PROJECTIVE PRIME IDEALS AND LOCALISATION IN PI-RINGS

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1. *Introduction*

The results here generalise [**2**, Proposition 4.3] and [**9**, Theorem 5.11]. We shall prove the following.

THEOREM A. *Let R be a Noetherian PI-ring. Let P be a non-idempotent prime ideal of R such that P_R is projective. Then P is left localisable and R_P is a prime principal left and right ideal ring.*

We also have the following theorem.

THEOREM B. *Let R be a Noetherian PI-ring. Let M be a non-idempotent maximal ideal of R such that M_R is projective. Then M has the left AR-property and M contains a right regular element of R.*

Thus the results show an intriguing relationship between properties on the two sides of the ring. An easy example (Example 4.1) shows that in Theorem A the right Ore condition need not hold with respect to $\mathscr{C}(P)$. A further example (Example 4.2) demonstrates that the assumption 'maximal ideal' cannot be weakened to 'prime ideal' in relation to the left AR-property.

One of the results required along the way is of independent interest. We prove (Lemma 3.1) that in a Noetherian ring an ideal with zero right annihilator and satisfying (a weak form of) the right AR-property contains a right regular element.

Our methods require Theorem A to be first proved for a maximal ideal. Extending the result to a general prime ideal presents a technical challenge. Since it is not yet known whether the cliques in a Noetherian PI-ring are localisable, a direct localisation approach is not available to us. We sidestep this difficulty by employing a trick of Goodearl and Stafford. This device guarantees that the prime ideal being examined extends to a prime ideal which belongs to a localisable clique in a polynomial extension of the given Noetherian PI-ring. With the authors' permission an account of this method is included here.

2. *Preliminaries and notation*

All rings have an identity element and all subrings considered are assumed to have the same identity.

Let R be a ring with an overring T and let I be an ideal of R. Then I is said to be *invertible in T* if there exists a subset S of T such that $SI = IS = R$. In this case the

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set $\{s \in T \mid Is \subseteq R\} = \{s \in T \mid sI \subseteq R\}$ and we will call this set I^{-1}. We simply say I is *invertible* if the overring is clear from the context. An ideal I is said to have the *right Artin–Rees property* if given a right ideal E there exists a natural number n such that $E \cap I^n \subseteq EI$. The *left Artin–Rees property* is defined analogously. If I is an ideal in a Noetherian ring which is invertible in some overring then by [**10**, Corollary 2.5, page 101], I has the left and right Artin–Rees property.

We denote by $\mathscr{C}(I)$ the set of elements which are regular mod I. If we need to emphasise the ring we will write $\mathscr{C}_R(I)$. Thus $\mathscr{C}(0)$ denotes the set of regular elements of R. If R is Noetherian and P is a maximal ideal it is well known that $\mathscr{C}(P) = \mathscr{C}(P^n)$ for all $n \geqslant 1$. We denote by N the nilpotent radical of R.

If U_R is a right R-module then the set of right R-module homomorphisms from U_R to R_R is denoted by U^*. This is in fact a left R-module under the natural action. If V is any subset of U then $U^*(V)$ consists of finite sums $\sum_{i=1}^n f_i(x_i)$ where $f_i \in U^*$ and $x_i \in V$. It can be seen that if K is a right ideal of R then $K^*(K)$ is a two-sided ideal of R with $K \subseteq K^*(K)$.

We denote the *reduced rank* of a finitely generated right module U over a right Noetherian ring R by $\rho(U)$. For the definition of reduced rank and some of its properties see [**4**, Chapter 2].

We denote the Krull dimension (in the sense of Gabriel–Rentschler) of a right R-module M over a ring R by $\operatorname{Kdim}(M_R)$. For more details of this Krull dimension and some of its properties see [**10**, Chapter 6].

The dual basis lemma states that a module U_R is projective if and only if there exist families $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, $\{u_\lambda\}_{\lambda\in\Lambda}$, $\sigma_\lambda \in U^*$, $u_\lambda \in U$ (sometimes referred to as a *dual basis for* U_R) such that for each $u \in U$, $u = \sum_\alpha u_\alpha(\sigma_\alpha(u))$ and $\sigma_\alpha(u) = 0$ for all but finitely many α. It can be easily deduced from the dual basis lemma that if $I \subseteq K$ are ideals of R with K_R projective and $I = KI$ then K/I is a projective right R/I module.

Let R be a Noetherian ring. If P and Q are prime ideals of R then we say there is a *link from Q to P* written $Q \rightsquigarrow P$ if there is an ideal A of R such that $Q \cap P \supsetneqq A \supseteq QP$ and $(Q \cap P)/A$ is torsion-free as a right R/P-module and as a left R/Q-module. A set X of prime ideals of R is said to be *right link closed* if whenever P and Q are prime ideals of R with $Q \rightsquigarrow P$ and $P \in X$ then $Q \in X$. A left and right link closed set of prime ideals of R is called a *clique* if no proper subset is both left and right link closed.

