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Differential equations with natural matrices coefficients

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Abstract. In this paper we present the main results of the master thesis in applied mathematics of the second named author, which was supervised by the first named author. Such results, and for instance this paper, concerns to some differential and algebraic results involving natural matrices. The problem of solving differential equations is very ancient and is very important to get explicit solutions of differential equations to be applied in physics and other areas. In this paper, as well in the master thesis, we study the differential and algebraic structure of linear differential equations with natural matrix coefficients and generalizations. These results are original and important for researchers interested in differential algebra and applications of differential equations.

1. Introduction

In this paper is applied by the first time the concept of natural matrix with differential equations. Combining some results of the both authors is presented this paper, the first paper in this line, with the aim of impacting the scientific community in differential algebra. The results of this research include that the differential Galois of linear differential systems with p -modified natural matrices, being p a polynomial, is isomorphic to the multiplicative group (C^*, \cdot) . This is a very important result in differential algebra because give us tools to solve more general differential equations arising in mathematical physics. For example, Hamiltonian systems with variational equations linked to natural matrices.

The concept of natural matrix was introduced in 1998 by Ramírez and Gorrostola in [1]. Further developments concerning natural matrices correspond to [2-5], where the authors used Pasting and reversing in natural numbers [6,7] as well digital root [8].

The aim of this work is to study from an algebraic point of view some properties of differential equations with natural matrices as coefficients. In particular we study the exact solvability of such differential equations as well their differential Galois group. A standard reference to do a starting point is [9], in where an explicit method to transform linear differential systems of 2×2 matrices in second order linear differential equations is presented. Further references that can help to understand the results presented here are [10-13].

The methodology presented in this paper corresponds to the scientific method in pure mathematics, that is, the validation of results corresponds to the proof of such results through direct or indirect mathematical proofs. The main aim of the paper is to show new results involving differential equations and natural matrices to determine explicit solutions and differential Galois group of such systems, In particular, in agreement with [14-16] and [17], differential Galois theory can be applied to general



dynamical systems (classic and quantum), therefore our models of differential equations with natural matrix coefficients also can be applied in classical and quantum mechanics. It is our motivation in the starting of this study, but this paper is only the starting point to achieve it.

We recall that a natural matrix on n rows and m columns is a matrix such that its components is a sequence of natural numbers from 1 to nm whether it can be written as a vector with nm components. In particular we consider square natural matrices.

On the other hand, there exists a theory to study the solvability of polynomials through the solvability of its group of symmetries, the so-called Galois group. It is well known that Galois theory is an interesting and difficult topic in abstract algebra, see [12] and references therein. The simplest Galois theory for polynomials corresponds to field of coefficients being rational numbers (the field of quotients for integers). Given a polynomial, the extension corresponds to the rational numbers adding the non-rational roots. For instance, the Galois Group of such polynomial is the group of automorphisms leaving fixed the rational numbers. Thus, the action of the Galois group for a polynomial is given by the permutations group, also called the symmetric group.

In an analogous way as for polynomials, there exist a Galois theory for linear differential equations. This theory was invented by Picard and Vessiot, for this reason the differential Galois theory is known as Picard-Vessiot theory. In contrast with classical Galois theory, the action of the differential Galois group in Picard-Vessiot theory is given by an algebraic group of matrices. This means that differential Galois group for a differential equation can be finite or not. Books and papers are devoted to differential Galois theory and its applications, for example, see [14] and [17] for applications to integrability of planar polynomial vector fields, [15] for applications in quantum mechanics and [16] for applications in Hamiltonian systems.

The simplest differential Galois theory for linear differential equations is when the differential field corresponds to the rational functions over the complex numbers (the field of quotients for polynomials with complex coefficients), The extension, called Picard-Vessiot extension, contains the nonrational solutions of the differential equation and the rational functions. Thus, the differential Galois group is the group of differential automorphisms such that restricted to the rational functions coincide with the identity function.

