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Testing GeoGebra as an effective tool to improve the understanding of the concept of limit on engineering students

Testing GeoGebra as an effective tool to improve the understanding of the concept of limit on engineering students

J C Aréchiga Maravillas[1], S Arceo Díaz[1] J Salazar-Torres[2], and D A Sierra Andrade[1]

[1] Instituto Tecnológico de Colima, Tecnológico Nacional de México, México
[2] Facultad de Ciencias Básicas y Biomédicas, Universidad Simón Bolívar, Colombia

E-mail: santiago.arceo@itcolima.edu.mx

Abstract. The impact GeoGebra on the teaching of the concept of limit was analyzed. Two groups of engineering students, studying differential calculus, served as control and test groups. The traditional teaching, based on examples solved by hand, was given to the control group while a series of activities involving the usage of the mathematical software GeoGebra were applied in an attempt of improving the degree of assimilation on the concept of limits.

1. Introduction

The study of Calculus is one of the most important topics for engineering students and, within it, the concept of limit is necessary for them to learn the definition of the derivative of a function. Evidence collected in past years points that the current way in which students learn about who to calculate the limit of a function is, mostly, based on the mechanical repetition of well defined guidelines and algebraic manipulations, without allowing the time necessary for them to develop a significant learning on the concepts (most of them are not even capable of giving an informal definition on the concept of limits). This lack of a true understanding on the mathematical definition of limits later causes difficulties for students to fully understand other very important concepts related to calculus, such as the definition of the integral as the limit of a Riemann sum.

Other authors on the past have employed GeoGebra to help students to understand mathemathical concepts [1, 2]. Quinlan [3] created an applet that allows students to calculate the limit of a function by means of the graph of a succession of numbers. Criscuolo and Gnudi [4] describe the results of applying GeoGebra to middle school students during a course whose objective was helping them to develop their mathematical and metha-cognitive ability. Their conclusion suggests that GeoGebra represents a very useful tool in subverting the typical disadvantages of students in mathematical learning, as the difficulty of visualizing the function's plots or lack of ability to follow the algebraic steps without making errors in the calculation. Mustafa [5] applied a strategy to test the effectivity of GeoGebra as a tool to help students to develop their mathematical ability in evaluating limits and continuity of different functions, the results over different test done by students using the software were compared to a control group that used more traditional procedures. The results of this study showed that students using

GeoGebra got better average scores than those within the control group. Something similar was done by Bustos [6] with high school students, adding a pre-evaluation of mathematical ability from which it was obtained that both the control and test groups were equally proficient, and got to the same conclusion that pointed out GeoGebra is effective in improving the comprehension on the concept of limits and the ability of students to calculate them.

Similarly, some other software have been used to teach students the concepts and mathematical procedures related to the limit of a function. Macias and Silva [7] employed a didactic strategy based on directed test using the software Derive on two randomly selected groups of students, control and test. Each test was composed by four sections, showing the objective, instructions and two blank spaces in which students could write down their results and conclusions, and involved the conceptual learning of limits and the development of algebraic calculations and plots. The statistical validity of results was tested by employing Fisher's test, with an significance of 5% and these results suggested that there was enough evidence to conclude on the improvement in results of the test group. Pantoja, López, Ortega and Hernadez [8] used the software WinPlot and a selection of audiovisual tools within an Instructional Design directed to teach students of four different schools on the concepts of limits and the continuity of functions and their results suggested that these tools can be seen as an effective learning tool.

Following all these previous studies, this work proposes the design and application of a compendium of activities that have as main objective to teach the concept of the limit of a function by using GeoGebra to engineering students of the Tecnológico Nacional de México. Each of the activities described below introduces the limit of a function as a succession of approximations on the values of a function, as the independent variable tends to a desired value and, afterwards, presents the formal definition of limit to the student using the concept of mathematical neighborhood. This approach to the concept of limit was taught to a test group of students in mechatronic engineering and their scores were compared against a control group of students of the same career. The working hypothesis is that students in the test group, using GeoGebra as a learning tool, would show better scores that those students in the control group. The statistical validity of results was tested by a statistical Z-test and by the bootstrap technique. The methodology and results are shown below.

