São Paulo Journal of Mathematical Sciences **3**, 2 (2009), 283–298

On the q-meromorphic Weyl algebra

Rafael Díaz

Facultad de Administración, Universidad del Rosario, Bogotá, Colombia
E-mail address: ragadiaz@gmail.com

Eddy Pariguan

Departamento de Matemáticas, Pontificia Universidad Javeriana, Bogotá, Colombia
E-mail address: epariguan@javeriana.edu.co

Abstract. We introduce a q-analogue MW_q for the meromorphic Weyl algebra, and study the normalization problem and the symmetric powers $\mathrm{Sym}^n(MW_q)$ for such algebra from a combinatorial viewpoint.

1. Introduction

Pioneered by Euler, Jacobi, and Jackson among others, the results and applications of q-calculus [4, 10] have grown both in depth and scope, touching by now most branches of mathematics, including partition theory [3], combinatorics [30, 31], number theory [26], hypergeometric functions [4], quantum groups [25], knot theory [21], q-probabilities [28], Gaussian q-measure [20], Feynman q-integrals [13, 14], homological algebra [5, 24], and category theory [9]. Our goal in this work is to bring yet another mathematical object into the field of q-calculus, namely, we provide a q-analogue for the meromorphic Weyl algebra MW introduced in [15]. Roughly speaking MW is the algebra generated by x^{-1} and the derivative ∂. The q-analogue MW_q of the meromorphic Weyl algebra is essentially the algebra generated by x^{-1} and the q-derivative ∂_q. We focus on the normal polynomials for MW_q which arise in the problem of writing arbitrary monomials in MW_q as linear combination of monomials written in normal form; we provide both explicit formulae and a combinatorial interpretation for the normal polynomials. We also study the symmetric powers of MW_q using the methodology developed in [15] and further applied in [16, 19].

Let us say a few words on q-combinatorics. As explained by Zeilberger in [31] a combinatorial interpretation for a sequence $n_0, n_1, n_2, ...$ of non-negative integers, is a sequence of finite sets $x_0, x_1, x_2, ...$ such that $|x_k| = n_k$ for $k \in \mathbb{N}$. Each sequence of non-negative integers admits a wide variety of combinatorial interpretations; the art of combinatorics consists in finding patterns that yield, systematically, combinatorial interpretations for families of sequences of non-negative integers.

The field of q-combinatorics provides another approach for the study of natural numbers by combinatorial methods. Let $\mathbb{N}[q]$ be the semi-ring of polynomials in the variable q with coefficients in $\mathbb{N}$. Instead of working with sequences of finite sets the main object of study in q-combinatorics are sequences $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), ...$ of pairs (x, ω) where x is a finite set and $\omega : x \longrightarrow \mathbb{N}[q]$ is an arbitrary map. The cardinality of such a pair (x, ω) is defined to be

$$|x, \omega| = \sum_{i \in x} \omega(i) \in \mathbb{N}[q].$$

Notice that the cardinality $|x, \omega|$ of the pair (x, ω) is not an integer, but rather a polynomial in the variable q with non-negative integer coefficients. We say that a sequence of pairs $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), \cdots$ provides a combinatorial interpretation for a sequence of non-negative integers $n_0, n_1, n_2, \cdots$ if $|x_k, \omega_k|(1) = n_k$ for $k \in \mathbb{N}$, where $|x_k, \omega_k|(1)$ is the evaluation of the polynomial $|x_k, \omega_k|$ at 1. Of course the additional value of q-combinatorics comes from the fact that it is suited to handle not just sequences in $\mathbb{N}$, but more generally sequences in $\mathbb{N}[q]$. We say that a sequence $(x_0, \omega_0), (x_1, \omega_1), (x_2, \omega_2), \cdots$ provides a combinatorial interpretation for a sequence of polynomials $p_1, p_2, p_3, \cdots$ in $\mathbb{N}[q]$ if $|x_k, \omega_k| = p_k$ for $k \in \mathbb{N}$. One of the most prominent examples is the q-combinatorial interpretation for the q-analogues $[n]! \in \mathbb{N}[q]$ of the factorial numbers $n!$ given by

$$[n]! = \prod_{k=1}^{n} [k] \quad \text{where} \quad [k] = 1 + \cdots + q^{k-1}.$$

Consider the pair (S_n, i_n) where S_n is the set of permutations of $[[1, n]] = \{1, 2, \cdots, n\}$ and $i_n : S_n \longrightarrow \mathbb{N}[q]$ is the map given by $i_n(\sigma) = q^{|I_n(\sigma)|}$ where

$$I_n(\sigma) = \{(i, j) \ \mid \ 1 \leq i < j \leq n \ \text{ and } \ \sigma(i) > \sigma(j)\}.$$

An inductive argument [3, 14] shows that $|S_n, i_n| = [n]!$, therefore the sequence (S_n, i_n) provides a combinatorial interpretation for $[n]!$.

