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## Some tastings in Morales-Ramis theory

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**Abstract.** In this paper we present a short material concerning to some results in Morales-Ramis theory, which relates two different notions of integrability: Integrability of Hamiltonian systems through Liouville Arnold theorem and integrability of linear differential equations through differential Galois theory. As contribution, we obtain the abelian differential Galois group of the variational equation related to a bi-parametric Hamiltonian system.

### 1. Introduction

The Morales-Ramis theory is a powerful tool for showing the nonintegrability of Hamiltonian systems. To understand the Morales-Ramis theory, we need to introduce two different notions of integrability: the integrability of Hamiltonian systems in Liouville sense and the integrability of linear differential equations in Picard-Vessiot sense. Further developments of Morales-Ramis theory, with contributions of the first author, can be found in [1-5]. This work summarizes some results of these papers among others. Our main contribution corresponds to the obtaining the abelian differential Galois group of the variational equation related to a bi-parametric Hamiltonian system.

#### 1.1. Integrability of Hamiltonian systems

Let us consider a  $n$  degrees of freedom Hamiltonian  $H$ . The equations of the flow of the Hamiltonian system, in a system of canonical coordinates,  $x_1, \dots, x_n, y_1, \dots, y_n$  are written Equation (1).

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} \quad (1)$$

And they are known as Hamilton equations. We recall that the Poisson brackets between  $f(x_1, x_2, x_3, x_4)$  and  $g(x_1, x_2, x_3, x_4)$  is given by Equation (2).

$$\{f, g\} = \sum_{k=1}^n \left( \frac{\partial f}{\partial y_k} \frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial y_k} \right) \quad (2)$$

We say that  $f$  and  $g$  are in involution when  $\{f, g\} = 0$  also we say in this case that  $f$  and  $g$  commute under the Poisson bracket. In this way, we can write the Hamilton equations as follows:



$$\dot{x} = \{H, x\}, \quad \dot{y} = \{H, y\} \quad (3)$$

A Hamiltonian  $H$  in  $\mathbb{C}^{2n}$  is called integrable in the sense of Liouville if there exist  $n$  independent first integrals of the Hamiltonian system in involution [6,7]. We will say that  $H$  is integrable by terms of rational functions if we can find a complete set of integrals within the family of rational functions. Respectively, we can say that  $H$  is integrable by terms of meromorphic functions if we can find a complete set of integrals within the family of meromorphic functions [8,9].

We denote by  $X_H$  the Hamiltonian vector field, that is, the right-hand side of the Hamilton equations. In a general way, we deal with non-linear Hamiltonian systems. For suitability, without loss of generality, we can consider Hamiltonian systems with two degrees of freedom that is a Hamiltonian  $H$  in  $\mathbb{C}^4$ . Let  $\Gamma$  be an integral curve of  $X_H$ , being parametrized by  $\gamma: t \rightarrow (q_1(t), q_2(t), p_1(t), p_2(t))$  the first variational equation (VE) along  $\Gamma$  is given by the Equation (4).

$$\dot{\xi} = \text{Hes}(H(\gamma(t))) * \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \xi, \quad \dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4)^T, \quad \xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T \quad (4)$$

### 1.2. Picard-Vessiot theory

The Picard-Vessiot theory is the Galois theory of linear differential equations. In the classical Galois theory, the main object is a group of permutations of the roots, while in the Picard-Vessiot theory is a linear algebraic group. For other applications of the Picard Vessiot theory due to the first author can be found in [10] and [11]. In the remainder of this paper we only work, as particular case, with linear differential equations of second order (see Equation (5)).

$$y'' + ay' + by = 0, \quad a, b \in \mathbb{C}(x) \quad (5)$$

Suppose that  $y_1, y_2$  is a fundamental system of solutions of the differential equation. This means that  $y_1, y_2$  are linearly independent over  $\mathbb{C}$  and every solution is a linear combination of these two. Let  $L = \mathbb{C}(x)[y_1, y_2] = \mathbb{C}(x)[y_1, y_2, y_1', y_2']$ , that is the smallest differential field containing to  $\mathbb{C}(x)$  and  $\{y_1, y_2\}$ . The group of all differential automorphisms of  $L$  over  $\mathbb{C}(x)$  is called the Galois group of  $L$  over  $\mathbb{C}(x)$  and denoted by  $\text{Gal}(L/\mathbb{C}(x))$ . This means that for  $\sigma: L \rightarrow L$ ,  $\sigma(a') = \sigma'(a)$  and  $\sigma(a) = a, \forall a \in \mathbb{C}(x)$ . If  $\sigma \in \text{Gal}(L/\mathbb{C}(x))$  then  $\sigma y_1, \sigma y_2$  is another fundamental system of solutions of the linear differential equation. Hence there exists a matrix  $A \in \text{GL}(2, \mathbb{C})$  such that, Equation (6).

