

## Nonintegrability of the Armbruster–Guckenheimer–Kim Quartic Hamiltonian Through Morales–Ramis Theory\*

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**Abstract.** We show the nonintegrability of the three-parameter Armbruster–Guckenheimer–Kim quartic Hamiltonian using Morales–Ramis theory, with the exception of the three already known integrable cases. We use Poincaré sections to illustrate the breakdown of regular motion for some parameter values.

**Key words.** Hamiltonian, integrability of dynamical systems, differential Galois theory, Legendre equation, Schrödinger equation

**AMS subject classifications.** 37J30, 70H06, 12H05

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**1. Introduction.** Mechanical and physical systems whose energy is conserved are usually modeled by Hamiltonian systems with  $n$  degrees of freedom. These systems have a Hamiltonian function  $H$  as a first integral and we say that the Hamiltonian system is integrable in the Liouville sense (often referred to as complete integrability) if there exist  $n - 1$  additional smooth first integrals in involution  $\{I_j, H\} = 0$ ,  $j = 1, \dots, n - 1$ , which are functionally independent; see [1].

There are many physics and mathematical papers with examples of numerical solutions of Hamiltonian systems with two or more degrees of freedom giving evidence of nonintegrability. In the present paper, using Morales–Ramis theory [17] and some new contributions to this, we examine rigorously the nonintegrability of the Hamiltonian of two degrees of freedom with Armbruster–Guckenheimer–Kim (AGK) potential given by

$$(1) \quad H(x, y) = \frac{1}{2} (p_x^2 + p_y^2) - \frac{\mu}{2} (x^2 + y^2) - \frac{a}{4} (x^2 + y^2)^2 - \frac{b}{2} x^2 y^2,$$

where  $\mu$ ,  $a$ , and  $b$  are real parameters.

In 1989 Simonelli and Gollub [23] carried out experimental work with surface waves in square and rectangular containers subject to vertical oscillations. Their methods allowed measurements of both stable and unstable fixed points (sinks, sources, and saddles), and the

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nature of the bifurcation sequences was clearly established and reported. Later, motivated by these experiments reported in [23], Armbruster, Guckenheimer, and Kim [8] derived a dynamical model starting with a normal form given by four first order differential equations depending on several parameters. Their system provides a general description of the codimension two bifurcation problem. To restrict the model they performed a rescaling of the variables and parameters. When one of the scaling parameters  $\varepsilon = 0$ , the system of equations becomes Hamiltonian with the Hamiltonian function giving by (1) which has  $D_4$ -symmetry. Armbruster, Guckenheimer, and Kim [8] found a *large* parameter region of chaotic behavior. They argue that the existence of chaos in a given region suggests that the Hamiltonian (1) is nonintegrable.

As early as 1966 a similar potential was proposed by Andrie [7] as a dynamical model for a stellar system with an axis and a plane of symmetry given by

$$V = \frac{1}{2} \left( C^2 \bar{\omega}^{-2} - A^2 \xi^2 - B^2 z^2 + \frac{1}{2} a \xi^4 + \frac{1}{2} \gamma z^4 + 2\beta \xi^2 z^2 \right),$$

where  $\xi = \bar{\omega} - \bar{\omega}_0$ ,  $A$ ,  $B$ ,  $\alpha$ ,  $\gamma$  are positive constant,  $\beta$  is a constant, and  $C$  is the value of the angular momentum integral. This potential has often been used in the study of galactic dynamics; see, among others, [10].

In particular, if  $\mu = -1$  in (1) we have the Hamiltonian studied by Llibre and Roberto [15]. These authors examined its  $C^1$  nonintegrability in the sense of Liouville and Arnold; see [1]. More recently, Elmandouh [11] studied the dynamics of rotation of a nearly axisymmetric galaxy rotating with a constant angular velocity around a fixed axis using (1) in a rotating reference frame. He also examined the nonintegrability by means of Painlevé analysis.

The goal of this present paper is to show that the Hamiltonian (1) is, in fact, nonintegrable except for the cases  $b = 0$ ,  $a = -b$ , and  $b = 2a$  mentioned above. The potential of the Hamiltonian (1) can be written in the form of the sum of two homogeneous components

$$V(x, y) = -\frac{\mu}{2} (x^2 + y^2) - \frac{a}{4} (x^2 + y^2)^2 - \frac{b}{2} x^2 y^2 = V_{min} + V_{max},$$

where  $V_{min}$  and  $V_{max}$  are quadratic and quartic polynomial, respectively.

Maciejewski and Przybylska [16] and Bostan, Combet, and Din [9] have made advances on the Morales–Ramis theorem for homogeneous potentials. In particular [9] has found a way of applying the Morales–Ramis theorem for homogeneous potentials in a very efficient way. For all Darboux points, Bostan, Combet, and Din [9] implemented an algorithm that allows one to determinate necessary integrability conditions of homogeneous potentials up to degree 9. We use this algorithm to obtain necessary integrability conditions for the quartic potential. Then using an old result of Hietarinta [12] and Yoshida [27] for polynomial potentials the necessary integrability conditions for AGK potential is obtained. The above necessary conditions imply integrability by rational functions. The general Morales–Ramis theorem (see [17]) complements the study by giving necessary conditions for rational or meromorphic integrability, depending on the kind of singularity of the normal variational equation.

The structure of the present paper is as follows. In section 2, we briefly review some preliminary notions and results of Morales–Ramis theory to be used later. The main results will be given in section 3. In section 4 we use Poincaré sections to exhibit the breakdown in regular motion suggesting nonintegrability.

**2. Preliminaries.** In this section we describe some basic facts concerning the Morales–Ramis theory.

**2.1. Differential Galois theory.** Picard–Vessiot theory is the Galois theory of linear differential equations, also known as differential Galois theory. We present here some of its main definitions and results and refer the reader to [25] for a wide theoretical background.

We start by recalling some basic notions on algebraic groups and, afterward, Picard–Vessiot theory is introduced.

An algebraic group of matrices  $2 \times 2$  is a subgroup  $G \subset \mathrm{GL}(2, \mathbb{C})$  defined by means of algebraic equations in its matrix elements and in the inverse of its determinant. It is well known that any algebraic group  $G$  has a unique connected normal algebraic subgroup  $G^0$  of finite index. In particular, the *identity connected component*  $G^0$  of  $G$  is defined as the largest connected algebraic subgroup of  $G$  containing the identity. In the case that  $G = G^0$  we say that  $G$  is a *connected group*. Moreover, if  $G^0$  is abelian we say that  $G$  is *virtually abelian*.

Now, we briefly introduce Picard–Vessiot theory.

First, we say that  $(\mathbb{K}, ')$ —or, simply,  $\mathbb{K}$ —is a *differential field* if  $\mathbb{K}$  is a commutative field of characteristic zero, depending on  $x$ , and  $' = d/dx$  is a derivation on  $\mathbb{K}$  (that is, satisfying  $(a + b)' = a' + b'$  and  $(a \cdot b)' = a' \cdot b + a \cdot b'$  for all  $a, b \in \mathbb{K}$ ). We denote by  $\mathcal{C}$  the *field of constants* of  $\mathbb{K}$ , defined as  $\mathcal{C} = \{c \in \mathbb{K} \mid c' = 0\}$ .

We will deal with second order linear homogeneous differential equations, that is, equations of the form

$$(2) \quad y'' + b_1 y' + b_0 y = 0, \quad b_1, b_0 \in \mathbb{K},$$

and we will be concerned with the algebraic structure of their solutions. Moreover, throughout this work, we will refer to the current differential field as the smallest one containing the field of coefficients of this differential equation.

