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Recursive Variance Reduction in Reliability Analysis

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Abstract

Network reliability deals with reliability metrics of large classes of multicomponent systems. Recursive Variance Reduction (RVR) is a powerful pointwise estimation method, widely applied in network reliability analysis. In this paper, RVR is extended to arbitrary Stochastic Binary Systems, with minor requirements. Additionally, its variance is again lower than Crude Monte Carlo (CMC), in this general context.

Keywords: Stochastic Binary System, Network Reliability, Recursive Variance Reduction, Crude Monte Carlo.

1 Motivation

RVR is a powerful method for network reliability estimation, awarded by the scientific community in 1995 [1]. Historically, its applicability has been diverse, always in the context of networks. In this paper, we generalize RVR concept to Stochastic Binary Systems (SBS). The proofs are precisely the ones from [1], with minor changes in notation.

2 Global Notations and Definitions

- $\mathcal{G} = (\mathcal{E}, p, \Phi)$: SBS;
- $\mathcal{E}$: elements of $\mathcal{G}$, also called *links*;
- m: the number of elements of $\mathcal{E}$;
- $\mathcal{G} - l$: the SBS $(\mathcal{E} - l, p, \Phi_{-l})$ with $\Phi_{-l}(X) = \Phi(X)$; in order to simplify the notation we will write Φ instead of Φ_{-l} if no confusion arise.
- $\mathcal{G}/l$: the SBS $(\mathcal{E} - l, p, \Phi_{/l})$ with $\Phi_{/l}(X) = \Phi(X \cup \{l\})$; in order to simplify the notation we will write Φ instead of $\Phi_{/l}$ if no confusion arise.

- A subset $\mathcal{E}^{'}$ of $\mathcal{E}$ is said to be a *pathset* if $\Phi(\mathcal{E}^{'}) = 1$.
- A subset C of $\mathcal{E}$ is a *cutset* of $\mathcal{G}$ if $\Phi(C) = 0$;
- $|A|$: the cardinality of the set A;
- x_l: the binary random variable "state of link l", defined by

$$x_l = \begin{cases} 1 & \text{if link } l \text{ is up (operational),} \\ 0 & \text{if link } l \text{ is down (failed);} \end{cases}$$

- q_l: the failure probability of link l, that is, $q_l = \Pr\{x_l = 0\}$;
- $\overline{x}$ denotes the real $1 - x$;
- $X = (x_1, \ldots, x_m)$: the random SBS state vector;
- $\mathcal{G}_X$: the subset of $\mathcal{E}$ obtained by removing each failed link in X;
- $R(\mathcal{G}) = \Pr\{\mathcal{G}_X \text{ is a path set}\}$: the *reliability* of the SBS;
- $Y = 1 - \Phi(X)$;
- $Q(\mathcal{G}) = \Pr\{Y = 1\} = \mathrm{E}\{Y\}$: the *unreliability* parameter of SBS $\mathcal{G}$;
- $\widehat{W}$ denotes the sample mean based on r.v. W:

$$\widehat{W} = \frac{1}{N} \sum_{i=1}^{N} W^{(i)}$$

  where N is the fixed sample size and $W^{(1)}, \ldots, W^{(N)}$ are s-independent and identically distributed r.v. with distribution function of r.v. W;
- $\mathbf{1}_E$: the indicator function of the event E;
- $\overline{E}$: complementary event of the event E.

3 Crude Monte Carlo Technique

The unbiased crude Monte Carlo estimator of the unreliability parameter $Q(\mathcal{G})$ is a sample mean $\widehat{Y}$. More precisely,

$$\widehat{Y} = \frac{1}{N} \sum_{i=1}^{N} Y^{(i)} = \frac{1}{N} \sum_{i=1}^{N} (1 - \Phi(X^{(i)})) \tag{1}$$

where $X^{(1)}, \ldots, X^{(N)}$ constitute a random sample of X. The variance of this estimator is

$$\mathrm{Var}\left\{\widehat{Y}\right\} = \mathrm{Var}\{Y\}/N = Q(\mathcal{G})\overline{Q(\mathcal{G})}/N \tag{2}$$

and it is estimated by the unbiased estimator

$$\widehat{V} = \widehat{Y}(1-\widehat{Y})/(N-1) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \left(Y^{(i)} - \widehat{Y} \right)^2 . \tag{3}$$

