UNIVERSIDAD DE LA REPÚBLICA

Facultad de Ingeniería

Tesis para optar al Título de
Doctor en Ingeniería Eléctrica

ALMOST GLOBAL STABILITY OF DYNAMICAL SYSTEMS

(Casi estabilidad global de sistemas dinámicos)

Autor:
Pablo Monzón
Directores de Tesis:
Jorge Lewowicz
Fernando Paganini

Montevideo, Uruguay
2006

UNIVERSIDAD DE LA REPÚBLICA ORIENTAL DEL URUGUAY

INSTITUTO DE INGENIERíA ELÉCTRICA

Los abajo firmantes certificamos que hemos leído el presente trabajo titulado "**Almost global stability of dynamical systems (Casi estabilidad global de sistemas dinámicos)**" hecho por **Pablo Monzón** y encontramos que el mismo satisface los requerimientos curriculares que la Facultad de Ingeniería exige para la tesis del título de **Doctor en Ingeniería Eléctrica**.

Fecha: 3 de julio de 2006

Director de Tesis: ______________________

Dr. Jorge Lewowicz

Director de Tesis: ______________________

Dr. Fernando Paganini

Tribunal examinador: ______________________

Dr. José Luis Mancilla

____

Dr. Alejandro Romanelli

____

Dr. Enrique Ferreira

____

Dr. Martín Sambarino

____

Prof. Rafael Canetti

____

Dr. Roberto Markarian

ISSN: 1688-2776 (electronic version)

Pablo Monzón (monzon@fing.edu.uy)

Tesis de Doctorado en Ingeniería Eléctrica

Facultad de Ingeniería

Universidad de la República

Montevideo, Uruguay, 2006.

UNIVERSIDAD DE LA REPÚBLICA ORIENTAL DEL URUGUAY

Fecha: **3 de julio de 2006**

Autor: **Pablo Monzón**

Titulo: **Almost global stability of dynamical systems (Casi estabilidad global de sistemas dinámicos)**

Instituto: **Ingeniería Eléctrica**

Grado: **Doctor en Ingeniería Eléctrica** (**Dr. Ing.**)

.

para Paula

Contents

List of Figures

Acknowledgements

A Paula, mi compañera de vida, mi orgullo, mi alegría. Mi vida no es la misma desde que estamos juntos.

A mi familia: Ariel, Clara, Aldo y Diego. Si bien están lejos de los sistemas no lineales, siempre han seguido con cariño los pasos de mi carrera académica.

A mis tutores y directores: Rafael Canetti, Roberto Markarian, Jorge Lewowicz y Fernando Paganini, porque aceptaron sin dudar embarcarse en esta idea.

A la gente del IIE, en particular a Rafael Canetti, Álvaro Giusto, Néstor Macé, Enrique Ferreira, Andre Fonseca y Eduardo Máscolo, por las tardes de seminario, las discusiones de oficina y las conversaciones de café.

A Michel Artenstein por distraerme interesantemente de la Tesis.

A la gente del IMERL, que siempre está dispuesta a escuchar, en particular Eleonora Catsígeras por realizar sugerencias para el contraejemplo en dimensión 3.

A Anders Rantzer, por leer los primeros frutos de este trabajo y realizar sugerencias y recomendaciones.

A Eduardo Sontag, por estar abierto a las preguntas.

A José Luis Mancilla, por aceptar rápidamente la invitación.

A Rafael Potrie por subirse con ganas al proyecto.

A Julia, que consiguió hasta las artículos más difíciles.

A la secretaria del IIE, a la Sección Comisiones y a las muchachas del Consejo, por el apoyo continuo.

A mis amigos de siempre, porque siempre están ahí.

Abstract

In this Thesis we have put together many results recently appeared in the control community, related with the concepts of almost global stability of dynamical systems and density functions. We have also made some contributions on this direction: we have incorporated the idea of monotone measures and studied its relationship with almost global stability; we have blended these new ideas with the classical Ponincaré-Bendixson Theory for planar systems and we have developed converse results on the direction of proving an equivalence between the existence of density functions and almost global stability. Closing this work, we have analyzed the almost global synchronization of sinusoidally coupled oscillators, where we have stated how some conditions on the interconnection graph ensures the almost global stability of the synchronized state.

Resumen

En la presente Tesis hemos puesto en un solo trabajo recientes aportes aparecidos en la comunidad de control relacionados con el concepto de casi estabilidad global de sistemas dinámicos y las funciones de densidad. También hemos aportado conocimientos originales en dicha línea, a saber: la incorporación de la idea de medida monótona y su relación con la casi estabilidad global; la vinculación de dichas ideas con resultados clásicos como la Teoría de Poincaré-Bendixson para sistemas planos y el desarrollo de resultados recíprocos que van en la línea de probar que la casi estabilidad global es equivalente a la existencia de funciones de densidad. A modo de cierre de la Tesis, hemos incluido el análisis completo de un sistema de osciladores acoplados, donde hemos estudiado las condiciones de interconexión que aseguran la casi estabilidad global del estado de sincronización colectiva.

Chapter 1

Outline and contributions

A main problem in Control Theory has been the stability of a given dynamical system, usually described by an ordinary differential equation. The classical concept of stability was the Lyapunov one: a trajectory is *stable* if all the trajectories which start close enough to it, remain close for all future times. *Asymptotical stability* (a.s.), or *attraction*, means that, besides stability, the nearby trajectories converge to the given one. In the control context, a plant operating at some equilibrium point of our mathematical model will be stable if small perturbations can not lead the system to undesirable situations. In fact, we want that the system can recover its original operating point after the appearance of the perturbation. This means that the operating point corresponds to an asymptotically stable equilibrium point. By perturbations we refer to unprecise initial conditions, uncertain parameters or unmodelled dynamics. Usually, we will have a nominal model and a perturbed or uncertain model.

The classical tool for the study of stability is the Lyapunov Theory, first presented by the russian mathematician at the end of the XIX century (Lyapunov, 1892; Khalil, 1996). He related the idea of stability to the existence of a particular function associated to the system (Lyapunov function). His first theorems have triggered an entire area of the mathematics and have deeply impacted on the engineers. Control engineers have used these ideas in many ways for analysis and synthesis of control systems and still use them for new developments.

A first natural question related with the attraction property is the *region of attraction*. This is the set of all starting points whose trajectories converge to a given attractor and it is a very important mathematical object. Lyapunov Theory allows us to estimate the region of attraction using a level set of a Lyapunov functions, but sometimes, this estimation can be very poor. In many problems, we are interested not only in attraction but also in a non small region of attraction. If we have perturbations that slightly modify our initial conditions, but they still are in the region of attraction, we are sure that the trajectories will eventually be very close to the wanted operating point. The relationship between the region of attraction of our nominal model and the perturbed model will let us make some conclusions on the latter using our knowledge of the former. A particular case that is *nice* is when the region of attraction includes the whole state space. Global asymptotical stability (g.a.s.), as it is called, implies that all the trajectories converge to the origin and that, sooner o later, our system will behave near the wanted equilibrium point. Of course, this requires, among other things, that no other equilibrium point exists, and that can be a very hard restriction. In fact, from an engineer point of view, we will be glad if most of the trajectories converge to the origin, in the sense that the problematic set is small enough that can be neglected. This idea leads us to the concept of almost global stability (a.g.s.) and the study of some classical problems from a new point of view. A.g.s. has been studied for the last years by some researchers of the control community and lately has attracted the interest of the mathematicians. There are a lot of theoretical and practical problems, both for synthesis and analysis of control systems, and there are also *pure math* problems. In this Thesis, we have tried to explore this stability concept and make some contributions.

We have structured this Thesis in several chapters, following a logical sequence for the presentation. In Chapter 3 we introduce the idea of almost global stability and its characterization through density functions. We have put here the main results that have appeared in the technical literature in the last few years. In Chapter 4, we present a relaxation of the previous ideas using monotone measures, that is, measures that increase or decrease along the flow. We also present a result that combines these new ideas with the classical Poincaré-Bendixson Theory. A battery of converse results are proved in Chapter 5, including some relationships between density and Lyapunov functions. In Chapter 6 we present the analysis of almost global properties of a class of nonlinear dynamical systems that appear in several engineering and biological problems.

Original results are included where they fit, so they are spread all along the Thesis. In order to highlight these original results, we have taken the following premise: *every non original result (theorem, proposition, example, etc.) includes a reference to where it has been published.* In Appendix A we have included some original results that even when they have not lead to the sought goal, we think they may have some relevance by themselves.

The main contributions of this Thesis are:

- The proof of necessary conditions for almost global stability (Chapter 5).
- The introduction of monotone measures for dynamical systems and the study of its relationships with almost global stability and density functions (Chapter 4, Section 4.1).
- A result for planar systems that combines almost global stability with the classical Poincaré-Bendixson Theory (Chapter 4, Section 4.2).
- A general relationship between density functions and Lyapunov functions (Chapter 5, Section 5.5.1).
- The analysis of the almost global properties of sinusoidally weakly coupled oscillators (Chapter 6).

Some of the previous ideas have been presented at international conferences and reported in specialized journals:

- P. Monzón "On necessary conditions for almost global stability", en Proceedings of the 41st IEEE Conference on Decision and Control, p.p.4270-4271, Las Vegas, December 2002
- P. Monzón, "On necessary conditions for almost global stability", IEEE Transactions on Automatic Control, 48:44, pp.631-634, April, 2003.
- P. Monzón, "Almost global stability of planar systems", Congreso Latinoamericano de Control Automático, AUT048, May 2004, La Habana.
- P. Monzón, "Monotone measures and almost global stability of dynamical systems", Sixteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS2004), Katholieke Universiteit Leuven, Leuven, Belgium, TP3-3, July 2004.

- P. Monzón, "Monotone measures for dynamical systems", Congresso Brasileiro de Automática, September 2004, Gramado, Brasil.
- P. Monzón, "Almost global attraction in planar systems", en Systems & Control Letters, 54 (8), pp. 753-758, Aug, 2005.
- P. Monzón, F. Paganini, "Global considerations on the Kuramoto model of sinusoidally coupled oscillators", in Proceedings of the Joint 44th IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC) 2005, pp.3923-3928, Sevilla, December 2005.

Chapter 2

Resumen y contribuciones

Uno de los principales problemas que aborda la teoría del control es la estabilidad de los puntos de equilibrios de un sistema dinámico, descrito usualmente por un conjunto de ecuaciones diferenciales ordinarias. El concepto clásico de estabilidad fue introducido por el matemático ruso Alexander Lyapunov, quien a fines del siglo XIX presentó su teoría y abrió nuevos caminos en la matemática y la ingeniería. Según Lyapunov, un punto de equilibrio es *estable* si todas las trayectorias que se inician suficientemente cerca de él permanecen cerca de él en todo instante futuro. La estabilidad es asintótica, y hablamos de *atracción*, cuando además de la estabilidad tenemos la propiedad que todas las trayectorias cercanas convergen al punto de equilibrio cuando el tiempo tiende a infinito.

En el contexto del control, el punto de operación de una planta física se corresponde usualmente con un punto de equilibrio del modelo matemático que estemos utilizando. Este punto será estable si pequeñas perturbaciones no llevan el sistema a situaciones de funcionamiento no deseadas. Por perturbaciones entendemos tanto condiciones iniciales aproximadas o no conocidas de arranque del sistema, como ruido, etc. Nos interesa en general que el sistema sea capaz de recuperar su estado original luego de la perturbación. En el modelo matemático, esto se traduce en que el sistema presente un punto de equilibrio asintóticamente estable. Muchas de estas perturbaciones también pueden ser incorporadas al modelo matemático y usualmente tendremos dos modelos, uno nominal, con el cual trabajamos, y uno incierto, que incorpora el desconocimiento que podemos tener de la realidad. Los trabajos de Lyapunov introdujeron las primeras herramientas para tratar de manera sistemática el tema de la estabilidad. Analizando sistemas mecánicos y generalizando la idea de energía, Lyapunov introdujo condiciones suficientes de estabilidad, basadas en la existencia de cierta clase de funciones, que hoy llamamos funciones de Lyapunov. El ingeniero de control usa estas ideas tanto para el análisis de sistemas como para la síntesis de controladores y aún hoy, en el siglo XXI, siguen apareciendo resultados basados directamente en las ideas de Lyapunov.

Si bien la propiedad de atracción de un punto de equilibrio es deseada, también interesa conocer la denominada *región de atracción*, que podríamos definir como el área de influencia del atractor. Tener una buena idea de esta región nos brinda cierta tranquilidad ante la presencia de perturbaciones que muevan respecto del sistema del punto de operación. Si la perturbación es *pequeña*, entonces el sistema no se sale de la región de atracción y al transcurrir el tiempo recuperamos el punto de operación. El *mejor* caso se da cuando la región de atracción consiste en todo el espacio de estados posibles y ahí tenemos la denominada *estabilidad global*. La propia teoría de Lyapunov permite estimar el tamaño de la región de atracción utilizando curvas de nivel definidas por funciones de Lyapunov. En este sentido, podemos buscar funciones de Lyapunov que no sólo indiquen la estabilidad, sino que también nos den una buena estimación de la región de atracción. La propia teoría tiene sus limitaciones, ya que no siempre es posible encontrar funciones de Lyapunov definidas *globalmente*, ya sea porque no sabemos cómo buscarlas o porque simplemente no pueden existir. En estos contextos, podemos usar una idea alternativa: la estabilidad casi global.

Desde un punto de vista ingenieril, nos alcanza con poder asegurar que prácticamente todas las trayectorias van a parar al punto de operación deseado, en el entendido de que las que no lo hacen, conforman un conjunto de tamaño o medida *despreciable*. Este concepto se denomina estabilidad casi global y ha sido estudiada en los últimos años por la comunidad de control, a partir de la introducción de una condición suficiente que sigue la línea de Lyapunov:

Teorema (Rantzer, 2001a) Consideremos el sistema

$$\dot{x} = f(x)$$

con $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $f(0) = 0$. Supongamos que existe una función de densidad $\rho \in C^1(\mathcal{R}^n \setminus \{0\}, [0, +\infty))$ tal que

$$\nabla.(\rho f)(x) > 0 \;\; c.t.p. \;\; , \quad \int_{\{\|x\|>\epsilon\}} \rho(x)dx < \infty \;\; \forall \epsilon > 0$$

Entonces, el conjunto de trayectorias que no son atraídas al origen tiene medida de Lebesgue nula. $\triangle$

En términos matemáticos, la estabilidad casi global implica que el conjunto de trayectorias que al tender el tiempo a infinito no convergen al atractor tiene medida de Lebesgue nula. Varios trabajos teóricos y prácticos se han disparado a partir de este resultado y vinculando estas ideas nuevas con conceptos más clásicos. En este sentido hemos intentado contribuir a través del trabajo desarrollado en esta Tesis.

Las principales contribuciones de la Tesis son:

1) La introducción de la idea de medidas monótonas para sistemas dinámicos y el estudio de sus relaciones con la estabilidad casi global y las funciones de densidad (Capítulo 4)

2) La presentación de condiciones necesarias para la estabilidad casi global de sistemas dinámicos (Capítulo 5). Se probó que ésta propiedad implica necesariamente que existan funciones de densidad y medidas monótonas. La técnica de prueba desarrollada requiere que el sistema presente, además de la estabilidad casi global, la propiedad de estabilidad local. Se trabajó primero sobre sistemas lineales, luego sobre sistemas globalmente estables, llegando finalmente a sistemas estables casi globales.

   El siguiente Teorema representa el estilo de los resultados mencionados en los items anteriores:

   Teorema Consideremos el sistema

$$\dot{x} = f(x)$$

con $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$. Supongamos que el origen es un punto localmente estable, tal que el conjunto de trayectorias que no converge al origen tiene medida de Lebesgue nula. Entonces, existe una medida μ definida sobre los conjuntos Borelianos de $\mathcal{R}^n$ tal que

a) $\mu(Y) = 0$ si Y tiene medida de Lebesgue nula;

b) $\mu(Y) < +\infty$ si $Y \subset B^c(0, \epsilon)$, $\epsilon > 0$.

c) para todo Y con $0 < \mu(Y) < \infty$, se tiene $\mu[f^t(Y)] > \mu(Y)$, siendo $f^t(Y)$ el conjunto de puntos que en tiempo t se alcanzan partiendo del conjunto Y, moviéndose a través de las trayectorias del sistema.

(Las propiedades a), b) y c) definen lo que llamamos una *medida monótona, acotada al infinito*) $\triangle$.

La técnica básica de la prueba consiste en mapear las trayectorias de la region de atracción del sistema dado en las trayectorias de un sistema lineal auxiliar, para el cual probamos que existen medidas monótonas y funciones de densidad. Este mapeo permite entonces *pasar* las medidas monótonas y densidades del sistema lineal al no lineal.

3) Resultados que combinan las ideas de estabilidad casi global con la teoría de Poincaré-Bendixson para sistemas planos (Capítulo 4).

4) Una relación general entre funciones de Lyapunov y funciones de densidad (Capítulo 5), usando técnicas similares a las empleadas en los puntos 1) y 2).

5) El análisis detallado de propiedades casi globales de osciladores sinusoidales débilmente acoplados, denominado *sistema de Kuramoto*, de interés en diversas áreas de la ingeniería y la biología (Capítulo 6). Cada oscilador modela un agente independiente y el acoplamiento señalado describe la interacción entre dichos agentes. La dinámica del i-ésimo agente tiene la siguiente forma:

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i)$$

donde θ_i corresponde a la fase de dicho oscilador y $\mathcal{N}_i$ representa el conjunto de agentes que interactúan con él (sus *vecinos*). En base a la vinculación del modelo de Kuramoto con grafos dirigidos que detallan la interacción de los agentes, fue posible encontrar condiciones que deben imponerse al grafo de interconexión respectivo (tanto condiciones topológicas como espectrales) para que el sistema presente la propiedad de *sincronización casi global.* Esta propiedad, en la cual todos los osciladores presentan la misma fase, refleja en general el surgimiento de una conducta colectiva como fruto de la influencia mutua de los agentes individuales.

A modo de ejemplo, presentamos los siguientes Teoremas que ilustran el estilo de los resultados obtenidos:

Teorema Consideremos el sistema de Kuramoto de n agentes cuya interacción está descrita por un grafo G. Entonces, si G es completo, el conjunto de condiciones iniciales que no lleva a la sincronización colectiva tiene medida de Lebesgue nula. $\triangle$

Teorema Consideremos el sistema de Kuramoto de n agentes cuya interacción está descrita por un grafo G. Entonces, si G es un árbol sin ciclos, el conjunto de condiciones iniciales que no lleva a la sincronización colectiva tiene medida de Lebesgue nula. $\triangle$

Teorema Consideremos el sistema de Kuramoto de al menos 4 agentes, cuya interacción está descrita por un grafo G. Entonces, si G es un ciclo, el conjunto de condiciones iniciales que no lleva a la sincronización colectiva tiene medida de Lebesgue no nula. $\triangle$

Algunas de las ideas anteriores han sido presentadas Congresos internacionales y reportados en Revistas del área del control:

- P. Monzón, On necessary conditions for almost global stability, en *Proceedings of the 41st IEEE Conference on Decision and Control*, p.p.4270-4271, Las Vegas, December 2002
- P. Monzón, On necessary conditions for almost global stability, *IEEE Transactions on Automatic Control*, 48:44, pp.631-634, April, 2003.
- P. Monzón, Almost global stability of planar systems, *Congreso Latinoamericano de Control Automático*, AUT048, May 2004, La Habana.

- P. Monzón, Monotone measures and almost global stability of dynamical systems, *Sixteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, Katholieke Universiteit Leuven, Leuven, Belgium, TP3-3, July 2004.

- P. Monzón, Monotone measures for dynamical systems, *Congresso Brasileiro de Automática*, September 2004, Gramado, Brasil.

- P. Monzón, Almost global attraction in planar systems, en *Systems & Control Letters*, 54 (8), pp. 753-758, Aug, 2005.

- P. Monzón, F. Paganini, Global considerations on the Kuramoto model of sinusoidally coupled oscillators, in *Proceedings of the Joint 44th IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC) 2005*, pp.3923-3928, Sevilla, December 2005.

Dentro de las perspectivas de trabajo futuro, se prevé abordar líneas más bien teóricas, referidas a la conexión entre propiedades locales, casi globales y globales de estabilidad, el levantamiento de la hipótesis de estabilidad local para los teoremas recíprocos, la formulación de este tipo de problemas en el contexto de sistemas dinámicos discretos y la incorporación al sistema de acciones de control. En una línea de trabajo más aplicada, pretendemos desarrollar una clasificación osciladores débilmente acoplados que presentan la propiedad de sincronización casi global, a través de una categorización de los grafos subyacentes.

Chapter 3

Almost global stability

In this Chapter we present the basic concepts and definitions of almost global stability of dynamical systems and the results that characterize it. We will introduce the works of Anders Rantzer reported in several papers appeared in the last years, starting with the basic article (Rantzer, 2001a) and complemented with other related reports and publications. We present the main Theorems and Lemmas in their original versions. In some cases, we have also included the proofs, mainly following the line of the referred article, with slight modifications just to clarify the presentation or to be consistent with this Thesis.

3.1 Basic definitions

Lyapunov stability has been the core of control theory for more than hundred years, since the appearance of the works of the russian mathematic at the end of the 19^{th} century (Lyapunov, 1892).

Definition 3.1. *Given a system of differential equations*

$$\dot{x} = f(x) \tag{3.1.1}$$

with $x \in \mathcal{R}^n$, $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ and $f(0) = 0$, we say that the origin is asymptotically stable (a.s.) in the sense of Lyapunov (Khalil, 1996) if

1. *it is stable, i.e. for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\text{if } \|x_0\| < \delta(\epsilon) \text{ then } \left|\left|f^t(x_0)\right|\right| < \epsilon \quad \forall t \geq 0$$

2. *there exists $\delta_0 > 0$ such that* if $\|x_0\| < \delta_0$ then $\left|\left|f^t(x_0)\right|\right| \to 0$ as $t \to +\infty$

where $f^t(x_0)$ denotes the time t of the trajectory of system (3.1.1) which starts at x_0.

♠

Lyapunov characterized local asymptotical stability of the origin by the existence of a function

$$V : D \subset \mathcal{R}^n \to [0, +\infty)$$

of class C^1, defined on a neighborhood D of the origin, such that $V(0) = 0$, $V(x) > 0$ if $x \neq 0$ and

$$\dot{V}(x) = \lim_{h \to 0} \frac{V\left(f^h(x)\right) - V(x)}{h} = \nabla V(x).f(x) < 0$$

for $x \neq 0$.

Sometimes we refer to an asymptotically stable equilibrium point as an *attractor*. The region of attraction of the origin is the set R of all the initial conditions that have trajectories which converge to the origin:

$$R = \left\{ x \in \mathcal{R}^n \mid \lim_{t \to +\infty} f^t(x) = 0 \right\}$$

When $R = \mathcal{R}^n$, we have *global* asymptotical stability (g.a.s.), and this situation is nice from the engineering point of view, since no matter how far from the origin we start, we will eventually converge to it. Nevertheless, there are several cases where global properties can not be established. For example, when we have other equilibrium points than the attractor, we know that some trajectories do not converge to it. Or even if we have proved local asymptotic stability, it may be a hard task to prove global stability. Asking for almost global properties is an intermediate stage for the second case and a final goal for the first one.

Definition 3.2. *We said that the dynamical system (*3.1.1*) is* almost global stable *(a.g.s.) if almost all the trajectories converge to the origin, that is, the set of points that are not attracted by the origin has zero Lebesgue measure.*

♠

In other words, the system is a.g.s. if the set R^c

$$R^c = \left\{ x \in \mathcal{R}^n \mid \lim_{t \to +\infty} f^t(x) \neq 0 \ \text{ or does not exist} \right\}$$

has zero Lebesgue measure. We emphasize that the definition above presents a concept of stability weaker than the classical global asymptotic stability but can complement well the local stability property.

3.2 A classical approach

A way to prove global stability is through a Lyapunov function that works globally, that is, if we can take $D = \mathcal{R}^n$ in the definition of V (plus some technical conditions, like radial unboundedness (Khalil, 1996)). There are systematic ways to construct a Lyapunov function for some classes of systems, but it is not, in general, an easy task. When we have several equilibrium points, we can no longer search for a global Lyapunov function. One possible way to prove a.g.s. of the origin is showing that it is the only attractor set and that no trajectory goes away to infinity. There are several concepts and Theorems that help in that direction, like the ideas of ω and α limit sets and LaSalle's result [which are going to be introduced later in this Thesis]. In some cases, these tools can also be applied to establish a.g.s., as is shown in Chapter 6, where we analyze the almost global synchronization of sinusoidally coupled oscillators.

3.3 A new approach

In the last few years, another way to establish almost global properties was presented to the control community by Anders Rantzer (Rantzer, 2001a). The key contribution was the introduction of some particular functions that for a.g.s. systems play a role similar of that of Lyapunov functions for a.s. systems: the density functions.

Definition 3.3. *Given a dynamical system $\dot{x} = f(x)$, a density function for this system is a scalar function $\rho : \mathcal{R}^n \backslash \{0\} \to [0, +\infty)$, of class C^1, integrable on the outside of a ball centered at the origin, and such that the following divergence condition is satisfied*

$$\nabla.(\rho f)(x) > 0 \ \textit{almost everywhere}\ (a.e.) \tag{3.3.1}$$

♠

A density function gives us a system-related way of measure sets in $\mathcal{R}^n$, as will follow from Lemma 3.3. Here is the main result of (Rantzer, 2001a):

Theorem 3.1. *(Rantzer, 2001a) Given the equation*

$$\dot{x} = f(x) \tag{3.3.2}$$

where $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $f(0) = 0$ and $\frac{\|f(x)\|}{\|x\|}$ is globally bounded, suppose there exists a density function $\rho : \mathcal{R}^n - \{0\} \to [0, +\infty)$. Then almost all the trajectories converge to the origin, i.e. the origin is almost globally stable.

The proof is based on the following lemmas. The first one is a recurrence-like property on measure spaces and the second one is similar to the classical Liouville Theorem.

Lemma 3.2. *(Rantzer, 2001a) Consider a measure space $(X, \mathcal{A}, \mu)$, a set $P \subset X$ of finite measure and a measurable map*[1] *$T : X \to X$. Suppose that*

$$\mu\left(T^{-1}Y\right) \leq \mu(Y) \quad \textit{for all measurable } Y \subset X \tag{3.3.3}$$

Define Z as the set of elements of P that return infinitely many times to P. Then

$$\mu\left(T^{-1}Z\right) = \mu(Z)$$

Proof: Z is a measurable set. This follows from the identity

$$Z = P \bigcap \left[\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} T^{-k}P\right]$$

Since $P \in \mathcal{A}$ and T is measurable, $Z \in \mathcal{A}$.