The rest of this work is organized as follows. In Section 2 we summarize some facts on the meromorphic Weyl algebra; we do not include proofs since all the stated results are consequences, setting $q = 1$, of the corresponding q-analogue results proved in the subsequent sections. The main results of this work are given in Sections 3 and 4 where we introduce MW_q the q-analogue of the meromorphic Weyl algebra, discuss its basic properties, provide a couple of representations for it, study the normal polynomials that arise in the process of writing monomials in MW_q in normal form, and begin the study of the symmetric powers $\mathrm{Sym}^n(MW_q)$ of the q-meromorphic Weyl algebra.

2. **The meromorphic Weyl algebra**

The Weyl algebra is the associative algebra over the field of complex numbers $\mathbb{C}$ given by

$$W = \mathbb{C}\langle x, y\rangle / \langle yx - xy - 1\rangle$$

where $\mathbb{C}\langle x, y\rangle$ is the free associative algebra over $\mathbb{C}$ generated by formal variables x and y, and $\langle yx - xy - 1\rangle$ is the ideal generated by $yx - xy - 1$. The Weyl algebra comes with a natural representation

$$\rho : W \longrightarrow End(\mathbb{C}[x]),$$

where $\mathbb{C}[x]$ is the vector space of polynomials in the variable x and $End(\mathbb{C}[x])$ is the algebra of endomorphisms of $\mathbb{C}[x]$, which explain why it appears so often in many branches of mathematics and physics. The map ρ is given on the generators of W by

$$\rho(x)f = xf \quad \text{and} \quad \rho(y)f = \frac{\partial f}{\partial x}.$$

Notice that in the definition above the letter x on the left-hand side is a non-commutative variable, while on the right-hand side the letter x denotes the generator of $\mathbb{C}[x]$. This sort of abuse of notation is common in the literature and we hope it causes no confusion.

The meromorphic Weyl algebra MW is the associative algebra over $\mathbb{C}$ given by

$$MW = \mathbb{C}\langle x, y\rangle / \langle yx - xy - x^2\rangle.$$

MW comes with a natural representation ρ which justifies its name. Let $C^\infty(\mathbb{R}^*)$ be the space of smooth complex valued functions on the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The representation

$$\rho : MW \longrightarrow \mathrm{End}(C^\infty(\mathbb{R}^*))$$

is defined by letting the generators of MW act on $f \in C^\infty(\mathbb{R}^*)$ as follows:

$$\rho(x)f = x^{-1}f \quad \text{and} \quad \rho(y)f = -\frac{\partial f}{\partial x}.$$

An integral analogue of the Weyl algebra is obtained by considering the operators $l(x)$ and $l(y)$ acting on $f \in C^{\infty}(\mathbb{R})$ as follows:

$$l(x)f = xf \quad \text{and} \quad l(y)f = \int_0^x f(t)dt.$$

It is not hard to see that l extends naturally to yield a representation

$$l : \mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle \longrightarrow \mathrm{End}(C^{\infty}(\mathbb{R}))$$

of the algebra

$$\mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle,$$

which is isomorphic to the meromorphic Weyl algebra via the isomorphism

$$t : MW \longrightarrow \mathbb{C}\langle x, y\rangle/\langle yx - xy + y^2\rangle$$

given on generators by $t(x) = y$ and $t(y) = x$. Thus the map $\iota : MW \longrightarrow \mathrm{End}(C^{\infty}(\mathbb{R}))$ given on generators by

$$\iota(x)f = \int_0^{\infty} f(t)dt \quad \text{and} \quad \iota(y)f = xf$$

defines a representation of the meromorphic Weyl algebra.

We will use the following notation. For $A = (A_1, \cdots, A_n) \in (\mathbb{N}^2)^n$ where $A_i = (a_i, b_i)$, we set $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$, and $|A| = (|a|, |b|) = (a_1 + \cdots + a_n, b_1 + \cdots + b_n)$.

The normal coordinates $N(A, k)$ of the monomial $\prod_{i=1}^n x^{a_i} y^{b_i} \in MW$ are given by

$$\prod_{i=1}^n x^{a_i} y^{b_i} = \sum_{k=0}^{|b|} N(A, k) x^{|a|+k} y^{|b|-k}.$$

For $k > |b|$ we set $N(A, k) = 0$.

Given vector $a = (a_1, \cdots, a_n)$ then for $i \in [[1, n-1]]$ we let $a_{>i}$ be the vector $(a_{i+1}, \cdots, a_n)$. The increasing factorial [29] is given by

$$n^{(k)} = n(n+1)(n+2)\cdots(n+k-1)$$

for $n \in \mathbb{N}$ and $k \geq 1$ an integer. In the statement of the Theorem 1 the notation $p \vdash k$ means that p is a vector $(p_1, \cdots, p_{n-1}) \in \mathbb{N}^{n-1}$ such that $|p| = \sum_{i=1}^{n-1} p_i = k$.