The simulation algorithm consists of repeating independently N times the following experiment. A sample of each variable x_l is taken in order to form a sample of vector state X. An implementation of Φ is called to decide resulting subset of $\mathcal{G}$ is a pathset. The estimation of $Q(\mathcal{G})$ is the frequency of subset that are not path sets. The procedure *CMC* (for Crude Monte Carlo) can be expressed as follows:

Procedure CMC

1. Initialization : $\widehat{Y} = 0$.
2. For each experiment $n = 1, \ldots, N$ do
   - 2.1 For each link $l = 1, \ldots, m$ do
     - sample $\mathcal{U}$ from $Uniform(0,1)$;
     - If ($\mathcal{U} \in [0, q_l[$) Then $x_l = 0$ Else $x_l = 1$.
   - 2.2 Evaluate structure function $\Phi(X)$ and add $1 - \Phi(X)$ to $\widehat{Y}$.
3. Compute the estimate of $Q(\mathcal{G})$: $\widehat{Y} = \widehat{Y}/N$.
4. Compute the estimate of $\mathrm{Var}\left\{\widehat{Y}\right\}$: $\widehat{V} = \widehat{Y}(1-\widehat{Y})/(N-1)$.

4 Recursive Variance Reduction Algorithm

4.1 Basics

An unbiased estimator $\widehat{U}$ is more accurate than another unbiased estimator $\widehat{V}$ iff the variance of $\widehat{U}$ is smaller than $\widehat{V}$ [2]. If $\widehat{U}$ and $\widehat{V}$ are sample mean estimators we obtain

$$\mathrm{Var}\left\{\widehat{U}\right\} = \mathrm{Var}\left\{U\right\}/N \text{ and } \mathrm{Var}\left\{\widehat{V}\right\} = \mathrm{Var}\left\{V\right\}/N.$$

Consequently, it suffices to look at the variances of r.v. U and V to compare $\widehat{U}$ and $\widehat{V}$. In particular, a sample mean based on a r.v. that has same expectation as Y and smaller variance is a more accurate estimator than the standard (crude) one.

In first step of the proposed method we show how to obtain a r.v. with same expectation as r.v. Y and smaller variance.

This r.v. will be expressed as a function of the probability that the c elements of a given cutset in $\mathcal{G}$ are down and of c r.v. $Y_1, \ldots, Y_c$. Each Y_i is such that $\mathrm{E}\left\{Y_i\right\} = Q(\mathcal{G}_i)$ where $\mathcal{G}_i$ is a SBS smaller than $\mathcal{G}$. The following proposition formalizes this step.

Let Z be a r.v defined by

$$Z = Q_C + \overline{Q_C} \sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i \tag{4}$$

where

- $C = \{l_1, l_2, \ldots, l_{|C|}\}$ is a fixed cutset in $\mathcal{G}$;
- Q_C is the probability of event A_C which is the event that all elements in C are in failed state;
- for $1 \leq i \leq |C|$, $Y_i = 1 - \Phi(X_i)$ and X_i is the random vector state defined by
  $$\Pr\{X_i = x\} = \Pr\{(X = x)|B_i\}, \text{ for } x \in \{0,1\}^m$$
  where B_1 denotes the event that link l_1 is up and for $2 \leq i \leq |C|$, B_i the event that all elements in set $\{l_1, l_2, \ldots, l_{i-1}\}$ are failed and l_i is operational;
- $(J_i)_{1 \leq i \leq |C|}$ is a sequence of disjoint intervals whose union gives $[0,1]$, the length of each J_i being $\Pr\{B_i\} / \overline{Q_C}$;
- $\mathcal{U}$ is a random variable with uniform distribution on $[0,1]$ s-independent of all previously defined r.v.

