Antitrust policies in network environments

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Abstract

We study the effectiveness of two different antitrust policies by characterizing the network structure of market-sharing agreements that arises under those settings. Market-sharing agreements prevent firms from entering each other's market. The set of these agreements defines a collusive network, which is pursued by antitrust authorities. We show that under a constant probability of inspection and a penalty equal to firm's limited liability, firms form collusive alliances where all of them are interconnected. In contrast, when the probability of inspection reacts to prices firms form collusive cartels where they are not necessarily fully interconnected. These results suggest that more competitive structures can be sustained in the second case than in the first one. Notwithstanding, antitrust laws may have a pro-competitive effect in both scenarios, as they give firms in large alliances more incentives to cut their agreements at once.

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1. Introduction

Cartels are present among us even if we are not able to see them. Collusive practices create concern to the public in general. Thus antitrust authorities devote considerable time and efforts in order to discover and prosecute them.

Among different collusive practices, market-sharing agreements is one of them. Using these agreements firms divide up a market and agree not to enter each other's territory. In the present article, market-sharing agreement are modeled as bilateral agreements and the set of these reciprocal agreements gives rise to a collusive network among firms.

One goal of antitrust authorities is to weaken firms' incentive to form and maintain collusive agreements over time in order to increase market efficiency. This article addresses this point by studying the effectiveness of antitrust policy to deter the formation of collusive agreements in a network context.

Toward that ends, this article examines the collusive network structure that arises under the presence of two different antitrust policies. One of these policies is defined by a fixed probability of inspection, and a penalty equal to the firm's limited liability; that is equal to the firm's total profits. The other policy is characterized by a probability of inspection that depends on prices and a fixed fine.

In the present network framework, the probability of a firm being detected depends on the network structure since it depends on the number of agreements that each firm has signed. That is, the probability of firm i's being detected depends not only on whether firm i is inspected but also on whether any firm that has formed an agreement with i is inspected. If a firm is inspected and a market-sharing agreement exists, all firms involved are penalized. However, the firm in consideration may be detected without being inspected because any firm that has an agreement with it, is inspected.

When the antitrust authority reacts to prices, the previous relationship is strengthened because the probability of inspection is not fixed but depends on the agreements that each firm has signed. In this case the probability that a firm being caught depends not only on the agreements that this firm has signed but also on the number of agreements that its criminal partners have signed.

We provide the stable network characterization in the two antitrust settings. First of all, it worth saying that in the absence of the antitrust authority, a network is stable if its collusive cartels are large enough. We show that when an antitrust authority is considered, however, structures that are more competitive can be sustained through bilateral agreements. Furthermore, we show that under certain conditions, the two policies have a pro-competitve impact. As the probability of inspection increases, firms in large cartels have more incentives to renege on all their collusive agreements at once, and it might lead to break-down collusion.

Comparing both policies, when the probability of inspection is fixed, member's cartels are fully interconnected, i.e., components are complete. However, when the probability of inspection is sensitive to prices, components are not necessary complete. This result implies that more competitive structures may arise with respect to the first case.

This article brings together elements from the literature of social networks, collusion (particularly, market-sharing agreements), and law enforcement. Networks is currently a very active field of research. Prominent contributions to this literature include, among others, Jackson and Wolinsky (1996), Goyal (1993), Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005). In particular, in first paper, the formation and stability of social networks are modeled when agents choose to maintain or destroy links using the notion of pair-wise stability. We follow Jackson and Wolinsky (1996) and Jackson and van den Nouweland (2004) to characterize the stable and the strongly stable networks.

Asides from these theoretical articles, there is also more and more literature that applies the theory of economic networks to models of oligopoly. In particular, the present article is closely related to Belleflamme and Bloch (2004) and Roldán (2010). Belleflamme and Bloch (2004) have analyzed the collusive network of market-sharing agreements among firms, but they do not consider the existence of antitrust authorities. Therefore, their results may be limited under those circumstances. Roldán (2010), however, studies how the presence of an antitrust authority affects the market-sharing agreements made by firms and examines the network structure that arises when each firm takes into account the cost, imposed by competition authorities, from signing these collusive agreements. She shows that under the presence of an antitrust authority more competitive structures are possible to sustain over time. The present article compares the effect of different antitrust policies in the collusive network structure.

Regarding the collusion literature, after the seminal contribution of Stigler (1950), collusive cartels have been extensively studied. For an excellent reference of this literature, see Vives (2001). As the present article, there are a number of articles that study the effect of antitrust policy on cartel behavior. Block et al. (1981) is a first systematic attempt to estimate the impact of antitrust enforcement on horizontal minimum price fixing. Their model explicitly considers the effect of antitrust enforcement on the decision of firms to fix prices collusively. They show that a cartel's optimal price is an intermediate price (between the competitive price and the cartel's price in absence of antitrust authority) and that this intermediate price depends on the levels of antitrust enforcement efforts and penalties.[4]

However, the interest for studying the effect of the antitrust policy on the collusive behavior has recently reemerged. For example, Harrington (2004, 2005) explores how detection affects cartel pricing when detection and penalties are endogenous. Firms want to raise prices but not suspicions that they are coordinating their behavior. Harrington (2005), assumes that the probability of detection is sensitive to price changes, he shows that the steady-state price is decreasing in the damage and in the probability of detection. These results are in line with the results of the present article in the sense of the pro-competitive effect of the antitrust policy. More recently, Motchenkova et al (2010) analyze the effectiveness of antitrust policy in a repeated oligopoly model when both fines and detection probabilities are endogenous. The main difference between the present article and those articles is that this article studies the impact of the antitrust policy on the collusive behavior in a social network framework.

The remaining of the article is organized as follows. Section 2 presents the basic model to be study. Section 3 characterizes the set of pair-wise stable networks when the probability of inspection is fixed and the penalty is equal to firm's limited liability; while Section 4 characterizes the network structure under an antitrust authority that reacts to prices. Section

[4]Besanko and Spulber (1989, 1990) with a different approach, use a game of incomplete information where the firms' common cost is private information and neither the antitrust authority nor the buyers observe the cartel formation. They find that the cartel's equilibrium price is decreasing in the fines. LaCasse (1995) and Polo (1997) follow this approach.

5 concludes. All proof are relegated to the Appendix.

2. The model

We consider a simple model to study the interaction between an (antitrust) authority and a finite number of firms which choose whether or not to form pairwise collusive agreements. The set of these collusive links gives rise to a network. We are interested in the structure of the collusive network that emerges under the presence of the authority. This analysis is carried out in two different settings which are described below.