Now we define the sequence of sets

$$Z_n = \cup_{k=1}^{n} T^{-k}Z \quad , \quad Z_0 = \emptyset$$

Observe that Z_n contains the elements of the space whose trajectories reach Z in at most n steps. By induction we will prove that

$$\mu\left(T^{-1}Z\right) \geq \mu\left(Z_n \cap Z\right) + \mu\left(T^{-n-1}Z \cap Z_n^c\right) \tag{3.3.4}$$

For $n = 0$, it trivially holds. Let us assume that (3.3.4) holds for $h \geq 0$. Then

$$\mu\left(T^{-1}Z\right) \geq \mu\left(Z_h \cap Z\right) + \mu\left(T^{-h-1}Z \cap Z_h^c\right)$$

By the additive property of the measure we have

$$\mu\left(T^{-1}Z\right) \geq \mu\left(Z_h \cap Z\right) + \mu\left(T^{-h-1}Z \cap Z_h^c \cap Z\right) + \mu\left(T^{-h-1}Z \cap Z_h^c \cap Z^c\right)$$

Putting the things in Z together we get

$$\mu\left(T^{-1}Z\right) \geq \mu\left\{\left[Z_h \cup \left(T^{-n-1}Z \cap Z_h^c\right)\right] \cap Z\right\} + \mu\left(T^{-h-1}Z \cap Z_h^c \cap Z^c\right)$$

Using (3.3.3),

$$\mu\left(T^{-1}Z\right) \geq \mu\left\{\left[Z_h \cup \left(T^{-n-1}Z \cap Z_h^c\right)\right] \cap Z\right\} + \mu\left[T^{-1}\left(T^{-h-1}Z \cap Z_h^c \cap Z^c\right)\right]$$

[1] *By* measurable map *we mean that the inverse image of every measurable set is measurable.*

Then, by definition of Z_{h+1}, we obtain the expression

$$\mu\left(T^{-1}Z\right) \geq \mu\left(Z_{h+1} \cap Z\right) + \mu\left(T^{-h-2}Z \cap Z_{h+1}^{c}\right)$$

So, from (3.3.4) we can write

$$\mu(Z) \geq \mu\left(T^{-1}Z\right) \geq \sup_{n} \mu\left(Z_n \cap Z\right)$$

Observe that

$$Z = \left(\cup_{n=1}^{\infty} T^{-n}Z\right) \cap P$$

This is true because if $x \in Z \subset P$, then by definition of Z we know that there is a first returning time N and then

$$\bar{x} = T^{N}(x) \in P \Rightarrow \bar{x} \in Z$$

then $x \in T^{-N}Z \cap P$. Conversely, if $x \in P$ and exists n_0 such that $x \in T^{-n_0}Z$, then

$$\bar{x} = T^{n_0}(x) \in Z$$

and since $\bar{x}$ returns infinitely many times to Z, so does x. We have also that

$$Z_n \nearrow \cup_{n=1}^{\infty} T^{-n}Z$$

then

$$\sup_{n} \mu\left(Z_n \cap Z\right) = \mu\left[\left(\cup_{n=1}^{\infty} T^{-n}Z\right) \cap Z\right] = \mu(Z)$$

From the inequality

$$\mu(Z) \geq \mu\left(T^{-1}Z\right) \geq \mu(Z)$$

we obtain the thesis.

■

Lemma 3.3. *(Rantzer, 2001a) Let $f \in C^1(D, \mathcal{R}^n)$ where $D \subset \mathcal{R}^n$ is open; consider the system $\dot{x} = f(x)$ and let $\rho \in C^1(D, \mathcal{R})$ be integrable. For a measurable set Z, assume that*

$$f^{\tau}(Z) = \{f^{\tau}(x) \;\; x \in Z\}$$

is a subset of D for all $0 \leq \tau \leq t$. Then[2]

$$\int_{f^t(Z)} \rho(x)dx - \int_{Z} \rho(x)dx = \int_0^t \int_{f^{\tau}(Z)} \left[\nabla.(\rho f)\right](x)dxd\tau \tag{3.3.5}$$

[2] *For $\rho \equiv 1$, we obtain the classical Liouville's Formula (Mañé, 1987).*

Proof: First of all, consider a matrix function $M : \mathcal{R} \to \mathcal{R}^{n\times n}$ of class C^1, such that $M(0) = I$. Then it is true that

$$\frac{\partial}{\partial t}\det\left[M(t)\right]\bigg|_{t=0} = tr\left[M'(0)\right]$$

where $tr(M')$ stands for the trace of the matrix M', that is, the sum of the diagonal elements. Recalling that the determinant of a given matrix is a multilinear function of its entries, it follows that $\det\left[M(t)\right]$ is a differentiable function. By the assumptions, we can write M as

$$M(t) = \begin{bmatrix} 1+\alpha_{11}(t) & \alpha_{12}(t) & \cdots & \alpha_{1n}(t) \\ \alpha_{21}(t) & 1+\alpha_{22}(t) & \cdots & \alpha_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}(t) & \alpha_{n2}(t) & \cdots & 1+\alpha_{n1}(t) \end{bmatrix}$$

with α_{ij} class C^1 functions and

$$\alpha_{ij}(0) = 0 \quad i = 1,\ldots,n, \quad j = 1,\ldots n$$

Then $\det\left[M(t)\right]$ can be written as

$$\det\left[M(t)\right] = 1 + \sum_{k=1}^{n}\alpha_{kk}(t) + \cdots$$

where we have not written the rest of the terms, but we note that they contain products of at least two different α_{ij} with $i \neq j$. Then, by direct differentiation and evaluation at 0, we obtain

$$\frac{\partial}{\partial t}\det\left[M(t)\right]\bigg|_{t=0} = \sum_{k=1}^{n}\alpha'_{kk}(0) = tr\left[M'(0)\right]$$

Consider now a given solution $f^t(x)$ and denote by $D_x f^t(x)$ the respective Jacobian, that is, the variation of $f^t(x)$ with respect of the initial condition x. Observe that if we put $M(t) = D_x f^t(x)$, we are in the previous situation since $M(0) = I$ and $M(t)$ is a differentiable function since f is C^1. Then

$$\begin{aligned} \tfrac{\partial}{\partial t}\det\left[D_x f^t(x)\right]\big|_{t=0} = tr\left[\tfrac{\partial}{\partial t}D_x f^t(x)\big|_{t=0}\right] &= tr\left\{D_x\left[\tfrac{\partial}{\partial t}f^t(x)\right]\big|_{t=0}\right\} \\ &= tr\left[D_x f(x)\right]\right] = \nabla.f(x) \end{aligned}$$

Consider the function $\rho_t(x) = \rho\left[f^t(x)\right].\det\left[D_x f^t(x)\right]$. Note that $\rho(x) = \rho_0(x)$. Then

$$\frac{\partial}{\partial t}\rho_t(x)\bigg|_{t=0} = \frac{\partial}{\partial t}\rho\left[f^t(x)\right].\det\left[D_x f^t(x)\right]\bigg|_{t=0} + \rho\left[f^t(x)\right].\frac{\partial}{\partial t}\left\{\det\left[D_x f^t(x)\right]\right\}\bigg|_{t=0}$$

By the chain rule and the previous observation,

$$\left.\frac{\partial}{\partial t}\rho_t(x)\right|_{t=0} = \nabla\rho(x).f(x) + \rho(x).\nabla.f(x) = \nabla.\left[\rho f\right](x)$$

We extend the result to the expression

$$\left.\frac{\partial}{\partial t}\rho_t(x)\right|_{t=\tau} = \lim_{h\to 0}\left[\frac{\rho_{\tau+h}(x) - \rho_\tau(x)}{h}\right] \tag{3.3.6}$$

with

$$\rho_{\tau+h}(x) = \rho\left[f^{\tau+h}(x)\right].\det\left[D_x f^{\tau+h}(x)\right]$$

Let us define $z = f^\tau(x)$. By properties of the solution of a differential equation,

$$f^{\tau+h}(x) = f^h(z) \quad , \quad D_x f^{t+h}(x) = D_z f^h(z).D_x f^\tau(x)$$

Then

$$\rho_{\tau+h}(x) = \rho\left[f^h(z)\right].\det\left[D_z f^h(z)\right].\det\left[D_x f^\tau(x)\right] = \rho_h(z).\det\left[D_x f^\tau(x)\right]$$

and $\rho_t(x) = \rho_0(z).\det\left[D_x f^t(x)\right]$. The limit (3.3.6) can be written as

$$\lim_{h\to 0}\left\{\frac{\left[\rho_h(z) - \rho_0(z)\right].\det\left[D_x f^\tau(x)\right]}{h}\right\} = \left.\frac{\partial}{\partial t}\rho_t(z)\right|_{t=0}.\det\left[D_x f^\tau(x)\right]$$

So

$$\left.\frac{\partial}{\partial t}\rho_t(x)\right|_{t=\tau} = \nabla.\left[\rho f\right]\left[f^\tau(x)\right].\det\left[D_x f^\tau(x)\right] \tag{3.3.7}$$

Consider the left side of identity (3.3.5). We make a change of variables $x = f^t(z)$ in order to put both integrals into the same domain Z:

$$\int_{f^t(Z)}\rho(x)dx = \int_Z \rho\left[f^t(z)\right].\det\left[D_z f^t(z)\right]dz = \int_Z \rho_t(z)dz$$

Then by (3.3.7), the left side in (3.3.5) becomes

$$\begin{aligned}\textstyle\int_Z\left\{\rho_t(z) - \rho_0(z)\right\}dz = & \textstyle\int_Z\left\{\int_0^t \frac{\partial}{\partial\tau}\rho_\tau(z)d\tau\right\}dz = \\ & \textstyle\int_Z\left\{\int_0^t \nabla.\left[\rho f\right]\left[f^\tau(z)\right].\det\left[D_z f^\tau(z)\right]d\tau\right\}dz\end{aligned}$$

Exchanging the order of integration and unmaking the change of variables we obtain

$$\int_{f^t(Z)}\rho(x)dx - \int_Z \rho(x)dx = \int_0^t\int_{f^\tau(Z)}\nabla.\left[\rho f\right](x)dxd\tau$$

■

Before going to the proof of the main result, we recall the definition of the ω and α-limit sets:

Definition 3.4. *Let $x \in \mathcal{R}^n$, the ω-limit of x is the set*

$$\omega(x) = \left\{ y \in \mathcal{R}^n \mid \exists t_n \to +\infty \ \ s.t. \ \lim_{n \to +\infty} f^{t_n}(x) = y \right\}$$

♠

$\omega(x)$ contains all the points that are *close* to the positive orbit of x for arbitrarily large times. In the same way is defined the α-limit set, except the time diverges to the past. If the trajectory through x is bounded to the future (past), the ω-limit (α-limit) is non empty, invariant, bounded, closed and connected (Khalil, 1996). For all $y \in \{f^t(x)\}$, it is true that $\omega(x) = \omega(y)$, $\alpha(x) = \alpha(y)$.

Proof of Theorem 3.1:

First of all we must note that due to the global boundedness of $f(x)/\|x\|$ we know that the system is complete, that is, the trajectories are defined for every real time t.

Let ϵ be an arbitrary small positive number. Consider the set

$$Z_\epsilon = \left\{ x \in \mathcal{R}^n \mid \limsup_{t \to +\infty} \|f^t(x)\| > \epsilon \right\}$$

If Z_ϵ has zero Lebesgue measure for every $\epsilon > 0$, then the set

$$Z = \bigcup_{n=1}^{\infty} Z_{\epsilon_n} \quad , \quad \epsilon_n = \frac{1}{n}$$

also has zero Lebesgue measure and if $x \notin Z$, it is true that

$$\lim_{t \to +\infty} f^t(x) = 0$$

In order to show that Z_ϵ has zero Lebesgue measure, we will show that

$$\int_{f^t(Z_\epsilon)} \rho(x)dx = \int_{Z_\epsilon} \rho(x)dx \quad \text{for some } \ t \geq 0$$

If $0 \notin \bar{Z}_\epsilon$, we can use inequality (3.3.1) and identity (3.3.5). A particular case is when Z_ϵ is an invariant set, i.e. $f^t(Z_\epsilon) = Z_\epsilon$.

Consider first the case where the origin is not only almost globally stable but also locally stable in the sense of Lyapunov (l.s.). Then, given $\epsilon > 0$ we can find a positive number δ such that

$$\sup_{t\geq 0} \|f^t(x)\| < \epsilon \quad \forall \|x\| \leq \delta$$

Being 0 locally stable, if for a given $x \in \mathcal{R}^n$, $0 \in \omega(x)$, then $\omega(x) = \{0\}$. Define $D = [B(0,\delta)]^c$, a set within is valid Lemma 3.3, and consider the following measurable set

$$M = \bigcap_{n=1}^{\infty} \left\{ x \mid \sup_{t\geq 0} \|f^{n+t}(x)\| > \epsilon \right\}$$

It is clear that $M \subset D$ and that M contains the points which their ω-limit sets are disjoint with $\{0\}$. Since the trajectories are defined for all t, $f^t(M) = M$ for all nonzero t. Then, M has zero Lebesgue measure. Observe that we have not used the sign definition of ρ and it is needless in the locally stable case. Then, almost all the trajectories in D have

$$\limsup_{t\geq 0} \|f^t(x)\| < \epsilon$$

Since this is true for every $\epsilon > 0$, we can conclude that almost all the trajectories go to the origin.

We consider now the more general situation of almost global stability. We can not follow the previous way, since we can not isolate the origin from the points that may go so far from it, and then we may have some problems in the definition of the set D. We will use Lemma 3.2. Let $X = \mathcal{R}^n$ and define $T : \mathcal{R}^n \to \mathcal{R}^n$ as the time 1 of the flow:

$$T(x) = f^1(x)$$

Introduce the measure μ

$$\mu(X) = \int_X \rho(x)dx$$

for every measurable set of $\mathcal{R}^n$. We are using here the sign definition of ρ. By the integral assumption on ρ, every measurable set that does not contain the origin in its adherence has finite measure. We note that μ and T satisfy

$$\mu\left(T^{-1}Y\right) \leq \mu(Y)$$

for every Y with $\mu(Y) = +\infty$. Consider a set Y such that $0 \notin \bar{Y}$. Then, $\mu(Y) < +\infty$ and we can find a small enough δ such that Y, $f^1(Y) \subset B^c(0,\delta)$.

With $D = B^c(0, \delta)$ we can apply Lemma 3.3:

$$\mu\left(T^{-1}Y\right) \leq \mu(Y) < \infty$$

Finally, consider a set Y with $0 \in \bar{Y}$ and $\mu(Y) < +\infty$. We can not directly apply Lemma 3.3 since we can not isolate the set from $\{0\}$. We use the following decomposition:

$$Y = \bigcup_{n=1}^{\infty}\left[Y \cap B^c\left(0, \frac{1}{n}\right)\right] \equiv \bigcup_{n=1}^{\infty} Y_n$$

The sequence $\{Y_n\}_{n \in \mathcal{N}}$ is increasing and

$$\mu(Y) = \sum_{n=1}^{\infty} \mu(Y_n)$$

Each Y_n verifies that $0 \notin \bar{Y}_n$, then

$$\mu\left(T^{-1}Y_n\right) \leq \mu(Y_n) < \infty$$

We have that

$$\mu\left(T^{-1}Y\right) = \mu\left[T^{-1}\bigcup_{n=1}^{\infty} Y_n\right] = \mu\left[\bigcup_{n=1}^{\infty} T^{-1}Y_n\right] \leq \sum_{n=1}^{\infty} \mu\left(T^{-1}Y_n\right) \leq \sum_{n=1}^{\infty} \mu\left(Y_n\right) = \mu(Y)$$

So μ and T satisfy the hypothesis of Lemma 3.2. Let ϵ be an arbitrary positive number and define

$$P = \{x \in \mathcal{R}^n \mid \|x\| > \epsilon\}$$

Then $\mu(P) < +\infty$ and the set $Z \subset P$ of points that return infinite times to P satisfies

$$\mu\left(T^{-1}Z\right) = \mu(Z)$$

That is

$$\int_{-1}^{0}\int_{f^{\tau}(Z)} \nabla.\left[\rho f\right](x) dx d\tau = 0$$

Then, by the sign condition (3.3.1), Z has zero Lebesgue measure and almost all points in P never return to P for large times. This is the same that

$$\limsup_{n \geq 0} \|f^n(x)\| \leq \epsilon \quad \text{for almost every } x \in P$$

Since ϵ is an arbitrary number, we conclude for almost $x \in \mathcal{R}^n$,

$$\limsup_{n \geq 0} \|f^n(x)\| = 0$$

Consider now an arbitrarily large time t and its integer part $[t]$. Let C be a global bound of $f(x)/\|x\|$. Then, by the comparison lemma (Khalil, 1996), we can compare the trajectories of the following two systems

$$\dot{x} = f(x) \quad \text{ and } \quad \dot{y} = g(y) = C.y$$

with the same initial condition $f^{[t]}(x)$ with an arbitrary x. Since $\|f(x)\| \leq C.\|x\|$, it follows that

$$\|f^t(x)\| \leq e^{C(t-[t])}.\|f^{[t]}(x)\| \leq e^C.\|f^{[t]}(x)\| \quad , \quad [t] \leq t < [t]+1$$

Then, given $\epsilon > 0$, if we choose n such that

$$\|f^m(x)\| \leq \frac{\epsilon}{e^C} \quad \forall m \geq n$$

we have, for every $t \geq n$,

$$\|f^t(x)\| \leq \epsilon \text{ for almost all } x \in \mathcal{R}^n$$

■

3.4 Extensions of the main result

We present here some extensions of Theorem 3.1. The first one is just a relaxation on the requirements on the vector field.

Theorem 3.4. *(Rantzer, 2001a) Given the equation (3.3.2) with $f \in C^1(\mathcal{R}^n \setminus \{0\}, \mathcal{R}^n)$, $f(x)/\|x\|$ is locally bounded around $x = 0$, $\rho(x).f(x)/\|x\|$ in integrable on*

$$\{x \in \mathcal{R}^n \mid \|x\| \geq 1\}$$

and

$$\nabla.\left[\rho f\right](x) > 0 \quad a.e.$$

Then, for almost all initial states x, the trajectory $f^t(x)$ exists for every non-negative time t and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Proof: Define the auxiliary functions

$$\rho_0(x) = \left[\frac{e^{-\|x\|}}{1+\rho^2(x)} + \frac{\|f(x)\|^2}{\|x\|^2}\right]^{-\frac{1}{2}}.\rho(x) \quad , \quad f_0(x) = \frac{\rho(x).f(x)}{\rho_0(x)}$$

Then

$$\frac{\|f_0(x)\|}{\|x\|} = \frac{\rho(x)}{\rho_0(x)}.\frac{\|f(x)\|}{\|x\|} \approx \frac{\|f(x)\|^2}{\|x\|^2}$$

which locally bounded around $x = 0$ and for large x,

$$\rho_0(x) \approx \frac{\rho(x)\|f(x)\|}{\|x\|}$$

which is integrable for $\|x\| \geq 1$. Then ρ_0 is integrable in that domain and $f_0(x)/\|x\|$ is globally bounded.

So f_0 and ρ_0 are in the conditions of Theorem 3.1 and almost all the trajectories $f_0^t(x)$ tends to zero as $t \to \infty$.

Observe that the two systems $\frac{dx}{dt} = f_0(x)$ and $\frac{dy}{d\tau} = f(y)$ are related through the time-change

$$\tau = \int_0^t \frac{\rho\left[f_0^u(x)\right]}{\rho_0\left[f_0^u(x)\right]} du$$

since

$$\frac{d}{dt} f_0^t(x) = f_0\left[f_0^t(x)\right] = f\left[f_0^t(x)\right] . \frac{\rho\left[f_0^t(x)\right]}{\rho_0\left[f_0^t(x)\right]} = \frac{d}{d\tau} f_0^{\tau(t)}(x) . \frac{d\tau}{dt}$$

■

We can use density functions to prove the almost global stability of a set, not just an equilibrium point.

Theorem 3.5. *(Rantzer, 2001b) Let M be a manifold and a subset $A \subset M$. Consider the equation (3.3.2) with f a C^1 complete vector field on M such that $f(x) = 0$ for every $x \in A$. Let ρ be a C^1 function defined on $M \setminus A$ and integrable outside every neighborhood of A. If*

$$\nabla . \left[\rho . f\right](x) > 0 \quad , \quad a.e.\ x \in M \setminus A$$

then A is almost globally stable.

Proof: As in Theorem 3.1, for every Borelian subset Y of M we define

$$\mu(Y) = \int_Y \rho(x) dx$$

Then $\mu\left(T^{-1}Y\right) \leq \mu(Y)$. We can apply Lemma 3.2 to the set

$$P = \{x \in M \mid d(x, M) > \epsilon\}$$

where d denotes the distance from x to M. Then P has finite μ-measure and then the set Z of recurrent points has zero Lebesgue measure. As in Theorem 3.1 it implies that almost all the trajectories converge to the set A.

■

With the previous idea, the results for almost global stability can be extended to manifolds and to invariant, almost attractor sets.

3.5 Examples

Example 3.1. *(Rantzer, 2001a) Consider the planar system*

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix}$$

It has two equilibria at $[0,0]^T$ *and* $[2,0]^T$. *Some trajectories and the vector field are shown in figure* 3.1. *Consider the candidate density function* $\rho(x) = \|x\|^{-\alpha}$, *where* α *must be chosen in order to fulfill the integral and the sign conditions.*

$$\nabla.\left[\rho.f\right](x) = \nabla\rho.f + \rho.\nabla.f = -\alpha\|x\|^{-\alpha-2}.x^T.f + 4.\|x\|^{-\alpha}(x_1 - 1) =$$

$$= -\alpha\|x\|^{-\alpha-2}.(x_1 - 2).\left(x_1^2 + x_2^2\right) + 4.\|x\|^{-\alpha}(x_1 - 1) = \|x\|^{-\alpha}.\left[(4-\alpha).x_1 + 2\alpha - 4\right]$$

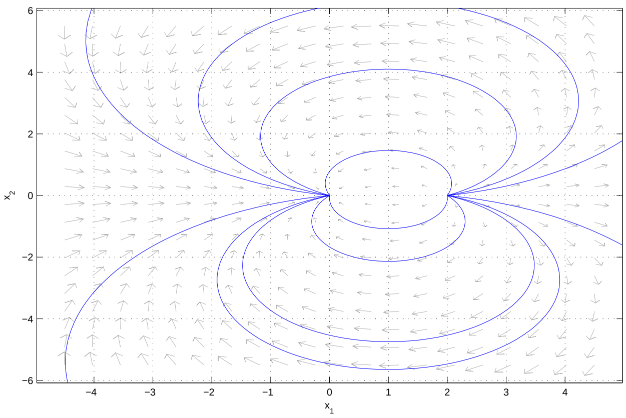

Figure 3.1: Phase portrait of the system of Example 3.1.

Taking $\alpha = 4$ *we get* $\nabla.\left[\rho.f\right](x) = 4.\|x\|^{-\alpha}$ *and almost all the trajectories goes to the origin. The exceptions are the unstable point* $[2,0]^T$ *and a branch of its unstable manifold.*

♣

Example 3.2. *Consider the scalar system*

$$\dot{\theta} = a - b\sin(\theta) \quad a < b \tag{3.5.1}$$

It corresponds, for example, to a first order phase-locked loop (Paganini and Kofman, 1989) or to a weightless pendulum with friction and external torque (Khalil, 1996). For $a < b$, *it has two equilibrium points in the interval* $[0, 2\pi)$: θ_e *stable and* θ_u *unstable.*

We introduce the periodic density function shown in figure 3.2. *It has been designed in order to fulfill the divergence requirement.*

$$\frac{\partial}{\partial\theta}\rho(\theta).\left[a - b\sin(\theta)\right] = \frac{\partial\rho(\theta)}{\partial\theta}.\left[a - b\sin(\theta)\right] - b.\rho(\theta).\left[\cos(\theta)\right]$$

Function ρ must be periodic, of class C^1 for every $\theta \neq \theta_e$ and integrable around θ_e.

We have taken

$$\rho(\theta) = \frac{g(\theta)}{a - b\sin(\theta)}$$

with $g(\theta_u) = 0$ in order to make the function continuous at the saddle point and $g(0) = g(2\pi)$. Finally we remark that

$$\frac{g(\theta)}{a - b\sin(\theta)}$$

is integrable around θ_e.

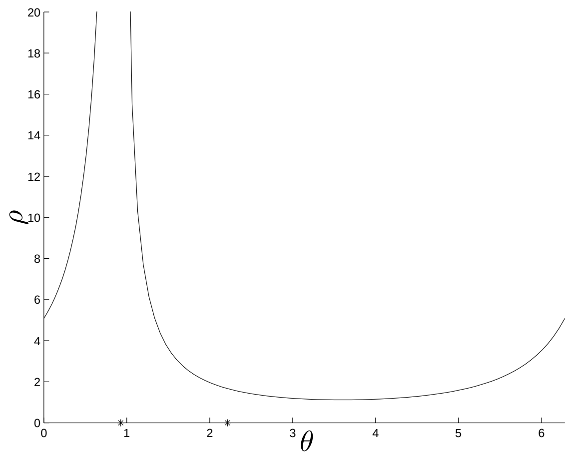

Figure 3.2: Density function for the example 3.2.

♣

Example 3.3. *(Prajna and Rantzer and Parrilo, 2004) Consider the system*

$$\begin{aligned} \dot{x}_1 &= -6x_1x_2 - x_1^2x_2 + 2x_2^2 \\ \dot{x}_2 &= x_2.\left[2.2x_1^2 - 4.9x_2^2\right] \end{aligned}$$

which has a continuum of equilibrium points on the line $x_2 = 0$. It follows that

$$\rho(x) = \frac{1}{x_1^2 + x_2^2}$$

is a density function for the system. Then the origin is almost globally stable. The phase portrait is shown in figure 3.3. *The only trajectories that are not attracted by the origin are the equilibrium points of the x_1 axis.*

♣

Example 3.4. *The following set of equations*

$$\begin{aligned} \dot{\theta}_1 &= K.\sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= K.\sin(\theta_3 - \theta_2) \\ \dot{\theta}_3 &= K.\sin(\theta_1 - \theta_3) \end{aligned}$$

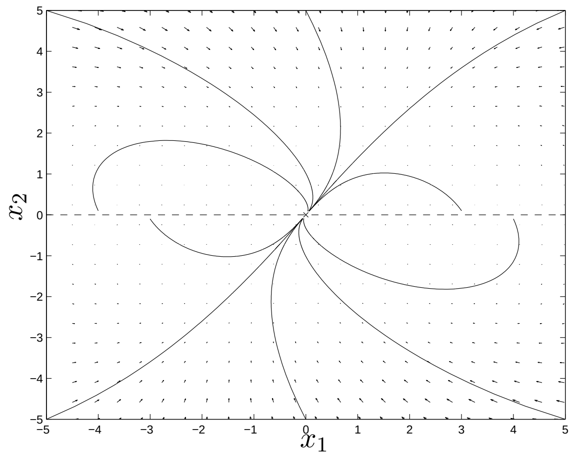

Figure 3.3: Phase portrait of the system of Example 3.3.

describes the dynamic of three oscillators coupled unidirectionally on a ring[3]*. The angles θ_i belong to the interval $[0, 2\pi)$ and the state space is the three-dimensional torus $\mathcal{T}^3$. The diagonal D of the torus, i.e., the set of points with $\theta_1 = \theta_2 = \theta_3$, is called the* consensus *set of the system and it is a closed curve in the state space. The function*

$$\rho(\theta) = \frac{1}{3 - \cos(\theta_2 - \theta_1) - \cos(\theta_3 - \theta_2) - \cos(\theta_1 - \theta_3)}$$

is continuously differentiable in $\mathcal{T}^3 \setminus D$ and is integrable on the complement of every neighborhood of D. It is also true that (see Chapter 6*, Section* 6.6*)*

$$\nabla . [\rho . f](\theta) > 0 \quad , \ a.e.$$

Then the consensus set is almost globally stable.

