Analysis on Lamb's Problem

by

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis. This thesis has also not been submitted for any degree in any university previously.

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Abstract

We implement the master relationship in [12], Laplace-Fourier path in [13], and determinant of a surface wave in [14] together to form a LY algorithm and apply this algorithm to solve the Lamb's problem completely. We obtain an explicit solution formula for the Lamb's problem in the space-time variable $\vec{x}$-t. The solution formula is given in terms of the fundamental solutions of the d'Alembert wave equations in 3-D and 2-D by the Kirchhoff's formula and Hadamard's formula . Complicated 2D-3D coupling wave structures on the surface present in the surface wave solution formula. This shows that the wave structures given in the paper are much richer than the Rayleigh wave discussed in the original articles, [19, 9]. Further computation and estimates of the solution formula would also be discussed in this article and then gain results consistent with the theory in seismology.

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Contents

CHAPTER 1

Derivation of Solution Formula

1.1 Introduction

The presence of a unbounded boundary in a multi-D space domain will completely change the natures of problems without any boundary completely such as the compressible Euler equation, compressible and incompressible Navier-Stokes equation, Maxwell equation, etc.. Without understanding the basic wave natures around boundary, the general practice "to find robust estimates" among the researchers in the modern PDE for initial value problem may fail. The necessity for a new input to gain insights on the wave natures around the boundary arouses. Such an input would become a sharp tool to help the general practice to continue when an initial-boundary value problem encountered.

A good candidate for such a new input is the construction of an explicit solution formula of the Green's function for a constant coefficient problem in a half space domain. The advantage of an explicit solution formula of the Green's function is that one can represent the solution of a linear or nonlinear problem by the Duhamel's principle in terms of the Green's function so that the singular structure (in the space-time variable) of the Green's function around the boundary will pass to the solution. This may give the sufficient ansatz structure around boundary so that one can focus on how to obtain sharper estimates for the linear or nonlinear problems encountered. Thus, explicit formulae in the space-time variable might be very useful.

Our primary interest is to develop a general methodology to obtain an explicit solution formula of the Green's function for a half space problem. For the sake of making this paper interesting to the majority of mathematical disciplines in science, we choose a rather classical unsolved mathematical problem, which is commonly known in mathematics, physics, and engineering communities, to demonstrate the effectiveness of the new methodology, the LY algorithm, which is a structured program for a general class of PDEs. The details of the algorithm will be given in Chapter 2.

We choose the Lamb's problem as a source of ideas to practice the LY algorithm. The Lamb's problem is an initial-boundary value problem for a linear elasticity problem in 3-D half space with a free boundary condition. This problem is an important mathematical model to study the natural phenomenon, "earthquake". The free boundary value problem for linear elasticity was initiated by Lord Rayleigh. In [19], he investigated the motion of waves on the surface by considering a linear elastic equation for an isotropic elastic medium in a three-dimensional half-space $\mathbb{R}^3_+$ with a free boundary condition at $x = 0$; and in [9] Lamb proposed the initial boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{\rho}\nabla \cdot \left(\lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\right), \\
&\quad \vec{x} \equiv (x, y, z) \in \mathbb{R}^3_+ \equiv \{(x, y, z) : x > 0, y, z \in \mathbb{R}\},\ t \geq 0, \\
&\begin{pmatrix} (2\mu + \lambda)\partial_x & \lambda\partial_y & \lambda\partial_z \\ \mu\partial_y & \mu\partial_x & 0 \\ \mu\partial_z & 0 & \mu\partial_x \end{pmatrix} \mathbf{u}(0, y, z, t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
&\begin{cases} \mathbf{u}(\vec{x}, 0) = \vec{\Phi}(\vec{x}) \\ \partial_t \mathbf{u}(\vec{x}, 0) = \vec{\Psi}(\vec{x}), \end{cases}
\end{aligned} \tag{1.1.1}$$

where $\mathbf{u} = \mathbf{u}(\vec{x}, t) \in \mathbb{R}^3$ is the displacement vector; and $\vec{\Phi}$ and $\vec{\Psi}$ are the given initial data.

The elastic properties of isotropic materials are characterized by density ρ (constant) and Lamé constants $\lambda > 0$ and $\mu > 0$. Instead of solving the initial boundary value problem (1.1.1), Rayleigh considered a special solution $\mathbf{u}(x, y, z, t) = e^{pt-rx-ify-igz}\mathbf{v}(f, g, p)$ (a wave train solution in y-z plane) to fit the boundary condition, the speed of surface wave motion was obtained in terms of the wave numbers (f, g) in y-z plane and the Lamé constants, i.e. $p = \Omega(f, g, \lambda, \mu)$, which is a dispersion relationship. This surface wave motion was named after him as the Rayleigh wave in physics; and indeed such a surface wave motion is a generic physics phenomena. In [9], Lamb continued to investigate the structure of the solution of the initial boundary value problem for (1.1.1) in the transform variables and related the Rayleigh wave to the phenomenon in seismology, the earthquake. This problem became a well-known problem, the Lamb's problem, in the seismology, geophysics, mechanical engineering, etc.. One can find related references for the Rayleigh wave and the Lamb's problem in research articles and textbooks in physics, geophysics, mechanical engineering such as [10, 1, 17, 2, 4, 8, 11, 3, 18, 20, 22].

The system (1.1.1) is a hyperbolic system in 3-D half space domain. Though there were many works for linear hyperbolic systems in half space domain, for examples, [6, 16, 21], the definite structures of the surface wave for the system (1.1.1) were never been obtained before 2011. The first key step towards this definite surface wave structure was obtained in [12]. The fundamental solution was used to convert an initial-boundary value problem into a problem with an inhomogeneous boundary value problem together with zero initial data so that an intrinsic relationship among the boundary data in terms of transform variables was discovered. This step also works for the system (1.1.1). One can use the fundamental solution to convert the system into the following form:

$$\frac{\partial^2}{\partial t^2}\mathbf{a} - \frac{\lambda+2\mu}{\rho}\nabla(\nabla\cdot\mathbf{a}) + \frac{\mu}{\rho}\nabla\times(\nabla\times\mathbf{a}) = 0, \tag{1.1.2a}$$

$$\mathbf{a}(\vec{x},0) = \partial_t\mathbf{a}(\vec{x},0) = \vec{0}, \tag{1.1.2b}$$

$$\begin{pmatrix} (2\mu+\lambda)\partial_x & \lambda\partial_y & \lambda\partial_z \\ \mu\partial_y & \mu\partial_x & 0 \\ \mu\partial_z & 0 & \mu\partial_x \end{pmatrix}\mathbf{a}(0,y,z,t) = \mathbf{g}_b(y,z,t), \tag{1.1.2c}$$

and $\mathbf{g}_b(y,z,t)$ is a given function in terms of initial data $\mathbf{u}(x,y,z,0)$ and $\mathbf{u}_t(x,y,z,0)$. Then, one introduces the transforms

$$\begin{cases} \hat{\mathbf{a}}(x,i\eta,i\zeta,t) = \mathbb{F}[\mathbf{a}](x,i\eta,i\zeta,t) & \equiv \iint_{\mathbb{R}^2}\mathbf{a}(x,y,z,t)e^{-iy\eta-iz\zeta}dydz, \\ \tilde{\mathbf{a}}(x,i\eta,i\zeta,s) = \mathbb{L}[\mathbf{a}](x,i\eta,i\zeta,s) & \equiv \int_0^\infty \hat{\mathbf{a}}(x,i\eta,i\zeta,t)e^{-st}dt, \\ \mathbb{J}[\mathbf{a}](\xi,i\eta,i\zeta,s) & \equiv \int_0^\infty \tilde{\mathbf{a}}(x,i\eta,i\zeta,s)e^{-x\xi}dx. \end{cases} \tag{1.1.3}$$

Then, similar to [12], by (1.1.2a), (1.1.2b), and

$$\lim_{x\to\infty}\mathbb{L}[\mathbf{a}](x,i\eta,i\zeta,s) < \infty \text{ for each given } (\eta,\zeta,s)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$$

together, one obtains an intrinsic algebraic relationship between the Dirichlet data $\mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s)$ and Neummann data $\mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s)$, which is the master relationship:

$$\mathfrak{M}(i\eta,i\zeta,s;\mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s),\mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s)) = \vec{0},$$

where the system $\mathfrak{M}$ is linear in $\mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s)$ and $\mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s)$. This system of linear equations and the transform for (1.1.2c) together give rise the explicit solution of $(\mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s),\mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s))$ in the transform variables as follows

$$\begin{pmatrix} \mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s) \\ \mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s) \end{pmatrix}_{6\times 1} = \begin{pmatrix} S_{11}(i\eta,i\zeta,s) & S_{12}(i\eta,i\zeta,s) & S_{13}(i\eta,i\zeta,s) \\ S_{21}(i\eta,i\zeta,s) & S_{22}(i\eta,i\zeta,s) & S_{23}(i\eta,i\zeta,s) \\ S_{31}(i\eta,i\zeta,s) & S_{32}(i\eta,i\zeta,s) & S_{33}(i\eta,i\zeta,s) \\ S_{41}(i\eta,i\zeta,s) & S_{42}(i\eta,i\zeta,s) & S_{53}(i\eta,i\zeta,s) \\ S_{61}(i\eta,i\zeta,s) & S_{62}(i\eta,i\zeta,s) & S_{63}(i\eta,i\zeta,s) \end{pmatrix}_{6\times 3} \mathbb{L}[\mathbf{g}_b](i\eta,i\zeta,s). \tag{1.1.4}$$

Here, the symbols S_{ij} are given by explicit rational functions,

$$S_{ij} = \frac{S^n_{ij}(i\eta,i\zeta,s,\xi_T,\xi_L)}{S^d_{ij}(i\eta,i\zeta,s,\xi_T,\xi_L)},$$

in $i\eta$, $i\zeta$, s, and roots $\xi_L(i\eta,i\zeta,s)$, $\xi_T(i\eta,i\zeta,s)$ of characteristic polynomial $p(\xi,i\eta,i\zeta,s)$ of the system (1.1.2a), where

$$\begin{cases} p(\xi,i\eta,i\zeta,s) \equiv p_T(\xi,i\eta,i\zeta,s)^2 p_L(\xi,i\eta,i\zeta,s); \\ p_T(\xi,i\eta,i\zeta,s) \equiv \left(\mu\xi^2 - \mu\left(\eta^2+\zeta^2\right) - s^2\rho\right), \\ p_L(\xi,i\eta,i\zeta,s) \equiv \left((\lambda+2\mu)\xi^2 - (\lambda+2\mu)(\eta^2+\zeta^2) - s^2\rho\right), \\ \xi_T = \sqrt{\eta^2+\zeta^2+s^2/c_T^2},\ c_T = \sqrt{\frac{\mu}{\rho}}:\ \text{speed of S-wave}, \\ \xi_L = \sqrt{\eta^2+\zeta^2+s^2/c_L^2},\ c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}:\ \text{speed of P-wave}. \end{cases}$$

The symbols ξ_T and ξ_L implicitly represent the differential equations at $x=\infty$, and similarly one can identify the denominator $S^d_{ij}(i\eta,i\eta,s,\xi_T,\xi_L)$ as an implicit balance between PDE at $x=\infty$ and boudary condition. It is a common sense in science to make every quanitities into the same UNIT in order to make comparison. Now, the only unit in the problem within our imagination is polynomial. This concept leads to characteristic-non characteristic decomposition to regularize the denominator $S^d_{ab}(i\eta,i\zeta,s,\xi_T,\xi_L)$ into polynomial in the LY algorithm. One decomposes the symbols S_{ab} into the form:

$$S_{ab} = \sum_{0\le m,n\le 1} \left(c_{ab;mn}(i\eta,i\zeta) + \frac{\mathfrak{n}_{ab;mn}(i\eta,i\zeta,s)}{\mathfrak{D}(i\eta,i\zeta,s)} \right) \left(\frac{\partial_s\xi_L}{s}\right)^m \left(\frac{\partial_s\xi_T}{s}\right)^n, \tag{1.1.5}$$

where $c_{ab;mn}(i\eta, i\zeta)$ and $\mathfrak{n}_{ab;mn}(i\eta, i\zeta, s)$ are polynomials in $i\eta$, $i\zeta$, and s of degree less than 4, and the polynomial $\mathfrak{D}(i\eta, i\zeta, s)$ can be realized as an explicit balance between PDE at $x = \infty$ and the boundary condtion in the same unit. We also define it as the determinant of the Rayleigh wave:

$$\mathfrak{D}(i\eta, i\zeta, s) = \prod_{j=1}^{3}(s^2 + c_j^2\ (\eta^2 + \zeta^2)), \tag{1.1.6}$$

where

$$\begin{cases} c_1 = \sqrt{\frac{8\mu}{3\rho} - \frac{2\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{2\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \\ c_2 = \sqrt{\frac{8\mu}{3\rho} - \frac{(-1-i\sqrt{3})\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{(-1+i\sqrt{3})\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \\ c_3 = \sqrt{\frac{8\mu}{3\rho} - \frac{(-1+i\sqrt{3})\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{(-1-i\sqrt{3})\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \\ A = \mu^3\rho^6(17\lambda - 11\mu)(\lambda + 2\mu)^2, \\ Q = \mu^6\rho^{12}(\lambda + 2\mu)^3\left(11\lambda^3 + 4\lambda^2\mu - 9\lambda\mu^2 - 10\mu^3\right). \end{cases} \tag{1.1.7}$$

The critical value σ^* of the Poisson ratio $\frac{\lambda}{2(\mu+\lambda)}$ is

$$\sigma^* \equiv \frac{1}{12}\left(2 + \frac{\sqrt[3]{194 + 57\sqrt{114}}}{2^{2/3}} - \frac{55}{\sqrt[3]{2\left(194 + 57\sqrt{114}\right)}}\right) = 0.263082\ldots. \tag{1.1.8}$$

All c_i are real numbers when $\frac{1}{2}\lambda/(\lambda + \mu) \le \sigma^*$. Two c_i are complex numbers when $\frac{1}{2}\lambda/(\mu + \lambda) > \sigma^*$.

By completing the LY algorithm given in Section 3, we obtain the solution formula of the surface wave:

Theorem 1.1.1. *[Surface Wave Formula] For the problem* (1.1.2)*, the surface wave* $(\mathbf{a}(0, y, z, t), \mathbf{a}_x(0, y, z, t))$ *would has three expressions due to different Lame constants.*

The solution formula:

Case. $\frac{1}{2}\lambda/(\mu+\lambda) < \sigma^*$.

$$\begin{pmatrix} \mathbf{a}(0,y,z,t) \\ \mathbf{a}_x(0,y,z,t) \end{pmatrix} = \left\{ \sum_{0\le m,n\le 1} \left(c_{ij;mn}(\partial_y,\partial_z) + \sum_{k=1}^{3} \mathbb{U}_1(y,z,c_k t) \underset{(y,z,t)}{*} N_{ij;mn;k1}(\partial_y,\partial_z) \right) \underset{(y,z,t)}{*} \mathbb{W}_L^m(y,z,t) \underset{(y,z,t)}{*} \mathbb{W}_T^n \underset{(y,z,t)}{*} \right\}_{6\times 3} \underset{(y,z,t)}{*} \mathbf{g}_b(y,z,t). \quad (1.1.9)$$

Case. $\frac{1}{2}\lambda/(\mu+\lambda) = \sigma^*$.

$$\begin{pmatrix} \mathbf{a}(0,y,z,t) \\ \mathbf{a}_x(0,y,z,t) \end{pmatrix} = \left(\sum_{\substack{0\le m,n\le 1 \\ 1\le k\le 3}} \mathbb{U}_0(y,z,c_k t) \underset{(y,z,t)}{*} \mathbb{W}_L^m(y,z,t) \underset{(y,z,t)}{*} \mathbb{W}_T^n \underset{(y,z,t)}{*} N_{ij;mn;k0}(\partial_y,\partial_z,t) \right)_{6\times 3} \mathbf{g}_b(y,z,t)$$
$$+ \left\{ \sum_{0\le m,n\le 1} \left(c_{ij;mn}(\partial_y,\partial_z) + \sum_{k=1}^{3} \mathbb{U}_1(y,z,c_k t) \underset{(y,z,t)}{*} N_{ij;mn;k1}(\partial_y,\partial_z) \right) \underset{(y,z,t)}{*} \mathbb{W}_L^m(y,z,t) \underset{(y,z,t)}{*} \mathbb{W}_T^n \underset{(y,z,t)}{*} \right\}_{6\times 3} \underset{(y,z,t)}{*} \mathbf{g}_b(y,z,t). \quad (1.1.10)$$

Here, $c_{ij,mn}(i\eta,i\zeta)$, $N_{ij;mn;k1}(i\eta,i\zeta)$, and $N_{ij;mn;k0}(i\eta,i\zeta,t)$ are all polynomials in (η,ζ) with degree less or equal to 4 generated by the LY algorithm, and $N_{ij;mn;k1}$ is a polynomial in t of degree 1. The functions $\mathbb{W}_L(y,z,t)$ and $\mathbb{W}_T(y,z,t)$ are defined by

$$\begin{cases} \mathbb{W}_L(y,z,t) \equiv 2\mathbb{W}_0(0,y,z,c_L t), \\ \mathbb{W}_T(y,z,t) \equiv 2\mathbb{W}_0(0,y,z,c_T t). \end{cases}$$

Here, $\mathbb{W}_0(x,y,z,t)$, $\mathbb{U}_0(y,z,t)$, and $\mathbb{U}_1(y,z,t)$ are the solutions of the d'Alembert *wave equations in 3-D and 2-D as follows*

$$\begin{cases} (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\mathbb{W}_0 = 0, \\ \mathbb{W}_0(x,y,z,0) = 0, \\ \partial_t \mathbb{W}_0(x,y,z,0) = \delta(x)\delta(y)\delta(z), \end{cases} \quad \begin{cases} (\partial_t^2 - \partial_y^2 - \partial_z^2)\mathbb{U}_0 = 0, \\ \partial_t \mathbb{U}_0(y,z,0) = 0, \\ \mathbb{U}_0(y,z,0) = \delta(y)\delta(z), \end{cases} \quad \begin{cases} (\partial_t^2 - \partial_y^2 - \partial_z^2)\mathbb{U}_1 = 0, \\ \partial_t \mathbb{U}_1(y,z,0) = \delta(y)\delta(z), \\ \mathbb{U}_1(y,z,0) = 0; \end{cases} \tag{1.1.11}$$

and where $\underset{(y,z,t)}{*}$ *is the convolution operator over the domain* $\mathbb{R}^2 \times \mathbb{R}_+$.

Case. $\frac{1}{2}\lambda/(\mu+\lambda) > \sigma^*$**.**

$$\begin{pmatrix} \mathbf{a}(0,y,z,t) \\ \mathbf{a}_x(0,y,z,t) \end{pmatrix} = \left\{ \sum_{0\le m,n\le 1} \left(c_{ij;mn}(\partial_y,\partial_z) + \sum_{k=1}^{3} \mathbb{U}_1(y,z,c_k t) \underset{(y,z,t)}{*} N_{ij;mn;k1}(\partial_y,\partial_z) \right) \underset{(y,z,t)}{*} \mathbb{W}_L^m(y,z,t) \underset{(y,z,t)}{*} \mathbb{W}_T^n \underset{(y,z,t)}{*} \right\}_{6\times 3} \underset{(y,z,t)}{*} \mathbf{g}_b(y,z,t). \tag{1.1.12}$$

Here for c_1 which is a real number, the symbols are similar defined as the other two cases. While for c_2 and c_3 which are complex numbers, the symbol $\mathbb{U}_1$ is not the solution of wave equation. But we can still conclude that this formula is also valid i.e it can be reverse to physical domain.

Remark 1.1.2. Here, the functions $\mathbb{W}_0(x,y,z,t)$, $\mathbb{U}_0(y,z,t)$, and $U_1(y,z,t)$ are the kernel functions given by the Kirchhoff's formula and the Hadamard's formula by the method of descendent. They are generalized functions, the spherical delta function etc. Thus, the kernel functions of $(\mathbf{a}(0,y,z,t), \mathbf{a}_x(0,y,z,t))$ are finite combinations of differential operators and generalized functions.

We will not spell out all the polynomials $c_{ij;mn}$ and $N_{ij;mn;kl}$ in the paper, since they can be generated explicitly by a program in the mathematica 8.0. Thus, we will not list all polynomials except some typical polynomials, since to display all the polynomials or not will not affect the rigorous integrity of this paper.

Remark 1.1.3. The Rayleigh wave described by the geophysics community is the native 2-D wave structure. It is corresponding to the surface waves in (1.1.12) and (1.1.10) with $(m,n) = (0,0)$. Theorem 1.1.1 gives the generic surface wave patterns. It gives waves on the surface possessing a complicated 2D-3D wave nature.

Remark 1.1.4. In the third case $\pm c_1$ represents the only real roots of 1.1.6. Thus we see that when the poisson ratio is greater than the critical value there would be four complex root of 1.1.6, but these roots would have similar cancelations as in case 1.

