Non-Linear and Non-Smooth dynamics study in sustainable development systems

Estudio de la dinámica No-lineal y No-suave en sistemas de desarrollo sostenible

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To my parents Patricia and Misael for a life full of love and patience
To my sisters Adriana and Sandra for always hold me in their hearts
To my nephews Juan Miguel and Guadalupe who are my inspiration

Resumen

El eje temático de este trabajo es estudiar la interacción dinámica a largo plazo entre la explotación de recursos naturales y el crecimiento de la población en una sociedad que depende económicamente de la agricultura y los recursos naturales. Se presentan diferentes modelos de desarrollo y se analizan utilizando teoría de bifurcaciones y simulaciones en el espacio de estados con el objetivo de encontrar valores de los parámetros que permitan un comportamiento a largo plazo ya sea con la extinción de los recursos o con valores positivos de estos. Las simulaciones muestran que los sistemas dependen fuertemente de los parámetros tecnológicos y sociales los cuales inducen diferentes tipos de bifurcaciones. Hopf, silla-nodo, puntos de bifurcación de codimension dos, comportamiento caótico, y bifurcaciones de ciclos limite aparecieron en los sistemas. Fenómenos no suaves como deslizamiento y pseudo equilibrios fueron obtenidos cuando una reserva de recursos renovables es protegida de la explotación humana. Para un escenario sostenible se concluyo que se deben introducir al sistema varias acciones de sostenibilidad con el objetivo de permanecer en los equilibrios no triviales o en un comportamiento oscilatorio.

Palabras clave: Desarrollo sostenible; modelos dinámicos; teoría de bifurcación; sistemas no suaves; deslizamiento.

Abstract

The central thematic of this work is the study of the long-run dynamic interaction between the exploitation of natural resources and population growth in a society that is economically dependant on renewable resources and agriculture. Different mathematical models of development are presented and analysed through bifurcation theory and state space simulations in order to obtain parameter settings that yield positive values of resources and population in the long-run and parameter settings in which resources become extinct. Simulations show that systems are highly dependant on social and technological parameters which induce different types of bifurcations. Hopf, saddle-node, codimension-two bifurcations points, chaotic behaviour, and bifurcation of limit cycles appeared in the systems. Non-smooth phenomena such as sliding and pseudo equilibrium are also present when a reserve of renewable resources is protected from human exploitation. Analysis conclude that for a sustainable scheme, several sustainability actions must be introduced in the system in order to keep non-trivial equilibrium or stationary oscillations among the dimensions.

Keywords: Sustainable development; dynamic models; bifurcation theory; non-smooth systems; sliding.

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1 Introduction

The concept of sustainable development implies worldwide responsibility and shift to more sustainable lifestyles and patterns of consumption and production to obtain the harmony among society, economy, and nature. This concept incorporates current and future global environmental concerns and must consider the role of ethics and values in determining choices affecting local and global environmental conditions. For example, the way science and new technologies are influencing the development of global environmental, or the way issues associated with global ecosystems influence cultural changes. Consequently, nowadays when we study global and local development it is necessary to consider issues of ethics, such as: the haves and the have-nots; the rich and the poor; consumption patterns and hunger; the rights of future generations; and our responsibility toward the planet, other species, and all forms of life ([Serageldin, 1998]; [WCED, 1987]).

Developed and rich countries have contributed by far the greatest amount of greenhouse gases, general pollution, natural resources exhaustion, and environment degradation. But nowadays, the fastest growth in emissions and natural resources exploitation are in the developing countries, which are projected to overtake the developed and industrial countries. Nevertheless, in many cases undeveloped and developing countries, whose natural resources are exploited not only by themselves but also by others countries with highest technological progress and capital accumulation, are the result of dealings with world powers. For this reason, poor countries have less or no chance of appropriate developing unless rich countries reduce the enormous proportion of environment degradation that they contribute to the global impact.

1.1. Technological progress for development

Technological progress has become in development's base. For example, medical and agricultural uses of biotechnology have helped to prolong human life by reducing suffering and hunger; or it has allowed humanity to develop science of prediction, mitigation, and prevention of severe effects of natural disaster. In brief, the entire world cannot live without technology and it is necessary that all countries especially developing countries have access to technology and capital.

The increase in food production can only come from land area increase, yield increase, and decrease of losses since contributions from aquaculture and synthetic food are negligible ([Serageldin, 1998]). The increase of yield in the agricultural sector was the main objective of the green revolution in order to improve the nutrition of poor people. For this purpose, new technologies including biotechnology were developed and transferred worldwide. New seed varieties were created allowing more than one harvest a year and resistance and tolerance of plants diseases, animal pests,

irregular irrigation, poor soils, and many other factors. At the same time the increase in the number of harvests allowed an increase in employment and thus in income. Nevertheless, all impacts were not positive; the technologies and its benefits were not distributed equitably: technology transfer was among rich, and poor were left behind. Others negative effects were the reduced use of seed biodiversity and increment of single crop farming because people abandoned traditional ones increasing the exhaustion of the soil, the use of agrochemical products and its environmental effects destroying future production possibilities, the cost of agricultural machinery, and many other negative effects that make this kind of agronomy unsustainable ([Altieri, 1999]).

On the other hand, agroecology emerged as sustainable agriculture linking ecology, culture, economy, and society with the objective of reducing environmental and social negative effects with less use of external supplies than agriculture based in biotechnology. Agroecology is focused in both production and ecological sustainability of the production system. For this purpose, optimal and sustainable conditions for animals and plants's growth must be created in order to build healthy environments with soil preservation and viable food and farming communities with knowledge and handling of local available resources ([Altieri, 1999]).