Let us suppose that  $y_1, y_2$  is a basis of solutions of (2), i.e.,  $y_1, y_2$  are linearly independent over  $\mathcal{C}$  and every solution is a linear combination over  $\mathcal{C}$  of these two solutions. Let  $\mathbb{L} = \mathbb{K}\langle y_1, y_2 \rangle = \mathbb{K}(y_1, y_2, y_1', y_2')$  be the differential extension of  $\mathbb{K}$  such that  $\mathcal{C}$  is the field of constants for  $\mathbb{K}$  and  $\mathbb{L}$ . We say that  $\mathbb{L}$ , the smallest differential field containing  $\mathbb{K}$  and  $\{y_1, y_2\}$ , is the *Picard–Vessiot extension* of  $\mathbb{K}$  for the differential equation (2).

The group of all the differential automorphisms of  $\mathbb{L}$  over  $\mathbb{K}$  that commute with the derivation  $'$  is called the *differential Galois group* of  $\mathbb{L}$  over  $\mathbb{K}$  and is denoted by  $\mathrm{DGal}(\mathbb{L}/\mathbb{K})$ . This means, in particular, that for any  $\sigma \in \mathrm{DGal}(\mathbb{L}/\mathbb{K})$ ,  $\sigma(a') = (\sigma(a))'$  for all  $a \in \mathbb{L}$  and  $\sigma(a) = a$  for all  $a \in \mathbb{K}$ . Thus, if  $\{y_1, y_2\}$  is a fundamental system of solutions of (2) and  $\sigma \in \mathrm{DGal}(\mathbb{L}/\mathbb{K})$ , then  $\{\sigma y_1, \sigma y_2\}$  is also a fundamental system. This implies the existence of a nonsingular constant matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

such that

$$\sigma \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

This fact can be extended in a natural way to a system

$$\sigma \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(y'_1) & \sigma(y'_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} A_\sigma,$$

which leads to a faithful representation  $\text{DGal}(\mathbb{L}/\mathbb{K}) \rightarrow \text{GL}(2, \mathbb{C})$  and makes it possible to consider  $\text{DGal}(\mathbb{L}/\mathbb{K})$  isomorphic to a subgroup of  $\text{GL}(2, \mathbb{C})$  depending (up to conjugacy) on the choice of the fundamental system  $\{y_1, y_2\}$ .

One of the fundamental results of the Picard–Vessiot theory is the following theorem (see [13, 14]).

**Theorem 2.1.** *The Galois group  $\text{DGal}(\mathbb{L}/\mathbb{K})$  is an algebraic subgroup of  $\text{GL}(2, \mathbb{C})$ .*

We say that (2) is *integrable* if the Picard–Vessiot extension  $\mathbb{L} \supset \mathbb{K}$  is obtained as a tower of differential fields  $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \dots \subset \mathbb{L}_m = \mathbb{L}$  such that  $\mathbb{L}_i = \mathbb{L}_{i-1}(\eta)$  for  $i = 1, \dots, m$ , where either

- (i)  $\eta$  is *algebraic* over  $\mathbb{L}_{i-1}$ , that is,  $\eta$  satisfies a polynomial equation with coefficients in  $\mathbb{L}_{i-1}$ ;
- (ii)  $\eta$  is *primitive* over  $\mathbb{L}_{i-1}$ , that is,  $\eta' \in \mathbb{L}_{i-1}$ ;
- (iii)  $\eta$  is *exponential* over  $\mathbb{L}_{i-1}$ , that is,  $\eta'/\eta \in \mathbb{L}_{i-1}$ .

If  $\eta$  is obtained as a combination of (i), (ii), and (iii), we say that  $\eta$  is *Liouvillian*. Usually in terms of differential algebra’s terminology we say that (2) is integrable if the corresponding Picard–Vessiot extension is Liouvillian. Moreover, the following theorem holds.

**Theorem 2.2 (Kolchin).** *Equation (2) is integrable if and only if  $\text{DGal}(\mathbb{L}/\mathbb{K})$  is virtually solvable, that is, its identity component  $(\text{DGal}(\mathbb{L}/\mathbb{K}))^0$  is solvable.*

Let us notice that for a second order linear differential equation  $y'' = r(x)y$  with coefficients in  $\mathbb{K}$ , the only connected nonsolvable group is  $SL(2, \mathbb{C})$ ; see [13, 17, 25].

The following result can be found in [5].

**Theorem 2.3 (Acosta-Humánez et al. [5]).** *The Legendre differential equation*

$$(3) \quad (1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left( \nu(\nu + 1) - \frac{\tilde{\mu}^2}{1 - x^2} \right) y = 0$$

is integrable if and only if either

- 1.  $\tilde{\mu} \pm \nu \in \mathbb{Z}$  or  $\nu \in \mathbb{Z}$ , or
- 2.  $\pm \tilde{\mu}, \pm(2\nu + 1)$  belong to one of the following six families:

Case	$\tilde{\mu} \in$	$\nu \in$	$\tilde{\mu} + \nu \in$
(a)	$\mathbb{Z} + \frac{1}{2}$	$\mathbb{C}$	
(b)	$\mathbb{Z} \pm \frac{1}{3}$	$\frac{1}{2}\mathbb{Z} \pm \frac{1}{3}$	$\mathbb{Z} + \frac{1}{6}$
(c)	$\mathbb{Z} \pm \frac{2}{5}$	$\frac{1}{2}\mathbb{Z} \pm \frac{1}{5}$	$\mathbb{Z} + \frac{1}{10}$
(d)	$\mathbb{Z} \pm \frac{1}{3}$	$\frac{1}{2}\mathbb{Z} \pm \frac{2}{5}$	$\mathbb{Z} + \frac{1}{10}$
(e)	$\mathbb{Z} \pm \frac{1}{5}$	$\frac{1}{2}\mathbb{Z} \pm \frac{2}{5}$	$\mathbb{Z} + \frac{1}{10}$
(f)	$\mathbb{Z} \pm \frac{2}{5}$	$\frac{1}{2}\mathbb{Z} \pm \frac{1}{3}$	$\mathbb{Z} + \frac{1}{6}$

We recall that the singularities of Legendre equation (3) are of the regular type.

**Theorem 2.4** (Acosta-Humánez and Blázquez-Sanz [3]). *Assume  $\mathbb{K} = \mathbb{C}(e^{it})$ . The differential Galois group of extended Mathieu differential equation*

$$\ddot{y} = (a + b \sin(t) + c \cos(t))y, \quad |b| + |c| \neq 0,$$

*is isomorphic to  $\mathrm{SL}(2, \mathbb{C})$ .*

Let us remark that the variable change  $x = e^{it}$  makes the differential equation algebraic, such that  $x = \infty$  is an irregular singularity.

**2.2. Morales–Ramis theory.** A Hamiltonian  $H$  in  $\mathbb{C}^{2n}$  is called *integrable in the sense of Liouville* if there exist  $n$  independent first integrals of the Hamiltonian system in involution. We say that  $H$  is *integrable in terms of rational functions* if we can find a complete set of integrals within the family of rational functions. Respectively, we can say that  $H$  is *integrable in terms of meromorphic functions* if we can find a complete set of integrals within the family of meromorphic functions. Morales–Ramis theory relates integrability of Hamiltonian systems in the Liouville sense with the integrability of Picard–Vessiot theory in terms of differential Galois theory; see [17, 18, 19, 20].