2. Results and contributions

In this section we present the results of the paper. We start considering linear differential 2×2 systems where the coefficient is a rationally modified natural matrix, that is, a natural matrix multiplied by a rational function. The differential system is as follows in Equation (1).

$$\frac{dx}{dt} = p(t)(x + 2y), \frac{dy}{dt} = p(t)(3x + 4y), p(t) \in \mathbb{C}(t). \quad (1)$$

We observe that $p(t)$ is a rational function in where $\mathbb{C}(t)$ represents the differential field of rational functions. The system given in Equation (1) can be transformed in the following differential equation, that is, Equation (2).

$$\frac{d^2y}{dt^2} - (5p(t) + \frac{1}{p(t)} \frac{dp(t)}{dt}) \frac{dy}{dt} - 2p(t)^2 y(t) = 0. \quad (2)$$

The differential equation given in Equation (2) can be transformed to the reduced differential equation, Equation (3).

$$\frac{d^2z}{dt^2} = r(t)z, r(t) = \frac{1}{4p(t)^2} (33p(t)^4 + \frac{3}{p(t)^2} (\frac{dp(t)}{dt})^2 - 2p(t)p''(t)) \quad (3)$$

In general, the study of solvability of these equations are not easy task. For this reason, we restrict ourselves to the case natural matrices instead of rationally modified natural matrices. That is, we consider $p(t)$ as constant (its derivative is zero).

We observe that the rank of the $n \times n$ natural matrix for $n > 1$ is 2. Thus, we arrive to the following result.

Theorem 1. The differential Galois group of a linear differential system with natural matrices coefficients is either: a subgroup of exponential torus or is a subgroup of the additive group.

Proof. Let A be a $n \times n$ natural matrix. If $n = 2$, then we obtain as differential group of differential Equations (1) and Equation (2) the exponential torus, while for differential equation given in Equation (3) we obtain the multiplicative group. Moreover, in general, for $n > 2$ this equation can be solved by quadratures using the exponential function. This means that we can compute the solutions and differential Galois group through the exponential of natural matrices, which is made via their Jordan forms. The characteristic polynomial has the form $P(\lambda) = \lambda^{n-2}Q(\lambda)$, where Q is a quadratic polynomial. Therefore, the differential Galois of the linear differential system with natural matrices coefficients is a subgroup of the exponential torus or is a subgroup of the additive group.

The result of Theorem 1 is very important for people working in differential algebra because gives an explicit characterization of the differential Galois of a linear differential equation. Researchers in differential algebra and differential Galois theory are interested in the obtaining of explicit examples of differential equations with their differential Galois group. With our Theorem 1 we provide them a plenty of examples because they depend of the polynomial that can be placed there. To illustrate the theorem 1, we present the following example.

Example 1. Consider the polynomial $p(t) = t$, then the system given in Equation (1) becomes in Equation (4) as follows.

$$\frac{dx}{dt} = tx + 2ty, \frac{dy}{dt} = (3tx + 4ty), \quad (4)$$

Now, combining Equation (2) with Equation (4) we have Equation (5).

$$\frac{d^2y}{dt^2} - (5t + \frac{1}{t}) \frac{dy}{dt} - 2t^2y(t) = 0. \quad (5)$$

Finally, combining Equation (3) with Equation (5) we obtain Equation (6) as follows.

$$\frac{d^2z}{dt^2} = r(t)z, r(t) = \frac{1}{4t^2} \left(33t^4 + \frac{3}{t^2} \right). \quad (6)$$

The basis of solutions for Equation (6) is given by Equation (7) as follows.

$$y_1 = e^{t^2(\sqrt{\{33\}+5})/4}, y_2 = e^{-t^2(\sqrt{\{33\}-5})/4}. \quad (7)$$

Thus, we observe that the differential Galois for Equation (5) and Equation (6) is the 2-dimensional exponential torus, such as was expected by the theorem 1.

3. Conclusions

In this paper we characterized the differential Galois of linear differential systems with natural matrices coefficients. We made the case with constant coefficients and the case in where the coefficients are rationally modified natural matrices should be analysed in further researches.

Some experiments were made using polynomials to modify the natural matrices, we obtained integrability through exponential functions in such cases. The interested reader can try to proof the integrability in a more general sense.

We hope that this paper can be the starting point to study classical and quantum mechanics using natural matrices. We motivate to the reader to study linear and non-linear differential systems in where variational equations can be a natural matrix. As far as we know, it is the first time in where is applied natural matrix to differential equations. For this reason, we cannot contrast this work with another ones.

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