Sustainable forest and resources management is another important issue to take into account when we refer to sustainable development because humanity is currently consuming the world's natural resources at an outrageous rate. The use of these resources are increasing pressure on ecosystems and changing our environment from local to global scales threaten our natural support systems and future development; for example, the harvesting of forest or the burning of fossil fuels. Therefore, the sustainable use of resources and sustainable development of societies are among the most important goals in international, national, and regional governance approaches.

In the new millennium, there are a big number of actions that we have to take in order to sustain humanity; namely, we must restrict population growth to limits that can be born on Earth with its resources stock; it means that birth and death rates need to be stabilized as soon as possible. We also need a better distribution of resources in places where they are needed most; and we must stop growth in pollution. For this purpose, socio-political changes and cultural values are needed to change current resource consumption patterns.

1.2.   Modelling sustainable development

Nowadays sustainable development is extensively described and studied, but very few works are dedicated to mathematical modelling techniques and numerical simulation. According to [Clark et al., 1995], mathematical models, namely non-linear models can provide a point of entry into system's evolution supplying some reasonable correspondence between models and reality. Wherefore, once the mathematical model is well-known and understood, it can be used as a decision tool, and control actions can be designed in order to achieve economic and environmental sustainability; for this purpose, numerical simulations linked to mathematical models are used as another powerful tool allowing input and output analysis and the comparison of the results with others

and real data. The existing models describing sustainable development are mainly linear and have very poor numerical simulation and no stability analysis of equilibrium points. Due to the high variable interaction among social, economical, and environmental issues it is advisable to develop non-linear models that describe in a organized manner the participation of each component part in the whole behaviour. Non-linear phenomena such as bifurcation and chaos can appear; or non-smooth phenomena such as sliding, pseudo equilibria, and non-smooth bifurcations when non-linear models are replaced by non-smooth models.

1.3. Previous work

Most of the existing non-linear models applied to sustainable societies are related to the Lotka-Volterra predator-prey model with population as the predator and natural resources as the prey. In general, these models are described for sets of smooth differential equations $ODEs$ representing the dynamic interaction between the exploitation of natural resources and population growth. Mainly results are focussed in study the effect of institutional reforms as sensitivity analysis of equilibrium points by changing parameter values ([Brander & Taylor, 1998]; [Dalton & Coats, 2000], [Reuveny & Decker, 2000], [Dalton et al., 2005], [D'Alessandro, 2007]). Rise and fall of Easter Island was the case of study and several important qualitative results were obtained; for example, technological progress in harvesting makes resource to fall ([Brander & Taylor, 1998]) and technologies that prevent harvesting make resource stock to survive ([Dalton et al., 2005]); mortality and birth rate have opposite effect on resource stock and moderate and low fertility generates smooth trajectories and avoid population and resource extinction ([Reuveny & Decker, 2000]).

 [Brander & Taylor, 1998] developed a simple predator-prey model to give an intuitive reasoning on the rise and fall of the Easter Island. They did not consider the change of institutional reforms and technology with respect to time; nevertheless, they induced disturbances in the parameters which can give some notions of its effects on the equilibrium values of the states. [Dalton & Coats, 2000] found that modification of the economic institutions on the Island could have damped the feast-and-famine cycles by affecting the trajectory to the equilibrium but not the equilibrium levels of the state. [Reuveny & Decker, 2000] capture the notion of population management by acting in the fertility function. They concluded that the reduced fertility rates could prevent the extinction of population and resource. On the other hand, they examine how carrying capacity, resource stock, and harvesting affect the global behaviour of the system when they have logarithmic and exponential non-autonomous forms, sometimes resulting in system collapse. Taxes on resource exploitation were introduced by [Pezzey & Anderies, 2003]. Results showed that higher subsistence requirement destabilizes the system and no policy can prevent overshoot. All numerical and analytical results show resource and population reaction when conservation policies were applied. [Prskawetz et al., 1999] develop a three sector demoeconomic model where not only resource use but also labour income were taxed and then these taxes were used to finance education and increase

technology.

All the works aforementioned present long-run dynamics with resource exhaustion followed by positive growth. [D'Alessandro, 2007] introduced the so-called *critical depensation growth function* which allows the system to be irreversible. It means that there is a critical level of renewable resource such that below that level the rate of regeneration becomes negative and the exhaustion will be inevitable. On the other hand, he disintegrated the ecological complex in two different resources in order to construct a more general system resulting in multiple internal equilibria. Finally, comparative dynamics were carried out and applicability to both cultural and climate change and technological progress was evaluated. This model is extensively studied in chapter 2.

Discontinuous systems

The above non-linear models have continuous vector field, namely, they are smooth. Now, when modelling it is also often to obtain models in which the smooth evolution of the system change abruptly. These kind of systems are called discontinuous and depending on their features, different methods are required. For example, in mechanics there are *impact models* whose dynamic phenomena occurs in a quite diversify time scale, then, discontinuous event for these systems are neither constant nor pre-specify. Other class are that of *periodically systems* where the discontinuity in the state space is generated by a periodic exogenous shock on the system, so discontinuous event is constant and priory fixed. A third kind of discontinuous system is that of *piecewise smooth* systems that is often called Filippov systems which is the class of systems discussed in this work and are describe by standard differential equations

$$\dot{x} = f\left(x\left(t\right)\right), \quad x \in R^n,$$

where the function f is discontinuous.