Let  $V$  be a meromorphic homogeneous potential. We say that  $c \in \mathbb{C}^n \setminus \{0\}$  is a *Darboux point* if there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that

$$\frac{\partial}{\partial q_j} V(c) = \alpha c_j \quad \forall j = 1, \dots, n.$$

**Theorem 2.5** (Morales-Ruiz and Ramis [17, 19, 20]). *Let  $H$  be a Hamiltonian in  $\mathbb{C}^{2n}$ , and let  $\gamma$  be a particular solution such that the variational equation associated to the system has regular (resp., irregular) singularities at the points of  $\gamma$  at infinity. If  $H$  is integrable by terms of meromorphic (rational) functions, then the differential Galois group of the variational equation is virtually abelian.*

**Theorem 2.6** (Morales-Ruiz and Ramis [17, 19, 20]). *Let  $V \in \mathbb{C}(q_1, q_2)$  be a homogeneous rational function of homogeneity degree  $k \neq -2, 0, 2$ , and let  $c \in \mathbb{C}^2 \setminus \{0\}$  be a Darboux point of  $V$ . If the potential  $V$  is integrable, then for any eigenvalue  $\lambda$  of the Hessian matrix of  $V$  at  $c$ , the pair  $(k, \lambda)$  belongs to Table 1 for some  $j \in \mathbb{Z}$ .*

**2.3. Galoisian integrability of Schrödinger equation.** The analysis of the integrability of the Schrödinger equation using differential Galois theory has been studied in [4, 6], using as main tools the *Kovacic algorithm* and *Hamiltonian algebrization*; wave functions and spectrum of Schrödinger equations with potentials known as shape invariant were obtained.

It is well known that the one-dimensional stationary nonrelativistic Schrödinger equation is given by

$$(4) \quad \Psi'' = (V(y) - \lambda)\Psi,$$

where  $V$  is the potential and  $\lambda$  is called energy.

**Table 1**  
Morales–Ramis table.

$k$	$\lambda$	$k$	$\lambda$
$\mathbb{Z}^*$	$\frac{1}{2}jk(jk + k - 2)$	-3	$-\frac{25}{8} + \frac{1}{8}(\frac{12}{5} + 6j)^2$
$\mathbb{Z}^*$	$\frac{1}{2}(jk + 1)(jk + k - 1)$	3	$-\frac{1}{8} + \frac{1}{8}(2 + 6j)^2$
-5	$-\frac{49}{8} + \frac{1}{8}(\frac{10}{3} + 10j)^2$	3	$-\frac{1}{8} + \frac{1}{8}(\frac{3}{2} + 6j)^2$
-5	$-\frac{49}{8} + \frac{1}{8}(4 + 10j)^2$	3	$-\frac{1}{8} + \frac{1}{8}(\frac{6}{5} + 6j)^2$
-4	$-\frac{9}{2} + \frac{1}{2}(\frac{4}{3} + 4j)^2$	3	$-\frac{1}{8} + \frac{1}{8}(\frac{12}{5} + 6j)^2$
-3	$-\frac{25}{8} + \frac{1}{8}(2 + 6j)^2$	4	$-\frac{1}{2} + \frac{1}{2}(\frac{4}{3} + 4j)^2$
-3	$-\frac{25}{8} + \frac{1}{8}(\frac{3}{2} + 6j)^2$	5	$-\frac{9}{8} + \frac{1}{8}(\frac{10}{3} + 10j)^2$
-3	$-\frac{25}{8} + \frac{1}{8}(\frac{6}{5} + 6j)^2$	5	$-\frac{9}{8} + \frac{1}{8}(4 + 6j)^2$

A special class of potentials is called Pöschl–Teller (see [22, 24]) which transforms (4) into

$$(5) \quad \Psi'' = \left( -\frac{\ell(\ell + 1)}{\cosh^2 y} + n^2 \right) \Psi, \quad y \in \mathbb{R},$$

where  $n, \ell \in \mathbb{Z}^+$ . It is possible to algebrize this equation by means of the change of variables  $z = \tanh y$ . This substitution transforms the Schrödinger equation (5) into

$$(6) \quad (1 - z^2) \frac{d^2 \Psi}{dz^2} - 2z \frac{d\Psi}{dz} + \left( \ell(\ell + 1) - \frac{n^2}{1 - z^2} \right) \Psi = 0,$$

which is a differential equation for the associated Legendre function. Two linearly independent solutions of (6) are given by the associated Legendre functions of the first and second kinds, respectively.

**3. Main results.** We start by giving the known integrable cases of the Hamiltonian (1) with associated differential equations

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ p_x &= \mu x + a(x^2 + y^2)x + bxy^2, \\ p_y &= \mu y + a(x^2 + y^2)y + bx^2y. \end{aligned}$$

There exist values of the parameters for which the system has additional symmetry and (1) is completely integrable:

- (i) If  $b = 0$ , by considering polar coordinates  $x = r \cos \theta$  and  $y = r \sin \theta$ , the Hamiltonian (1) becomes

$$(7) \quad H(x, y) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \frac{\mu}{2}r^2 - \frac{a}{4}r^4,$$

which does not depend on  $\theta$ . Since  $p_\theta$  is a second independent first integral the angular momentum  $r^2\dot{\theta} = x\dot{y} - \dot{x}y$  is conserved.

(ii) If  $a = -b$ , system (7) reduces to two identical uncoupled Duffing's oscillators,

$$\ddot{x} = \mu x + ax^3 \quad \text{and} \quad \ddot{y} = \mu y + ay^3,$$

which can be integrated by quadrature; see [26].

(iii) In case  $b = 2a$ , we employ the symplectic transformation

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(u - v), \quad y = \frac{1}{\sqrt{2}}(u + v), \\ p_x &= \frac{1}{\sqrt{2}}(p_u - p_v), \quad p_y = \frac{1}{\sqrt{2}}(p_u + p_v) \end{aligned}$$

and one easily finds that the Hamiltonian (1) becomes

$$(8) \quad H(u, v) = \frac{1}{2}(p_u^2 + p_v^2) - \frac{\mu}{2}(u^2 + v^2) - \frac{a}{2}(u^4 + v^4).$$

The transformed Hamiltonian system decouples into two single degree of freedom oscillators

$$\ddot{u} - \mu u - 2au^3 = 0 \quad \text{and} \quad \ddot{v} - \mu v - 2av^3 = 0,$$

and each of them is integrable by quadrature.

Let us remark that the first two cases were previously discovered by Armbruster, Guckenheimer, and Kim [8]. Elmandouh [11] also cited these two cases and added the case  $b = 2a$  when the angular velocity  $\omega = 0$ .

**3.1. Application of Bostan–Combot–Safey El Din algorithm.** The AGK Hamiltonian has the following polynomial potential which can be grouped into two homogeneous potentials of grades two and four:

$$(9) \quad V(x, y) = -\frac{\mu}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - \frac{b}{2}x^2y^2 = V_{min} + V_{max}.$$

An old result of Hietarinta [12, Corollary of Theorem 3'] proved again by Yoshida [27] and Nakagawa [21] states the following.