With the solution formula of $\mathbf{a}(0,y,z,t)$ and $\mathbf{a}_x(0,y,z,t)$ given in Theorem 1.1.1, one has the solution formula for $\mathbf{a}(\vec{x},t)$ with $\vec{x} \in \mathbb{R}^3_+$ by the first Green's identity:

Corollary 1.1.5 (Interior wave formula)**.** *The solution formula of the problem* (1.1.2) *is*

$$\begin{aligned}
\mathbf{a}(\vec{x},t) = &- \int_0^t \iint_{\vec{x}_*\in\partial\mathbb{R}^3_+} \frac{1}{\rho}\mathbb{G}_1(\vec{x}-\vec{x}_*,t-\tau)\mathrm{g}_b(\vec{x}_*,\tau)d\vec{x}_*d\tau \\
&+ \int_0^t \iint_{\vec{x}_*\in\partial\mathbb{R}^3_+} \left(\mathbb{G}_1(\vec{x}-\vec{x}_*,t-\tau)_{x_*} \begin{pmatrix} \frac{\lambda+2\mu}{\rho} & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & \frac{\mu}{\rho} \end{pmatrix} \mathbf{a}(\vec{x}_*,\tau) \right) d\vec{x}_*d\tau \\
&- \int_0^t \iint_{\vec{x}_*\in\partial\mathbb{R}^3_+} \left(\mathbb{G}_1(\vec{x}-\vec{x}_*,t-\tau) \begin{pmatrix} 0 & \frac{\mu\partial_{y_*}}{\rho} & \frac{\mu\partial_{z_*}}{\rho} \\ \frac{\lambda\partial_{y_*}}{\rho} & 0 & 0 \\ \frac{\lambda\partial_{z_*}}{\rho} & 0 & 0 \end{pmatrix} \mathbf{a}(\vec{x}_*,\tau) \right) d\vec{x}_*d\tau
\end{aligned} \tag{1.1.13}$$

with $\mathbf{a}(0,y_*,z_*,\tau)$ *given by Theorem 1.1.1, where* $\mathbb{G}_1(\vec{x},t)$ *is the fundamental* *solution of* (1.1.1)

$$
\begin{aligned}
&\mathbb{G}_1(\vec{x},t) \equiv (\mathbf{G}_{ij}(\vec{x},t))_{3\times 3},\\
&\mathbf{G}_{ij}(\vec{x},t) \equiv \frac{1}{4\pi\rho}\left(3\hat{x}_i\hat{x}_j-\delta_{ij}\right)\frac{1}{|\vec{x}|^3}\left[H\left(t-\frac{|\vec{x}|}{c_L}\right)-H\left(t-\frac{|\vec{x}|}{c_T}\right)\right]t\\
&\qquad\qquad + \frac{1}{4\pi\rho c_L^2}\hat{x}_i\hat{x}_j\frac{1}{|\vec{x}|}\delta\left(t-\frac{|\vec{x}|}{c_L}\right)-\frac{1}{4\pi\rho c_T^2}\left(\hat{x}_i\hat{x}_j-\delta_{ij}\right)\frac{1}{|\vec{x}|}\delta\left(t-\frac{|\vec{x}|}{c_T}\right)\\
&\mathbb{G}_0(\vec{x},t) \equiv \partial_t\mathbb{G}_1(\vec{x},t),\\
&\qquad \hat{x}_i \equiv \frac{x_i}{|\vec{x}|},\ \ H(r) \equiv \begin{cases} 1\, \textit{for}\ r>0,\\ 0\, \textit{for}\ r\le 0,\end{cases}\\
&\qquad \vec{n} \equiv (1,0,0)^T.
\end{aligned}
\tag{1.1.14}
$$

With Corollary 1.1.5, one has the solution formula for the Lamb's problem:

Corollary 1.1.6 (Lamb's problem)**.** *The solution formula for problem* (1.1.1) *is given by*

$$
\begin{cases}
\mathbf{A}(\vec{x},t) \equiv \iiint_{\mathbb{R}^3_+} \mathbb{G}_0(\vec{x}-\vec{x}_*,t)\Phi(\vec{x}_*)+\mathbb{G}_1(\vec{x}-\vec{x}_*,t)\Psi(\vec{x}_*)d\vec{x}_*,\\
\mathbf{g}_b(y,z,t) \equiv -\begin{pmatrix} (2\mu+\lambda)\partial_x & \lambda\partial_y & \lambda\partial_z\\ \mu\partial_y & \mu\partial_x & 0\\ \mu\partial_z & 0 & \mu\partial_x \end{pmatrix}\mathbf{A}(0,y,z,t),\\
\mathbf{u}(\vec{x},t) \equiv \mathbf{A}(\vec{x},t)+\mathbf{a}(\vec{x},t),
\end{cases}
\tag{1.1.15}
$$

where $\mathbf{a}(\vec{x},t)$ *is the solution given by Corollary 1.1.5 with the given inhomogeneous boundary data* $\mathrm{g}_b(y,z,t)$ *as the one given in* (1.1.15)*.*

The ingredients of the LY algorithm in Section 3 are given in a logical order below:

1. A fundamental solution: to shift initial data to boundary data.

2. A *master relationship*: An intrinsic algebraic relationship among the full boundary data in the transform variables for a solution of differential equation with zero initial data.

3. Algebraic solution of the full boundary data in the transform variables.

4. An algebraic *characteristic-non characteristic decomposition* of the symbols of boundary data: To decompose the symbols into a polynomial in $\partial_s\xi_L/s$, and $\partial_s\xi_T/s$ over the ring spanned by rational functions in η, ζ, and s. The denominator of the rational function gives the determinant of the surface wave.

5. *Laplace-Fourier path*: A path in the complex plane for the Laplace variable s consistent of the spectrum with respect to all wave numbers. This is an instrument to invert the symbols $\partial_s\xi_L/s$ and $\partial_s\xi_T/s$ into waves in the $\vec{x}$-t domain.

The combination of ingredients (1), (2), and (3) were introduced in [12] for the purpose to study multi-D viscous shock profile stability. The ingredient (5) was introduced in [13] to invert the symbols of the full boundary data in the transform variables into data in the space-time variable $\vec{x}$-t. The ingredient (4) was introduced to realize the surface wave for a linearized compressible Navier-Stokes equation [14].

With above five components, the solution formula for any 2×2 hyperbolic system in a 2-D half-space domain was obtained in [5] with any arbitrary well-posed boundary condition.

In Section 2, the preliminaries materials are given. In Section 3, we will give the LY algorithm to conclude Theorem 1.1.1 as the main program of the paper; and design Sections 4,5,6 as the subroutines for completing some details described in the LY algorithm.

The LY algorithm is also successfully implemented to the study the pointwise 2-D wave scattering over viscous shock profile, [15].

1.2 Preliminary

In this section, we first list some basic properties of Laplace and Fourier Transform.

Proposition 1.2.1. *For any $g, h \in C_c^\infty[0,\infty)$ with the property $\partial_t^k g(0) = 0$ for $k \in \mathbb{N} \cup \{0\}$, then their Laplace transforms $G(s) \equiv \int_0^\infty e^{-st} g(t)dt$ and $F(s) \equiv \int_0^\infty e^{-st} f(t)dt$ satisfy*

$$\begin{cases} \partial_s G(s) = -\displaystyle\int_0^\infty te^{-st}(g(t) + c_0\delta(t))dt \text{ for any } c_0 \in \mathbb{R}, \\ G(s)F(s) = \displaystyle\int_0^\infty e^{-st} g(t) * f(t)dt, \ \left(g(t) * f(t) \equiv \displaystyle\int_0^t g(t-\tau)f(\tau)d\tau\right), \\ \displaystyle\int_0^\infty e^{-st} g^{[n]}(t)dt = s^n G(s), \\ \int_0^\infty e^{-st} f^{[n]}(t)dt = s^n F(s) - \sum_{l=0}^{n-1} s^l \partial_t^{n-1-l} f(0). \end{cases} \tag{1.2.1}$$

Proposition 1.2.2. *For any $g \in C_c^\infty[0,\infty)$ and $G(s) \equiv \int_0^\infty e^{-st} g(t)dt$,*

$$g(t) = \frac{1}{2\pi i} \int_{Re(s)=0} e^{st} G(s)ds.$$

Proposition 1.2.3. *For any $G(s)$ analytic in $s \in Re(s) \le 0$ with the property that there exists c_0 such that $\int_{Re(s)=x} |G(s)|ds < c_0$ for all $x > 0$, then*

$$\frac{1}{2\pi i} \int_{Re(s)=0} e^{st} G(s)ds = 0 \text{ for all } t < 0;$$

and $G(s)$ is the Laplace transform of the function $g(t)$ given by

$$g(t) = \frac{1}{2\pi i} \int_{Re(s)=0} e^{st} G(s)ds.$$

Proposition 1.2.4. *Suppose that* $g \in L^\infty(0,\infty)$ *and its Laplace transform* $G(s) = \int_0^\infty e^{-st} g(t)dt$ *is a rational function of s. Then,*

$$\operatorname*{Res}_{\substack{G(s^*)=0 \\ Re(s^*)>0 \\ s=s^*}} G(s) = 0.$$

Proposition 1.2.5 (Fourier Transform)**.** *The Fourier transform of the solutions of the d'Alembert wave equations given* (1.1.11) *are*

$$\begin{cases} \iiint_{\mathbb{R}^3} e^{-ix\hat{\xi}-iy\eta-iz\zeta} \mathbb{W}_0(x,y,z,t)dxdydz = \dfrac{\sin(\sqrt{\hat{\xi}^2+\eta^2+\zeta^2}\; t)}{\sqrt{\hat{\xi}^2+\eta^2+\zeta^2}}, \\ \iint_{\mathbb{R}^2} e^{-iy\eta-iz\zeta} \mathbb{U}_0(y,z,t)dydz = \cos(\sqrt{\eta^2+\zeta^2}\; t), \\ \iint_{\mathbb{R}^2} e^{-iy\eta-iz\zeta} \mathbb{U}_1(y,z,t)dydz = \dfrac{\sin(\sqrt{\eta^2+\zeta^2}\; t)}{\sqrt{\eta^2+\zeta^2}}. \end{cases}$$

Proposition 1.2.6. *Let f be a function defined on* $\mathbb{R}^2$*, and satisfies*

$$f(\vec{x}) = f(|\vec{x}|) \quad \text{for } x \in \mathbb{R}^2,$$

then

$$f(\vec{x}) = \frac{1}{2\pi}\int_0^\infty F(k) J_0(k|\vec{x}|) k dk,$$

where $F(k)$ *is the Fourier transform of* f*,* J_0 *is the Bessel function of the first kind of order 0.*

Proposition 1.2.7 (Orthogonality of Bessel function)**.** *Let* J_ν *be the Bessel function of the first kind of order* ν*, then*

$$\int_0^\infty J_\nu(kr) J_\nu(k'r) r dr = \frac{\delta(k-k')}{k}$$

1.3 LY Algorithm

The LY algorithm will be given as a logical sequence to yield Theorem 1.1.1.

i. **Fundamental Solution, Shift Initial Data, and Transforms**

With the fundamental solution $\mathbb{G}_0(\vec{x},t)$ and $\mathbb{G}_1(\vec{x},t)$ of (1.1.1) given in (1.1.14), one can shift the initial data in (1.1.1) to the boundary data to obtain the new function $\mathbf{a}(\vec{x},t)$ as follows:

$$\begin{cases}\mathbf{A}(\vec{x},t) \equiv \iiint_{\mathbb{R}^3_+} \mathbb{G}_0(\vec{x}-\vec{x}_*,t)\Phi(\vec{x}_*) + \mathbb{G}_1(\vec{x}-\vec{x}_*,t)\Psi(\vec{x}_*)d\vec{x}_*,\\ \mathbf{a}(\vec{x},t) \equiv \mathbf{u}(\vec{x},t) - \mathbf{A}(\vec{x},t).\end{cases} \tag{1.3.1}$$

The function $\mathbf{A}(\vec{x},t)$ is a given function in terms of $\vec{\Psi}$ and $\vec{\Phi}$; and the variable $\mathbf{a}$ satisfies (1.1.2) with the inhomogenous boundary condition at $x=0$ given in (1.1.2c) with $\mathbf{g}_b(y,z,t)$ defined by

$$\mathbf{g}_b(y,z,t) \equiv -\begin{pmatrix}(2\mu+\lambda)\partial_x & \lambda\partial_y & \lambda\partial_z\\ \mu\partial_y & \mu\partial_x & 0\\ \mu\partial_z & 0 & \mu\partial_x\end{pmatrix}\mathbf{A}(0,y,z,t). \tag{1.3.2}$$

We reserve two notions for the Dirichlet data and Neumann data:

$$\begin{cases}\mathbf{D}(i\eta,i\zeta,s) \equiv \mathbb{L}[\mathbf{a}](0,i\eta,i\zeta,s),\\ \mathbf{N}(i\eta,i\zeta,s) \equiv \mathbb{L}[\mathbf{a}_x](0,i\eta,i\zeta,s).\end{cases} \tag{1.3.3}$$

The system (1.1.2) in the transform variables is

$$\mathbf{M}\cdot\mathbb{J}[\mathbf{a}] = \begin{pmatrix}-(2\mu+\lambda)\xi & -(\mu+\lambda)i\eta & -(\mu+\lambda)i\zeta\\ -(\mu+\lambda)i\eta & -\mu\xi & 0\\ -(\mu+\lambda)i\eta & 0 & -\mu\xi\end{pmatrix}\mathbf{D} + \begin{pmatrix}-(2\mu+\lambda) & 0 & 0\\ 0 & -\mu & 0\\ 0 & 0 & -\mu\end{pmatrix}\mathbf{N}, \tag{1.3.4}$$

where

$$\mathbf{M} = \begin{pmatrix}\rho s^2-(2\mu+\lambda)\xi^2+\mu\eta^2+\mu\zeta^2 & -(\lambda+\mu)i\xi\eta & -(\lambda+\mu)i\xi\zeta\\ -(\lambda+\mu)i\xi\eta & \rho s^2-\mu\xi^2+(2\mu+\lambda)\eta^2+\mu\zeta^2 & (\lambda+\mu)\eta\zeta\\ -(\lambda+\mu)i\xi\zeta & (\lambda+\mu)\eta\zeta & \rho s^2-\mu\xi^2+\mu\eta^2+(2\mu+\lambda)\zeta^2\end{pmatrix},$$

and boundary condition in the transform variable is

$$\begin{pmatrix}0 & i\eta\lambda & i\zeta\lambda\\ i\eta\mu & 0 & 0\\ i\zeta\mu & 0 & 0\end{pmatrix}\mathbf{D} + \begin{pmatrix}(2\mu+\lambda) & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{pmatrix}\mathbf{N} = \mathbb{L}[\mathbf{g}_b]. \tag{1.3.5}$$

By multiplying $\mathbf{M}^{-1}$ to (1.3.4), one has

$$\mathbb{J}[\mathbf{a}] = \mathbf{M}^{-1}\left(\begin{pmatrix} -(2\mu+\lambda)\xi & -(\mu+\lambda)i\eta & -(\mu+\lambda)i\zeta \\ -(\mu+\lambda)i\eta & -\mu\xi & 0 \\ -(\mu+\lambda)i\eta & 0 & -\mu\xi \end{pmatrix}\mathbf{D} + \begin{pmatrix} -(2\mu+\lambda) & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\mu \end{pmatrix}\mathbf{N}\right)$$
$$= \frac{adj(\mathbf{M})}{p(\xi,i\eta,i\zeta,s)}\left(\begin{pmatrix} -(2\mu+\lambda)\xi & -(\mu+\lambda)i\eta & -(\mu+\lambda)i\zeta \\ -(\mu+\lambda)i\eta & -\mu\xi & 0 \\ -(\mu+\lambda)i\eta & 0 & -\mu\xi \end{pmatrix}\mathbf{D} + \begin{pmatrix} -(2\mu+\lambda) & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\mu \end{pmatrix}\mathbf{N}\right)$$
$$\equiv \mathbf{soln}(\xi,i\eta,i\zeta,s;\mathbf{D},\mathbf{N}), \quad (1.3.6)$$

where $p(\xi,i\eta,i\zeta,s)$ is the determinant of the matrix $\mathbf{M}$ and $p_T(p_L)$ is the characteristic polynomial for the transverse (longitudinal) wave:

$$p(\xi,i\eta,i\zeta,s) = det(\mathbf{M}) = p_T(\xi,i\eta,i\zeta,s)^2 p_L(\xi,i\eta,i\zeta,s), \qquad (1.3.7)$$

where

$$\begin{cases} p_T(\xi,i\eta,i\zeta,s) = \left(\mu\xi^2 - \mu\left(\eta^2+\zeta^2\right) - s^2\rho\right), \\ p_L(\xi,i\eta,i\zeta,s) = \left((\lambda+2\mu)\xi^2 - (\lambda+2\mu)(\eta^2+\zeta^2) - s^2\rho\right). \end{cases} \qquad (1.3.8)$$

Remark 1.3.1. The parameter x does not show up in $\mathbb{J}[a]$ due to the fact that the initial data was set to be zero. It is a very important initial step in this program.

ii. **Well-Posedness,** ***Master Relationship*****, and solution of boundary data in transform variables**

The function **soln** is rational function in ξ so that one can perform the inverse transform in the x-variable for given $(\eta,\zeta,s) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$:

$$\mathbb{L}[\mathbf{a}](x,i\eta,i\zeta,s) = \sum_{p(\xi_*,i\eta,i\zeta,s)=0} e^{\xi_* x}\operatorname*{Res}_{\xi=\xi_*}\mathbf{soln}(\xi,i\eta,i\zeta,\mathbf{D},\mathbf{N}). \quad (1.3.9)$$

The well-posedness assumption:

For each $(\eta,\zeta,s) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$, the solution $\mathbf{a}(\vec{x},t)$ satisfies

$$\limsup_{x\to\infty} |\mathbb{L}[\mathrm{a}](x,i\eta,i\zeta,s)| < \infty. \qquad (1.3.10)$$

This well-posedness assumption will exclude the exponential growth component in (1.3.9). It gives rise to the following Master Relationship:

Definition 1.3.2 (Master Relationship)**.** For each $(\eta, \zeta, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$,

$$\vec{0} = \mathfrak{M}(i\eta, i\zeta, s; \mathbf{D}, \mathbf{N}) \equiv \operatorname*{Res}_{\xi=\xi_*} \mathbf{soln}(\xi, i\eta, i\zeta, s; \mathbf{D}, \mathbf{N}) \Big|_{\substack{p(\xi_*, i\eta, i\zeta, s)=0 \\ Re(\xi_*)>0}} . \tag{1.3.11}$$

The master relationship (1.3.11) and the boundary condition (1.3.5) form an algebraic system for $\mathbf{D}$ and $\mathbf{N}$. One obtains the solution

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{N} \end{pmatrix} (i\eta, i\zeta, s) = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \\ S_{51} & S_{52} & S_{53} \\ S_{61} & S_{62} & S_{63} \end{pmatrix} \mathbb{L}[\mathbf{g}_b](i\eta, i\zeta, s), \tag{1.3.12}$$

where the entries S_{ij} is a rational function in $i\eta$, $i\zeta$, s, ξ_T, and ξ_L:

$$\begin{cases} S_{ij} = S^n_{ij}(i\eta, i\zeta, s, \xi_T, \xi_L) / S^d_{ij}(i\eta, i\zeta, s, \xi_T, \xi_L), \\ \xi_T \equiv \sqrt{\eta^2 + \zeta^2 + \frac{\rho}{\mu} s^2} = \sqrt{\eta^2 + \zeta^2 + s^2 / c_T{}^2}, \\ \xi_L(i\eta, i\zeta, s) \equiv \sqrt{\eta^2 + \zeta^2 + \frac{\rho}{2\mu+\lambda} s^2} = \sqrt{\eta^2 + \zeta^2 + s^2 / c_L{}^2}, \\ p_T(\xi_T, i\eta, i\zeta, s) = p_L(\xi_L, i\eta, i\zeta, s) = 0, \end{cases} \tag{1.3.13}$$

and $S^n_{ij}(i\eta, i\zeta, s, a, b)$, $S^d_{ij}(i\eta, i\zeta, s, a, b)$ are polynomials in η, ζ, s, a, and b.

Remark 1.3.3. In Section 4, we will give the expression of matrix $(S_{ij})_{6\times 3}$. At least, we will write down the rational function $S_{22}(i\eta, i\zeta, s, a, b)$ i.e. $S^n_{22}(i\eta, i\zeta, s, a, b) / S^d_{22}(i\eta, i\zeta, s, a, b)$ explicitly.

iii. **Characteristic-non characteristic decomposition of the symbols S_{ij}, Determinant of Rayleigh Wave**

The symbols $1/S^d_{ij}(i\eta, i\zeta, s, \xi_T(i\eta, i\zeta, s), \xi_L(i\eta, i\zeta, s))$ are not local analytic function in $(i\eta, i\eta, s)$ around $(0,0,0)$. Algebraic manipulations using the specific form of the characteristic polynomials p_T and p_L are carried out to obtain the decomposition of the following form:

$$S_{ij}(i\eta, i\zeta, s, \xi_T, \xi_L) = \sum_{0\le m,n\le 1} \left(c_{ij;mn}(i\eta, i\zeta) + \frac{\mathfrak{n}_{ij;mn}(i\eta, i\zeta, s)}{\mathfrak{D}(i\eta, i\zeta, s)} \right) \left(\frac{\partial_s \xi_L}{s} \right)^m \left(\frac{\partial_s \xi_T}{s} \right)^n, \tag{1.3.14}$$

where $\mathfrak{D}(i\eta, i\zeta, s)$ and $\mathfrak{n}_{ij;mn}(i\eta, i\zeta, s)$ are polynomials in η, ζ, and s only; and the degrees of the polynomials in s satisfy

$$deg(\mathfrak{n}_{ij}(i\eta, i\zeta, s)) < deg(\mathfrak{D}(i\eta, i\zeta, s)),$$

and $c_{ij;mn}(i\eta, i\zeta)$ is polynomials in η and ζ with degree ≤ 2.

