) \\ \vdots \qquad\qquad \vdots \\ D^{\alpha_n}y_n(x) = f_n(x, y_1, y_{2,\cdots}y_n), \end{gathered} \quad [1]$$

where, D^{α_i} is the α_i order derivative in the sense of Caputo and $0 < \alpha_i < 1$, with initial conditions $y_i(0) = d_i,\ i = 1,2,\cdots n$.

Ordinary differentiation and integration is a common branch of calculus that have un limited application at various field of studies, fractional calculus constitute the completion of integration and ordinary differentiation which have been proved to be an important tool that is used in computing and modelling of physical components in telecommunication companies, mobile companies and for industrial practitioners, it plays a vital role in science, aerodynamics, engineering, and control systems, see (Kilbas, Srivastava, & Trujillo 2006; Agarwal, Andrade, & Cuevas, 2020) and (Agarwal, Lakshmikantham, & Nieto 2010; Podlubny, 1999; Samko, Kilbas, & Marichev, 1993). The process of simulation and modelling of physical systems lead to requirement of fractional derivatives that involves the solution of FDEs. Based on this fact, many researchers have work on numerical, analytic, and explicit methods for solving FDEs, few among the once that exist are in (Ray, Chaudhuri, & Bere, 2006; Yang, Xiao, & Su, 2010; Odibat, 2011), this are the analytical and numerical methods for solving

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FDEs. Podlubny in (Podlubny, 1999) make use of some properties related to Riemann-Liouville derivative and the Grunwald-Letnikov and introduce a numerical technique for solving arbitrary order derivative. The authors in (Diethelm, Ford, & Freed, 2020) proposed predictor-corrector a numerical approach for solving FDEs.

There are many difficulties that may arise when solving FDEs like convergent problem, singularity difficulties in solving complex systems which lead to no solution. Because of these issues, the authors in (Doha, Bhrawy, & Ezz-Eldien, 2011; Doha, Bhrawy, & Ezz-Eldien, 2012; Saadatmandi, & Dehghan, 2010) came up with operational matrix method that decomposed the FDEs into system of algebraic equations. The author in (Ibrahim, 2020) and the authors in (Ibrahim, & Rababah, 2020; Rababah, & Ibrahim, 2016a; Rababah, & Ibrahim, 2016b; Rababah, & Ibrahim, 2018) proposed different technique that involve numerical approximation of differential equations and curves respectively, which is an important issue in-techniques for solving fractional differential equations.

To tackle this problem, we introduce the application of the Legendre operational matrix of fractional order derivative that provide approximate solutions to the system of FDEs in Eq. (1). The paper scheduled as, Mathematical preliminaries and definitions is provided in Section 2. Legendre operational matrix is introduced in this Section 3 to solve system of FDEs. In section 4, the result obtained is used to show the effectiveness of the method by considering an example. We conclude in section 5.

2. Mathematical Preliminaries

Definitions with basic properties of fractional calculus are introduced in this section. Regarding the modelling of a physical systems, the Riemann-Liouville definition has some short comings (Podlubny, 1999). For this reason, the Caputo's definition is more guarantee in practical point of view, based on this fact, we are going to make use of it.

Definition 2.1 Consider the Caputo fractional derivative D^{α} as:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \qquad n-1<\alpha<n,\ n\epsilon\,\mathbb{N} \tag{2}$$

The Caputo fractional derivative has the following properties.

$$D^{\alpha}C=0, \qquad (C\ is\ constant) \tag{3}$$

$$D^{\alpha}t^{\beta}=\begin{cases}0, & \beta\in\mathbb{N}\cup\{0\}\quad and\quad \beta<\lceil\alpha\rceil\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, & \beta\in\mathbb{N}\cup\{0\}\ and\ \beta\geq\lceil\alpha\rceil\quad or\ \beta\notin\mathbb{N}\ and\ \beta>\lfloor\alpha\rfloor,\end{cases} \tag{4}$$

where $\lceil\alpha\rceil$ stand for the least integer $\geq\alpha$ and $\lfloor\alpha\rfloor$ stand for the greatest integer $\leq\alpha$.

Observe that, the operator defining Eq (2) is linear, since,

$$D^{\alpha}\big(\lambda f(t)+\mu g(t)\big)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t). \tag{5}$$

Where $\lambda\ and$ μ are constants.

Definition 2.2 Consider the Legendre polynomials on the interval [-1, 1], which can be generated from the recurrence formulae.

$$L_{i+1}(t) = \frac{2i+1}{i+1} t L_i(t) - \frac{i}{i+1} L_{i-1}(t), \qquad i = 1,2,\cdots,$$

where $L_0(t) = 1$ *and* $L_1(t) = t$. Transforming the Legendre polynomial on the interval $[-1,1]$ to the interval $[0,1]$ requires the change of variable $t = 2x - 1$, The process is called the shifted Legendre polynomials given by $L_i(2x-1)$ and indicated by $P_i(x)$.