**Proposition 3.1.** *Suppose that the Hamiltonian with a nonhomogeneous polynomial potential,*

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V_{min}(q_1, q_2) + \cdots + V_{max}(q_1, q_2),$$

*admits an additional rational first integral  $\Phi = \Phi(q_1, q_2, p_1, p_2)$ . Here  $V_{min}$  is the lowest degree part of the potential and  $V_{max}$  the highest degree part. Then, each of the two Hamiltonians,*

$$H_{min} = (p_1^2 + p_2^2) + V_{min}(q_1, q_2), \quad H_{max} = (p_1^2 + p_2^2) + V_{max}(q_1, q_2),$$

*also admits an additional rational first integral.*

In our case  $V_{min} = -\frac{\mu}{2}(x^2 + y^2)$  is obviously integrable for any value of  $\mu$ . It remains to determine for which values of the parameters the quartic potential  $V_{max}$  is integrable. Homogeneous potentials have been worked before and after the seminal Morales–Ramis theorem [9], [16]; in particular, Bostan, Combot, and Din [9] take the potential to a polar form by making  $x = r \cos \theta$ ,  $y = r \sin \theta$  and reducing them to the form

$$(10) \quad V(r, \theta) = r^k F(e^{i\theta}).$$

This change makes the evaluation of the Morales–Ramis table (Table 1) for the necessary condition of integrability for homogeneous potentials easier. Bostan, Combot, and Din are able to build a nice Maple algorithm using the following theorem.

**Theorem 3.2 (Bostan, Combot, and Din [9]).** *Let  $V \in \mathbb{C}(x, y)$  be a homogeneous potential with polar representation  $(F, k)$  and let  $\Lambda$  be the following set:*

$$\tilde{\Lambda}(F, k) := \left\{ k - \frac{z^2 F''(z)}{F(z)} : z \neq 0, F'(z) = 0, F(z) \neq 0 \right\}.$$

*Let  $Sp_D(\Delta^2 V)$  denote the union of the sets  $Sp(\Delta^2 V(c))$  taken over all Darboux points  $c \in D$  of  $V$ ; then*

$$k(k - 1) \cup Sp_D(\Delta^2 V) = k(k - 1) \cup \tilde{\Lambda}.$$

*Moreover, if  $V$  is integrable, then  $\tilde{\Lambda} \in E_k$ , where  $E_k$  is the so-called Morales–Ramis table.*

This result allows the construction of a powerful algorithm which allows quick evaluation of integrability even to homogeneous potential of degree 9.

Applying the Bostan–Combot–Safey El Din algorithm given in [9], we have found that the necessary values for integrability are  $b = 0$ ,  $b = 2a$ , and  $b = -a$ , which are the same as the well-known integrable cases described at the beginning of this section. The algorithm also informs us the spectrum of the Hessian matrix for each necessary condition of rational integrability yields

- $b = 0$ , the Darboux points have Hessian spectrums [12, 4];
- $b - a = 0$ , the Darboux points have Hessian spectrums [12, 12] or [12, 0];
- $a + b = 0$ , the Darboux points have Hessian spectrums [12, 12] or [12, 0].

We note that the trivial eigenvalue of the Hessian  $\lambda = 12$  appears also as a nontrivial eigenvalue.

Now applying Proposition 3.1 since the harmonic oscillator is trivially integrable we get the necessary condition for the integrability of the quartic potential. The procedure described at the beginning of this section showing the separability of the potential for the classical values implies the following result.

**Theorem 3.3.** *The AGK potential (1) is rationally integrable if and only if  $b = 0$ ,  $b = 2a$ , and  $b = -a$ .*

Note that for  $\mu = 0$  we have a homogeneous quartic potential whose necessary conditions for rational integrability are the same values given in Theorem 3.3.



**3.2. Application of Morales–Ramis theorem.** The Hamiltonian has square symmetry, that is, it is invariant under a rotation by  $\pi/2$ , and it is also time reversible. So, the planes of symmetry are given by planes which are invariant under the action of the dihedral group  $D_4$ :

- $\Gamma_1 = \{(x, y, p_x, p_y) : y = p_y = 0\}$ ,
- $\Gamma_2 = \{(x, y, p_x, p_y) : x = p_x = 0\}$ ,
- $\Gamma_3 = \{(x, y, p_x, p_y) : y = x, p_y = p_x\}$ ,
- $\Gamma_4 = \{(x, y, p_x, p_y) : y = -x, p_y = p_x\}$ ,
- $\Gamma_5 = \{(x, y, p_x, p_y) : y = x, p_y = -p_x\}$ ,
- $\Gamma_6 = \{(x, y, p_x, p_y) : y = -x, p_y = -p_x\}$ .

Let us observe that if we restrict the Hamiltonian (1) to the invariant planes of symmetry  $\Gamma_{1,2}$  and respectively  $\Gamma_{3,4,5,6}$ , we get the same form. Thus, it is enough to consider the following two cases:

$$(11a) \quad h = \frac{p_x^2}{2} - \frac{\mu}{2}x^2 - \frac{a}{4}x^4, \quad H|_{\Gamma_1} = h,$$

$$(11b) \quad \widehat{h} = \frac{p_x^2}{2} - \frac{\mu}{2}x^2 - \frac{2a+b}{4}x^4, \quad H|_{\Gamma_3} = 2\widehat{h}.$$

We note that each one of these expressions corresponds to a Hamiltonian with one degree of freedom, and also setting the change  $a \mapsto 2a+b$  into (11a) we obtain (11b). So, throughout the paper we will compute the conditions of integrability of the *normal variational equations* over  $\Gamma_1$  and  $\Gamma_3$ , and also we substitute  $a \mapsto 2a+b$  into the particular solution  $x(t)$  obtained in  $\Gamma_1$  to get them in  $\Gamma_3$ .

In order to apply the Morales–Ramis theory our strategy consists of selecting a non-equilibrium particular solution in the invariant planes  $\Gamma_1$  and  $\Gamma_3$ . The following step is to obtain an integral curve for the Hamiltonian (11a). Thus, to perform our study we take into account that a Hamiltonian is a conservative system and we can fix the energy level for  $\Gamma_1$ ,  $h$ ; that is,

$$(12) \quad \frac{p_x^2}{2} - \frac{\mu}{2}x^2 - \frac{a}{4}x^4 = h.$$

To solve it we rewrite (12) as

$$(13) \quad \frac{dx}{dt} = \pm \sqrt{2h + \mu x^2 + \frac{a}{2}x^4}.$$

We observe that (13) for any value of  $h$ ,  $a$ , and  $\mu$  corresponds to the well-known *incomplete elliptic integral of first kind* (see [2]), whose solutions are not always *elementary functions*. In order to obtain a manageable equation with not-too-complicated base fields, and get explicit solutions for (13), we should restrict the values of  $h$ . Furthermore, to be able to compute Galois groups for the variational equations along such particular solutions, we should stay with these solutions. In particular we choose the energy level  $h = 0$ ; then the restricted last equation becomes

$$(14) \quad \frac{dx}{dt} = \pm \sqrt{\mu x^2 + \frac{a}{2}x^4}.$$

Now, observe that (14) has three equilibrium points given by  $x = 0, \pm i\sqrt{\frac{2\mu}{a}}$ . To avoid triviality, we exclude these points and assume that  $x(t)$  is not a constant.

In a similar way, to obtain algebraic solutions over a suitable differential field in (13) with  $ah \neq 0$ , we can choose the energy level  $h = \frac{\mu^2}{4a}$ . Thus, the restricted equation becomes

$$\pm \frac{dx}{dt} = \frac{\mu}{\sqrt{2a}} + \sqrt{\frac{a}{2}}x^2.$$

Observe that the last equation has two equilibrium points given by  $x = \pm i\sqrt{\frac{\mu}{a}}$ . To avoid triviality, we exclude these points and assume that  $x(t)$  is not a constant. Recall that similar results are obtained for the respective  $\hat{h}$  in  $\Gamma_3$ .