Definition 1.3.4. The polynomial $\mathfrak{D}(i\eta, i\zeta, s)$ is defined as the determinant for Rayleigh surface wave given by (1.1.1). The terms $\partial_s \xi_L / s$ and $\partial_s \xi_T / s$ are defined as the symbols of the interior wave on the surface.

This notion and decomposition were initiated in [14]; and one can realize $1/\mathfrak{D}(i\eta, i\zeta, s)$ as the symbol of the differential operator $\mathfrak{D}(\partial_y, \partial_z, \partial_t)^{-1}$. It leads to the consideration of the roots of $\mathfrak{D}(i\eta, i\zeta, s)$ to recover the wave motion structure. The structures with convolution to $\partial_s \xi_L / s$ and $\partial_s \xi_T / s$ in (1.3.14) were never been recognised in any physics, geophysics, or mechanical engineering literatures. The determinant $\mathfrak{D}(i\eta, i\zeta, s)$ is:

$$\mathfrak{D}(i\eta, i\zeta, s) = \\ s^6 + 8\frac{\mu}{\rho}\left(\eta^2+\zeta^2\right)s^4 + 8\left(\frac{\mu}{\rho}\right)^2 \frac{3\lambda+4\mu}{\lambda+2\mu}\left(\eta^2+\zeta^2\right)^2 s^2 + 16\left(\frac{\mu}{\rho}\right)^3 \frac{\lambda+\mu}{\lambda+2\mu}\left(\eta^2+\zeta^2\right)^3. \tag{1.3.15}$$

The determinant $\mathfrak{D}$ can be factorized

into

$$\mathfrak{D} = \prod_{j=1}^{3}(s^2 + c_j^2\ (\eta^2 + \zeta^2)), \tag{1.3.16}$$

where c_j are listed below:

$$\begin{cases} c_1 = \sqrt{\frac{8\mu}{3\rho} - \frac{2\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{2\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \\ c_2 = \sqrt{\frac{8\mu}{3\rho} - \frac{(-1-i\sqrt{3})\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{(-1+i\sqrt{3})\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \\ c_3 = \sqrt{\frac{8\mu}{3\rho} - \frac{(-1+i\sqrt{3})\sqrt[3]{A-3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)} - \frac{(-1-i\sqrt{3})\sqrt[3]{A+3\sqrt{3}\sqrt{Q}}}{3\rho^3(\lambda+2\mu)}}, \end{cases} \tag{1.3.17}$$

and

$$\begin{cases} A = \mu^3\rho^6(17\lambda - 11\mu)(\lambda + 2\mu)^2, \\ Q = \mu^6\rho^{12}(\lambda + 2\mu)^3\left(11\lambda^3 + 4\lambda^2\mu - 9\lambda\mu^2 - 10\mu^3\right). \end{cases}$$

The classification table of the values c_i^2 is:

		c_1^2	c_2^2	c_3^2
$\frac{\lambda}{2(\mu+\lambda)} < \sigma^*$	$Q < 0$	*positive*	*positive*	*positive*
$\frac{\lambda}{2(\mu+\lambda)} = \sigma^*$	$Q = 0$	*positive*	*positive*	*positive*
$\frac{\lambda}{2(\mu+\lambda)} > \sigma^*$	$Q > 0$	*positive*	*complex*	*complex*

(1.3.18)

The critical value σ^*, which is given in (1.1.8), has been found in different articles from the secular equation produced by Rayleigh.

Remark 1.3.5. In Section 5, we will use the Euclid algorithm to perform the decomposition (1.3.14). The polynomials $c_{ij;mn}$ and $\mathfrak{n}_{ij;mn}$ can be computed explicitly by Mathematica 8.0; and they are polynomials of degree less or equal 4.

iv. **Inversion of surface wave propagator $\mathbb{L}^{-1}[S_{ij}]$.**

By the decomposition (1.3.14), the operator $\mathbb{L}^{-1}[S_{ij}]$ is decomposed into

$$\mathbb{L}^{-1}[S_{ij}](y,z,t) = \sum_{0\le m,n\le 1}\left\{\left(c_{ij;mn}(\partial_y,\partial_z) + \mathbb{L}^{-1}\left[\frac{\mathfrak{n}_{ij;mn}(i\eta,i\zeta,s)}{\mathfrak{D}(i\eta,i\zeta,s)}\right]\right) \underset{(y,z,t)}{*} \mathbb{L}^{-1}\left[\left(\frac{\partial_s\xi_L}{s}\right)^m\right] \underset{(y,z,t)}{*} \mathbb{L}^{-1}\left[\left(\frac{\partial_s\xi_T}{s}\right)^n\right]\right\}. \tag{1.3.19}$$

In this decomposition (1.3.19), the operator $\left(c_{ij;mn}(\partial_y, \partial_z) + \mathbb{L}^{-1}\left[\frac{\mathfrak{n}_{ij;mn}(i\eta,i\zeta,s)}{\mathfrak{D}(i\eta,i\zeta,s)}\right]\right)$ is identified as the native 2-D wave (Rayleigh wave); and the operators $\mathbb{L}^{-1}[\partial_s\xi_L/s]$ and $\mathbb{L}^{-1}[\partial_s\xi_T/s]$ are identified as the 3-D body waves on surface.

A **The inversion of the Rayleigh Wave (Native 2-D wave)**

a **Complex Rayleigh roots (When $\frac{1}{2}\lambda/(\mu+\lambda) > \sigma^*$).**

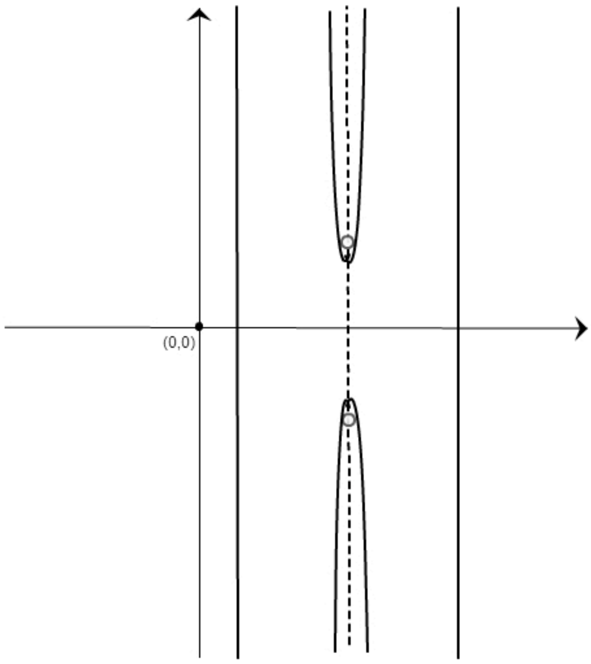

Figure 1.1: Poles with positive real part

To obtain $\mathbb{L}^{-1}\left[\frac{\mathfrak{n}_{ij;mn}(i\eta,i\zeta,s)}{\mathfrak{D}(i\eta,i\zeta,s)}\right]$ one needs to compute the poles of $\frac{\mathfrak{n}_{ij;mn}(i\eta,i\zeta,s)}{\mathfrak{D}(i\eta,i\zeta,s)}$ in s. The pole of the rational functions are at the zeros of $\mathfrak{D}$. Then, from the table (1.3.18) one has that when the Poisson ratio satisfies $\frac{\lambda}{2(\mu+\lambda)} > \sigma^*$, the coefficients c_2^2 and c_3^2 are complex conjugates. Thus the poles would be two couples of conjugates and symmetric with respect to y axis, which are apart from the imaginary axis.

Case1. The figure above explains the poles in the right half space. In this case, the Bromwich integral should be integrated along the path to the right of the poles. However, as the poles of c_2 and c_3 come from the rationalization of the determinant of the formula, the residue of these two poles would be zero. Thus the integral path of Bromwich integral can be switched to the line on the left. More precisely, we can use the imaginary axis as the integral path. Then we see this integral would only contain c_1 part and the integral along branch cut.

Case2. For the two poles in the left half space, one can compare their coefficients with the coefficients of the two poles in the right half space. Then, as the symmetric property of these poles, the contribution of the two conjugate poles in the left half space would be canceled just like the two in the right half space.

Then we can conclude that there would be no instability terms in the solution formula and thus our formula would still be valid in the case $\frac{\lambda}{2(\mu+\lambda)} > \sigma^*$.

b **Real roots (When $\frac{1}{2}\lambda/(\lambda+\mu) \leq \sigma^*$).**

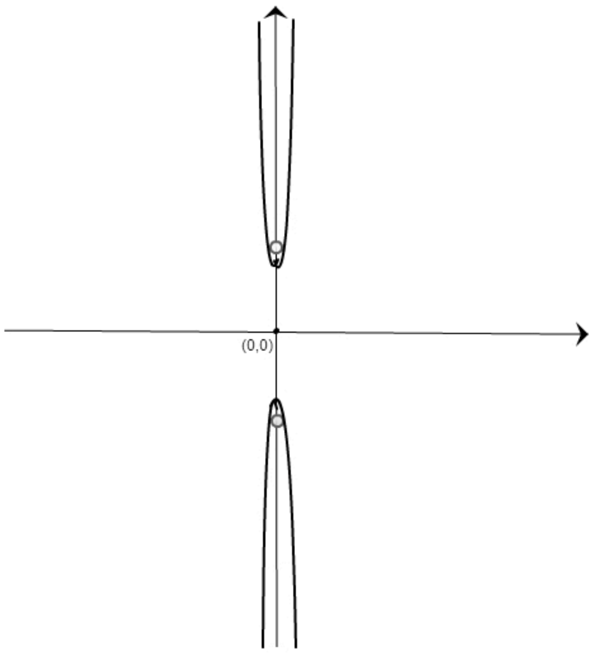

Figure 1.2: real roots

When the Poisson ratio $\lambda/(2(\mu+\lambda)) \le \sigma_*$, all roots s of $\mathfrak{D}=0$ are pure imaginary number. For any $K>0$,

$$\frac{1}{2\pi i}\int_{Re(s)=K} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}_{ij}}ds = \sum_{k=1}^{3}\left(\operatorname*{Res}_{s=ic_k\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}_{ij}} + \operatorname*{Res}_{s=-ic_k\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}_{ij}}\right)$$
$$= \sum_{k=1}^{3} N_{ij;mn;k0}(i\eta,i\zeta,t)\cos(c_k\sqrt{(\eta^2+\zeta^2)}t) + N_{ij;mn;k1}(i\eta,i\zeta)\frac{\sin(c_k\sqrt{(\eta^2+\zeta^2)}t)}{\sqrt{\eta^2+\zeta^2}}. \tag{1.3.20}$$

Case. $\frac{1}{2}\lambda/(\lambda+\mu) < \sigma^*$.

When the Poisson ratio $\frac{1}{2}\lambda/(\lambda+\mu) < \sigma^*$, one has that $N_{ij;mn;k0}=0$ and that $N_{ij;mn;k1}$ is a polynomial in $i\eta$ and $i\zeta$ with degree ≤ 3 so that

$$\frac{1}{2\pi i}\int_{Re(s)=K_1} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}ds = \sum_{k=1}^{3} N_{ij;mn;k1}(i\eta,i\zeta)\frac{\sin(c_k\sqrt{(\eta^2+\zeta^2)}t)}{\sqrt{(\eta^2+\zeta^2)}}. \tag{1.3.21}$$

From this and Proposition 1.2.5, one has

$$\mathbb{L}^{-1}\left[\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}\right] = \frac{1}{8\pi^3 i}\iint_{\mathbb{R}^2} e^{iy\eta+iz\zeta}\left(\int_{Re(s)=0} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}ds\right)d\eta d\zeta$$
$$= \sum_{k=1}^{3} N_{ij;mn;k1}(\partial_y,\partial_z)\mathbb{U}_1(y,z,c_k t). \tag{1.3.22}$$

Case. $\frac{1}{2}\lambda/(\lambda+\mu)=\sigma^*$

The coefficient $N_{ij;mn;kl}$ is a polynomial in $i\eta$ and $i\zeta$ of degree ≤ 4 and in particular $N_{ij;mn;k0}$ is polynomial of degree 1 in t such that

$$\frac{1}{2\pi i}\int_{Re(s)=K} e^{st}\,\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}ds$$
$$= \sum_{k=1}^{3}\left(N_{ij;mn;k0}(i\eta,i\zeta,t)\cos(c_k\sqrt{\eta^2+\zeta^2}t) + N_{ij;mn;k1}(i\eta,i\zeta)\frac{\sin(c_k\sqrt{\eta^2+\zeta^2}t)}{\sqrt{\eta^2+\zeta^2}}\right). \tag{1.3.23}$$

From this and Proposition 1.2.5,

$$\mathbb{L}^{-1}\left[\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}\right] = \frac{1}{8\pi^3 i}\iint_{\mathbb{R}^2} e^{iy\eta+iz\zeta}\left(\int_{Re(s)=0} e^{st}\frac{\mathfrak{n}_{ij;mn}}{\mathfrak{D}}ds\right)d\eta d\zeta$$
$$= \sum_{k=1}^{3} N_{ij;mn;k0}(\partial_y,\partial_z,t)\mathbb{U}_0(y,z,c_k t) + \sum_{k=1}^{3} N_{ij;mn;k1}(\partial_y,\partial_z)\mathbb{U}_1(y,z,c_k t). \tag{1.3.24}$$

Note that the polynomial $N_{ij;mn;k,0}(i\eta,i\zeta,t)$ as a polynomial of degree one in t is due to double roots of c_k. This resonance causes that there is a linear growth factor t for the case $\frac{1}{2}\lambda/(\lambda+\mu)=\sigma^*$.

Remark 1.3.6. As in the third case, the main part is the c_1 part and the computation of c_1 part is same as the computations in other two cases, we will only show the inverse transform of the case $\frac{1}{2}\lambda/(\mu+\lambda)\leq\sigma^*$.

B **The inversion of the 3-D interior wave on surface (Native 3-D wave)**

To perform the inversion $\mathbb{L}^{-1}[\partial_s\xi_L/s]$ and $\mathbb{L}^{-1}[\partial_s\xi_T/s]$ we will need to introduce the *Laplace-Fourier* paths as follows.

For each fixed (η,ζ) the *Laplace-Fourier paths* for ξ_T and ξ_L are defined as follows:

$$\begin{cases} \Gamma_T^+ \equiv \{s = s_T^+(\hat{\xi}, i\eta, i\zeta) | \xi_T(i\eta, i\zeta, s_T^+(\hat{\xi}, i\eta, i\zeta)) = i\hat{\xi},\ \hat{\xi} \in \mathbb{R}^+\}, \\ \Gamma_T^- \equiv \{s = s_T^-(\hat{\xi}, i\eta, i\zeta) | \xi_T(i\eta, i\zeta, s_T^-(\hat{\xi}, i\eta, i\zeta)) = i\hat{\xi},\ \hat{\xi} \in \mathbb{R}^+\}, \\ \Gamma_L^+ \equiv \{s = s_L^+(\hat{\xi}, i\eta, i\zeta) | \xi_L(i\eta, i\zeta, s_L^+(\hat{\xi}, i\eta, i\zeta)) = i\hat{\xi},\ \hat{\xi} \in \mathbb{R}^+\}, \\ \Gamma_L^- \equiv \{s = s_L^-(\hat{\xi}, i\eta, i\zeta) | \xi_L(i\eta, i\zeta, s_L^-(\hat{\xi}, i\eta, i\zeta)) = i\hat{\xi},\ \hat{\xi} \in \mathbb{R}^+\}. \end{cases} \tag{1.3.25}$$

There are two paths R_L and R_T:

$$\begin{cases} R_T = i\mathbb{R} \backslash \left(\Gamma_T^+ \cup \Gamma_T^-\right), \\ R_L = i\mathbb{R} \backslash \left(\Gamma_L^+ \cup \Gamma_L^-\right). \end{cases} \tag{1.3.26}$$

We re-decompose the path integral along $Re(s) = 0$ as follows.

$$\int_{Re(s)=0} e^{st} \frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s} ds = \left(\int_{\Gamma_T^+} + \int_{\Gamma_T^-} + \int_{R_T} \right) e^{st} \frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s} ds, \tag{1.3.27}$$

$$\int_{Re(s)=0} e^{st} \frac{\partial_s \xi_L(i\eta, i\zeta, s)}{s} ds = \left(\int_{\Gamma_L^+} + \int_{\Gamma_L^-} + \int_{R_L} \right) e^{st} \frac{\partial_s \xi_L(i\eta, i\zeta, s)}{s} ds. \tag{1.3.28}$$

From the above combinations of path integrals, one defines

$$\begin{cases} \mathscr{W}_T(y, z, t) \equiv \frac{1}{8\pi^3 i} \iint_{\mathbb{R}^2} e^{i\eta y + i\zeta z} \int_{\Gamma_T^+ \cup \Gamma_T^-} e^{st} \frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s} ds d\eta d\zeta \\ \mathscr{W}_L(y, z, t) \equiv \frac{1}{8\pi^3 i} \iint_{\mathbb{R}^2} e^{i\eta y + i\zeta z} \int_{\Gamma_L^+ \cup \Gamma_L^-} e^{st} \frac{\partial_s \xi_L(i\eta, i\zeta, s)}{s} ds d\eta d\zeta, \\ \mathscr{R}_T(y, z, t) \equiv \frac{1}{8\pi^3 i} \iint_{\mathbb{R}^2} e^{i\eta y + i\zeta z} \int_{R_T} e^{st} \frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s} ds d\eta d\zeta, \\ \mathscr{R}_L(y, z, t) \equiv \frac{1}{8\pi^3 i} \iint_{\mathbb{R}^2} e^{i\eta y + i\zeta z} \int_{R_L} e^{st} \frac{\partial_s \xi_L(i\eta, i\zeta, s)}{s} ds d\eta d\zeta. \end{cases} \tag{1.3.29}$$

Lemma 1.3.7. *The notions $\mathscr{W}_T(y,z,t)$, $\mathscr{W}_L(y,z,t)$, $\mathscr{R}_T(y,z,t)$, $\mathscr{R}_L(y,z,t)$ given in* (1.3.29) *satisfy*

$$\begin{cases} \mathscr{W}_T(y, z, t) = \mathscr{R}_T(y, z, t) = \mathbb{W}_0(0, y, z, c_T t), \\ \mathscr{R}_L(y, z, t) = \mathscr{W}_L(y, z, t) = \mathbb{W}_0(0, y, z, c_L t). \end{cases} \tag{1.3.30}$$

We will leave the proof of Lemma 1.3.7 in Section 6.

v. **The completion of the LY algorithm**

The decomposition in (1.3.19), the inversions in (1.3.22), (1.3.24), and Lemma 1.3.7 together conclude Theorem 1.1.1. This gives the final composition of the explicit solution formula of the surface wave, $(\mathbf{a}(0,y,z,t), \mathbf{a}_x(0,y,z,t))$ in terms of the given inhomogeneous term $\mathbf{g}_b(y,z,t)$.

1.4 Master Relationship, boundary condition, and matrix $(S_{ij})_{6\times 3}$

The *master relationship* (1.3.11) with $\xi_* \in \{\xi_L, \xi_T\}$ will pose 6 equations, but there are only 3 linearly independent equations. The free boundary conditions in transform variables

$$\begin{pmatrix} 0 & i\eta\lambda & i\zeta\lambda \\ i\eta\mu & 0 & 0 \\ i\zeta\mu & 0 & 0 \end{pmatrix}\mathbf{D} + \begin{pmatrix} (2\mu+\lambda) & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}\mathbf{N} = \mathbb{L}[\mathbf{g}_b] \qquad (1.4.1)$$

give another 3 linearly independent equations. One has the following linear system for **D** and **N**:

$$\begin{pmatrix} -i\zeta\left(\left(\zeta^2+\eta^2\right)\mu+s^2\rho\right) & \zeta\eta(\lambda+2\mu)\xi_L & \zeta^2(\lambda+2\mu)\,\xi_L & -i\zeta(\lambda+2\mu)\xi_L & \zeta\eta\mu & \zeta^2\mu \\ \left(\zeta^2+\eta^2\right)\mu\,\xi_T & i\eta\left(\left(\zeta^2+\eta^2\right)(\lambda+2\mu)+s^2\rho\right) & i\zeta\left(\left(\zeta^2+\eta^2\right)(\lambda+2\mu)+s^2\rho\right) & \left(\zeta^2+\eta^2\right)(\lambda+2\mu) & i\eta\mu\,\xi_T & i\zeta\mu\,\xi \\ -i\eta\left(\left(\zeta^2+\eta^2\right)\mu+s^2\rho\right) & \left(\eta^2(\lambda+2\mu)+s^2\rho\right)\,\xi_T & \zeta\eta(\lambda+2\mu)\,\xi_T & -i\eta(\lambda+2\mu)\,\xi_T & \left(\eta^2\mu+s^2\rho\right) & \zeta\eta\mu \\ 0 & i\lambda\eta & i\lambda\zeta & \lambda+2\mu & 0 & 0 \\ i\mu\eta & 0 & 0 & 0 & \mu & 0 \\ i\mu\zeta & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{N} \end{pmatrix}_{6\times 1} = \begin{pmatrix} \vec{0} \\ \mathbb{L}[\mathbf{g}_b] \end{pmatrix}_{6\times 1}. \tag{1.4.2}$$

The first three rows are due to the *master relationship*, the last three rows are the boundary conditions given in (1.4.1). The linear system gives the full boundary data:

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{N} \end{pmatrix}(i\eta, i\zeta, s) = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \\ S_{41} & S_{42} & S_{43} \\ S_{51} & S_{52} & S_{53} \\ S_{61} & S_{62} & S_{63} \end{pmatrix} \mathbb{L}[\mathbf{g}_b](i\eta, i\zeta, s), \tag{1.4.3}$$

where each entry S_{ij} is a rational function in $i\eta$, $i\zeta$, s, ξ_T, and ξ_L:

$$\begin{cases} S_{ij} = S^n_{ij}(i\eta, i\zeta, s, \xi_T, \xi_L)/S^d_{ij}(i\eta, i\zeta, s, \xi_T, \xi_L), \\ \xi_T \equiv \sqrt{\eta^2 + \zeta^2 + \frac{\rho}{\mu}s^2} = \sqrt{\eta^2 + \zeta^2 + s^2/{c_T}^2}, \\ \xi_L(i\eta, i\zeta, s) \equiv \sqrt{\eta^2 + \zeta^2 + \frac{\rho}{2\mu+\lambda}s^2} = \sqrt{\eta^2 + \zeta^2 + s^2/{c_L}^2}, \\ p_T(\xi_T, i\eta, i\zeta, s) = p_L(\xi_L, i\eta, i\zeta, s) = 0, \end{cases} \tag{1.4.4}$$

For example, the entry S_{22} is

$$\begin{aligned} S_{22} &= \frac{-s^4\rho^2\xi_T + 4\zeta^2\left(\zeta^2+\eta^2\right)\mu^2\left(\xi_L - \xi_T\right) + \mu\left(-4s^2\zeta^2\rho\xi_T - s^2\eta^2\rho\xi_T + 4s^2\zeta^2\rho\xi_L\right)}{\left(\left(\zeta^2+\eta^2\right)\mu + s^2\rho\right)\left(4s^2\left(\zeta^2+\eta^2\right)\mu\rho + s^4\rho^2 - 4\left(\zeta^2+\eta^2\right)\mu^2\left(-\zeta^2-\eta^2+\xi_L\xi_T\right)\right)} \\ &\equiv \frac{S^n_{22}(i\eta, i\zeta, s, \xi_T, \xi_L)}{S^d_{22}(i\eta, i\zeta, s, \xi_T, \xi_L)}; \end{aligned} \tag{1.4.5}$$

and we use this entry as an example to compute the polynomials $c_{ij;mn}$, $\mathfrak{n}_{ij;mn}$, and $N_{ij;mn;kl}$.