A set Y of prime ideals of R is said to satisfy the *intersection condition* if given any one-sided ideal I of R with $I \cap \mathscr{C}(P) \neq \varnothing$ for all $P \in Y$ then $I \cap (\bigcap_{P\in Y} \mathscr{C}(P)) \neq \varnothing$.

Let δ be a multiplicatively closed subset of R. The set δ is called a *right Ore set* if given $a \in R$ and $c \in \delta$ there exist $a_1 \in R$ and $c_1 \in \delta$ such that $ac_1 = ca_1$. Left Ore sets are defined analogously. A prime ideal P is said to be *right localisable* if $\mathscr{C}(P)$ is a right Ore set.

For any unexplained terminology we refer the reader to [**4**] or [**5**].

3. *The main theorem*

We shall start by looking at the special case of right projective non-idempotent maximal ideals.

It is well known that in a commutative Noetherian ring an ideal I contains a regular element if and only if I has zero annihilator. The next result may be viewed as a generalisation of this fact.

LEMMA 3.1. *Let R be a Noetherian ring. Let X be an ideal with the right AR-property such that $r(X) = 0$. Then X contains a right regular element.*

Proof. The AR-property gives the following 'commutativity' property for X. If Y is any ideal of R then there is a positive integer t such that $X^t Y \subseteq Y \cap X^t \subseteq YX$.

Since R is Noetherian we can find a chain of ideals

$$0 = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \dots \subsetneq B_{n-1} \subsetneq B_n = R$$

and prime ideals $P_1, \dots P_n$ such that P_i is a maximal left annihilator prime of ${}_R(R/B_{i-1})$, $B_i = \{r \in R \mid P_i r \subseteq B_{i-1}\}$ for $i = 1, \dots, n$ and $P_n = B_{n-1}$. For full details of this see [**4**, Chapter 13].

By [**4**, Lemmas 13.3, 13.4], we need to show that X is not contained in any P_i. Suppose that $X \subseteq P_i$ for some i. Then $XB_i \subseteq B_{i-1}$, so that $P_1 P_2 \dots P_{i-1} X B_i = 0$. The 'commutativity' property for X gives that there is a positive integer k with $X^k P_1 P_2 \dots P_{i-1} B_i = 0$. Because $r(X) = 0$ we get $P_1 P_2 \dots P_{i-1} B_i = 0$. Hence $P_2 \dots P_{i-1} B_i \subseteq B_1$, so that $P_3 \dots P_{i-1} B_i \subseteq B_2$, and so on. Thus $P_{i-1} B_i \subseteq B_{i-2}$, and hence $B_i \subseteq B_{i-1}$, which is a contradiction.

REMARK 3.2. Note that in the above proof we only require X to have the property that for any *prime ideal* P of R there exists a positive integer t such that $P \cap X^t \subseteq PX$. It is in fact the case that this property is equivalent to the right AR-property in the PI case.

COROLLARY 3.3. *An invertible ideal of a Noetherian ring contains a regular element.*

Proof. As noted in Section 2 such an ideal has the left and right AR-property and clearly has zero left and right annihilators so this can be proved by an easy modification of the proof of Lemma 3.1.

The next result is standard to all practitioners of Jategaonkars's localisation theory. We include its proof for completeness.

LEMMA 3.4. *Let R be a Noetherian ring satisfying the strong second layer condition. Let M be a right localisable maximal ideal. Then M has the right AR-property.*

Proof. Let X be a finitely generated right R-module with an essential submodule Y such that $YM = 0$. Since M is maximal we have $M = r(Y)$. Since Y is essential we have $\text{ass}(X) = \{M\}$. By [**5**, Theorem 11.4], X is annihilated by a product of primes in the right link closure of $\{M\}$. However, since M is right localisable, by [**5**, Theorem 12.21], $\{M\}$ is right link closed and so $XM^n = 0$ for some integer $n \geqslant 1$. It follows by [**4**, Lemma 11.2] that M has the right AR-property.

LEMMA 3.5. *Let R be a Noetherian ring. Suppose that M is a maximal ideal of R which is also a minimal prime ideal of R and which is right R-projective. Then M is idempotent.*

Proof. First note that in any Noetherian semi-prime ring an ideal which is both a maximal ideal and a minimal prime must be a direct summand of the ring and hence idempotent.

Let N be the nilpotent radical of R. We prove the result by induction on K, the index of nilpotency of N. The case $k = 1$ is trivial by the above. Now assume that $k \geqslant 2$. Suppose that M is not idempotent. Because M_R is projective we know that $M^*(M)$ is an idempotent ideal of R which contains M. However M is a non-idempotent maximal ideal of R. Therefore $M^*(M) = R$. Because M/N is both a maximal ideal and a minimal prime ideal of the semi-prime Noetherian ring R/N, it follows that M/N is a direct summand of R/N. In particular M/N is idempotent, so that $M = M^2 + N$. Thus $MN^{k-1} = M^2N^{k-1}$. Since $M^*(M) = R$, multiplying this on the left by M^* gives $N^{k-1} = MN^{k-1}$ so that M/N^{k-1} is a projective right R/N^{k-1}-module. By the induction hypothesis $M = M^2 + N^{k-1} = M^2 + MN^{k-1} = M^2$. This contradiction completes the induction.