Theorem 1. For $(A, k) \in (\mathbb{N}^2)^n \times \mathbb{N}$ the following identity holds

$$N(A, k) = \sum_{p \vdash k} \binom{b}{p} \prod_{i=1}^{n-1} (|a_{>i}| + |p_{>i}|)^{(p_i)},$$

where

$$\binom{b}{p} = \prod_{i=1}^{n-1} \binom{b_i}{p_i}.$$

The numbers $N(A,k)$ have a nice combinatorial meaning. Let $E_1,\dots,E_n,F_1,\dots,F_n$ be disjoint sets such that $|F_i| = a_i$, $|E_i| = b_i$ for $i \in [[1,n]]$, and set $E = \sqcup E_i$, $F = \sqcup F_i$. Let M_k be the set whose elements are maps $f : F \longrightarrow \{$ subsets of $E\}$ such that:

- $f(x) \cap f(y) = \emptyset$ for $x, y \in F$;
- if $y \in f(x),\ x \in F_i,\ y \in E_j$, then $j < i$;
- $\sum_{a\in F} |f(a)| = k$.

The sets M_k provide a combinatorial interpretation for the numbers $N(A,k)$, that is

$$|M_k| = N(A,k).$$

Figure 1 illustrates the combinatorial interpretation for $N(((2,3),(3,3),(3,4)),6)$: it shows an example of a map contributing to $N(((2,3),(3,3),(3,4)),6)$.

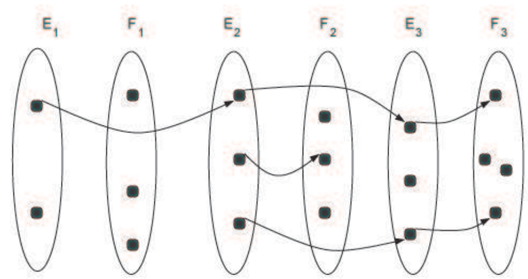

FIGURE 1. Combinatorial interpretation of $N(((2,3),(3,3),(3,4)),6)$.

Applying Theorem 1, specialized in the representation ρ, to $x^{-t} \in C^\infty(\mathbb{R}^*)$ we obtain for $(a,b,t) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}_+$ the following identity:

$$\prod_{i=1}^{n} (t + |a_{>i}| + |b_{>i}|)^{(b_i)} = \sum_{p \vdash k} \binom{b}{p} \prod_{i=1}^{n-1} (|a_{>i}| + |p_{>i}|)^{(p_i)} t^{(|b|-k)}.$$

This identity is thus an easy corollary of Theorem 1; however guessing or even proving it directly could be a bit of a pain. Applying Theorem 1, specialized in the representation ι, to x^t we get another quite intriguing identity:

$$\frac{1}{\prod_{i=1}^{n}(t+|a_{>i}|+|b_{\geq i}|+1)^{(a_i)}} = \sum_{p\vdash k}\binom{b}{p}\prod_{i=1}^{n-1}\frac{(|a_{>i}|+|p_{>i}|)^{(p_i)}}{(t+|b|-k+1)^{(|a|+k)}}.$$

A fundamental yet not fully appreciated fact in algebra is that one can associate with each associative algebra A a family of associative algebras $Sym^n(A)$ indexed by the natural numbers $n \in \mathbb{N}$. Formally, let $\mathbb{C}$-alg be the category of associative complex algebras. For $n \geq 1$ consider

$$\mathrm{Sym}^n : \mathbb{C}\text{-}alg \longrightarrow \mathbb{C}\text{-}alg$$

the functor sending an algebra A into its n-th symmetric power given by

$$\mathrm{Sym}^n(A) = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \mid a_i \in A,\ \sigma \in S_n\rangle.$$

Given $a_1 \otimes ... \otimes a_n \in A^{\otimes n}$ we denote by $\overline{a_1 \otimes ... \otimes a_n}$ the corresponding element in $\mathrm{Sym}^n(A)$. The rule for the product of m elements in $\mathrm{Sym}^n(A)$, see [15], is given as follows: let $a_{ij} \in A$ for $(i,j) \in [[1,m]] \times [[1,n]]$, then we have that

$$n!^{m-1}\prod_{i=1}^{m}\overline{\bigotimes_{j=1}^{n} a_{ij}} = \sum_{\sigma\in\{1\}\times S_n^{m-1}} \overline{\bigotimes_{j=1}^{n}\prod_{i=1}^{m} a_{i\sigma_i^{-1}(j)}},$$

where 1 denotes the identity permutation.

To our knowledge the symmetric powers have been fully studied only for a few algebras: for the algebra of polynomials whose symmetric powers may be identified with the algebra of symmetric polynomials; and for the algebra of matrices whose symmetric powers may be identified with the so called Schur algebras [15]. The symmetric powers of the Weyl algebra and its q-analogues are studied in [15, 16], the symmetric powers of the linear Boolean algebras are studied in [19].