1.5 Characteristic-non characteristic decomposition

The polynomials $S^d_{22}(i\eta, i\zeta, s, X, Y)$ and $S^n_{22}(i\eta, i\zeta, s, X, Y)$ given in (1.4.5) are assumed to be relative prime polynomials in X and Y over the coefficient ring $\mathbb{C}[\eta, \zeta, s]$. The denominator S^d_{22} contains roots ξ_L and ξ_T. These two roots are not local analytic in the variables $(i\zeta, i\zeta, s)$ around $(0, 0, 0)$. Due to this defect, they are classified as characteristic roots in [13]; and one will need to remove them from the denominators by simple algebraic manipulations introduced in [14]. The algebraic manuplications can be achieved as follows.

By the Euclid Algorithm, one can find polynomials $Q_1(i\eta, i\zeta, s, X, Y)$, $Q_T(i\eta, i\zeta, s, X, Y)$,

$Q_L(i\eta, i\zeta, s, X, Y)$, and $R(i\eta, i\zeta, s)$ such that

$$\begin{aligned}&Q_1(i\eta, i\zeta, s, X, Y)S_{22}^d(i\eta, i\zeta, s, X, Y)\\&+Q_T(i\eta, i\zeta, s, X, Y)p_T(i\eta, i\zeta, s, X)+Q_L(i\eta, i\zeta, s, X, Y)p_L(i\eta, i\zeta, s, Y)=R(i\eta, i\zeta, s).\end{aligned} \tag{1.5.1}$$

By this identity, one has

$$\begin{aligned}&S_{22}(i\eta, i\zeta, s, X, Y)\\&=\frac{Q_1(i\eta, i\zeta, s, X, Y)S_{22}^n(i\eta, i\zeta, s, X, Y)}{R(i\eta, i\zeta, s)-Q_T(i\eta, i\zeta, s, X, Y)p_T(i\eta, i\zeta, s, X)-Q_L(i\eta, i\zeta, s, X, Y)p_L(i\eta, i\zeta, s, Y)}.\end{aligned} \tag{1.5.2}$$

By the property that $deg(p_T)=deg(p_L)=2$ in ξ, there exist $q_T(i\eta, i\zeta, s, X, Y)$ and $q_L(i\eta, i\zeta, s, X, Y)$ such that

$$\begin{aligned}&Q_1(i\eta, i\zeta, s, X, Y)S_{22}^n(i\eta, i\zeta, s, X, Y)\\&\quad=q_T(i\eta, i\zeta, s, X, Y)p_T(i\eta, i\zeta, s, X)+q_L(i\eta, i\zeta, s, X, Y)p_L(i\eta, i\zeta, s, X)\\&\qquad+n_{00}(i\eta, i\zeta, s)+n_{01}(i\eta, i\zeta, s)X+n_{10}(i\eta, i\zeta, s)Y+n_{11}(i\eta, i\zeta, s)XY.\end{aligned} \tag{1.5.3}$$

Then, by (1.5.2), (1.5.3), and $p_T(i\eta, i\zeta, s, \xi_T)=p_L(i\eta, i\zeta, s, \xi_L)=0$ together to yield that

$$S_{22}(i\eta, i\zeta, s, \xi_T, \xi_L)=\frac{n_{00}+n_{01}\xi_T+n_{10}\xi_L+n_{11}\xi_T\xi_L}{R(i\eta, i\zeta, s)}. \tag{1.5.4}$$

This gives rise to

$$\begin{cases}R &=\left(\zeta^2+\eta^2+\frac{s^2\rho}{\mu}\right)\left(16\left(\zeta^2+\eta^2\right)^3\mu^3(\lambda+\mu)+8s^2\left(\zeta^2+\eta^2\right)^2\mu^2(3\lambda+4\mu)\rho\right)\\ &\quad+\left(\zeta^2+\eta^2+\frac{s^2\rho}{\mu}\right)\left(8s^4\left(\zeta^2+\eta^2\right)\mu(\lambda+2\mu)\rho^2+s^6(\lambda+2\mu)\rho^3\right),\\ n_{00} &=-\frac{R}{\mu\left(\zeta^2+\eta^2+\frac{s^2\rho}{\mu}\right)}\\ &\quad+\eta^2\left(4\left(\zeta^2+\eta^2\right)^2\mu^2(3\lambda+2\mu)+16s^2\left(\zeta^2+\eta^2\right)\mu(\lambda+\mu)\rho+3s^4(\lambda+2\mu)\rho^2\right),\\ n_{11} &=-4\eta^2\mu(\lambda+2\mu)\left(\left(\zeta^2+\eta^2\right)\mu+s^2\rho\right),\\ n_{01} &=n_{10}=0.\end{cases} \tag{1.5.5}$$

Then, substitute $\xi_L \partial_s \xi_L = s$ and $\xi_T \partial_s \xi_T = s$ together with (1.5.5) into (1.5.4) to yield that

$$S_{22}(i\eta, i\zeta, s, \xi_T, \xi_L) = \sum_{0 \le m,n \le 1} \left(c_{22;mn} + \frac{\mathfrak{n}_{22;mn}(i\eta, i\zeta, s)}{\mathfrak{D}(i\eta, i\zeta, s)} \right) \left(\frac{\partial_s \xi_L}{s} \right)^m \left(\frac{\partial_s \xi_T}{s} \right)^n . \tag{1.5.6}$$

One has

$$\begin{cases} \mathfrak{n}_{22;00} &= \mathfrak{n}_{22;11} = 0, \\ \mathfrak{n}_{22;10} &= -\dfrac{4\eta^2\mu\left(\left(\zeta^2+\eta^2\right)\mu + s^2\rho\right)\left(\left(\zeta^2+\eta^2\right)(\lambda+2\mu)+s^2\rho\right)}{\rho^4}, \\ \mathfrak{n}_{22;01} &= \dfrac{\eta^2\mu\left(4\left(\zeta^2+\eta^2\right)^2\mu^2(3\lambda+2\mu)+16s^2\left(\zeta^2+\eta^2\right)\mu(\lambda+\mu)\rho+3s^4(\lambda+2\mu)\rho^2\right)}{(\lambda+2\mu)\rho^4}, \\ c_{22;01} &= -\dfrac{1}{\rho}. \end{cases} \tag{1.5.7}$$

The divisor $\mathfrak{D}$ is independent of i, j,

$$\begin{aligned} &\mathfrak{D}(i\eta, i\zeta, s) \\ &= s^6 + 8\frac{\mu}{\rho}\left(\eta^2+\zeta^2\right)s^4 + 8\left(\frac{\mu}{\rho}\right)^2 \frac{3\lambda+4\mu}{\lambda+2\mu}\left(\eta^2+\zeta^2\right)^2 s^2 + 16\left(\frac{\mu}{\rho}\right)^3 \frac{\lambda+\mu}{\lambda+2\mu}\left(\eta^2+\zeta^2\right)^3 . \end{aligned} \tag{1.5.8}$$

The polynomial $\mathfrak{D}(i\eta, i\zeta, s)$ is defined as the determinant of the Rayleigh wave; and it can be factorized as a product of six linear polynomials in s:

$$\begin{aligned} \mathfrak{D} = &\left(s + ic_1\sqrt{\eta^2+\zeta^2}\right)\left(s - ic_1\sqrt{\eta^2+\zeta^2}\right)\left(s + ic_2\sqrt{\eta^2+\zeta^2}\right) \\ &\left(s - ic_2\sqrt{\eta^2+\zeta^2}\right)\left(s + ic_3\sqrt{\eta^2+\zeta^2}\right)\left(s - ic_3\sqrt{\eta^2+\zeta^2}\right), \end{aligned} \tag{1.5.9}$$

where the square of c_k are listed in (1.3.17).

1.6 Realization of decomposition

1.6.1 Inversion of the Rayleigh Wave

In this subsection, we list the integral $(2\pi i)^{-1}\int_{Re(s)=0+} e^{st}\mathfrak{n}_{22;10}/\mathfrak{D}ds$ to give an example to compute the polynomial $N_{22;10;kl}$.

By the exact expression in (1.5.7) for polynomials $\mathfrak{n}_{22;mn}$, one has

$$\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}} = -\frac{1}{\rho^4}\frac{4\eta^2\mu\left(\left(\zeta^2+\eta^2\right)\mu+s^2\rho\right)\left(\left(\zeta^2+\eta^2\right)(\lambda+2\mu)+s^2\rho\right)}{\prod_{i=1}^3\left(s^2+c_i^2(\eta^2+\zeta^2)\right)}. \tag{1.6.1}$$

Case. $\frac{1}{2}\lambda/(\lambda+\mu)<\sigma^*$.

In this case, three surface wave speeds c_1, c_2, c_3 are positive and distinct; and one has

$$\begin{cases} \underset{s=ic_k\sqrt{(\eta^2+\zeta^2)}}{\text{Res}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = -\frac{4\mu(\mu-\rho c_k^2)(\lambda+2\mu-\rho c_k^2)\eta^2}{\rho^4\prod_{i\neq k}(c_i^2-c_k^2)2ic_k\sqrt{\eta^2+\zeta^2}}e^{ic_k\sqrt{(\eta^2+\zeta^2)}t}, \\ \underset{s=-ic_k\sqrt{(\eta^2+\zeta^2)}}{\text{Res}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = -\frac{4\mu(\mu-\rho c_k^2)(\lambda+2\mu-\rho c_k^2)\eta^2}{\rho^4\prod_{i\neq k}(c_i^2-c_k^2)(-2ic_k\sqrt{\eta^2+\zeta^2})}e^{-ic_k\sqrt{(\eta^2+\zeta^2)}t}. \end{cases} \tag{1.6.2}$$

This yields that

$$\frac{1}{2\pi i}\int_{Re(s)=K} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}}ds = \sum_{k=1}^3\left(-\frac{4\eta^2\mu(\mu-\rho c_k^2)(\lambda+2\mu-\rho c_k^2)}{c_k\rho^4\prod_{i\neq k}(c_i^2-c_k^2)}\right)\frac{\sin(c_k\sqrt{(\eta^2+\zeta^2)}t)}{\sqrt{\eta^2+\zeta^2}}. \tag{1.6.3}$$

This results in

$$\begin{cases} N_{22;10;0k}=0, \\ N_{22;10;1k}=\left(-\frac{4\eta^2\mu(\mu-\rho c_k^2)(\lambda+2\mu-\rho c_k^2)}{c_k\rho^4\prod_{i\neq k}(c_i^2-c_k^2)}\right). \end{cases} \tag{1.6.4}$$

Case. $\frac{1}{2}\lambda/(\lambda+\mu)=\sigma^*$.

In this case two surface wave speeds coincide: $c_1, c_2, c_3>0$ and $c_2=c_3$. From

(1.6.1) one has

$$\begin{cases} \operatorname*{Res}_{s=ic_1\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = & -\frac{4\eta^2\mu(\mu-\rho c_1^2)(\lambda+2\mu-\rho c_1^2)}{\rho^4(c_2^2-c_1^2)^2 2ic_1\sqrt{\eta^2+\zeta^2}}e^{ic_1\sqrt{(\eta^2+\zeta^2)}t}, \\ \operatorname*{Res}_{s=-ic_1\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = & -\frac{4\eta^2\mu(\mu-\rho c_1^2)(\lambda+2\mu-\rho c_1^2)}{\rho^4(c_2^2-c_1^2)^2 2ic_1\sqrt{\eta^2+\zeta^2}}e^{-ic_1\sqrt{(\eta^2+\zeta^2)}t}, \\ \operatorname*{Res}_{s=ic_2\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = & -\frac{e^{ic_2\sqrt{\eta^2+\zeta^2}}t4\eta^2\mu\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4 c_2^2\left(-c_1^2+c_2^2\right)} \\ & +e^{ic_2\sqrt{\eta^2+\zeta^2}}\left(-\frac{8i\eta^2\mu\left(\lambda+3\mu-2\rho c_2^2\right)}{\sqrt{\zeta^2+\eta^2}\rho^3\left(-c_1^2c_2+c_2^3\right)}\right) \\ & +e^{ic_2\sqrt{\eta^2+\zeta^2}}\left(\frac{i\eta^2\mu\left(c_1^2-3c_2^2\right)\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\sqrt{\zeta^2+\eta^2}\rho^4c_2^3\left(c_1^2-c_2^2\right)^2}\right) \\ \operatorname*{Res}_{s=-ic_2\sqrt{(\eta^2+\zeta^2)}} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}_{ij}} = & -\frac{4e^{-i\sqrt{\zeta^2+\eta^2}c_2}t\eta^2\mu\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4 c_2^2\left(-c_1^2+c_2^2\right)} \\ & +e^{-ic_2\sqrt{\eta^2+\zeta^2}}\left(\frac{8i\eta^2\mu\left(\lambda+3\mu-2\rho c_2^2\right)}{\sqrt{\zeta^2+\eta^2}\rho^3\left(-c_1^2c_2+c_2^3\right)}\right) \\ & -e^{-ic_2\sqrt{\eta^2+\zeta^2}}\left(\frac{i\eta^2\mu\left(c_1^2-3c_2^2\right)\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\sqrt{\zeta^2+\eta^2}\rho^4c_2^3\left(c_1^2-c_2^2\right)^2}\right). \end{cases} \tag{1.6.5}$$

This gives rise to

$$\begin{aligned} &\frac{1}{2\pi i}\int_{Re(s)=0+} e^{st}\frac{\mathfrak{n}_{22;10}}{\mathfrak{D}}ds \\ =& -\frac{4\eta^2\mu\left(\mu-\rho c_1^2\right)\left(\lambda+2\mu-\rho c_1^2\right)}{\rho^4c_1\left(c_1^2-c_2^2\right)^2}\frac{\sin\left(c_1\sqrt{\eta^2+\zeta^2}t\right)}{\sqrt{\eta^2+\zeta^2}} \\ & -\frac{8t\eta^2\mu\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4c_2^2\left(-c_1^2+c_2^2\right)}\cos\left(c_2\sqrt{\eta^2+\zeta^2}t\right) \\ & -\frac{16\eta^2\mu\left(\lambda+3\mu-2\rho c_2^2\right)}{\rho^3c_2\left(c_1^2-c_2^2\right)^2}\frac{\sin\left(c_2\sqrt{\eta^2+\zeta^2}t\right)}{\sqrt{\eta^2+\zeta^2}} \\ & -\frac{2\eta^2\mu\left(c_1^2-3c_2^2\right)\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4c_2^3\left(c_1^2-c_2^2\right)^2}\frac{\sin\left(c_2\sqrt{\eta^2+\zeta^2}t\right)}{\sqrt{\eta^2+\zeta^2}}. \end{aligned} \tag{1.6.6}$$

This results in

$$\begin{cases} N_{22;10;01} = & 0, \\ N_{22;10;11} = & -\frac{4\eta^2\mu\left(\mu-\rho c_1^2\right)\left(\lambda+2\mu-\rho c_1^2\right)}{\rho^4 c_1\left(c_1^2-c_2^2\right)^2}, \\ N_{22;10;03} = N_{22;10;02} = & -\frac{8t\eta^2\mu\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4 c_2^2\left(-c_1^2+c_2^2\right)}, \\ N_{22;10;13} = N_{22;10;12} = & -\frac{16\eta^2\mu\left(\lambda+3\mu-2\rho c_2^2\right)}{\rho^3 c_2\left(c_1^2-c_2^2\right)^2} \\ & -\frac{2\eta^2\mu\left(c_1^2-3c_2^2\right)\left(\mu-\rho c_2^2\right)\left(\lambda+2\mu-\rho c_2^2\right)}{\rho^4 c_2^3\left(c_1^2-c_2^2\right)^2}. \end{cases} \tag{1.6.7}$$

1.6.2 Proof of Lemma 1.3.7.

It is sufficient to give the proof for $\mathscr{R}_T$ and $\mathscr{W}_T$, only.

For the path integral over the path R_T given in (1.3.26), one has

$$\frac{1}{2\pi i}\int_{R_T} e^{st}\frac{\partial s\xi_T\left(i\eta,i\zeta,s\right)}{s}ds = \frac{1}{2\pi i}\int_{R_T}\frac{e^{st}\frac{\rho}{\mu}}{\sqrt{\frac{\rho s^2}{\mu}+\eta^2+\zeta^2}}ds = \frac{1}{2}\sqrt{\frac{\rho}{\mu}}J_0\left(t\sqrt{\frac{\mu\left(\eta^2+\zeta^2\right)}{\rho}}\right). \tag{1.6.8}$$

By this identity together with substituting $\sqrt{(\eta^2+\zeta^2)} = \hat{r}$ into $\mathscr{R}_T$ defined in (1.3.29), and by the Propositions 1.2.6 and 1.2.7 together, one has

$$\begin{aligned} \mathscr{R}_T(y,z,t) =& \mathbb{F}^{-1}\left[\frac{1}{2}\sqrt{\frac{\rho}{\mu}}J_0\left(t\sqrt{\frac{\mu\left(\eta^2+\zeta^2\right)}{\rho}}\right)\right] \\ =& \frac{1}{2\pi}\frac{1}{2}\sqrt{\frac{\rho}{\mu}}\int_0^{+\infty} J_0\left(\hat{r}t\sqrt{\frac{\mu}{\rho}}\right)J_0\left(\hat{r}\sqrt{y^2+z^2}\right)\hat{r}d\hat{r} \\ =& \frac{1}{4\pi t c_T^2}\delta\left(tc_T-\sqrt{y^2+z^2}\right) \\ =& \mathbb{W}_0\left(0,y,z,c_T t\right). \end{aligned} \tag{1.6.9}$$

In order to compute the wave $\mathscr{W}_T$, one identifies the two branches of the *Laplace-Fourier path*s Γ_T^+ and Γ_T^- introduced in (1.3.25):

$$\Gamma_T^+ = \left[i\sqrt{\frac{\mu}{\rho}(\eta^2+\zeta^2)},+\infty\right), \quad \Gamma_T^- = \left(-\infty,-i\sqrt{\frac{\mu}{\rho}(\eta^2+\zeta^2)}\right].$$

Then, it follows

$$
\begin{aligned}
&\int_{\Gamma_T^+} e^{st}\frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s}ds \\
=&\int_0^{+\infty} \frac{e^{s_T^+(\hat{\xi}, i\eta, i\zeta)t}}{s_T^+(\hat{\xi}, i\eta, i\zeta)} \frac{\partial \xi_T\left(i\eta, i\zeta, s_T^+(\hat{\xi}, i\eta, i\zeta)\right)}{\partial s} \frac{\partial s_T^+(\hat{\xi}, i\eta, i\zeta)}{\partial \hat{\xi}} d\hat{\xi} \\
=&\int_0^{+\infty} \frac{e^{s_T^+(\hat{\xi}, i\eta, i\zeta)t}}{s_T^+(\hat{\xi}, i\eta, i\zeta)} i d\hat{\xi},
\end{aligned}
\tag{1.6.10}
$$

and

$$
\begin{aligned}
&\int_{\Gamma_T^-} e^{st}\frac{\partial_s \xi_T(i\eta, i\zeta, s)}{s}ds \\
=&\int_0^{+\infty} \frac{e^{s_T^-(\hat{\xi}, i\eta, i\zeta)t}}{s_T^-(\hat{\xi}, i\eta, i\zeta)} \frac{\partial \xi_T\left(i\eta, i\zeta, s_T^-(\hat{\xi}, i\eta, i\zeta)\right)}{\partial s} \frac{\partial s_T^-(\hat{\xi}, i\eta, i\zeta)}{\partial \hat{\xi}} d\hat{\xi} \\
=&\int_0^{+\infty} \frac{e^{s_T^-(\hat{\xi}, i\eta, i\zeta)t}}{s_T^-(\hat{\xi}, i\eta, i\zeta)} i d\hat{\xi}.
\end{aligned}
\tag{1.6.11}
$$

This gives rise to that

$$
\begin{aligned}
&\int_{\Gamma_T^+\cup\Gamma_T^-} \frac{e^{st}}{s}\partial_s \xi_T(i\eta, i\zeta, s)\, ds \\
=&\int_0^{+\infty} \frac{e^{s_T^+(\hat{\xi}, i\eta, i\zeta)t}}{s_T^+(\hat{\xi}, i\eta, i\zeta)} i d\hat{\xi} + \int_0^{+\infty} \frac{e^{s_T^-(\hat{\xi}, i\eta, i\zeta)t}}{s_T^-(\hat{\xi}, i\eta, i\zeta)} i d\hat{\xi} \\
=&\int_0^{+\infty} \frac{e^{i\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}t}}{i\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}} i d\hat{\xi} + \int_0^{+\infty} \frac{e^{-i\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}t}}{-i\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}} i d\hat{\xi} \\
=&2\int_0^{+\infty} \frac{i\sin\left(\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}t\right)}{\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}} d\hat{\xi} \\
=&\int_{-\infty}^{+\infty} \frac{i\sin\left(\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}t\right)}{\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}} d\hat{\xi}.
\end{aligned}
\tag{1.6.12}
$$

This identity and Proposition 1.2.5 conclude that the component $\mathscr{W}_T$ satisfies

$$\begin{aligned}\mathscr{W}_T(y,z,t) =& \frac{1}{8\pi^3 i}\iint_{\mathbb{R}^2} e^{i\eta y+i\zeta z}\int_{\Gamma_T^+\cup\Gamma_T^-} e^{st}\frac{\partial_s \xi_T\left(i\eta, i\zeta, s\right)}{s} ds d\eta d\zeta \\ =& \frac{1}{8\pi^3 i}\iint_{\mathbb{R}^2} e^{i\eta y+i\zeta z}\int_{-\infty}^{+\infty}\frac{i\sin\left(\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}t\right)}{\sqrt{\frac{\rho}{\mu}\left(\hat{\xi}^2+\eta^2+\zeta^2\right)}} d\hat{\xi} d\eta d\zeta \\ =& \mathbb{W}_0\left(0,y,z,c_T t\right).\end{aligned} \tag{1.6.13}$$

The lemma follows for $\mathscr{W}_T$ and $\mathscr{R}_T$.