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} P_i(x) - \frac{i}{i+1} P_{i-1}(x), \quad i = 1,2,\cdots, \qquad [6]$$

where $P_0(x) = 1$ *and* $P_1(x) = 2x - 1$

The formulae for the shifted Legendre polynomials $P_i(x)$ of degree i is expressed in analytical form as:

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)x^k}{(i-k)!\,(k!)^2}. \qquad [7]$$

Observe that, $P_i(0) = (-1)^i$ *and* $P_i(1) = 1$.

The orthogonality condition is.

$$\int_0^1 P_i(x)P_j(x)\,dx = \begin{cases} \dfrac{1}{2i+1}, & for\ i = j \\ 0, & for\ i \neq j. \end{cases} \qquad [8]$$

A function $y(x) \in L_2[0,1]$ can be given in form of shifted Legendre polynomials as

$$y(x) = \sum_{k=1}^{\infty} c_k P_k(x),$$

where c_k is

$$c_k = (2k+1) \int_0^1 y(x)\, P_k(x) dx, \qquad k = 0,1,2,\cdots$$

By considering the first (n + 1) terms

$$y(x) = \sum_{k=0}^{n} c_k\, P_k(x) = \boldsymbol{C}\Phi(\boldsymbol{x}),$$

where C and $\Phi(\boldsymbol{x})$ are the shifted Legendre coefficient and vectors respectively

$$\boldsymbol{C} = [c_0, c_1, \cdots c_n]. \qquad [9]$$

$$\Phi(\boldsymbol{x}) = [P_0(x), P_1(x), \cdots, P_n(x)]. \qquad [10]$$

3. Operational Matrix for Legendre

Differentiating the vector $\Phi(\boldsymbol{x})$ lead to

$$\frac{d\Phi(\boldsymbol{x})}{dx} = \boldsymbol{D}^{(1)}\Phi(\boldsymbol{x}) \quad [11]$$

Where $\boldsymbol{D}^{(1)}$ is the $(n+1) \times (n+1)$ operational matrix of derivatives

$$\boldsymbol{D}^{(1)} = (d_{ij}) = \begin{cases} 2(2j+1), & for\ j = i-k \\ 0, & otherwise, \end{cases} \quad [12]$$

where $\begin{cases} k = 1,3,\dots,m, & for\ m\ odd \\ k = 1,3,\dots,m-1, & for\ m\ even \end{cases}$

However, the generalization of the operational matrix of derivative given in (12) to arbitrary fractional order say, $\alpha > 0$, was proved in (Saadatmandi & Dehghan, 2010) and is given by.

$$D^{\alpha}\Phi(x) \simeq \mathbf{D}^{(\alpha)}\Phi(x), \quad [13]$$

where $\mathbf{D}^{(\alpha)}$ is the $(m+1) \times (m+1)$ operational matrix of order α

$$\mathbf{D}^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,m,k} \end{pmatrix}.$$

Also, $\theta_{i,j,k}$ is given by.

$$\theta_{i,j,k} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!\,(l+j)!}{(i-k)!\,k!\,\Gamma(k-\alpha+1)(j-l)!\,(l!)^2(k+l-\alpha+1)} \quad [14]$$

3.1 Using the Operational Matrix to Solve Linear System of FDEs

Considering the system of FDEs in Eq. (15), we make use of the shifted Legendre operational matrix to solve such problem.

$$\begin{cases} D^{\alpha_1}y_1 + \sum_{j=1}^{k_1} a_{1,j}\, y_j = g_1(x), & \lceil\alpha_1\rceil = m_1, \\ D^{\alpha_2}y_2 + \sum_{j=1}^{k_2} a_{2,j}\, y_j = g_2(x), & \lceil\alpha_2\rceil = m_2, \\ \vdots \quad \vdots \quad \vdots & \vdots \\ D^{\alpha_n}y_n + \sum_{j=1}^{k_n} a_{n,j}\, y_j = g_n(x), & \lceil\alpha_n\rceil = m_n, \end{cases} \quad [15]$$

with initial condition $y_i^{j}(0) = d_{i,j}, \quad i = 1,2,\cdots,n, \quad j = 0,1,\cdots,m-1,$

where k_i, $a_{i,j}$, and $d_{i,j}$ are constants. Here the D^{α_i} stand for the Caputo derivative of order α_i. Lets $y_i(x)$ and $g_i(x)$ in Eqs. (16) and (17) be the approximate Shifted Legendre polynomials