♣

3.6 Some properties of density functions

In this Section, we analyze some particular properties of density functions that can help in the search of that kind of functions.

3.6.1 Change of coordinates

Assume that we have a system $\dot{x} = f(x)$ and we consider a change of coordinates of the form[4]

$$x = \phi(z)$$

[3] *This kind of systems are deeply studied in Chapter* 6.

[4] This Section follows the presentation of (Angeli, 2003)

with ϕ a diffeomorphism of $\mathcal{R}^n$. Since

$$\dot{x} = D_z\phi(z).\dot{z}$$

the new description of the system becomes

$$\dot{z} = [D_z\phi]^{-1}[\phi(z)].f[\phi(z)]$$

It is well known that if we have a Lyapunov function V, the new function

$$\bar{V}(z) = V[\phi(z)]$$

verifies the condition

$$\dot{\bar{V}}(z) = \nabla\bar{V}.\dot{z} = \nabla V.D_z\phi[\phi(z)].[D_z\phi]^{-1}[\phi(z)].f[\phi(z)] = \dot{V}[\phi(z)] \leq 0$$

Then, $\bar{V}(z)$ is a Lyapunov function for the new system description. We consider now a density function $\rho(x)$ and we want to construct a density function for the new system using ρ and ϕ. Since for a given Borel set Z

$$\int_Z \rho(x)dx = \int_{\phi^{-1}(Z)} \rho[\phi(z)]\,|D_z\phi(z)|\,dz$$

where $|D_z|$ denotes the determinant of the matriz D_z. Then, it is natural to define

$$\bar{\rho}(z) = \rho[\phi(z)].|D_z\phi(z)|$$

This procedure preserves the sign definition of the divergence, since it preserves the value of the integrals. This idea will be used later in Chapter 5.

3.6.2 Non positive definite density functions

Since ρ and f are C^1 functions, the divergence condition (3.3.1) should be understood in the sense that $\nabla.(\rho f)$ only could vanish at a zero Lebesgue measure set. Since this divergence condition together with Lemma 3.3 imply the growth along the flow of the sets where ρ is integrable, it follows that ρ can not be integrable in the whole space $\mathcal{R}^n$, which is the largest invariant set.

In Theorem 3.1, for the special case of a local stable equilibrium point at the origin, the sign definition of the function ρ was not needed. In this context, it is natural to ask about the existence of density functions that take positive and negative values. The following set of results shows that every density function should be positive everywhere, even in the presence of local stability of the origin.

Theorem 3.6. *(Angeli, 2003) Consider a nonlinear system* $\dot{x} = f(x)$*. Assume that the system is complete and let a function* $\rho \in C^1(\mathcal{R}^n \setminus \{0\}, \mathcal{R})$ *with* $\nabla.(\rho f) > 0$ *almost everywhere. If* ρ *is integrable over*

$$\mathcal{N} = \{x \in \mathcal{R}^n \mid \rho(x) \leq 0\}$$

then $\mathcal{N}$ *has zero Lebesgue measure, i.e.,* $\rho(x) > 0$ *almost everywhere.*

Proof: First of all we will see that the set $\mathcal{N}$ is negatively invariant, that is, $f^{-t}(\mathcal{N}) \subset \mathcal{N}$ for every $t \geq 0$. By continuity of ρ and f we know that

$$\nabla. [\rho.f](x) = \dot{\rho}(x). [\nabla.f(x)] \geq 0 \quad \forall x \in \mathcal{R}^n \setminus \{0\}$$

So $-\dot{\rho}(x) \leq \rho(x). [\nabla.f(x)]$. Then

$$-\dot{\rho}(x) = \frac{\partial \rho}{\partial t}\left[f^{-t}(x)\right]\Big|_{t=0} \leq \rho(x). [\nabla.f(x)]$$

and

$$\rho\left[f^{-t}(x)\right] \leq \rho(x).e^{\int_0^t \nabla.f\left[f^{-\tau}(x)\right]d\tau}$$

where we have used the *comparison lemma* (Khalil, 1996). We can conclude that if $\rho(x) \leq 0$, $\rho\left[f^{-t}(x)\right] \leq 0$ for all $t \geq 0$.

Since ρ is integrable over N, we can apply Lemma 3.3:

$$\int_{f^{-\tau}(N)} \rho(x)dx - \int_N \rho(x)dx = -\int_0^t \int_{f^{-\tau}(N)} \nabla. [\rho.f](x)dxd\tau \leq 0$$

which is non positive due to the sign definition of the divergence. On the other hand, since N is negatively invariant and ρ is non positive over N, we have

$$\int_{f^{-\tau}(N)} \rho(x)dx - \int_N \rho(x)dx \geq 0$$

Then N has zero Lebesgue measure.

∎

The next result is just a minor extension of Theorem 3.6. We have added the local stability condition, trying specifically to answer the question about the existence of non definite density functions.

Proposition 3.7. *Consider the system (*3.3.2*). Assume that it is complete and that the origin is a locally stable equilibrium point. Let* ρ *be a (possible non positive) density function for (*3.3.2*) such that is integrable outside arbitrary neighborhoods of the origin. Then* ρ *is positive almost everywhere.*

Proof: Let $A \subset \mathcal{R}^n$ be a non zero Lebesgue measure set such that $\rho < 0$ on A and there exists $\epsilon > 0$ with

$$A \subset B^C(0, \epsilon)$$

From stability of the origin we can find $\delta > 0$ such that

$$M = \bigcup_{t<0} f^t(A) \subset B^C(0, \delta)$$

Then ρ in integrable on M and M is negative invariant ($f^t(M) \subseteq M$, $t \leq 0$). As in Theorem 3.6, for each $x \in \mathcal{R}^n \setminus \{0\}$ consider the auxiliar differentiable function

$$G_x(t) = \rho \left[f^t(x) \right] .e^{\int_0^t \nabla . f[f^\tau(x)] d\tau}$$

We have that

$$G'_x(t) = e^{\int_0^t \nabla . f[f^\tau(x)] d\tau} . \left[\dot{\rho} \left[f^t(x) \right] + \nabla . f \left[f^t(x) \right] . \rho \left[f^t(x) \right] \right] =$$

$$e^{\int_0^t \nabla . f[f^\tau(x)] d\tau} . \left[\nabla . \rho f \right] \left[f^\tau(x) \right] > 0$$

for almost every $x \in \mathcal{R}^n$. So G_x is non-decreasing and if $\rho(x) < 0$, $\rho \left[f^t(x) \right] < 0$ for all $t \leq 0$. Then ρ is negative on M since it is negative on A. It is true that for negative t,

$$\int_{f^t(M)} \rho(x) dx - \int_M \rho(x) dx \geq 0$$

From (3.3.5),

$$\int_{f^t(M)} \rho(x) dx - \int_M \rho(x) dx < 0$$

Then M has zero Lebesgue measure and this can not happen because $A \subset M$. Since ρ is continuous, then it must be non-negative almost everywhere. Positivity of the divergence of ρf implies that ρ can not be zero on a non-zero Lebesgue measure set. We conclude that ρ must be positive almost everywhere.

■

Then, although the direct result of Theorem 3.1 admits non positive density functions when we have local stability, the previous result shows that for complete systems, all density functions are essentially positive. The next Proposition shows that in the context of Rantzer's direct result, all density functions are positive almost everywhere.

Proposition 3.8. *Consider the dynamical system (3.3.2), with $x = 0$ a locally stable equilibrium point, and a function $\rho \in C^1(\mathcal{R}^n \setminus \{0\}, \mathcal{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathcal{R}^n : |x| \geq 1\}$ and*

$$[\nabla.(f\rho)](x) > 0 \quad \text{for almost all } x$$

Then ρ is positive almost everywhere.

Proof: As in Theorem 3.4, consider the functions

$$\rho_0(x) = \left[\frac{e^{-|x|}}{1 + |\rho(x)|^2} + \frac{|f(x)|^2}{|x|^2}\right]^{1/2} \rho(x)$$

and $f_0(x) = \frac{f(x)\rho(x)}{\rho_0(x)}$. Then $f_0(x)/|x|$ is globally bounded and the system $\dot{y} = f_0(y)$ is complete[5]. Moreover, ρ_0 is integrable outside arbitrary neighborhoods of the origin and we have that

$$\nabla.[\rho_0 f_0](x) = \nabla.[\rho f](x) > 0 \quad a.e.$$

So ρ_0 and f_0 are in the hypothesis of Proposition 3.7. Then ρ_0, and hence ρ, are positive almost everywhere.

■

3.6.3 Density functions for systems with negative definite divergence

The next result imposes a very hard restriction for a density function for a particular class of dynamical systems.

Proposition 3.9. *(Angeli, 2003) Assume that $\nabla.f < 0$ in some positive invariant open set $U \subset \mathcal{R}^n$. Let x_e and $W^s_{x_e} \subset U$ be respectively a saddle point in U and its stable manifold[6]. Then, any C^1 function ρ with $\rho(x) > 0$ almost everywhere in U and satisfying*

$$\nabla.[\rho.f] > 0 \quad \text{for almost every } x \in U$$

must vanish ($\rho(x) = 0$) for all $x \in W^s_{x_e} \cap U$.

Proof: By continuity of ρ and f in U, we have

$$\nabla.[\rho.f](x) = \dot{\rho}(x) + \rho(x).\nabla.f(x) \geq 0 \quad \text{for all } x \in U \tag{3.6.1}$$

[5] Is just a time re-parameterization of the original system.

[6] Definitions of stable, unstable and center manifolds can be found in (Guckenheimer and Holmes, 1983).

Since x_e is an equilibrium point, $\dot{\rho}(x_e) = 0$ and

$$\rho(x_e).\nabla.f(x_e) \geq 0$$

The sign definition of ρ and $\nabla.f$ implies the $\rho(x_e) = 0$.

Consider the stable manifold $W^s_{x_e}$ of x_e and assume there exists a point $x \in W^s_{x_e}$ with $\rho(x) > 0$. Since

$$\lim_{t \to +\infty} f^t(x) = x_e$$

Using the continuity of ρ we obtain

$$0 = \rho(x_e) = \lim_{t \to +\infty} \rho\left[f^t(x)\right]$$

and then there exists a point $\bar{x} \in f^t(x)$ with

$$\rho(\bar{x}) > 0 \quad , \quad \dot{\rho}(\bar{x}) < 0$$

But $0 \leq \dot{\rho}(\bar{x}) + \rho(\bar{x}).\nabla.f(\bar{x}) < 0$ and we get an absurd.

■

For almost globally stable dynamical systems with negative definite divergence, a density function must vanish along the stable manifold of existing saddle points. In this context. we can not make algorithms to find density functions, since in general, the involved stable manifolds are not known in a closed form and must be traced by numerical simulation. The next example, with significative relevance on power systems analysis, shows a case where global Lyapunov stability can not be achieved, but almost global stability exists for wide range of values of the parameters.

Example 3.5. *(Lesieutre, 1997) The OMIB (one-machine-infinite-bus) is the model a single salient-pole generator connected through a lossless transmission line to an infinite bus. The describing equations are*

$$\begin{cases} \dot{\bar{\delta}} & = & \omega \\ M\dot{\omega} & = & -D\omega + P_m - P_{e1}\sin(\bar{\delta}) + P_{e2}\sin(2\bar{\delta}) \end{cases} \tag{3.6.2}$$

where δ is the angle of the machine, ω is the velocity, M is the inertia of the rotor, D is a friction coefficient, P_m is the power externally supplied to the rotor and P_{e1} and P_{e2} are parameters that reflects the active power delivered by the machine to the infinite bus.

This model belongs to a well known class of second order systems. A complete study of this class can be found in (Andronov, 1949). The origin is a locally asymptotically stable equilibrium point. It happens that for small values of the friction coefficient D, the *region of attraction of the origin is bounded. When D increases, the region grows and there exists a critical value D_{crit} such that for bigger values, the origin is almost globally stable. In this case, the only trajectories that are not attracted by the origin are the saddle point of coordinates $(\pi, 0)$ and its stable manifold. The phase space is a cylinder, but we sketch the situation in the plane in Figures* 3.4*. So, the system presents a global bifurcation related to the parameter D (Guckenheimer and Holmes, 1983; Perko, 1991). Two significative different situations occurs when D reaches it critical value. A possible way to detect this bifurcation is using a density function, since if we can find a density function, we know we are in the a.g.s case. But for this system, the divergence of the field is $\nabla.f(\delta, \omega) = -D < 0$. Then, we are in the case of Proposition* 3.9 *and then it is not algorithmically possible to find a density function.*

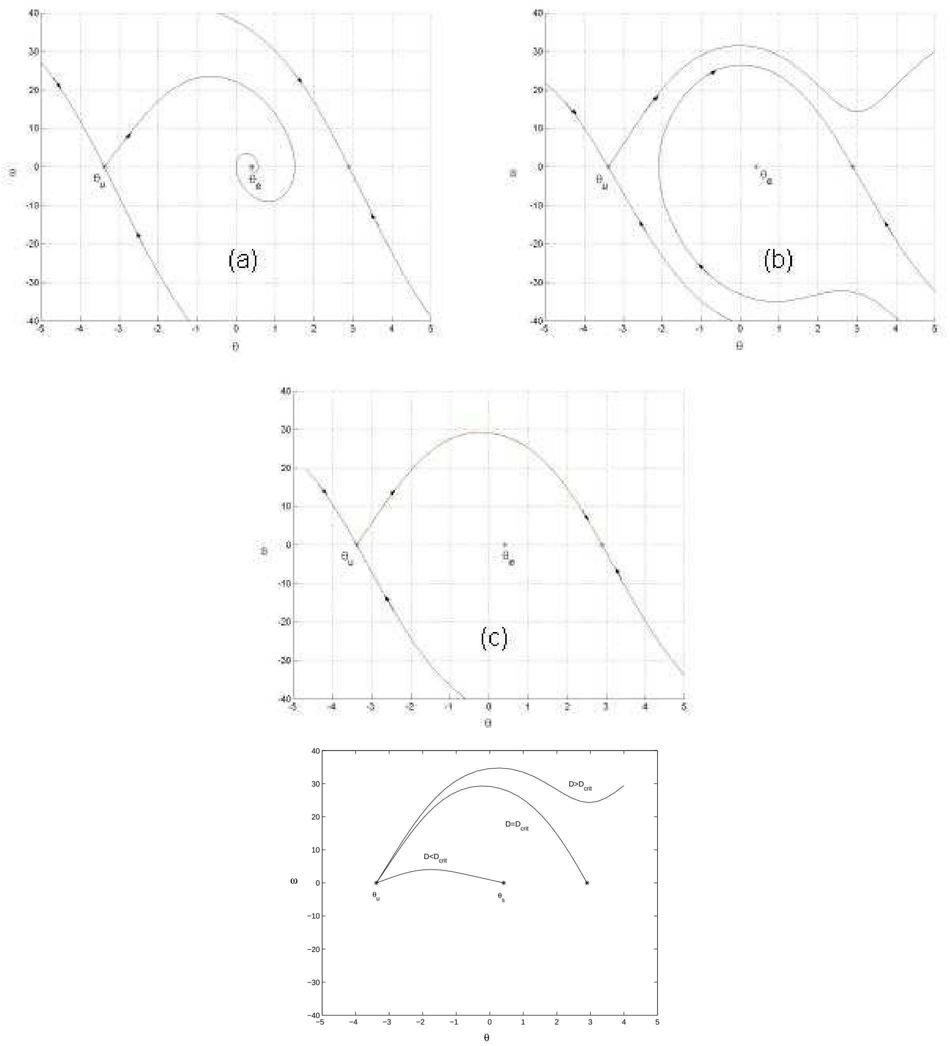

Figure 3.4: Phase portrait of the system of Example 3.5: (a) $D > D_{crit}$, (b) $D > D_{crit}$, (c) $D = D_{crit}$.

♣

Chapter 4

Monotone measures

In the previous Chapter we have seen how the existence of a density function implies the almost global stability of the system. Using a density function we can construct a measure over $\mathcal{R}^n$ that grows along the trajectories, due to the sign definition of the divergence, and is finite for sets that can be isolated from $\{0\}$, due to the integrability condition. Actually, these are the meanings of the identity (3.3.5), the divergence condition (3.3.1) and the integrability requirement. So we can re-state Theorem 1 in (Rantzer, 2001a) just in terms of measures, as we will show in the next Section.

The use of measures to study dynamical systems is the core of the ergodic theory. The great development of this theory was in the study of invariant measures and preserving maps, i.e., measures that do not change when sets evolve along the dynamical system. A lot of work was devoted to prove existence and properties of such kind of measures and it has become an important field of mathematics. In the present Chapter, we explore the relationship between monotone (increasing or decreasing) measures and the stability of dynamical systems.

4.1 Definitions

Consider the following version of Theorem 3.1.

Theorem 4.1. *Given the equation $\dot{x} = f(x)$ where $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $f(0) = 0$ and $\frac{\|f(x)\|}{\|x\|}$ is globally bounded, suppose there exists a Borel measure μ in $\mathcal{R}^n$, such that:*

- $\mu\left[B^c(0,\epsilon)\right] < +\infty$ *for every* $\epsilon > 0$.
- *for every Borel set Y with $0 < \mu(Y) < +\infty$ and for every $t > 0$*

$$\mu\left[f^t(Y)\right] > \mu(Y) \tag{4.1.1}$$

Then the origin is almost globally stable.

♣

We want to emphasize that this approach can drive us to a new set of results that we want to explore. Inequality (4.1.1) is crucial and this kind of behavior will be used along the Chapter, so we introduce the following definition.

Definition 4.1. *Given a vector field $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, a Borel measure μ is said to be monotone if for all Borel set Y it is true that*

- *if $\mu(Y) = 0$ then $\lambda(Y) = 0$ (being λ the Lebesgue measure). We use the standard notation $\mu \gg \lambda$.*
- *for every non-zero and finite μ-measure set Y*

$$\mu\left[f^t(Y)\right] - \mu(Y)$$

has definite sign for all $t > 0$.

♠

We say that a monotone measure μ is increasing if the measure of every set, with positive finite measure, grows along the flow. In the same way we define a decreasing measure. Observe that from the above definition, the measure of any invariant set, that is, a set Y satisfying $f^t(Y) = Y$ for all t, must be 0 or $+\infty$. As a consequence, the measure of the whole space should be infinite. In the same way, if for a given μ-measurable set Y with finite measure there exists $t > 0$ such that

$$\mu\left[f^t(Y)\right] = \mu(Y)$$

then $\mu(Y) = 0$ and Y has zero Lebesgue measure. We will deal with two particular kinds of monotone Borel measures: the ones *bounded over compact sets* and the ones *bounded at infinity* (for every $\epsilon > 0$, the exterior of the ball of radius ϵ centered at the origin has finite measure μ). The previous definitions can be extended for systems on a manifold in a direct way.

From the proof of Theorem 3.1, it is clear that every density function for a given system induces a growing measure bounded at infinity. On the other side, a decreasing measure bounded on compact sets recovers the idea that every set *shrinks to the attractor*. We present an illustrative example.

Example 4.1. *Consider the two dimensional torus and the system described by the equations*

$$\left.\begin{array}{rcl}\dot{\Phi}_1 & = & \sin(\Phi_2) - \sin(\Phi_1) \\ \dot{\Phi}_2 & = & \sin(\Phi_1) - \sin(\Phi_2)\end{array}\right. \Leftrightarrow \dot{\Phi} = F(\Phi)$$

with $\Phi_1 + \Phi_2 = 2\pi$, which is a particular case of two coupled oscillators that appears in some biological systems (Strogatz, 2000). Consider the functions

$$\begin{array}{rcl}\rho(\Phi) & = & \frac{1}{1-\cos(\Phi_1)} \\ l(\Phi) & = & \frac{1}{1+\cos(\Phi_1)}\end{array}$$

It follows that

$$\begin{array}{rcl}\nabla.\left[\rho.F\right](\Phi) & = & \frac{2}{1-\cos(\Phi_1)} \\ \nabla.\left[l.F\right](\Phi) & = & -\frac{2}{1+\cos(\Phi_1)}\end{array}$$

Then ρ and l induce, respectively, an increasing and a decreasing measure.

♣

For the system $\dot{x} = f(x)$, we introduce the set $\mathcal{M}_f$ of all increasing Borel measures. It is easy to see that $\mathcal{M}_f$ is a convex cone. In the next Chapter, we will introduce conditions that ensure that $\mathcal{M}_f$ is non-empty.

4.2 Monotone measures in $\mathcal{R}^2$

We present here a result for two dimensional spaces that combines the ideas of monotone measures with the Poincaré-Bendixson Theory, which characterizes the possible ω and α limit sets for a given trajectory[1].

It can be proved that if the trajectory $\{f^t(x)\}$ is bounded for $t \to \pm\infty$, $\omega(x)$ ($\alpha(x)$) is a non-empty, compact, connected and invariant set (Khalil, 1996) and we can talk of the ω (α) -limit of the whole trajectory through x.

Theorem 4.2. *Consider the system $\dot{x} = f(x)$, where $f \in C^1\left(\mathcal{R}^2, \mathcal{R}^2\right)$. Assume that there is a finite number of fixed points in any compact set of $\mathcal{R}^2$. Suppose there exists a measure $\mu \gg m$, with m the Lebesgue measure, satisfying that for every bounded and measurable set Y, $\mu(Y) < \infty$, and if $0 < \mu(Y) < \infty$, there exists $t \neq 0$ such that*

$$\mu\left[f^t(Y)\right] \neq \mu(Y) \tag{4.2.1}$$

Then, almost global stability (a.g.s.) of the origin implies local asymptotical stability (l.a.s.).

[1]The definition of the ω and α limit set was introduced in Chapter 3, Definition 3.4. More about these sets can be found in (Khalil, 1996).

Proof: The Poincaré-Bendixson Theorem (Perko, 1991) states that for a given point $x \in \mathcal{R}^2$ whose positive orbit is bounded, $\omega(x)$ can be only

1. a singular point
2. a closed orbit
3. singular points p_1, p_2, ..., p_n and regular orbits γ such that $\alpha(\gamma) = p_i$, $\omega(\gamma) = p_j$, $i, j = 1, \ldots, n$.

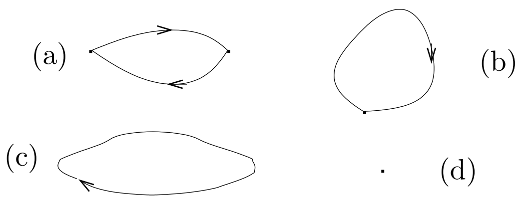

Figure 4.1: *Possible structures for a non empty ω-limit set.*

The same result follows for the α-limit set (Khalil, 1996; Roxin and de Spinadel, 1976). Typical ω-limit (α-limit) possible sets for a point x are shown in figure 4.1. The hypothesis (4.2.1) about μ implies that the only possible situation for a non empty ω-limit (α-limit) set is a single fixed point, case (d) in figure 4.1, since cases (a), (b) and (c) contains an invariant non-zero Lebesgue measure set. This will be an important fact.

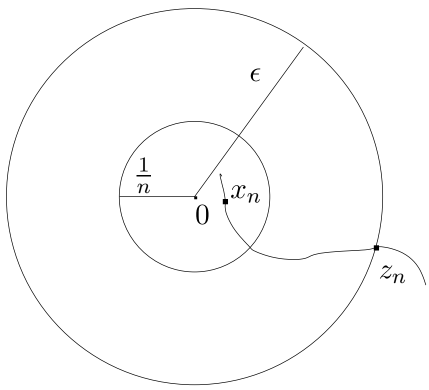

Figure 4.2: *Finding the sequence $\{x_n\}$.*

We will prove the thesis by contradiction. Suppose that the origin is not a locally asymptotically stable fixed point. Then, there is an $\epsilon > 0$, small enough to ensure that $x = 0$ is the only singular point inside the open ball $B(0, \epsilon)$, such that for every non-zero $n \in \mathcal{N}$ we can find a $x_n \in \mathcal{R}^2$ with

$$\|x_n\| < \frac{1}{n} \quad , \quad \sup_{t \geq 0} \{\|f^t(x_n)\|\} > \epsilon \tag{4.2.2}$$

as in figure 4.2. Define z_n as the first intersection of the trajectory $\{f^t(x_n)\}$ with the sphere $S(0, \epsilon)$. Then we obtain a sequence $\{z_n\}_{n \in \mathcal{N}}$ of points with norm equal to ϵ, whose trajectories *to the past* come close to the origin. Since $S(0, \epsilon)$ is a compact set, we can find a sub-sequence, which we still call $\{z_n\}$, converging to a point $z \in S(0, \epsilon)$. We affirm that

$$\alpha(z) = \{0\}$$

If it is not the case, there is a positive real a such that the trajectory never goes inside the ball $B(0, a)$. Then, since $x = 0$ is the only singular point in $B(0, \epsilon)$, the trajectory $\{f^t(z)\}$ leaves the ball $B(0, \epsilon)$. The situation is shown in figure 4.3.

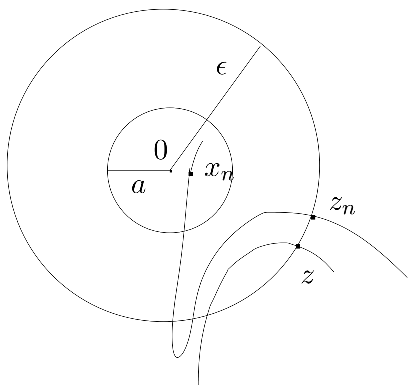

Figure 4.3: *Case:* $\alpha(z) \neq \{0\}$.

Then every trajectory starting close enough to z will accompany $\{f^t(z)\}$ to the past out of the ball and there exists a non-zero natural N_1 such that for every $n > N_1$, the trajectory $\{f^t(z_n)\}$ leaves the ball for some negative t. On the other hand, there is a non-zero natural N_2 such that for every $n > N_2$

$$\inf_{t \leq 0} \{\|f^t(z_n)\|\} < \frac{1}{n}$$

Then, for every $n > \max\{N_1, N_2\}$, the negative trajectory through z leaves $B(0, \epsilon)$ before it gets close to the origin, but this can not occur since z_n was

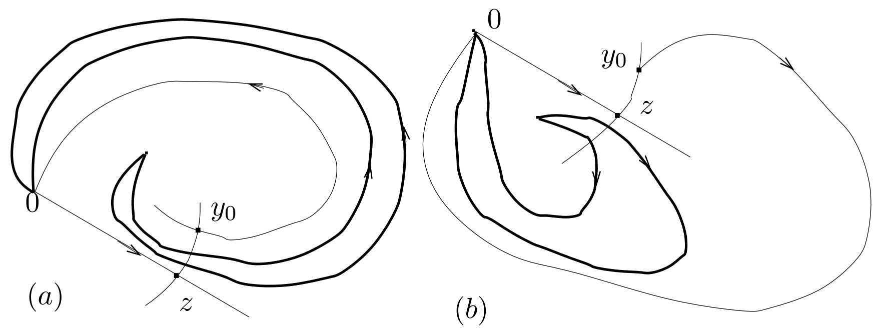

Figure 4.4: *Non zero measure invariant sets.*

defined in a way such that the piece of trajectory from x_n to z_n is totally in the inside of the closed ball $\overline{B(0,\epsilon)}$. Then $\alpha(z) = \{0\}$.
Now consider a transversal section to the trajectory through z. On this section, we can find a point y_0, arbitrarily close to z, whose ω-limit is the origin, since this kind of trajectories form a dense set due to the a.g.s assumption. Then, as in the Poincaré-Bendixson Theorem, we can construct a closed path with the negative trajectory through z, a piece of the transversal section and the positive trajectory through y_0. This path limits a closed region of the plane, with a finite number of fixed points inside it. The first situation we can have is the one shown in figure 4.4-(a). On the transversal section, we can find two points whose ω-limit sets are the origin and their α-limit sets are some singular point (could be other than the origin). The trajectories through these points are like the bold ones in figure 4.4-(a). The other case is shown in 4.4-(b) and the result is the same of the case (a). In both situations, the sets limited by the bold trajectories are invariant and have non zero Lebesgue measure. This is an absurd and then the origin is a locally stable fixed point.