CHAPTER 2

Computation and Estimates

2.1 Introduction

In the previous chapter we derive the solution formula of the Lamb's problem. Especially, we obtain the formula of the boundary data. Then we can reverse each term into time-space domain and combine them by convolution with respect to time and space variables. However, as our main goal is to construct the Green's function of Lamb's problem, we need to combine the boundary operators with the interior radiation waves, then our formula would have several drawbacks:

- The solution formula (1.1.15) expressed in matrix form for Lamb's problem is not clear enough for further analysis.
- There are too many convolutions and this would result in difficulty for estimates.

Thus now we need to recombine the formula for further estimates. For simplicity we only consider the case when Poisson ratio is smaller than the critical value and we also suppose the formula is independent of variable z and thus the half space system will become a 2-D system. In the 2-D system we can avoid 3-

D Kirchhoff's formula and thus the computation would be much more feasible; moreover the wave structure on the boundary (dimension 1) will remain existed. The 2-D system can be written as

$$\begin{aligned}
&\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{\rho}\nabla \cdot \left(\lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\right), \\
&\quad \vec{x} \equiv (x,y) \in \mathbb{R}^2_+ \equiv \{(x,y) : x > 0, y \in \mathbb{R}\},\ t \geq 0, \\
&\begin{pmatrix} (2\mu+\lambda)\partial_x & \lambda\partial_y \\ \mu\partial_y & \mu\partial_x \end{pmatrix} \mathbf{u}(0,y,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&\begin{cases} \mathbf{u}(\vec{x},0) = \vec{\Phi}(\vec{x}) \\ \partial_t \mathbf{u}(\vec{x},0) = \vec{\Psi}(\vec{x}), \end{cases}
\end{aligned} \tag{2.1.1}$$

Then we can consider the case only horizontal impulsion load initially and rewrite the 2-D system as below,

$$\begin{cases} \rho v_{tt} = (2\mu+\lambda)v_{yy} + \mu v_{xx} + (\lambda+\mu)u_{xy} \\ \\ \rho u_{tt} = (\mu+\lambda)v_{xy} + \mu u_{yy} + (2\mu+\lambda)u_{xx} \end{cases} \tag{2.1.2}$$

with boundary condition

$$\begin{cases} v_x + u_y = 0 \\ \\ (2\mu+\lambda)u_x + \lambda v_y = 0 \end{cases} \tag{2.1.3}$$

and initial condition

$$\begin{cases} u_t(x,y,0) = 0, & u(x,y,0) = 0 \\ \\ v_t(x,y,0) = \delta(x-x_0, y-y_0), & v(x,y,0) = 0 \end{cases} \tag{2.1.4}$$

Theorem 2.1.1. *The solution of system (2.1.2) with free boundary condition (2.1.3) and horizontal load (2.1.4) can be expressed in transform domain as*

$$\hat{u} = \frac{i\eta\left(e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}} - e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}\right)}{2s^2}$$

$$- \frac{N^u_{LL}\exp\left(-\frac{(x_0+x)\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2s^2\mathscr{D}}$$

$$- \frac{N^u_{TT}e^{-\frac{(x_0+x)\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2s^2\mathscr{D}} \tag{2.1.5}$$

$$- \frac{N^u_{LT}\exp\left(-\frac{x_0\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}} - \frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}\right)}{s^2\mathscr{D}}$$

$$- \frac{N^u_{TL}\exp\left(-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}} - \frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2\sqrt{\mu}s^2\mathscr{D}}$$

$$\hat{v} = \frac{\frac{\sqrt{\eta^2\mu+\rho s^2}e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{\sqrt{\mu}} - \frac{\eta^2\sqrt{\lambda+2\mu}e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}}{2s^2}$$

$$- \frac{N^v_{LL}\exp\left(-\frac{(x_0+x)\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2s^2\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\mathscr{D}}$$

$$- \frac{N^v_{TT}e^{-\frac{(x_0+x)\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\sqrt{\mu}s^2\mathscr{D}} \tag{2.1.6}$$

$$- \frac{N^v_{LT}\exp\left(-\frac{x_0\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}} - \frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}\right)}{s^2\mathscr{D}}$$

$$- \frac{N^v_{TL}\exp\left(-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}} - \frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2\sqrt{\mu}s^2\mathscr{D}}$$

The reflective operators $N^{u(v)}_{L(T)L(T)}$ can be found in (2.3.8) and (2.3.11).

2.2 Fundamental Solution of Elastic Equation

Then we consider the equation (2.1.2) in full space, i.e we want to study the fundamental solution of (2.1.2):

$$\begin{cases} \rho v_{tt} = (2\mu+\lambda)v_{yy} + \mu v_{xx} + (\lambda+\mu)u_{xy} \\ \rho u_{tt} = (\mu+\lambda)v_{xy} + \mu u_{yy} + (2\mu+\lambda)u_{xx} \end{cases} \tag{2.2.1}$$

with initial condition

$$\begin{cases} u_t(x,y,0) = 0, & u(x,y,0) = 0 \\ \\ v_t(x,y,0) = \delta(x,y), & v(x,y,0) = 0 \end{cases} \tag{2.2.2}$$

Lemma 2.2.1. *the solution of (2.2.1) with initial data (2.2.2) is:*

$$\begin{cases} \hat{u^f} = \dfrac{i\eta\left(e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}} - e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}\right)}{2s^2} \\ \\ \hat{v^f} = \dfrac{\frac{\sqrt{\eta^2\mu+\rho s^2}e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{\sqrt{\mu}} - \frac{\eta^2\sqrt{\lambda+2\mu}e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}}{2s^2} \end{cases} \tag{2.2.3}$$

Proof. Make Fourier transform of y and Laplace transform of t we can change the (2.2.1) into symbols as:

$$\begin{cases} s^2\rho\hat{v} - \rho\delta(x) = -(2\mu+\lambda)\eta^2\hat{v} + (\mu+\lambda)i\eta\hat{u}_x + \mu\hat{v}_{xx} \\ \\ s^2\rho\hat{u} = (\mu+\lambda)i\eta\hat{v}_x - \mu\eta^2\hat{u} + (2\mu+\lambda)\hat{u}_x x \end{cases} \tag{2.2.4}$$

This is an inhomogeneous second order ODE with source term $\delta(x)$. So we need to study the homogeneous one first:

$$\begin{cases} s^2\rho\hat{v} = -(2\mu+\lambda)\eta^2\hat{v} + (\mu+\lambda)i\eta\hat{u}_x + \mu\hat{v}_{xx} \\ \\ s^2\rho\hat{u} = (\mu+\lambda)i\eta\hat{v}_x - \mu\eta^2\hat{u} + (2\mu+\lambda)\hat{u}_{xx} \end{cases} \tag{2.2.5}$$

we can rewrite (2.2.5) into a matrix form if we denote u_x by m and v_x by n:

$$(\hat{m},\hat{n},\hat{u},\hat{v})_x = (\hat{m},\hat{n},\hat{u},\hat{v}) \begin{pmatrix} 0 & -\frac{(\mu+\lambda)i\eta}{\mu} & 1 & 0 \\ -\frac{(\mu+\lambda)i\eta}{2\mu+\lambda} & 0 & 0 & 1 \\ \frac{\rho s^2+\mu\eta^2}{2\mu+\lambda} & 0 & 0 & 0 \\ 0 & \frac{\rho s^2+(2\mu+\lambda)\eta^2}{\mu} & 0 & 0 \end{pmatrix} \tag{2.2.6}$$

Then the eigenvalues and correspond eigenfunctions of (2.2.6) can be list below:

$$\begin{cases} \sigma_1 = -\frac{\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}, & E_1 = \left\{-\frac{i\eta(\lambda+2\mu)}{\eta^2(\lambda+2\mu)+\rho s^2}, -\frac{\sqrt{\mu}\sqrt{\eta^2\mu+\rho s^2}}{\eta^2(\lambda+2\mu)+\rho s^2}, \frac{i\eta\sqrt{\mu}\sqrt{\eta^2\mu+\rho s^2}}{\eta^2(\lambda+2\mu)+\rho s^2}, 1\right\} \\ \sigma_2 = \frac{\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}} & E_2 = \left\{-\frac{i\eta(\lambda+2\mu)}{\eta^2(\lambda+2\mu)+\rho s^2}, \frac{\sqrt{\mu}\sqrt{\eta^2\mu+\rho s^2}}{\eta^2(\lambda+2\mu)+\rho s^2}, -\frac{i\eta\sqrt{\mu}\sqrt{\eta^2\mu+\rho s^2}}{\eta^2(\lambda+2\mu)+\rho s^2}, 1\right\} \\ \sigma_3 = -\frac{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}, & E_3 = \left\{-\frac{i}{\eta}, -\frac{\mu}{\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}, \frac{i\left(\eta^2\mu+\rho s^2\right)}{\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}, 1\right\} \\ \sigma_4 = \frac{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}, & E_4 = \left\{-\frac{i}{\eta}, \frac{\mu}{\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}, -\frac{i\left(\eta^2\mu+\rho s^2\right)}{\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}, 1\right\} \end{cases} \tag{2.2.7}$$

Then we can diagonalize (2.2.6) and then the two positive eigenvalues would lead to instability on the right hand-side and thus, for $x > 0$, the coefficients of these two terms would be forced to be zero. Then we have

for $x > 0$

$$\begin{cases} E_1 \cdot (\hat{m},\hat{n},\hat{u},\hat{v}) = Ae^{-\frac{\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}x} \\ E_2 \cdot (\hat{m},\hat{n},\hat{u},\hat{v}) = 0 \\ E_3 \cdot (\hat{m},\hat{n},\hat{u},\hat{v}) = Be^{-\frac{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}x} \\ E_4 \cdot (\hat{m},\hat{n},\hat{u},\hat{v}) = 0 \end{cases} \tag{2.2.8}$$

and then we can solve $(\hat{m}, \hat{n}, \hat{u}, \hat{v})$ as:

$$\begin{cases} \hat{m} = \dfrac{iB\eta\left(\eta^2(\lambda+2\mu)+\rho s^2\right)e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} - \dfrac{iA\eta\left(\eta^2(\lambda+2\mu)+\rho s^2\right)e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\rho s^2} \\ \hat{n} = \dfrac{e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(-A\eta^4\lambda\mu-2A\eta^4\mu^2-A\rho^2 s^4-A\eta^2\lambda\rho s^2-3A\eta^2\mu\rho s^2\right)}{2\sqrt{\mu}\rho s^2\sqrt{\eta^2\mu+\rho s^2}} \\ \qquad + \dfrac{B\eta^2\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \\ \hat{u} = \dfrac{i\eta e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(A\eta^2\lambda\sqrt{\mu}+2A\eta^2\mu^{3/2}+A\sqrt{\mu}\rho s^2\right)}{2\rho s^2\sqrt{\eta^2\mu+\rho s^2}} - \dfrac{iB\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \\ \hat{v} = \dfrac{e^{-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(A\eta^2\lambda+2A\eta^2\mu+A\rho s^2\right)}{2\rho s^2} + \dfrac{\left(-B\eta^2\lambda-2B\eta^2\mu\right)e^{-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \end{cases} \tag{2.2.9}$$

Similarly for $x < 0$

$$\begin{cases} E_1 \cdot (\hat{m}, \hat{n}, \hat{u}, \hat{v}) = 0 \\ E_2 \cdot (\hat{m}, \hat{n}, \hat{u}, \hat{v}) = Ce^{\frac{\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}x} \\ E_3 \cdot (\hat{m}, \hat{n}, \hat{u}, \hat{v}) = 0 \\ E_4 \cdot (\hat{m}, \hat{n}, \hat{u}, \hat{v}) = De^{\frac{\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}x} \end{cases} \tag{2.2.10}$$

Then we have

$$\begin{cases} \hat{m} = \dfrac{iD\eta\left(\eta^2(\lambda+2\mu)+\rho s^2\right)e^{\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} - \dfrac{iC\eta\left(\eta^2(\lambda+2\mu)+\rho s^2\right)e^{\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\rho s^2} \\ \hat{n} = \dfrac{e^{\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(C\eta^4\lambda\mu+2C\eta^4\mu^2+C\rho^2 s^4+C\eta^2\lambda\rho s^2+3C\eta^2\mu\rho s^2\right)}{2\sqrt{\mu}\rho s^2\sqrt{\eta^2\mu+\rho s^2}} \\ \qquad -\dfrac{D\eta^2\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}e^{\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \\ \hat{u} = \dfrac{i\eta e^{\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(-C\eta^2\lambda\sqrt{\mu}-2C\eta^2\mu^{3/2}-C\sqrt{\mu}\rho s^2\right)}{2\rho s^2\sqrt{\eta^2\mu+\rho s^2}} + \dfrac{iD\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}e^{\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \\ \hat{v} = \dfrac{e^{\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}\left(C\eta^2\lambda+2C\eta^2\mu+C\rho s^2\right)}{2\rho s^2} + \dfrac{\left(-D\eta^2\lambda-2D\eta^2\mu\right)e^{\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2\rho s^2} \end{cases} \tag{2.2.11}$$

From 2.2.4 we note that there is only one dirac-delta function in the system, i.e when we balance the two side of each equation in (2.2.4), we have the continuous condition for $\hat{m}, \hat{u}, \hat{v}$ and jump condition for $\hat{n}$:

$$\begin{cases} \hat{m}(0+) = \hat{m}(0-) \\ \hat{n}(0+) = \hat{n}(0-) - \frac{\rho}{\mu} \\ \hat{u}(0+) = \hat{u}(0-) \\ \hat{v}(0+) = \hat{v}(0-) \end{cases} \tag{2.2.12}$$

Then we can solve the coefficients as

$$\begin{cases} A = \dfrac{\rho\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}(\eta^2(\lambda+2\mu)+\rho s^2)} \\ B = \dfrac{\rho}{\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}} \\ C = \dfrac{\rho\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}(\eta^2(\lambda+2\mu)+\rho s^2)} \\ D = \dfrac{\rho}{\sqrt{\lambda+2\mu}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}} \end{cases} \tag{2.2.13}$$

Then we obtain lemma 2.2.1. □

2.3 Solution in Half Space

Now we turn to the half space problem. We consider the problem

$$\begin{cases} \rho v^r_{tt} = (2\mu+\lambda)v^r_{yy} + \mu v^r_{xx} + (\lambda+\mu)u^r_{xy} \\ \\ \rho u^r_{tt} = (\mu+\lambda)v^r_{xy} + \mu u^r_{yy} + (2\mu+\lambda)u^r_{xx} \end{cases} \tag{2.3.1}$$

with boundary condition

$$\begin{cases} v^r_x + u^r_y = g_1 \\ \\ (2\mu+\lambda)u^r_x + \lambda v^r_y = g_2 \end{cases} \tag{2.3.2}$$

where

$$\begin{cases} g_1 = -(v^f_x + u^f_y) *_{(x,y)} \delta(x-x_0, y-y_0) \mid_{x=0} \\ \\ g_2 = -((2\mu+\lambda)u^f_x + \lambda v^f_y) *_{(x,y)} \delta(x-x_0, y-y_0) \mid_{x=0} \end{cases} \tag{2.3.3}$$

and initial condition

$$\begin{cases} u^r_t(x,y,0) = 0, \quad u^r(x,y,0) = 0 \\ \\ v^r_t(x,y,0) = 0, \quad v^r(x,y,0) = 0 \end{cases} \tag{2.3.4}$$

Then $v^r + v^f$ and $u^r + u^f$ would satisfy the original problem 2.1.2. Then we have

$$
\begin{cases}
g_1 = -\dfrac{\rho e^{-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\mu} \\[2em]
g_2 = \dfrac{i\eta(\lambda+2\mu-1)\sqrt{\eta^2\mu+\rho s^2}e^{-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\sqrt{\mu}s^2} - \dfrac{i\eta\sqrt{\lambda+2\mu}\left(\eta^2(\lambda+2\mu-1)+\rho s^2\right)e^{-\frac{x_0\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}}}{2s^2\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}
\end{cases}
\tag{2.3.5}
$$

Now notice that (2.3.1) is homogeneous equation thus from the computation of fundamental solution we know the solution of (2.3.1) can be expressed as (2.2.9). Then substitute (2.2.9) into the boundary condition (2.3.2) we can solve the unknowns in (2.2.9) and derive the solution of (2.3.1) with boundary condition (2.3.2) and zero initial data (2.3.4).

$$
u^r = u^r_{LL} + u^r_{TT} + u^r_{LT} + u^r_{TL} \tag{2.3.6}
$$

where

$$
\begin{cases}
u^r_{LL} = -\dfrac{N^u_{LL}\exp\left(-\frac{(x_0+x)\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2s^2\mathscr{D}} \\[2em]
u^r_{TT} = -\dfrac{N^u_{TT}e^{-\frac{(x_0+x)\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2s^2\mathscr{D}} \\[2em]
u^r_{LT} = -\dfrac{N^u_{LT}\exp\left(-\frac{x_0\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}} - \frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}\right)}{s^2\mathscr{D}} \\[2em]
u^r_{TL} = -\dfrac{N^u_{TL}\exp\left(-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}} - \frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2\sqrt{\mu}s^2\mathscr{D}}
\end{cases}
\tag{2.3.7}
$$

Where we have the coefficients here

$$
\begin{cases}
N^u_{LL} = & i\eta(\lambda+2\mu)\left(2\eta^2\mu+\rho s^2\right)\left(\eta^2(\lambda+2\mu-1)+\rho s^2\right) \\
N^u_{TT} = & i\eta\left(\rho^2 s^4(-(\lambda+2\mu))-2\eta^2\mu\rho s^2(\lambda+2\mu)\right) \\
& -i\eta\left(2\eta^2\sqrt{\mu}(\lambda+2\mu-1)\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\right) \\
N^u_{LT} = & i\eta^3\mu(\lambda+2\mu)\left(\eta^2(\lambda+2\mu-1)+\rho s^2\right) \\
N^u_{TL} = & i\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\left(\rho s^2(\lambda+4\mu-1)\right) \\
& +i\eta\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\left(2\eta^2\mu(\lambda+2\mu-1)\right) \\
\mathscr{D} = & -4\eta^2\mu^{3/2}\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2} \\
& +\lambda\left(2\eta^2\mu+\rho s^2\right)^2+2\mu\left(2\eta^2\mu+\rho s^2\right)^2
\end{cases}
\tag{2.3.8}
$$

We also have the solution formula of v^r.

$$v^r = v^r_{LL} + v^r_{TT} + v^r_{LT} + v^r_{TL} \tag{2.3.9}$$

where

$$\begin{cases} v^r_{LL} = -\dfrac{N^v_{LL}\exp\left(-\frac{(x_0+x)\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2s^2\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\mathscr{D}} \\ v^r_{TT} = -\dfrac{N^v_{TT}e^{-\frac{(x_0+x)\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}}}{2\sqrt{\mu}s^2\mathscr{D}} \\ v^r_{LT} = -\dfrac{N^v_{LT}\exp\left(-\frac{x_0\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}-\frac{x\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}\right)}{s^2\mathscr{D}} \\ v^r_{TL} = -\dfrac{N^v_{TL}\exp\left(-\frac{x_0\sqrt{\eta^2\mu+\rho s^2}}{\sqrt{\mu}}-\frac{x\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}}{\sqrt{\lambda+2\mu}}\right)}{2\sqrt{\mu}s^2\mathscr{D}} \end{cases} \tag{2.3.10}$$

where we have the coefficients as

$$\begin{cases} N^v_{LL} = & \eta^2(\lambda+2\mu)^{3/2}\left(2\eta^2\mu+\rho s^2\right)\left(\eta^2(\lambda+2\mu-1)+\rho s^2\right) \\ N^v_{TT} = & \sqrt{\eta^2\mu+\rho s^2}\left(\rho^2 s^4(\lambda+2\mu)+2\eta^2\mu\rho s^2(\lambda+2\mu)\right) \\ & +\sqrt{\eta^2\mu+\rho s^2}\left(2\eta^2\sqrt{\mu}(\lambda+2\mu-1)\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2}\right) \\ N^v_{LT} = & \eta^2\sqrt{\mu}(\lambda+2\mu)\sqrt{\eta^2\mu+\rho s^2}\left(\eta^2(\lambda+2\mu-1)+\rho s^2\right) \\ N^v_{TL} = & \eta^2(\lambda+2\mu)\sqrt{\eta^2\mu+\rho s^2}\left(2\eta^2\mu(\lambda+2\mu-1)+\rho s^2(\lambda+4\mu-1)\right) \\ \mathscr{D} = & -4\eta^2\mu^{3/2}\sqrt{\lambda+2\mu}\sqrt{\eta^2\mu+\rho s^2}\sqrt{\eta^2(\lambda+2\mu)+\rho s^2} \\ & +\lambda\left(2\eta^2\mu+\rho s^2\right)^2+2\mu\left(2\eta^2\mu+\rho s^2\right)^2 \end{cases} \tag{2.3.11}$$

Combining lemma (2.2.1), (2.3.7) and (2.3.10) we can prove the theorem 2.1.1.