Let $\mathrm{Sym}^n(MW)$ be the n-symmetric power of the meromorphi5c Weyl algebra. An explicit formulae for the product of m elements in $\mathrm{Sym}^n(MW)$ is provided next. We denote the element

$$\overline{x^{a_1}y^{b_1} \otimes ... \otimes x^{a_n}y^{b_n}} \in \mathrm{Sym}^n(MW) \quad \text{by} \quad \overline{\prod_{j=1}^{n} x_j^{a_j} y_j^{b_j}}.$$

Theorem 2. For each map $(a,b) : [[1,m]]\times[[1,n]] \longrightarrow \mathbb{N}^2$ the following identity holds in $\mathrm{Sym}^n(MW)$:

$$(n!)^{m-1}\prod_{i=1}^{m}\overline{\prod_{j=1}^{n} x_j^{a_{ij}} y_j^{b_{ij}}} =$$

$$= \sum_{\sigma,k,p} \left(\prod_{l=1}^{m-1} \prod_{j=1}^{n} \binom{b_j^\sigma}{p^j} (|(a_j^\sigma)_{>l}| + |p^j_{>l}|)^{(p_l^j)} \right) \overline{\prod_{j=1}^{n} x_j^{|a_j^\sigma|+k_j} y_j^{|b_j^\sigma|-k_j}}.$$

In the formula above we are using the following conventions:
$\sigma \in \{1\} \times S_n^{m-1}$, $k \in \mathbb{N}^n$ is such that $k_j \leq |b_j^\sigma|$, $p = (p^1, ..., p^n) \in (\mathbb{N}^{m-1})^n$, $p^j = (p_1^j, ..., p_{m-1}^j)$, $a_j^\sigma = (a_{1\sigma_1^{-1}(j)}, ..., a_{m\sigma_m^{-1}(j)})$, and
$b_j^\sigma = (b_{1\sigma_1^{-1}(j)}, ..., b_{m\sigma_m^{-1}(j)})$

The next example shows the high computational power required to compute even the simplest products in the symmetric powers of the meromorphic Weyl algebra.

Example 3. For $n = 2, m = 2$ we have

$$2(x_1y_1^2x_2^2y_2^2)(x_1^2y_1x_2y_2^2) =$$
$$= x_1^3y_1^4x_2^3y_2^4 + 6x_1^3y_1^4x_2^4y_2^3 + 8x_1^3y_1^4x_2^5y_2^2 + +8x_1^4y_1^3x_2^4y_2^3 + 20x_1^4y_1^3x_2^5y_2^2 +$$
$$+ 6x_1^5y_1^2x_2^3y_2^4 + 12x_1^5y_1^2x_2^5y_2^2 + x_1^3y_1^4x_2^4y_2^4 + 2x_1^3y_1^4x_2^5y_2^3 + 6x_1^3y_1^4x_2^6y_2^2 +$$
$$+ 2x_1^4y_1^3x_2^4y_2^4 + 4x_1^4y_1^3x_2^5y_2^3 + 12x_1^4y_1^3x_2^6y_2^3 + 6x_1^5y_1^2x_2^4y_2^4 + 12x_1^5y_1^2x_2^5y_2^3 +$$
$$+ 36x_1^5y_1^2x_2^6y_2^2.$$

3. **The q-meromorphic Weyl algebra**

In this section we introduce the q-meromorphic Weyl algebra and discuss some of its basic properties. Let us first review a few basic notions of q-calculus; the interested reader may consult [10, 11, 20] for further information. Let $M(\mathbb{R}^*)$ be the space of complex value functions defined on the punctured real line $\mathbb{R} \setminus \{0\}$ and fix a positive real number $0 < q < 1$. The q-derivative

$$\partial_q : M(\mathbb{R}^*) \longrightarrow M(\mathbb{R}^*)$$

is given by

$$\partial_q f = \frac{I_q f - f}{(q-1)x},$$

where $I_q f(x) = f(qx)$ for $x \in \mathbb{R}^*$.

Definition 4. The q-meromorphic Weyl is the algebra given by

$$MW_q = \mathbb{C}\langle x, y\rangle[q]/\langle yx - qxy - x^2\rangle,$$

where $\mathbb{C}\langle x, y\rangle[q]$ is the free associative algebra generated by the non-commuting variables x, y and the commutative variable q.

Notice that in the definition above q is used as a formal variable rather than a number. It should always be clear from the context whether we are using q as a formal variable or as a number. Next result explains how the algebra MW_q arises in q-calculus. For our next result we make use of the q-Leibnitz rule

$$\partial_q(fg) = f\partial_q g + I_q g\partial_q f.$$

Theorem 5. a The map $\rho : MW_q \longrightarrow \mathrm{End}(M(\mathbb{R}^*))$ given on generators by

$$\rho(x)f = x^{-1}f, \quad \rho(y)f = -q^{-1}\partial_{q^{-1}}f, \quad \text{and} \quad \rho(q)f = qf$$

for $f \in M(\mathbb{R}^*)$ defines a representation of MW_q.