Collusive Networks. With $\mathcal{N} := \{1, ..., n\}$ we denote a set of n identical risk neutral firms. Following BB (2004), we associate market i to firm i. All markets are identical. When only one firm is active in a market, other firms find profitable to enter that market. The firms, however, may enter into a reciprocal market sharing agreement (henceforth, agreement). An agreement between firms i and j restrains firm i from entering firm j's market and viceversa. For each pair of distinct firms i and j, we denote with $a_{ij} = 1$ the existence of an agreement between them; otherwise $a_{ij} = 0$.[5] A (collusive) **network** g is thus summarized by the agreement matrix $\mathcal{A} := (a_{ij})$. With $\mathcal{G}$ we denote the set of all possible networks.
The **neighborhood** of firm i in network g is the set of firms with whom firm i has entered into an agreement; that is $\mathcal{N}_i := \{j \in \mathcal{N} - i : a_{ij} = 1\}$. With m_i we denote the degree of firm i; that is the number of firms belonging to $\mathcal{N}_i$. Any firm i for which $m_i \geq 1$ is a **collusive** firm. We assume, as usual, that the profit a firm gets from being active in market i is decreasing in the number of active firms

$$\pi(n_i) \geq \pi(n_i + 1), \tag{A1}$$

where n_i is the number of active firms in market i. We also further assume that the profit function $\pi(.)$ satisfy the decreasing ratio property

$$\frac{\pi(n_i - 1)}{\pi(n_i)} \geq \frac{\pi(n_i)}{\pi(n_i + 1)}. \tag{A2}$$

Notice that A2 implies that the profit function $\pi(.)$ is convex. That is

$$\pi(n_i - 1) - \pi(n_i) \geq \pi(n_i) - \pi(n_i + 1),$$

because $\pi(n_i + 1) \leq \pi(n_i)$. Since a firm collects profits from both its own market and all other markets in which the firm enters, we can write firm i's total profit in the *absence* of the authority as

$$\Pi^i(g) = \pi(n_i) + \sum_{k, a_{ik}=0} \pi(n_k). \tag{1}$$

[5]By definition, we set $a_{ii} = 1$ for any firm $i \in \mathcal{N}$. Note also that by the reciprocal nature of an agreement $a_{ij} = a_{ji}$.

The Antitrust Authority. Before firms choose their agreements, the authority commits to an antitrust police. More precisely, each firm i is inspected with probability $\gamma_i \in (0,1)$. For simplicity, we assume that inspection ends up in detection without error. Thus, conditional on inspection, a collusive firm is *directly* detected with probability one. But collusive firms can also be *indirectly* detected. That is, in the presence of an agreement between firms i and j if one of them is inspected, the other will also be detected with probability one. Moreover, for simplicity, we assume that the chain of indirect detections has lenght equal to one; that is if firm i is indirectly detected, no firm having a link with firm i will be detected by the authority. A detected firm pays a fine equal to $f_i > 0$. To sum up, the authority's policy scheme is a tuple $(\gamma_i, f_i)_{i \in \mathcal{N}}$. Assuming that inspection across firms is independent, i.e., inspection of any firm has no effect on or relationship with the inspection of any other firm, a collusive firm's detection probability is

$$\beta_i(g) = 1 - (1-\gamma_i) \prod_{k \neq i, a_{ik}=1} (1-\gamma_k). \tag{2}$$

Note that the detection probability of firm i depends positively on its degree m_i. Thus, intuitively, m_i captures the vulnerability of the firm. Note also that, because firms can be detected not only directly but also indirectly, the detection probability is higher than the inspection probability. Finally, the (expected) payoff of a collusive firm in network g is

$$E_i(g) = \Pi_i(g) - \beta_i(g) f_i, \tag{3}$$

whereas a firm with no agreements gets a payoff just equal to $\pi(n) + \sum_{k \neq i, a_{ik}=0} \pi(n_k)$.

Stable Networks. To study the set of collusive networks that emerge under the presence of an authority, we borrow our solution concept from JW. Thus, from the set $\mathcal{G}$ of all networks we select those which are 'strategically stable.' To formally define the notion of stability, the following definitions are useful. First, with ij we denote the existence of an agreement between firms i and j in network g; that is we write ij when $a_{ij} = 1$. Similarly, with $g+ij$ we denote the network obtained by replacing $g_{ij} = 0$ by $g_{ij} = 1$. Second, we say that network $g' \in \mathcal{G}$ is obtainable from network $g \in \mathcal{G}$ via deviations by $\mathcal{S} \subset \mathcal{N}$ if (a) $ij \in g'$ and $ij \notin g$, implies that $\{i,j\} \subset \mathcal{S}$; and (b) $ij \in g$ and $ij \notin g'$ implies that $\{i,j\} \cap \mathcal{S} \neq \emptyset$. Then network g is **pairwise stable** if the set $\mathcal{S}$ contains at most two firms and those firms in $\mathcal{S}$ are restricted to eliminate just one agreement or forming an additional one. In other words, this concept considers only deviations on a single link at a time.

A **complete** network g^c is a network in which every pair of distinct firms are joined by a link; that is $a_{ij} = 1$ for all $i, j \in \mathcal{N}$. We say that firm i is **isolated** if it is not linked to any other firm; that is if $a_{ij} = 0$ for all $j \in \mathcal{N}$. An **empty** network g^e is a network in which every firm is isolated. A **path** between firms i and j in network g is a list of distinct firms $i_1, i_2, ..., i_K$ such that $i_k i_{k+1} \in g$ for $k \in \{1, ..., K-1\}$, $i_1 = i$ and $i_K = j$. A network is **connected** if every pair of distinct firms $i, j \in \mathcal{N}$ can be joined by a path. A (connected) **component** g' of network g is a **maximal** connected piece of g. A component g' is complete if every pair of distinct firms in g' are joined by a link. Note that, from these definitions, an isolated firm is not considered a component. The size of a complete component, that is the number of firms in it, is equal to $m_i(g') + 1$. Finally, with $\mathcal{G}^* \subset \mathcal{G}$ we denote the set of pair-wise stable networks.

3. Collusive Networks and Antitrust Policy: Part I

In this section we study the set of pairwise stable collusive networks under the following policy scheme $(\gamma_i, f_i)_{i\in\mathcal{N}} = (\gamma, \Pi_i(g))_{i\in\mathcal{N}}$. Simply put, we assume that firms are inspected with the common and constant probability γ while fines are set equal to their total profits.

Concerning the inspection process, the reader may think of it as a situation in which the authority has a given available budget that allows it to inspect every firm with constant probability $\gamma \in (0, 1)$. Or, more generally, γ may be seen as the (common) belief entertained by the firms with respect to their chances of being inspected. So, firm i's probability of detection can now be written as:

$$\beta_i(g) = 1 - (1-\gamma)^{m_i+1}, \tag{4}$$

where, by definition, $m_i = n - n_i$.