REMARK 3.6. Lemma 3.17 will show that the above result also holds with 'prime' replacing 'maximal'.

LEMMA 3.7. *Let R be a semi-prime Noetherian PI-ring. Let M be a maximal ideal. Suppose that M_R is projective. Then M is either idempotent or invertible.*

Proof. Suppose that M is not idempotent. Then M is not a direct summand of R, and because R is semi-prime it follows that M is not a minimal prime of R. Thus $M \cap \mathscr{C}(0) \neq \varnothing$. Let Q be the (semi-simple Artinian) quotient ring of R. It is standard that we can identify M^* with the set $M^l = \{q \in Q \mid qM \subseteq R\}$. Because M_R is projective it follows from the dual basis lemma that $1 \in MM^l$ and that M^lM is an idempotent ideal of R which contains M. However M is a non-idempotent maximal ideal of R, so we have $M^lM = R$. Thus M is left invertible and, by [**6**, Theorem 3.5], M is also right invertible.

THEOREM 3.8. *Let R be a Noetherian PI-ring. Let M be a non-idempotent maximal ideal such that M_R is projective. Let P be a minimal prime of R with $P \subseteq M$. Then $P = MP$.*

Proof. Note that since M is not idempotent by Lemma 3.5 we have $P \subsetneq M$. Also P/MP is a left R/M-module and hence is Artinian. Thus by Lenagan's theorem [**8**, Proposition], $(P/MP)_R$ is Artinian. Consider P/MP as a right R/P-module. By taking a composition series we see that P/MP is annihilated by a product of maximal ideals of R/P, all of which contain a regular element of R/P since R/P is not Artinian. Thus there exists $c \in \mathscr{C}(P)$ such that $Pc \subseteq MP$. This gives

$$M^*(P)\,c \subseteq M^*(MP) \subseteq M^*(M)\,P \subseteq RP = P.$$

Thus $M^*(P) \subseteq P$ and so $P = MP$ using the projectivity of M and the dual basis lemma.

COROLLARY 3.9. *Let R be a Noetherian PI-ring. Suppose that M is a non-idempotent maximal ideal and M_R is projective. Suppose that P is a minimal prime of R with $P \subseteq M$. Then $P = \bigcap_{n=1}^{\infty} M^n$. In particular P is unique. Also M/P is an invertible ideal of R/P.*

Proof. Let $I = \bigcap_{n=1}^{\infty} M^n$. By Theorem 3.8 we have $P = MP$ and so $P \subseteq I$. Also, since $P = MP$, M/P is a non-idempotent maximal ideal of R/P which is projective as a right R/P-module so that, by Lemma 3.7, M/P is invertible. Thus, by [**7**, Corollary 3.2], $\bigcap_{n=1}^{\infty}(M/P)^n = 0$ from which it follows that $I \subseteq P$ in R. Thus $\bigcap_{n=1}^{\infty} M^n = I = P$.

THEOREM 3.10. *Let R be a Noetherian PI-ring. Let M be a non-idempotent maximal ideal such that M_R is projective. Let N be the nilpotent radical of R. Then $N = MN$.*

Proof. Let A be the ideal of R such that $N \subseteq A$ and A/N is the Artin radical of R/N. Because A/N is a direct summand in R/N there is an ideal B of R such that $A \cap B = N$ and $A + B = R$. Let P be a minimal prime of R with $P \subseteq M$. Because R/P has zero Artin radical we must have $A \subseteq P$. Hence $A \subseteq M$ and $M^*(A) \subseteq R$. We will now show that $M^*(A) = A$. It is enough to show that $M^*(AB) \subseteq N$ since $A = A^2 + AB$ so that $M^*(A) = M^*(A)A + M^*(A)B \subseteq A + N = A$. We have $A = A^2 + N$ so that $M^*(AB) = M^*(A^2B) + M^*(NB)$ where $M^*(A^2B) = M^*(A)AB \subseteq RN = N$. Thus it is enough to show that $M^*(NB) \subseteq N$.

Set $S = R/MN$ and $W = N/MN$. Then W is an ideal of S which is left Artinian (because $MW = 0$) and so W_S is Artinian by [**8**, Proposition]. However $WN = 0$ so that we can consider W to be a right R/N-module. Let U be a simple right R/N-module. Then either U is torsion or U embeds in R/N and hence U embeds in A/N and so $U(B/N) = 0$. We have $B/N = g(R/N)$ for some central idempotent $g \in R/N$. Let c be a regular (indeed, arbitrary) element of R/N. Then $wcg = wgc$ for all $w \in W$. Because W has a composition series as a right R/N-module and because every simple right R/N-module is annihilated either by g or by some regular element of R/N, there is a regular element c of R/N such that $0 = Wgc = W(B/N)c = WBc$. Working back in R this gives $NBc \subseteq MN$ for some $c \in \mathscr{C}(N)$. Therefore $M^*(NB)c = M^*(NBc) \subseteq M^*(MN) \subseteq M^*(M)N \subseteq N$ and so $M^*(NB) \subseteq N$ as required.