This expression can separate the vertical and horizontal displacement into four parts respectively and each part can be viewed as a reflection wave with particular incoming impulsion (transverse wave or longitude wave) and outgoing radiation wave (transverse wave or longitude wave). However, when study the property of the solution, these formulas have to be combined together to generate some cancelation. For instance, according to the above formulae, the term s^2 in the denominator represents Newton potential and the influence domain would be infinite if we study $u_L L^r$ independently. On the other hand if we combine these formulas together then there would be some cancelations so that outside some cone the solution would be zero, which fits the classic theory in elastic equation.

2.4 Poisson Solid and Solution Behavior on the Surface

In seismology Poisson solid is a ideal type of elastic material with its Lame constants to be $\mu = \lambda = \rho = 1$. One can simply observe that Poisson solid has Poisson ratio smaller than the critical value. We will use this special and classic model to study the behavior of our solution on the boundary in the following. In fact we have

Theorem 2.4.1. *For initial data is of the form*

$$\delta(x - x_0) Heaviside(1 - |y|)$$

where $x_0 \geq$, the solution behavior on the boundary can be expressed as below

$$|\left((u_b - u_b^s) *_y Heaviside(1 - |y|)\right)| \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.1}$$

where

$$
\begin{aligned}
u_b^s =& \frac{1}{12+8\sqrt{3}} \partial_x \frac{\log\left(c_1^2\left(t-x_0\right)\left(t+x_0\right)-2c_1 ty+x_0^2+y^2\right)}{2c_1\sqrt{1-c_1^2}} \\
&- \frac{1}{12+8\sqrt{3}} \partial_x \frac{\log\left(\left(c_1 t-c_1 x_0+y\right)\left(c_1\left(t+x_0\right)+y\right)+x_0^2\right)}{2c_1\sqrt{1-c_1^2}} \\
&+ \frac{1}{4}\partial_x \frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)-6c_1 ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}} \\
&- \frac{1}{4}\partial_x \frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)+6c_1 ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}
\end{aligned}
\tag{2.4.2}
$$

and $c_1^2 = 2 - \frac{2}{\sqrt{3}}$.

Theorem 2.4.2. *For initial data* $u\left(x,y,0\right)$ *has compact support, denoted by D, in the the half space, the solution behavior on the boundary* u_b *can be expressed as below*

$$
|u_b - \iint\limits_D u_b^s\left(-x_0, y-y_0\right) u\left(x_0,y_0,0\right) dx_0 dy_0| \le O(1)\frac{1}{\sqrt{t}} \tag{2.4.3}
$$

where

$$
\begin{aligned}
u_b^s(x_0,y) =&\frac{1}{12+8\sqrt{3}}\partial_x\frac{\log\left(c_1^2\left(t-x_0\right)\left(t+x_0\right)-2c_1ty+x_0^2+y^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&-\frac{1}{12+8\sqrt{3}}\partial_x\frac{\log\left(\left(c_1t-c_1x_0+y\right)\left(c_1\left(t+x_0\right)+y\right)+x_0^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&+\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)-6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}\\
&-\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)+6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}
\end{aligned}
\tag{2.4.4}
$$

and $c_1^2 = 2 - \frac{2}{\sqrt{3}}$.

Theorem 2.4.2 is a consequence of theorem 2.4.1. In fact when initial data is of compact support on the half space, then for initial data restricted on any fixed x_0 we have the structure in theorem 2.4.1. Now x_0 is uniformly bounded as the initial data has compact support, thus our estimate in theorem 2.4.1 is uniformly with respect to x_0. Thus when integrate with respect to x_0 we can prove theorem 2.4.2.

2.4.1 Initial Impulsion absorption

From previous study we know that the initial impulsion would translate freely until hitting the boundary. In other words, the initial impulsion would be separated into two parts: one will move towards the rear of the half space and would have a 2-D wave behavior and the other part would collide with the boundary and generate reflection waves and surface wave. Thus our first task is to study the time asymptotic behavior of the solution in free space which will completely absorb the initial impulsion. We have derived this part in the previous sections for general Lame constants and when we substitute $\mu = \lambda = \rho = 1$ into the formula we will obtain

$$\begin{cases} u^f = \dfrac{i\eta e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)} - i\eta e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{2s^2} \\ \\ v^f = \frac{1}{2} i\eta \left(\dfrac{i\sqrt{\eta^2+s^2} e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{\eta s^2} - \dfrac{i\eta e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{s^2\sqrt{\eta^2+\frac{s^2}{3}}} \right) \end{cases} \tag{2.4.5}$$

In fact this can be reverse to time-space domain:

$$\begin{cases} \mathscr{F}^{-1}[\mathscr{L}^{-1}[\dfrac{e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{3\sqrt{\eta^2+\frac{s^2}{3}}}]] = \dfrac{1}{\sqrt{3}\sqrt{3t^2-x^2-y^2}} \\ \\ \mathscr{F}^{-1}[\mathscr{L}^{-1}[e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}]] = -sgn[x]\partial_x \dfrac{\sqrt{3}}{\sqrt{3t^2-x^2-y^2}} \\ \\ \mathscr{L}^{-1}[\frac{1}{s^2}] = t \\ \\ \mathscr{F}^{-1}[i\eta] = \partial_y \end{cases} \tag{2.4.6}$$

Thus we have

$$u^f = \begin{cases} \dfrac{xy\left(-\frac{1}{\sqrt{t^2-\frac{x^2}{3}-\frac{y^2}{3}}}-\frac{24}{6\sqrt{t^2-x^2-y^2}+2\sqrt{9t^2-3(x^2+y^2)}}+\frac{3}{\sqrt{t^2-x^2-y^2}}\right)}{6(x^2+y^2)}, & \sqrt{x^2+y^2}\le t; \\ \dfrac{xy\left(5\left(x^2+y^2\right)-18t^2\right)}{(x^2+y^2)^2\sqrt{9t^2-3(x^2+y^2)}}, & t\le\sqrt{x^2+y^2}\le\sqrt{3}t \end{cases}$$

$$v^f = \begin{cases} \dfrac{t^2(x-y)(x+y)}{\sqrt{3}(x^2+y^2)\sqrt{t^2-x^2-y^2}\sqrt{3t^2-x^2-y^2}\left(\sqrt{t^2-x^2-y^2}+\sqrt{t^2-\frac{x^2}{3}-\frac{y^2}{3}}\right)} \\ +\dfrac{y^2}{2(x^2+y^2)\sqrt{t^2-x^2-y^2}}+\dfrac{x^2}{6(x^2+y^2)\sqrt{t^2-\frac{x^2}{3}-\frac{y^2}{3}}}, & \sqrt{x^2+y^2}\le t; \\ \dfrac{t^2\left(y^2-x^2\right)}{2(x^2+y^2)^2\sqrt{t^2-\frac{x^2}{3}-\frac{y^2}{3}}}+\dfrac{x^2}{6(x^2+y^2)\sqrt{t^2-\frac{x^2}{3}-\frac{y^2}{3}}}, & t\le\sqrt{x^2+y^2}\le\sqrt{3}t. \end{cases} \tag{2.4.7}$$

Then it is obvious that u^f and v^f would have uniformly decay rate of $\frac{1}{\sqrt{t}}$ if initial data has compact support. This part would completely absorb the initial impulsion and will act as a source impulsion when collide with the boundary.

2.4.2 Initial Impulsion Restricted on the Boundary

Now we turn to the reflection waves and surface wave. Firstly we assume the initial data is totally on the boundary.

We substitute $\mu=\lambda=\rho=1$ into the formula we obtained previously:

$$
\begin{cases}
u^r_{LL} = \dfrac{3i\eta\left(2\eta^2+s^2\right)^2 e^{\frac{(-x_0-x)\sqrt{3\eta^2+s^2}}{\sqrt{3}}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\[2em]
u^r_{TT} = \dfrac{i\eta\left(3s^4+6\eta^2s^2+4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)e^{(-x_0-x)\sqrt{\eta^2+s^2}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\[2em]
u^r_{LT} = -\dfrac{3i\eta^3\left(2\eta^2+s^2\right)e^{-\frac{x_0\sqrt{3\eta^2+s^2}}{\sqrt{3}}-x\sqrt{\eta^2+s^2}}}{s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\[2em]
u^r_{TL} = -\dfrac{i\sqrt{3}\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\left(4\eta^2+4s^2\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)-\frac{x\sqrt{3\eta^2+s^2}}{\sqrt{3}}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)}
\end{cases}
\tag{2.4.8}
$$

Now we assume a initial transverse load on the free surface i.e

$$
\begin{cases}
u_t(x,y,0) = 0, & u(x,y,0) = 0 \\[1em]
v_t(x,y,0) = \delta(x, y-y_0), & v(x,y,0) = 0
\end{cases}
\tag{2.4.9}
$$

Then our solution formula of u can be expressed by superposition of (2.4.7) and (2.4.10)with $x_0 = 0$ as our initial data is restricted on the boundary. u^f has a exact expression so we only need to deal with u^r.

$$\begin{cases} u_{LL}^r = \dfrac{3i\eta\left(2\eta^2+s^2\right)^2 e^{\frac{(-x)\sqrt{3\eta^2+s^2}}{\sqrt{3}}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\ \\ u_{TT}^r = \dfrac{i\eta\left(3s^4+6\eta^2s^2+4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)e^{(-x)\sqrt{\eta^2+s^2}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\ \\ u_{LT}^r = -\dfrac{3i\eta^3\left(2\eta^2+s^2\right)e^{-x\sqrt{\eta^2+s^2}}}{s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\ \\ u_{TL}^r = -\dfrac{i\sqrt{3}\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\left(4\eta^2+4s^2\right)e^{-\frac{x\sqrt{3\eta^2+s^2}}{\sqrt{3}}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \end{cases} \tag{2.4.10}$$

Then we can recombine u^r as

$$\begin{cases} u_L^r = \dfrac{i\eta\left(12\eta^4+3s^4+12\eta^2s^2-4\sqrt{3}s^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)e^{-\frac{x\sqrt{3\eta^2+s^2}}{\sqrt{3}}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \\ \\ u_T^r = \dfrac{i\eta\left(-12\eta^4+3s^4+4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)e^{-x\sqrt{\eta^2+s^2}}}{2s^2\left(3(2\eta^2+s^2)^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \end{cases} \tag{2.4.11}$$

Then combine u^f we have that

$$\begin{cases} u_L^r + u_L^f = -\dfrac{2i\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{4\sqrt{3}\eta^4+\sqrt{3}s^4+4\sqrt{3}\eta^2s^2-4\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}} \\ \\ u_T^r + u_T^f = \dfrac{i\sqrt{3}\eta\left(2\eta^2+s^2\right)e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{4\sqrt{3}\eta^4+\sqrt{3}s^4+4\sqrt{3}\eta^2s^2-4\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}} \end{cases} \tag{2.4.12}$$

Then rationalize above formula and do the partial fraction we have:

$$
\begin{cases}
u_L^r + u_L^f = & -\frac{i\eta\left(\sqrt{\eta^2+s^2}-\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{4\sqrt{\eta^2+\frac{s^2}{3}}(4\eta^2+s^2)} \\
& -\frac{3i\eta\sqrt{\eta^2+s^2}e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{4s^2\sqrt{\eta^2+\frac{s^2}{3}}} - \frac{i\sqrt{3}\eta\sqrt{3\eta^2+s^2}e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{4s^2\sqrt{\eta^2+\frac{s^2}{3}}} \\
& -\frac{i\eta\left(4\sqrt{3}\sqrt{\eta^2+s^2}-7\sqrt{\eta^2+s^2}-\sqrt{3}\sqrt{3\eta^2+s^2}+2\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{2\left(\sqrt{3}-3\right)\left(\left(\sqrt{3}-3\right)s^2-4\eta^2\right)\sqrt{\eta^2+\frac{s^2}{3}}} \\
& +\frac{i\eta\left(5\sqrt{3}\sqrt{\eta^2+s^2}+9\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}+3\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{12\sqrt{\eta^2+\frac{s^2}{3}}\left(4\eta^2+\left(3+\sqrt{3}\right)s^2\right)} \\
u_T^r + u_T^f = & \frac{3i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}-2\sqrt{\eta^2+s^2}+\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{2\left(\sqrt{3}-3\right)\left(\left(\sqrt{3}-3\right)s^2-4\eta^2\right)\sqrt{\eta^2+s^2}} \\
& +\frac{3i\eta\left(\sqrt{\eta^2+s^2}-\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{4\sqrt{\eta^2+s^2}(4\eta^2+s^2)} \\
& -\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}+3\sqrt{\eta^2+s^2}-\sqrt{3}\sqrt{3\eta^2+s^2}+3\sqrt{3\eta^2+s^2}\right)e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{4\sqrt{\eta^2+s^2}\left(4\eta^2+\sqrt{3}s^2+3s^2\right)} \\
& +\frac{i\sqrt{3}\eta\sqrt{3\eta^2+s^2}e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{4s^2\sqrt{\eta^2+s^2}} + \frac{3i\eta e^{|x|\left(-\sqrt{\eta^2+s^2}\right)}}{4s^2}
\end{cases}
\tag{2.4.13}
$$

Now as we want to estimate the solution on the boundary, we let $x = 0$ and then the formula for u_b in transform variable domain would be simplified as

$$\hat{u}_b = \frac{3i\eta}{8\eta^2+2s^2} - \frac{2i\sqrt{3}\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}}{s^4+4\eta^2 s^2}$$

$$- \frac{i\sqrt{3}\left(1+\sqrt{3}\right)\eta}{4\left(2\sqrt{3}\eta^2+6\eta^2+3s^2\right)} + \frac{i\left(3+2\sqrt{3}\right)\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}}{3s^4+2\left(3+\sqrt{3}\right)\eta^2 s^2}$$

$$+ \frac{i\sqrt{3}\left(\sqrt{3}-1\right)\eta}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)} + \frac{i\left(2\sqrt{3}-3\right)\eta\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}}{3s^4-2\left(\sqrt{3}-3\right)\eta^2 s^2}$$

$$= \frac{\frac{i\sqrt{3}\left(\sqrt{3}-1\right)\eta\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)} + \frac{i\left(7-4\sqrt{3}\right)\eta\sqrt{3\eta^2+s^2}}{8\left(\sqrt{3}-2\right)\eta^2+2\left(\sqrt{3}-3\right)s^2}}{\sqrt{\eta^2+s^2}}$$

$$+ \frac{\frac{3i\eta\sqrt{\eta^2+s^2}}{8\eta^2+2s^2} - \frac{3i\sqrt{3}\eta\sqrt{3\eta^2+s^2}}{8\eta^2+2s^2}}{\sqrt{\eta^2+s^2}}$$

$$+ \frac{\frac{i\left(7+4\sqrt{3}\right)\eta\sqrt{3\eta^2+s^2}}{8\left(2+\sqrt{3}\right)\eta^2+2\left(3+\sqrt{3}\right)s^2} - \frac{i\sqrt{3}\left(1+\sqrt{3}\right)\eta\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2+6\eta^2+3s^2\right)}}{\sqrt{\eta^2+s^2}} \tag{2.4.14}$$

Then we need to reverse this formula into time-space domain. First we need to know some inverse transform of basic functions in our formula.

Lemma 2.4.3.

$$\begin{aligned} \mathscr{F}^{-1}[Bessel[0, c_T\eta t]] &= \frac{H(c_T t - |y|)}{\sqrt{c_T^2 t^2 - y^2}} \\ \mathscr{F}^{-1}[\hat{f}\hat{g}] &= f *_y g \\ \mathscr{F}^{-1}[\frac{Sin[c_1\eta t]}{c_1\eta}] &= \frac{H(c_1 t - |y|)}{2c_1} \end{aligned} \tag{2.4.15}$$

Proof. □

• Analysis on c_1 term

Now we first consider the terms

$$\frac{\frac{i\sqrt{3}\left(\sqrt{3}-1\right)\eta\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}+\frac{i\left(7-4\sqrt{3}\right)\eta\sqrt{3\eta^2+s^2}}{8\left(\sqrt{3}-2\right)\eta^2+2\left(\sqrt{3}-3\right)s^2}}{\sqrt{\eta^2+s^2}} \tag{2.4.16}$$

The first part would be simplified as a 1-D wave which propagate along the surface:

$$\begin{aligned}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{i\sqrt{3}\left(\sqrt{3}-1\right)\eta\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}}{\sqrt{\eta^2+s^2}}]] &= \mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\sqrt{3}\left(\sqrt{3}-1\right)\eta}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}]] \\ &= -\frac{\sqrt{3}\left(\sqrt{3}-1\right)}{12}\frac{\delta(y+c_1t)-\delta(y-c_1t)}{2c_1}\end{aligned} \tag{2.4.17}$$

where $c_1^2=\frac{6-2\sqrt{3}}{3}$.

However, when consider the other part in (2.4.16) we need detail of the convolution of 1-D wave and 2-D wave.

Lemma 2.4.4.

$$\begin{aligned}&\mathscr{F}^{-1}[\eta^2\frac{Sin[c_1\eta t]}{c_1\eta} *_t Bessel[0,\eta t]] \\ =&-\partial_y\left(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}}\right) \\ &-\partial_y\left(\frac{-log(t-c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})+log(t+c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})}{2c_1\sqrt{(1-c_1^2)}}\right).\end{aligned} \tag{2.4.18}$$

Proof. 1. When $y>t$ we will observe that in 2.4.42the forward cone and backward cone will not intersect hence the convolution would be 0.