In the economic literature of legal enforcement, fines are usually assumed to be socially costless. Therefore, when the authority seeks to deter collusion, fines –at least in the case of risk neutral firms, no legal errors and bankruptcy problems– should be set as high as possible in order to minimize inspection costs. It seems clear then that, under this principle, fines do not reflect neither the illegal gains obtained by the firms nor the harm caused by them. Several authors, see for instance, Motta and Polo (2003), Rey (2003) and Spagnolo (2004), study antitrust issues assuming that penalties are independent of the damages imposed by the firms. In contrast, Harrington (2004, 2005) and Motchenkova (2008) examine similar issues under the assumption that fines are, at least in some degree, sensitive to both illegal gains and the social harm caused by offenders. In particular, Harrington (2004,2005) considers a penalty function that depends on the price level set by the cartel. Since, in our case, fines are given by

$$f_i(g) = \pi(n_i) + \sum_{k, a_{ik}=0} \pi(n_k), \tag{5}$$

our setting is specially amenable to Harrington's analysis. More precisely, similar to Harrington (2004, 2005), one component of the fine $\pi(n_i)$ is determined by the collusive activity undertaken by firm i, while the other $\sum_{k,a_{ik}=0} \pi(n_k)$ is, in some sense, independent of firm i's collusive behavior.

Payoffs. Since $f_i(g) = \Pi_i(g)$, it is easily seen that the (expected) payoff of firm i for concluding an agreement with firm j is:

$$E_i(g+ij) - E_i(g) = \Big(\beta_i(g+ij)\Pi_i(g+ij) - \beta_i(g)\Pi_i(g)\Big),$$

and from expressions (3), (2) and (5), we get that:

$$E_i(g+ij) - E_i(g) = (1-\gamma)^{m_i+1} M_i(n_i, n_j, n_k, \gamma) \quad \forall k : g_{i,k} = 0, \tag{6}$$

where $M_i(n_i, n_j, n_k, \gamma) := \pi(n_i - 1) - \pi(n_i) - \pi(n_j) - \gamma\big(\pi(n_i-1) + \sum_{k\neq j, a_{ik}=0} \pi(n_k)\big)$.

Note that in the absence of an authority, the extra payoff firm i gets from an agreement depends only on its own number of agreements and that of the potential partner. However, in the presence of the authority, it also depends on the number of agreements of all other firms which are not firm i's partner.[6]

Since firm i will enter into an agreement with firm j if, and only if, $E_i(g+ij) - E_i(g)$ is positive, we focus our attention on $M_i(n_i, n_j, n_k, \gamma)$ and, hereafter, we refer to this function as the marginal return function. This function captures the main consequences for firm i of adding a link with firm j. First, as firm j will not enter in firm i's market, the profits of the latter must necessarily increase. More precisely, firm i's gain in its own market is given by $\pi(n_i - 1) - \pi(n_i)$. Second, by the reciprocal nature of the agreement, firm i will refrain from entering firm j's market. This leads to a decrease in the profit of firm i equal to $\pi(n_j)$. Finally, as firm j, the new partner of firm i, is inspected with probability γ, firm i suffers an increase in the expected fine equal to $\gamma\big(\pi(n_i - 1) + \sum_{k \neq j, a_{ik}=0} \pi(n_k)\big)$.

Lemma 1 below describes the main properties of the marginal return function.

Lemma 1 (Properties of Marginal Returns)**.** *The marginal return function $M_i(n_i, n_j, n_k, \gamma)$ is increasing in n_j, increasing in n_k, decreasing in γ and it has an ambiguous relationship with n_i.*

The main intuition behind this result is simple. First, it seems clear that a more effective antitrust police diminishes the attractiveness of collusive agreements. Second, when the number of active firms in market j, firm i's potential partner, increases, firm i's lost profit in market j decreases. Hence, the marginal returns become higher. Third, when the number of active firms in all those market in which firm i is active increases, i.e. an increase in n_k, the increase in the expected fine due to an extra link diminishes. Finally, the relationship between marginal returns and n_i is ambiguous. There are indeed to countervaling forces driving this result. On the one hand, as by Equation A2, the profit function is concave a decrease in n_i increases the marginal returns. More formally, as convexity implies that the difference $\pi(n_i - 1) - \pi(n_i)$ rises as n_i diminishes, the marginal return of an extra link increases. But, on the other hand, as firm j is inspected, and hence firm i's detected, with probability γ, firm i retains only a fraction, i.e., $(1-\gamma)$ of the extra gains due to the new link.

The main result of this section is:

Proposition 1 (Pair-wise Stable Networks)**.** *Network $g \in \mathcal{G}^*$ if and only if it can be partitioned into a set of isolated firms and complete components, g_1,...,g_L of different sizes $m(g_l) \neq m(g_{l'})$, $\forall l, l'$ such that neither an isolated firm has an incentive to form a link with another isolated one nor a firm that belongs to the smallest component has an incentive to cut a link with a firm inside it.*

Antitrust Policies and Stable Networks. In this setting, given the network characterization in Proposition 1, we examine the type of stable networks that can be sustained at different levels of the antitrust enforcement. First, it seems intuitive that if the efficacy of the authority is 'small', the complete network is always pair-wise stable. It is direct to formalize this

[6] The expressions above are valid when $m_i \geq 1$. However, when firm i is isolated, i.e. when $m_i = 0$, we have that $E_i(g+ij) - E_i(g) = \pi(N-1)(1-\gamma)^2 - \pi(N) - \pi(n_j) - \sum_{k \neq j, g_{ki}=0} \pi(n_k)(1-(1-\gamma)^2)$.

intuition. For that, let us define

$$\hat{\gamma} := 1 - \frac{2\pi\,(2)}{\pi\,(1)}.$$

We then have:

Proposition 2 (Ineffective Policy)**.** *The complete network $g^c \in \mathcal{G}^*$ if and only if $\gamma \leq \hat{\gamma}$.*

At the other extreme, intuition suggests that a sufficiently aggresive antitrust police breaks down any interconected network completely. Put it differently: a sufficient effective policy results in the empty network. This intuition is also easily formalized. For that, let us define

$$\bar{\gamma}(n) := 1 - \left(\frac{n\pi(n)}{\pi(n-1) + (n-2)\pi(N)}\right)^{0.5},$$

for all $n \in \{3, 4, ...\}$ and $\bar{\gamma}(n) < 1$.