Therefore $M^*(A) = A$ and so $A = MA$. We have $A + B = R$ and $A \cap B = N$ so that $N = AB + BA$. In order to show that $MN = N$ it is now enough to show that $N = AB + MBA$. We shall therefore work in the ring R/AB. Note that M/AB is projective as a right R/AB-module because $AB = MAB$.

From now on we shall suppose without loss of generality that $AB = 0$. Thus $A + B = R$ and $A \cap B = N$, so that $B = eR$ and $A = R(1-e)$ for some idempotent element e of R. We can identify R with the 2×2 upper triangular matrix ring

$$\begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$$

where $S = eRe$, $W = eR(1-e)$, $T = (1-e)R(1-e)$. We have $N = BA = eR(1-e)$, so that

$$N = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}$$

in the matrix representation of R. Therefore $R/N \cong S \oplus T$, so that S and T are semi-prime. Note that

$$B = \begin{pmatrix} S & W \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & W \\ 0 & T \end{pmatrix}.$$

Because $A \subseteq M$ we must have

$$M = \begin{pmatrix} V & W \\ 0 & T \end{pmatrix}$$

for some maximal ideal V of S.

We shall now show that V is right projective, equivalently that eMe is right eRe-projective. Note first that $Re = eRe$ and so $ReR = eR$. Thus $(ReR)_R$ is projective. The projectivity of eMe now follows, since as right eRe-modules, eMe is a direct summand of Me and by [**2**, Corollary 2.12], Me is projective.

Therefore V_S is projective and not idempotent. By Lemma 3.7, V is an invertible ideal of S and so has the right AR-property. Thus M has the right AR-property modulo N. However $A = R(1-e)$ so that $A^2 = A$. Hence

$$N = BA = BA^2 \subseteq BAM = NM.$$

For any prime ideal Q of R there exists an integer $s \geqslant 1$ such that $Q \cap M^s \subseteq QM + N$. Since $N = NM \subseteq QM$ this gives $Q \cap M^s \subseteq QM$. It follows by Lemma 3.1 (noting Remark 3.2 and since $M^*(M) = R$) that M contains a right regular element d say. We have $N \cong dN$ as right R-modules so we have $\rho(N/dN) = 0$. Since $dN \subseteq MN$ this gives $\rho(N/MN) = 0$ and so there exists $d_1 \in \mathscr{C}(N)$ such that $Nd_1 \subseteq MN$. Hence $M^*(N)\,d_1 = M^*(Nd_1) \subseteq M^*(MN) = M^*(M)\,N \subseteq N$ and so $M^*N \subseteq N$. This gives $MN = N$ as required.

Theorem 3.11. *Let R be a Noetherian PI-ring. Suppose that M is a non-idempotent maximal ideal and that M_R is projective. Let Q be a prime ideal of R. Then there exists a positive integer t such that $Q \cap M^t \subseteq MQ$. It follows that M is left localisable.*

Proof. By Corollary 3.9, M contains a unique minimal prime P_1 say. Suppose that R contains other minimal primes $P_2, P_3, \ldots, P_n$. (We deal with the possibility that P_1 is the only minimal prime later.) Then we have $M + (P_2 \cap P_3 \cap \ldots \cap P_n) = R$ and, of course, $N = P_1 \cap P_2 \cap \ldots \cap P_n$, where N is the nilpotent radical of R. This gives $Q \cap P_1 = M(Q \cap P_1) + (P_2 \cap P_2 \cap \ldots \cap P_n)(Q \cap P_1)$ and so $Q \cap P_1 \subseteq M(Q \cap P_1) + N = M(Q \cap P_1) + MN = M(Q \cap P_1)$ (since $N = MN$ by Theorem 3.10). By Corollary 3.9, M/P_1 is invertible and so has the left AR-property. Thus there exists a positive integer t such that

$$Q \cap M^t \subseteq MQ + P_1. \qquad (1)$$

This gives $Q \cap M^t \subseteq MQ + (Q \cap P_1) = MQ + M(Q \cap P_1) = MQ$. (Notice that this follows immediately from (1) using the facts that $N = MN$ and $N \subseteq Q$ when $N = P_1$, that is, when P_1 is the only minimal prime of R.) It is now easy to deduce that if $M \rightsquigarrow Q$ then $M = Q$ which implies that M is left localisable by [**5**, Theorem 12.21].

Theorem 3.12 (Theorem B in Section 1). *Let R be a Noetherian PI-ring. Let M be a non-idempotent maximal ideal of R such that M_R is projective. Then M has the left AR-property and M contains a right regular element of R.*

Proof. That M has the left AR-property is immediate from Theorem 3.11 and Lemma 3.4.