■

Observations:

- First note that if the measure μ is monotone, then the condition (4.2.1) is fulfilled.

- The almost global stability assumption can be relaxed. It is enough to ask that the set of trajectories attracted by the origin is dense in the plane; that is, given a point $x \in \mathcal{R}^2$ and $\delta > 0$, there is a point $y \in \mathcal{R}^2$ such that

$$\|x - y\| < \delta \quad \text{and} \quad \lim_{t\to+\infty} f^t(y) = 0$$

Example 4.2. *Consider the system of two coupled oscillators, a very particular case of the general Kuramoto model (Kuramoto, 1984).*

$$\begin{aligned}\dot{\theta}_1 &= K.\sin(\theta_2 - \theta_1)\\ \dot{\theta}_2 &= K.\sin(\theta_1 - \theta_2)\end{aligned}$$

Let us call $\Phi_1 = \theta_2 - \theta_1$ *and* $\Phi_2 = \theta_1 - \theta_2 = -\Phi_1$. *We can re-write the system as*

$$\dot{\Phi} = f(\Phi) \tag{4.2.3}$$

with

$$\begin{aligned}f_1(\Phi) &= -2.K.\sin(\Phi_1)\\ f_2(\Phi) &= -f_1(\Phi)\end{aligned}$$

As in Example 3.4, *we will look for a density function to establish the almost global stability of the consensus. Consider the candidate function*

$$\rho(\Phi) = \frac{1}{1 - \cos(\Phi_1)}$$

which is C^1 *in* $\mathcal{T}^2 \setminus \{\Phi = 0\}$. *A direct calculation gives*

$$\nabla.[\rho.f](\Phi) = \frac{2}{1 - \cos(\Phi_1)} \tag{4.2.4}$$

which is positive almost everywhere in the torus. Then ρ *is a density function for the system and then the set* $\{\Phi = 0\}$, *i.e. the consensus set, is almost global stable, in the sense that almost every trajectory converge to it.*

Since we have already established the almost global attraction, we can apply Theorem 4.2 *in order to test the local stability of the consensus. Since the function*

$$l(\Phi) = \frac{1}{1 + \cos(\Phi_1)}$$

satisfies

$$\nabla.[l.f](\Phi) = -\frac{2}{1 + \cos(\Phi_1)}$$

it induces a decreasing Borel measure and we are in the hypothesis of Theorem 4.2. *Then the consensus is almost globally and locally asymptotically stable.*

♣

We apply the previous result in order to characterize the behavior at infinity of an almost globally stable system.

Theorem 4.3. *Consider the complete nonlinear system* $\dot{x} = f(x)$ *with* $f \in C^1(\mathcal{R}^2, \mathcal{R}^2)$. *Assume that the set* $f^{-1}(\{0\})$ *is finite in* $\mathcal{R}^2$ *and that there is a monotone measure* μ *bounded at infinity. If the set*

$$A = \{x \in \mathcal{R}^2 \mid \lim_{t \to +\infty} \|f^{-t}(x)\| = +\infty\}$$

is dense in $\mathcal{R}^2$ *then the* ∞ *is a locally asymptotically stable point to the past.*

Proof: We have to show that given an arbitrary positive number M, there is a positive number K, depending on M, such that

$$\text{if } \|x\| > K \ \Rightarrow \ \|f^{-t}(x)\| > M \ \ \forall t \geq 0$$

and that K can be chosen just that $\|f^{-t}(x)\| \to +\infty$.

Instead of that, we will compactify the plane using the stereographic projection in order to work on the compact Riemann sphere. Doing this, we obtain a dynamical system on the sphere with an a.g.s. equilibrium point at the south pole S (corresponding to the origin of the plane) and an equilibrium point at the north pole N (corresponding to the infinity of the plane). We know that N attracts a dense set of trajectories to the past and that we can define a Borel measure μ over the sphere in a way that given any non zero Lebesgue measure neighborhood Y of N with $S \notin \bar{Y}$, it verifies $0 < \mu(Y) < \infty$ and for every $t > 0$,

$$\mu\left[f^t(Y)\right] > \mu(Y)$$

Then we consider the reversed system over the sphere

$$\dot{x} = -f(x)$$

and we obtain that N attracts a dense set of trajectories. We can reconstruct the proof of Theorem 4.2, denying the Thesis and getting the existence of the bold trajectories of figure 4.4. If the set A enclosed by this curves has finite measure μ we get an absurd, just as in the previous proof. So, the question we must answer is if $S \in \bar{A}$. But if it was the case, S would be the α-limit of the bold trajectories and then S could not attract almost all the trajectories of the original system. In order to see this, consider again the closed path constructed with the negative trajectory of z, the positive trajectory of y_0 and a piece of the transversal section through z. We draw again the picture in figure 4.5.
Then all the trajectories started outside this closed path must enter it to reach S and these can be done only through the piece of transversal section, which can be made arbitrarily small because of the dense assumption on the set of trajectories attracted by the north pole N to the past.

■

The counter-reciprocal version of the previous Theorem is very interesting.

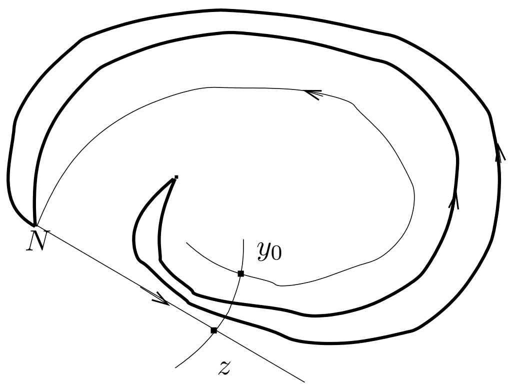

Figure 4.5: *Situation on Theorem* 4.3.

Corollary 4.4. *Consider the nonlinear system* $\dot{x} = f(x)$ *with* $f \in C^1\left(\mathcal{R}^2, \mathcal{R}^2\right)$. *Assume that the set* $f^{-1}(\{0\})$ *is finite in* $\mathcal{R}^2$ *and that there is a monotone Borel measure* μ *bounded at infinity. If there is at least one trajectory that goes to infinity to the future, then the set of trajectories that go to infinity to the past is not dense in* $\mathcal{R}^2$.

♣

We present again Example 3.1.

Example 4.3. *(Rantzer, 2001a) Consider the planar system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -6x_2 + 2x_1x_2 \end{bmatrix}$$

It has four equilibria at $(0,0)$, $(2,0)$ *and* $(3, \pm\sqrt{3})$. *We note that the axis* $\{x_2 = 0\}$ *is an invariant set.*

Then, if we consider the initial condition $(x_{10}, 0)$ *with* $x_{10} > 2$, *we find out that the trajectory goes to infinity. Besides that, it was shown in (Rantzer, 2001a) that the system admits a density function*

$$\rho(x_1, x_2) = \left[x_1^2 + x_2^2\right]^{-2}$$

$$\nabla.\left[\rho f\right](x_1, x_2) = 16.x_2^2.\left[x_1^2 + x_2^2\right]^{-4}$$

Observe that the local stability of the origin and the existence of the monotone Borel measure prevent the existence of limit cycles.

Then, we can conclude that there exist trajectories that do not go to infinity to the past and then they must go to another equilibrium point.

♣

Of course, the previous result of Example 4.3 could have been obtained through other ways. For example we can classify the equilibrium points and realize that the only divergent trajectory is the one we have found. Moreover, the trajectories that are not attracted by the origin are this one and the stable manifold of $(2, 0)$, as can be seen in figure 4.6.

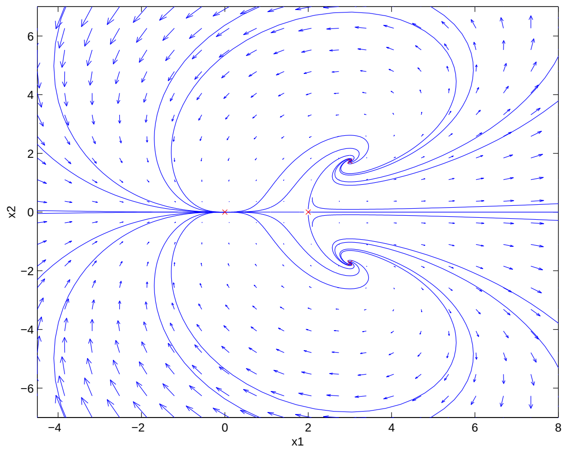

Figure 4.6: *System of Example* 4.3.

The main result of this section is deeply grounded on the topological consequences of the dimension 2. The following example shows that in dimension 3 we can have a measure satisfying (4.2.1) but the origin can be a.g.s. and not locally stable.

Example 4.4. *Consider the following*[2] *dynamical system defined on* $\mathcal{R}^3$*:*

$$\begin{cases} \dot{x} &= & x^2 - y^2 \\ \dot{y} &= & 2xy \\ \dot{z} &= & -z \end{cases}$$

On the z *direction we have the decoupled dynamic*

$$z(t) = e^{-t}.z_0$$

and at the plane $z = 0$ *the dynamic has the phase portrait depicted in figure* 4.7*. As can be proved analytically, the trajectories on* $z = 0$ *are circumferences with the center on the line* $x = 0$*. So the origin* $(0, 0, 0)$ *is an almost globally stable equilibrium point but not locally stable.*

We will see that the Lebesgue measure λ *verifies* (4.2.1)*. Consider a bounded nonzero measure set* $C \subset \mathcal{R}^3$*. It can be covered by a bounded rectangle* $A \times B$*, with* $A \subset \mathcal{R}^2$

[2] *We particularly thanks Eleonora Catsígeras for her help and suggestions about this Example.*

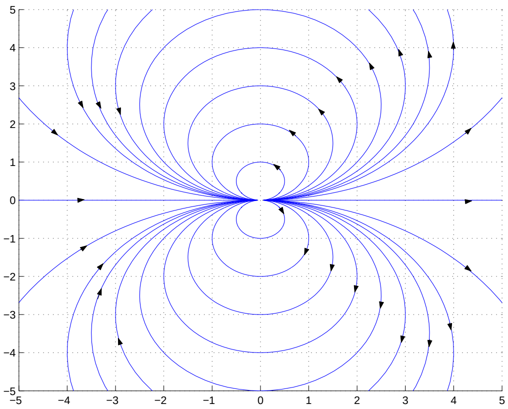

Figure 4.7: System of Example 4.4 on $z = 0$.

and $B \subset \mathcal{R}$ bounded Borel sets. It is clear that if we denote by F the field on $\mathcal{R}^3$, we have

$$\lambda \left[F^t(A \times B)\right] = m \left[f^t(A)\right] .e^{-t}|B|$$

where $|B|$ stands for the length of the interval B and m denotes the Lebesgue measure on $\mathcal{R}^2$. The numbers $m\left[f^t(A)\right]$ are bounded since almost all the trajectories converge to the $(0,0)$ in the plane $z = 0$. Then there is a time t big enough such that

$$\lambda \left[F^t(A \times B)\right] < \lambda[A \times B]$$

and λ verifies (4.2.1).

♣

Chapter 5

Converse results

In this Chapter we present some converse results for Theorems 3.1 and 4.1. We start with some simple particular cases and we move on to the more general situation. First of all we present a converse result for linear Hurwitz systems and general g.a.s. systems. Then, we show that almost global stability of the origin plus local stability imply the existence of a monotone measure. After that, we analyze how the previous result can be extended to ensure the existence of a density function.

5.1 Linear Hurwitz systems

Consider the following Proposition:

Proposition 5.1. *(Rantzer, 2001a) Let $V(x) > 0$ for $x \neq 0$ and*

$$\nabla V \, . \, f < \alpha^{-1}.(\nabla .f).V$$

for almost all x and for some $\alpha > 0$. Then $\rho(x) = V^{-\alpha}(x)$ satisfies

$$\nabla .\left[\rho f\right](x) > 0 \quad , \quad a.e.$$

In particular, if P is a positive definite matrix satisfying

$$A^T P + PA < \left[\alpha^{-1} \, . \, trace(A)\right] .P$$

then $\rho(x) = \left[x^T P x\right]^{-\alpha}$ satisfies

$$\nabla .\left[\rho f\right](x) > 0 \quad , \quad a.e.$$

for $f(x) = Ax$.

Proof: Consider the function $\rho(x) = V^{-\alpha}(x)$. Then

$$\begin{aligned} \nabla\left[\rho f\right](x) &= \dot{\rho}(x) + \rho(x).\left[\nabla .f\right](x) \\ &= -\alpha .V^{-(\alpha+1)}(x).\nabla V(x).f(x) + V^{-\alpha}(x).\left[\nabla .f\right](x) \end{aligned}$$

Then

$$\nabla\left[\rho f\right](x) = \alpha.V^{-(\alpha+1)}(x).\left[\alpha^{-1}.\left(\nabla.f\right).V - \nabla V.f\right](x) > 0 \ , \quad a.e. \tag{5.1.1}$$

For the linear case, the proof is quite similar.

■

As an immediate consequence, we obtain a converse result for linear systems. Consider the canonical asymptotically stable linear system[1],

$$\dot{y} = g(y) = -y \tag{5.1.2}$$

It admits a quadratic Lyapunov function $V(y) = y^T P y$, $P = P^T > 0$ such that its derivative

$$\dot{V}(y) = -2y^T P y$$

is negative definite. Consider the candidate density function

$$\rho(y) = [V(y)]^{-\alpha}$$

with $\alpha > 0$ big enough to ensure the integrability of ρ. Let us analyze the identity

$$\nabla.(\rho g)(y) = \nabla\rho(y).g(y) + \rho(y).\nabla.g(y)$$

Since

$$\nabla\rho(y) = -\alpha V^{-(\alpha+1)}(y)\nabla V(y)$$

$$\nabla.g(y) = -n$$

$$\nabla V(y).g(y) = -2y^T P y$$

we obtain

$$\nabla.(\rho g)(y) = -\alpha V^{-(\alpha+1)}(y).\left(-2y^T P y\right) + V^{-\alpha}(y).\left(-n\right)$$

So

$$\nabla.(\rho g)(y) = -V^{-(\alpha+1)}(y).\left[-2\alpha y^T P y + n y^T P y\right] = (2\alpha - n)V^{-(\alpha+1)}(y).y^T P y$$

which is positive definite if α is chosen big enough. The same procedure can be extended to any Hurwitz system and any Lyapunov function, since the basic fact is that the divergence of the field is strictly negative.

[1]It represents a general asymptotically stable linear system $\dot{y} = Ay$ with A Hurwitz, i.e. all the eigenvalues of A lying in the open left half complex plane.

For a general nonlinear asymptotically stable system it is not always possible to obtain a density function just inverting a given Lyapunov function as is shown in Example 5.1.

Example 5.1. *Consider the second order system*

$$\dot{y} = f(y) \Leftrightarrow \begin{cases} \dot{y}_1 &= -y_1(1+y_2^2) \\ \dot{y}_2 &= -y_2(1-y_1^2) \end{cases}$$

which admits the Lyapunov function $\mathcal{V}(y) = \frac{1}{2}y^T y$. *If we put* $\rho(x) = [\mathcal{V}(x)]^{-\alpha}$, *we have that*

$$\nabla.\left(\rho f\right)(y) = [\mathcal{V}(y)]^{-\alpha-1}\left(y_1^2 + y_2^2\right)\left[\alpha - 1 + \frac{y_1^2 - y_2^2}{2}\right]$$

which has undefined sign for every positive α. *Then no* α *can be found to get a density function of the form* $\rho(y) = [V(y)]^{-\alpha}$, *at least with the quadratic Lyapunov function we have chosen.*

♣

In general, the construction of a density function associated to a Lyapunov function is not straightforward as in the linear case and is presented in Chapter 5.5.1.

5.2 Two useful Lemmas

Now, we present two Lemmas that are going to be very important in order to prove converse results for almost global stability. We must mention that a result similar to Lemma 5.2 can be found in (Grune, 1999) in a very different context.

We will use the auxiliar canonical asymptotically stable linear system

$$\dot{y} = g(y) = -y \tag{5.2.1}$$

Lemma 5.2. *Consider the system*

$$\dot{x} = f(x) \tag{5.2.2}$$

with $f \in C^1\left(\mathcal{R}^n, \mathcal{R}^n\right)$ *and* $x = 0$ *being an asymptotically stable equilibrium point, such that*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

*is a Hurwitz matrix. Consider also the linear system (*5.2.1*). Denote by* R *the open subset of* $\mathcal{R}^n$ *which is the region of attraction of the origin. Then, there* *exists a continuous function* $h_1 : R \to \mathcal{R}^n$*, satisfying that for every* $x \in R$ *and every* τ *such that* $f^\tau(x)$ *is defined, the following is true*

$$h_1 \circ f^\tau(x) = g^\tau \circ h_1(x) \qquad (5.2.3)$$

Moreover, if the system is complete, h_1 *is an homeomorphism.*

Proof: Again $f^t(x_0)$ will denote the trajectory at time t for system (5.2.2), starting at x_0. In the same way, $g^t(y_0)$ will refer to the trajectories of the linear system (5.2.1).

Since A is a Hurwitz matrix, the nonlinear system admits a quadratic local Lyapunov function of the form $\mathcal{V}(x) = x^T P x$, with $P = P^T > 0$ and $A^T P + PA < 0$. Consider the ellipsoid

$$\mathcal{E} = \left\{ x \in R \mid x^T P x = \delta \right\}$$

with δ small enough such that $\mathcal{E}$ is included in the domain of definition of $\mathcal{V}$. It must be clear that all the trajectories of the region of attraction of the origin intersect the ellipsoid just once, since they converge to the origin and the ellipsoid is a level curve of the function $\mathcal{V}$, which decreases along the trajectories. Let $H : \mathcal{R}^n \to \mathcal{R}^n$ be a C^∞ diffeormophism carrying the ellipsoid $\mathcal{E}$ to the sphere

$$\mathcal{S} = \{ y \in \mathcal{R}^n \mid \|y\| = 1 \}$$

Function H can be taken in a way such that the orientation of those manifolds is preserved, i.e.,

$$det \left[\frac{\partial H}{\partial x}(x) \right] > 0$$

For every point $x \in R$, define $t(x)$ as the time corresponding to the intersection of the trajectory through x with the ellipsoid, that is $f^{t(x)}(x) \in \mathcal{E}$. We define $h_1 : R \to \mathcal{R}^n$ as follows

$$h_1(x) = g^{-t(x)} \left[H \left(f^{t(x)}(x) \right) \right]$$

Figure 5.1 shows the construction process for h_1. For $x = 0$, we put $h(0) = 0$. Points in the interior of the ellipsoid must flow *to the past* to reach it, having a corresponding negative time $t(x)$. Consider a given point x and a given time τ such that $f^\tau(x)$ exists. Then we have that

$$t\left[f^\tau(x) \right] = t(x) - \tau$$

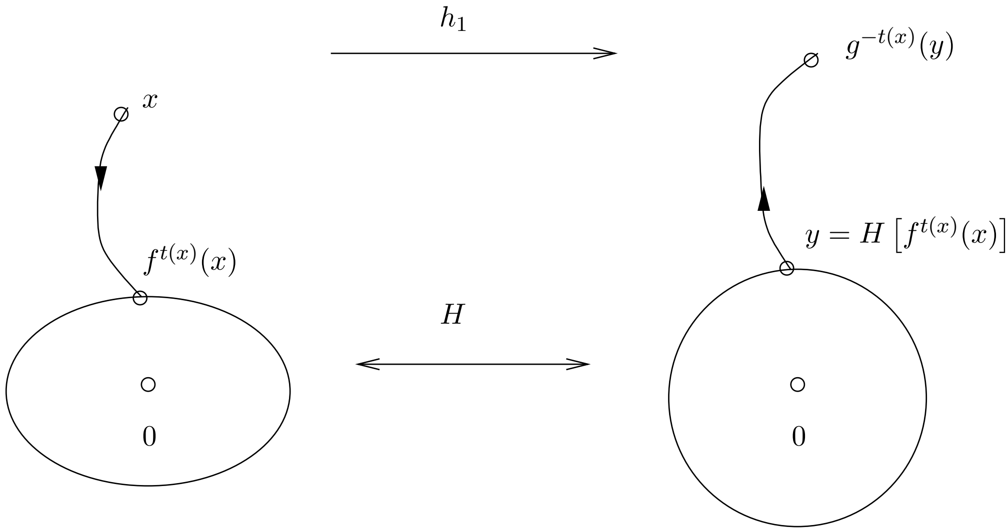

Figure 5.1: *Definition of function* h_1.

So

$$h_1\left[f^{\tau}(x)\right] = g^{-t[f^{\tau}(x)]}\left[H\left[f^{t[f^{\tau}(x)]}\left(f^{\tau}(x)\right)\right]\right]$$

$$= g^{-t(x)+\tau}\left[H\left(f^{t(x)}(x)\right)\right] = g^{\tau}\left[h_1(x)\right]$$

We can express this as follows

$$h_1 \circ f^{\tau}(x) = g^{\tau} \circ h_1(x)$$

for all $x \in R$, for all $\tau \in \mathcal{R}$ such that $f^{\tau}(x)$ exists.

By construction, h_1 is an open function, i.e. the image of an open set is also open. Note that the hole process is reversible, so the inverse of h_1 exists and is continuous. If the trajectories are defined for all real t, then the function h_1 is a homeomorphism between the region of attraction R and $\mathcal{R}^n$ and it is a continuous conjugacy between the nonlinear and the linear system.

■

Lemma 5.3. *Consider the system (*5.2.2*) with* $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ *and* $x = 0$ *being an asymptotically stable equilibrium point with region of attraction* R*, such that the Jacobian matrix at the origin is a Hurwitz matrix. Consider also the linear system (*5.2.1*). Then there exists a continuous function* $h_2 : R\backslash\{0\} \to \mathcal{R}^n$*, satisfying that for every non zero* $x \in R$ *and every* τ *such that* $f^{\tau}(x)$ *is defined, the following is true*

$$h_2 \circ f^{\tau}(x) = g^{-\tau} \circ h_2(x) \tag{5.2.4}$$

Proof: Like in the previous Lemma, we obtain a Lyapunov level surface from the local exponential stability hypothesis. Then we define

$$h_2(x) = g^{t(x)}\left[H\left(f^{t(x)}(x)\right)\right] \tag{5.2.5}$$

Figure 5.2 shows the construction process for h_2. Observe that in this case, we

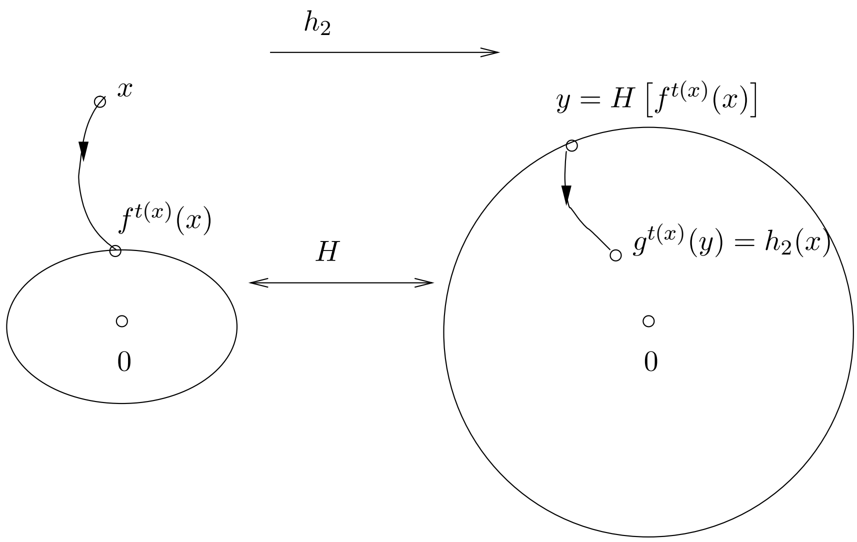

Figure 5.2: *Definition of function h_2.*

move forward in time after we change from the nonlinear system to the linear one. The following facts are true. Their proofs are like in Lemma 5.2.

- The exterior of the ellipsoid $\mathcal{E}$ is mapped in the interior of the sphere $\mathcal{S}$;
- More general, the outside of a ball centered at the origin is mapped into the inside of a ball centered at the origin;
- For every $x \in R$ and for every $\tau \in \mathcal{R}$ such that $f^\tau(x)$ is defined $h_2 \circ f^\tau(x) = g^{-\tau} \circ h_2(x)$.
- Our construction let us to continuously extend h_2 to the border of R, by setting $h_2(x) = 0$ if $x \in \partial R$

■

Remarks:

1) The previous Lemmas give us a way to map the trajectories of the region of attraction of the nonlinear system into the trajectories of the linear one, with the possible exception of the trajectory through the origin. The function h_1 is a *time-preserving* correspondence and h_2 is a *time-reversing* one.

2) The function h_1 is as differentiable as the field f and this is an important fact as we will mention later. Moreover, it satisfies the following condition

$$det \left[\frac{\partial h_1}{\partial x}(x) \right] > 0 \;\; \forall x \neq 0$$

since the flow preserves the orientation too. This is also true for h_2, for $x \neq 0$.

3) Local exponential stability of the origin is used only to obtain the ellipsoid $\mathcal{E}$, which is a surface diffeormorphic to the unit sphere $\mathcal{S}$. It can be replaced by any level surface of a local Lyapunov function, as long as it can be proved that this level surface is homeomorphic to the sphere. The existence of a Lyapunov function for an asymptotically stable system is ensured by Massera's result (Massera, 1949). The fact that a compact level surface is diffeomorphic to the unit sphere is true for surfaces of dimensions 1 and 2 and is guaranteed by the h-Cobordism Theorem of Smale (Smale, 1962; Milnor, 1965) for surfaces of dimensions equal or greater than 5, while for dimension 4 only an homeomorphism can be ensured (Freedman, 1982). So the hypothesis of the Lemmas 5.2 and 5.3 can be relaxed, asking for the origin to be a local attractor with no particular restriction on its linear approximation, and requiring that the dimension of the space be different from 4 (i.e. the Lyapunov level surface should have dimension different from 3). It seems to be a correct proof of the Poincaré Conjecture (Perelman, 2003).

5.3 Monotone measures

With the functions h_1 and h_2 introduced in the previous section, we can prove converse results for almost globally stable systems.

Proposition 5.4. *Consider the system (*5.2.2*) with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being an almost globally stable fixed point, such that the Jacobian matrix at the origin is a Hurwitz matrix. Then*

1. *there exists an increasing Borel measure μ bounded at infinity.*

2. *there exists a decreasing Borel measure finite on compacts.*

Proof: We can prove the result either with the auxiliary functions h_1 or h_2. Observe that the existence of these functions is guaranteed by hypothesis. The process is the following: we will define a monotone Borel measure for the non-linear system using a monotone Borel measure for the linear system.