Proof. We must prove that

$$\rho(y)\rho(x)f = q\rho(x)\rho(y)f + \rho(x^2)f.$$

Since $\partial_{q^{-1}}x^{-1} = -qx^{-2}$ we find that

$$\begin{aligned}
\rho(y)\rho(x)f &= \rho(y)(x^{-1}f) = -q^{-1}\partial_{q^{-1}}(x^{-1}f) \\
&= -q^{-1}(q^{-1}x)^{-1}\partial_{q^{-1}}f - q^{-1}f\partial_{q^{-1}}(x^{-1}) \\
&= -x^{-1}\partial_{q^{-1}}f + x^2 f \\
&= q\rho(x)\rho(y)f + \rho(x^2)f.
\end{aligned}$$

□

Recall [10] that the Jackson integral of a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$\int_0^x f(t)d_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x).$$

A non-fully exploited feature of the Jackson integral is that it satisfies a twisted form of the Rota-Baxter identity [9, 12, 29]; indeed one can show that

$$\begin{aligned}
\left(\int_0^x f(s)d_q s\right)\left(\int_0^x g(t)d_q t\right) &= \int_0^x \left(\int_0^t f(s)d_q s\right) g(t)d_q t + \\
&+ \int_0^x f(t)\left(\int_0^{qt} g(s)d_q s\right) d_q t.
\end{aligned}$$

It is not hard to check that the Jackson integral is a right inverse operator for the q-derivative, that is

$$\partial_q \int_0^x f(t)d_q t = f(x).$$

From the q-Leibniz rule and the fundamental theorem of q-calculus one obtains the q-integration by parts formula

$$\int_0^x I_q f \partial_q g d_q t = f(x)g(x) - f(0)t(0) - \int_0^x g \partial_q f d_q t.$$

In particular setting

$$f(x) = x \quad \text{and} \quad g(x) = \int_0^x f(t) d_q t$$

we obtain the relation

$$x \int_0^x f d_q t = q \int_0^x t f d_q t + \int_0^x \int_0^t f d_q s d_q t.$$

Let $I(\mathbb{R})$ be a space of functions on the real line closed under Jackson integration and under multiplication by polynomial functions. The previous considerations give the following result.

Theorem 6. The map

$$\iota : MW_q \longrightarrow \operatorname{End}(I(\mathbb{R}))$$

given on generators by

$$\iota(x)f = \int_0^x f d_q t, \quad \iota(y)f = xf, \quad \text{and} \quad \iota(q)f = qf,$$

for $f \in I(\mathbb{R})$ defines a representation of MW_q.

We order the generators of MW_q as $q < x < y$. A monomial in MW_q of the form $q^a x^b y^c$ is said to be in normal form. One can show that the set monomials in normal form is a basis for MW_q. Recall from the introduction that we are writing $[n] = 1 + ... + q^{n-1}$ for an integer $n \geq 1$.

Lemma 7. For $n \geq 1$ the identity $yx^n = q^n x^n y + [n]x^{n+1}$ holds in MW_q.

Proof. For $n = 1$ we get $yx = qxy + x^2$. By induction we have that

$$\begin{aligned} yx^{n+1} &= yx^n x = (q^n x^n y + [n]x^{n+1})x = q^n x^n (yx) + [n]x^{n+1}x = \\ &= q^{n+1}x^{n+1}y + [n+1]x^{n+2}. \end{aligned}$$

□

Definition 8. Let $(a, b) \in \mathbb{N}$ and $0 \leq k \leq a$. The normal coordinates $c(a, b, k)$ are the elements of $\mathbb{N}[q]$ given by the following identity in MW_q:

$$y^a x^b = \sum_{k=0}^{a} c(a, b, k) x^{b+k} y^{a-k}.$$

For $k > a$ we set $c(a,b,k) = 0$. Notice that by definition $c(0,b,k) = \delta_{0,k}$ where δ is Kronecker's delta function.

Proposition 9. The following identities hold in MW_q:

(1) $c(a+1,b,k) = c(a,b,k)q^{b+k} + c(a,b,k-1)[b+k-1]$ for $1 \leq k \leq a$.
(2) $c(a+1,b,0) = c(a,b,0)q^b$.
(3) $c(a+1,b,a+1) = c(a,b,a)[b+a]$.

Proof. By Lemma 7 and Definition 8 we have

$$yx^b = \sum_{k=0}^{1} c(1,b,k)x^{b+k}y^{1-k} = q^b x^b y + [b]x^{b+1},$$

thus $c(1,b,0) = q^b$ and $c(1,b,1) = [b]$. On the other hand we compute

$$\begin{aligned} y^{a+1}x^b &= \sum_{k=0}^{a} c(a,b,k)(yx^{b+k})y^{a-k} \\ &= \sum_{k=0}^{a} c(a,b,k)(q^{b+k}x^{b+k}y + [b+k]x^{b+k+1})y^{a-k} \\ &= c(a,b,0)q^b x^b y^{a+1} + \sum_{k=1}^{a} c(a,b,k)q^{b+k}x^{b+k}y^{a+1-k} \\ &\quad + \sum_{k=1}^{a} c(a,b,k-1)[b+k-1]x^{b+k}y^{a+1-k} + \\ &\quad + c(a,b,a)[b+a]x^{a+b+1}. \end{aligned}$$

By definition we have that

$$y^{a+1}x^b = \sum_{k=0}^{a+1} c(a+1,b,k)x^{b+k}y^{a+1-k}.$$

Therefore we have shown that

$$\begin{aligned} &\sum_{k=0}^{a+1} c(a+1,b,k)x^{b+k}y^{a+1-k} = c(a,b,0)q^b x^b y^{a+1} \\ &+ \sum_{k=1}^{a} \Big(c(a,b,k)q^{b+k} + c(a,b,k-1)[b+k-1]\Big)\, x^{b+k}y^{a+1-k} \\ &+ c(a,b,a)[b+a]x^{a+b+1}. \end{aligned}$$

Considering this equality termwise gives the desired identities.