Next, we present a rigorous proof of the nonintegrability of Hamiltonian (1) using the Morales–Ramis theory. Since this theorem gives conditions for integrability in terms of the differential Galois group of the variational equations along a particular solution, we compute the linearized system of (7) along a nonconstant solution of the dynamical system given above.

In a general frame we assume  $\gamma(t) = (x(t), y(t), p_x(t), p_y(t)) \in \Gamma_{1,3}$  is a nonstationary solution of (7); then the variational equations  $\dot{\xi} = A(t)\xi$  of (7) along  $\gamma(t)$  are given by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu + 3ax(t)^2 + (a+b)y(t)^2 & 2(a+b)x(t)y(t) & 0 & 0 \\ 2(a+b)x(t)y(t) & \mu + (a+b)x(t)^2 + 3ay(t)^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}.$$

Assume  $\gamma(t) = (x(t), 0, p_x(t), 0) \in \Gamma_1$  is a nonstationary solution of (7); then the variational equations  $\dot{\xi} = A(t)\xi$  of (7) along  $\gamma(t)$  are given by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu + 3ax(t)^2 & 0 & 0 & 0 \\ 0 & \mu + (a+b)x(t)^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}.$$

The second order variational system is composed of two uncoupled Schrödinger equations:

$$(15) \quad \ddot{\xi}_1 = (3ax^2(t) + \mu) \xi_1,$$

$$(16) \quad \ddot{\xi}_2 = ((a+b)x^2(t) + \mu) \xi_2.$$

It is well known that the tangential variational equations do not provide conditions of non-integrability of Hamiltonian systems with two degrees of freedom; see [19]. It is easy to see that when we replace  $2a$  instead  $b$  into (16), the variational normal and tangential equations are the same. For this reason, from now on we let out the tangential equations.

We now assume  $\gamma(t) = (x(t), x(t), p_x(t), p_x(t)) \in \Gamma_3$  is a nonstationary solution of (7); then the variational equation of (7) along  $\gamma(t)$  becomes

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu + (4a + b)x(t)^2 & 2(a + b)x(t)^2 & 0 & 0 \\ 2(a + b)x(t)^2 & \mu + (4a + b)x(t)^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}.$$

In this case the normal equation yields

$$(17) \quad \ddot{\xi}_2 = ((2a - b)x(t)^2 + \mu) \xi_2.$$

The computation is carried out considering several combinations of the signs of the parameters  $\mu$  and  $a$  with  $\mu \neq 0$ , and distinguishing between cases  $a = 0$  and  $a \neq 0$ . We also assume without loss of generality that the constants of integration of the normal variational equations are zero.

**3.3. Case  $a = 0$ : Rational nonintegrability.** In this case the normal variational equation (16) takes the form

$$(18) \quad \ddot{\xi}_2 = (bx^2(t) + \mu) \xi_2,$$

respectively.

A simple computation shows that  $x(t) = \cos(\sqrt{\mu}t)$  with  $h = -1/2$  is a particular solution of the variational equations on the symmetry plane  $\Gamma_1$ . Therefore we obtain that (18) along  $x(t)$  takes the form

$$\ddot{\xi}_2 = (b \cos^2(\sqrt{\mu}t) + \mu) \xi_2.$$

Taking into account that  $\cos^2(\sqrt{\mu}t) = \frac{1}{2}(1 + \cos(2\sqrt{\mu}t))$ , the last equation becomes

$$(19) \quad \ddot{\xi}_2 = \left( \frac{b}{2} + \mu + \frac{b}{2} \cos(2\sqrt{\mu}t) \right) \xi_2,$$

which is a Mathieu equation. In a similar way, we obtain an equivalent result for  $\Gamma_3$  by replacing  $b \mapsto -b$ .

We obtain for  $b \neq 0$  and  $\mu \neq 0$ , according to Theorem 2.4, that (19) is nonintegrable; see [3]. Therefore, we have proved the following result.

**Theorem 3.4.** *The Hamiltonian system (7) with  $a = 0$ ,  $b \neq 0$ , and  $\mu \neq 0$  is nonintegrable through rational first integrals.*

Now in the case  $\mu = 0$ ,  $a = 0$ ,  $h = 0$ , and  $\hat{h} \neq 0$  the normal variational equations (16) and (17) are nonintegrables because they correspond to quantic harmonic oscillators with zero energy level for both planes  $\Gamma_1$  and  $\Gamma_3$ ; see [3, 6]. Finally, the cases  $h = \hat{h} = 0$  are not considered because we obtain for both planes  $\Gamma_1$  and  $\Gamma_3$  that the normal variational equations (16) and (17) are trivially integrables. The first one corresponds to a differential equation with constant coefficients and the second one corresponds to a Cauchy–Euler equation, which are always integrable with the abelian differential Galois group; see [4, 6]. Thus the Morales–Ramis theorem does not give any obstruction to the integrability.

**Table 2**  
Particular solutions over  $\gamma(t)$ .

	$h = 0$	$h = \frac{\mu^2}{4a}$
1.	$x(t) = \pm i \frac{\sqrt{\frac{2\mu}{a}}}{\cosh(\sqrt{\mu}t)}$	$x(t) = \pm i \sqrt{\frac{\mu}{a}} \tanh\left(\sqrt{\frac{\mu}{2}}t\right)$
2.	$x(t) = \pm \frac{i\sqrt{\frac{2\mu}{a}}}{\cos(i\sqrt{\mu}t)}$	$x(t) = \pm i \sqrt{\frac{\mu}{a}} \tan\left(i\sqrt{\frac{\mu}{2}}t\right)$
3.	$x(t) = \pm \frac{\sqrt{\frac{2\mu}{a}}}{\sin(i\sqrt{\mu}t)}$	$x(t) = \pm \sqrt{\frac{\mu}{a}} \cot\left(i\sqrt{\frac{\mu}{2}}t\right)$
4.	$x(t) = \pm \frac{\sqrt{\frac{2\mu}{a}}}{\sinh(\sqrt{\mu}t)}$	$x(t) = \pm \sqrt{\frac{\mu}{a}} \coth\left(\sqrt{\frac{\mu}{2}}t\right)$

**3.4. Case  $a \neq 0$ : Meromorphic nonintegrability.** We consider  $\gamma(t) = (x(t), 0, p_x(t), 0) \in \Gamma_1$  which is a nonstationary solution of (7). Next, we solve (14) for  $h = 0$  and  $h = \frac{\mu^2}{4a}$ , which give us possible particular solutions over  $\gamma(t)$ , shown in Table 2.

These solutions are complex functions where also  $t$  is a complex number. Moreover, we note that it is not possible to get algebraic particular solutions for  $h \neq 0$  with  $\mu = 0$ , thus we exclude this case. So, we consider only one algebraic particular solution, given in the previous table, for each one of the energy levels ( $h = 0$ ,  $h \neq 0$ ), to construct the normal variational equations. The remaining cases are obtained through the transformations  $t \mapsto it$  and  $t \mapsto \frac{\pi}{2} - t$ . Furthermore, the particular solutions over  $\Gamma_3$  can be obtained by the change  $a \mapsto 2a + b$  into the particular solutions over  $\Gamma_1$ .