Proposition 3 (Effective Policy)**.** *Let $n \in \{3, 4, ...\}$. Then, the empty network $g^e \in \mathcal{G}^*$ if and only if $\gamma > \bar{\gamma}(n)$.*

To get further intuition about Proposition 3, the following observations are useful. Since in an empty network all firms are isolated, stability requires that no two firms find attractive to create a link. That happens precisely when $\gamma > \bar{\gamma}(N)$. In other words, $\bar{\gamma}(N)$ is the smallest probability of inspection above which firms have no incentive to form an agreement given that all other firms remain isolated. Moreover, observe that $\bar{\gamma}(N)$ is strictly decreasing in N. That is, as N increases, the "loot" becomes less "attractive" (i.e., $\bar{\gamma}(N)$ is decreasing in N), and therefore the threshold will get smaller. Furthermore, straightforward computations show that $\bar{\gamma}(N) < \hat{\gamma}$. Consequently, we have:

Corollary 1 (Existence)**.** *Let $\gamma \in (\bar{\gamma}(n), \hat{\gamma}]$. Then, both the empty network g^e and complete network g^c belong to $\mathcal{G}^*$.*

Formally, Corollary 1 shows that the empty and complete network always exists; in other words, it shows that set $\mathcal{G}^*$ is not empty. First, for $\gamma \leq \hat{\gamma}$, the complete network $g^c \in \mathcal{G}^*$. Second, for $\gamma > \bar{\gamma}$, $g^e \in \mathcal{G}^*$. And given that $\bar{\gamma}(n) < \hat{\gamma}$, for $\gamma \in (\bar{\gamma}(n), \hat{\gamma}]$, g^e and g^c arise as pair-wise stable configurations.

It is interesting to observe that given our penalty function, when a firm is caught in a collusive agreement it loses profits not only from its own market but also from those other markets in which the firm is actively competing. Therefore firms in smaller components have a higher opportunity cost of forming a link in comparison to those firms belonging to bigger alliances. That is, a firm inside a small alliance does not have much to gain and has a lot to lose when one more link is made. More precisely, by signing an agreement, it gains $(1-\gamma)\pi(n_i - 1) - \pi(n_i)$, that gets smaller as the alliance is smaller and it loses not only the access to profits on foreign market j, $\pi\,(n_j)$, but it also loses, its expected gains in foreing markets[7]

$$\gamma \sum_{k:g_{ik}=0} \pi\,(n_k)\,,$$

[7] Remember that the number of active firms is greater in smaller components.

therefore, firms in smaller components are more sensitive to the antitrust enforcement. This intuition can be easily formalized as follows. Let $\gamma(n_i)$ be the smallest probability at which firm i is indifferent whether to form a new link or not. In other words, $\gamma(n_i)$ is such that $M_i(n_i, n_j, n_k, \gamma(n_i)) = 0$. It is direct to show that:

$$\gamma(n_i) = \frac{\pi(n_i - 1) - 2\pi(n_i)}{\pi(n_i - 1) + \sum\limits_{k:g_{ik}=0} \pi(n_k)}$$

Clearly, by definition, $M_i(n_i, n_j, n_k, \gamma) < 0$ for $\gamma > \gamma(n_i)$ and firm i will have no incentives to form a new agreement. Hence the following result follows immediately:

Proposition 4 (Smaller versus Bigger Components)**.** *Let firm $i \in g_i$ and $j \in g_j$ with $m(g_i) < m(g_j)$. Then $\gamma(n_i) < \gamma(n_j)$.*

Proposition 4 shows that as γ increases, the authority first tears down small components. In other words, smaller components are more sensitive to antitrust policy. In the limit, firms must decide to form a very large alliance (complete network) or no alliance at all (empty network).

4. Collusive Networks and Antitrust Policy: Part II

In the previous section we studied the case in which the policy scheme consisted of a fixed detection probability and a fine equal to the total profits obtained by the collusive firm. This type of policy is usually applied in many different countries. For instance, in both the EU and the US, current laws establish not only a ceiling to the maximum fine but also try to relate the penalty to cartel's consequences. These policies often arises as a result of a concern about firms' ability to pay the fines and the presence of potential legal errors. On the one hand, the presence of legal errors may deter some socially desirable behavior, like some forms of legal cooperation between firms which may be misjudged as collusive. In this case, fines must be related to the harm caused or the gains produced to the cartel's members. On the other hand, fines that are too high may put at risk the firms' ability to continue competing in the future.

In this section, we study the set of pair-wise stable networks under the following policy scheme $(\gamma_i, f_i)_{i\in\mathcal{N}} = (\gamma(n_i), f)_{i\in\mathcal{N}}$. Simply put, fines are a fixed sum of money and firm i's probability of inspection depends on the number of active firms in market i. This policy has a simple interpretation. On the one hand, the fixed fine f may be interpreted as the money equivalent of certain number of years in prison. In any case, the point is that f is a fixed independent penalty of the total payoff got by the firm. On the other hand, the simple idea behind $\gamma(n_i) \in (0, 1)$ is as follows. The smaller the number of active firms in the market, the higher the inspection probability. That is

$$\gamma(n_i - 1) \geq \gamma(n_i). \quad \text{(A3)}$$

Assumption A3 is easy to justify: when a firm has more agreements, prices are higher and the authority receives a noisy signal indicating a higher probability of collusive behavior. A typical example of such signals are costumers' complaints in the presence of sudden price increases. We also assume that the probability function $\gamma(.)$ satisfy the decreasing ratio property

$$\frac{\gamma(n_i - 1)}{\gamma(n_i)} \geq \frac{\gamma(n_i)}{\gamma(n_i + 1)}. \tag{A4}$$

Notice that Assumption A4 implies that the probability function $\gamma(.)$ is convex. That is

$$\gamma(n_i - 1) - \gamma(n_i) \geq \gamma(n_i) - \gamma(n_i + 1),$$

Assumption A4 is the other side of the coin of Assumption A2. As the profit function satisfies the decreasing ratio property also do prices. Hence, it seems natural to assume A4. Using (2) we have that the detection probability can be written as:

$$\beta_i(g) = 1 - (1 - \gamma(n_i)) \prod_{k \neq i, a_{ik}=1} (1 - \gamma(n_k)). \tag{7}$$

Note that the detection probability of firm i increases as both the number of active firms in its own market, n_i, and the number of active firms in all those markets in which firm i does not enter, n_k, decreases.

Payoffs. It is now is easily seen that the (expected) payoff of firm i for concluding an agreement with firm j is:

$$E_i(g + ij) - E_i(g) = \Big(\Pi_i\,(g + ij) - \Pi_i\,(g)\Big) - \Big(\beta_i(g + ij) - \beta_i(g)\Big) f,$$

and from expressions (3) and (7) we get that:

$$E_i(g + ij) - E_i(g) = W_i(n_i, n_j) - C_i(n_i, n_j, n_k, \gamma) \equiv Z_i((n_i, n_j, n_k, \gamma) \quad \forall k : g_{i,k} = 1, \tag{8}$$

where:

$$W_i(n_i, n_j) := \Big(\pi(n_i - 1) - \pi(n_i) - \pi(n_j)\Big),$$

and:

$$C_i(n_i, n_j, n_k, \gamma) := f\Big[\gamma(n_i - 1) - \gamma(n_i) + \gamma(n_j - 1)(1 - \gamma(n_i - 1))\Big] \prod_{k \neq i, a_{ik}=1} (1 - \gamma(n_k)).$$

Note that the difference between this policy with the first we have examined. Now the extra payoff that firm i gets due to an agreement depends on the number of agreements of its partners and not, as in the first case, of the number of agreement of other firms which are not firm i's partner.