By Corollary 3.9 there is a unique minimal prime ideal P of R with $P \subseteq M$. We shall suppose that $P \neq 0$, for otherwise R is a prime ring and the result follows immediately from Goldie's theorem. By Theorem 3.8 and Theorem 3.11 we know that $P = MP$ and that we can left-localise R at M. Let R_M denote the corresponding local ring. Then $R_M M$ is the Jacobson radical of R_M and $R_M P = R_M MP = R_M M \cdot R_M P$. Therefore $R_M P = 0$, by Nakayama's lemma. Hence $cP = 0$ for some $c \in \mathscr{C}(M)$.

We start to construct a left affiliated series for R by taking P_1 to be a maximal left annihilator prime in R with $c \in P_1$ (note that $cP = 0$ and that we are supposing that $P \neq 0$, so that $RcR \subseteq l(P) \neq R$). Set $B_1 = r(P_1)$. Then $cB_1 = 0$ with $c \in \mathscr{C}(M) \subseteq \mathscr{C}(P)$, so that $B_1 \subseteq P$. Also the fact that $c \in P_1$ and the maximality of M give $R = M + P_1$, so that $B_1 = MB_1 + P_1 B_1 = MB_1$. Therefore $M/B_1 = M/MB_1$ is right R/B_1-projective. Hence we can do a similar thing in R/B_1. Suppose that $P/B_1 \neq 0$; let $P_2 \supseteq B_1$ be such that P_2/B_1 is a maximal left annihilator prime of R/B_1 which contains some $d \in \mathscr{C}(M/B_1)$ with $d(P/B_1) = 0$; take $B_2 \supseteq B_1$ with $B_2/B_1 = r(P_2/B_1)$; then $B_2 \subseteq P$, and so on. The B_i are strictly increasing and contained in P. Eventually we get $B_{n-1} = P$ for some n, and then we take $P_n = P$ and $B_n = R$. Each of $P_1, \ldots, P_{n-1}$ contains an element of $\mathscr{C}(M)$, and so the maximality of M gives that M is not contained in any of them. Also M is not contained in $P_n = P$. Therefore M is not contained in any of $P_1, \ldots, P_n$, and so M contains a right regular element of R (see [**4**, Lemmas 13.3 and 13.4]).

Next we shall show how to extend some of the results for maximal ideals to the case of general right projective non-idempotent prime ideals.

LEMMA 3.13. *Let R be a Noetherian ring. Let P be a non-idempotent prime ideal of R with P_R projective. Let $\delta \subseteq \mathscr{C}(P)$ be a two-sided Ore set of regular elements. Then PR_δ is not idempotent.*

Proof. Since δ is a two-sided Ore set we have $PR_\delta = R_\delta P$ and so if $(PR_\delta)^2 = PR_\delta$ we have $PR_\delta = P^2 R_\delta$ which gives $P \subseteq P^2 R_\delta$ and so there exists $c \in \delta \subseteq \mathscr{C}(P)$ such that $Pc \subseteq P^2$. Thus $P^*(P)\, c = P^*(Pc) \subseteq P^*(P^2) = P^*(P)\, P \subseteq P$, and so we have $P^*(P) \subseteq P$. This gives $P = P^2$, a contradiction.

The next lemma is standard.

LEMMA 3.14. *Let P be a prime ideal of a Noetherian ring R. Let δ be a two-sided Ore set of regular elements such that $P \cap \delta = \emptyset$ and PR_δ is a left localisable prime ideal of R_δ. Then P is left localisable.*

The next result we need was proved around 1985 by Goodearl and Stafford but has never been published before. Our account of the proof is based on an informal note produced by Warfield.

THEOREM 3.15 (Goodearl–Stafford lemma). *Let R be a Noetherian PI-ring, let P be a prime ideal of R, and let X denote the clique of the prime ideal $P[t]$ in the polynomial ring $R[t]$. Set $\mathscr{C}(X) = \bigcap \{\mathscr{C}(Q) : Q \in X\}$. Let K be a right ideal of $R[t]$ such that $K \cap \mathscr{C}(Q)$ is non-empty for all $Q \in X$. Then $K \cap \mathscr{C}(X)$ is non-empty, and consequently $\mathscr{C}(X)$ is an Ore set in $R[t]$ and $P[t]$ extends to a maximal ideal in the corresponding partial quotient ring $R[t]_{\mathscr{C}(X)}$.*

Proof. The first step is to show that all the primes in X are induced from primes of R (that is, have the form $Q_0[t]$ for some prime ideal Q_0 of R). Let $Q \in X$. It is enough to suppose that there is a link from $P[t]$ to Q and to show that Q is induced from a prime of R. Thus we suppose that there is an ideal I of $R[t]$ such that $P[t]Q \subseteq I \subsetneq P[t] \cap Q$ and such that $(P[t] \cap Q)/I$ is torsion-free as a left $R[t]/P[t]$-module and as a right $R[t]/Q$-module. Set $Q_0 = Q \cap R$ and $I_0 = I \cap R$. Then $PQ_0 \subseteq I_0 \subseteq P \cap Q_0$, and $(P \cap Q_0)/I_0$ is torsion-free as a left R/P-module and as a right R/Q_0-module; but we may have $I_0 = P \cap Q_0$.