We recall that  $\mathcal{T}_\ell$ , the  $\ell$ th triangular number, is defined by

$$\mathcal{T}_\ell = 1 + 2 + \cdots + \ell = \frac{1}{2}\ell(\ell + 1), \quad \ell \in \mathbb{Z}^+.$$

Since  $a \in \mathbb{C}^*$ , instead of working with the space of parameters  $\{(a, b, \mu) \in \mathbb{C}^3\}$  we work with  $\mathcal{W} := \{\kappa \in \mathbb{C} : \kappa = b/a, a \neq 0\}$ . We define the following subsets of  $\mathcal{W}$  with  $\mu \neq 0$  given as

$$\Lambda_1 := \{\kappa : \kappa = \mathcal{T}_\ell - 1\}, \quad \Lambda_2 := \left\{ \kappa : \kappa = 2 \frac{1 - \mathcal{T}_\ell}{1 + \mathcal{T}_\ell} \right\}.$$

We define by  $\Lambda := \Lambda_1 \cap \Lambda_2$  to introduce our main result.

**Theorem 3.5.** *Assume  $a\mu \neq 0$ . If  $\kappa \notin \Lambda$ , then the Hamiltonian (1) is nonintegrable through meromorphic first integrals.*

*Proof.* We consider only the case  $h = \hat{h} = 0$  in the invariant planes  $\Gamma_1$  and  $\Gamma_3$ , respectively.

We begin with the variational equation near  $\Gamma_1$ . We use (11a) with  $h = 0$  to get  $x(t)$  which is given in the Table 2. Substituting the expression for  $x(t)$  in (16) with  $\kappa = b/a$ , we obtain

$$(20) \quad \ddot{\xi}_{21} = \mu \left( -\frac{2(1 + \kappa)}{\cosh^2(\sqrt{\mu}t)} + 1 \right) \xi_{21},$$

where we assume coefficients in the differential field  $\mathbb{K} = \mathbb{C}(\tanh(\sqrt{\mu}t))$ . Through the change  $\tau = \sqrt{\mu}t$  and letting  $\xi' = \frac{d\xi}{dz}$ , (20) is transformed into the Schrödinger equation with Pöschl–Teller potential

$$(21) \quad \xi''_{21} = \left( -\frac{2(1+\kappa)}{\cosh^2(\tau)} + 1 \right) \xi_{21},$$

which is transformed into the Legendre's differential equation with parameters  $\ell$  and  $\tilde{\mu}$  such that  $2(1+\kappa) = \ell(\ell+1)$  and  $\tilde{\mu} = \pm 1$ . Since the second parameter is an integer, by Theorem 2.3, we know that (20) is integrable if and only if  $1 \pm \ell \in \mathbb{Z}$  or  $\ell \in \mathbb{Z}$ , because we are only getting case 1 of Theorem 2.3. So,  $\ell(\ell+1)$  must be an integer, which implies that (20) has Liouvillian solutions if and only if  $\kappa \in \Lambda_1$ . Moreover, the differential Galois group  $\text{DGal}(\mathbb{L}/\mathbb{K})$  of this normal variational equation (20) for  $\kappa \in \Lambda_1$  is isomorphic to the additive group  $\mathbb{C}$ , which is a connected abelian group, while for  $\kappa \notin \Lambda_1$  the differential Galois group  $\text{DGal}(\mathbb{L}/\mathbb{K})$  is an algebraic subgroup isomorphic to  $\text{SL}(2, \mathbb{C})$  and thus nonabelian. Therefore, by the Morales–Ramis theorem we can conclude that if  $\kappa \notin \Lambda_1$ , then the Hamiltonian AGK is not integrable through meromorphic first integrals.

The normal variational equations for the second, third, and fourth cases of the particular solutions  $x(t)$  are transformed into (20) through the combination of the changes of variables  $t \mapsto it$  and  $t \mapsto \frac{\pi}{2} - t$ . So we get similar results of integrability of (20) and consequently similar conditions of nonintegrability for the Hamiltonian AGK.

From here we consider the variational equation near  $\Gamma_3$ . We use (11b) to obtain  $x(t)$  which is for  $\hat{h} = 0$  given in Table 2 with  $2a + b$  instead of  $a$ . Now we substitute  $x(t)$  into the normal equation (17). It has the form

$$(22) \quad \ddot{\xi}_{21} = \mu \left( -\frac{2(2-\kappa)/(2+\kappa)}{\cosh^2(\sqrt{\mu}t)} + 1 \right) \xi_{21}.$$

As previously, we rescaled the independent variable to get a Schrödinger equation with Pöschl–Teller potential, which can be transformed into a Legendre's differential equation. From (5), we can conclude that  $\frac{2-\kappa}{2+\kappa} = \mathcal{T}_\ell$  or equivalently  $\kappa = 2\frac{1-\mathcal{T}_\ell}{1+\mathcal{T}_\ell}$ . A similar analysis applied to the above case leads to the nonintegrability conditions through meromorphic first integrals for the Hamiltonian AGK with  $\kappa \notin \Lambda_2$ .

Finally, we conclude that the Hamiltonian AGK (1) is nonintegrable through meromorphic first integrals for  $\kappa \notin \Lambda$ . ■

We observe that  $\Lambda = \Lambda_1 \cap \Lambda_2 = \{-1, 0, 2\}$ . Furthermore, the values  $b = 0, 2a, -a$  given by Theorem 3.3 correspond to  $\kappa = 0, 2, -1$ , respectively. Then, the former satisfy the necessary condition for meromorphic integrability.

We would now like to make some remarks. In order to apply Morales–Ramis theory as nonintegrability criteria, it is enough to exhibit an orbit for which the normal variational equation is not abelian. Taking this into account and since  $\tilde{\mu}$  is not constant, we have only been considering the proof of the last theorem for both cases  $h = \hat{h} = 0$ . Moreover, if we replace any particular solution for  $h = \frac{\mu^2}{4a}$  into the normal variational equation (16) and carry

out combinations of the change of variables  $t \mapsto it$  and  $t \mapsto \frac{\pi}{2} - t$ , we obtain the normal variational equation

$$(23) \quad \xi''_{22} = \left( -\frac{2(1+\kappa)}{\cosh^2(\tau)} + 2\kappa + 1 \right) \xi_{22},$$

which is transformed into the Legendre’s differential equation with parameters  $\ell$  and  $\tilde{\mu}$  such that  $2(1+\kappa) = \ell(\ell+1)$  and  $\tilde{\mu}^2 = 2\kappa+1$ . We get equivalent conclusions for  $\hat{h} = \frac{\mu^2}{4(2a+b)}$  in the variational equation (17).