Since firm i will enter into an agreement with firm j if, and only if, $E_i(g + ij) - E_i(g)$ is positive, we focus our attention on $Z_i(n_i, n_j, n_k, \gamma)$ and, hereafter, we refer to this function as the marginal return function. This function captures the main consequences for firm i of adding a link with firm j. First, as firm j will not enter in firm i's market, the profits of the latter must necessarily increase. More precisely, firm i's gain in its own market is given by $\pi(n_i - 1) - \pi(n_i)$. Second, by the reciprocal nature of the agreement, firm i will refrain from entering firm j's market. This leads to a decrease in the profit of firm i equal to $\pi(n_j)$. Finally, as firm j, the new partner of firm i, is inspected with probability $\gamma(n_j - 1)$, firm i suffers an increase in the expected fine equal to $C_i(n_i, n_j, n_k, \gamma)$.

Lemma 2 below describes the main properties of the marginal return function.

Lemma 2 (Properties of Marginal Returns)**.** *The marginal return function $M_i(n_i, n_j, n_k, \gamma)$ is increasing in n_j, decreasing in n_k, and it has an ambiguous relationship with n_i.*

The intuition behind this result is not so simple as that of Lemma 1. First, when the number of active firms in market j increases firm i's lost profit in market j decreases. Besides, as an increase in n_j leads to a lower $\gamma(n_j)$, the increase in the probability of being detected diminishes. Second, when the number of active firms in all those market in which firm i does not enter (i.e., markets in which the firm has an agreement) increases, the cost of making an agreement with j increases. The reason is as follows. As the number of competitors in those markets raises, the detection probability due an agreement with firm j also increases. Finally, the relationship between marginal returns and n_i is ambiguous. Although there are several countervailing forces driving this result we stress the following two. On the one hand, by Equation A2, i.e., the convexity of the profit function, a decrease in n_i increases the marginal returns. More formally, as convexity implies that the difference $\pi(n_i - 1) - \pi(n_i)$ rises as n_i diminishes, the marginal return of an extra link increases. But, on the other hand, as firm j is inspected with a higher probability, firm i is also detected with higher probability leading to inconclusive results.

Collusive Networks: Characterization. Observe that a necessary condition for $Z_i \geq 0$ is that $W_i \geq 0$. Hence it must be that

$$\pi\left(n_i - 1\right) - \pi\left(n_i\right) - \pi\left(n_j\right) > 0$$
$$\pi\left(n_j - 1\right) - \pi\left(n_j\right) - \pi\left(n_i\right) > 0$$

Therefore, we can write Z_i as:

$$\pi\left(n-1\right) - 2\pi\left(n\right) - f\Big[\gamma(n-1) - \gamma(n) + \gamma(n-1)(1-\gamma(n-1))\Big] \prod_{k\neq i, a_{ik}=1} (1-\gamma(n_k))$$

Let us now define $\underline{n}^*$ as the minimum number of active firms in a market such that $Z_i \geq 0$. Alternatively, let us denote $\bar{m}^* = N - \underline{n}^*$ as the maximal number of agreements such that this condition holds. Now, let us denote with $\bar{n}^*$ the maximal number of active firms in a market such that $\pi\left(n-1\right) - 2\pi\left(n\right) > 0$. Alternatively, let us define $\underline{m}^* = N - \bar{n}^*$ as the minimal number of agreements that a firm needs to have such that the condition holds. Finally, observe that $\underline{n}^* < \bar{n}^*$.

Our main result is:

Proposition 5 (Pair-wise Stable Networks)**.** *A pair-wise stable network can be partitioned into a set of isolated firms and a number of components each of which has the same number of links or agreements such that:*

1 *When in a pair-wise stable network* $\max_{l=1,2,...,p}\left\{m\left(g_l\right)\right\} = \bar{m}^* + 1$*, largest component is complete. Moreover, if there exist more than one component, they are also complete and their sizes could be equal to the size of the largest component. Components whose sizes are less than the size of the largest component have different sizes. In each component each firm has at most* $\bar{m}^*$ *agreements.*

2 When in a pair-wise stable network, $\max_{l=1,2,\ldots,p}\{m(g_l)\} > \bar{m}^* + 1$*, this largest component is not fully interconnected. Moreover, if there exist more than one component, they might not be complete and their sizes could be equal to the size of the largest component. Components whose sizes are greater than* $\bar{m}^* + 1$ *are not complete and their members will have* $\bar{m}^*$ *agreements. On the contrary, components whose sizes are smaller or equal to* $\bar{m}^* + 1$ *are fully interconnected with different sizes and their members will have at most* $\bar{m}^*$ *agreements.*

Item 2 of this proposition shows a contrasting result with that of 1. In particular, it shows that a subset of $\mathcal{G}$ is composed of not fully interconnected symmetric components. The following propositions charactherize further the nature of collusive networks. In particular:

Proposition 6 (Ineffective Antitrust Policy)**.** *The complete network* $g^c \in \mathcal{G}^*$ *if, and only if,*

$$\big(1-\gamma(2)\big)^{n-2}\Big(\big(1-\gamma(1)\big)^2 - \big(1-\gamma(2)\big)\Big).$$

Similarly, at the other extreme:

Proposition 7 (Effective Antitrust Policy)**.** *Let* $\underline{n}^* = n$*. Then the empty network is the unique pair-wise stable network.*

5. Discussion and Concluding Remarks

This article has studied the set of collusive stable networks under two different policy frameworks. Naturally, the presence of an antitrust policy weakens firm's incentives to participate in collusive agreements since it reduces the net expected benefit from signing them. In the current network framework, the channels through which antitrust policy impacts on competition has its peculiarities. Firms, considering whether or not to sign an agreement, take into account the probability of being discovered rather than the probability of being inspected, and the first probability positively depends on the number of agreements that each firm has signed.

In one of our cases, the probability of inspection is constant and the penalty imposed by the authority is equal to the total profits of a guilty firm. In such a case, that penalty depends on the network configuration as a whole. On the other hand, in the other case being studied, the probability of inspection depend on the network configuration but the penalty imposed by the authority is fixed.

For the first case, we have shown that pair-wise stable networks can be decomposed into a set of isolated firms and complete components of different sizes. Moreover, we concluded that, when the authority set a fixed and sufficiently small probability of inspection, the complete network is pair-wise stable. But as the authority increases γ, are smaller alliances the first to be destroyed or detected. In turn, the set of isolated firms expands. Therefore, as γ increases, the empty network, g^e, tends to emerge as the only pair-wise stable network. Recalling that in an empty network, all firms are active in all markets, we conclude that this policy has pro-competitive effects.