2. When $t>y>c_1t$ the forward cone and backward cone has intersection and

so the integral can be rewritten as:

$$
\begin{aligned}
&\mathscr{F}^{-1}[\eta^2 \frac{Sin[c_1\eta t]}{c_1\eta} *_t Bessel[0,\eta t]] \\
&= -\partial_y^2 \int_0^t \int_{-\infty}^{+\infty} \frac{H(s-|\bar{y}|)}{\sqrt{s^2-\bar{y}^2}} \frac{H(y-\bar{y}+c_i(t-s)) - H(y-\bar{y}) - c_i(t-s)}{2c_i} d\bar{y}ds \\
&= -\partial_y^2 (\int_{\frac{y-c_1t}{1-c_1}}^{\frac{y+c_1t}{1+c_1}} \int_{y-c_1(t-s)}^{s} \frac{1}{2c_1\sqrt{s^2-\bar{y}^2}} d\bar{y}ds + \int_{\frac{y+c_1t}{1+c_1}}^{t} \int_{y-c_1(t-s)}^{y+c_1(t-s)} \frac{1}{2c_1\sqrt{s^2-\bar{y}^2}} d\bar{y}ds) \\
&= -\partial_y^2 (\int_{\frac{y-c_1t}{1-c_1}}^{\frac{y+c_1t}{1+c_1}} \frac{ArcCos[\frac{c_1s-c_1t+y}{s}]}{2c_1} ds + \int_{\frac{y+c_1t}{1+c_1}}^{t} \frac{-ArcSin[\frac{c_1s-c_1t+y}{s}] + ArcSin[\frac{-c_1s+c_1t+y}{s}]}{2c_1} ds) \\
&= -\partial_y (\frac{log(y-c_1t) - log(y+c_1t)}{2c_1\sqrt{(1-c_1^2)}}) \\
&\quad -\partial_y (\frac{-log(t-c_1y+\sqrt{(c_1^2-1)(y^2-t^2)}) + log(t+c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})}{2c_1\sqrt{(1-c_1^2)}})
\end{aligned}
\tag{2.4.19}
$$

3. When $c_1 t > y > 0$ we have:

$$
\begin{aligned}
&\mathscr{F}^{-1}[\eta^2 \frac{Sin[c_1\eta t]}{c_1\eta} *_t Bessel[0,\eta t]] \\
&= -\partial_y^2 \int_0^t \int_{-\infty}^{+\infty} \frac{H(s-|\bar{y}|)}{\sqrt{s^2-\bar{y}^2}} \frac{H(y-\bar{y}+c_i(t-s)) - H(y-\bar{y}) - c_i(t-s)}{2c_i} d\bar{y}ds \\
&= -\partial_y^2 (\int_0^{\frac{-y+c_1t}{1+c_1}} \int_{-s}^{s} \frac{1}{2c_1\sqrt{s^2-\bar{y}^2}} d\bar{y}ds + \int_{\frac{-y+c_1t}{1+c_1}}^{\frac{y+c_1t}{1+c_1}} \int_{y-c_1(t-s)}^{s} \frac{1}{2c_1\sqrt{s^2-\bar{y}^2}} d\bar{y}ds \\
&\quad + \int_{\frac{y+c_1t}{1+c_1}}^{t} \int_{y-c_1(t-s)}^{y+c_1(t-s)} \frac{1}{2c_1\sqrt{s^2-\bar{y}^2}} d\bar{y}ds) \\
&= -\partial_y^2 (\int_0^{\frac{-y+c_1t}{1+c_1}} \frac{\pi}{2c_1} ds + \int_{\frac{-y+c_1t}{1+c_1}}^{\frac{y+c_1t}{1+c_1}} \frac{ArcCos[\frac{c_1s-c_1t+y}{s}]}{2c_1} ds \\
&\quad + \int_{\frac{y+c_1t}{1+c_1}}^{t} \frac{-ArcSin[\frac{c_1s-c_1t+y}{s}] + ArcSin[\frac{-c_1s+c_1t+y}{s}]}{2c_1} ds) \\
&= -\partial_y (\frac{log(-y+c_1t) - log(y+c_1t)}{2c_1\sqrt{(1-c_1^2)}}) \\
&\quad -\partial_y (\frac{-log(t-c_1y+\sqrt{(c_1^2-1)(y^2-t^2)}) + log(t+c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})}{2c_1\sqrt{(1-c_1^2)}})
\end{aligned}
\tag{2.4.20}
$$

Now considering that $\mathscr{F}^{-1}[\frac{Sin[c_1\eta t]}{c_1\eta} *_t Bessel[0,\eta t]]$ is an even function with respect to x, we have $\partial_y \mathscr{F}^{-1}[\frac{Sin[c_1\eta t]}{c_1\eta} *_t Bessel[0,\eta t]]$ is an odd function and hence we can write out the formula for $y < 0$. thus we finish the proof

□

Lemma 2.4.5.

$$\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\left(\frac{(c_T^2 - c_1^2)\,\eta^2}{(c_1^2\eta^2 + s^2)\,\sqrt{\eta^2\mu + \rho s^2}} + \frac{1}{\sqrt{\eta^2\mu + \rho s^2}}\right)]] \\
&= -\,\partial_y(1 - c_1^2)(\frac{log(|-y + c_1 t|) - log(|y + c_1 t|)}{2c_1\sqrt{(1 - c_1^2)}}) \\
&\quad + \frac{t}{y^2 - c_1^2 t^2}\sqrt{1 - c_1^2} Heaviside[|y| > t] + \frac{1}{\sqrt{t^2 - y^2} + t\sqrt{1 - c_1^2}}
\end{aligned} \tag{2.4.21}$$

Proof. According to 2.4.4 we have

$$
\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\left(\frac{\left(c_T^2-c_1^2\right)\eta^2}{\left(c_1^2\eta^2+s^2\right)\sqrt{\eta^2\mu+\rho s^2}}+\frac{1}{\sqrt{\eta^2\mu+\rho s^2}}\right)]]\\
=&-\partial_y(1-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}})\\
&-\partial_y(1-c_1^2)(\frac{-log(t-c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})+log(t+c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})}{2c_1\sqrt{(1-c_1^2)}})\\
&+\frac{1}{\sqrt{t^2-y^2}}\\
=&-\partial_y(1-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}})\\
&+\partial_y(1-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}})Heaviside[|y|>t]\\
&-\partial_y(1-c_1^2)(\frac{-log(t-c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})+log(t+c_1y+\sqrt{(c_1^2-1)(y^2-t^2)})}{2c_1\sqrt{(1-c_1^2)}})\\
&+\frac{1}{\sqrt{t^2-y^2}}\\
=&-\partial_y(1-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}})\\
&+\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[|y|>t]+\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}
\end{aligned}
\tag{2.4.22}
$$

□

Now we can apply 2.4.5 to our formula 2.4.16 and we have the corollary:

Corollary 2.4.6.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\left(7-4\sqrt{3}\right)\sqrt{3\eta^2+s^2}}{8\left(\sqrt{3}-2\right)\eta^2+2\left(\sqrt{3}-3\right)s^2}]]$$

$$\begin{aligned}
=&-\frac{1}{12}\left(-9+5\sqrt{3}\right)\partial_y(3-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(3-c_1^2)}})\\
&+\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{t}{y^2-c_1^2t^2}\sqrt{3-c_1^2}Heaviside[|y|>\sqrt{3}t]\\
&+\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}
\end{aligned} \tag{2.4.23}$$

Corollary 2.4.7.

$$\mathscr{F}^{-1}[\frac{\sqrt{3}\left(\sqrt{3}-1\right)\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}]]$$

$$\begin{aligned}
=&-\frac{1}{12}\left(\sqrt{3}-3\right)\partial_y(1-c_1^2)(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}})\\
&+\frac{1}{12}\left(\sqrt{3}-3\right)\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[|y|>t]\\
&+\frac{1}{12}\left(\sqrt{3}-3\right)\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}
\end{aligned} \tag{2.4.24}$$

Lemma 2.4.8.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\left(7-4\sqrt{3}\right)\sqrt{3\eta^2+s^2}}{8\left(\sqrt{3}-2\right)\eta^2+2\left(\sqrt{3}-3\right)s^2}]]$$

$$\begin{aligned}
=&\mathscr{F}^{-1}[\frac{\sqrt{3}\left(\sqrt{3}-1\right)\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}]] \\
&-\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{t}{y^2-c_1^2t^2}\sqrt{3-c_1^2}Heaviside[t<|y|<\sqrt{3}t] \\
&+\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}} \\
&-\frac{1}{12}\left(\sqrt{3}-3\right)\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}
\end{aligned} \tag{2.4.25}$$

Proof. Combine corollary 2.4.6 and corollary 2.4.7 one can find the coefficients for the term

$$-\frac{1}{12}\left(\sqrt{3}-3\right)\partial_y(1-c_1^2)\left(\frac{log(|-y+c_1t|)-log(|y+c_1t|)}{2c_1\sqrt{(1-c_1^2)}}\right)$$

are the same. Then we can draw the conclusion. □

So now our formula for c_1 can be written in the form

$$\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{i\sqrt{3}(\sqrt{3}-1)\eta\sqrt{\eta^2+s^2}}{4(2\sqrt{3}\eta^2-6\eta^2-3s^2)}+\frac{i(7-4\sqrt{3})\eta\sqrt{3\eta^2+s^2}}{8(\sqrt{3}-2)\eta^2+2(\sqrt{3}-3)s^2}}{\sqrt{\eta^2+s^2}}]] \\
=&2\mathscr{F}^{-1}[\frac{i\eta\sqrt{3}\left(\sqrt{3}-1\right)}{4\left(2\sqrt{3}\eta^2-6\eta^2-3s^2\right)}]]+\frac{1}{\sqrt{t^2-y^2}}*_{y,t}N(y,t) \\
=&-2\frac{\sqrt{3}\left(\sqrt{3}-1\right)}{12}\frac{\delta(y+c_1t)-\delta(y-c_1t)}{2c_1}+\partial_y\frac{1}{\sqrt{t^2-y^2}}*_{y,t}N(y,t)
\end{aligned} \tag{2.4.26}$$

where

$$
\begin{aligned}
N(y,t) = & -\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{t}{y^2-c_1^2t^2}\sqrt{3-c_1^2}Heaviside[t<|y|<\sqrt{3}t] \\
& +\frac{1}{12}\left(-9+5\sqrt{3}\right)\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}} \\
& -\frac{1}{12}\left(\sqrt{3}-3\right)\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}
\end{aligned}
\tag{2.4.27}
$$

Now to investigate the time asymptotic property of this formula we need to make some assumptions on the initial data. In fact we assume the initial data has a compact support on the surface:

$$
u(x,y,0) = \delta(x)Heaviside(1-|y|). \tag{2.4.28}
$$

Lemma 2.4.9. *For any fixed y,*

$$
\partial_y \frac{1}{\sqrt{t^2-y^2}} *_{(t,y)} \left(\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}\right) *_y H(1-|y|) \tag{2.4.29}
$$

would has a decay rate of $\frac{\log(t)}{t}$.

Proof. We need to compute the integration

$$
\partial_y \int_0^t ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \tag{2.4.30}
$$

As y is fixed, thus for large t one can find a small number α such that

$$
\begin{aligned}
& \int_0^{\alpha t} ds \int d\eta \partial_y \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
& \leq \int_0^{\alpha t} ds \int d\eta \frac{O(1)}{t} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
& \leq \frac{O(1)}{t}\left(\int_0^1 ds \int d\eta \frac{1}{\sqrt{3s^2-\eta^2}} + \int_1^{\alpha t} ds \int d\eta \frac{1}{\left(\sqrt{3-c_1^2}s\right)}\right) \\
& \leq \frac{O(1)\log(t)}{t}
\end{aligned}
\tag{2.4.31}
$$

$$
\begin{aligned}
&\int_{\alpha t}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \partial_\eta \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&= \int_{\alpha t}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{\eta}{\sqrt{3s^2-\eta^2}\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)^2} \\
&\leq \frac{O(1)}{t} \int_{\alpha t}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{1}{\sqrt{3s^2-\eta^2}} \\
&\leq \frac{O(1)}{t} \int_{0}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{1}{\sqrt{3s^2-\eta^2}} \\
&\leq \frac{O(1)}{t}
\end{aligned} \tag{2.4.32}
$$

Thus we have for fixed y,

$$
\partial_y \int_0^t ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(y-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \leq \frac{O(1)\log(t)}{t} \tag{2.4.33}
$$

□

However this decay rate is not uniformly in y. This is because of that the choice of α in the previous proof is not uniform, α would change when y changes. To estimate the uniformly decay rate of the convolution 2.4.29 we need to consider the behavior along the cone, i.e $y = kt$. For the case $k \neq 1$ one can use the similar method to prove 2.4.33. While when $k = 1$, the situation will change.

Lemma 2.4.10. *If* $\frac{y-b}{t} = 1$ *for some constant b, then* $\partial_y \frac{1}{\sqrt{t^2-y^2}} *_{(t,y)} \left(\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}\right) *_y H(1-|y|)$ *has decay rate of* $\frac{1}{\sqrt{t}}$

Proof. For simplicity, we let $b = 0$. In fact, as the initial data is of compact support $[-1, 1]$, we need to compute the integration

$$
\begin{aligned}
&\partial_y \int_{-1}^{1} d\alpha \int_0^t ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
=&\partial_y \int_{-1}^{1} d\alpha \int_0^{10} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&+\partial_y \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&+\partial_y \int_{-1}^{1} d\alpha \int_{\sqrt{t}}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)}
\end{aligned} \tag{2.4.34}
$$

where $|\alpha| \leq 1$.

For the first part, as $s < 10$, one can rewrite it as

$$
\partial_y \int_{-1}^{1} d\alpha \int_0^{10} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-(\eta-\alpha)^2}\right)} \tag{2.4.35}
$$

For each fixed s, the integration can be viewed as the 2-D wave convolve with a compact support initial data at time s, where we put the differential operator to the initial data. According to the classic theory of 2-D wave equation we have

$$
\partial_y \int_{-1}^{1} d\alpha \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-(\eta-\alpha)^2}\right)} \leq \frac{O(1)}{\sqrt{t-s}}
$$

Thus

$$
\partial_y \int_{-1}^{1} d\alpha \int_0^{10} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-(\eta-\alpha)^2}\right)} \leq \frac{O(1)}{\sqrt{t}}
$$

For the third part we have

$$
\begin{aligned}
&\partial_y \int_{-1}^{1} d\alpha \int_{\sqrt{t}}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&= \int_{-1}^{1} d\alpha \int_{\sqrt{t}}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \partial_\eta \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&= \int_{-1}^{1} d\alpha \int_{\sqrt{t}}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{\eta}{\sqrt{3s^2-\eta^2}\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)^2} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{\sqrt{t}}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\sqrt{3s^2-\eta^2}} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{0}^{t} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\sqrt{3s^2-\eta^2}} \\
&\leq \frac{O(1)}{\sqrt{t}}
\end{aligned}
\tag{2.4.36}
$$

Finally for the second part we have

$$
\begin{aligned}
&\partial_y \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&= \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \partial_\eta \frac{1}{\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)} \\
&= \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int d\eta \frac{1}{\sqrt{(t-s)^2-(t-\alpha-\eta)^2}} \frac{\eta}{\sqrt{3s^2-\eta^2}\left(\sqrt{3-c_1^2}s+\sqrt{3s^2-\eta^2}\right)^2} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int d\eta \frac{1}{\sqrt{\eta+\alpha-s}} \frac{1}{\sqrt{\sqrt{3}s-\eta}} \frac{1}{s\sqrt{s}} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int_{s-\alpha}^{\sqrt{3}s} d\eta \left(\frac{1}{\sqrt{\eta+\alpha-s}} + \frac{1}{\sqrt{\sqrt{3}s-\eta}}\right) \frac{1}{\sqrt{\eta+\alpha-s}+\sqrt{\sqrt{3}s-\eta}} \frac{1}{s\sqrt{s}} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \int_{s-\alpha}^{\sqrt{3}s} d\eta \left(\frac{1}{\sqrt{\eta+\alpha-s}} + \frac{1}{\sqrt{\sqrt{3}s-\eta}}\right) \frac{1}{\sqrt{s}} \frac{1}{s\sqrt{s}} \\
&\leq \frac{O(1)}{\sqrt{t}} \int_{-1}^{1} d\alpha \int_{10}^{\sqrt{t}} ds \frac{1}{s\sqrt{s}} \\
&\leq \frac{O(1)}{\sqrt{t}}
\end{aligned}
\tag{2.4.37}
$$

□

Lemma 2.4.11.

$$\begin{aligned}
&| \mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{i\sqrt{3}(\sqrt{3}-1)\eta\sqrt{\eta^2+s^2}}{4(2\sqrt{3}\eta^2-6\eta^2-3s^2)}+\frac{i(7-4\sqrt{3})\eta\sqrt{3\eta^2+s^2}}{8(\sqrt{3}-2)\eta^2+2(\sqrt{3}-3)s^2}}{\sqrt{\eta^2+s^2}}]] *_y Heaviside(1-|y|) \\
&\quad -(-2\frac{\sqrt{3}\left(\sqrt{3}-1\right)}{12}\frac{\delta(y+c_1t)-\delta(y-c_1t)}{2c_1}) *_y Heaviside(1-|y|) \,| \\
&\quad \leq O(1)\frac{1}{\sqrt{t}}
\end{aligned} \tag{2.4.38}$$

Proof. (2.4.38) is a direct consequence of lemma 2.4.8 and lemma 2.4.10. □

•Analysis on c_2 and c_3 terms

We will consider

$$\frac{\frac{3i\eta\sqrt{\eta^2+s^2}}{8\eta^2+2s^2}-\frac{3i\sqrt{3}\eta\sqrt{3\eta^2+s^2}}{8\eta^2+2s^2}}{\sqrt{\eta^2+s^2}} \tag{2.4.39}$$

of (2.4.14) and the other term would have similar property.

In (2.4.39) the first term represents a 1-D wave with speed $c_2 = 2$ which would be greater than the speed of both transverse and longitude waves. According to seismology research this case cannot happen so we need to show the cancelation of such 1-D wave in our formula.

Lemma 2.4.12. *For c_2 and c_3 we have*

$$\mathscr{F}^{-1}[\mathbb{i}\eta\frac{Sin[c_i\eta t]}{c_i\eta} *_t Bessel[0,\eta t]]$$
$$= \begin{cases} 0, & y > c_i t; \\ -\frac{sgn[y]\pi}{2c_i\sqrt{(c_i^2-1)}}, & c_i t > |y| > t; \\ -\frac{sgn[y]}{2c_i\sqrt{(c_i^2-1)}}(-\tan^{-1}\left(\frac{\sqrt{(c_i^2-1)(t-y)(t+y)}}{yc_i-t}\right) - \tan^{-1}\left(\frac{\sqrt{(c_i^2-1)(t-y)(t+y)}}{yc_i+t}\right)+\pi), & t > |y| > \frac{t}{c_i}. \\ -\frac{sgn[y]}{2c_i\sqrt{(c_i^2-1)}}(\tan^{-1}\left(\frac{\sqrt{(c_i^2-1)(t-y)(t+y)}}{t-yc_i}\right) - \tan^{-1}\left(\frac{\sqrt{(c_i^2-1)(t-y)(t+y)}}{yc_i+t}\right)), & \frac{t}{c_i} > |y|. \end{cases} \tag{2.4.40}$$

$$\mathscr{F}^{-1}[\mathbb{i}\eta\frac{Sin[c_i\eta t]}{c_i\eta} *_t Bessel[0,c_L\eta t]]$$
$$= \begin{cases} 0, & y > c_i t; \\ -\frac{sgn[y]\pi}{2c_i\sqrt{(c_i^2-c_L^2)}}, & c_i t > |y| > c_L t; \\ -\frac{sgn[y]}{2c_i\sqrt{(c_i^2-c_L^2)}}(-\tan^{-1}\left(\frac{\sqrt{(c_i^2-c_L^2)(c_L^2t^2-y^2)}}{yc_i-c_L^2t}\right) - \tan^{-1}\left(\frac{\sqrt{(c_i^2-c_L^2)(c_L^2t^2-y^2)}}{yc_i+c_L^2t}\right)+\pi), & c_L t > |y| > \frac{t}{c_i}. \\ -\frac{sgn[y]}{2c_i\sqrt{(c_i^2-c_L^2)}}(\tan^{-1}\left(\frac{\sqrt{(c_i^2-c_L^2)(c_L^2t^2-y^2)}}{c_L^2t-yc_i}\right) - \tan^{-1}\left(\frac{\sqrt{(c_i^2-c_L^2)(c_L^2t^2-y^2)}}{yc_i+c_L^2t}\right)), & \frac{t}{c_i} > |y|. \end{cases} \tag{2.4.41}$$

Proof.

$$\begin{aligned} &\mathscr{F}^{-1}[\eta^2\frac{Sin[c_i\eta t]}{c_i\eta} *_t Bessel[0,\eta t]] \\ =& -\partial_y^2\int_0^t\int_{-\infty}^{+\infty}\frac{H(s-|\bar{y}|)}{\sqrt{s^2-\bar{y}^2}}\frac{H(y-\bar{y}+c_i(t-s))-H(y-\bar{y})-c_i(t-s)}{2c_i}d\bar{y}ds \\ =& -\partial_y\int_0^t\int_{-\infty}^{+\infty}\frac{H(\bar{y}+s)-H(\bar{y}-s)}{\sqrt{s^2-\bar{y}^2}}\frac{\delta(y-\bar{y}+c_i(t-s))-\delta(y-\bar{y})-c_i(t-s)}{2c_i}d\bar{y}ds \\ =& -\partial_y\int_0^t\frac{H(y+c_i(t-s)+s)-H(y+c_i(t-s)-s)}{2c_i\sqrt{s^2-(y+c_i(t-s))^2}}ds \\ &+\partial_y\int_0^t\frac{H(y-c_i(t-s)+s)-H(y-c_i(t-s)-s)}{2c_i\sqrt{s^2-(y-c_i(t-s))^2}}ds \end{aligned} \tag{2.4.42}$$

1. When $y > c_i t$ the integral function is equal to 0 hence the integral is 0.

2. When $c_it > y > t$ we have

$$\begin{aligned}
&\Bigg(-\int_0^t \frac{H(y+c_i(t-s)+s)-H(y+c_i(t-s)-s)}{2c_i\sqrt{s^2-(y+c_i(t-s))^2}}ds \\
&\quad+\int_0^t \frac{H(y-c_i(t-s)+s)-H(y-c_i(t-s)-s)}{2c_i\sqrt{s^2-(y-c_i(t-s))^2}}ds\Bigg) \\
&=-\int_0^t -\frac{H(y-c_i(t-s)+s)-H(y-c_i(t-s)-s)}{2c_i\sqrt{s^2-(y-c_i(t-s))^2}}ds \\
&=\int_{\frac{c_it-y}{c_i+1}}^{\frac{c_it-y}{c_i-1}} \frac{1}{2c_i\sqrt{((1+c_i)s+(y-c_it))((1-c_i)s-(y-c_it))}}ds \\
&=\frac{\pi}{2c_i\sqrt{(c_i^2-1)}}
\end{aligned} \tag{2.4.43}$$

3. When $t > y > 0$ we have

$$\begin{aligned}
&\Bigg(-\int_0^t \frac{H(y+c_i(t-s)+s)-H(y+c_i(t-s)-s)}{2c_i\sqrt{s^2-(y+c_i(t-s))^2}}ds \\
&\quad+\int_0^t \frac{H(y-c_i(t-s)+s)-H(y-c_i(t-s)-s)}{2c_i\sqrt{s^2-(y-c_i(t-s))^2}}ds\Bigg) \\
&=(\int_{\frac{c_it-y}{c_i+1}}^{t} \frac{1}{2c_i\sqrt{((1+c_i)s+(y-c_it))((1-c_i)s-(y-c_it))}}ds \\
&\quad-\int_{\frac{c_it+y}{c_i+1}}^{t} \frac{1}{2c_i\sqrt{((1-c_i)s+(y+c_it))((1+c_i)s-(y+c_it))}}ds) \\
&=\begin{cases}
\dfrac{-\tan^{-1}\left(\frac{\sqrt{\left(c_i^2-1\right)(t-y)(t+y)}}{yc_i-t}\right)-\tan^{-1}\left(\frac{\sqrt{\left(c_i^2-1\right)(t-y)(t+y)}}{yc_i+t}\right)+\pi}{2c_i\sqrt{c_i^2-1}}, & y > \frac{t}{c_i} \\[2ex]
\dfrac{\tan^{-1}\left(\frac{\sqrt{\left(c_i^2-1\right)(t-y)(t+y)}}{t-yc_i}\right)-\tan^{-1}\left(\frac{\sqrt{\left(c_i^2-1\right)(t-y)(t+y)}}{yc_i+t}\right)}{2c_i\sqrt{c_i^2-1}}, & y < \frac{t}{c_i}
\end{cases}
\end{aligned} \tag{2.4.44}$$

□

Now considering that $\mathscr{F}^{-1}[\frac{Sin[c_i\eta t]}{c_i\eta} *_t Bessel[0,\eta t]]$ is an even function with respect to y, we have $\partial_y \mathscr{F}^{-1}[\frac{Sin[c_i \eta t]}{c_i \eta} *_t Bessel[0, \eta t]]$ is an odd function and hence we can write out the formula for $y < 0$. Thus we have:

Lemma 2.4.13. *For c_2 and c_3 we can derive:*

$$\begin{aligned}&\mathscr{F}^{-1}[\eta^2 \frac{Sin[c_i \eta t]}{c_i \eta} *_t Bessel[0, \eta t]]\\=&\frac{t^2}{\left(t^2 c_i^2 - y^2\right)\sqrt{t^2 - y^2}} + \frac{\pi}{2c_i \sqrt{(c_i^2 - 1)}}(\delta(y - c_i t) + \delta(y + c_i t)).\end{aligned} \tag{2.4.45}$$

Proof. □

Similarly we can deduce the results for general positive real number c_L which is smaller than c_2 and c_3.