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma y_1 \\ \sigma y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (6)$$

- Theorem 1. The Galois group  $G = \text{Gal}(L/\mathbb{C}(x))$  is an algebraic subgroup of  $\text{GL}(2, \mathbb{C})$ . Moreover, the Galois group of a reduced linear differential equation  $\xi'' = r\xi$ ,  $r \in \mathbb{C}(x)$  is an algebraic subgroup of  $\text{SL}(2, \mathbb{C})$ .
- Theorem 2. A linear differential equation is solvable integrable by terms of, Liouvillian functions, if and only if the connected component of the identity element of its Galois group is a solvable group.

### 2. Morales-Ramis theory

We want to relate integrability of Hamiltonian systems to Picard-Vessiot theory. The following theorems treat this problem.

- Theorem 3. Morales-Ramis [12]. Let  $H$  be a Hamiltonian in  $\mathbb{C}^{2n}$ , and  $\gamma$  a particular solution such that the normal variational equation (NVE) has regular (resp. irregular) singularities at the points of  $\gamma$  at infinity. Then, if  $H$  is completely integrable by terms of meromorphic (resp. rational) functions, then the Identity component of Galois Group of the NVE is abelian.

To understand completely this technical result, it is required a formal study of concerning to differential Galois theory and Morales-Ramis theory. We can illustrate this theorem through the following examples, but only for a basic level. Example, consider the Hamiltonian presented in Equation (7).

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 - 2q_1^2 - 6q_1q_2^2 \quad (7)$$

The Hamiltonian equations, Equation (8), are:

$$\dot{q}_1 = p_1, \dot{q}_2 = p_2, \quad \dot{p}_1 = 6q_1^2 + 6q_2^2, \dot{p}_2 = 12q_1q_2 \quad (8)$$

Taking the invariant plane  $q_2 = p_2 = 0$  we have  $\ddot{q}_1 = 6q_1^2$  a solution for this equation is  $q_1(t) = \frac{1}{t^2}$  and the variational equation is  $12\xi_1 = t^2\ddot{\xi}_1$ , which corresponds to a Cauchy-Euler equation, thus, the Galois group is abelian due to the Hamiltonian system is integrable. The following examples were taken from [12]. Consider the Hamiltonian of the Equation (9):

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 - Q(q_1)\frac{q_2^2}{2} + \beta(q_1, q_2)q_2^3, \quad (9)$$

where  $Q(q_1)$  is a polynomial and  $\beta(q_1, q_2)$  is a function of two variables with continuous partial derivative and  $\lim_{q_2 \rightarrow 0} \frac{\partial^j \beta(q_1, q_2)}{\partial q_2} < \infty$ ,  $0 \leq j \leq 2$ . The Hamilton equations, Equation (10) and Equation (11), are:

$$\dot{q}_1 = p_1, \dot{p}_1 = Q'(q_1)\frac{q_2^2}{2} - \frac{\partial \beta(q_1, q_2)}{\partial q_1} q_2^3 \quad (10)$$

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = Q(q_1)q_2 - \frac{\partial \beta(q_1, q_2)}{\partial q_2} q_2^3 - 3\beta(q_1, q_2)q_2^2 \quad (11)$$

Taking the invariant plane  $q_2 = p_2 = 0$  we have  $q_1(t) = at + b$  and the variational equation, Equation (12), is:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix}, \quad (12)$$

where  $\theta = Q(q_1)$ , then  $Q(q_1)\xi_2 = \ddot{\xi}_2$ . If  $Q(q_1)$  is a polynomial then the Galois group is not abelian, although in some case is solvable [1,4], hence the Hamiltonian system is not integrable by Morales-Ramis Theorem. Example considerer the Hamiltonian of the Equation (13).