Firstly suppose that $I_0 \neq P \cap Q_0$. Then $(P \cap Q_0)/I_0$ gives a link from P to Q_0. Hence there is a link from $P[t]$ to $Q_0[t]$. However $Q_0[t] \subseteq Q$ and there is a link from $P[t]$ to Q. Because $R[t]$ is a Noetherian PI-ring it follows, by [**5**, Corollary 12.6], that $Q = Q_0[t]$.

Now suppose that $I_0 = P \cap Q_0$. For the remainder of this paragraph we shall suppose without loss of generality that $I_0 = 0$. Thus R is a semi-prime ring with $P \cap Q_0 = 0$, so that at least one of P or Q_0 is a minimal prime ideal of R. Suppose that P is a minimal prime of R (the case in which Q_0 is a minimal prime is similar). Because P is a minimal prime of the semi-prime Noetherian ring R, a standard argument gives that $\mathscr{C}_R(P)$ is an Ore subset of R and hence also of $R[t]$. Because $\mathscr{C}_R(P) \subseteq \mathscr{C}_{R[t]}(P[t])$ and $P[t]$ is linked to Q, it follows, by [**5**, Lemma 12.17], that $\mathscr{C}_R(P) \subseteq \mathscr{C}_{R[t]}(Q)$. Hence $\mathscr{C}_R(P) \subseteq \mathscr{C}_R(Q_0)$, so that $Q_0 \subseteq P$. However $P \cap Q_0 = 0$ and P is a minimal prime of R. Therefore $P = Q_0 = 0$. In particular, this shows that the zero ideal of $R[t]$ is linked to Q, so that $Q = 0 = Q_0[t] = P[t]$.

At this stage we know that all the prime ideals of $R[t]$ which belong to the clique X are induced from primes of R. We now consider the right ideal K given in the statement of the theorem. We wish to show that $K \cap \mathscr{C}(X)$ is non-empty. As in the proof of [**11**, Lemma 6], there is a positive integer n and elements $c_1, \dots, c_n$ of K such that for each $Q \in X$ there exists i such that $c_i \in \mathscr{C}(Q)$. It is now enough to do the following.

Let X_1 and X_2 be subsets of X; suppose that $K \cap \mathscr{C}(X_1)$ and $K \cap \mathscr{C}(X_2)$ are non-empty; we must show that $K \cap \mathscr{C}(X_1 \cup X_2)$ is non-empty. Let $f \in R[t]$. We shall use $L(f)$ (respectively $A(f)$) to denote the right ideal of R which consists of 0 together with all the leading (respectively anti-leading) coefficients of elements of $fR[t]$. Let Q be a prime ideal of R. It can be shown, using the theory of prime Goldie rings, that the three following conditions are equivalent.

(1) $f \in \mathscr{C}(Q[t])$;
(2) $L(f) \cap \mathscr{C}(Q)$ is non-empty;
(3) $A(f) \cap \mathscr{C}(Q)$ is non-empty.

We shall use this in conjunction with the fact that all the elements of X are induced from primes of R. We fix $c_1 \in K \cap \mathscr{C}(X_1)$ and $c_2 \in K \cap \mathscr{C}(X_2)$. For each $g \in R[t]$ with $c_2 g \neq 0$, we can choose a positive integer k large enough to ensure that the leading coefficient of $c_2 g$ is equal to that of $(c_1 + c_2 t^k) g$. Because $L(c_2)$ is generated by the leading coefficients of a finite number of such elements $c_2 g$, there is a positive integer k such that $L(c_2) \subseteq L(c_1 + c_2 t^k)$. For every $Q[t] \in X_2$ we have $c_2 \in \mathscr{C}(Q[t])$, that is $L(c_2) \cap \mathscr{C}(Q)$ is non-empty, so that $L(c_1 + c_2 t^k) \cap \mathscr{C}(Q)$ is non-empty. It follows that $c_1 + c_2 t^k \in \mathscr{C}(X_2)$. Similarly, by increasing k if necessary, we can ensure that $A(c_1) \subseteq A(c_1 + c_2 t^k)$ and hence that $c_1 + c_2 t^k \in \mathscr{C}(X_1)$. Therefore for some k we have $c_1 + c_2 t^k \in K \cap \mathscr{C}(X_1 \cup X_2)$.

We have proved that the set X of prime ideals of $R[t]$ satisfies the intersection condition. The rest of the theorem follows from [**10**, Theorem 4.3.17 and Theorem 4.3.18].