$$y_i(x) = \sum_{k=0}^{m} c_{i,k}\, P_k(x) = \mathbf{C}_i \Phi(x) \qquad [16]$$

$$g_i(x) = \sum_{k=0}^{m} g_{i,k}\, P_k(x) = \mathbf{G}_i \Phi(x), \qquad [17]$$

where the vector $C_i = [c_{i,0},\ c_{i,1}, \cdots, c_{i,m}]$ can be identified vector, while the vector $G_i = [g_{i,0},\ g_{i,1}, \cdots, g_{i,m}]$ is familiar. Making use of (13) on (16) lead to

$$D^{\alpha_i} y_i(x) \simeq \mathbf{C}_i \mathbf{D}^{(\alpha_i)} \Phi(x), \quad i = 1,2,\cdots,n. \qquad [18]$$

Calculating the residual of (15) with the help of (13) and (16) yields

$$R_m^i(x) = \left(\mathbf{C}_i \mathbf{D}^{(\alpha_i)} + \sum_{j=1}^{k_i} a_{i,j}\, \mathbf{C}_j - \mathbf{G}_i \right) \Phi(x), \quad i = 1,2,\cdots,n. \qquad [19]$$

Applying tau method led to $(m - m_i + 1)$ linear equations

$$\langle R_m^i(x), P_j(x) \rangle = \int_0^1 R_m^i\,(x) P_j(x) dx = 0, \quad j = 0,1,\cdots,m_i - 1. \qquad [20]$$

Using (13) and (16), one can obtain

$$y_i^j(0) = \mathbf{C}_i \mathbf{D}^{(j)} \Phi(0) = d_{i,j}, \quad j = 0,1,\cdots,M_i - 1. \qquad [21]$$

(20) and (21) generate $n(m+1)$ set of linear equations, our goal is to obtain the solution $y_i(x)$ for $i = 1,2,\cdots,n$. This can be achieved by solving for the unknown vector $\mathbf{C}_i$.

4. Numerical Example

In this section, the explicit and numerical result obtained from the previous section will be used to solve some linear systems of FDE.

Example 4.1. Consider the system of linear FDEs.

$$\begin{aligned} D^{\alpha_1} y_1(x) &= y_1(x) + y_2(x) \\ D^{\alpha_2} y_2(x) &= -y_1(x) + y_2(x) \end{aligned} \qquad [22]$$

with initial conditions

$$y_1(0) = 0, \ \ y_2(0) = 1. \qquad [23]$$

Then, with $m = 8$ the approximation of $y_1(x)$ and $y_2(x)$ will be.

$$y_i(x) = \sum_{k=0}^{8} c_{i,k}\, P_k(x) = \mathbf{C}_i \Phi(x) \quad i = 1,2.$$

Let us solve this problem first with $\alpha_1 = \alpha_2 = 1$, and exact solution as

$$y_1(x) = e^x sinx \text{ and } y_2(x) = e^x cosx.$$

So, we have

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 18 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 22 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 18 & 0 & 26 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 22 & 0 & 30 & 0 \end{pmatrix}$$

Using (19) we rewrite (22) as

$$\begin{cases} R_8^1(x) = (\mathbf{C}_1\mathbf{D}^{(1)} - \mathbf{C}_1 - \mathbf{C}_2)\Phi(x) \\ R_8^2(x) = \left(\mathbf{C}_2\mathbf{D}^{(1)} + \mathbf{C}_1 - \mathbf{C}_2\right)\Phi(x). \end{cases}$$

By using Eqs. (20) and (21), we obtained 18 equations that leads to

$\mathbf{C}_{1,0} = 0.9093306736$, $\mathbf{C}_{1,1} = 1.13407384$, $\mathbf{C}_{1,2} = 0.2363240605$, $\mathbf{C}_{1,3} = 0.009901654498$,

$\mathbf{C}_{1,4} = -0.001959877229$, $\mathbf{C}_{1,5} = -0.0002975652235$, $\mathbf{C}_{1,6} = -0.00001723840079$,

$\mathbf{C}_{1,7} = -0.2863271581 \times 10^{-6}$, $\mathbf{C}_{1,8} = 0.2511255047 \times 10^{-7}$, $\mathbf{C}_{2,0} = 1.378024614$, $\mathbf{C}_{2,1} = 0.2720079791$, $\mathbf{C}_{2,2} = -0.1402860310$, $\mathbf{C}_{2,3} = -0.03758092173$,

, $\mathbf{C}_{2,4} = -0.003401450684$, $\mathbf{C}_{2,5} = 0.00008112711784$, $\mathbf{C}_{2,6} = 0.9793894685 \times 10^{-5}$, $\mathbf{C}_{2,7} = 0.1039703672 \times 10^{-5}$, $\mathbf{C}_{2,8} = 0.4420102768 \times 10^{-7}$.