Let us first use h_1. The domain of h_1 is almost all the space, due to the almost global stability property of the system. Let R be the region of attraction of the origin. It is open, invariant and connected (Khalil, 1996). Then, every Borel set $Y \subset \mathcal{R}^n$ can be split into two sets:

$$Y_R = Y \cap R \quad , \quad Y_{R^c} = Y \cap R^c$$

Observe that for every $t \geq 0$ such that $f^t(Y)$ exists,

$$\left[f^t(Y)\right]_R = f^t(Y_R) \quad , \quad \left[f^t(Y)\right]_{R^c} = f^t\left(Y_{R^c}\right)$$

Consider a scalar function $\sigma \in C^1\left(\mathcal{R}^n \setminus \{0\}, [0, +\infty)\right)$. For a given Borel set $Y \subset \mathcal{R}^n$ define

$$\mu(Y) = \int_{h_1(Y_R)} \sigma(y)dy \tag{5.3.1}$$

It is clear that μ is a Borel measure, since R^c has zero Lebesgue measure and h_1 is one to one.

Consider a given time t such that $f^t(x)$ exists for every $x \in Y$. Then

$$\mu\left[f^t(Y)\right] - \mu(Y) = \int_{h_1[f^t(Y_R)]} \sigma(y)dy - \int_{h_1(Y_R)} \sigma(y)dy$$

Using (5.2.3) we obtain

$$\mu\left[f^t(Y)\right] - \mu(Y) = \int_{g^t[h_1(Y_R)]} \sigma(y)dy - \int_{h_1(Y_R)} \sigma(y)dy$$

If we can apply Lemma 3.3,

$$\mu\left[f^t(Y)\right] - \mu(Y) = \int_0^t \int_{g^\tau[h_1(Y_R)]} \nabla.\left[\sigma.g\right](y)dyd\tau$$

If σ is a density function for the linear field, and if we can apply Lemma 3.3, we see that μ is an increasing measure. Moreover, consider an arbitrary $\epsilon > 0$ and assume that $Y \subset B^c(0, \epsilon)$. Then

$$\mu(Y) = \int_{h_1(Y_R)} \sigma(y)dy < +\infty$$

since h_1 is an open function and σ is integrable outside arbitrary neighborhoods of the origin. So μ is a monotone increasing Borel measure bounded at infinity.

If we choose σ such that $\nabla.\left[\sigma.g\right](y) < 0 \quad a.e.$, we conclude that μ is a decreasing measure, and if σ is integrable over compacts sets, μ turns out to be bounded over compact sets. It is enough to take σ as a Lyapunov function since

$$\nabla.\left[\sigma.g\right](y) = \dot{\sigma}(y) + \nabla.g(y).\sigma(y) = \dot{\sigma}(y) - n.\sigma(y)$$

We can repeat the previous arguments using function h_2. We only show the construction of an increasing measure. As before

$$\mu(Y) = \int_{h_2(Y_R)} \sigma(y)dy$$

If we can apply Lemma 3.3

$$\mu\left[f^t(Y)\right] - \mu(Y) = \int_{g^{-t}[h_2(Y_R)]} \sigma(y)dy - \int_{h_2(Y_R)} \sigma(y)dy =$$

$$= -\int_0^t \int_{g^{-\tau}[h_2(Y_R)]} \nabla.\left[\sigma.g\right](y)dyd\tau$$

due to (5.2.4). So if $\nabla.\left[\sigma.g\right] < 0$ almost everywhere and σ is integrable over compact sets, then μ is a monotone Borel measure bounded over compacts. If σ is a Lyapunov function for the linear system, μ is an increasing measure bounded at infinite (this last assertion comes from the reversing time property of h_2).

■

Due to the remarks mentioned after Lemmas 5.2 and 5.3 we can write this more general result:

Theorem 5.5. *Let the system $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, $n \neq 4$, and $x = 0$ an almost globally stable equilibrium point with local stability. Then*

1. *there exists a monotone Borel measure μ bounded at infinity.*

2. *there exists a monotone Borel measure μ finite on compact sets.*

♣

5.4 Density functions

In the previous Section we have proved that almost global stability of the origin, plus local stability, ensures the existence of a monotone measure. This is not yet a full converse theorem for Rantzer's original result (3.1). We will first show how a direct use of Lemma 5.2 leads us to a density function when we have global stability and then we will focus on the more general case.

5.4.1 G.a.s. systems

When the field f is twice differentiable, we get that h_1 of Lemma 5.2 is twice differentiable too. We will see that when the origin is globally asymptotically stable, the monotone Borel measure μ bounded over compacts we have constructed comes from a density function.

Proposition 5.6. *Consider the system (*5.2.2*) with $f \in C^2(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being a globally asymptotically stable fixed point, such that the Jacobian matrix at the origin is Hurwitz. Then, there exists a density function $\bar{\rho} \in C^1(\mathcal{R}^n - \{0\}, [0, +\infty))$ for system (*5.2.2*).*

Proof: We know there exists a density function for the linear system 5.2.1. Using that function, we construct a density function for the nonlinear system. Let us call ρ a density function for linear system (5.2.1), positive definite and such that

$$\nabla . [\rho g](y) > 0 \quad , \quad \text{a.e.}$$

In this case, the function from Lemma 5.2 is defined on the whole space $\mathcal{R}^n$, since we assume global attraction. In a natural way we define

$$\bar{\rho}(x) = \rho[h_1(x)] . \left| \frac{\partial h_1}{\partial x}(x) \right| \tag{5.4.1}$$

which is non-negative like ρ and differentiable in $\mathcal{R}^n \setminus \{0\}$ since h_1 is of class C^2.

We will show that $\nabla . [\bar{\rho} f](x) > 0$ almost everywhere. First of all, we note that for any measurable set $\mathcal{Z}$, whose closure does not contain the origin, it is true that

$$\int_{\mathcal{Z}} \bar{\rho}(x)dx = \int_{\mathcal{Z}} \rho[h_1(x)] . \left| \frac{\partial h_1}{\partial x}(x) \right| dx = \int_{h_1(\mathcal{Z})} \rho(y)dy$$

Since through h_1 the outside of a ball centered at the origin is mapped in the outside of another ball centered at the origin, $\bar{\rho}$ is integrable outside any ball centered at the origin, since ρ is.

Equation (5.2.3) implies that for every $x \in \mathcal{R}^n$,

$$\left|\frac{\partial h_1}{\partial x}\left[f^t(x)\right]\right| . \left|\frac{\partial h_1}{\partial x}(x)\right| = \left|\frac{\partial g^t}{\partial x}\left[h_1(x)\right]\right| . \left|\frac{\partial h_1}{\partial x}(x)\right|$$

From Lemma 3.3,

$$\nabla.(\bar{\rho}f)(x) = \frac{\partial}{\partial t}\left\{\bar{\rho}\left[f^t(x)\right] . \left|\frac{\partial f^t}{\partial x}(x)\right|\right\}\Bigg|_{t=0}$$

and we have

$$\nabla.(\bar{\rho}f)(x) = \frac{\partial}{\partial t}\left\{\rho\left[h_1\left(f^t(x)\right)\right] . \left|\frac{\partial h_1}{\partial x}\left[f^t(x)\right]\right| . \left|\frac{\partial f^t}{\partial x}(x)\right|\right\}\Bigg|_{t=0}$$

$$\nabla.(\bar{\rho}f)(x) = \frac{\partial}{\partial t}\left\{\rho\left[g^t\left(h_1(x)\right)\right] . \left|\frac{\partial g^t}{\partial x}\left[h_1(x)\right]\right| . \left|\frac{\partial h_1}{\partial x}(x)\right|\right\}\Bigg|_{t=0} =$$

$$\nabla.(\bar{\rho}f)(x) = \left|\frac{\partial h}{\partial x}(x)\right| .\nabla.(\rho g)\left[h_1(x)\right] > 0 \ \ a.e.$$

■

Again, we can relax the local requirements of the previous Theorem, due to the remarks we have made after the introduction of the function h_1. If we only assume local stability of the origin, h_1 turns out to be a diffeomorphism when the dimension of the space is different from 4 or 5.

Theorem 5.7. *Let the system*

$$\dot{x} = f(x) \tag{5.4.2}$$

with $f \in C^2(\mathcal{R}^n, \mathcal{R}^n)$, $n \neq 4$ *and* 5 *and* $x = 0$ *a globally asymptotically stable fixed point. Then there exists a density function* $\bar{\rho} \in C^1(\mathcal{R}^n - \{0\}, [0, +\infty))$ *for the system (*5.4.2*).*

♣

5.4.2 A.g.s. systems

The previous construction does not work when we are dealing with almost global stable systems with local stability. Why? Because function h_1 is not defined for all the points of $\mathcal{R}^n$ but only to points on the region of attraction R. In order to solve that problem, we will analyze the way we can define the density function for the nonlinear system at the points that are not attracted. We will proceed as in the g.a.s. case, but we will exploit the degree of freedom we have in the choice of the density function for the linear system. The following Theorem shows that a.g.s. implies the existence of a *density-like* function, in the sense that we could not prove continuity at every point of the first derivative. The proof was developed together with Rafael Potrie.

Theorem 5.8. *Consider the system (5.2.2) with $f \in C^2(\mathcal{R}^n, \mathcal{R}^n)$ and $x = 0$ being an almost globally stable fixed point, such that the Jacobian matrix at the origin is Hurwitz. Then, there exists a density $\bar{\rho}$ differentiable and with continuous derivative up to a set of zero Lebesgue measure. Also, this density can be constructed such that it is zero at the complement of the basin of attraction.*

Proof: As in Proposition 5.6, define a candidate for density function as

$$\overline{\rho}(x) = \rho(h_1(x)) \left| \frac{\partial h_1}{\partial x}(x) \right|$$

where ρ is a density for the field $\dot{y} = -y$. We want to see that defined this way, $\bar{\rho}$ can be extended to the complement of the basin of attraction, the set R^c. We know that given $\{x_n\}_{n\in\mathcal{N}} \subset R$ such that que $x_n \to z \in R^c$, we have that the corresponding times $t(x_n) \to +\infty$, so $h_1(x_n) \to \infty$: This means that if we make $\rho(y) \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right| \to 0$ as $y \to \infty$ we will achieve continuity of $\overline{\rho}$ in R^c. Also we can see that:

$$\frac{\partial \overline{\rho}}{\partial x}(x) = \frac{\partial \rho}{\partial y}(h_1(x)) \frac{\partial h_1}{\partial x}(x) \left| \frac{\partial h_1}{\partial x}(x) \right| + \rho(h_1(x)) \nabla \left| \frac{\partial h_1}{\partial x}(x) \right|$$

And this equation holds for all $x \in R^c$. So, if we bound ρ and $\frac{\partial \rho}{\partial y}$ adequately we can make $\overline{\rho}$ of class C^1. This may not be possible, since we only have control on ρ and we have to handle two inequalities, one to make the derivative continuous and another one to ensure that ρ is a density for the system $\dot{y} = -y$. Let

$$j(r) = r^2 \sup_{\|y\| \le r} \left\{ \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right|, \left\| \nabla \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right| \right\|, \frac{1}{d(h_1^{-1}(y), \mathcal{R}^c)} \right\}$$

Function j is well defined, because $D_r = \{y : |y| \le r\}$ is compact, and the functions defined continuous (the one that could be doubted is the distance but $\mathcal{R}^c$ is closed and $h_1^{-1}(D_r) \cap R^c = \emptyset$ then $\inf\{d(h_1^{-1}(y), R^c) > 0$). Now, we define $\beta : \mathcal{R}^n \to \mathcal{R}$, C^∞, increasing, with $\beta(0) = 0$ and such that outside some neighborhood of 0 satisfies $\beta(y) > j(\|y\|)$ (and is constant in $\|y\| =$ constant). Then, β is a Lyapunov function for $\dot{y} = -y$. Also, we can consider β to be convex in each direction. We prove that if $\alpha > n$, the function $\rho = \beta^{-\alpha}$ will be a density for $\dot{y} = -y$. It's enough to see (because of Proposition 5.1) that

$$-\alpha.\nabla\beta(y)y < \nabla \cdot (-y)\beta(y) \Rightarrow \alpha.\nabla\beta(y)y > n\beta(y)$$

But since the function is radial (its gradient is parallel to the function $g(y) = -y$) we can verify the property in one direction. We see that

$$n.\beta(x_1, 0, \ldots, 0) < \alpha.x_1 \frac{\partial \beta}{\partial x_1}(x_1, 0, \ldots, 0)$$

And it is easy to see that for a real valued convex function w such that $w(0) = 0$ we have that $w(x) \leq xw'(x)$. So, if $\alpha > n$ we can have what we wanted. Now, given $\{x_n\}_{n \in \mathcal{N}} \subset R$ such that $x_n \to z \in R^c$ we have

$$\overline{\rho}(x_n) = \rho(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| = \beta^{-\alpha}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

So, $\overline{\rho}(x) = 0 \ \forall x \in R^c$. To see that this function is differentiable we consider $z \in R^c$ and any sequence $\{x_n\}_{n \in \mathcal{N}} \subset R$ (there is no problem considering the sequence in R since it is zero in its complement) such that $x_n \to z$ and we have

$$\frac{\|\overline{\rho}(x_n) - \overline{\rho}(z)\|}{\|x_n - z\|} = \frac{\|\overline{\rho}(x_n)\|}{\|x_n - z\|} = \frac{\beta^{-\alpha}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right|}{\|x_n - z\|} \to 0$$

Taking $\alpha > 2$

$$\beta^{-1}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

$$\frac{\beta^{-1}(h_1(x_n))}{\|x_n - z\|} < \frac{1}{\|h_1(x_n)\|^2} \underbrace{\frac{d(h_1^{-1}(h_1(x_n)), \mathcal{R}^c)}{\|x_n - z\|}}_{<1} \to 0$$

So, we have that $\overline{\rho}$ is differentiable in all $\mathcal{R}^n$, the continuity of its derivative in R is immediate, but to have continuity in all $\mathcal{R}^n$ we need that for some α

$$\frac{\partial \rho}{\partial y}(y) = (-\alpha \beta^{-\alpha-1}(y)) \nabla \beta(y)$$

goes to zero as fast as ρ when $y \to \infty$, since in this case

$$\frac{\partial \overline{\rho}}{\partial x}(x_n) = \frac{\partial \rho}{\partial y}(h_1(x_n)) \frac{\partial h_1}{\partial x}(x_n) \left| \frac{\partial h_1}{\partial x}(x) \right| + \rho(h_1(x_n)) \nabla \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

Unfortunately, this is not true in general since we can construct a convex function such that its derivative is larger than any power of the function in a sequence going to infinity. However, if β can be constructed satisfying that property then the density will be C^1.

■

Once again, we remark that we can relax the local hypothesis of the stability of the origin. The full extension of the previous result is a future line or research.

Example 3.1 shows how for a system satisfying the hypothesis of Theorem 5.8 it can be found a density function which is strictly positive, i.e. that does not vanish at the complement of the region of attraction.

5.5 Applications

Concluding this Chapter, we present some results that show how the previous converse Theorems can be used.

5.5.1 Density functions and Lyapunov functions

The Lyapunov stability is a property of the system stronger than the almost global stability. We have seen that some times we just can invert a Lyapunov function in order to obtain a density function, but this is not a general procedure.

Proposition 5.1 deduces a sufficient inequality condition for a density function. In order to obtain a density function we must check if the number α is enough to establish the integrability condition. Observe that the previous result can not be directly applied to a given Lyapunov function, since in this case we only can affirm that $\nabla V.f$ is non positive, but a priori we have no control on the sign and value of the term $(\nabla .f).V$. But when the divergence of the field f has definite sign, the inequality is enough to construct a density function from a Lyapunov function. A particular case is when the system is linear and the associated matrix A is Hurwitz, as we have seen at the beginning of this Chapter.

Suppose that V is a global Lyapunov function for a globally asymptotically stable system. Inequality (5.1.1) can be re-written as

$$-(\nabla V.f).V^{-(\alpha+1)}.\left[\frac{(\nabla .f).V}{-(\nabla V.f)}-\alpha\right]>0 \quad , \quad a.e.$$

with $-(\nabla V.f)>0$ for $x\neq 0$ since V is a Lyapunov function. In order to satisfy the divergence condition for ρ we must have

$$\alpha>\frac{(\nabla .f).V}{-(\nabla V.f)} \quad , \quad a.e. \tag{5.5.1}$$

The previous inequality takes different forms when we add hypothesis on the field f. As an example, consider the following result for homogeneous systems[2].

Theorem 5.9. *(Rantzer, 2003) Let the zero equilibrium of the system* $\dot{x} = f(x)$ *be asymptotically stable, with* $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ *homogeneous of degree* l *with respect to* Δ_λ^r*. Then, there exists a homogeneous function* $\rho \in C^\infty(\mathcal{R}^n \setminus \{0\}, \mathcal{R}^n)$ *with negative degree of homogeneity which satisfies the following properties*

(i) $\rho(x) > 0$*, for all* $x \neq 0$*,*

(ii) $\nabla.(\rho.f)(x) > 0$*, for all* $x \neq 0$*,*

(iii) $\rho(x).f(x)/\|x\|_h$ *is integrable on* $\{x \,|\, \|x\|_h \geq 1\}$*, with* $\|.\|_h$ *the homogeneous norm associated to* Δ_λ^r*.*

♣

From homogeneity of the system, condition (5.5.1) on α turns out to be

$$\alpha > \max_{\{\|x\|=1\}} \frac{(\nabla.f(x)).V(x)}{-[\nabla V(x).f(x)]}$$

and Theorem 5.9 answers the question we had stated at the beginning of this Chapter, in the sense that in this case we can obtain a density function just inverting a Lyapunov function.

What can we do when we do not have an homogeneous system? In the proof of Proposition 5.6 we could have used the function h_2 obtained by Lemma 5.3 in order to obtain a density function for the nonlinear system. Recall that we have tried to measure the growth of a given set along the nonlinear system using a monotone measure for the linear system. Due to the properties of the function h_2, particularly the time-reversing aspect, the way we measure the growth of the sets in the linear case should be different to the one we used in Proposition 5.6. The following Theorem establishes a general relationship between density functions and Lyapunov functions.

Proposition 5.10. *Let the system*

$$\dot{x} = f(x) \tag{5.5.2}$$

with $f \in C^2(\mathcal{R}^n, \mathcal{R}^n)$ *with no finite escape time, with* $x = 0$ *a globally asymptotically stable fixed point with local exponential stability. Then every Lyapunov function induces a density function for the system.*

[2]Basic definitions about homogeneous systems can be found in (Sepulchre and Aeyels, 1996)

Proof: Let us study what happens if we use the function h_2 instead of h_1 in the construction of the density function for the nonlinear system. We define the function

$$\bar{\rho}(x) = \sigma\left[h_2(x)\right].\left|\frac{\partial h_2}{\partial x}(x)\right| \tag{5.5.3}$$

where $\sigma : \mathcal{R}^n \to \mathcal{R}^n$ is a continuous function to be defined later. Recall that $\bar{\rho}$ must be continuous and differentiable for every $x \neq 0$, integrable on the outside of every $B(0,\epsilon)$ with $\epsilon > 0$ and satisfying the condition $\nabla.(\bar{\rho}f)(x) > 0$ a.e.

Since h_2 maps the exterior of any ball centered at the origin into the interior of another ball centered at the origin, the function σ should be integrable over bounded sets. The condition on the divergence of $\bar{\rho}f$ results in

$$\nabla.(\bar{\rho}f)(x_0) = \frac{\partial}{\partial t}\left\{\sigma\left[g^{-t}\left(h_2(x)\right)\right].\left|\frac{\partial g^{-t}}{\partial x}\left[h_2(x_0)\right]\right|.\left|\frac{\partial h_2}{\partial x}(x_0)\right|\right\}\Bigg|_{t=0}$$

$$\nabla.(\bar{\rho}f)(x_0) = -\left|\frac{\partial h_2}{\partial x}(x_0)\right|.\nabla.(\sigma g)\left[h_2(x_0)\right]$$

So, in order to get a density function for the nonlinear system, it is enough that $\nabla.(\sigma g) < 0$ for every $x \in \mathcal{R}$. This is the case if σ is a Lyapunov function for the linear system. Then every Lyapunov function of the linear system induces a density function for the nonlinear system.

So it remains to show that every Lyapunov function for the nonlinear system induces a Lyapunov function for the linear system. But this is true since from the hypothesis, the function h_1 is a diffeomorphism between the linear system and the nonlinear one. Then, if we have a Lyapunov function $\bar{\mathcal{V}}$ for the nonlinear system, we have the function

$$\mathcal{V}(y) = \bar{\mathcal{V}}\left[h_1^{-1}(y)\right]$$

which is a Lyapunov function for the linear system, since

$$\frac{\partial}{\partial t}\left[\mathcal{V}\left[g^t(y)\right]\right]\Bigg|_{t=0} = \frac{\partial}{\partial t}\left[\bar{\mathcal{V}}\left[h_1^{-1}\left(g^t(y)\right)\right]\right]\Bigg|_{t=0} = \frac{\partial}{\partial t}\left[\bar{\mathcal{V}}\left[f^t\left(h_1^{-1}(y)\right)\right]\right]\Bigg|_{t=0}$$

So,

$$\frac{\partial}{\partial t}\left[\mathcal{V}\left[g^t(y)\right]\right]\Bigg|_{t=0} = \dot{\bar{\mathcal{V}}}\left[h_1^{-1}(y)\right] < 0$$

where we have used the property: $f^t \circ h_1^{-1}(y) = h_1^{-1} \circ g^t(y)$.

■

Of course, this result admits the relaxations on the local stability requirements with the considerations on the dimension of the system.

5.5.2 Compact global attractor

The next result shows that the presence of a global compact attractor implies the existence of a monotone measure.

Theorem 5.11. *Let $\mathcal{A}$ be a global compact attractor of the system $\dot{x} = f(x)$, with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$. Then, there exist*

- *an increasing monotone measure bounded at infinity;*
- *a decreasing monotone measure bounded over compact sets (outside $\mathcal{A}$).*

Proof: Since $\mathcal{A}$ is a global attractor, it is an invariant set for the dynamical system. As we have done in Theorem 4.3, we consider the reversed system $\dot{z} = -f(z) = \tilde{f}(z)$. For this system, the infinite is an asymptotic equilibrium point. Its region of attraction is $R = \mathcal{R}^n \setminus \mathcal{A} \cup \{\infty\}$. In this context, we can apply Lemma 5.2 and obtain a function $h_1 : R \to \mathcal{R}^n$ such that $h_1(\infty) = 0$ and

$$h_1 \circ \tilde{f}^t = g^t \circ h_1 \quad \forall t$$

where g is the identity vector field, as in Lemma 5.2. We introduce two mutually singular measures in $\mathcal{R}^n$: $\tilde{\mu}_1$ and $\tilde{\mu}_2$:

$$\tilde{\mu}_1(Z) = \begin{cases} \infty & \text{if} \quad \bar{Z} \cap \mathcal{A} \neq \emptyset \\ 0 & \text{if} \quad \bar{Z} \cap \mathcal{A} = \emptyset \end{cases}$$

and

$$\tilde{\mu}_2(Z) = \mu\left[h_1(Z \cap R)\right]$$

for a given Borelian set $Z \subset \mathcal{R}^n$, where μ is a monotone measure for the linear system. We define $\tilde{\mu}$ as:

$$\tilde{\mu}(Z) = \tilde{\mu}_1(\bar{Z} \cap \mathcal{A}) + \tilde{\mu}_2(Z \cap R)$$

For arbitraries positive time t and Borelian set Z, it follows that

$$\tilde{\mu}\left[f^t(Z)\right] = \tilde{\mu}\left[\tilde{f}^{-t}(Z)\right] = \tilde{\mu}_1\left[\tilde{f}^{-t}(Z) \cap \mathcal{A}\right] + \tilde{\mu}_2\left[\tilde{f}^{-t}(Z) \cap R\right]$$

Recall that $\mathcal{A}$ and R are invariant sets. Then,

$$\tilde{f}^{-t}(Z) \cap \mathcal{A} = \tilde{f}^{-t}(Z \cap \mathcal{A}) \quad , \quad \tilde{f}^{t}(Z) \cap R = \tilde{f}^{-t}(Z \cap R)$$

and $\tilde{\mu}_1\left[\tilde{f}^{-t}(Z \cap \mathcal{A})\right] = \tilde{\mu}_1\left(Z \cap \mathcal{A}\right)$. So,

$$\tilde{\mu}\left[f^t(Z)\right] = \tilde{\mu}_1\left[Z \cap \mathcal{A}\right] + \mu\left(h_1\left[\tilde{f}^{-t}(Z \cap R)\right]\right) =$$

$$= \tilde{\mu}_1 [Z \cap \mathcal{A}] + \mu \left(g^{-t} [h_1(Z \cap R)]\right)$$

If we choose μ to be a decreasing monotone measure bounded on compact sets for the linear system, we obtain

$$\tilde{\mu} \left[f^t(Z)\right] \geq \tilde{\mu}_1 [Z \cap \mathcal{A}] + \mu [h_1(Z \cap R)] = \tilde{\mu}(Z)$$

where the strick inequality is present when $\bar{Z} \cap A = \emptyset$. The proposed measure $\tilde{\mu}$ is an increasing monotone measure. Observe that if $Z = B^c(0, \epsilon)$, with positive ϵ such that $\mathcal{A} \cap B^c(0, \epsilon) = \emptyset$, then function h_1 maps $Z \cap R$ inside a ball centered at the origin, which implies that $\tilde{\mu}$ is bounded at infinity.

We modify the previous construction in order to prove the existence of a decreasing measure bounded on compact sets. First of all, we choose $\tilde{\mu}_1$ as an invariant measure for the dynamical system restricted to $\mathcal{A}$.

$$\tilde{\mu} \left[f^t(Z \cap \mathcal{A})\right] = \tilde{\mu}(Z \cap \mathcal{A})$$

We can make this choice because $\mathcal{A}$ is invariant and compact (Chernov and Markarian, 2003). Then, as before

$$\tilde{\mu} \left[f^t(Z)\right] = \tilde{\mu}_1 [Z \cap \mathcal{A}] + \mu \left(g^{-t} [h_1(Z \cap R)]\right)$$

If μ is an increasing measure, for every bounded set Z we have

$$\tilde{\mu} \left[f^t(Z)\right] < \tilde{\mu}(Z)$$

Moreover, μ is bounded over compact sets, because h_1 carries a compact set to the outside of a ball centered at the origin.

$\blacksquare$

Chapter 6

Sinusoidally coupled oscillators

In this Chapter, we present the analysis of a nonlinear system that appears in several contexts involving synchronization, such as biology, communications, microwaves, etc. Some local properties of this system are well known, so we explore new properties and we try to add more knowledge and to establish some global or almost global results. Our first goal was to find a density function for the system, but it happened to be a hard task, specially for high dimensions. So, we modified the approach, using more classical techniques. Nevertheless, we include a particular Section about the searching of a density function and the problems we have faced.