Filippov systems for modelling systems in ecology has had an increasing interest during the last 14 year. Some papers as [Krivan, 1996], [Krivan, 1998], [Krivan & Sikder, 1999], [Krivan & Eisner, 2003], and [Srinivasu & Gayatri, 2005], have studied the dynamics of population of predators feeding on two different preys or switching between different habitats. [Dercole et al., 2007] studied the long-term behaviour of population communities described by Filippov systems. The analysis is carried out through bifurcation analysis of the model with respect to the parameters. Results showed that non-linear and non-smooth phenomena can explain the consequences of exploitation or protection.

1.4.   Main objectives of the thesis

The main objective of this work is to analyse the dynamics of sustainable development, giving new insights on the modelling through systems of non-linear differential equations. For this reason, the following are the specific objectives of present work:

- Obtain mathematical models (non-linear and/or non-smooth) in sustainable development, including variables in economy, natural resources, and population.

- Perform numerical simulations in the state space for the mathematical models.

- Analyse the invariant sets of the mathematical models.

- Perform bifurcation analysis in the mathematical models with standard and tested mathematical software such as MatLab and MAPLE

- Conclude on the mathematical results obtained.

1.5. Structure of the document

Chapter 2 presents a planar non-linear system with interacting population and natural resources. The study include analysis of basins of attraction for different initial conditions, one-parameter bifurcations when a single parameter is varied, and two-parameter bifurcation when two parameters changes simultaneously. Chapter 3 introduce a third differential equation which represents the dynamics of endogenous technological progress, it includes the study of basins of attraction and bifurcation analysis. In Chapter 4 models are replaced by piecewise smooth models with one and two discontinuities, then, non-smooth phenomena are obtained and studied. The document ends with a summary of the main results found in the research and offer suggestions for future work in chapter 5.

2 Renewable resources and population dynamics

Most of the existing models that study non-linear behaviour of the interaction between population growth and the exploitation of natural resources are sets of ordinary differential equations (ODEs) related to the Lotka-Volterra predator-prey model. This kind of interactions may be explained by using *causal diagrams* which are useful tools in modelling from system dynamics (see for example [Courchamp et al., 1999] and [Aracil & Gordillo, 1997]). They illustrate the feedback structure of the system and help to build the desired models. In this sense, renewable resource and population growth can be represented by the two simple causal diagrams in Figure **2-1**.

Positive feedback cycle (Figure 2.1(a)) in resources dynamics can be explained as follows: an increase in resources population, causes an increment in its reproductive capacities[1]. When population increases birth rate of renewable resources (for example trees) increases too, resulting an accumulation of resources which is limited by the carrying capacity.

(a)   (b)
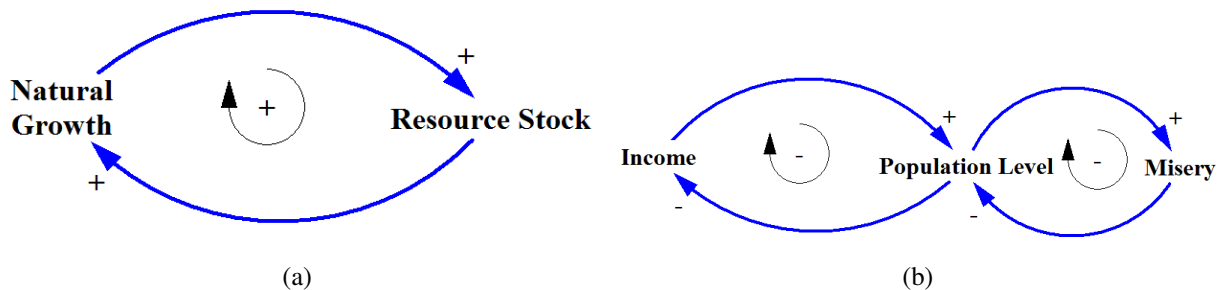
Figure 2-1: Isolated causal diagrams: (a) is the causal diagram for renewable resource growth and (b) is causal diagram for population growth.

On the other hand, population dynamics is suppose to follows the negative feedback cycles in Figure 2.1(b). According to [Malthus, 1798], an increase in per-capita income leads to increasing population size, but an increasing population reduces per-capita income augmenting the misery. Consequently consumption falls back which is consistent with causal diagram 2.1(b).

Economic activities in the primary sector[2] as deforestation are against the environment because humans harvest forest and jungles in order to produce income for surviving and ignore the importance

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[1]For instance, according to plant succession theory more trees can host more insects and birds which are largely responsible of pollination.

[2]Primary sector of the economy extracts or harvests products from the earth, and most of times are considered

of conserving natural resource for future generations. This complex relation may be represented by combining resource and population dynamics. The resulting causal diagram in Figure **2-2** presents a population that produces income not only from renewable resources but also from agriculture. An increasing population involves and augment in the resources requirement and agricultural production. At the same time when more resources are required the stock will decrease due to the increase in harvesting.
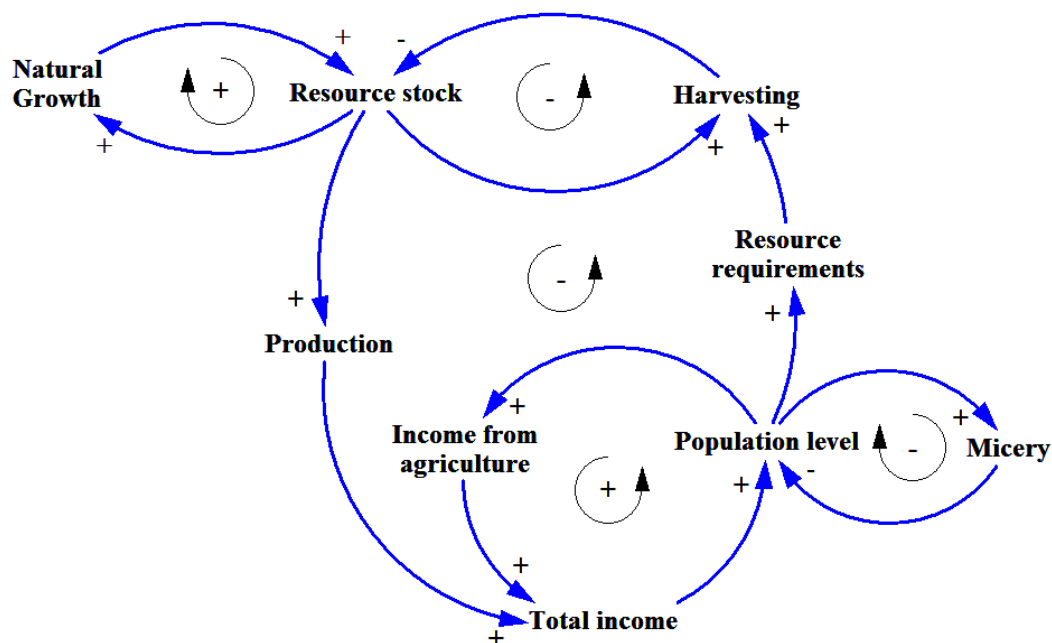