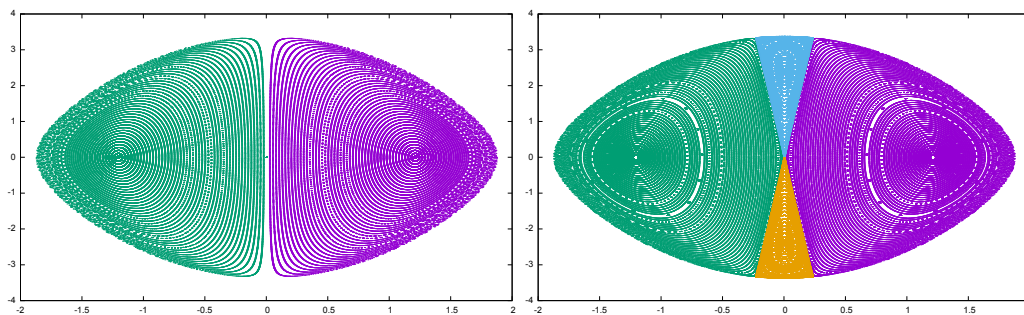
Finally, we remark that for  $\mu = 0$  and  $h = \hat{h} = 0$ , the normal variational equations (16) and (17) correspond to the Cauchy–Euler equation for  $\Gamma_1$  and  $\Gamma_3$ , respectively, given by

$$\ddot{\eta}_1 = \frac{2(1+\kappa)}{t^2} \eta_1, \quad \ddot{\eta}_3 = 2\frac{2-\kappa}{(2+\kappa)t^2} \eta_3,$$

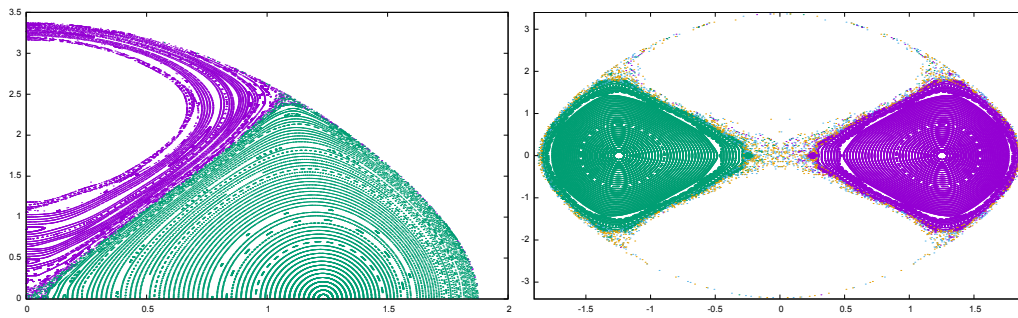
which are always integrable with the abelian differential Galois group; see [4, 6]. Thus the Morales–Ramis theorem does not give any obstruction to the integrability. Furthermore for  $h \neq 0$  and  $\hat{h} \neq 0$  the particular solution is an elliptic integral of first kind that is not algebraic. In this case the variational equations are not in a suitable differential field to apply the Morales–Ramis theorem.

**4. Numerical experiments.** In this section, we present a quick view of the system behavior through the Poincaré section technique. We have chosen the Poincaré section ( $y = 0, p_y > 0$ ). The values of the parameters were chosen to view the system behavior at the well-known integrable cases, and then, by moving the values of one of the parameters ( $a$  or  $b$ ), carry the system to a region where KAM tori start being destroyed. The energy and  $\mu$  are maintained constant for each set of experiments.

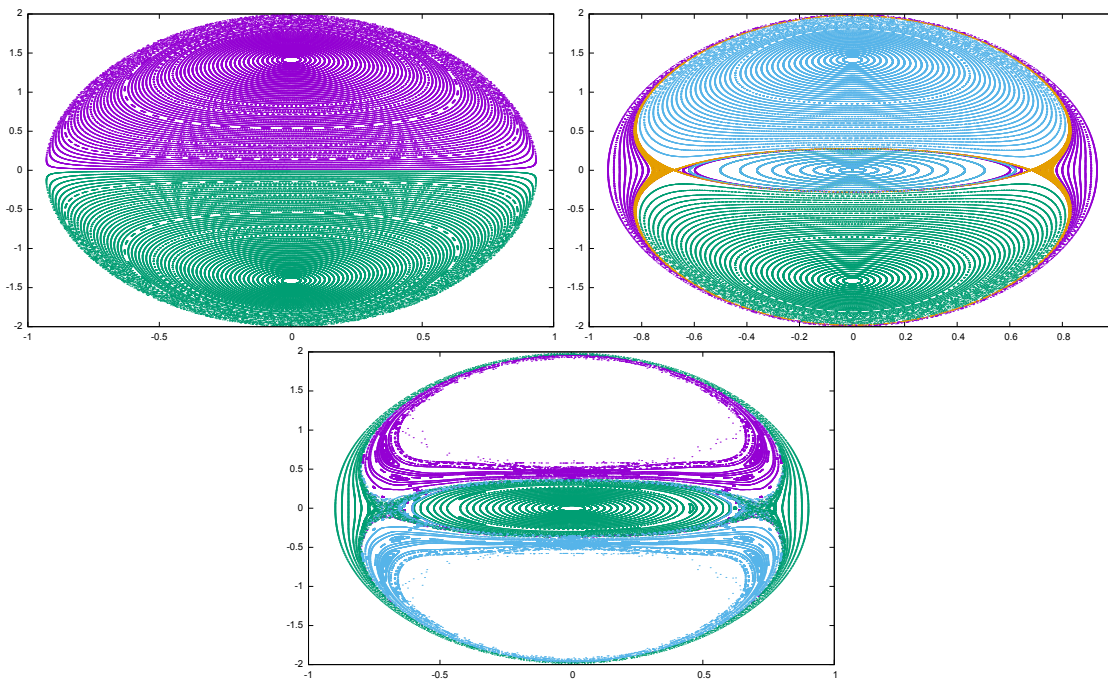
The first example stemmed from the search of Armbruster, Guckenheimer, and Kim reported in Figure 5 of [8], which has the following set of parameters:  $h = 5.7, \mu = -5, a = 1$ , and  $b = 0.5$ . We here have worked with  $b$  going from zero to five. Figures 1 and 2 show the evolution from the integrable case  $a = 1$  and  $b = 0$  to  $a = 1$  and  $b = 0.5$ , where the superior part of the section is completely chaotic. This is the case of Figure 5 of [8], where they have



**Figure 1.** Poincaré section  $(x, p_x)$ : The left figure shows the integrable case  $b = 0, a = 1, \mu = 5$ , and  $h = 5.7$ , while the right figure shows the same values of the parameters as the last one with the exception of  $b = 0.01$ . Note the wedge of KAM tori which appears in the vertical direction.



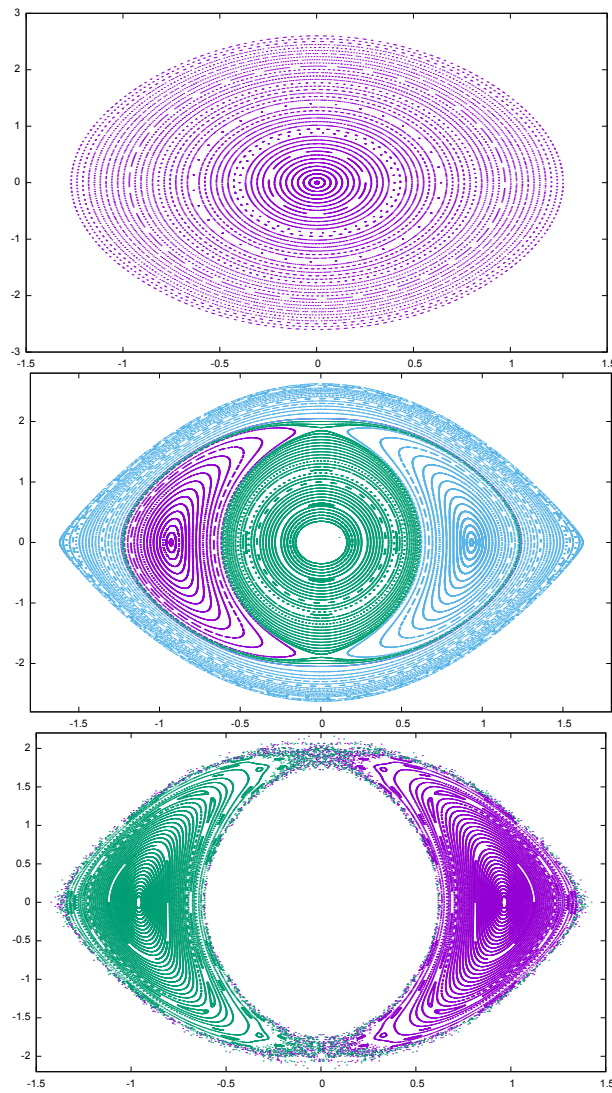
**Figure 2.** Poincaré section  $(x, p_x)$ :  $b = 0.3$  and remaining parameters as in Figure 1 (left); the vertical tori break through escape in the open phase space; we reach  $b = 0.5$  (right), and now we have the same conditions as [8, Figure 5]; the horizontal empty region show by [8] is filled with remnant tori.