THEOREM 3.16. *Let R be a Noetherian PI-ring. Let P be a non-idempotent prime ideal of R such that P_R is projective. Then P is left localisable.*

Proof. Let $S = R[t]$. Let X denote the clique of the prime ideal $P[t]$ in $R[t]$. Let $\delta = \bigcap_{Q \in X} \mathscr{C}(Q)$. By Theorem 3.15, δ is a (two-sided) Ore set in S. Let

$$I = \{s \in S \mid sc = 0 \text{ for some } c \in \delta\}.$$

Then $I = \{s \in S \mid cs = 0 \text{ for some } c \in \delta\}$ since δ is a two-sided Ore set.

Consider $T = S_\delta$ which is an overring of S/I. By Theorem 3.15 we know that $P[t]\,T$ is a maximal ideal of T. If $P[t]/I$ is idempotent then, letting $W = P[t]$, we have $Wc \subseteq W^2$ for some $c \in \delta$. Thus $W^*(Wc) \subseteq W$ with $c \in \mathscr{C}(W)$. Then $W^*(W) \subseteq W$ which gives that $W = P[t]$ is idempotent and so P is idempotent, a contradiction. Thus $P[t]/I$ is not idempotent. By Lemma 3.13, $P[t]\,T$ is a non-idempotent ideal of T. Also $P[t]\,T$ is right projective and thus is left localisable by Theorem 3.11. It follows by Lemma 3.14 that $P[t]/I$ is a left localisable prime ideal of S/I. Given this, it is straightforward to deduce that $P[t]$ is a left localisable prime ideal of S.

Let Q be a prime ideal of R with $P \rightsquigarrow Q$. It can be checked that $P[t] \rightsquigarrow Q[t]$ as prime ideals of $R[t]$. Since $P[t]$ is left localisable by [**5**, Theorem 12.21], $P[t] = Q[t]$ and so $P = Q$. Hence if $P \rightsquigarrow Q$ then $P = Q$ and so, again by [**5**, Theorem 12.21], P is a left localisable ideal of R.

As we now know that we can left localise at a right projective non-idempotent prime ideal P we shall proceed to investigate the structure of the corresponding local ring R_P.

LEMMA 3.17. *Let R be a Noetherian PI-ring. Let P be a non-idempotent prime ideal of R such that P_R is projective. Then P is not minimal.*

Proof. Keeping the notation established in Theorem 3.16, by Lemma 3.13, $P[t]\,T$ is a non-idempotent maximal ideal of T which is projective as a right T-module. Therefore, by Lemma 3.5, $P[t]\,T$ is not minimal and thus neither is $P[t]/I$. Hence there is a prime ideal Q of $R[t]$ with $I \subseteq Q \subsetneq P[t]$. Then $Q \cap R$ is prime and $Q \cap R \subseteq P$. If $Q \cap R = P$ then $P[t] = (Q \cap R)[t] \subseteq Q$, a contradiction.

LEMMA 3.18. *Let R be a Noetherian PI-ring. Let P be a non-idempotent prime ideal of R such that P_R is right projective. Let Q be a minimal prime with $Q \subseteq P$. Then $Q = PQ = \bigcap_{n=1}^{\infty} P^n = \{r \in R \mid cr = 0$ for some $c \in \mathscr{C}(P)\}$. In particular Q is unique.*

Proof. Let $I = \{r \in R \mid cr = 0 \text{ for some } c \in \mathscr{C}(P)\}$. It is standard that $\bigcap_{n=1}^{\infty} P^n \subseteq Q$ and that $\mathscr{C}(P) \subseteq \mathscr{C}(Q)$ so that $I \subseteq Q$.

Let $W = Q/PQ$ so that W is an ideal of R/PQ and let $\alpha = \operatorname{Kdim}(R/P)$. We have $PW = 0$, so $\operatorname{Kdim}({}_R W) \leqslant \alpha$. By [**5**, Theorem 13.15], $\operatorname{Kdim}(W_R) \leqslant \alpha$. Since $WQ = 0$ we have $\operatorname{Kdim}(W_{R/Q}) \leqslant \alpha$. However $Q \subsetneq P$ by Lemma 3.17 and so $\operatorname{Kdim}(R/Q) > \alpha$. This means that W is torsion as a right (R/Q)-module, so there exists $c \in \mathscr{C}(Q)$ such that $Qc \subseteq PQ$. Hence $P^*(Q)\,c = P^*(Qc) \subseteq P^*(PQ) = P^*(P)\,Q \subseteq RQ = Q$. It follows that $P^*(Q) \subseteq Q$ so that $Q \subseteq PQ$. Thus $Q = PQ$. Hence $I \subseteq Q \subseteq \bigcap_{n=1}^{\infty} P^n \subseteq Q$. Since $Q = PQ$, after left localising at P, we have by Nakayama's lemma that $Q \subseteq I$. Thus $\bigcap_{n=1}^{\infty} P^n = Q = I$.

THEOREM 3.19 (Theorem A in Section 1). *Let R be a Noetherian PI-ring. Let P be a non-idempotent prime ideal of R such that P_R is projective. Then P is left localisable and R_P is a prime principal left and right ideal ring.*

Proof. Let $I = \{r \in R \mid cr = 0 \text{ for some } c \in \mathscr{C}(P)\}$. Thus R_P is the overring of the ring R/I obtained by left localising at P/I.