□

Notice that the first identity from Proposition 9 together with the initial conditions $c(0,b,k)=\delta_{0,k}$ completely determine the function $c(a,b,k)$. We shall use this fact in the proof of Theorem 11. Our next result shows that $c(a,b,a)$ is the q-analogue of the increasing factorial.

Lemma 10. (1) $c(a,b,0)=q^{ab}$.
(2) $c(a,b,a)=[b][b+1]\dots[b+a-1]=[b]^{(a)}$.

Proof. Clearly $c(1,b,0)=q^b$. Moreover by Proposition 9 we have that

$$c(a+1,b,0)=c(a,b,0)q^b=q^{ab}q^b=q^{(a+1)b}.$$

For $a=1$ we have $c(1,b,1)=[b]^{(1)}=[b]$, and using again Proposition 9 we get

$$c(a+1,b,a+1)=c(a,b,a)[b+a]=[b]^{(a)}[b+a]=[b]^{(a+1)}.$$

□

We are ready to discuss the combinatorial interpretation of the normal polynomials $c(a,b,k)$. Let $P_k[[1,a]]$ be the set of subsets of $[[1,a]]$ with k elements. We define a q-weight

$$\omega_b : P_k[[1,a]] \longrightarrow \mathbb{N}[q]$$

which sends $A\in P_k[[1,a]]$ into

$$\omega_b(A)=[b]^{(k)}q^{(a-k)b}q^{\sum_{i\in A^c}|A_{<i}|}.$$

Theorem 11. For $(a,b)\in\mathbb{N}\times\mathbb{N}_+$ and $0\le k\le a$, we have that $c(a,b,k)=|P_k[[1,a]],\omega_b|$.

Proof. We have to show that

$$c(a,b,k)=|P_k(a),\omega_b|=[b]^{(k)}q^{(a-k)b}\sum_{A\in P_k[[1,a]]}q^{\sum_{i\in A^c}|A_{<i}|}.$$

Let $\overline{c}(a,b,k)$ be given by the right hand side of formula above for $a\ge 1$ and $\overline{c}(0,b,k)=\delta_{0,k}$. We must show that $\overline{c}(a,b,k)=c(a,b,k)$. Since $\overline{c}(0,b,k)=c(0,b,k)$, it is enough to show that $\overline{c}(a,b,k)$ satisfies, for $1\le k\le a$, the recursion

$$\overline{c}(a+1,b,k)=\overline{c}(a,b,k)q^{b+k}+\overline{c}(a,b,k-1)[b+k-1].$$

Sets $A \in P_k[[1,a+1]]$ are classified in two blocks according to whether $a+1 \notin A$ or $a+1 \in A$. Thus we obtain that

$$\overline{c}(a+1,b,k) = |P_k(a+1),\omega_b| = [b]^{(k)} q^{(a-k+1)b} \sum_{A \in P_k[[1,a+1]]} q^{\sum_{i\in A^c} |A_{<i}|}$$

is equal to the sum of two terms

$$\left([b]^{(k)} q^{(a-k)b} \sum_{A\in P_k[[1,a]]} q^{\sum_{i\in A^c}|A_{<i}|} \right) q^{b+k} \ +$$

$$\left([b]^{(k-1)} q^{(a-k+1)b} \sum_{A\in P_{k-1}[[1,a]]} q^{\sum_{i\in A^c}|A_{<i}|} \right) [b+k-1].$$

Thus the numbers $\overline{c}(a,b,k)$ satisfy the required recursion. □

Let us remark that writing $A \in P_k[[1,a]]$ as $A = \{t_1 < t_2 < \cdots < t_k\}$, using the elementary identity

$$\sum_{i\in A^c} |A_{<i}| = \sum_{s=1}^{k} s(t_{s+1} - t_s - 1)$$

and setting $t_{k+1} = a+1$ we obtain that:

$$c(a,b,k) = [b-1]^{(k)} q^{(a-k)b} \sum_{1\le t_1 < \cdots < t_k \le a} q^{\sum_{s=1}^{k} s(t_{s+1}-t_s-1)}.$$

4. **Normal polynomials and symmetric powers of** MW_q

In this section we find explicit formulae for the normal polynomials of the algebra MW_q. We also begin the study of the symmetric power of that algebra.

Definition 12. Let $A = (A_1, \cdots, A_n) \in (\mathbb{N}^2)^n$ with $A_i = (a_i, b_i)$. The normal polynomial $N(A,k,q) \in \mathbb{N}[q]$ is defined by the following identity in MW_q:

$$\prod_{i=1}^{n} x^{a_i} y^{b_i} = \sum_{k=0}^{|b|} N(A,k,q) x^{|a|+k} y^{|b|-k}.$$

For $k > |b|$ we set $N(A,k,q) = 0$.

Recall from Section 2 that the notation $p \vdash k$ means that p is a vector $(p_1, \cdots, p_{n-1}) \in \mathbb{N}^{n-1}$ such that $|p| = \sum_{i=1}^{n-1} p_i = k$. Our next result is obtained using Definition 8 several times.