Consequently, the solutions $y_i(x)$ for $i = 1,2$ are calculated and the Fig.1 and Fig.2 shows that the proposed method provide a solution that is equivalent to the exact solution with minimum error.

Fig.3 shows the approximate solution of system (22) for $\alpha_1 = 0.7$ and $\alpha_2 = 0.9$. The solution

Fig.1: y1(x) and exact y1(x) when α1=α2=1 for example 4.1.

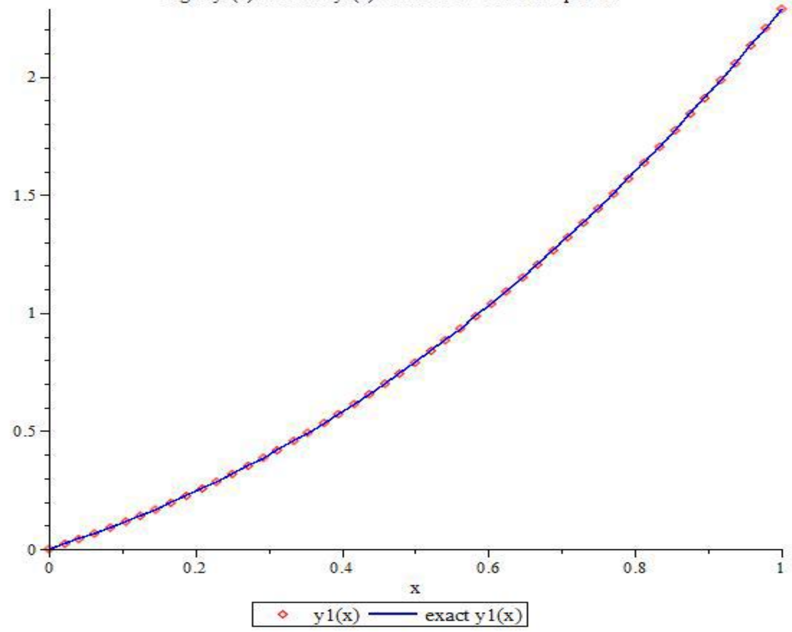
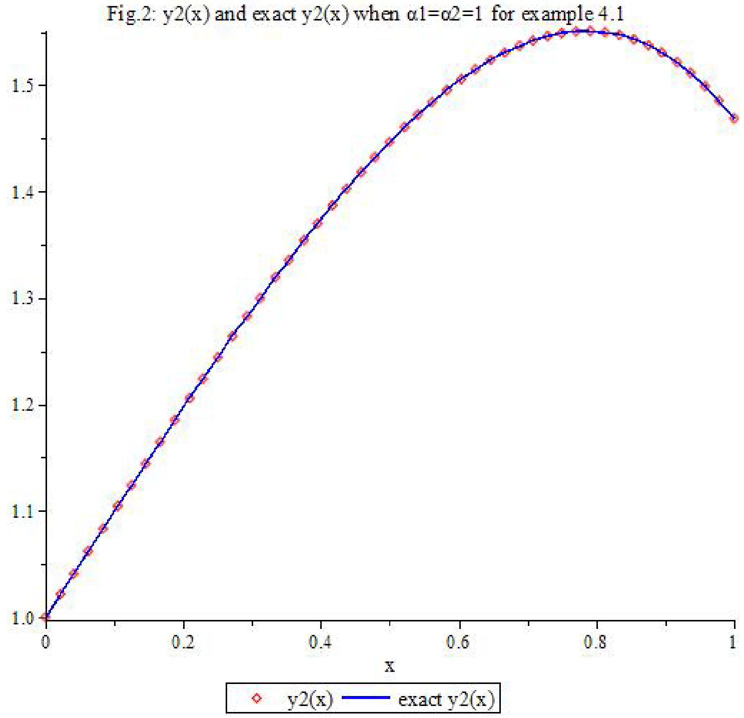


Fig.2: y2(x) and exact y2(x) when α1=α2=1 for example 4.1



obtained is equivalent with the solution in Monami, & Odibat (2007)