On the other hand, when the authority reacts to costumers' complains, i.e., when γ depends on n, components are not necessary complete in pair-wise stable networks. Besides, it is possible to define a lower bound on the size of components. This bound is related with the number of isolated firms and it has important impacts on the level of competition in the market. As the minimal number of firms active in a market that are necessary for make an agreement profitable increases, it is possible to expect more competitive structure.

An important policy implication of the present formulation is that the organization of the illegal behavior matters. That is, the analysis of the optimal deterrence of market-sharing agreements has to take into account the organizational structure of collusive firms. Furthermore, without considering the effects of the organizational structure, empirical studies may overestimate the contribution of efforts devoted to investigate and prosecute collusive agreements.[8] Finally, determining the optimal antitrust policy in a network context is part of our agenda for future research.

A. Proofs

Proof of Lemma 1. Recall that

$$M_i(n_i, n_j, n_k, \gamma) := \pi(n_i - 1) - \pi(n_i) - \pi(n_j) - \gamma\big(\pi(n_i - 1) + \sum_{k \neq j, a_{ik}=0} \pi(n_k)\big).$$

To save on notation we write $M_i \equiv M_i(n_i, n_j, n_k, \gamma)$. Then: (a) Since by A1 $\pi(.)$ is a decreasing function, an increase in n_j decreases $\pi(n_j)$ and hence it increases M_i; (b) Since by A1 $\pi(.)$ is a decreasing function, an increase in n_k decreases $\sum_{k \neq j, a_{ik}=0} \pi(n_k)$ and hence it increases M_i; (c) An increase in γ clearly diminishes M_i; and (d) From A2 a decrease in n_i increases $\pi(n_i - 1) - \pi(n_i)$ but at the same time it increases $\gamma(\pi(n_i - 1)$. □

Proof of Proposition 1. We first provide necessary conditions. Sufficiency is shown at last.
Necesity: We follow a sequence of steps.
(a) $n_i = n = n_j$. As network g is pairwise stable, when $g_{ij} = 1$ the following two conditions must simultaneously hold

$$\begin{aligned}(1-\gamma)\pi(n_i - 1) &\geq \pi(n_i) + \pi(n_j) + \gamma \sum_{k: g_{ik}=0} \pi(n_k)\,. \\ (1-\gamma)\,\pi(n_j - 1) &\geq \pi(n_j) + \pi(n_i) + \gamma \sum_{k: g_{jk}=0} \pi(n_k)\end{aligned}$$

Given that by A1 $\pi(.)$ is decreasing in n, the following are a pair of necessary conditions that must be satisfied for the above inequalities to hold:

$$\begin{aligned}\pi(n_i - 1) &> \pi(n_j) \\ \pi(n_j - 1) &> \pi(n_i)\end{aligned}$$

From the first inequality, $n_i - 1 < n_j$, and from the second one, $n_j - 1 < n_i$. Hence:

$$n_j - 1 < n_i < n_j + 1 \Leftrightarrow n_i = n_j$$

and the result follows.

[8]Some empirical papers that estimate the deterrent effect of the policy are, among others, Buccirossi and Spagnolo,2005; Connor, 2006; Zimmerman and Connor, 2005

(b) **Any component** $g' \in g$ **is complete**. Suppose not. Then, there are three firms i, j, l in the component such that $g_{ij} = g_{jl} = 1$ and $g_{il} = 0$. As g is stable, from part (a) we know that $n_i = n_j \equiv n$ and $n_j = n_l \equiv n$, then $n_i = n_j = n_l \equiv n$. From the stability conditions in part (a) we have that $M_i \geq 0$ for $g_{ij} = 1$:

$$\underbrace{\frac{\pi(n)}{\pi(n+1)}}_{A} \geq \underbrace{\frac{2}{(1-\gamma)} + \frac{\gamma \sum_{k:g_{ik}=0, i\neq k} \pi^i(n_k(g))}{(1-\gamma)\pi(n+1)}}_{B}$$

Similar conditions must hold for the other firms. Given that $g_{il} = 0$ and g' is stable one or both conditions hold for $h = i$ and/or $h = l$:

$$\underbrace{\frac{\pi(n-1)}{\pi(n)}}_{D} < \underbrace{\frac{2}{(1-\gamma)} + \frac{\gamma \sum_{k:g_{hk}=0, h\neq k} \pi^h(n_k(g))}{(1-\gamma)\pi(n)}}_{E}$$

By A2 it follows that

$$A \leq D$$

and from stability

$$B \leq A \leq D < E$$

However, from A1 it is direct to see that:

$$B > E,$$

leading to a contradiction. Therefore g' must be a complete component.

(c) **Components must have different sizes**. Take two firms i, j in component g' and a firm l in g''. Suppose that $m(g') = m(g'')$. Therefore, we have $n_i = n_j = n_l \equiv n$. The stability of g implies that $M_i \geq 0$ and $M_j \geq 0$. Hence:

$$\underbrace{\frac{\pi(n)}{\pi(n+1)}}_{A} \geq \underbrace{\frac{2}{(1-\alpha)} + \frac{\alpha \sum_{k:g_{ik}=0, i\neq k} \pi^i(n_k(g))}{(1-\alpha)\pi(n+1)}}_{B}$$

For $h = i$ and/or $h = l$, the following condition holds:

$$\underbrace{\frac{\pi(n-1)}{\pi(n)}}_{D} < \underbrace{\frac{2}{(1-\alpha)} + \frac{\alpha \sum_{k:g_{hk}=0,\ h\neq k} \pi^i(n_k(g))}{(1-\alpha)\pi(n)}}_{E}$$

By A2

$$A \leq D$$

and since the network is stable

$$B \leq A \leq D < E$$

However, from A1 it is direct to see that:

$$B > E$$

leading to a contradiction.
(d) Components must have different sizes Let $i \in g_h^*$ and suppose it does not have incentives to cut a link with a firm inside its component. Then:

$$\frac{\pi\left(N-m\left(g_h^*\right)\right)}{\pi\left(N-m\left(g_h^*\right)+1\right)}>\frac{2}{(1-\gamma)}+\frac{\gamma\left[\left(m\left(g_l\right)+1\right)\pi\left(N-m\left(g_l\right)\right)+\sum\limits_{k:g_{ik}=0}\pi\left(n_k\right)\right]}{(1-\gamma)\,\pi\left(N-m\left(g_h^*\right)+1\right)} \tag{8}$$