6.1 Preliminaries

In the 1970s, Kuramoto proposes a model to describe a population of weakly coupled oscillators, following the works of A. T. Winfree on collective synchronization of biological systems (Kuramoto, 1984; Kuramoto, 1975). Each individual oscillator is described by its phase and the coupling between two individuals is a function of the phase difference. The general Kuramoto model takes the following form (Strogatz, 2000):

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \quad , \; i = 1, \ldots, N$$

where Γ_{ij} are the *interaction* functions that model the coupling and N is the total number of oscillators. Since $\theta \in [0, 2\pi)$, the corresponding state space is the N-dimensional torus $\mathcal{T}^N$. This model has turned to be suitable for describing many different systems in biology, physics and engineering

(Kuramoto, 1984; Strogatz, 2000; Dussopt, 1999). We consider the particular case of sinusoidally coupled oscillators,

$$\dot{\theta}_i = \omega_i + \frac{K}{N}.\sum_{j\in\mathcal{N}_i} \sin(\theta_j - \theta_i) \tag{6.1.1}$$

where $\mathcal{N}_i$ refers to the set of index of agents that affect the behavior of agent i -the *neighbors* of i- and K is a the strength of the coupling. We will assume that all the agents have the same natural frequency. With a suitable shift, we can re-write the previous model as

$$\dot{\theta}_i = \frac{K}{N}.\sum_{j\in\mathcal{N}_i} \sin(\theta_j - \theta_i) \tag{6.1.2}$$

Example 6.1. *[Van der Pol] Consider the circuit shown in Figure* 6.1*. There is a nonlinear resistor whose voltage-current characteristic appears in Figure* 6.2*. This kind of circuit was deeply studied in the beginning of the XX^{th} century by Balthasar Van der Pol, while he was analyzing the existence of free and forced oscillations of a triode oscillator (Van der Pol, 1934). There is also a nice survey in (Le Corbeiller, 1935). Usually, the nonlinear resistor is assumed to have the following voltage-current relationship*

$$i_{NR}(v) = -\alpha v + \beta v^3$$

with positive constants α and β (it can be thought as a truncated Taylor development of an odd function). The driving current has a constant component that defines the operating point, and a sinusoidal driving component. If we write the Kirchoff's Laws for the circuit and take the first derivative, we arrive to the so-called Van der Pol's equation*:*

$$\frac{d^2v}{dt^2} - \frac{d}{dt}\left(\alpha v - \beta v^3\right) + \omega_0^2 v = \omega_0 V_0 \sin(\omega t) \tag{6.1.3}$$

where $\omega_0 = \frac{1}{LC}$ is the natural frequency of the circuit and V_0 is a constant related to the input.

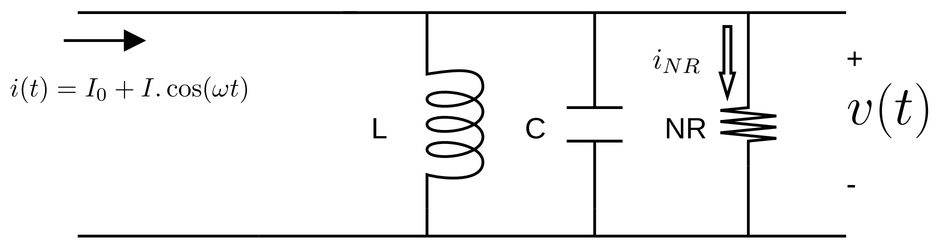

Figure 6.1: Non linear circuit of the Van der Pol equation (Example 6.1). NR is a nonlinear resistor. Its voltage-current description is shown in figure 6.2.

Having in mind the phasor analysis for linear circuits, we can search for the existence of a periodic solution of the form[1]

$$v(t) = \Re e\left[V(t)e^{-j\omega t}\right]$$

[1] *In (Van der Pol, 1934), the proposed solution had the following form: $v(t) = b_1(t).\sin(\omega t) + b_2(t).\cos(\omega t)$. We follow here the presentation of (York, 1993).*

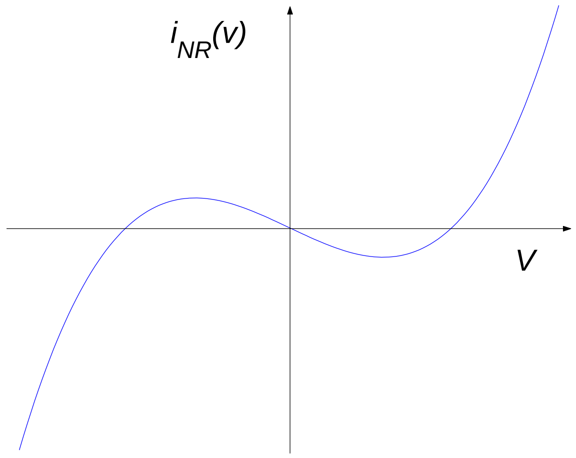

Figure 6.2: Voltage-current characteristic of the nonlinear resistor of figure 6.1 (Example 6.1).

with $V(t) = |V(t)|.e^{-j\psi(t)}$. *We neglect the high harmonic generation of the term* v^3 *and assume a* slow time variation *of the amplitud and the phase of* $V(t)$ *and that* ω *is close to* ω_0. *We obtain the two equations:*

$$\begin{aligned} \frac{d}{dt}|V(t)| &= \frac{1}{2}\left[\alpha - \beta|V(t)|^2\right].|V(t)| - \frac{\omega V_0}{2}\cos\left[\psi(t)\right] \\ \frac{d}{dt}\psi(t) &= d + l\sin\left[\psi(t)\right] \end{aligned} \tag{6.1.4}$$

Here, $d = \omega_0 - \omega$ *stands for the frequency deviation and* $l = \frac{\omega}{2}.\frac{\beta}{2}.V_0$ *is the* locking coefficient.

If V_0 *is small enough, the two equations can be decoupled. The modulus of* $V(t)$ *converge to a nonzero value and the dynamic of the system can be described by the single equation*

$$\dot{\psi}(t) = d + l\sin\left[\psi(t)\right]$$

which for $l > d$ *resembles (*6.1.1*). Van der Pol proved that the system can be forced to have an oscillation of the frequency of the input, so the system* locked *to the input.*

♣

To describe the interaction between agents, we can draw a digraph (directed graph) G, which has the agents as nodes. There is a link from node i to node k if $k \in \mathcal{N}_i$. Along this work, we will deal with connected graphs. Although, we will say something about the case with non-connected graph.

We want to emphasize the following aspects of system (6.1.2):

- The dynamic depends only on the phase difference of the oscillators. Then, there are several properties that are invariant under translations on the torus. For example, if $\bar{\theta}$ is an equilibrium point, so is $\bar{\theta} + c.\mathbf{1}_N$ for every $c \in [0, 2\pi)$ [2].

- As was done by Kuramoto (Kuramoto, 1984), we associate the individual oscillator phases to points running around the circle of radius 1 in the complex plane. Then, each oscillator can be described by a unitary phasor $V_i = e^{j\theta_i}$.

- We name by **consensus** or **synchronization** the state where all the phase differences are zero, i.e. the diagonal of the state space. Every consensus state is of the form $\bar{\theta} = c.\mathbf{1}_N$, with $c \in [0, 2\pi)$. We have a closed curve of consensus points. Observe that at a consensus point, all the associated phasors coincide.

- We say we have *partial synchronization* when all the phasors all parallel but they are not synchronized; i.e. most of the phases takes the value 0 (taking a suitable reference), but there are m agents with phase $\pm\pi$, for some $0 < 2m \leq N$.

- The other equilibrium points have non-parallel phasors and we refer to them as *non-synchronized* equilibrium points.

The key question we try to answer in this work is whether or not the system behavior of (6.1.2) reaches *consensus*. Recently, the Kuramoto model has received the attention of control theorists interested in the coordination and consensus of multi-agent systems (see (Jadbabaie, 2004) and references there in). We focus on the global properties of the consensus equilibrium point of many agents described by two different versions of the Kuramoto model. Since the system has many equilibrium points, we can not talk about global stability, or global attraction. But we may wonder on almost global stability, that is, the set of initial conditions that not lead to synchronization has zero Lebesgue measure. From an engineering point of view, this is a nice property (Rantzer, 2001a).

The rest of the Chapter is organized as follows. We introduce some general properties of the equilibrium points of (6.1.2) that will help us all along the Chapter. In order to study the stability of the synchronized set, we perform a Center Manifold analysis. We continue with the classic symmetric sine model (Kuramoto, 1984; Strogatz, 2000) in which the mutual interaction between

[2] $\mathbf{1}_N$ denotes the column vector in $\mathcal{R}^N$ with all the elements equal to one

agents depends on the sine of the phase difference between them. This model was studied for an arbitrary interaction topology in (Jadbabaie, 2004), and La Salle's Invariance principle was invoked to show convergence to the consensus equilibria. However, as we will show in Section 6.4, the characterization of these equilibria in (Jadbabaie, 2004) is incomplete, so the resulting almost global stability claims are not valid in the general case. Indeed, we characterize situations where the system has other attractors besides the consensus point. Nevertheless, in the *all to all* Kuramoto case, we are able to show that there are no other attractors and hence obtain almost global stability. Closing this part, we present the analysis of a system where the symmetric interaction is not *all to all*. We also show in Section 6.5 how some of these results extend to the case of non-symmetric interaction, where the coupling is unidirectional. We focus on the well-studied special case of a ring of coupled oscillators (Ermentrout, 1985), whose local stability was studied extensively in (Rogge, 2004). Later, we make some comments about density functions for the Kuramoto model. Finally, we present some conclusions.

6.2 General properties

The following results are true for the general dynamic (6.1.2).

Proposition 6.1. *At any equilibrium point $\bar{\theta}$ of (*6.1.2*), it must be true that the phasors*

$$\sum_{h\in\mathcal{N}_i} V_h \quad , \quad V_i$$

are parallel in the complex plane, for every i.

Proof: For $i = 1, \ldots, N$, consider the number

$$\alpha_i = \sum_{h\in\mathcal{N}_i} \frac{V_h}{V_i} = \sum_{h\in\mathcal{N}_i} e^{j(\bar{\theta}_h - \bar{\theta}_i)} =$$

$$= \sum_{h\in\mathcal{N}_i} \cos(\bar{\theta}_h - \bar{\theta}_i) + j. \sum_{h\in\mathcal{N}_i} \sin(\bar{\theta}_h - \bar{\theta}_i)$$

Since $\bar{\theta}$ is an equilibrium point, α_i is a real number and

$$\sum_{h\in\mathcal{N}_i} V_h = \alpha_i . V_i$$

■

Important consequences of Proposition 6.1 will be presented in further sections. Nevertheless, we can write some direct corollaries.

Corollary 6.2. *If for a given agent i, $\mathcal{N}_i = \{k\}$, with $i \neq k$ then, at an equilibrium point $\bar{\theta}$*

$$\bar{\theta}_i = \bar{\theta}_k \quad \text{or} \quad \bar{\theta}_i = \bar{\theta}_k + \pi$$

■

Corollary 6.3. *If for a given agent i, its respective degree*[3] *is odd, then, at an equilibrium point, there must be at least one agent with associated phasor parallel to V_i.*

■

To conclude this Section, we introduce the concept of *phase-locking solution.* We say that a solution $\theta(t)$ is *phase-locking* when the phase difference between any two agents remains constant in time. It follows that for $i = 1, \ldots, N$,

$$\dot{\theta}_i = \Omega$$

and $\theta_i(t) = \Omega.t + \theta_{0i}$. For the particular case of $\Omega = 0$, we have the equilibrium points described above. Phase-locking solutions with $\Omega \neq 0$ correspond to closed periodic orbits in $\mathcal{T}^N$ and play important roles in many contexts, such pace generators or muscular contractions in biology (Ermentrout, 1985), cyclic pursuit problems (Marshall, 2004) or circular polarization generation with antennas (Dussopt, 1999). More about phase-locking solutions will be said in Section 6.5.

6.3 Center manifold analysis

In this Section we present a review of some basic aspects of the center manifold theory, based on (Guckenheimer and Holmes, 1983; Khalil, 1996) that will help us in our analysis of sinusoidally coupled oscillators. More developed analysis can be found in the given references. Consider the system in $\mathcal{R}^n$ described by

$$\dot{x} = f(x) \quad , \quad f(0) = 0 \tag{6.3.1}$$

with differentiable f, so that the system can be written as

$$\dot{x} = Ax + \tilde{f}(x) \tag{6.3.2}$$

where

$$A = \frac{\partial f}{\partial x}(0) \quad , \quad \tilde{f}(x) = f(x) - Ax$$

[3] *The degree of the agent i ($deg(i)$) is the numbers of neighbors of agent i. It is equal to $\#\mathcal{N}_i$.*

and

$$\tilde{f}(0) = 0 \quad , \quad \frac{\partial \tilde{f}}{\partial x}(0) = 0$$

We will recall two important results: Hartman-Groβman Theorem and Center Manifold Theorem (Guckenheimer and Holmes, 1983; Khalil, 1996).

Theorem 6.4. *[Hartman-Groβman] If A has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighborhood U of the origin locally taking orbits of the nonlinear flow f^t of (6.3.1) to those of the linear flow associated to $\dot{x} = Ax$. The homeomorphism preserves the sense of the orbits and can also be chosen to preserve parametrization by time.*

■

Theorem 6.5. *[Center Manifold] Let f be a C^r vector field on $\mathcal{R}^n$ with $f(0) = 0$ and let $A = \frac{\partial f}{\partial x}(0)$. Divide the spectrum of A into three parts σ_s, σ_c, σ_u with*

$$\Re e(\lambda) \begin{cases} < 0 & \text{if } \lambda \in \sigma_s \\ = 0 & \text{if } \lambda \in \sigma_c \\ > 0 & \text{if } \lambda \in \sigma_u \end{cases}$$

Let the generalized eigenspaces of σ_s, σ_c, σ_u be E^s, E^c, E^u respectively. Then, there exist C^r stable and unstable invariant manifolds W^s and W^u tangent to E^s and E^u at 0 and a C^{r-1} manifold W^c tangent to E^c at 0. The manifolds W^s, W^c, W^u are all invariants for the flow of f. The stable and unstable manifolds are unique, but W^c need not be. W^c is called a center manifold.

■

By Theorem 6.4, it is clear that if A has an eigenvalue with positive real part, then the equilibrium point is unstable. Similarly, if A is a Hurwitz matrix, then the origin is asymptotically stable. So, we will consider the case where A has k eigenvalues on the imaginary axis and $n-k$ eigenvalues with negative real part. In this case, the mere position of the eigenvalues is not enough to establish or to deny the stability property of the equilibrium point. Theorem 6.5 is there for us[4]. Consider the change of variables $x = Tu$, with T an invertible matrix, such that

$$T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

A_1 and A_2 are the generalized eigenspaces associated to the eigenvalues with zero and negative real parts respectively. If we split vector u in two parts:

$$u = \begin{bmatrix} y \\ z \end{bmatrix}$$

[4]The rest of this Section follows the presentation of (Khalil, 1996)

with $y \in \mathcal{R}^k$ and $z \in \mathcal{R}^{n-k}$, we may re-write equation (6.3.2) as

$$T\dot{u} = ATu + \tilde{f}\,(Tu) \Rightarrow \dot{u} = T^{-1}ATu + T^{-1}\tilde{f}\,(Tu)$$

or

$$\left\{ \begin{array}{rcl} \dot{y} & = & A_1 y + g_1(y,z) \\ \dot{z} & = & A_2 z + g_2(y,z) \end{array} \right. \tag{6.3.3}$$

with

$$\begin{bmatrix} g_1(y,z) \\ g_2(y,z) \end{bmatrix} = T^{-1}\tilde{f}\left(T\begin{bmatrix} y \\ z \end{bmatrix}\right)$$

and $g_i(0,0) = 0$, $\frac{\partial g_i}{\partial y}(0,0) = 0$, $\frac{\partial g_i}{\partial z}(0,0) = 0$, $i = 1,2$. We know there is a center manifold. It can be proved ((Khalil, 1996); Theorem 4.1) that there exists $\delta > 0$ and a smooth function

$$h : B(0,\delta) \subset \mathcal{R}^k \to \mathcal{R}^{n-k}$$

with $h(0) = 0$ and $\frac{\partial h}{\partial y}(0) = 0$ such that $z = h(y)$ defines a center manifold. So, if the initial conditions satisfy $z_0 = h(y_0)$, then the system evolves along the center manifold. In the following, we assume we know a center manifold W^c given by $y = h(z)$. If we start at W^c, $\dot{z} = \frac{d}{dt}h(y) = \frac{\partial h}{\partial y}(y)\dot{y}$. Then

$$A_2 h(y) + g_2\left(y, h(y)\right) = \frac{\partial h}{\partial y}(y).\left[A_1 y + g_1(y, h(y))\right] \tag{6.3.4}$$

or

$$A_2 h(y) - \frac{\partial h}{\partial y}(y).A_1 y = \frac{\partial h}{\partial y}(y).g_1(y, h(y)) - g_2\left(y, h(y)\right) \tag{6.3.5}$$

Identity (6.3.4) can be viewed as a partial differential equation that must satisfy $h(y)$, and shows a way to find a center manifold. Now, we introduce the variable $w = z - h(y)$, which represents the *deviation* of the actual point (y,z) from the center manifold. Observe that precisely at W^c, we have $w \equiv 0$, so $\dot{w} = 0$ (and we recover (6.3.4)). Outside W^c, the derivative of w is

$$\dot{w} = \dot{z} - \frac{\partial h}{\partial y}(y)\dot{y} = A_2.\left[w + h(y)\right] + g_2\left(y, w + h(y)\right) - \frac{\partial h}{\partial y}(y).\left[A_1 y + g_1(y, w + h(y))\right]$$

Using (6.3.5), we can set

$$\dot{w} = A_2.w + g_2\left(y, w + h(y)\right) - g_2\left(y, h(y)\right) - \frac{\partial h}{\partial y}(y).\left[g_1(y, w + h(y)) - g_1(y, h(y))\right]$$

We define

$$\left\{ \begin{array}{l} N_1(y,w) = g_1(y, w + h(y)) - g_1(y, h(y)) \\ \\ N_2(y,w) = g_2(y, w + h(y)) - g_2(y, h(y)) - \frac{\partial h}{\partial y}(y).N_1(y,w) \end{array} \right.$$

Recalling the properties of g_1 and g_2, it follows that

$$N_i(0,w) = 0 \quad ; \quad \frac{\partial N_i}{\partial w}(0,0) = 0 \quad , \; i = 1,2$$

and then, we can find $\rho > 0$ such that if $\max\{\|y\|, \|w\|\} < \rho$,

$$\|N_i(y,w)\|_2 \leq k_i \|w\| \quad , \; i = 1,2 \quad , \; \forall w$$

Putting all things together, we have an alternative description of system (6.3.2):

$$\begin{cases} \dot{y} &= A_1 y + g_1(y, h(y)) + N_1(y,w) \\ \dot{w} &= A_2 w + N_2(y,w) \end{cases} \tag{6.3.6}$$

Since A_2 is Hurwitz, N_2 can be viewed as a perturbation of a nominal asymptotically stable system. Then, for initial conditions close enough to $y = 0$, $w = 0$, the function $V_2(w) = w^T P w$, with $P = P^T > 0$ and $A_2^T P + P A_2 < 0$, decreases along the trajectories of (6.3.6) and then, the system approaches the center manifold W^c. Sometimes, this property is called *transversal stability*, since it refers to what happens in a direction transversal to the center manifold (the w part of the state). In order to prove stability of the origin, we must study the dynamic along the center manifold, i.e., we must focus on the (reduced) system:

$$\dot{y} = A_1 y + g_1(y, h(y)) \tag{6.3.7}$$

where we have dropped the high order term $N_1(y,w)$ because we want to study local perturbations of the origin.

To conclude this Section, we show how we have applied the center manifold analysis to a consensus equilibrium point $\bar{\theta}$ of system (6.1.2). Due to the invariance of the system under translations parallel to vector $\mathbf{1}_N$, it is clear that a first order approximation around a given equilibrium point $\bar{\theta}$ will have the zero eigenvalue, with associated eigenvector $\mathbf{1}_N$. We use the notation of (Khalil, 1996). Re-write the system as

$$\frac{d}{dt}(\theta - \bar{\theta}) = A.(\theta - \bar{\theta}) + f(\theta) - A.(\theta - \bar{\theta}) = A.(\theta - \bar{\theta}) + \tilde{f}(\theta)$$

where the matrix A is a first order approximation around the equilibrium point $\bar{\theta}$ and $A.\mathbf{1}_N = 0$. We can find a change of coordinates $\theta - \bar{\theta} = T.\begin{bmatrix} y \\ z \end{bmatrix}$ that takes the system to the form:

$$\begin{cases} \dot{y} &= g_1(y,z) \\ \dot{z} &= A_2 z + g_2(y,z) \end{cases} \tag{6.3.8}$$

where $y \in \mathcal{R}$, $z \in \mathcal{R}^{N-1}$, $(0,0) \in \mathcal{R} \times \mathcal{R}^{N-1}$ is an equilibrium point of (6.3.8),

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} = T^{-1}AT$$

where A_2 is a $(N-1) \times (N-1)$ matrix and

$$\begin{bmatrix} g_1(y,z) \\ g_2(y,z) \end{bmatrix} = T^{-1}\tilde{f}\left(T\begin{bmatrix} y \\ z \end{bmatrix}\right)$$

Matrix T can be taken with vector $\mathbf{1}_N$ as first column, so

$$\mathbf{1}_N = T\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We will show that a center manifold is given locally by $z = h(y)$, with $h : \mathcal{R} \to \mathcal{R}^{N-1}$ defined by $h(y) \equiv 0$. This manifold is formed by all the consensus or synchronized states. We know that h must satisfy the partial differential equation (Khalil, 1996)

$$A_2h(y) + g_2\left(y, h(y)\right) = \frac{\partial h}{\partial y}(y).\left[g_1(y, h(y))\right]$$

So, we only have to prove that

$$g_2(y,0) \equiv 0 \quad , \ \forall y$$

But

$$\begin{aligned} \begin{bmatrix} g_1(y,0) \\ g_2(y,0) \end{bmatrix} &= T^{-1}\tilde{f}\left(T\begin{bmatrix} y \\ 0 \end{bmatrix}\right) = T^{-1}\tilde{f}\left(y.\mathbf{1}_N\right) \\ &= T^{-1}\left[f(y.\mathbf{1}_N) - y.A.\mathbf{1}_N\right] = 0 \end{aligned}$$

for all y. Then, if A_2 is Hurwitz, we obtain the so-called *transversal stability* and the stability of $\bar{\theta}$ can be assessed looking at the reduced dynamic

$$\dot{y} = h(z) = 0$$

which is stable for all y, but not asymptotically stable. So, if A_2 is Hurwitz, if we start the system with an initial condition close enough to $\bar{\theta}$, the trajectory will approach the center manifold (the consensus set); it will converge to a consensus point (possibly different from $\bar{\theta}$). Then, the stability properties of $\bar{\theta}$ refer to the whole set of consensus equilibrium points, which is a closed curve in $\mathcal{T}^N$.

6.4 The symmetric Kuramoto Model

6.4.1 Dynamics

The dynamic of a given agent depends on the sine of its phase differences with its neighbors. Symmetry is characterized by:

$$i \in \mathcal{N}_k \Rightarrow k \in \mathcal{N}_i$$

As in (Jadbababie, 2003), we can build a directed graph G with the agents as nodes and the edges representing the relationships between agents. We only put one link between neighbors, with arbitrary orientation. Let e be the number of edges. We construct the incidence matrix $B_{N\times e}$ as follows:

$$B_{il} = \begin{cases} 1 & \text{if edge } l \text{ reaches node } i \\ -1 & \text{if edge } l \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

When $\mathcal{N}_i = \{1, \dots, N\} \setminus \{i\}$ for every i, we have the *complete* or *all to all* case. In matrix notation, the dynamic can be written as

$$\dot{\theta} = -\frac{K}{N}.B.\sin\left(B^T\theta\right) \tag{6.4.1}$$

Equation (6.4.1) does not depend on the particular orientation we have chosen for the links. We can further simplify the notation by eliminating the factor $\frac{K}{N}$; this amounts to renormalizing time. So, we get

$$\dot{\theta} = -B.\sin\left(B^T\theta\right) \tag{6.4.2}$$

First of all, we show that the only phase-locking solutions of a symmetric system are the ones with $\Omega = 0$.

Lemma 6.6. *The only phase-locking solutions of system (6.4.2) are equilibrium points.*

Proof: Symmetry implies that the sum of all the phases is a constant magnitud of the system:

$$\frac{d}{dt}\sum_{i=1}^{N}\theta_i = \sum_{i=1}^{N}\dot{\theta}_i = \mathbf{1}^T\dot{\theta} = -\mathbf{1}^T B.\sin(B^T\theta) = 0$$

since $B^T.\mathbf{1} = 0$. At a phase-locking solution, $\dot{\theta} = \Omega.\mathbf{1}$. Then

$$0 = \mathbf{1}^T\dot{\theta} = \Omega.\mathbf{1}^T.\mathbf{1} = N.\Omega$$

So, $\Omega = 0$ and we have an equilibrium point.

■

6.4.2 Stability analysis

Local stability of the consensus point for system (6.4.2) was studied in (Jadbabaie, 2004) using La Salle's invariance principle (Khalil, 1996). The function

$$U(\theta) = e - \mathbf{1}_e^T \cos(B^T \theta) \tag{6.4.3}$$

is non-negative, and such that the system can be written in the gradient form

$$\dot{\theta} = -\nabla U;$$

In particular this implies that

$$\dot{U}(\theta) = -\|\dot{\theta}\|^2,$$

Hence the function is non-increasing along the trajectories. Since $U \equiv 0$ at the consensus set, it is a local Lyapunov function for the consensus set, meaning that if we start near enough to this set, we will converge to it.

Since the state space is compact, every trajectory has a non-empty ω-limit set (Guckenheimer and Holmes, 1983). La Salle's result (Khalil, 1996) ensures that every trajectory goes to the set $W = \left\{\theta \mid \dot{U}(\theta) = 0\right\}$, i.e., goes to an equilibrium point. In particular, this proves that the system admits no closed curves and we recover the conclusion of Lemma 6.6. In order to establish almost global attraction of the consensus set, it must be true that this set is the only attractor. The local analysis of all the equilibrium points of (6.4.2) done in (Jadbabaie, 2004) is incomplete, and therefore the conclusion of almost global stability is in general not true, as is shown in the next Example.

Example 6.2. *Consider the case with $N = 6$ in which the dynamics of the agents are as follows:*

$$\dot{\theta}_i = [\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i)]$$

Here the configuration is circular; we identify θ_7 and θ_0 with θ_1. Consider the equilibrium point showed in Fig. 6.3. Using an approach that will be presented later, it can be shown that this configuration is locally attractive.

♣

We thus see that guaranteeing asymptotic consensus is more involved; in the following subsections we provide some theory that may help classify these other equilibria, and also show that in the complete graph case, there is indeed almost global stability of the consensus set.

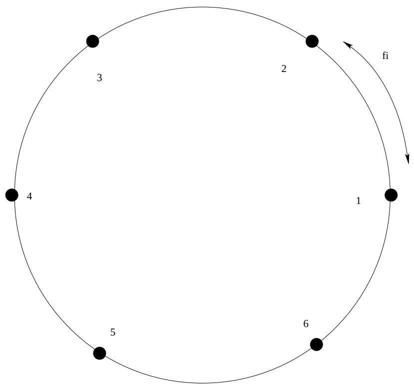

Figure 6.3: Stable non-consensus equilibrium for the Kuramoto model of Example 6.2.