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 - \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 q_1)^2} + \lambda_0 - \lambda_1 q_2^2 - \lambda_2 q_1 q_2^2 - \lambda_3 q_1^2 q_2^2 + \beta(q_1, q_2)q_2^3, \quad (13)$$

where  $\lambda_i \in \mathbb{C}$ , with  $\lambda_3 \neq 0$ ,  $\beta(q_1, q_2)$  is a function of two variable with continuous partial derivative and  $\lim_{q_2 \rightarrow 0} \frac{\partial^j \beta(q_1, q_2)}{\partial q_2} < \infty$ ,  $0 \leq j \leq 2$ . The Hamilton equations, Equation (14) and Equation (15), are:

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = \frac{-4\lambda_3\lambda_4}{(\lambda_2 + 2\lambda_3q_1)^3} + \lambda_2q_2^2 + 2\lambda_3q_1q_2^2 - \frac{\partial\beta(q_1, q_2)}{\partial q_1} q_2^3 \tag{14}$$

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = 2\lambda_1q_2 + 2\lambda_2q_1q_2 + 2\lambda_3q_2q_1^2 - \frac{\partial\beta(q_1, q_2)}{\partial q_2} q_2^3 - 3\beta(q_1, q_2)q_2^2 \tag{15}$$

Taking  $q_2 = p_2 = 0$  and setting  $H(q_1, 0, p_1, 0) = h$  we see that Equation (16) and Equation (17).

$$h = \frac{1}{2}P_1^2 - \frac{\lambda_4}{(\lambda_2 + 2\lambda_3q_1)^2} + \lambda_0 \tag{16}$$

$$\dot{q}_1 = P_1 = \left(2h + \frac{\lambda_4}{(\lambda_2 + 2\lambda_3q_1)^2} - 2\lambda_0\right)^{\frac{1}{2}} \tag{17}$$

Now, we pick  $h = \lambda_0$ , thus we have Equation (18) and Equation (19).

$$\frac{dq_1}{dt} = \left(\frac{\lambda_4}{(\lambda_2 + 2\lambda_3q_1)^2}\right)^{\frac{1}{2}} \tag{18}$$

$$\lambda_2q_1 + \lambda_3q_1^2 = \pm(\lambda_4)^{\frac{1}{2}}t + c \tag{19}$$

For instance, the variational equation is, Equation (20).

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix}, \alpha = \frac{24\lambda_3^2\lambda_4}{(\lambda_2 + 2\lambda_3q_1)^4}, \beta = 2\lambda_1 + 2\lambda_2q_1 + 2\lambda_3q_1^2 \tag{20}$$

Then, Equation (21).

$$(2\lambda_1 + 2\lambda_2q_1 + 2\lambda_3q_1^2)\xi_2 = \ddot{\xi}_2, \tag{21}$$

replacing in Equation (21), we have Equation (22).

$$p(t)\xi_2 = \ddot{\xi}_2, \tag{22}$$

where  $p(t) = 2\lambda_1 + 2(\pm(\lambda_4)^{\frac{1}{2}}t + c)$  and consequently, the Galois group is not abelian hence the Hamiltonian system is not integrable. Example. Consider the following Hamiltonian, Equation (23).

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{aq_1^3 + bq_1^2q_2 + cq_2^3} \tag{23}$$

Where  $a, b, c \in \mathbb{C}$  The Hamilton equations are, Equation (24) and Equation (25):

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = \frac{3aq_1^2 + 2bq_1q_2}{(aq_1^3 + bq_1^2q_2 + cq_2^3)^2} \tag{24}$$

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = \frac{bq_1^2 + 3cq_2^2}{(aq_1^3 + bq_1^2q_2 + cq_2^3)^2}, \tag{25}$$

taking the invariant plane,  $q_1 = p_1 = 0$  we have Equation (26) and Equation (27).

$$\dot{p}_2 = \frac{3}{cq_2^4}, \quad \ddot{q}_2 = \frac{3}{cq_2^4} \tag{26}$$

$$q_2(t) = \left(-\frac{25}{2c}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \tag{27}$$

$Z(t) = (0, q_2(t), 0, p_2(t))$ , the variational equation is Equation (28).

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}, \quad \gamma = -\frac{4b}{25t^2}, \beta = \frac{24}{-25t^2} \tag{28}$$

$\dot{q}_1 = p_1, \dot{p}_1 = -\frac{4b}{25t^2}q_1$  then  $\ddot{q}_1 = -\frac{4b}{25t^2}q_1$  is a Cauchy-Euler equation, the Galois group is the identity, this group is abelian but we cannot state that the Hamiltonian system is integrable. The previous theorem of Morales-Ramis was extended by Morales-Ramis-Simó.