**Figure 3.** Poincaré section  $(x, p_x)$ : the integrable case  $b = 2a$  with  $a = 1$ ,  $b = 2$ ,  $\mu = 5$ ,  $h = 2$  (top left); note that the tori have rotated to the vertical axis; for  $b = 2.8$  there is a major bifurcation in the vertical direction (top right) and for  $b = 6$  there appears to be chaos in the escaping trajectories (bottom).

shown the chaotic layer only. The intermediary figures show how a meager top and bottom region of KAM tori,  $b = 0.1$ , evolves to a partially chaotic region (0.3), and finally to regions of no tori at all since trajectories become unbounded. Some KAM tori remain in the horizontal regions, which we show in Figure 2.

In Figure 3 the integrable case  $a = 1$ ,  $b = -1$  with  $h = 3.5$  and  $\mu = 5$  is shown. Note now that the section of the tori have rotated  $\pi/2$  and this indicates a different way of breaking down. In fact, in Figure 3,  $b = 2.8$ , and keeping the remaining parameters the same, we see

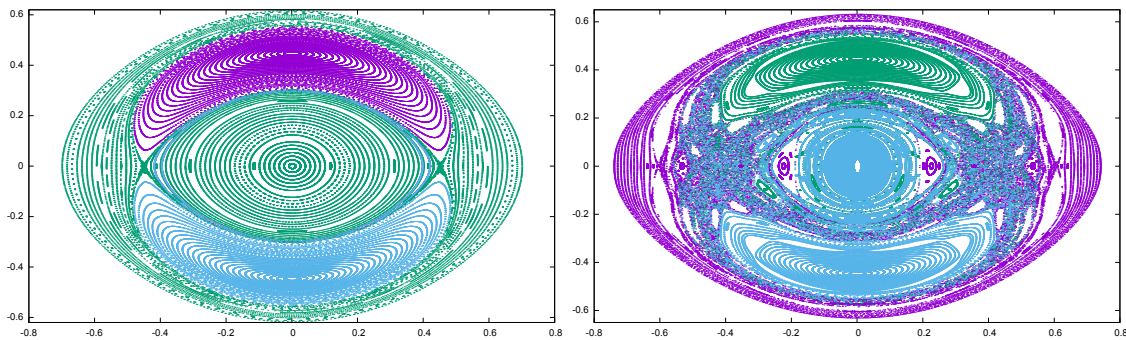


**Figure 4.** Poincaré section  $(x, p_x)$ : the integrable case  $b = -a$  with  $a = 1$ ,  $b = -1$ ,  $h = 3.5$ , and  $\mu = 5$  (top) and we can see the symmetrical KAM tori; taking  $a$  to  $a = 1.8$  (middle) there is a horizontal bifurcation and the central tori start escaping; with  $a = 2.08$  (bottom) the central part becomes more chaotic and most of the bordering trajectories escape.

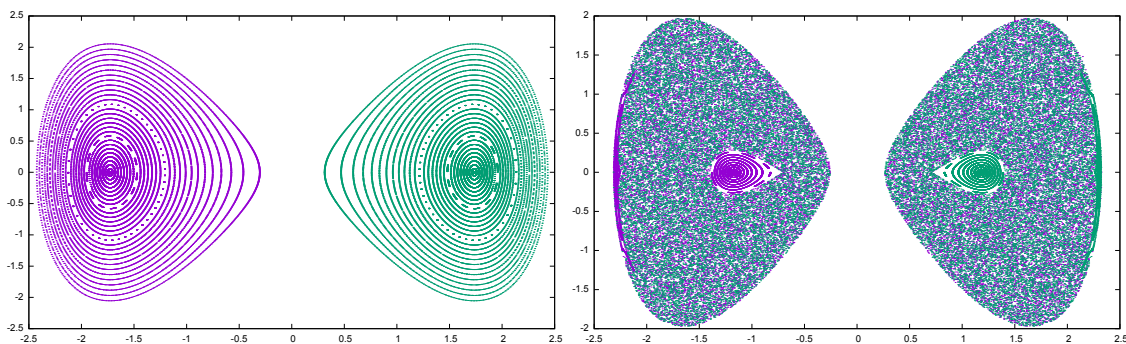
a stable periodic orbit at the origin and two unstable ones at each side of the figure. Finally for  $b = 6$ , the top and bottom cuts of the KAM tori break down with the internal trajectories escaping to nonclosed regions of the phase space. This behavior can be seen in Figure 3. So far, although the route to chaos seems different for the two cases, it seems that escape trajectories play an important role.

The last integrable case is  $b = -a$  can be seen in Figure 4. We have chosen as the starting value  $a = 1$  and  $b = 1$ ; the remaining values are  $h = 3.5$  and  $\mu = 5$ . The cuts of the Poincaré section start centered and for  $a = 1.8$  there is a horizontal bifurcation and the orbits in the





**Figure 5.** Poincaré section  $(x, p_x)$  with  $h = 0.2$ ,  $\mu = 1$  starting at the integrable case  $a = 1$ ,  $b = -1$ . The integrable section is symmetrical as the top of Figure 4 with  $a = 1$ ,  $b = -2.5$  (left) there is a vertical bifurcation and with  $b = -5$  (right) the system is quite chaotic but without escape trajectories as in the case with  $\mu = 5$ .



**Figure 6.** Poincaré section  $(x, p_x)$  with  $h = -0.1$ ,  $\mu = -3$  starting at the integrable case  $a = -1$ ,  $b = 1$  and the second  $a = -1.1$ ,  $b = 1$ . Note how rapidly chaos appears with a small change in the  $a$  parameter.

central region starts to escape (Figure 4, middle). For  $a = 2.05$  the central region is more eroded and all the bordering trajectories escape (Figure 4, bottom).

Finally we have an example of chaotic behavior that contradicts all the previous experiments. As can be seen in Figure 5, there are no escaping orbits among the chaotic ones. We started with  $a = 1$ ,  $b = -1$  and the value of the Hamiltonian  $h = 0.2$  and  $\mu = 1$  remained constant through the evolution of  $b$  from  $-1$  to  $-5$ . The section in  $a = 1$  and  $b = -1$  is similar to Figure 4, top. When  $b = -2.5$  there is already a major bifurcation in the vertical direction. Finally when  $b = -5$  there is chaos at large with some islands still left. Apparently the value of  $\mu$  is responsible for this behavior. The last example is given in Figure 6 with  $h = -0.1$  and  $\mu = -3$ , where integrable case  $a = -1$ ,  $b = 1$  (left), and chaos when  $a = -1.1$ ,  $b = 1$  (right) are shown.

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