Assume however that $j \in g_l$ for $m(g_l) > m(g_h^*)$ does have an incentive to cut a link with a firm inside its component. Then:

$$\frac{\pi\left(N-m\left(g_l\right)\right)}{\pi\left(N-m\left(g_l\right)+1\right)}<\frac{2}{(1-\gamma)}+\frac{\gamma\left[\left(m\left(g_h^*\right)+1\right)\pi\left(N-m\left(g_h^*\right)\right)+\sum\limits_{k:g_{jk}=0}\pi\left(n_k\right)\right]}{(1-\gamma)\,\pi\left(N-m\left(g_l\right)+1\right)} \tag{9}$$

Since, by A1 $\pi(.)$ is decreasing in n, it follows that RHS(8)>RHS(9). By A2, the LHS(8)<LHS(9). Therefore, if i does not have an incentive to cut a link with a firm inside its component, LHS(8)>RHS(8), and hence LHS(9)>RHS(9), which contradicts (9).
Sufficiency. Consider a network g that can be decomposed into a set of isolated firms and distinct complete components, g_1,...,g_L of different sizes $m(g_l) \neq m(g_{l'})$,$\forall l, l'$. Isolated players have no incentive to create a link with another isolated one. As long as a firm i, which belongs to the smallest component, does not have incentives to cut a link with a firm inside its component, then, no firm inside a component has incentives to cut a link. Additionally, given that $m(g_l) \neq m(g_{l'})$,$\forall l, l'$, there do not exist two firms belonging to different components that have an incentive to form an agreement between themselves. □

Proof of Proposition 2. If g^c is pairwise stable, then

$$(1-\gamma)\,\pi(1) \geq 2\pi(2) \tag{9}$$

By rewriting the last condition, we get $\gamma \leq \hat{\gamma} = 1 - \frac{2\pi(2)}{\pi(1)}$. If $\gamma \leq \hat{\gamma} = 1 - \frac{2\pi(2)}{\pi(1)}$, then $(1-\gamma)\,\pi(1) \geq 2\pi(2)$. Therefore, g^c is pairwise stable. □

Proof of Proposition 3. Assume that $n \geq 3$. If g^e is pair-wise stable then

$$(1-\gamma)^2\left[\pi(n-1)+(n-2)\,\pi(n)\right]<\pi(n)+\pi(n)+(n-2)\,\pi(n) \tag{10}$$

and, by straightforward calculations

$$\gamma > 1-\left[\frac{n\pi(n)}{\left[\pi(n-1)+(n-2)\,\pi(n)\right]}\right]^{\frac{1}{2}}=\bar{\gamma}(n)$$

If $\gamma > \bar{\gamma}(n)$, then (10) holds. Therefore, g^e is pair-wise stable. *square*

Proof of Proposition 4. For simplicity, we assume just two complete components g_1 and g_2. For each firm $i \in g_1$, n_1 is the number of active firms in its market, and for each firm $j \in g_2$, n_2 is the number of active firms in its market. Let us define $\gamma(n_i) := \frac{\pi(n_i-1)-2\pi(n_i)}{\pi(n_i-1)+\sum_{k\neq j, gi=0}\pi(n_k)}$.

We are interested in knowing whether $\gamma(n_1) > \gamma(n_2)$. That is,

$$\frac{\pi\left(n_1-1\right)-2\pi\left(n_1\right)}{\pi\left(n_1-1\right)+\left(n-n_2+1\right)\pi\left(n_2\right)}>\frac{\pi\left(n_2-1\right)-2\pi\left(n_2\right)}{\pi\left(n_2-1\right)+\left(n-n_1+1\right)\pi\left(n_1\right)}$$

By solving the last expression, we get that $A > B$ where

$$A := (n - n_1 + 1)\,\pi(n_1)\,\pi(n_1 - 1) - 2\pi(n_1)\,\pi(n_2 - 1) - 2(n - n_1 + 1)\,[\pi(n_1)]^2 \tag{11}$$

and

$$B := (n - n_2 + 1)\,\pi(n_2)\,\pi(n_2 - 1) - 2\pi(n_2)\,\pi(n_1 - 1) - 2(n - n_2 + 1)\,[\pi(n_2)]^2 \tag{12}$$

In order to decide the sense of the inequality, we rearrange the above expression into the following two parts:

$$(n - n_1 + 1)\,\pi(n_1)\,[\pi(n_1 - 1) - 2\pi(n_1)] \geq (n - n_2 + 1)\,\pi(n_2)\,[\pi(n_2 - 1) - 2\pi(n_2)]$$

$$\pi(n_1)\,\pi(n_2 - 1) \leq \pi(n_2)\,\pi(n_1 - 1)$$

If $n_1 > n_2$, then (i) $(n - n_1 + 1) < (N - n_2 + 1)$; (ii) since individual profits are decreasing in n, $\pi(n_1) < \pi(n_2)$; (iii) since individual profits are log-convex in n, $[\pi(n_1 - 1) - 2\pi(n_1)] < [\pi(n_2 - 1) - 2\pi(n_2)]$. Therefore:

$$(n - n_1 + 1)\,\pi(n_1)\,[\pi(n_1 - 1) - 2\pi(n_1)] < (n - n_2 + 1)\,\pi(n_2)\,[\pi(n_2 - 1) - 2\pi(n_2)] \tag{13}$$

Additionally, if $n_1 > n_2$, then, by the log-convexity assumption, $\frac{\pi(n_2-1)}{\pi(n_2)} > \frac{\pi(n_1-1)}{\pi(n_1)}$. Hence

$$\pi(n_1)\,\pi(n_2 - 1) > \pi(n_2)\,\pi(n_1 - 1) \tag{14}$$

Therefore, if, $n_1 > n_2$, by (13) and (14), then

$$\gamma(n_1) < \gamma(n_2)$$

and we are done. □

Proof of Lemma 2. Note that with respect to n_j:

$$\frac{\triangle Z_i}{\triangle n_j} = \frac{\triangle W_i}{\triangle n_j} - \frac{\triangle C_i}{\triangle n_j}. \tag{13}$$

Tedious algebraic manipulations lead to:

$$\frac{\triangle W_i}{\triangle n_j} = \big(\pi(n_j) - \pi(n_j + 1)\big) > 0 \tag{14}$$

by A1 of the profit function. Similarly:

$$\frac{\triangle C_i}{\triangle n_j} = -\big(1 - \gamma(n_i - 1)(\gamma(n_j - 1) - \gamma(n_j)\big) \prod_{k \neq i, a_{ik}=1} (1 - \gamma(n_k)) < 0 \tag{15}$$

hence $\frac{\triangle Z_i}{\triangle n_j} > 0$

Now with respect to n_k:

$$\frac{\triangle Z_i}{\triangle n_k} = \frac{\triangle W_i}{\triangle n_k} - \frac{\triangle C_i}{\triangle n_k} = -\frac{\triangle C_i}{\triangle n_k}. \tag{16}$$

since $\dfrac{\triangle W_i}{\triangle n_k} = 0$. Tedious algebraic manipulations lead to:

$$\frac{\triangle C_i}{\triangle n_k} = \big(\gamma(n_i - 1) - \gamma(n_i)\big) + (\gamma(n_j - 1)(1 - \gamma(n_i - 1) \prod_{k \neq i, a_{ik}=1} (\gamma(n_k) - \gamma(n_k + 1)) > 0 \tag{17}$$

by A3. Finally, we have:

$$\frac{\triangle Z_i}{\triangle n_i} = \frac{\triangle W_i}{\triangle n_i} - \frac{\triangle C_i}{\triangle n_i}. \tag{18}$$