Lemma 2.4.14. *For c_2 and c_3 we can derive:*

$$\begin{aligned}&\mathscr{F}^{-1}[\eta^2 \frac{Sin[c_i \eta t]}{c_i \eta} *_t Bessel[0, c_L \eta t]]\\=&\frac{t^2}{\left(t^2 c_i^2 - y^2\right)\sqrt{c_L^2 t^2 - y^2}} + \frac{\pi}{2c_i \sqrt{(c_i^2 - c_L^2)}}(\delta(y - c_i t) + \delta(y + c_i t)).\end{aligned} \tag{2.4.46}$$

Proof. □

Corollary 2.4.15.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{3\sqrt{\eta^2 + s^2}}{8\eta^2 + 2s^2}]] = \frac{3\sqrt{t^2 - y^2}}{2\left(4t^2 - y^2\right)} - \frac{9\pi}{8\sqrt{3}}(\delta(y - 2t) + \delta(y + 2t)) \tag{2.4.47}$$

Corollary 2.4.16.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{3\sqrt{3}\sqrt{3\eta^2 + s^2}}{8\eta^2 + 2s^2}]] = \frac{3\sqrt{3}\sqrt{3t^2 - y^2}}{2\left(4t^2 - y^2\right)} - \frac{3\sqrt{3}\pi}{8}(\delta(y - 2t) + \delta(y + 2t)) \tag{2.4.48}$$

Lemma 2.4.17.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{3i\eta\sqrt{\eta^2+s^2}}{8\eta^2+2s^2}-\frac{3i\sqrt{3}\eta\sqrt{3\eta^2+s^2}}{8\eta^2+2s^2}}{\sqrt{\eta^2+s^2}}]] \tag{2.4.49}$$

$$=\partial_y(\frac{3\sqrt{t^2-y^2}}{2\left(4t^2-y^2\right)}-\frac{3\sqrt{3}\sqrt{3t^2-y^2}}{2\left(4t^2-y^2\right)})*_{(y,t)}\frac{1}{\sqrt{t^2-y^2}}$$

Proof. Applying corollary 2.4.15 and 2.4.16 we obtain (2.4.49). □

Again (2.4.49) is only a L^1 operator and we assume the initial data to be of compact support to study the time asymptotic property of this term.

Lemma 2.4.18.

$$\mid \mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{3i\eta\sqrt{\eta^2+s^2}}{8\eta^2+2s^2}-\frac{3i\sqrt{3}\eta\sqrt{3\eta^2+s^2}}{8\eta^2+2s^2}}{\sqrt{\eta^2+s^2}}]]*_y Heaviside(1-|y|)\mid \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.50}$$

Proof. Apply lemma 2.4.17 and the (2.4.18) can be similarly proven as we did for c_1 terms. □

Corollary 2.4.19.

$$\mid \mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{\frac{i\left(7+4\sqrt{3}\right)\eta\sqrt{3\eta^2+s^2}}{8\left(2+\sqrt{3}\right)\eta^2+2\left(3+\sqrt{3}\right)s^2}-\frac{i\sqrt{3}\left(1+\sqrt{3}\right)\eta\sqrt{\eta^2+s^2}}{4\left(2\sqrt{3}\eta^2+6\eta^2+3s^2\right)}}{\sqrt{\eta^2+s^2}}]]*_y Heaviside(1-|y|)\mid \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.51}$$

Lemma 2.4.20. *If the initial data is restricted on the boundary and has compact support on the boundary surface, then the behavior of the solution on the boundary would has the property below:*

$$\mid (u(0,y,t)-(-2\frac{\sqrt{3}\left(\sqrt{3}-1\right)}{12}\frac{\delta(y+c_1t)-\delta(y-c_1t)}{2c_1}))*_y Heaviside(1-|y|)\mid \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.52}$$

Proof. Applying corollary 2.4.19, lemma 2.4.18 and lemma 2.4.11 we can imply (2.4.52). □

2.4.3 Initial Impulsion in Interior

Now we consider the case when initial impulsion is in the interior domain i.e we suppose our initial data to be

$$u(x,y,0) = \delta(x - x_0, y - y_0) \tag{2.4.53}$$

Then on the boundary the reflection part of the solution has the formula as

$$\hat{u}_b^r = \frac{i\eta\left(3\left(2\eta^2+s^2\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)} + \left(6\eta^2 - 4\sqrt{3}\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}+3s^2\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}\right)}{2\left(12\eta^4+3s^4+12\eta^2s^2-4\sqrt{3}\eta^2\sqrt{\eta^2+s^2}\sqrt{3\eta^2+s^2}\right)} \tag{2.4.54}$$

Now rationalize the denominator and we can rewrite (2.4.54) as linear combination of two parts.

$$\begin{cases} \hat{u}_T^r = \dfrac{i\eta\left(6\sqrt{\eta^2+s^2}\left(-24\eta^6+3s^6+2\eta^2s^4-28\eta^4s^2\right)-24\sqrt{3}\left(\eta^2+s^2\right)^2\left(2\eta^2+s^2\right)\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{12s^2\sqrt{\eta^2+s^2}(4\eta^2+s^2)\left(8\eta^4+3s^4+12\eta^2s^2\right)} \\ \hat{u}_L^r = \dfrac{i\eta\left(\sqrt{3}\sqrt{3\eta^2+s^2}\left(2\eta^2+s^2\right)^3+4\eta^2\sqrt{\eta^2+s^2}\left(3\eta^2+s^2\right)\left(2\eta^2+s^2\right)\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{2s^2\sqrt{\eta^2+\frac{s^2}{3}}(4\eta^2+s^2)\left(8\eta^4+3s^4+12\eta^2s^2\right)} \end{cases} \tag{2.4.55}$$

Then by partial fraction we can separate (2.4.55) into simpler operators:

$$\begin{cases} \hat{u}_T^r = & -\frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{8s^2\sqrt{\eta^2+s^2}} \\ & +\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}+2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{24\left(2+\sqrt{3}\right)\sqrt{\eta^2+s^2}\left(c_1^2\eta^2+s^2\right)} \\ & +N_{2,3}^T \\ \hat{u}_L^r = & \frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8s^2\sqrt{\eta^2+\frac{s^2}{3}}} \\ & -\frac{i\eta\left(2\sqrt{3}\sqrt{\eta^2+s^2}+3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{24\sqrt{\eta^2+\frac{s^2}{3}}\left(s^2+c_1^2\eta^2\right)} \\ & +N_{2,3}^L \end{cases} \tag{2.4.56}$$

where we have N_1 and N_2 as below:

$$\begin{cases} N_{2,3}^T = & \frac{9i\eta\left(\sqrt{\eta^2+s^2}-\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{8\sqrt{\eta^2+s^2}(4\eta^2+s^2)} \\ & +\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}-2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{8\left(\sqrt{3}-2\right)\sqrt{\eta^2+s^2}\left(2\left(3+\sqrt{3}\right)\eta^2+3s^2\right)} \\ N_{2,3}^L = & \frac{i\eta\left(\sqrt{3}\sqrt{3\eta^2+s^2}-\sqrt{\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8\sqrt{\eta^2+\frac{s^2}{3}}(4\eta^2+s^2)} \\ & +\frac{i\eta\left(-7\sqrt{3}\sqrt{\eta^2+s^2}+12\sqrt{\eta^2+s^2}+2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8\left(\sqrt{3}-2\right)\sqrt{\eta^2+\frac{s^2}{3}}\left(2\left(3+\sqrt{3}\right)\eta^2+3s^2\right)} \end{cases} \tag{2.4.57}$$

In the previous section we show the cancelation of c_2 and c_3 terms. Now for $N^T_{2,3}$ and $N^L_{2,3}$ we can similarly prove that there would be no surface wave with speed c_2 or c_3. Then we can have some results without detail of proof.

Lemma 2.4.21.

$$| \mathscr{F}^{-1}[\mathscr{L}^{-1}[N^T_{2,3}]] *_y Heaviside(1-|y|) | \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.58}$$

Lemma 2.4.22.

$$| \mathscr{F}^{-1}[\mathscr{L}^{-1}[N^L_{2,3}]] *_y Heaviside(1-|y|) | \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.59}$$

For the c_1 terms we can use similar method as we did in previous section to study its time asymptotic property.

Lemma 2.4.23.

$$\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}+2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)}{24\left(2+\sqrt{3}\right)\left(c_1^2\eta^2+s^2\right)}]] \\
=&2\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}\right)}{24\left(2+\sqrt{3}\right)\left(c_1^2\eta^2+s^2\right)}]] \\
&-(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[\sqrt{3}t>|y|>t] \\
&-(\frac{7}{8\sqrt{3}}-\frac{1}{2})\left(\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}+\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}\right)
\end{aligned} \tag{2.4.60}$$

Proof. This is a consequence of lemma 2.4.5. □

Lemma 2.4.24.

$$
\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta e^{|x_0|\left(-\sqrt{\eta^2+s^2}\right)}}{\left(c_1^2\eta^2+s^2\right)}]]\\
=&\partial_x \frac{\log\left(c_1^2\left(t-x_0\right)\left(t+x_0\right)-2c_1ty+x_0^2+y^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&-\partial_x \frac{\log\left(\left(c_1t-c_1x_0+y\right)\left(c_1\left(t+x_0\right)+y\right)+x_0^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&+\frac{2x_0y}{\sqrt{t^2-x_0^2-y^2}\left(2\sqrt{1-c_1^2}t\sqrt{t^2-x_0^2-y^2}+\left(2-c_1^2\right)t^2+\left(c_1^2-1\right)x_0^2-y^2\right)}
\end{aligned}
\tag{2.4.61}
$$

Proof. □

Lemma 2.4.25. *we can apart c_1 term of u_T^r into combination of surface wave and L^1 operators:*

$$
\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}+2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{24\left(2+\sqrt{3}\right)\sqrt{\eta^2+s^2}\left(c_1^2\eta^2+s^2\right)}]]\\
=&\frac{1}{12+8\sqrt{3}}\partial_x \frac{\log\left(c_1^2\left(t-x_0\right)\left(t+x_0\right)-2c_1ty+x_0^2+y^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&-\frac{1}{12+8\sqrt{3}}\partial_x \frac{\log\left(\left(c_1t-c_1x_0+y\right)\left(c_1\left(t+x_0\right)+y\right)+x_0^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&+N_1^T
\end{aligned}
\tag{2.4.62}
$$

Proof. Apply lemma 2.4.23 and lemma 2.4.24 one have

$$
\begin{aligned}
&\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(\sqrt{3}\sqrt{\eta^2+s^2}+2\sqrt{3}\sqrt{3\eta^2+s^2}-3\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{24\left(2+\sqrt{3}\right)\sqrt{\eta^2+s^2}\,(c_1^2\eta^2+s^2)}]]\\
=&2\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(\sqrt{3}e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}\right)}{24\left(2+\sqrt{3}\right)(c_1^2\eta^2+s^2)}]]\\
&-\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[\sqrt{3}t>|y|>t]*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]\\
&-\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]\\
&+\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]\\
\\
=&\frac{1}{12+8\sqrt{3}}\partial_x\frac{\log\left(c_1^2\left(t-x_0\right)\left(t+x_0\right)-2c_1ty+x_0^2+y^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&-\frac{1}{12+8\sqrt{3}}\partial_x\frac{\log\left(\left(c_1t-c_1x_0+y\right)\left(c_1\left(t+x_0\right)+y\right)+x_0^2\right)}{2c_1\sqrt{1-c_1^2}}\\
&+\frac{1}{12+8\sqrt{3}}\frac{2x_0y}{\sqrt{t^2-x_0^2-y^2}\left(2\sqrt{1-c_1^2}t\sqrt{t^2-x_0^2-y^2}+\left(2-c_1^2\right)t^2+\left(c_1^2-1\right)x_0^2-y^2\right)}\\
&-\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[\sqrt{3}t>|y|>t]*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]\\
&-\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]\\
&+\partial_y(\frac{7}{8\sqrt{3}}-\frac{1}{2})\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{\sqrt{\eta^2+s^2}}]]
\end{aligned}
$$

.

(2.4.63)

The first two terms with log function are surface wave components and we can denote the remainder as N_1^T. □

We can do the same analysis on c_1 term of u_L^r.

Lemma 2.4.26. *we can apart c_1 term of u_L^r into combination of surface wave and L^1 operators:*

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[-\frac{i\eta\left(2\sqrt{3}\sqrt{\eta^2+s^2}+3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{24\sqrt{\eta^2+\frac{s^2}{3}}\left(s^2+c_1^2\eta^2\right)}]]$$

$$=\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)-6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}$$

$$-\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)+6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}$$

$$+N_1^L \tag{2.4.64}$$

Proof.

$$\mathscr{F}^{-1}[\mathscr{L}^{-1}[-\frac{i\eta\left(2\sqrt{3}\sqrt{\eta^2+s^2}+3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{24\sqrt{\eta^2+\frac{s^2}{3}}\left(s^2+c_1^2\eta^2\right)}]]$$

$$\begin{aligned}
=&2\mathscr{F}^{-1}[\mathscr{L}^{-1}[-\frac{i\eta e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8\left(c_1^2\eta^2+s^2\right)}]]\\
&-\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[\sqrt{3}t>|y|>t]*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]]\\
&-\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]]\\
&+\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]]
\end{aligned}$$

$$\begin{aligned}
=&\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)-6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}\\
&-\frac{1}{4}\partial_x\frac{\sqrt{3}\log\left(c_1^2\left(3t^2-x_0^2\right)+6c_1ty+3\left(x_0^2+y^2\right)\right)}{2c_1\sqrt{3-c_1^2}}\\
&+\frac{1}{4}\frac{2\sqrt{3}x_0y}{\sqrt{t^2-\frac{x_0^2}{3}-\frac{y^2}{3}}\left(6\sqrt{3-c_1^2}t\sqrt{3t^2-x_0^2-y^2}-3\left(c_1^2-6\right)t^2+\left(c_1^2-3\right)x_0^2-3y^2\right)}\\
&-\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{t}{y^2-c_1^2t^2}\sqrt{1-c_1^2}Heaviside[\sqrt{3}t>|y|>t]*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]]\\
&-\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{1}{\sqrt{t^2-y^2}+t\sqrt{1-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]]\\
&+\partial_y\frac{\left(3+2\sqrt{3}\right)}{24}\frac{1}{\sqrt{3t^2-y^2}+t\sqrt{3-c_1^2}}*_{(y,t)}\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{\sqrt{\eta^2+\frac{s^2}{3}}}]].
\end{aligned} \tag{2.4.65}$$

□

Lemma 2.4.27. *For N_1^T and N_1^L we have the time asymptotic structure when the initial data has compact support in variable y i.e the initial data is of the form*

$$\delta(x-x_0)Heaviside(1-|y|)$$

Then N_1^T and N_1^L would has a uniformly decay rate of $\frac{1}{\sqrt{t}}$

$$|\ \mathscr{F}^{-1}[\mathscr{L}^{-1}[N_1^T]] *_y Heaviside(1-|y|)\ | \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.66}$$

$$|\ \mathscr{F}^{-1}[\mathscr{L}^{-1}[N_1^L]] *_y Heaviside(1-|y|)\ | \leq O(1)\frac{1}{\sqrt{t}} \tag{2.4.67}$$

Proof. The proof is similar as what we did in previous sections. □

Now the we need to deal with the last two terms

$$\begin{aligned} &-\frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{8s^2\sqrt{\eta^2+s^2}} \\ &+\frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8s^2\sqrt{\eta^2+\frac{s^2}{3}}} \end{aligned} \tag{2.4.68}$$

These two terms cannot be studied independently because of the s^2 on the denominator which represents the Newton potential. In fact the unbounded influence domain due to the Newton potential would be restricted in a cone after cancelation in the summation.

Lemma 2.4.28.

$$
\begin{aligned}
&|\mathscr{F}^{-1}[\mathscr{L}^{-1}[-\frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+s^2}\right)}}{8s^2\sqrt{\eta^2+s^2}}]] *_y \mathit{Heaviside}(1-|y|)\\
&+\mathscr{F}^{-1}[\mathscr{L}^{-1}[\frac{i\eta\left(3\sqrt{\eta^2+s^2}+\sqrt{3}\sqrt{3\eta^2+s^2}\right)e^{x_0\left(-\sqrt{\eta^2+\frac{s^2}{3}}\right)}}{8s^2\sqrt{\eta^2+\frac{s^2}{3}}}]] *_y \mathit{Heaviside}(1-|y|)|\\
&\leq O(1)\frac{1}{\sqrt{t}}
\end{aligned}
\tag{2.4.69}
$$

Proof. In fact the term s^2 can be viewed as a limit case of $s^2+c^2\eta^2$ where $c=0$. Thus we can repeat the process in the previous work in lemma 2.4.26 and lemma 2.4.25 and combine our results for solution in free space in (2.4.7) to obtain the estimate in (2.4.69). □

Finally combining (2.4.7), lemma 2.4.21, lemma 2.4.22, lemma 2.4.28, lemma 2.4.27, lemma 2.4.26 and lemma 2.4.25 we can prove the theorem 2.4.1 and theorem 2.4.2.

CHAPTER 3

Conclusion and Discussion

3.1 Results

According to the previous chapters we have two main results i.e theorem 1.1.1, theorem 2.1.1 and theorem 2.4.2.

In chapter 1 theorem 1.1.1 gives out surface formula without any restriction of the poisson ratio, i.e the poisson ratio can be any constant between 0 and 0.5. Although our formula is valid in mathematical sense there are something strange when comparing our formula with the classic theory in seismology. More precisely, in case 1 and case 2, all the three poles are on the imaginary axis, thus the bromwich integral cannot avoid any pole and this may lead to some surface wave with speed greater than body waves', which contradicts classic seismology, which says surface wave speed should be smaller than body waves', and observation in real life [1] [8]. Thus some further study is needed on these formulas obtained in theorem 1.1.1 to make our conclusion consistent with classic theory.

In chapter 2 we have two main tasks. Firstly, we recombine the formula in transform space and obtained solution for particular initial boundary value in theorem 2.1.1. The recombination highly simplified our formula and allow us to reverse it to time-space domain. Secondly, we explained that the terms with wave speed greater than body waves' in 1.1.1 will be canceled and thus our conclusion will coincide with classic theory. Finally, we estimates our formulas in 2.1.1 and show that the c_1 terms (contain the surface wave part with wave speed c_1 which is smaller than body waves' speed) would be the main part. More precisely, on the surface, the formulas in 2.1.1 will have a uniformly decay rate $\frac{1}{\sqrt{t}}$ except for surface wave part which travels along the surface with speed c_1. It is well known in classic theory of seismology that the surface wave will be more destructive than body waves and our results are illustration of this theory.

3.2 Difficulties and Future Work

There are still some difficulty remain. Firstly, we can generate the Green's function of the Lamb's problem but it is hard to give out a proper estimates of the formulas even in 2-D case. More precisely (in 2-D case) the interaction of 2-D waves with different speed is not clear enough. Secondly, Lamb's problem assumes the homogeneous Lame constants i.e the elastic property of the materia is the same every where. It is natural to think of what happen if the media is inhomogeneous. One of the models to investigate the inhomogeneous problem is so called Love wave problem. In this model the half space is divide to 2 layers, which are parallel to the boundary, with different elastic property. Thus the upper layer would be a wave guide in which Love wave would translate along the surface. The main difference between Love wave and Rayleigh wave is that Love wave has dispersion property while Rayleigh wave not, so Love wave speed is not unique. Many study on this topic try to reduce the system into eigenvalue problems and draw some conclusions with assumptions about the form of the solution [1], while, to handle with initial boundary value problem, some further study and maybe some new tools are needed.

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