Theorem 13. For $(A,k) \in (\mathbb{N}^2)^n \times \mathbb{N}$ we have that

$$N(A,k,q) = \sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i),$$

where the partition p of k must be such that $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

It is not hard to show that the normal polynomial may also be computed via the identity

$$N(A,k,q) = \sum_{p \vdash k} \prod_{i=1}^{n-1} c(|b_{\leq i}| - |p_{<i}|, a_{i+1}, p_i),$$

where $0 \leq p_i \leq |b_{\leq i}| - |p_{<i}|$ for $i \in [[1, n-1]]$.

Applying Theorem 13, specialized in the representation ρ, to x^{-t} we obtain that if $(a,b,t) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}_+$ then

$$\prod_{i=1}^{n} [t + |b_{\geq i}| + |a_{>i}| - 1] = \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i) \right) [t + |b| - k - 1],$$

where $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

Using the alternative expression for $N(A,k,q)$ given above, one obtains that:

$$\prod_{i=1}^{n} [t + |b_{\geq i}| + |a_{>i}| - 1] = \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(|b_{\leq i}| - |p_{<i}|, a_{i+1}, p_i) \right) [t + |b| - k - 1],$$

where $0 \leq p_i \leq |b_{\leq i}| - |p_{<i}|$ for $i \in [[1, n-1]]$.

If instead of ρ we use the representation ι applied to x^t we get the identity:

$$\frac{1}{\prod_{i=1}^{n} [t + |a_{\geq i}| + |b_{\geq i}| + 1]^{(a_i)}} =$$
$$= \sum_{k=0}^{|b|} \left(\sum_{p \vdash k} \prod_{i=1}^{n-1} c(b_i, |a_{>i}| + |p_{>i}|, p_i) \right) \frac{1}{[t + |a| + |b|]^{(|a|+k)}},$$

where $0 \leq p_i \leq b_i$ for $i \in [[1, n-1]]$.

Also with the alternative expression for $N(A,k,q)$ we get:

$$\frac{1}{\prod_{i=1}^{n}[t+|a_{\geq i}|+|b_{\geq i}|+1]^{(a_i)}}=$$
$$=\sum_{k=0}^{|b|}\left(\sum_{p\vdash k}\prod_{i=1}^{n-1}c(|b_{\leq i}|-|p_{<i}|,a_{i+1},p_i)\right)\frac{1}{[t+|a|+|b|]^{(|a|+k)},}$$

where $0\leq p_i\leq |b_{\leq i}|-|p_{<i}|$ for $i\in[[1,n-1]]$.

Next we provide explicit formulae for the products of several elements in the n-th symmetric power $\mathrm{Sym}^n(MW_q)$ of the q-meromorphic Weyl algebra MW_q.

Theorem 14. For each map $(a,b):[[1,m]]\times[[1,n]]\longrightarrow\mathbb{N}^2$ the following identity holds in $\mathrm{Sym}^n(MW)$:

$$(n!)^{m-1}\prod_{i=1}^{m}\overline{\prod_{j=1}^{n}x_j^{a_{ij}}y_j^{b_{ij}}}=$$
$$=\sum_{\sigma,k,p}\left(\prod_{l=1}^{m-1}\prod_{j=1}^{n}c((b_j^{\sigma})_l,|(a_j^{\sigma})_{>l}|+|p_{>l}^{j}|,p_l^{j})\right)\overline{\prod_{j=1}^{n}x_j^{|a_j^{\sigma}|+k_j}y_j^{|b_j^{\sigma}|-k_j}}.$$

In the formula above we are using the following conventions:
$\sigma\in\{1\}\times S_n^{m-1}$, $k\in\mathbb{N}^n$ is such that $k_j\leq|b_j^{\sigma}|$, $p=(p^1,...,p^n)\in(\mathbb{N}^{m-1})^n$, $p^j=(p_1^j,...,p_{m-1}^j)$, $a_j^{\sigma}=(a_{1\sigma_1^{-1}(j)},...,a_{m\sigma_m^{-1}(j)})$, and
$b_j^{\sigma}=(b_{1\sigma_1^{-1}(j)},...,b_{m\sigma_m^{-1}(j)})$

The explicit computation of products in $\mathrm{Sym}^n(MW_q)$ is rather difficult as the following example shows.

Example 15. For $n=2,m=2$ we have

$$\begin{gathered}2(x_1y_1x_2^2y_2)(x_1^2y_1^2x_2y_2)=x_1y_1x_2^2y_2x_1^2y_1^2x_2y_2+x_1y_1x_2^2y_2x_1y_1x_2^2y_2^2=\\ q^3x_1^3y_1^3x_2^3y_2^2+q^2x_1^3y_1^3x_2^4y_2+(q^2+q)x_1^4y_1^2x_2^3y_2^2+(q+1)x_1^4y_1^2x_2^4y_2+\\ q^3x_1^2y_1^2x_2^4y_2^3+(q^2+q)x_1^2y_1^2x_2^5y_2^2+q^2x_1^3y_1x_2^4y_2^3+(q+1)x_1^3y_1x_2^5y_2^2.\end{gathered}$$

We close this work mentioning a couple of research problems. First, it would be interesting to study the Hochschild cohomology of the meromorphic and q-meromorphic Weyl algebras and their corresponding symmetric powers along the lines developed in [1, 2]. Second, using techniques introduced in [18] and further developed in [6, 7, 8, 9] we have constructed a categorification of the Weyl algebra, and more generally of the Kontsevich star product [27] for Poisson structures on $\mathbb{R}^n$. It would be interesting to study the categorification of the meromorphic and q-meromorphic Weyl algebras.