- Theorem 4. Morales-Ramis-Simo [13]. Let  $H$  be a Hamiltonian in  $C^{2n}$ , and  $\gamma$  a particular solution such that the NVE has regular (resp. irregular) singularities at the points of  $\gamma$  at infinity. Then, if  $H$  is completely integrable by terms of meromorphic (resp. rational) functions, then the identity component of Galois Group of any linearized high order variational equation is abelian.

The following examples illustrate the way to compute the second order variational equation considerer the Hamiltonian, Equation (29).

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}a_1q_1^2 + \frac{1}{2}a_2q_2^2 + \frac{1}{4}a_5q_1^4 + \frac{1}{4}a_3q_2^4 + \frac{1}{2}a_4q_1^2q_2^2, \tag{29}$$

where the Hamilton equations are given by Equation (30) and Equation (31).

$$\dot{q}_1 = p_1, \dot{p}_1 = -a_1q_1 - a_5q_1^3 - a_4q_1q_2^2 \tag{30}$$

$$\dot{q}_2 = p_2, \dot{p}_2 = -a_2q_2 - a_3q_2^3 - a_4q_1^2q_2 \tag{31}$$

Taking as invariant plane  $\Gamma = \{(q_1, q_2, p_1, p_2): q_2 = p_2 = 0\}$  we obtain first variational, Equation (32).

$$\dot{\xi}^{(1)} = A(t)\xi^{(1)}, \quad A(t) = \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ B_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}, \quad B_{2 \times 2} = \begin{pmatrix} c & 0 \\ 0 & \delta \end{pmatrix}, \quad c = -a_1 - 3a_5q_1^2, \tag{32}$$

where  $\xi^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}, \xi_4^{(1)})^T$ ,  $\delta = -a_2 - a_4 q_1^2$  and  $q_1 = q_1(t)$ , being  $(q_1(t), 0, q_1(t), 0)$  a particular solution of the Hamiltonian system over the invariant plane. The second variational equation is given by Equation (33)

$$\dot{\xi}^{(2)} = A(t)\xi^{(2)} + f(t), \quad f(t) = (0, 0, \rho, \mu)^T, \quad \rho = -3a_5 (\xi_1^{(1)})^2 - a_4 q_1 (\xi_1^{(1)})^2, \quad (33)$$

where  $\xi^{(2)} = (\xi_1^{(2)}, \xi_2^{(2)}, \xi_3^{(2)}, \xi_4^{(2)})^T$  and  $\mu = -2a_4 q_1 \xi_1^{(1)} \xi_2^{(1)}$ .

### 3. Contribution

The following proposition is our original contribution to this paper. Assume the Hamiltonian system given by Equation (34).

$$H = \frac{p_1^2 + p_2^2}{2} - \frac{1}{aq_1^m + bq_2^m}, \quad a \neq 0, m > 2 \quad (34)$$

The differential Galois group of the variational equation corresponding to the invariant plane  $q_2 = p_2 = 0$  and energy level  $h = 0$ , is virtually abelian. Furthermore, the Galois group is independent of the choice of  $a$  and  $b$ . Proof. The subsystem in invariant plane is Equation (35).

$$h = \frac{p_1^2}{2} - \frac{1}{aq_1^m}, \quad a \neq 0, m > 2, \quad (35)$$

then we obtain a particular solution for  $q_1$  given by Equation (36).

$$q_1(t) = \left(\frac{m+2}{\beta t}\right)^{\frac{2}{m+2}}, \quad \beta = \sqrt{2a}, \quad (36)$$

for instance, the variational equation is given by Equation (37).

$$\dot{\xi} = A(t)\xi, \quad A(t) = \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ B_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}, \quad B_{2 \times 2} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad c = \frac{m(m+1)}{aq_1^{m+2}}, \quad (37)$$

where  $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T$  thus, we arrive to the Cauchy-Euler equation, Equation (38).

$$\frac{d^2 \xi_1}{dt^2} = \frac{2m(m+1)}{(m+2)^2 t^2} \xi_1, \quad (38)$$

and for instance, the Galois group is always abelian.

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