Figure 2-2: Causal diagram for the interaction between renewable resources and population.

Now, the next step is to find the appropriate differential equations (ODEs) to represent the desired behaviour. A general structure for those two-dimensional systems is given by equation (2-1), where $\dot{S}$ and $\dot{L}$ represent change in stock of the renewable resource and change in population at time t respectively. The terms $G(S)$, $H(L, S)$, $F(L, S)$, and $E(L)$ have different shapes depending on the model. According to the causal diagram in Figure **2-2**, resource dynamics can be defined as the growth of the natural resource in absence of human habitation $G(S)$ minus the harvesting rate $H(L, S)$ (see for example [Brander & Taylor, 1998]) that also depends on the population level. On the other hand, population depends on income from economic activities $F(L, S)$ which is always positive and the population increase in the absence of income $E(L)$ that is necessarily negative.

$$\begin{cases} \dot{S} = G\left(S\right) - H\left(L, S\right) \\ \dot{L} = E\left(L\right) + F\left(L, S\right) \end{cases} \tag{2-1}$$

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raw materials for other industries, some activities associated with the primary sector include agriculture, mining, forestry, farming, hunting, and fishing.

This structure of ordinary differential equations (ODEs) has been extensively used to represent simple predator-prey models with two features: services supplied by environment as a unique natural resource and resource exhaustion followed by positive growth. These considerations imply that such models can not describe irreversible ecological change because both population and natural resources will never disappear. [D'Alessandro, 2007] changed this panorama by introducing two new assumptions. First, he introduced the dynamical properties of a renewable resource, called forest and a inexhaustible resource, called land in order to produce wood and corn respectively; second, he incorporated irreversibility in the dynamics of the renewable resource. These assumptions allow to have an equilibrium with positive population and exhaustion of a part of the ecological complex.

2.1.   Mathematical model

The Ricardo-Malthus model proposed by [D'Alessandro, 2007] consists of two economic activities that exploit two natural resources: forest and land. The first economic activity is the harvesting of wood according to the production function $H(L, S) = \alpha\beta LS$; this means that the harvesting rate is proportional to the current population and resources. The second economic activity is the agricultural production represented as a Cobb-Douglas production function. According to Ricardo (1772-1823) the per-capita fertility function depends on the caloric contribution of both goods[3]: wood and corn, then total fertility is given by $F(L, S) = \left(\gamma\lambda(1-\beta)^{\delta} L^{\delta-1} + \gamma\overline{\phi}\alpha\beta S\right) L$, where αS represents the unit labor requirement in the resource sector and βL is the resources requirement. The natural growth of the renewable resource is represented by the *critical depensation growth function* $G(S) = \rho(S/k - 1)(1 - S/K)S$. Finally, misery is the minimum per-capita caloric level needed to survive, then $E(L) = -\overline{\sigma}L$. The final set of two coordinated differential equations is system (2-2)(for details see [Brander & Taylor, 1998] and [D'Alessandro, 2007]).

$$\begin{cases} \dot{S} = \left[\rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right) - \alpha\beta L\right] S; \\ \dot{L} = \gamma\left(\lambda(1-\beta)^{\delta} L^{\delta-1} + \overline{\phi}\alpha\beta S - \overline{\sigma}\right) L. \end{cases} \tag{2-2}$$

2.2.   Parameters definition

The general parameters definition, standard parameters values, and units for system (2-2) are introduced with the following abbreviations, : mass unit of wood (*muw*), mass unit of corn (*muc*), mass unit of population (*mup*), and caloric unit (*cu*).

$\rho > 0$ is the regeneration rate of the resource $(dimensionless)$.

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[3]In most papers fertility function depends on wood, but for D'Alessandro it also depends on corn.

K is the maximum size of the resources that the environment can support, it means that if $S = K$ the growth rate of resources is zero (muw).

k is the resource value where the growth rate becomes negative and the renewable resource tends to the extinction (muw).

$\alpha > 0$ represents the available technology for harvesting $(mup^{-}1)$.

$\beta \in [0, 1]$ is a parameter of preference between one of the two subsistence goods: wood and corn $(dimensionless)$.

$\bar{\sigma} > 0$ is the natural level of calories required to survive in terms of corn (muc/mup).

$\gamma > 0$ is the caloric unit of corn (cu/muc).

$\bar{\phi} > 0$ is the caloric unit of wood in terms of corn (muc/muw).

$\lambda > 0$ represents land fertility (muc/mup^{δ}).