We will analyze the stability of the equilibrium points using Jacobian linearization. A first order approximation of the system at an equilibrium point $\bar{\theta}$ takes the form $\dot{\delta\theta} = A.\delta\theta$, with $\delta\theta = (\theta - \bar{\theta})$ and A the symmetric matrix $N \times N$ with entries

$$\begin{cases} a_{ii} = & -\sum_{k \in \mathcal{N}_i} \cos(\bar{\theta}_k - \bar{\theta}_i) = -\alpha_i \\ a_{hi} = & \begin{cases} \cos(\bar{\theta}_h - \bar{\theta}_i) & , \quad h \in \mathcal{N}_i \\ 0 & , \quad h \notin \mathcal{N}_i \end{cases} \end{cases}$$

with α_i defined as in Proposition 6.1.

The matrix A is symmetric, reflecting the bidirectional influence of the agents. It can be written as

$$A = -B.diag\left[\cos(B^T\bar{\theta})\right].B^T \tag{6.4.4}$$

At a consensus equilibrium point, A equals the Laplacian matrix $L = BB^T$ associated to the graph G (Biggs, 1993). As we have mentioned earlier, A always has the eigenvector $\mathbf{1}_N$ with zero eigenvalue. Recall that if G is a connected graph, then 0 is a simple eigenvalue of L.

Lemma 6.7. *Let $\bar{\theta}$ be an equilibrium point of (*6.4.2*), such that there is at least one $\alpha_i < 0$. Then, $\bar{\theta}$ is unstable.*

Proof: The numbers $-\alpha_i$ appear at the diagonal of the symmetric matrix A. If some $\alpha_i < 0$, A can not be negative definite nor semi-definite and so $\bar{\theta}$ is unstable.

■

Lemma 6.8. *Let $\bar{\theta}$ be an equilibrium point of (6.4.2), such that $\cos(\bar{\theta}_k - \bar{\theta}_i) > 0$ for every $k \in \mathcal{N}_i$. Then, $\bar{\theta}$ is stable.*

Proof: We know that if the graph G is connected, then 0 is a simple eigenvalue of $L = BB^T$ and is also the minimum eigenvalue of L (Biggs, 1993). Since

$$A = -B.diag\left[\cos(B^T\bar{\theta})\right].B^T$$

the fact that the matrix

$$diag\left[\cos(B^T\theta)\right]$$

is positive definite concludes the proof.

■

Example 6.3. *Lemma 6.8 covers the system shown in Example 6.2. In that case, the characteristic polynomial of the linear approximation has the roots 0 and -2 (simple), and $-\frac{1}{2}$ and $-\frac{3}{2}$ (double). Indeed, for large N, there can be equilibrium configurations with all neighboring angles lesser than $\pi/2$, and thus provide attractors other than the consensus set.*

♣

6.4.3 Complete system

In the case of complete (full mesh) graph, phase differences larger than $\pi/2$ always occur at a non-consensus equilibrium point. We are now ready to prove that the consensus set is the only attractor for the complete symmetric case.

Lemma 6.9. *Let $\bar{\theta}$ be an equilibrium point of (6.4.2)* ***for the complete case****. Then, we have three different types of equilibrium points:*

- synchronization*: $\sum_{i=1}^{N} V_i = N.V_1$;*
- partial consensus*: all V_i are parallel but not equal;*
- balanced *(non-synchronized): $\sum_{i=1}^{N} V_i = 0$.*

Proof: Let α be the sum of all the phasors. It is clear that for each $i = 1, \ldots, N$,

$$\alpha = \sum_{i=1}^{N} V_i = V_i.\left[1 + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{V_k}{V_i}\right] = V_i.[1 + \alpha_i]$$

with α_i is a real number defined as in Proposition 6.1. At a synchronization point, all the V_i coincide and then $\sum_{i=1}^{N} V_i = N.V_1$ and $\alpha_i = N - 1$ for $i = 1, \ldots, N$.

At partial consensus point, all V_i are parallel and we can take the reference such that there are m agents with $V_i = -1$, $(1 \leq 2m \leq N)$, and $N - m$ agents with $V_i = 1$. The first group contains the *unsynchronized* variables (an unsynchronized variable θ_h agrees with $m - 1$ variables and disagree with the other $N - m$.). In this case, $\sum_{i=1}^{N} V_i = N - 2m$.

Finally, consider an equilibrium point with V_i and V_k non parallel. Then

$$\alpha = (1 + \alpha_i).V_i = (1 + \alpha_k).V_k$$

It follows that

$$\alpha = \sum_{i=1}^{N} V_i = 0 \quad , \quad \alpha_i = -1 \quad , \quad i = 1, \ldots, N$$

■

Completeness is crucial in Lemma 6.9, as the next example shows.

Example 6.4. *Consider the non-complete system described by*

$$\begin{aligned}
\dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) \\
\dot{\theta}_2 &= \sin(\theta_1 - \theta_2) + \sin(\theta_3 - \theta_2) + \sin(\theta_4 - \theta_2) \\
\dot{\theta}_3 &= \sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3) \\
\dot{\theta}_4 &= \sin(\theta_2 - \theta_4)
\end{aligned}$$

If we focus on the equilibrium point given by

$$\bar{\theta}_1 = 0 \quad , \quad \bar{\theta}_2 = \frac{2\pi}{3} \quad , \quad \bar{\theta}_3 = \frac{4\pi}{3} \quad , \quad \bar{\theta}_4 = \frac{5\pi}{3}$$

which is shown in figure 6.4, we found that the sum of all the phasors is a non zero complex number.

♣

Theorem 6.10. *For the complete system (6.4.2), the consensus set is the only attractor.*

Proof: As we have seen in Lemma 6.9, for the complete case we have three different types of equilibrium points. Next, we study their local stability properties.

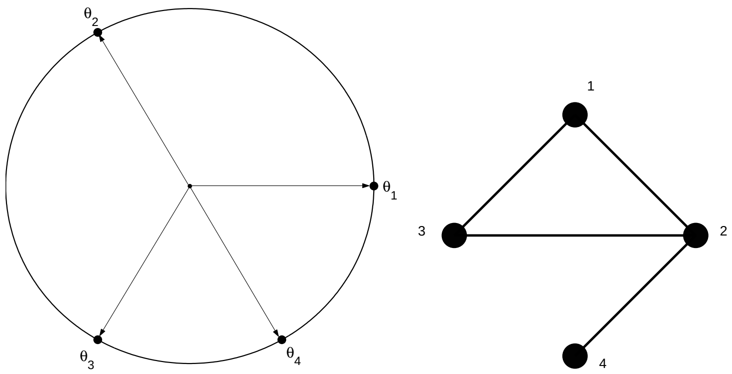

Figure 6.4: Non-complete system of Example 6.4.

- **Synchronization:** We use again the local Lyapunov function $U(\theta) = e - \mathbf{1}_e^T.\cos(B^T\theta)$. So, we have local asymptotical stability. In this case, the matrix A takes a very particular form:

$$A = \left[-N.I + \mathbf{1}^T\mathbf{1}\right]$$

which is symmetric and circulant. It is straightforward to show that its characteristic polynomial is

$$p(\lambda) = \lambda.(\lambda + N)^{N-1}$$

and A has 0 as a single eigenvalue and $-N$ as an eigenvalue with multiplicity $(N-1)$ (see, for example, (Marshall, 2004)).

- **Partial consensus:** consider a partial consensus point $\bar{\theta}$, with its corresponding m, $1 \leq 2m \leq N$. All the phase differences are 0 or $\pm\pi$. The numbers

$$\alpha_h = (m-1).\cos(0) + (N-m).\cos(\pi)$$
$$= m - 1 - N + m = -N + 2m - 1$$

are the same for every unsynchronized variable and we denote it by α_U. In the same way, the number

$$\alpha_S = (N-m-1).\cos(0) + m.\cos(\pi)$$
$$= N - 2m - 1$$

corresponds to every synchronized variable. Since α_U is always negative, $\bar{\theta}$ is unstable by Lemma 6.7.

- **Non-consensus:** let $\bar{\theta}$ be a balanced equilibrium point. Since by Lemma 6.9, $\alpha_1 = -1$, Lemma 6.7 implies that $\bar{\theta}$ is unstable.

■

Corollary 6.11. *For the complete case, the synchronized set is almost globally stable.*

Proof: As we have mentioned in Subsection 6.4.1, following (Jadbabaie, 2004), we can apply La Salle's Invariance result using the function U introduced in (6.4.3). From compactness of the state space, all the trajectories must converge to the largest invariant set contained in $\left\{\theta \mid \dot{U}(\theta) = 0\right\}$, i.e., must go to an equilibrium. From Theorem 6.10, the consensus set is the only attractor. Then, the only trajectories that are not attracted by the consensus point are the stable manifolds of the saddle equilibrium points, which conform a zero measure set.

■

6.4.4 Non-complete case

We have established the almost global attraction of the synchronized state in the complete case. We may wonder about what happens in the non-complete case. As we have seen in Example 6.4, there are non-complete systems with other stable equilibrium points besides the consensus set. So, we must remove carefully the completeness hypothesis, in order to be able to ensure stability properties of the synchronized state. Observe that this path leads us to the huge graph theory (Biggs, 1993). In this Section, we present some results in that direction.

In Theorem 6.10, for the complete case, we proved that a partial synchronized equilibrium point is unstable. This result is true for every graph. In the following proof, we use a different approach.

Proposition 6.12. *Let $\bar{\theta}$ be a partial consensus equilibrium point of (*6.4.2*), with graph G. Then $\bar{\theta}$ is unstable.*

Proof: Since we are dealing with a partial consensus equilibrium point, we can split the agents in two groups. Taking an appropriate reference, we only have phases 0 and π. Define the vector

$$v = \cos(\bar{\theta})$$

Then, v contains only the numbers 1 and -1. Consider the product $B^T v$. Since each row of B^T refers to an specific link of G, a component of this vector is 0 if the respective link connects two agents with the same phase, and is ± 2 if the link connects agents with different phases. The matrix

$$diag\left[\cos(B^T\bar{\theta})\right]$$

also has the value -1 at place (l,l) if the link related to the l-th row of B^T joins agents from different groups. Putting all this things together it turns out the identity

$$v^T Av = -v^T B.diag\left[\cos(B^T\bar{\theta})\right] B^T v = 4 \times c$$

where c is the positive number of links that join agents with different phases. Then, A must have a positive eigenvalue and $\bar{\theta}$ is an unstable equilibrium point.

■

The previous result is also true for non-connected graphs. If for a given graph G we can prove that the only equilibrium points correspond to partial or total consensus, we can ensure the almost global stability of the synchronized state. With an appropriate reference, a (partial or total) consensus state $\bar{\theta}$ is such that

$$\sin(B^T\bar{\theta}) = 0$$

In order to have only partial or total consensus equilibria, 0 must be the only solution of the equation

$$0 = B.u$$

Observe that for a connected graph, the matrix B, with N rows and e columns, has always rank $N-1$. Then, the previous equation has only the trivial solution when $e = N-1$, that is, it has full column rank. The only connected graphs with $N-1$ links are the trees without cycles. We can state the next general result.

Theorem 6.13. *Consider the system (*6.4.2*). If the associated graph G is a connected tree with no cycles, the consensus set is almost globally stable.*

■

Example 6.5. *A star graph is a connected tree graph that has a particular node, called the* hub*, which is related with all of the rest of the nodes, while all the rest of the nodes are related to the hub only. The graph can be drawn as a star and it models several examples of centralized interactions between agents. It is a particular* *case of Theorem* 6.13*. The synchronized state is an almost global attractor. Moreover, if we have two star graphs and we couple them through their hubs, as in figure* 6.5*, (or through any pair of agents), we obtain a new almost globally stable system (a kind of* synchronization preserving interconnection*). If we add one more link to a connected tree without cycles, we must have a cycle, and we may lose the almost global stability property, as in Example* 6.2 *and* 6.4*.*

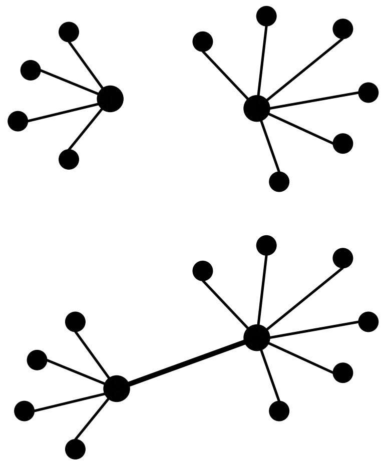

Figure 6.5: Two star graphs coupled through their hubs (Example 6.5).

♣

6.4.5 Non-connected case

To conclude this Section we want to say something about symmetric systems with associated non-connected graph G. In this case, we can split the dynamic into the sub-groups of agents related to the connected components of the graph G. These groups have decoupled dynamics. So, the best thing we can do is to apply the results of this Section to each sub-dynamic. Recall the classical result of graphs (Biggs, 1993).

Theorem 6.14. *If G is a disconnected graph, then the spectrum of G is the union of the spectra of the components of G.*

■

6.5 An example of non-symmetric graphs

Previous results do not directly extend to the general case of non-symmetric graphs (i.e., where $k \in \mathcal{N}_i$ does not imply that $i \in \mathcal{N}_k$).

In this regard, we mention the following:

- The Jacobian linearization is not symmetric, which implies that we are not dealing with a gradient system. As shown in (Rogge, 2004) for the ring example, there can be other periodic orbits in the system, non-trivial phase-locking solutions, where the phase differences converge but not the angles themselves.
- Proposition 6.1 and Corollary 6.2 are still valid.

In the rest of the Section, we analyze a particular non-symmetric system: sinusoidally coupled oscillators in a ring structure.

6.5.1 Dynamics

We focus on the study of the dynamics of N oscillators coupled in a ring structure, in a way that the system is described by the equations

$$\dot{\theta}_i = K. \sin\left(\theta_{i+1} - \theta_i\right) \tag{6.5.1}$$

$i = 1, \ldots, N$, $N+1 = 1$ (Ermentrout, 1985). Besides the consensus equilibrium points, we are also interested in the solution where all the oscillators are locked, in the sense that the phase differences between them remain constant in time; i.e. the *phase-locking solutions* introduced in Section 6.2 (Rogge, 2004). So, a particular phase-locking solution is characterized by a unique number $\Omega = \sin\Psi$, $0 \leq \Psi \leq 2\pi$, such that

$$\dot{\theta}_i = \sin(\Psi) \quad , \quad i = 1, \ldots, N$$

It follows that Ψ or $\pi - \Psi$ represents the distance between two consecutive oscillators. So

$$\theta_i(t) = \sin(\Psi).t + \theta_{i0} \quad , \quad i = 1, \ldots, N$$

represents a limit cycle in the N-Torus (or an equilibrium point if Ψ is 0 or π). Observe that the orbit of a phase-locking solution with non zero $\sin(\alpha)$ is invariant under translations with associated vector $c.\mathbf{1}$.

It is useful to re-write equations (6.5.1) in terms of the sequential phase differences

$$\Phi_i = \theta_{i+1} - \theta_i \quad , \quad i = 1, \ldots, N$$

The new description of the system is

$$\dot{\Phi}_i = K.\left[\sin\left(\Phi_{i+1}\right) - \sin\left(\Phi_i\right)\right] \tag{6.5.2}$$

$i = 1, \ldots, N$, $N + 1 = 1$. In this context, the phase-locking solutions of (6.5.1) are the equilibrium points of (6.5.2) and for a given phase-locking solution, the phase difference between consecutive oscillators can take only one of two possible values: an angle Ψ or its complement $\pi - \Psi$; when $\Psi = \pi/2$, there is only one value. It is clear that the restriction $\sum_{i=1}^{N} \Phi_i = 2k\pi$ must hold for some $k \in \mathcal{Z}$.

6.5.2 Stability analysis

A complete local analysis of the stability of equilibrium points and phase-locking solutions of (6.5.1) was done in (Rogge, 2004), using Jacobian linearization techniques combined with Gershsgorin's Theorem of localization of the eigenvalues of a given matrix. We propose a function V similar to function U introduced in (6.4.3). In this case, we have that $V(\Phi) = N - \sum_{i=1}^{N} \cos\left(\Phi_i\right)$, where Φ is the vector of all the cyclic phase differences. Using (6.5.2) we have that

$$\begin{aligned} \dot{V}(\Phi) &= \textstyle\sum_{i=1}^{N} \sin\left(\Phi_i\right).\dot{\Phi}_i \\ &= -\textstyle\sum_{i=1}^{N} \left[\sin^2\left(\Phi_i\right) - \sin\left(\Phi_{i+1}\right).\sin\left(\Phi_i\right)\right] \end{aligned}$$

Re-arranging terms we obtain

$$\dot{V}(\Phi) = -\frac{1}{2}\sum_{i=1}^{N}\left[\sin\left(\Phi_{i+1}\right) - \sin\left(\Phi_i\right)\right]^2$$

Then $\dot{V} \leq 0$ in the torus and every trajectory goes to the set where $\dot{V}$ vanishes. But

$$C = \left\{\dot{V} = 0\right\} = \left\{\sin\left(\Phi_{i+1}\right) = \sin\left(\Phi_i\right)\right\}$$

which contains the phase-locking solutions as its only invariants. So, we can affirm that almost all the trajectories in the torus converge to one of the stable phase-locking solutions. The study of the local properties shows that for $N = 2$ and 3, the non-trivial phase-locking solutions are unstable and the consensus set is the only attractor. Then, for $N = 2$ and 3 we have the almost global synchronization property. For higher dimensions, there are asymptotically stable limit cycles.

6.6 Density functions

The ideas of Anders Rantzer (Rantzer, 2001a) about density functions were our first approach in order to prove almost global stability for sinusoidally coupled oscillators. It turned out to be a hard task, (and sometimes impossible). We have several equilibria, and in some cases, many of these are saddle points, with non-trivial stable manifold. In (Angeli, 2003), David Angeli analyzed some hard restrictions that must be satisfied for a density function in the presence of a saddle equilibrium point with negative divergence (Proposition 3.9 of this Thesis). In that case, a given density function must vanish along the stable manifold of the equilibrium point. This is a big problem for a systematic search of density functions. We will show that for $N \geq 5$, both Kuramoto systems, the symmetric and the ring structure, satisfy Angeli's conditions.

We first consider the symmetric model.

Proposition 6.15. *For the symmetric system (*6.1.2*), there is a unstable equilibrium with non-trivial stable manifold and negative divergence.*

Proof: We have seen that there are several unstable equilibria for system. In particular, Theorem 6.12 shows that every partial consensus point is unstable. We will show that for $N \geq 5$, there exists a partial consensus with negative divergence. Consider a partial consensus equilibrium point and split the agents in two sets, C_0, with respective angles 0 and C_π, with respective angles π (taking a suitable reference). The divergence of the field (6.4.2) is the trace of matrix A introduced in (6.4.4):

$$\begin{aligned}\nabla.f(\bar{\theta}) &= trace\left[-B.diag\left[\cos(B^T\bar{\theta})\right].B^T\right] \\ &= -trace\left[diag\left[\cos(B^T\bar{\theta})\right].B^T.B\right]\end{aligned}$$

Since the diagonal elements of the matrix $B^T.B$ are all equal to 2, we can write

$$\nabla.f(\bar{\theta}) = -2.\mathbf{1}^T.\cos(B^T\bar{\theta}) = -2.(l_I - l_X)$$

where l_I is the total numbers of links interior to C_0 and C_π and l_X is the total number of links between the two sets. Denote by $deg(i)$ the degree of the agent i, i.e., the numbers of neighbors of agent i. Let d be the minimum degree in the associated graph G and let k the index of the respective agent ($d = deg(k)$). Consider the partial consensus with

$$C_0 = \{1, 2, \ldots, N\} \setminus \{k\} \quad , \quad C_\pi = \{k\}$$

(all the agents but agent k with phase 0 and agent k with phase π). It is clear that $l_X = d$. In this case, l_I is the numbers of links interior to the C_0. We have that

$$N.d \leq \sum_{i=1}^{N} deg(i) = \sum_{i \neq k} deg(i) + d = 2.(l_I + d) = 2.(l_I + l_X)$$

So,

$$(N-2).l_X \leq 2.l_I \Rightarrow (N-4).d \leq 2(l_I - l_X)$$

For $N > 4$, we have found a partial consensus equilibrium point with negative divergence.

■

We have a similar situation for oscillators coupled in a ring structure.

Proposition 6.16. *For the non-symmetric system (*6.5.2*), there is a unstable equilibrium with non-trivial stable manifold and negative divergence.*

Proof: If we denote the field by F, we have that

$$\nabla.F(\bar{\Phi}) = -\sum_{i=1}^{N} \cos(\bar{\Phi}_i)$$

Let $0 < 2m \leq N$ and consider the following equilibrium point:

$$\bar{\theta}_i = \pi, \quad i = 1, \ldots, m-1$$

$$\bar{\theta}_i = 0, \quad i = m, \ldots, N$$

which it is unstable (Rogge, 2004). We have only two links between this two sets. So, all the phase differences are 0, except two of them, which are π. Then

$$\nabla.F(\bar{\Phi}) = -\left[(N-2) - 2\right] = -(N-4)$$

Again, for $N > 4$ we have a saddle point with negative divergence.

■

The hard restrictions appears only for N greater than 4. We have found density functions for $N = 2$ and 3 for both, symmetric and ring coupled oscillators. We have just inverted the function U introduced in (6.4.3). We only present the analysis for a ring structure since for the symmetric case is quite similar. We are still working on the case $N = 4$.

6.6.1 The case $N = 2$

The system is

$$\dot{\Phi} = f(\Phi) \tag{6.6.1}$$

with

$$f_1(\Phi) = K.\left[\sin(\Phi_2) - \sin(\Phi_1)\right]$$

$$f_2(\Phi) = K.\left[\sin(\Phi_1) - \sin(\Phi_2)\right]$$

The angle α can take only the values 0 and $\pm\pi$, so there are not limit cycles. Observe that since $\Phi_1 + \Phi_2 = 2\pi$, if $\Phi_1 = 0$, then $\Phi_2 = 2\pi$ $(= 0)$. Consider the function

$$\rho(\Phi) = \frac{1}{1 - \cos(\Phi_1)}$$

which is C^1 in $\mathcal{T}^2 \setminus \{\Phi = 0\}$. A direct calculation gives

$$\nabla.\left[\rho.f\right](\Phi) = \frac{2}{1 - \cos(\Phi_1)} \tag{6.6.2}$$

which is positive almost everywhere in the torus. Then ρ is a density function for the system and we recover the almost global attraction of the set $\{\Phi = 0\}$, i.e. the consensus set.

This case of $N = 2$ gives a nice example where the ideas of Section 4.2 can be applied using monotone measures, sin ce we have the increasing measure induced by (6.6.2) and the decreasing measure associated to the function

$$l(\Phi) = \frac{1}{1 + \cos(\Phi_1)}$$

which satisfies

$$\nabla.\left[l.f\right](\Phi) = -\frac{2}{1 + \cos(\Phi_1)}$$

6.6.2 The case $N = 3$

The system is

$$\dot{\Phi} = F(\Phi)$$

with $F_i(\Phi) = K.\sin(\Phi_{i+1}) - \sin(\Phi_i)$. Possible values of α for this system are 0, $2\pi/3$ and $4\pi/3$. We have the the (full and partial) synchronized solutions and two limit-cycles. We will look for a density function

$$\rho(\Phi) : \mathcal{T}^3 \setminus \{0\} \to [0, +\infty)$$

of class C^1. Again, we use the candidate function

$$\rho(\Phi) = \frac{1}{3 - [\cos(\Phi_1) + cos(\Phi_2) + cos(\theta_3)]}$$

Then

$$\dot{\rho} = \frac{\sum_{i=1}^{3} \sin^2(\Phi_i) - \sum_{i=1}^{3} \sin(\Phi_i) . \sin(\Phi_{i+1})}{\left[3 - \sum_{i=1}^{3} cos(\Phi_i)\right]^2}$$

$$\nabla.F(\theta) = -[\cos(\Phi_1) + \cos(\Phi_2) + \cos(\Phi_3)]$$

Using the identities

$$\sin^2(\Phi_i) + \cos^2(\Phi_i) = 1$$

$$\cos(\Phi_{i+1}) . \cos(\Phi_i) - \sin(\Phi_{i+1}) . \sin(\Phi_i) = \cos(\Phi_{i+1} + \Phi_i)$$

and

$$\cos(\Phi_{i+1} + \Phi_i) = \cos(\Phi_{i-1})$$

$(i = 1, 2, 3,\ 3 + 1 = 1)$. Recall that $\Phi_1 + \Phi_2 + \Phi_3 = 2\pi$. We must check if the expression

$$\begin{aligned} p(\theta) = 6 - 4. [\cos(\Phi_1) + \cos(\Phi_2) + \cos(\Phi_3)] + \\ 2. [\cos(\Phi_2) . \cos(\Phi_1) + \cos(\Phi_3) . \cos(\Phi_2) + \cos(\Phi_3) . \cos(\Phi_1)] \end{aligned} \tag{6.6.3}$$

is positive almost everywhere. Observe that since $\cos^2(\Phi_i) \leq 1$, it is sufficient to establish the relative position in $\mathcal{R}^3$ of the cube

$$u_i^2 \leq 1 \quad , \quad i = 1, 2, 3$$

and the revolution cone

$$6 - 4.(u_1 + u_2 + u_3) + 2.(u_1 u_2 + u_2 u_3 + u_3 u_1) = 0$$

with vertex $[1; 1; 1]$ and axis parallel to $[1; 1; 1]$. It follows that the whole cube is *inside* the cone, so the function p is non-negative in the cube and then ρ is a density function. Then, the consensus state is almost globally stable.

6.7 Conclusions

In this Chapter, we have presented some global considerations for the Kuramoto model with sinusoidal influence functions. We first deduced some results for the general symmetric case. In this context, for the complete case, we have proved the almost global attraction of the synchronized state. We could extend this almost global property to some systems with non-complete graph. We also analyzed the non-symmetric case of coupled oscillators in a ring structure,

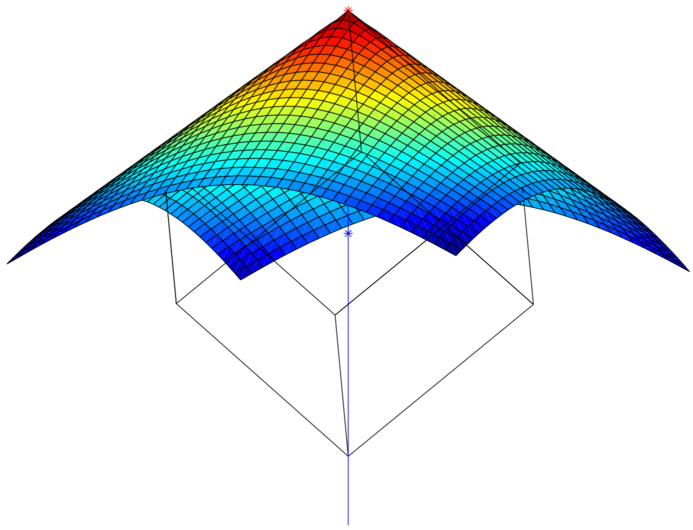

Figure 6.6: *Relative position for the case $N = 3$.*

where we have shown the almost global stability of the stable phase-locking solutions. In both cases, we explored the existence of density functions. We have proved that for more than four agents, both systems present the hard restrictions that make impossible a systematic search of a density function. We have found density functions for the cases of two and three agents. The case with four agents is more complicated and deserves a particular analysis.