Then Z has same expectation as Y and smaller variance. More precisely, we have

$$\mathrm{E}\{Z\} = Q(\mathcal{G}) = \mathrm{E}\{Y\} \tag{5}$$

$$\mathrm{Var}\{Z\} = (Q(\mathcal{G}) - Q_C)\overline{Q(\mathcal{G})} \leq Q(\mathcal{G})\overline{Q(\mathcal{G})} = \mathrm{Var}\{Y\}. \tag{6}$$

The following remarks will help us through the remainder of this work.

- $(B_i)_{1 \leq i \leq |C|}$, form a partition of $\overline{A_C}$ and consequently
  $$\sum_{i=1}^{i=|C|} \Pr\{B_i\} / \overline{Q_C} = \sum_{i=1}^{i=|C|} \Pr\{(\mathcal{U} \in J_i)\} = 1.$$
- Each of the random variables Y_i defined in the previous proposition verifies
  $$\mathrm{E}\{Y_i\} = \mathrm{E}\{1 - \Phi(X_i)\} = \mathrm{E}\{(1 - \Phi(X))|B_i\} = \mathrm{E}\{Y|B_i\}.$$
  Since B_i is the event that all components in set $\{l_1, l_2, \ldots, l_{i-1}\}$ are failed and l_i is operational,
  $$\mathrm{E}\{Y_i\} = Q(\mathcal{G}_i)$$
  where $\mathcal{G}_i = (\mathcal{G} - l_1 - \ldots - l_{i-1}) / l_i$.

The proof of Proposition 4.1 is given in the Appendix. The following proposition shows that if we replace in Z each Y_i by a r.v. $Y_i^{'}$ that has same expectation and at most same variance, the resulting r.v. Z' has variance smaller or equal to Z.

With same definitions and notations as Proposition 4.1 and $|C|$ random variables $Y_1^{'}, Y_2^{'}, \ldots Y_{|C|}^{'}$ such that

- $\mathrm{E}\left\{Y_i^{'}\right\} = \mathrm{E}\left\{Y_i\right\}$;
- $\mathrm{Var}\left\{Y_i^{'}\right\} \leq \mathrm{Var}\left\{Y_i\right\}$;
- $Y_i^{'}$ is s-independent of $\mathcal{U}$, for i, $i = 1, \ldots, |C|$,

the random variable $Z^{'}$

$$Z^{'} = Q_C + \overline{Q_C} \sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i^{'} \tag{7}$$

has same expectation as Z and smaller or equal variance. More precisely, we have

$$\mathrm{E}\left\{Z^{'}\right\} = Q(\mathcal{G}) = \mathrm{E}\left\{Y\right\} \tag{8}$$

$$\mathrm{Var}\left\{Z^{'}\right\} \leq \mathrm{Var}\left\{Z\right\} = (Q(\mathcal{G}) - Q_C)\overline{Q(\mathcal{G})} \leq Q(\mathcal{G})\overline{Q(\mathcal{G})} = \mathrm{Var}\left\{Y\right\}. \tag{9}$$

The previous proposition (proved in the Appendix) implies that the sample mean $\widehat{Z'}$ based on Z' is more accurate than $\widehat{Z}$. Next we present the process of recursively applying the previous ideas to obtain an even more accurate estimator.

Let Y_i', $1 \leq i \leq |C|$, be as follows:

$$Y_i' \text{ is } \begin{cases} 1 & \text{if } \Phi(\mathcal{E}_i) = 0; \\ 0 & \text{if } \Phi(\emptyset) = 1; \\ Z_i & \text{otherwise,} \end{cases}$$

where Z_i is a r.v. constructed from Y_i by the same process as was used in Proposition 4.1 to obtain a r.v. Z with same expectation as Y and smaller variance. It is clear that the r.v. Y_i', $1 \leq i \leq |C|$, verify the conditions of Proposition 4.1. We can then apply this proposition to affirm that the r.v. $Z^{'}$,

$$Z^{'} = Q_C + \overline{Q_C} \sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i^{'}$$

has same expectation as Y, smaller variance than Z and consequently than Y.