2. Methodology

The concept of limit was introduced by Newton and Leibnitz during the XVII Century [9]. In a very informal way, the limit of a function is can be defined by the Equation (1).

$$\lim_{x \to p} f(x) = L, \tag{1}$$

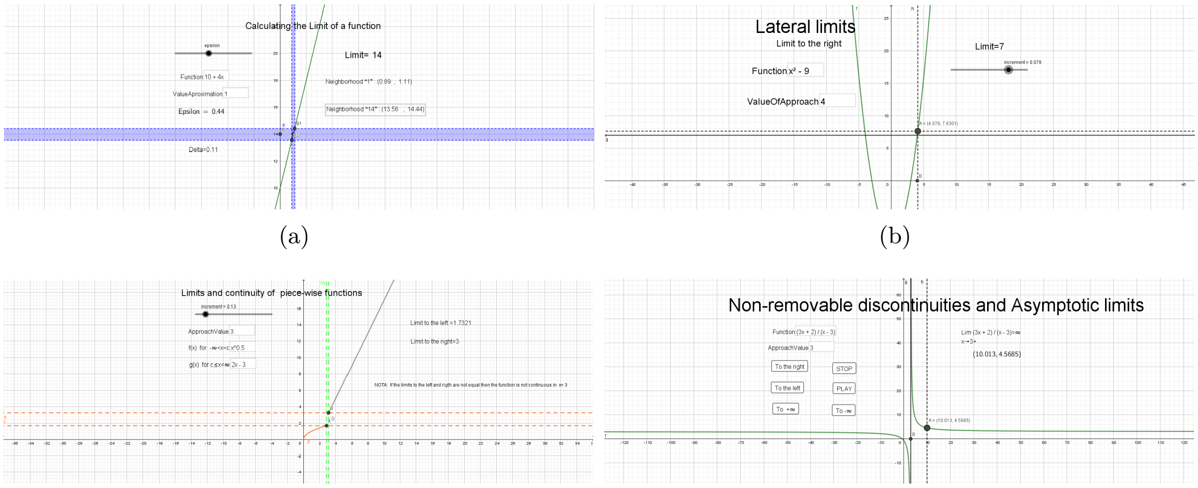
that reads the value of $f(x)$ becomes arbitrarily close to L, as the value of its independent variable, x becomes arbitrarily close to p. The authors of this study decided that the concept of limit would be presented to students in a very intuitive way, in which the usage of numerical examples and plots of each function could illustrate how $f(x)$ gets near to L as x approaches to p (see Figure 1). The methodology followed is described below.

The study was based on a quasi-experimental design [10] composed by two groups: a control group and a test group (see Figure 1). The teaching of the limit concept for the control group was approached in a traditional way, while the teaching of this concept for the test group was carried out by GeoGebra (Figure 1).

First, a diagnostic test was designed as an instrument that could be used to determine the degree of similarity in the mathematical proficiency of students. The specific mathematical abilities that are covered by the diagnostic test are:

- Basic arithmetic and algebraic ability.

- Comprehension about the concept of inequation and the absolute value of a function.
- Comprehension, and capacity of calculating, the inverse of a given function.



(a) (b) (c) (d)

Figure 1. Screenshots from each of the practices designed for the test group, with the exception of practice 2 due to its similarity to practice 1. On each case, the student is given a function and an approach value. The student can visualize what the limit of the function will be by moving the slider from left to right. The black dots mark the value of x and its corresponding f(x), while the dashed lines represent their absolute errors. (a) Practice 1 and Practice 2: Defining the limit of a continuous function. (b) Practice 3: Defining the side limits of continuous function. (c) Practice: Limits and continuity of piece-wise functions. (d) Practice 5: Functions with infinite limits.

The diagnostic test was comprised by questions of increasing difficulty, equally distributed in number, to ensure that each of the mathematical abilities mentioned above were given equal importance.

While the concept of limits and its calculation on a variety of cases was taught in the traditional way to students within the control group, students within the test group where introduced to it by doing five different guided practices (Figure 1(a) to Figure 1(d)) [11]:

- Basic notion and definition of the limit of a function: The purpose and basic usage of GeoGebra is presented to students and the concept and definition of the limit of a function is explained by graphically representing the proximity of $f(x)$ to L as x tends to p. (Figure 1(a))
- Basic properties of limits: GeoGebra is used to define multiple functions and their graphs and their limits are numerically calculated. Afterwards the limit of several combinations of functions are calculated to illustrate each of the properties of limits (Figure 1(a)).
- Lateral limits: By means of graphs and numerical calculations done in GeoGebra, it is taught to the student how the value of the function $f(x)$ approaches to its limit L, independently on the way in which the value of x approaches to p (either by values of x that progressively decrease until the become arbitrarily close to p or by values of x that are smaller than p but that progressively become larger), Figure 1(b)
- Infinite limits and asymptotic behavior of functions: Situations in which the calculation of a limit results in either a singularity or an infinitely growing, or decreasing, values for $f(x)$, Figure 1(c).

- Continuity of functions: The concept of a continuous function is explored and graphs and numerical calculations for continuous and discontinuous functions are done. Figure 1(d).

On each case, a collection of functions, extracted from the bibliography [12–15] were given to students. Then, by using the formula for each function as an input, it was asked to the students to determine the predicted value for the limit of f(x), as x tends to and specified value (by using the slider tool that represents the degree of approximation between x and the specified value). Afterwards, it was asked to the students to compare the predicted limit to the limit calculated analytically. Finally, each student should elaborate a personal and collective conclusion, discussed in class among students and professor, and it is asked that results are repeated by calculations done by hand.