Tedious algebraic manipulations lead to:

$$\frac{\triangle W_i}{\triangle n_i} = \big(\pi(n_i - 1) - \pi(n_i)\big) + \big(\pi(n_i) - \pi(n_i + 1)\big) \leq 0 \tag{19}$$

by A2. Similarly:

$$\frac{\triangle C_i}{\triangle n_i} = (A + B) \prod_{k \neq i, a_{ik}=1} (1 - \gamma(n_k)) \tag{20}$$

where:

$$A := \big(\gamma(n_i) - \gamma(n_i + 1)\big) - \big(\gamma(n_i - 1) - \gamma(n_i)\big) \leq 0 \tag{21}$$

by A4. And since:

$$B := \gamma(n_j - 1)\big(\gamma(n_i - 1) - \gamma(n_i)\big) \geq 0 \tag{22}$$

by A3. Hence $A + B$ can be either positive or negative. □

Proof of Proposition 5. Assume network g is pair-wise stable. For all $m > \bar{m}^*$ and for all $m < \underline{m}* Z_i < 0$. Therefore, in a pair-wise stable network, no firm has more than $\bar{m}^*$ agreements and no less than $\underline{m}*$.

(a) Firstly, given a component of size $m(g_l)$, the number $m(g_l) - 1$ represents the maximal number of agreement that every firm in g_l may have. Recall that $\bar{m}^*$ is the maximal number of agreement such that $Z_i \geq 0$. Since components are symmetric, every firm in g_l has the same number of agreements. That is, for all pair of firms i and j that belong to g_l, then $m_i = m_j = m$. Assume now that i and j belong to g_l but $g_{ij} = 0$, if $m < m(g_l) - 1 \leq \bar{m}^*$, firms i and j will have incentives to form an agreement between them as $m < \bar{m}^*$. But it is a contradiction with the assumption of pair-wise stability of network g. Then, if $m(g_l) - 1 \leq \bar{m}^*$ all components must be complete and its member will have at most $\bar{m}^*$ agreements.

Second, let us assume that there exist two largest components such that their sizes are equal to $\bar{m}^* + 1$. As we have shown, these components must be complete and every firm inside them has $\bar{m}^*$ agreements. No firm inside these components has incentives to sever a link with a firm in the component as each firm in them has $\bar{m}^*$ agreements. Let us consider now a link between two firms belonging to each component. These firms have no incentives to sign one more agreements, as long as each one has $\bar{m}^*$ agreements, and this number is the largest number of links such that $Z_i \geq 0$.

Finally, let us consider a firm $i \in g_1$ and $j \in g_2$ such that $\bar{m}^* > m(g_1) > m(g_2)$. The firm j that belongs to the smaller component, refuses to sign an agreement with i, since $n_i < n_j$ and then, $\pi(n_j - 1) - \pi(n_i) < 0$. As $n - \bar{n} > 1$, isolated players have no incentives to form any agreements.

(b) Assume $m(g_l') = \max_{l=1,2,\ldots,p} \{m(g_l)\} > \bar{m}^* + 1$. Therefore, $m(g_l') - 1 > \bar{m}^*$. Since $\bar{m}^*$ is the maximal number of links such that $Z_i \geq 0$, every firm in g_l' has $\bar{m}^*$ agreements and then g_l' will not be fully interconnected. If each firm in g_l' has lesser agreements than $\bar{m}^*$, it will have incentives to form one more link. If a firm inside this component has more agreements than $\bar{m}^*$ and/or the component is fully interconnected, every firm inside g_l' will have incentives to sever a link, as long as $m(g_l') - 1 > \bar{m}^*$, and thus g would not be pair-wise stable.

Let us assume now that another component g_l'' such that $\bar{m}^* + 1 < m(g_l'') \leq m(g_l') \max_{l=1,2,\ldots,p}\{m(g_l)\}$. As before, g_l'' will not be fully interconnected and its members will have $\bar{m}^*$ links. No firm inside g_l'' has incentives to cut a link as it has $\bar{m}^*$ agreements. Moreover, no firm in g_l'' has incentives to sign another agreements since it has $\bar{m}^*$ links. Finally, let us consider that g_l'' is such that $m(g_l'') \leq \bar{m}^* + 1$. Then, it follows the proof in part (a). As $n - \bar{n} > 1$, the isolated players have no incentives to form any agreements. □

Proof of Proposition 6. Assume $g_{ij} = 1$. Then, $n_i = n_j = n$. Therefore, Z_i can be written as:

$$\pi(n) - 2\pi(n+1) - \left[\pi(n) - \pi(n+1) - \prod_{j\text{st}g_{ij}=1} (1-\gamma(n+1)) \left[(1-\gamma(n))^2 - (1-\gamma(n+1))\right]\right]$$

We must prove now that when $n = 1$, this expression is negative. That is, we must prove that:

$$-\pi(2) + (1-\gamma(2))^{n-2}\left[(\pi(1) - \pi(n))(1-\gamma(1))^2 - (\pi(2) - \pi(n))(1-\gamma(2))\right] < 0$$

For that, let us verify that the sign of bracket expression is also negative. First of all, let us observe that $\left[(1-\gamma(2)) - (1-\gamma(1))^2\right] > 0$. Therefore:

$$\pi(n)\left[(1-\gamma(2)) - (1-\gamma(1))^2\right] < \pi(2)\left[(1-\phi(2)) - (1-\gamma(1))^2\right] < \pi(2)(1-\gamma(2)) - \pi(1)(1-\gamma(1))^2$$

Consequently, $\underline{n}^*$ must be greater than 1 and the complete network will never be stable. When the industry is "sufficiently" large, the empty network emerge as a pair-wise stable network. □.

Proof of Proposition 7. Let $n - \overline{n}^* = \underline{m}^*$. We interpret $\underline{m}^*$ as the minimal number of agreement that a firm has to have in order to form an additional one. Therefore, when $n - \overline{n}^* = 1$ any two firm has incentive to form an additional agreement. However, if $n - \overline{n}^* > 1$ any two firms need to have more than one agreement in order to make profitable to form an additional link. □

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