$\delta \in (0, 1)$ indicates that technological advance in agricultural sector for intensifying the use of land shows decreasing return to scale[4] $(dimensionless)$.

 [Brander & Taylor, 1998] adjusted parameter values according to the pattern of consumption, population growth, and resource degradation that led to the collapse of the island. [D'Alessandro, 2007] made some adjustments to the values in order to obtain the desire behaviour of the system. Values in Table **2-1** are used in the whole work and if changes are needed these will be specified.

Parameter	Value
β	0.3
α	0.0001
ρ	0.025
λ	12.95
δ	0.7
k	700
K	12000
γ	0.1
$\bar{\sigma}$	1.4
$\bar{\phi}$	3

Table 2-1: Parameter Values

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[4]In production when output increases by less than proportional change in input there is decreasing return to scale.

2.3.   Equilibrium points

Equilibrium points of an autonomous dynamical system is a solution of the system that does not change over time and can be found when equations in system (2-2) are simultaneously equal to zero. Since S and L represent resources and population, they must be non-negative and only the positive quadrant is considered in this study. For this system the number of equilibria depends on the specific parameter values presenting either six, five, or four steady states. Four trivial equilibria on the axes exist and correspond to the next solutions (L, S):

$$P1 = (0, 0);$$
$$P2 = (0, k);$$
$$P3 = (0, K);$$
$$P4 = \left(\left(\frac{\lambda (1-\beta)^\delta}{\overline{\sigma}} \right)^{1/(1-\delta)}, 0 \right).$$

Note that $P4 \neq P1$ if $\lambda \neq 0$ and $\beta \neq 1$.

Internal equilibria on the first quadrant also exist when system (2-3) presents one or two real and positive solutions. It is necessary to find out numerical solutions for these equilibria because it is not possible to find analytical ones. A useful tool in order to obtain additional information about these internal equilibria is to consider the interception of the nullclines $N1$ and $N2$ in equation (2-4), [Meiss, 2007];

$$\begin{cases} \rho \left(\frac{S}{k} - 1 \right) \left(1 - \frac{S}{K} \right) - \alpha \beta L = 0; \\ \lambda (1 - \beta)^\delta L^{\delta-1} + \overline{\phi} \alpha \beta S - \overline{\sigma} = 0; \end{cases} \tag{2-3}$$

$$N1 = -\left(\frac{\lambda(1-\beta)^\delta L^{\delta-1} - \overline{\sigma}}{\overline{\phi} \alpha \beta} \right); \\ N2 = \frac{\rho(S-k)(K-S)}{kK\alpha\beta}. \tag{2-4}$$

2.4.   Stability of equilibrium points

Non-linear models as system (2-2) may have complex solutions, and definitions of stability for linear systems are deficient and can not be applied. A better and general definition of stability refers to orbits that are close according to Lyapunov stability: and equilibrium is stable if orbits that start "nearby" stay "nearby". However, linear models help us understand the behaviour in a small neighbourhood of equilibria, then stability of equilibrium points in a set of smooth non-linear autonomous differential equations can be determined by the linearization theorem where the eigenvalues of the Jacobian matrix evaluated at the equilibrium point defines that stability. This is true under the assumption that in a small neighbourhood of the equilibrium the non-linear system

behaves as a linear system in the matrix form $\dot{x} = Ax$ where A is the Jacobian and x the states. The resulting linearized system is

$$\begin{bmatrix} \dot{S} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} -\frac{3\rho}{kK}S^2 + \frac{2(k+K)\rho}{kK}S - \rho - \alpha\beta L & -\alpha\beta S \\ \gamma\overline{\phi}\alpha\beta L & \gamma\left[\delta\lambda\left(1-\beta\right)^\delta L^{\delta-1} + \overline{\phi}\alpha\beta S - \overline{\sigma}\right] \end{bmatrix} \begin{bmatrix} S \\ L \end{bmatrix}.$$

$$(2\text{-}5)$$

Equilibrium is said to be hyperbolic when eigenvalues are real or complex-conjugate, and nonhyperbolic when are imaginary or at least one eigenvalue is zero. For two-dimensional system stability of equilibrium points is summarized in Table **2-2**.

Eigenvalue of A	Condition	Type of equilibria
λ_1, λ_2 real	both¡0	Stable node
λ_1, λ_2 real	both¿0	Unstable node
λ_1, λ_2 real	$\lambda_1\lambda_2$¡0	Saddle
λ_1, λ_2 complex	$Re\lambda_i$¡0	Stable focus
λ_1, λ_2 complex	$Re\lambda_i$¿0	Unstable focus
λ_1, λ_2 imaginary		Center
	$\lambda_1 = 0, \lambda_2 \neq 0$	Saddle-node
	$\lambda_1 = 0, \lambda_2 = 0$	Bogdanov-Takens

Table 2-2: Hyperbolic and non-hyperbolic equilibria in a two-dimensional non-linear system.

Trivial equilibria have invariable stability, so $P1$ and $P2$ are always *unstable nodes*, $P3$ a *saddle*, and $P4$ whose eigenvalues are real and negative for any positive parameters values ($\lambda_1 = -\rho - \alpha\beta \left(\frac{\lambda(1-\beta)^\delta}{\overline{\sigma}}\right)^{1/(1-\delta)}$, $\lambda_2 = \gamma\overline{\sigma}\left(\delta - 1\right)$) is always a *stable node*. On the other hand, since analytical solutions for internal equilibria are not available, analytical solutions for eigenvalues are not available too. Then it is necessary to make it numerically by replacing the numerical values of each equilibrium in the Jacobian matrix. Numerical simulations indicate that when both internal equilibria exist the lower one $P5$ is always a saddle while the higher one $P6$ has variable stability depending on parameter values as is studied in section 2.6.