Once the period in which each of the practices described above is done, a final exam, comprised by equally distributed open questions related to the specific topics of the five practices, was applied to students of both groups. The average score and number of successful students after this final exam are shown in next section.

3. Results

Tables 1 shows the difference in average score and the number of successful obtained by students after the final exam. The statistical relevance of two different working hypothesis was evaluated by means of the Z-test (see Figure 2) [16]:

- Hypothesis 1: The average score of the test group will be above the control group, Figure2.
- Hypothesis 2: The number of successful students after the final test of students in the test group will be larger than the number of students in the control group, Figure 2.

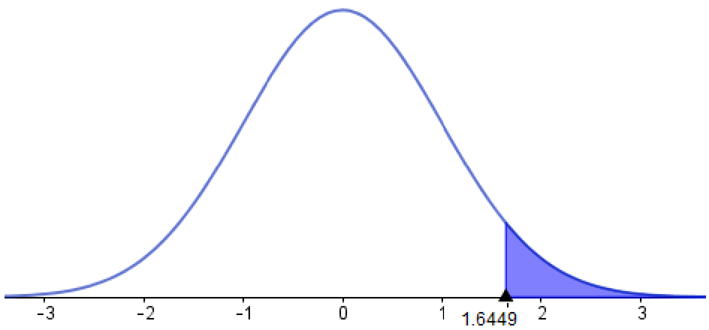


Figure 2. Z-test applied to the difference in average scores (for hypothesis 1) and the difference in the number of successful students (for hypothesis 2). In both cases the critical value for the Z-score, with a significance level of %5 is $Z_c = 1.66$.

According to the data, there was not enough evidence to support the hypothesis that the methodology that used GeoGebra induced a significant difference between the average scores of the test a control group. However, the same test, applied to the number of successful, allows to conclude that the improvement of students within the test group has some statistical relevance.

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Table 1. Number of students, mean score, standard deviation and number of successful students for the control and test groups after the final exam. The value of the Z-score obtained from the data for the difference in mean scores suggests that there is not enough evidence to support the first hypothesis ($Z = 1.21 < Z_c$), while it does support the second hypothesis ($Z = 1.88 > Z_c$).

Parameter	N	Mean	σ	Successful students
Control	42	55.21	23.75	16
Test	39	62.56	30.14	23

Due to the lack or more data, the bootstrap technique [17] was also used in an attempt to support, or deny, these results. Table 2 shows the estimated values for the mean and median after a thousand re-samplings of the original data, done with replacement. Either by comparing the mean or the median obtained for the test using the average score or the difference in the number of successful students , the scores for the control group are surpassed by those of the test group: the best estimates for the mean score of both groups show a difference of about 8 points, while the values obtained for a probability of 99% show a difference of 10 points. Even when the simulated values for the median of both groups show larger variations than those calculated for the mean score also support the assumption of improvement for the test group, as the difference of the medians from both groups, show a difference of 6 points with a probability of 99%.

Table 2. Results of the bootstrap randomization test after a thousand re-samplings of the original data.

			Control				
Parameter	Q1	Estimate	Q3	P95	P99	SD	IQR
Mean	52.57	55.21	57.43	61.07	63.83	3.53	4.86
Median	46.00	50.50	55.50	67.50	72.50	7.54	9.50
			Test				
Parameter	Q1	Estimate	Q3	P95	P99	SD	IQR
Mean	51.53	62.56	65.88	69.92	73.38	4.75	6.34
Median	70.00	70.00	73.00	75.00	78.00	4.43	3.00

4. Conclusions

The present work is a first attempt of including GeoGebra into the subject of differential calculus, for engineering students of the Tecnológico Nacional de México. The concept of the limit of a function was taught to two groups of students by either the traditional treatment (labeled as the control group) or by practices designed to use GeoGebra (the test group) and the null hypothesis (e.g that GeoGebra does not have any measurable impact on the learning of the concept of the limit of a function) was tested against the alternative hypotheses in two cases: "The average score of the test group will be above that of the control group" and that "The number of successful in the final test of students in the test group will be larger than that of students in the control group". The evidence collected by using the scores of the two groups after the final exam only supported the hypothesis that GeoGebra helped to improve the number of successful students, while no enough evidence was found for supporting that the difference in average scores between the two groups was statistically significant (although a difference of about 10 points was found after a thousand re-samplings, by using the bootstrap randomization test with a confidence level of 99%).

A future implementation of GeoGebra for teaching of the concept of the limit of a function on the near future is being planned to gather more evidence that could confirm of contradict the results presented on this work, along with the inclusion of other practices that could cover the complete list of topics within the subject of differential calculus.

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