Fig.3: y1(x) and y2(x) when α1=0.7,α2=0.9 for example 4.1

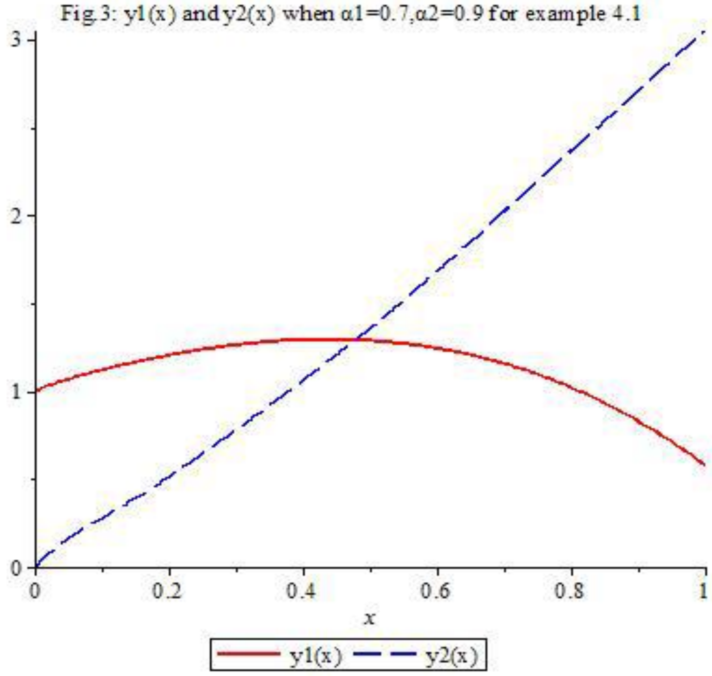
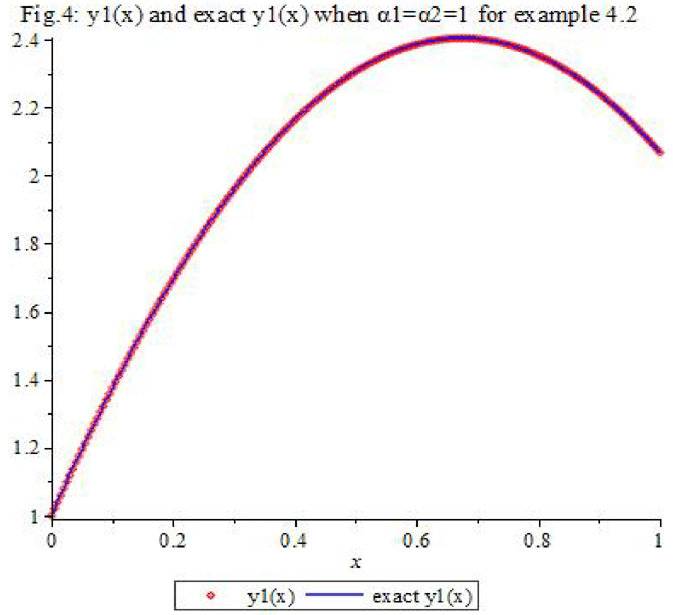


Fig.4: y1(x) and exact y1(x) when α1=α2=1 for example 4.2



Example 4.2. Consider the following linear system of FDEs.

$$\begin{aligned} D^{\alpha_1} y_1(x) + y_2(x) = 1, &\quad 0 < \alpha_1 \leq 1 \\ D^{\alpha_2} y_2(x) - y_1(x) = 4, &\quad 0 < \alpha_2 \leq 1 \end{aligned} \qquad [24]$$