If for each i, $1 \leq i \leq |C|$, Y_i' is a constant value 1 or 0, the process terminates. Otherwise, for all r.v. $Y_i' = Z_i$, we can apply the same idea recursively in order to construct a r.v that has same expectation and smaller variance. When all resulting systems are either up or down independently of links state the process terminates.

The recursive operator $F()$ that associates to $\mathcal{G} = (\mathcal{E}, p, \Phi)$ the r.v. resulting from the process described above to build an estimate of $Q(\mathcal{G})$ can be expressed as follows:

$$F(\mathcal{G}) = \begin{cases} 1 & \text{if } \mathcal{E} \text{ is a pathset ;} \\ 0 & \text{if } \mathcal{E} \text{ is a cutset;} \\ Q_{C(\mathcal{G})} + \overline{Q_{C(\mathcal{G})}} \displaystyle\sum_{i=1}^{|C(C(\mathcal{G}))|} \mathbf{1}_{(\mathcal{U}(C(\mathcal{G})) \in J_i)} F(\mathcal{G}_i) & \text{otherwise,} \end{cases} \tag{10}$$

where

- $C(\mathcal{G}) = \{l_1, l_2, \ldots, l_{|C(\mathcal{G})|}\}$ is a fixed cutset in $\mathcal{G}$;
- $Q_{C(\mathcal{G})}$ is the probability that all components in $C(\mathcal{G})$ are in failed state;
- $(J_i)_{1 \le i \le |C(\mathcal{G})|}$ is a sequence of disjoint intervals whose union gives $[0, 1]$, the length of each J_i being $\Pr\{B_i\} / \overline{Q_{C(\mathcal{G})}}$, where B_i denotes the event that all components in set $\{l_1, l_2, \ldots, l_{i-1}\}$ are failed and component l_i is operational;
- $\mathcal{G}_i = (\mathcal{G} - l_1 - \ldots - l_{i-1}) / l_i$;
- $\mathcal{U}(\mathcal{G})$ is a random variable with uniform distribution on $[0, 1]$, s-independent of all previously used r.v.

The resulting unbiased estimator of $Q(\mathcal{G})$ is

$$\widehat{F}(\mathcal{G}) = \sum_{i=1}^{N} F^{(i)}(\mathcal{G})/N \tag{11}$$

where $F^{(1)}(\mathcal{G}), \ldots, F^{(N)}(\mathcal{G})$ constitute a random sample of $F(\mathcal{G})$. The variance of this estimator is

$$\operatorname{Var}\left\{\widehat{F}(\mathcal{G})\right\} = \operatorname{Var}\{F(\mathcal{G})\} / N. \tag{12}$$

It is estimated by the unbiased estimator

$$\widehat{V}_F = \sum_{i=1}^{N} (F^{(i)}(\mathcal{G}) - \widehat{F}(\mathcal{G}))^2 / ((N-1)N). \tag{13}$$

(i) The exact value of the variance of the crude estimator $\widehat{Y}$ (1) is $Q(\mathcal{G})\overline{Q(\mathcal{G})}/N$ (2). For the new estimator $\widehat{F}(\mathcal{G})$ (11), we know an upper bound of its variance, that is by relation (9) of proposition (4.1) equal to $(Q(\mathcal{G}) - Q_{C(\mathcal{G})})\overline{Q(\mathcal{G})}/N$. It results that an estimate of $Q(\mathcal{G})$ by $\widehat{F}(\mathcal{G})$ is at least

$Q(\mathcal{G})/(Q(\mathcal{G}) - Q_{C(\mathcal{G})})$ times more efficient than an estimate by $\widehat{Y}$. The value $Q(\mathcal{G})/(Q(\mathcal{G}) - Q_{C(\mathcal{G})})$ is then a lower bound of the efficiency of the method. This bound, always greater than 1, depends on the chosen cutset and it is maximal when we use the cutset which has the largest probability $Q_{C(\mathcal{G})}$ of having all of its components failed.