Suppose an initial condition with population $L = 200$, renewable resource at its maximum level $S = 12000$, and parameter values are those in Table **2-2**. Under these conditions internal equilibrium $P5$ is a saddle ($\lambda_1 = 0{,}0416175$ and $\lambda_2 = -0{,}0307268$), and $P6$ is a unstable focus ($\lambda_1 = 0{,}0003323 + 0{,}0619820i$ and $\lambda_2 = 0{,}0003323 - 0{,}0619820i$). In this case the system converge to a stable limit cycle associated to $P6$, then $P4$ is say to be locally stable. This limit cycle in Figure **2-3** shows that densities of population and resources can oscillate periodically. Abundance of resources involves a rising population; but over time, resources are over-harvested producing an economic decline and eventually a reduction of population. After a famine period where both population and resources decline the system permits the regeneration of resources and consequently

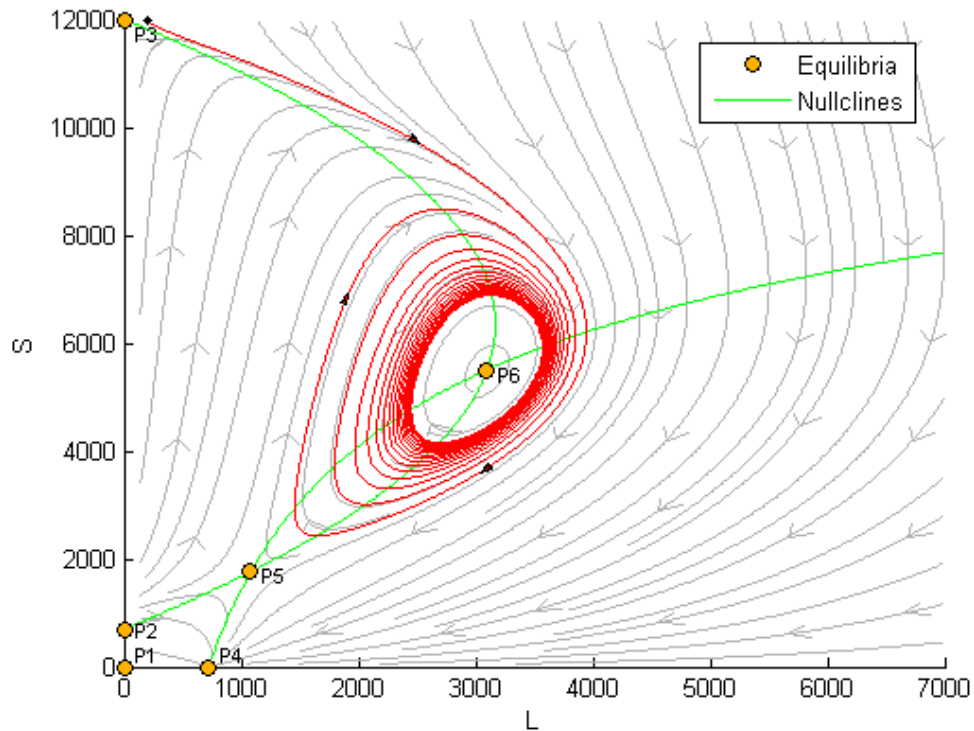and increase of population level. Finally, the over-exploitation repeats and so do feast and famine cycles.

Figure 2-3: Phase portrait of system (2-2), showing equilibrium points and nullclines. The point $P4$ is asymptotically stable, $P6$ is asymptotically unstable, and $P5$ is a saddle.

2.5. Sensitivity to initial conditions

An attractor is a set of the state space toward which all trajectories of a dynamical system move over time. At the same time each attractor has a *basin of attraction* which is a set of initial conditions that approaches to it in the long-run. Assuming a stationary environment in which all parameters of the model are constant, trajectories followed by the system are associated to one of the two attractors: the stable limit cycle or $P4$. The existence of multi attractors makes the system sensitive to the initial conditions. As it is mentioned in section 2.4, $P5$ is always a saddle which is defined by a positive and a negative manifold; hence, the separatrix between the two basins of attraction is associated to the stable manifold that is drawn in Figure **2-4**. **B1** is the region containing all initial conditions that converge to stable limit cycle, i.e. positive renewable resource and positive population, while **B2** contains all initial conditions that converge to $P4$.