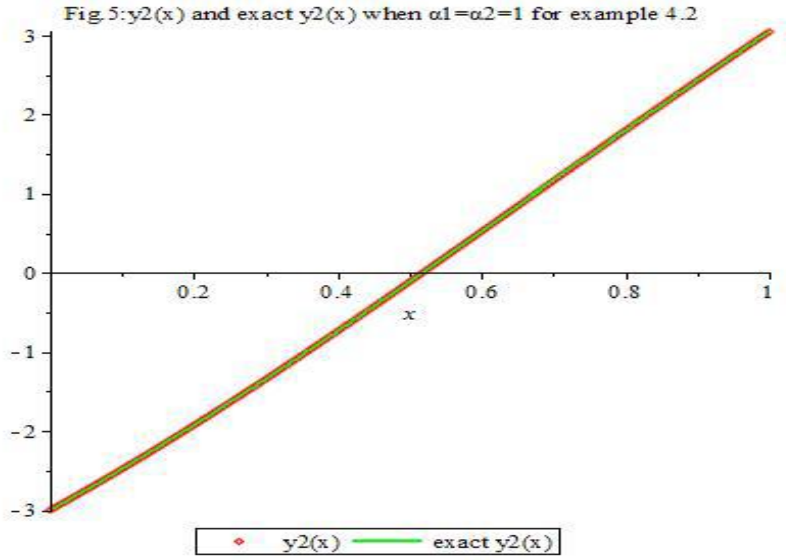
subject to the initial conditions

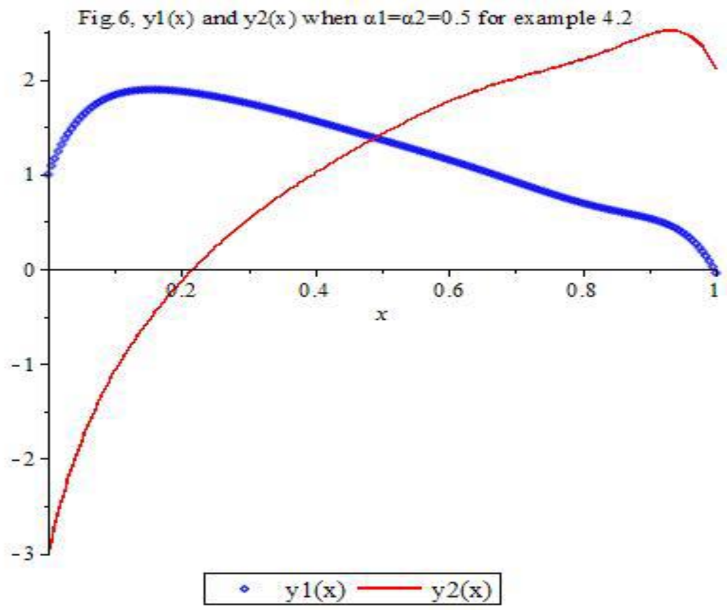
$$y_1(0) = 1, \;\; y_2(0) = -3. \qquad [25]$$

with exact solution, for $\alpha_1 = \alpha_2 = 1$ as

$$\begin{aligned} y_1(x) &= 5cosx + 4sinx - 4 \\ y_2(x) &= -4cosx + 5sinx + 1 \end{aligned} \qquad [26]$$

By considering the proposed method in section 3, the approximate solution $y_1(x)$ and $y_2(x)$ when $\alpha_1 = \alpha_2 = 1$ and $m = 8$ are depicted in fig. 4 and fig. 5 respectively. The approximate solution is equivalent with the exact solution with minimum error.





Setting $\alpha_1 = \alpha_2 = \frac{1}{2}$ with $m = 8$ and repeating the same process with the proposed method in section 3, we evaluate the solution of $y_1(x)$ and $y_2(x)$. Fig. 6 show the result. Since for the first two examples, we do not have the exact solution when α_1 and α_2 are in fractional form, hence, in the next example we purposely design a system where there is approximate known solution.

Example 4.3

$$y_1(x) = 5cosx + 4sinx - 4$$
$$y_2(x) = -4cosx + 5sinx + 1 \quad [27]$$

subject to the initial conditions

$$y_1(0) = 0, \ \ y_2(0) = 1. \quad [28]$$

if $\alpha_1 = \alpha_2 = 1$ and $f_1(x) = cosx + xsinx, \quad f_2(x) = -sinx - x^2 sinx$, the solution is given by $y_1(x) = sinx \quad y_2(x) = cosx.$
Using (19), we have

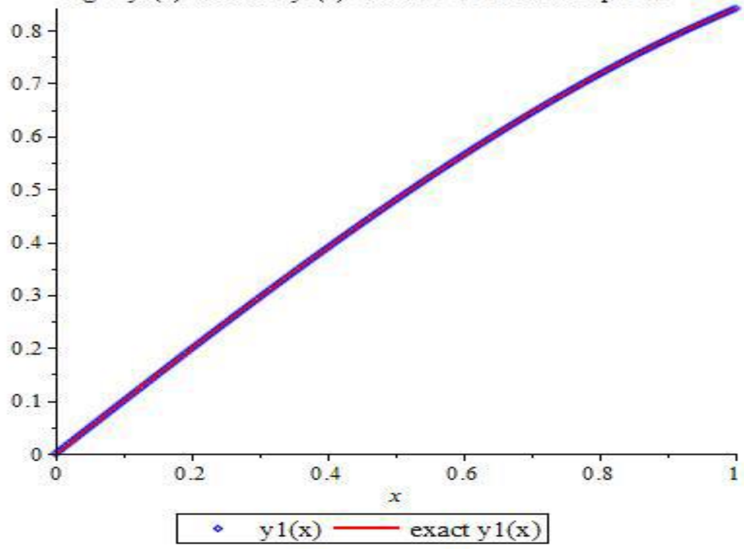
$$\begin{cases} R_8^1(x) = (\mathbf{C}_1\mathbf{D}^{(1)} + x\mathbf{C}_2)\Phi(x) \\ R_8^2(x) = (\mathbf{C}_2\mathbf{D}^{(1)} - x^2\mathbf{C}_1)\Phi(x) \end{cases}$$