Acknowledgements. Our thanks to Nicolas Andruskiewitsch, Takashi Kimura and Sylvie Paycha. We also thank a couple of anonymous referees for helpful suggestions and comments. The second author thanks the organizing committee for inviting her to participate in the "XVII Coloquio Latinoamericano de Algebra" held in Medell'in, Colombia, 2007.

References

[1] J. Alev, M. Farinati, T. Lambre, A. Solotar, Homologie des invariants d'une algèbre de Weyl sous l'action d'un groupe fini, J. Algebra 232 (2000) 564-577.

[2] J. Alev, T. Lambre, Homologie des invariants d'une algèbre de Weyl, K-Theory 18 (1999) 401-411.

[3] G. Andrews, The theory of partitions, Cambridge Univ. Press, Cambridge, 1938.

[4] G. Andrews, R. Askey, R. Roy, Special functions, Cambridge Univ. Press, Cambridge, 1999.

[5] M. Angel, R. Díaz, On the q-analogue of the Maurer-Cartan, Advan. Stud. Contemp. Math. 12 (2006) 315-322.

[6] H. Blandín, R. Díaz, Compositional Bernoulli numbers, Afr. Diaspora J. Math. 7, in press.

[7] H. Blandín, R. Díaz, On the combinatorics of hypergeometric functions, Adv. Stud. Contemp. Math. 14 (2007) 153-160.

[8] H. Blandín, R. Díaz, Rational combinatorics, Adv. Appl. Math. 40 (2008) 107-126.

[9] E. Castillo, R. Díaz, Rota-Baxter categories, Int. Electron. J. Algebra, in press.

[10] P. Cheung, V. Kac, Quantum calculus, Springer-Verlag, Berlin, 2002.

[11] A. De Sole, V. Kac, On integral representations of q-gamma and q-beta functions, Rend. Mat. Acc. Lincei 9 (2005) 11-29.

[12] R. Díaz, M. Páez, An identity in Rota-Baxter algebras, Sem. Lothar. Combin. 57 (2007) B57b.

[13] R. Díaz, E. Pariguan, An example of Feynman-Jackson integral, J. Phys. A 40 (2007) 1265-1272.

[14] R. Díaz, E. Pariguan, Feynman-Jackson integrals, J. Nonlinear Math. Phys. 13 (2006) 365-376.

[15] R. Díaz, E. Pariguan, Quantum symmetric functions, Comm. Alg. 33 (2005) 1947-1978.

[16] R. Díaz, E. Pariguan, Symmetric quantum Weyl algebra, Ann. Math. Blaise Pascal 11 (2004) 187-203.

[17] R. Díaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, Divulg. Mat. 15 (2007) 179-192.

[18] R. Díaz, E. Pariguan, Super, quantum and noncommutative species, preprint, arXiv:math.CT/0509674.

[19] R. Díaz, M. Rivas, Symmetric Boolean Algebras, preprint, arXiv:math.CO/0612134.

[20] R. Díaz, C. Teruel, q,k-generalized gamma and beta functions, J. Nonlinear Math. Phys. 12 (2005) 118-134.

[21] R. Fenn, V. Turaev, Weyl Algebras and Knots, J. Geom. Phys. 57 (2007) 1313-1324.

[22] S. Garoufalidis, T. Lê, D. Zeilberger, The Quantum MacMahon Master Theorem, Proc. Natl. Acad. Sci. 103 (2006) 13928-31.

[23] G. Gasper, M. Rahman, Basic hypergeometric series, Cambridge Univ. Press, Cambridge, 1990.

[24] M. Kapranov, On the q-analog of homological algebra, preprint, arXiv:math.QA/9611005.

[25] C. Kassel, Quantum Groups, Springer-Verlag, Berlin, 1995.

[26] T. Kim, On the q-extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007) 1458-1465.

[27] M. Kontsevich, Deformation Quantization of Poisson Manifolds, Lett. Math. Phys. 66 (2003)157-216.

[28] B. Kupershmidt, q-Probability: I. Basic Discrete Distributions, J. Nonlinear Math. Phys. 7 (2000) 73-93.

[29] G.-C. Rota, Gian-Carlo Rota on Combinatorics, J. Kung (Ed.), Birkhäuser, Boston, 1995.

[30] D. Zeilberger, Proof of the Refined Alternating Sing Matrix Conjecture, New York J. Math. 2 (1996) 59-68.

[31] D. Zeilberger, Enumerative and algebraic combinatorics, in: T. Gowers (Ed.), The Princeton Companion to Mathematics, Princeton Univ. Press, Princeton, 2008.