(ii) Suppose that we have to estimate a parameter $Q(\mathcal{G}')$ where $\Phi_{\mathcal{G}'}(S) = 0$ if $\Phi_{\mathcal{G}}(S) = 0$. Then, the cutset $C(\mathcal{G})$ of $\mathcal{G}$ chosen to start the construction of r.v. $F(\mathcal{G})$ (10) is also a cutset of $\mathcal{G}'$. Consequently, it can be used to start the construction of r.v. $F(\mathcal{G}')$ in order to estimate $Q(\mathcal{G}')$. In this case, an estimate of $Q(\mathcal{G}')$ by $\widehat{F}(\mathcal{G}')$ is at least $Q(\mathcal{G}')/(Q(\mathcal{G}') - Q_{C(\mathcal{G})})$ times more efficient than an estimate by the crude method.

4.2 Implementation

In this subsection we give the recursive algorithm that corresponds to the proposed method when the cutset chosen at each step is the set of first links that make Φ null.

The main procedure RVR (for Recursive Variance Reduction) consists of a loop that calls N times the recursive procedure F and collects the returned values to give an estimate of the desired measure $Q(\mathcal{G})$.

Procedure RVR

Input: SBS $\mathcal{G}$
Output: an estimate $\widehat{F}(\mathcal{G})$ of $Q(\mathcal{G})$ and an estimate $\widehat{V}$ of its variance

1. Main loop: For $i = 1, \ldots, N$ $F^{(i)} = F(\mathcal{G})$.
2. Compute $Q(\mathcal{G})$ estimate: $\widehat{F}(\mathcal{G}) = \sum_{i=1}^{N} F^{(i)}/N$.
3. Compute variance estimate: $\widehat{V}_F = \sum_{i=1}^{N} (F^{(i)} - \widehat{F}(\mathcal{G}))^2/((N-1)N)$.

Procedure $F()$ embodies the proposed recursive variance reduction scheme. When this procedure is called with parameters $\mathcal{G}$, it gives a pseudo-random trial of the variable $F(\mathcal{G})$ defined in (10).

In step 1 of this procedure the end recursion conditions are checked. More precisely, if Φ is 1 in the empty set return 0 (step 1.1) and if $\mathcal{E}$ is a cutset we return 1 (step 1.2). When neither of these conditions are fulfilled, we can always find at least one cutset of $\mathcal{G}$. In step 2 the chosen cutset C is, for simplicity, the first composed that make Φ null. In the third step, we compute the probability that all components in C are failed.

Since the uniformly distributed value $\mathcal{U}$ will belong only to a single interval J_i among all the possible J_k, $1 \leq k \leq |C|$, we only make a recursive call for the corresponding $\mathcal{G}_i$. In step 4 an uniform r.v. is used to determine a J_i, and in step 5 the corresponding graph and target set K_i are computed. Step 6 returns the value for $F(\mathcal{G})$ (after making a recursive call).

Procedure $F(\mathcal{G})$

Input: network $\mathcal{G}$,
Output: a random sample of r.v. $F(\mathcal{G})$

1. Check end recursion condition:
    1.1. Check if $\mathcal{G}$ is always connected: If $\Phi(\emptyset) = 1$ return(0).
    1.2. Check if the SBS is never a pathset: If $\Phi(\mathcal{E}) = 0$ return(1).
2. Find a cutset C : $C = \{l_1, \ldots, l_{|C|}\}$ the set of all links such that $\Phi(C) = 0$ and $\Phi(\{l_1, \ldots, l_i) = 1$ for $i < |C|$.
3. Compute the probability that all components in C are failed: $Q_C = \prod_{i=1}^{|C|} q_{l_i}$.
4. Sample $\mathcal{U}$ from $Uniform(0, 1)$ and select J_i such that $\mathcal{U} \in J_i$
5. Construct the corresponding network : $\mathcal{G}_i = (G - l_1 - l_2 - \ldots - l_{i-1})/l_i$.
6. Recursive step: return$(Q_C + \overline{Q_C} \times F(\mathcal{G}_i))$.