According to Figure **2-4**, there is a maximum level of population such that initial renewable resource is enough for people to survive, beyond this threshold of population, renewable resources
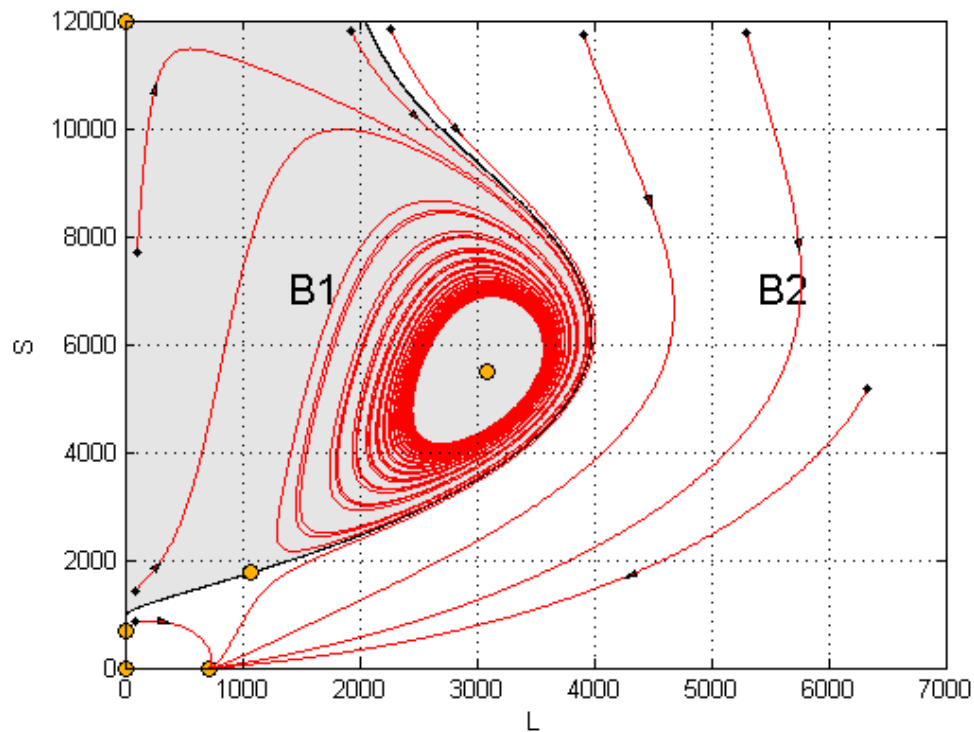
Figure 2-4: Basins of attraction in the planar system.

disappear in the long run and population stays at its minimum level $P4$ surviving from the other economic activity, i.e. agriculture.

2.5.1. Allee efect

Forest and other renewable resources are very vulnerable to extinction because of the excessive harvesting, this is well represented by models with *Allee effect*[5] as system (2-2). The separatrix between the two basins of attraction in Figure **2-4** shows that for some positive values of S, it is not possible to avoid resource exhaustion even in absence of population. This phenomenon is consequence of the Allee effect represented in the critical depensation growth function that describes the natural growth of the resource.

The natural growth rate of the resource is $G(S)$; hence, per capita growth rate is $\frac{G(S)}{S} = \rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right)$. According to Figure **2-5** the per-capita growth has a strongly dependence from resource stock. At low population sizes the Allee effect presents a reduced growth, and once the Allee threshold (k) is

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[5]Allee effect theory states that for very large populations, the reproduction and survival rates of individuals decrease with population density and may exist a threshold below it populations failure, i.e. strong Allee effect. Some examples of the Allee effect on plants and animals are reported in [Liermann & Hilborn, 2001] and [Ford, 2008] respectively.

crossed the per-capita growth becomes negative and the system can not regenerate, so exhaustion
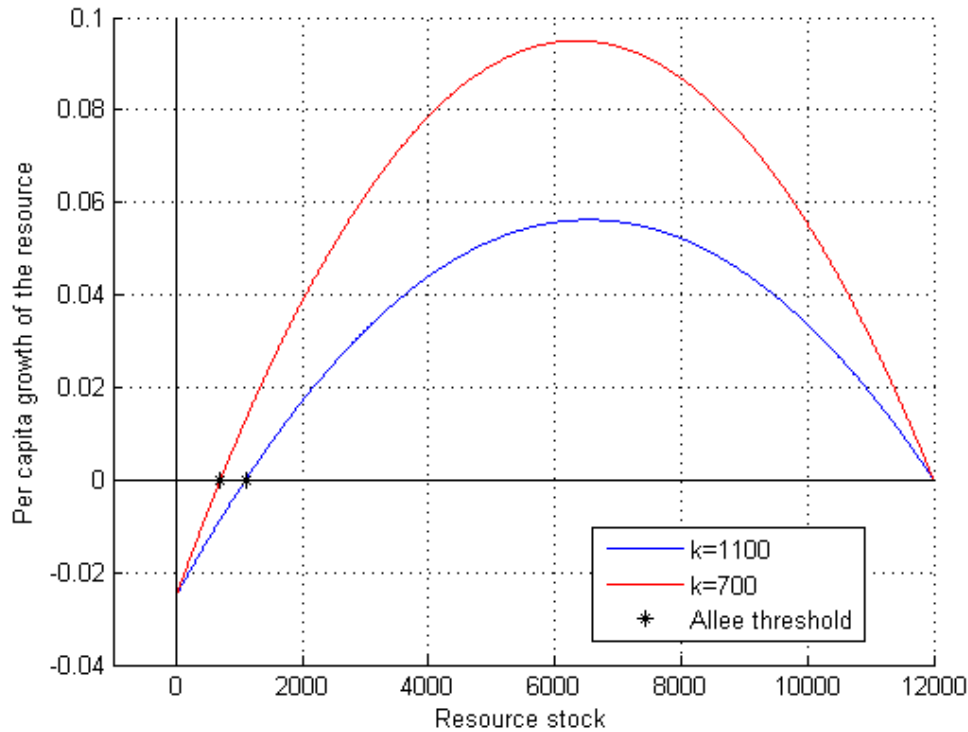is unavoidable.

Figure 2-5: Allee threshold.

2.6.   One-parameter bifurcation of equilibria

In this section sensitivity to parameter values is studied since vector field in population and resource
dynamics of system (2-2) depend on a set of parameters (Table **2-1**). [D'Alessandro, 2007], [Pezzey
& Anderies, 2003], and many other authors studied the possible long-run behaviour of population
and resources by developing comparative dynamics for different parameter values, obtaining im-
portant qualitative changes in the whole behaviour. A more systematic investigation of the possible
long-run behaviour of the interaction between population and natural resources can be obtained by
using bifurcation theory.