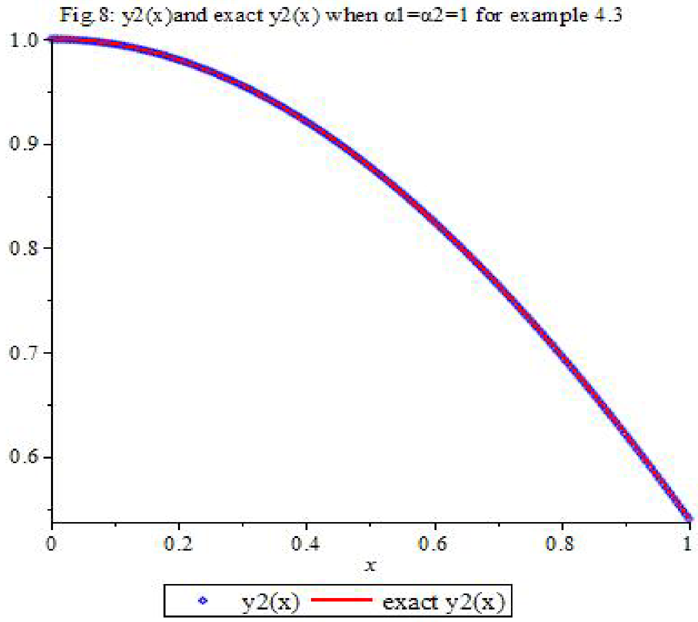
by considering the initial conditions and by expanding the RHS of (27) in series form, equating coefficients for $R_8^1(x) = series(f_1(x))$ and $R_8^2(x) = series(f_2(x))$, we get a system of linear equations. Solving the system of linear equations by Maple 18, we obtain the following

$$c_{1,0} = \frac{3707}{8064}, \quad c_{1,1} = \frac{51761}{120960}, \quad c_{1,2} = \frac{-4747}{120960}, \quad c_{1,3} = \frac{-457}{63360}, \quad c_{1,4} = \frac{25}{88704}, \quad c_{1,5} = \frac{1}{34944}, \quad c_{1,6} = \frac{-1}{1330560}, \quad c_{1,7} = \frac{-1}{17297280}, \quad c_{1,8} = 0, \quad c_{2,0} = \frac{305353}{362880}, \quad c_{2,1} = \frac{-10099}{43200}, \quad c_{2,2} = \frac{-143371}{1995840}, \quad c_{2,3} = \frac{5617}{1425600}, \quad c_{2,4} = \frac{1489}{2882880}, \quad c_{2,5} = \frac{-37}{2358720}, \quad c_{2,6} = \frac{-13}{9979200}, \quad c_{2,7} = \frac{1}{34594560}, \quad c_{2,8} = \frac{1}{518918400}.$$

Consequently the solutions $y_i(x)$ for $i = 1,2$ are calculated and the Fig.7 and Fig.8 shows that our solution is in good agreement with the exact solution.

Fig 7: y1(x) and exact y1(x) when α1=α2=1 for example 4.3





Now, If $\alpha_1 = \alpha_2 = 0.5$ and if we set the solution again to be $y_1(x) = sinx$ and $y_2(x) = cosx$. Then we should have the RHS is

$$f_1(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{8}{15}\frac{x^{\frac{5}{2}}}{\sqrt{\pi}} + \frac{32}{945}\frac{x^{\frac{9}{2}}}{\sqrt{\pi}} - \frac{128}{135135}\frac{x^{\frac{13}{2}}}{\sqrt{\pi}} + \frac{512}{34459425}\frac{x^{\frac{17}{2}}}{\sqrt{\pi}} - \frac{2048}{13749310575}\frac{x^{\frac{21}{2}}}{\sqrt{\pi}} + x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7 + \frac{1}{40320}x^9 - \frac{1}{3628800}x^{11}$$

and

$$f_2(x) = \frac{-4}{3}\frac{x^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{16}{105}\frac{x^{\frac{7}{2}}}{\sqrt{\pi}} - \frac{64}{10395}\frac{x^{\frac{11}{2}}}{\sqrt{\pi}} + \frac{256}{2027025}\frac{x^{\frac{15}{2}}}{\sqrt{\pi}} - \frac{1024}{654729075}\frac{x^{\frac{19}{2}}}{\sqrt{\pi}} - x^3 + \frac{1}{6}x^5 - \frac{1}{120}x^7 + \frac{1}{5040}x^9 - \frac{1}{362880}x^{11}$$