Let's do a quick calculation of the complexity of this algorithm in terms of evaluation of Φ. Step 1.1 and 1.2 is done in $O(1)$ time. Steps 2, 3 and 4 are clearly $O(|\mathcal{E}|)$. Step 5, the computation of $\mathcal{G}_i$, takes $O(1)$. Step 6 is the recursive call, which calls $F()$ for a single subgraph $\mathcal{G}_i$. Then the added complexity of all operations in the body of $F()$ previous to the recursive call is $O(|\mathcal{E}|)$. As the recursion depth is bounded by $|\mathcal{E}|$ (because at each recursive step the number of links of the considered is diminished at least by one), we have that the total complexity of a call to $F(\mathcal{G})$ is $O(|\mathcal{E}|^2)$.

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Appendix

Proof of Proposition 4.1.

(a) Since A_C and B_i, $1 \leq i \leq |C|$, are collectively exhaustive and mutually exclusive, the total expectation theorem [2] gives

$$Q(\mathcal{G}) = \mathrm{E}\left\{Y\right\} \quad = \quad Q_C \mathrm{E}\left\{Y|A_C\right\} + \sum_{i=1}^{i=|C|} \Pr\left\{B_i\right\} \mathrm{E}\left\{Y|B_i\right\}.$$

Since $\Pr\left\{Y = 0|A_C\right\} = 0$, we obtain $\mathrm{E}\left\{Y|A_C\right\} = 1$ and it results that

$$\begin{aligned} \mathrm{E}\left\{Y\right\} &= Q_C + \overline{Q_C} \sum_{i=1}^{i=|C|} \frac{\Pr\left\{B_i\right\}}{\overline{Q_C}} \mathrm{E}\left\{Y|B_i\right\} \\ &= Q_C + \overline{Q_C} \sum_{i=1}^{i=|C|} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U} \in J_i)}\right\} \mathrm{E}\left\{Y|B_i\right\}. \end{aligned}$$

Since $\mathcal{U}$ is s-independent of Y_i, for any i, we obtain

$$\mathrm{E}\left\{Y\right\} = Q_C + \overline{Q_C} \mathrm{E}\left\{\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i\right\} = \mathrm{E}\left\{Z\right\}. \tag{14}$$

(b) The variance of Z is

$$\mathrm{Var}\left\{Z\right\} \quad = \quad \overline{Q_C}^2 \mathrm{Var}\left\{\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i\right\}.$$

Since $\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i$ is a Bernoulli r.v. we have

$$\mathrm{Var}\left\{Z\right\} = \overline{Q_C}^2 \mathrm{E}\left\{\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i\right\} \left(1 - \mathrm{E}\left\{\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i\right\}\right). \tag{15}$$

By (14) we have

$$\mathrm{E}\left\{\sum_{i=1}^{i=|C|} \mathbf{1}_{(\mathcal{U} \in J_i)} Y_i\right\} = \frac{\mathrm{E}\left\{Y\right\} - Q_C}{1 - Q_C} = \frac{Q(\mathcal{G}) - Q_C}{1 - Q_C}.$$

By replacing in (15), we obtain

$$\mathrm{Var}\left\{Z\right\} = (Q(\mathcal{G}) - Q_C)\overline{Q(\mathcal{G})} \leq Q(\mathcal{G})\overline{Q(\mathcal{G})} = \mathrm{Var}\left\{Y\right\}. \tag{16}$$

Proof of Proposition 4.1.

(a) Since $\mathrm{E}\left\{Y_i^{'}\right\} = \mathrm{E}\left\{Y_i\right\}$ for each i, $1 \leq i \leq |C|$, we obtain $\mathrm{E}\left\{Z^{'}\right\} = \mathrm{E}\left\{Z\right\}$.