Chapter 7

Conclusions and future works

In this final Chapter, we summarize the main contributions of the Thesis and briefly describe possible directions of future research.

7.1 Main contributions

The main objective of the Thesis was the study of the almost global stability concept. We considered the works of Anders Rantzer as starting point for our research. In Chapter 3, we have reviewed the main concepts, introduced some new results and Examples and put together different ideas.

In Chapter 4, we have written Rantzer's ideas in terms of what we have called *monotone measures*. We have included a particular relationship between monotone measures (and density functions) and local Lyapunov asymptotical stability through a combination of this new concepts with the classical Poincaré-Bendixson Theory.

Chapter 5 is the core of the Thesis. A battery of new results were developed in order to state converse results for almost global stability. First of all, we have recalled the proof of the existence of a density function for a linear Hurwitz system and we have extended this result to monotone measures. After that, we have presented two different ways to map the trajectories of the region of attraction of a nonlinear system into the trajectories of a linear Hurwitz system. We have made some considerations about the connection between the validity of this result for every dimension and the Poincaré Conjecture, which seems to be recently proved. Using this maps, we have constructed a monotone measure for a nonlinear almost globally stable system using a monotone measure for the linear one. This idea can be directly used to construct a density function for a nonlinear global asymptotically stable system. The extension for the almost globally stable case is not trivial and we could only construct a continuous and differentiable function, but with derivative not necessarily continuous at the points outside the region of attraction, which is a zero Lebesgue measure set. At the end of the Chapter, we have presented some applications of the converse results. In particular, we have proved that we can construct a density function using a global Lyapunov function for a g.a.s system.

To conclude the Thesis, in Chapter 6 we have presented the whole analysis of the almost global stability of sinusoidally coupled oscillators. We have shown how the graph that describes the interconnection plays a major role in order to achieve the almost global property. We have also found density functions for the case of 2 and 3 oscillators and we have shown that for higher dimensions, a density function can not be algorithmically found.

Finally, at the Appendix, we have put some nice results that we have derived through our work and that we have discarded them because we have solved the same problems by shorter ways.

7.2 Future works

Several lines of research are suitable to go on after this Thesis. Some of them lead to pure mathematical problems, like the deeply study of monotone measures, which falls into the ergodic theory, the full extension of the converse result for almost globally stable systems, including the cases with no local stability, the relationship of almost global stability and local or global bifurcations, the extension to higher dimensions of the Poincaré-Bendixson based results and the analysis of almost global stability of time-varying systems.

Another interesting way is the analysis and synthesis of control systems using density functions. In this Thesis, we have not mentioned some steps given in that direction (Prajna and Rantzer and Parrilo, 2004; Angeli, 2004). We think that a nice thing to do is the extension of some Lyapunov based control properties to the case of almost global stability, using density functions.

Finally, a deeper analysis of sinusoidally coupled oscillators is by itself an interesting line of research, particularly because it is a multidisciplinary task, involving dynamical systems and graph theory, and it has many applications to biology systems and coordination problems. We see two different posible directions. One of them is the characterization of the family of graphs which describe almost global synchronizing coupled oscillators. The other line is the analysis of a second order model for coupled oscillators, where a control signal can be easily included.

Appendix

Appendix A

Non positive density functions

In this Appendix, we present some results we have developed in order to deny the existence of non-negative density functions. We put this results here because we have found a shorter path to prove the same properties (Proposition 3.8). Nevertheless, we think that some of the results included in this Appendix may have some interest.

The identity

$$\int_{f^t(\mathcal{Z})} \rho(x)dx - \int_{\mathcal{Z}} \rho(x)dx = \int_0^t \int_{f^\tau(\mathcal{Z})} \left[\nabla.(\rho f)\right](x)dxd\tau$$

proved in Lemma 1 in (Rantzer, 2001a) shows that $\nabla.\left[\rho f\right]$ measures the growth of a given initial volume, weighted with density ρ, along the flow. Positive definition of the sign of the divergence implies that if we can apply the Lemma to an invariant set, then this set has zero measure, as is shown in 3.6.

It was mentioned in Theorem 3.1 that the positive definition of the density function is not needed when the origin is a stable equilibrium point in the sense of Lyapunov. In this case, there is a positive distance from 0 to the invariant set of trajectories that don't converge to the origin and Lemma 3.3 can be applied. From positivity of the divergence we conclude that this set has zero Lebesgue measure. Moreover, every invariant set with strictly positive distance to the origin has zero Lebesgue measure.

The behavior of non-positive definite density functions was explored in (Angeli, 2003) and the core of these works are Theorems 3.6 and 3.9 in this Thesis. Consider a function

$$\rho : \mathcal{R}^n \setminus \{0\} \to \mathcal{R}$$

of class C^1, with

$$\nabla.\left[\rho(x)f(x)\right] > 0 \ \ a.e.$$

If the dynamical system $\dot{x} = f(x)$ is complete, it was shown that the set

$$\mathcal{N} = \{x \in \mathcal{R}^n \mid \rho(x) \leq 0\}$$

is negatively invariant and then we have two possibilities: $\mathcal{N}$ has zero Lebesgue measure or $0 \in \bar{\mathcal{N}}$ (and $\mathcal{N}$ has infinite ρ-weighted measure). So, a non-positive definite density function with non-zero measure set $\mathcal{N}$ must be negative arbitrarily close to the origin. The result can be extended as follows.

Lemma A.1. *Given a complete dynamical system $\dot{x} = f(x)$, define the sets*

$$\mathcal{N} = \{x \in \mathcal{R}^n \mid \rho(x) \leq 0\}$$

$$\mathcal{P} = \{x \in \mathcal{R}^n \mid \rho(x) \geq 0\}$$

Then, if ρ is integrable on $\mathcal{P}$, this set has zero Lebesgue measure. The same conclusion holds for $\mathcal{N}$.

Proof: The following inequality directly comes from the divergence condition.

$$\dot{\rho}(x) + [\nabla.f(x)].\rho(x) > 0 \ \ a.e. \ x \in \mathcal{R}^n$$

For each $x \in \mathcal{R}^n$, consider the function $G : \mathcal{R} \to \mathcal{R}$ defined as

$$G_x(t) = \rho\left[f^t(x)\right].e^{\int_0^t \nabla.f[f^\tau(x)]d\tau}$$

Then,

$$G'_x(t) = e^{\int_0^t \nabla.f[f^\tau(x)]d\tau}.\left[\dot{\rho}\left[f^t(x)\right] + [\nabla.f(x)].\rho(x)\right]$$

We have that for almost all $x \in \mathcal{R}^n$,

$$G'_x(t) \geq 0 \ \ \forall t \in \mathcal{R}$$

and G grows along the trajectories. Since $G_x(0) = \rho(x)$ we obtain the inequality

$$\rho\left[f^t(x)\right] \geq \rho(x).e^{-\int_0^t \nabla.f[f^\tau(x)]d\tau} \ \ t \geq 0 \tag{A.0.1}$$

For a non-negative $\rho(x) = G_x(0)$ and from completeness of the dynamical system, we conclude that

$$\mathcal{P} = \{x \in \mathcal{R}^n \mid \rho(x) \leq 0\}$$

is positive invariant. On the other hand, if $\rho(x) = G_x(0) \leq 0$,

$$\rho\left[f^t(x)\right].e^{\int_0^t \nabla.f[f^\tau(x)]d\tau} \leq \rho(x) \ \ t \leq 0 \tag{A.0.2}$$

and $\mathcal{N}$ is negatively invariant. Equation (A.0.2) is the same that appears in (Angeli, 2003), just written in another way.

If ρ is integrable on $\mathcal{P}$, we can apply Lemma 3.3 with $Z = P$. Then $\mathcal{P}$ must have zero Lebesgue measure or must satisfy that $0 \in \bar{\mathcal{P}}$ and so, ρ must take non-negative values on points arbitrarily close to the origin. In the same way, the set $\mathcal{N}$ must have zero measure or ρ must take non-positive values on points arbitrarily close to the origin.

■

The previous Lemma has direct consequences on the admisible structure of a density function. We can have positive density functions as the examples presented in (Rantzer, 2001a). But a non-positive definite density function can not have a definite sign in a neighborhood of the origin. So the only possibilities are negative definite density functions, or functions with both positive and negative values arbitrarily close to the origin. The following proposition shows that the first case is not possible.

Theorem A.2. *Consider a complete dynamical system $\dot{x} = f(x)$ with $f \in C^2$ such that 0 is a locally stable equilibrium point in the sense of Lyapunov. Then there is no function $\rho \in C^1(\mathcal{R}^n \setminus \{0\}, (-\infty, 0])$ integrable outside arbitrary neighborhoods of the origin and such that*

$$\nabla . [\rho(x) f(x)] > 0 \ \ a.e.$$

We have split the proof of Theorem A.2 into several Lemmas. Some of them are interesting by themselves.

Lemma A.3. *Consider the canonical Hurwitz system*

$$\dot{z} = g(z) = -z \tag{A.0.3}$$

Then there is no C^1 function $\rho : \mathcal{R}^n \setminus \{0\} \to (-\infty, 0]$ of the form

$$\rho(x) = -\varphi(\|x\|)$$

with $\varphi : (0, +\infty) \to (0, +\infty)$ of class C^1 and such that

$$\left| \int_{\{\|x\| \geq \epsilon\}} \rho(x) dx \right| < +\infty$$

for arbitrary $\epsilon > 0$ and

$$\nabla . [\rho(x) g(x)] > 0 \ \ a.e.$$

Proof: Since

$$\nabla.\left[\rho(x)g(x)\right] = \nabla\rho(x).g(x) + \left[\nabla.g(x)\right]\rho(x)$$

a direct calculus of the divergence gives, for $x \neq 0$,

$$0 < -\varphi'(\|x\|)\frac{x^T}{\|x\|}.(-x) + n.\varphi(\|x\|) \quad a.e.$$

Then

$$0 \leq y.\varphi'(y) + n.\varphi(y) \quad , \quad \forall y > 0 \tag{A.0.4}$$

Let us define the auxiliary C^1 function

$$\psi(y) = y^n.\varphi(y) \quad , \quad y > 0$$

It is true that

$$\psi'(y) = ny^{n-1}.\varphi(y) + y^n.\varphi'(y)$$

Then

$$\psi'(y) = y^{n-1}.\left[n.\varphi(y) + y.\varphi'(y)\right] > 0 \quad , \quad \forall y > 0$$

The function ψ turns out to be increasing and then

$$\psi(y) = y^n.\varphi(y) \geq \psi(1) = \varphi(1) \quad , \quad \forall y > 1$$

So

$$-\rho(x) = \varphi(\|x\|) \geq \frac{\varphi(1)}{\|x\|^n} \quad , \quad \forall \|x\| > 1$$

and this can not be true due to the integrability assumption for ρ.

∎

Lemma A.4. *Consider the dynamical system $\dot{x} = f(x)$ such that the field f satisfies*

$$U.f(x) = f(Ux)$$

for all $x \in \mathcal{R}^n$ and for all orthogonal matrix $U \in \mathcal{R}_{n\times n}$. Suppose there is a C^1 function $\rho : \mathcal{R}^n \setminus \{0\}$ such that

$$\left|\int_{\{\|x\|>\epsilon\}} \rho(x)dx\right| < +\infty$$

and

$$\nabla.\left[\rho(x)g(x)\right] > 0 \quad a.e.$$

Then, there exists $\varphi : (0, +\infty) \to \mathcal{R}$ of class C^1 such that $\tilde{\rho}(x) = \varphi(\|x\|)$ has the same sign of ρ and

$$\left|\int_{\{\|x\|\geq\epsilon\}} \tilde{\rho}(x)dx\right| < +\infty \quad \forall \epsilon > 0 \quad , \quad \nabla.\left[\tilde{\rho}(x)f(x)\right] > 0 \ a.e.$$

Proof: Consider the group of the orthogonal matrices SO, which is compact and admits a left and right invariant probability Borel measure μ, i.e., for every continuous function $\gamma : SO \to \mathcal{R}$ and every $\tilde{U} \in SO$,

$$\int_{SO} \gamma(U)d\mu(U) = \int_{SO} \gamma(\tilde{U}U)d\mu(U) = \int_{SO} \gamma(U\tilde{U})d\mu(U)$$

The measure μ is called *the Haar measure* of the compact group SO and it is unique (Rudin, 1991). For every $x \in \mathcal{R}^n \setminus \{0\}$ define the function $\gamma_x : SO \to \mathcal{R}$ as follows

$$\gamma_x(U) = \rho(Ux)$$

It is clear that γ_x is continuous as a function of U, since ρ is a continuous function of its argument. With this set up, we define

$$\varphi(y) = \int_{SO} \gamma_x(U)d\mu(U) = \int_{SO} \rho(Ux)d\mu(U)$$

with $x \in \mathcal{R}^n$ such that $y = \|x\|$. First of all we must check that the function is well defined, i.e., the value $\varphi(y)$ does not depend on the particular choice of the point x. Consider another point $\tilde{x} \neq x$ with the same norm. There is an orthogonal matrix $\tilde{U}$ such that

$$\tilde{U}x = \tilde{x}$$

Then, by the invariance of μ,

$$\int_{SO} \gamma_x(U)d\mu(U) = \int_{SO} \gamma_x(U\tilde{U})d\mu(U)$$

So

$$\int_{SO} \rho(U\tilde{U}x)d\mu(U) = \int_{SO} \rho(U\tilde{x})d\mu(U) = \int_{SO} \gamma_{\tilde{x}}(U)d\mu(U)$$

and φ is well defined. Now we will show that the function

$$\tilde{\rho}(x) = \varphi(\|x\|)$$

verifies the required conditions. First of all, it has the same sign of ρ since SO is compact and ρ is continuous. Consider the integral

$$\int_{\|x\|\geq\epsilon} \tilde{\rho}(x)dx = \int_{\|x\|\geq\epsilon} \left[\int_{SO} \rho(Ux)d\mu(U)\right] dx$$

for an arbitrary $\epsilon > 0$. By Fubini's theorem we can exchange the order of integration, because the change of variable $z = Ux$ with U orthogonal gives

$$\left|\int_{\|x\|\geq\epsilon} \rho(Ux)dx\right| = \left|\int_{\|x\|\geq\epsilon} \rho(z)dz\right| = M < +\infty \ , \ \forall U \in SO$$

and the second integral is finite by hypothesis. We can ensure the uniform convergence (in the x variable) of the integral that defines $\tilde{\rho}$. Then

$$\left|\int_{\|x\|\geq\epsilon} \tilde{\rho}(x)dx\right| = \left|\int_{SO}\left[\int_{\|x\|\geq\epsilon} \rho(Ux)dx\right] d\mu(U)\right| = M.\int_{SO} d\mu(U) = M$$

Consider the gradient of $\tilde{\rho}$:

$$\nabla\tilde{\rho}(x) = \frac{\partial}{\partial x}\left[\int_{SO} \rho(Ux)d\mu(U)\right] = \int_{SO} \nabla\rho(Ux)Ud\mu(U)$$

due to the uniform convergence. So

$$\nabla.\left[\tilde{\rho}(x)f(x)\right] = \left[\int_{SO} \nabla\rho(Ux)Ud\mu(U)\right] f(x) + \left[\nabla.f(x)\right]\int_{SO} \rho(Ux)d\mu(U) =$$

$$= \int_{SO} \left[\nabla\rho(Ux)U.f(x) + \left[\nabla.f(x)\right]\rho(Ux)\right] d\mu(U)$$

From the hypothesis,

$$\nabla.\left[\tilde{\rho}(x)f(x)\right] = \int_{SO} \left\{\nabla.\left[\rho(Ux)f(Ux)\right]\right\} d\mu(U) > 0 \quad , \quad a.e.$$

since $\nabla.\left[\rho(x)f(x)\right] > 0$ almost everywhere.

■

Lemmas A.3 and A.4 show that no non-positive definite density function can exist for the canonical Hurwitz system (A.0.3), since its field is invariant under the action of orthogonal matrices. Lemma 5.2 gives us a way of mapping the trajectories of the region of attraction of a dynamical system into the trajectories of the canonical Hurwitz system.

Proposition A.5. *Consider the complete dynamical system* $\dot{x} = f(x)$ *with* $f \in C^2$ *and* $x = 0$ *a locally stable equilibrium point. Let* $\rho : \mathcal{R}^n \setminus \{0\} \to \mathcal{R}$ *be a density function, i.e.,*

$$\left|\int_{\{\|x\|\geq\epsilon\}} \rho(x)dx\right| < +\infty \ \ \forall\epsilon > 0 \quad , \quad \nabla.\left[\rho(x)f(x)\right] > 0 \ \ a.e.$$

Then, there exists a function $\bar{\rho} : \mathcal{R}^n \setminus \{0\} \to \mathcal{R}$*, with the same sign of* ρ*, such that*

$$\left|\int_{\{\|y\|\geq\epsilon\}} \bar{\rho}(y)dy\right| < +\infty \ \ \forall\epsilon > 0 \quad , \quad \nabla.\left[\bar{\rho}(y)g(y)\right] > 0 \ \ a.e.$$

with $g(y) = -y$*.*

Proof: By Theorem 3.1, the origin is almost global stable. This implies that it is an almost global attractor. By Lemma 5.2, we obtain a particular diffeomorphism h_1 between the region of attraction of the origin and $\mathcal{R}^n$. Define $\bar{\rho}$ as

$$\bar{\rho}(y) = \rho\left[h_1^{-1}(y)\right].\left|\frac{\partial h_1^{-1}}{\partial y}(y)\right|$$

Then, the sign condition is verified. Consider an arbitrary $\epsilon > 0$. We have

$$\left|\int_{\{\|y\|\geq\epsilon\}} \bar{\rho}(y)dy\right| = \left|\int_{\{\|y\|\geq\epsilon\}} \rho\left[h_1^{-1}(y)\right].\left|\frac{\partial h_1^{-1}}{\partial y}(y)\right| dy\right| = \left|\int_{h_1^{-1}\{\|y\|\geq\epsilon\}} \rho(x)dx\right|$$

The last integral is bounded and this fact comes from the assumptions on ρ and the fact that h_1 is an open function. Finally, we check the divergence condition. Since

$$\nabla.\left[\bar{\rho}(x)g(x)\right] = \frac{\partial}{\partial t}\left\{\bar{\rho}\left[g^t(x)\right].\left|\frac{\partial g^t}{\partial x}(x)\right|\right\}\Big|_{t=0}$$

we have

$$\nabla.\left[\bar{\rho}(x)g(x)\right] = \frac{\partial}{\partial t}\left\{\rho\left(h_1^{-1}\left[g^t(y)\right]\right).\left|\frac{\partial h_1^{-1}}{\partial y}\left[g^t(y)\right]\right|.\left|\frac{\partial g^t}{\partial x}(x)\right|\right\}\Big|_{t=0}$$

From $h_1 \circ f^t = g^t \circ h_1$, comes the identity

$$\left|\frac{\partial h_1^{-1}}{\partial y}\left[g^t(y)\right]\right|.\left|\frac{\partial g^t}{\partial x}(x)\right| = \left|\frac{\partial f^t}{\partial x}\left[h_1^{-1}(y)\right]\right|.\left|\frac{\partial h_1^{-1}}{\partial y}(y)\right|$$

So,

$$\nabla.\left[\bar{\rho}(x)g(x)\right] = \left|\frac{\partial h_1^{-1}}{\partial y}(y)\right|.\nabla.\left[\rho\left[h_1^{-1}(y)\right] g\left[h_1^{-1}(y)\right]\right] > 0 \;\; a.e.$$

■

Now, we can proof the main result.

Proof of Theorem A.2:

Suppose there is a function $\rho \in C^1\left(\mathcal{R}^n \setminus \{0\}, (-\infty, 0]\right)$ integrable outside arbitrary neighborhoods of the origin and such that

$$\nabla.\left[\rho(x)f(x)\right] > 0 \;\; a.e.$$

Then, by Proposition A.5 we can construct a function $\bar{\rho}$ with the same sign and integrability condition of ρ and with

$$\nabla.\left[\bar{\rho}(y).g(y)\right] > 0 \;\; a.e.$$

Since ρ is negative definite, so is $\bar{\rho}$ and by Lemma A.4 we can find a function $\varphi : (0, +\infty) \to [0, +\infty)$, of class C^1, such that $\tilde{\rho}(y) = -\varphi(\|y\|)$ is a density function for the system $\dot{y} = g(y)$. But this would contradict Lemma A.3.

$\blacksquare$

Bibliography

Andronov, A. A.; Chaikin,C. E. (1949). *Theory of Oscillations*, Princeton University Press.

Angeli, D. (2003). Some remarks on density functions for dual Lyapunov methods, *Proceedings of the 42nd IEEE Conference on Decision and Control*, Hawaii, pp. 5080–5082.

Angeli, D. (2004). An almost global notion of input-to-state stability, *Trans. on Automatic Control* **47**(3): 410–421.

Arnold, V. (1974). *Équations Differentielles Ordinaires*, MIR, Moscu.

Biggs, N. (1993). *Algebraic Graph theory*, Cambridge University Press.

Chernov N., Markarian R. (1996). *Introduction to the Ergodic Theory of Chaotic Billiards*, IMPA, Brasil.

Cvetkovic D., Doob M., Sachs H. (1979). *Spectra of Graphs: Theory and Applications*, Academic Press, Ney York.

Dussopt, L.; Laheurte, J. (1999). Coupled oscillator array generating circular polarization, *IEEE Microwave Guided Wave Letters* **9**(4): 160–162.

Ermentrout, G. (1985). The behavior of ring coupled oscillators, *Journal of Mathematical Biology* **23**: 55–74.

Freedman, M. (1982). The topology of the four-dimensional manifolds, *J. of Diff. Geom.* **17**: 357–453.

Grune, L; Sontag, E.; Wirth, F. (1999). Asymptotic stability equals exponential stability, and ISS equals finite energy gain - if you twist your eyes., *Systems and Control Letters* (38): 127–134.

Guckenheimer, J. and Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences 42, Springer-Verlag.

Jadbababie, A.; Lin, J.; Morse, S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Transactions on Automatic Control* **48**(6): 988–1000.

Jadbabaie, A.; Barahona, M.; Motee, N. (2004). On the stability of the Kuramoto model of coupled nonlinear oscillators, *Proc. of the American Control Conference.*

Khalil, H. (1996). *Nonlinear Systems*, Prentice-Hall.

Kuramoto, Y. (1975). in *International symposium on mathematical problems in theoretical physics*, *Lecture notes in Physics* **39**: 420, Springer.

Kuramoto, Y. (1984). Cooperative dynamics of oscillator community, *Progress of Theoretical Physics Suppl* (79): 223–240.

Le Corbeiller; (1935). The non-linear theory of the maintenance of oscillations, *Journal of the Institute of Electrical Engineers* , vol 79, pp. 361-380.

Lesieutre, B.; Paganini F. (1997). Generic properties, one-parameter deformations and the BCU method, *IEEE Trans. on Circuits & Systems–I: Fundamental Theory and Applications* , vol 46, No. 6, pp. 760-763.

Lyapunov, A. (1892). *A general task about the stability of motion*, Ph.D. Thesis, University of Kharkov.

Mañé, R. (1987). *Ergodic Theory and Differentiable Dynamics*, Springer, Berlin.

Marshal, J.; Broucke, M.; Francis, B. (2004). Formations of vehicles in cyclic pursuit, *IEEE Transactions on Automatic Control* **49**(11): 1963–1974.

Massera, J. L. (1949). On Liapunoff's conditions of stability, *Annals of Mathematics* **50**: 705–721.

Milnor, J. (1965). *Lectures on the h-Cobordism Theorem*, Princeton University Press.

Monzón, P. (2002). On necessary conditions for almost global stability, *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, pp. 4270-4271.

Monzón, P. (2003). On necessary conditions for almost global stability, *Trans. on Automatic Control* **48**:44: 631–634, April.

Monzón, P. (2004a). Almost global stability of planar systems, *Congreso Latinoamericano de Control Automático*, La Habana, AUT048.

Monzón, P. (2004b). Monotone measures and almost global stability of dynamical systems, *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, Katholieke Universiteit Leuven, July, Leuven, Belgium, TP3-3.

Monzón,P. (2004c). Monotone measures for dynamical systems, *Congresso Brasileiro de Automática*, Gramado, Brasil.

Monzón P. (2005a). Almost global attraction in planar systems, *Systems and Control Letters* **42**(3): 753-758.

Monzón,P.; Paganini F. (2005b). Global considerations on the Kuramoto model of sinusoidally coupled oscillators, *Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference*, Sevilla, pp. 3923-3928.

Moshtagh N,; Jadbabaie A.; Daniilidis K. (2005). Distributed Geodesic Control Laws for Flocking of Nonholonomic Agents, *Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference*, Sevilla, pp. 2835-2838.

Paganini F. and Kofman D. (1989). *Phase-Locked Loop*, Internal Report, Facultad de Ingeniería-Udelar, Uruguay.

Perelman G.(2003). *Ricci Flow and Geometrization of Three-Manifolds*, Massachusetts Institute of Technology, Department of Mathematics, Simons Lecture Series.

Perko, L. (1991). *Differential Equations and Dynamical Systems*, Text in Applied Mathematics 7, Springer-Verlag.

Prajna, S.; Parrilo, P. Rantzer, A. (2004). Nonlinear control synthesis by convex optimization, *Trans. on Automatic Control* **41**(5): 310–314.

Rantzer, A. (2001a). A dual to Lyapunov's stability theorem, *Systems and Control Letters* **42**(3): 161–168.

Rantzer, A. Ceraggioli F. (2001b). Smooth blending of nonlinear controllers using density functions, In *Proceedings of the European Control Conference*.

Rantzer, A. (2002). A converse Theorem for density functions, *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas. pp. 1890-1891.

Rantzer, A.; Prajna S. (2003). On homogeneous density functions, *in* A. Rantzer and C. Byrnes (eds), *Directions in Mathematical Systems Theory and Control*, Springer Verlag, Berlin Heidelberg, pp. 261–274. ISBN 3-540-00065-8.

Rogge, J. A.; Aeyels, R. D. (2004). Stability of phase-locking solutions in a ring of unidirectionally coupled oscillators, *Journal of Physics A: Mathematical and General* **37**: 11135–11148.

Roxin, E. and de Spinadel, V. W. (1976). *Ordinary Differential Equations*, Editorial Universitaria, Buenos Aires.

Rudin, W. (1991). *Functional Analysis*, 2nd edn, McGraw-Hill.

Sepulchre, R. and Aeyels, D. (1996). Homogeneous Lyapunov functions and necessary conditions for stabilization., *Math. Control Signals and Systems.* **9**: 34–58.

Smale, S. (1962). On the structure of manifolds, *Amer. J. of Math.* **84**: 387–399.

Strogatz S. (1994). *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*, Perseus Book Publishing.

Strogatz, S. H. (2000). From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled nonlinear oscillators, *Physica D* (143): 1–20.

Van der Pol B. (1934). The nonlinear theory of electric oscillations, *Proc. of the Institute of Radio Engineers* **22** (9): 1051–1086.

York, R.A. (1993). Nonlinear Analysis of Phase Relationships in Quasi-Optical Oscillator Arrays, *IEEE Trans. on Microwave Theory and Techniques* **41**(10): 1799–1808.

Zadeh, L. A.; Desoer, C. A. (1963). *Linear System Theory*, McGraw-Hill, NY.