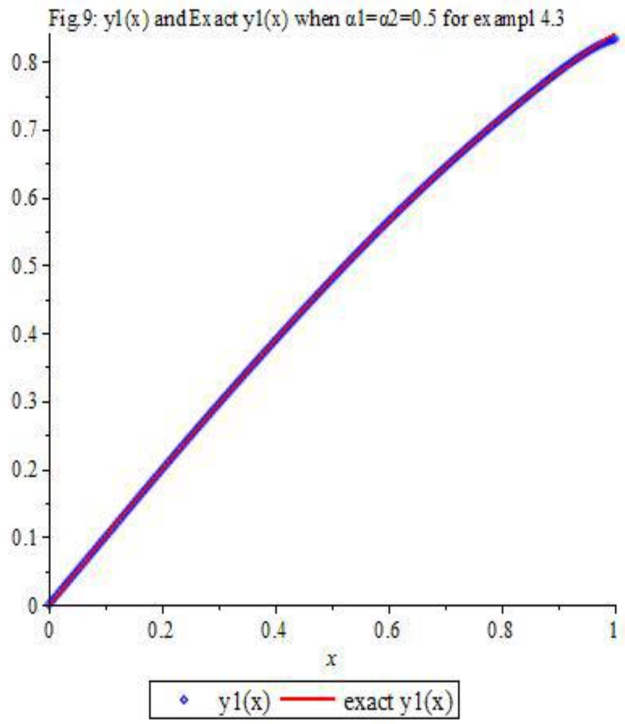
Using (19), we have

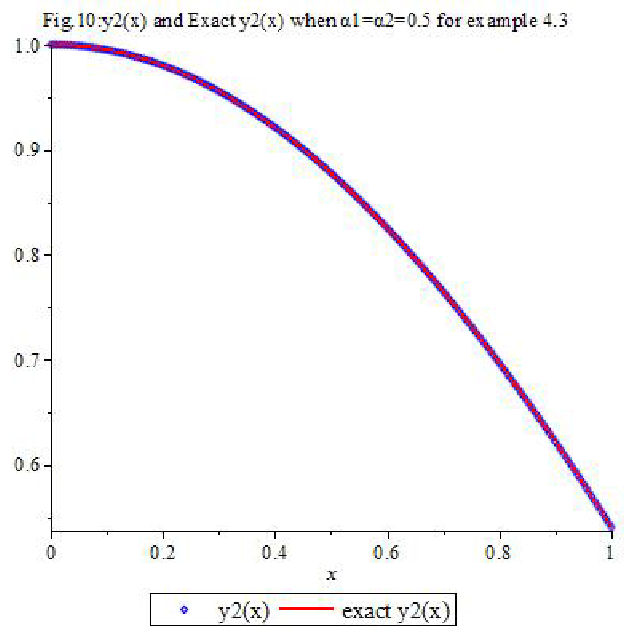
$$\begin{cases} R_8^1(x) = (\mathbf{C}_1\mathbf{D}^{(0.5)} + x\mathbf{C}_2)\Phi(x) \\ R_8^2(x) = \left(\mathbf{C}_2\mathbf{D}^{(0.5)} - x^2\mathbf{C}_1\right)\Phi(x), \end{cases}$$

by considering the initial conditions and by setting x as $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \cdots, \frac{15}{16}$ we get a system of linear equations. Solving the system of linear equation by Maple 18, we obtain the following

$$\begin{aligned} c_{1,0} &= 0.4596816778,\ c_{1,1} = 0.4276384606,\ c_{1,2} = -0.03980198886,\ c_{1,3} \\ &= -0.007860382622, c_{1,4} = -0.0009552411593,\ c_{1,5} \\ &= -0.001305466691,\ c_{1,6} = -0.00137331118,\ c_{1,7} = -0.001877028101,\ c_{1,8} \\ &= -0.0009555533489,\ c_{2,0} = 0.8415010942,\ c_{2,1} = -0.2337307134,\ c_{2,2} \\ &= -0.07177149475,\ c_{2,3} = 0.003996851546,\ c_{2,4} = 0.0005752980385,\ c_{2,5} \\ &= 0.0000160256798,\ c_{2,6} = -0.0000040218736,\ c_{2,7} \\ &= -0.00006562939305,\ c_{2,8} = -0.0000843410948. \end{aligned}$$

Consequently the solutions $y_i(x)$ for $i = 1,2$ are calculated and the Fig.9 and Fig.10 shows that our solution is in good agreement with the exact solution.





5. Conclusion

In this paper, we extend the application of Legendre operational matrix fractional order derivative to solve linear systems of FDEs. The main essence of the method is to use the operational matrices in collaboration with tau methods to decompose the system of FDEs to a system of algebraic equations. The results obtained is shown to be equivalent with the known exact solutions and are efficient and satisfactory compared to the existing results.

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