By Lemma 3.18, I is a prime ideal. Clearly P/I is a left localisable prime ideal of R/I. By [**3**, Theorem A], P/I is a right localisable prime ideal of R/I. Again by Lemma 3.18, $I = PI$. This gives that P/I is a right projective prime ideal of R/I and so $PR_P = R_P P$, the Jacobson radical of R_P, is right R_P-projective. Also $R_P P$ is not idempotent for otherwise, by Nakayama's lemma, we have $R_P P = 0$ giving $cP = 0$ for some $c \in \mathscr{C}(P)$ and thus that $P = I$. This gives, by Lemma 3.18, that P is minimal, a contradiction to Lemma 3.17. By Lemma 3.7, PR_P is an invertible ideal of R_P, in other words, $J(R_P)$ is invertible. The result now follows by [**7**, Theorem 2.6].

4. *Examples*

EXAMPLE 4.1. Let p be prime. Let R be the ring

$$\begin{pmatrix} \mathbb{Z}/p\mathbb{Z} & \mathbb{Z}/p\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}.$$

Consider

$$M = \begin{pmatrix} \mathbb{Z}/p\mathbb{Z} & \mathbb{Z}/p\mathbb{Z} \\ 0 & p\mathbb{Z} \end{pmatrix}.$$

Then $M = cR$ where

$$c = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

is right regular so $M_R \cong R_R$ and M_R is projective. Note that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = 0$$

so that c is not left regular. By Theorem A, $\mathscr{C}(M)$ is a left Ore set. Note that $\mathscr{C}(M)$ is not a right Ore set. We see this by taking

$$c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{C}(M) \quad \text{and} \quad r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $d, s \in R$ are such that $cs = rd$ it can be checked that $d \notin \mathscr{C}(M)$.

EXAMPLE 4.2. We shall give an example of a Noetherian PI-ring R with a non-idempotent right projective prime ideal P (so that P is left localisable by Theorem A) such that P does not satisfy the left AR-property and $r(P) \neq 0$ (so that P contains no right regular elements). In fact we shall have $PN = 0$ where N is the nilpotent radical of R and $N \neq 0$. This shows that the assumption that M is maximal in Theorem B is necessary.

Start by taking $T = F[X, Y]$ where F is a field. Set $M = XT + YT$. Then M is a maximal ideal of T. Set

$$S = \begin{pmatrix} T & M \\ T & T \end{pmatrix},$$

and $V = M_2(XT)$. Then V is a prime ideal of S. Identifying V^* with the largest subset of the quotient ring of S which left-multiplies V into S, we have

$$V^* = \begin{pmatrix} X^{-1}M & X^{-1}M \\ X^{-1}T & X^{-1}T \end{pmatrix},$$

$$V^*V = \begin{pmatrix} M & M \\ T & T \end{pmatrix}$$

and $VV^* = M_2(T)$ so that $1 \in VV^*$. Therefore V_S is projective. Set

$$W = \begin{pmatrix} M & M \\ T & T \end{pmatrix},$$

so that $V^*V = W$. Note that $V \subseteq W$ and $V(S/W) = 0$.

Set

$$R = \begin{pmatrix} S & S/W \\ 0 & S/W \end{pmatrix}$$

and

$$P = \begin{pmatrix} V & S/W \\ 0 & S/W \end{pmatrix}.$$

Then P is a non-idempotent prime ideal of R. We shall now show that P_R is projective. Let e_{ij} denote the 2×2 matrix with 1 in the (i, j)-position and zeros elsewhere. Set $U = Pe_{11}$. Then U is a right ideal of R because $V(S/W) = 0$. Also $P = U \oplus e_{12} R \oplus e_{22} R$. Thus, in order to show that P_R is projective, it is enough to show that U_R is projective. It is straightforward to use a dual basis for V_S to construct a dual basis for U_R (recall that $V^*V = W$ so that $f(V)(S/W) = 0$ for all $f \in V^*$). Therefore P_R is projective.

We have $S/W \cong T/M \cong F$, so that

$$N = \begin{pmatrix} 0 & S/W \\ 0 & 0 \end{pmatrix}.$$

Hence $N \neq 0$ and $PN = 0$. Suppose that $P^n \cap N \subseteq PN$ for some positive integer n. Then $P^n \cap N = 0$. It is easy to check that $N \subseteq P^n$, so that $N = 0$; this is a contradiction. Therefore P does not have the left AR-property.

Thus if R is a Noetherian PI-ring and P is a non-idempotent right projective prime ideal of R, we have shown that P is left localisable but we do not get some of the stronger results which hold when P is a maximal ideal. Braun conjectures that if $P^*(P) = R$ then P satisfies the left AR-property. We further believe that such a P contains a right regular element.

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Note added in proof, June 2001. A shorter proof of Theorem 3.10 can be given using the reduced rank of the modules $(M^k N)_R$.

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