(b) We have

$$
\begin{aligned}
\mathrm{Var}\left\{Z^{'}\right\} - \mathrm{Var}\left\{Z\right\} &= \overline{Q_C}^2 \Big(\sum_{i=1}^{i=|C|} (\mathrm{Var}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}\right\} - \mathrm{Var}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i\right\}) \\
&+ 2 \sum_{1\leq i<j\leq |C|} (\mathrm{Cov}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}, \mathbf{1}_{(\mathcal{U}\in J_j)} Y_j^{'}\right\} \\
&- \mathrm{Cov}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i, \mathbf{1}_{(\mathcal{U}\in J_j)} Y_j\right\})\Big).
\end{aligned}
$$

It suffices to show that for any i,

$$
\mathrm{Var}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}\right\} \leq \mathrm{Var}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i\right\} \tag{17}
$$

and

$$
\mathrm{Cov}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}, \mathbf{1}_{(\mathcal{U}\in J_j)} Y_j^{'}\right\} = \mathrm{Cov}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i, \mathbf{1}_{(\mathcal{U}\in J_j)} Y_j\right\}. \tag{18}
$$

As by hypothesis $\mathrm{Var}\left\{Y_i^{'}\right\} \leq \mathrm{Var}\left\{Y_i\right\}$ and $\mathrm{E}\left\{Y_i^{'}\right\} = \mathrm{E}\left\{Y_i\right\}$ we have

$$
\mathrm{E}\left\{Y_i^{'2}\right\} \leq \mathrm{E}\left\{Y_i^2\right\}. \tag{19}
$$

By the independence between r.v. $\mathcal{U}$ and $Y_i^{'}$ and relation (19) we obtain

$$
\mathrm{E}\left\{\mathbf{1}^2_{(\mathcal{U}\in J_i)} Y_i^{'2}\right\} = \mathrm{E}\left\{\mathbf{1}^2_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i^{'2}\right\} \leq \mathrm{E}\left\{\mathbf{1}^2_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i^2\right\}. \tag{20}
$$

Since r.v. $Y_i^{'}$ and Y_i have same expectation we obtain

$$
\mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\}^2 \mathrm{E}\left\{Y_i^{'}\right\}^2 = \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\}^2 \mathrm{E}\left\{Y_i\right\}^2. \tag{21}
$$

Finally relations (20) and (21) imply (17). It remains to prove relation (18). Since, for $i \neq j$,

$$
\Pr\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} \mathbf{1}_{(\mathcal{U}\in J_j)} = 1\right\} = 0
$$

we have

$$
\mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} \mathbf{1}_{(\mathcal{U}\in J_j)} Y_i^{'} Y_j^{'}\right\} = \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} \mathbf{1}_{(\mathcal{U}\in J_j)} Y_i Y_j\right\} = 0. \tag{22}
$$

By using independence between $\mathcal{U}$ and each $Y_i^{'}$, for $i = 1, \ldots, |C|$, we obtain

$$\mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)} Y_j^{'}\right\} = \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i^{'}\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)}\right\} \mathrm{E}\left\{Y_j^{'}\right\}. \tag{23}$$

In relation (23), if we replace $\mathrm{E}\left\{Y_k^{'}\right\}$ by $\mathrm{E}\left\{Y_k\right\}$ for $k = i, j$ we have

$$\mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i^{'}\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)}\right\} \mathrm{E}\left\{Y_j^{'}\right\} = \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)}\right\} \mathrm{E}\left\{Y_j\right\}. \tag{24}$$

Finally by independence between $\mathcal{U}$ and r.v. Y_i, for $i = 1, \ldots, |C|$, we obtain

$$\begin{aligned}
\mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i^{'}\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)} Y_j^{'}\right\} &= \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)}\right\} \mathrm{E}\left\{Y_i\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)}\right\} \mathrm{E}\left\{Y_j\right\} \\
&= \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_i)} Y_i\right\} \mathrm{E}\left\{\mathbf{1}_{(\mathcal{U}\in J_j)} Y_j\right\}.
\end{aligned}$$

The last result and relation (22) imply (18).