The following results describe and study one-parameter bifurcations (*codimension-one*) of equi-
libria and limit cycles in the parameter space. In particular, bifurcation analysis is applied to the
technological progress (α), preference (β), and other important parameters. Results show that two

local bifurcations[6] and two global bifurcation[7] occurs and help to explain new features of sustainable development dynamics. Details of existence of each bifurcation are explained in the next pages. As an extra result, existence of two-parameter bifurcations when varying two parameters is also studied.

2.6.1. Existence of Andronov-Hopf bifurcation

This local bifurcation associates equilibria with the birth of a limit cycle and typically occurs when an equilibrium has a pair of eigenvalues that cross the imaginary axis creating or destroying a periodic orbit. Some studies as [Anderies, 1998], [Prskawetz et al., 2003], [Prskawetz et al., 2000], and [Prskawetz et al., 1999] have found this bifurcation in different population and resources dynamic systems, but only [Zhaou & Liu, 2010] developed a strict mathematical analysis of this bifurcation by using Hopf theorem and normal form theory. Results of that work state that for some values of all parameters in system (2-2) Hopf bifurcation can exist and explain the emergence of limit cycles in model by [D'Alessandro, 2007]. The aim of this section is to develop and represent numerical analysis and simulations of the stationary state through graphic descriptions: plots in the coordinates space and bifurcation diagrams.

In section 2.4 stability of internal equilibria can be determined by the eigenvalues of the Jacobian matrix evaluated at each internal equilibria. According to the Hopf bifurcation theorem, at bifurcation point there is a simple pair of pure imaginary eigenvalues and no other with zero real parts. This stability can be explained as the transition from unstable to stable focus or viceversa as shown in Figure **2-6**.

Consider the boundary case where people do not develop and use any technology for harvesting, i.e $\alpha = 0$. The implications of this are that only agriculture is used for sustenance and resource level remains at its maximum size $S = K$ while people remains at $L = \left(\frac{\lambda(1-\beta)^{\delta}}{\bar{\sigma}}\right)^{1/(1-\delta)}$. As soon as the population begins to increase, so does harvesting technology. This phenomenon is explained in some papers related with theories of endogenous population growth (see for example [Kremer, 1993] and [Buserup, 1981]) where an increasing population means more potential inventors and more people using that technologies. Figure **2-6** presents continuation of equilibrium points when varying α. As expected for small values of α, e.g. harvesting by hand, renewable resources level decreases, but the system quickly approaches the equilibrium (stable node) because of the small population size. As α increases, e.g. the use of chainsaws, renewable resources are harvested more efficiently and ecological complex deteriorates faster while population keep growing. Consequently, convergence to the equilibrium is slower. An explanation for this behaviour is the transition from stable node (both eigenvalues are real and negative) to stable focus (eigenvalues are complex-conjugate with negative real parts). Finally, the Jacobian matrix A evaluated at Hopf bifurcation point ($\alpha = 0{,}00009975, L = 5525{,}4989, S = 3107{,}0926$) is

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[6]Local bifurcations occurs when a parameter change produces a change of the stability of an equilibrium.

[7]Global bifurcations occurs when invariant sets, such as periodic orbits collide with equilibria.
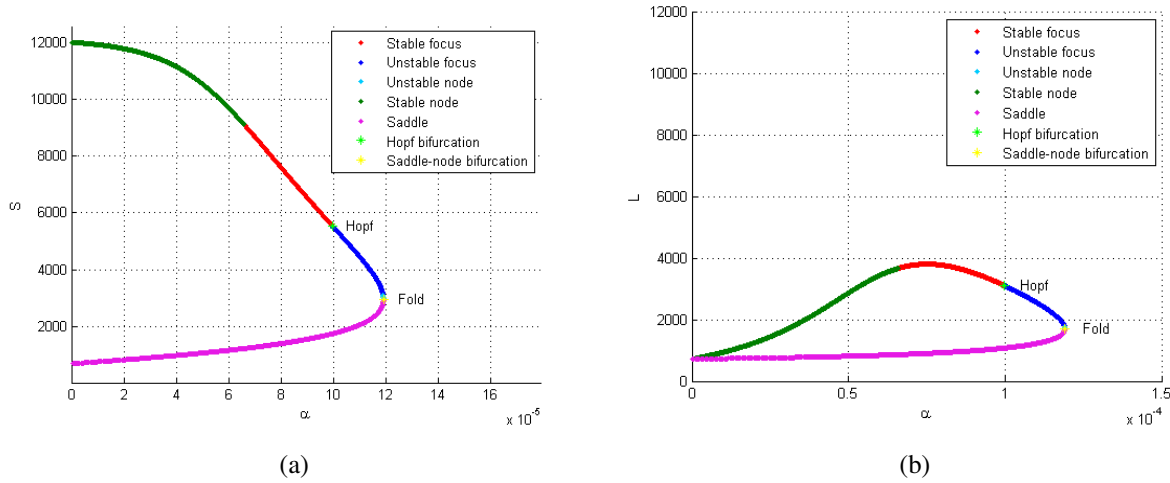
Figure 2-6: Continuation of equilibrium points for α: (a) presents the resources decrease and (b) the population increase when augmenting α.

$$A = \begin{bmatrix} 0{,}027122205456897 & -0{,}165350560477725 \\ 0{,}027893924301285 & -0{,}027120120136463 \end{bmatrix},$$

and both eigenvalues become purely imaginary ($\lambda_1 = 0{,}062263300151025i$ and $\lambda_2 = -0{,}062263300151025i$). Slightly increasing α beyond the bifurcation value, $P6$ turns a unstable focus and system may exhibit permanent oscillatory behaviour as a consequence of the lost of stability. Starting inside the cycle, oscillations in population and resources increase approaching the limit cycle, and starting outside de cycle, damped oscillations occur until the limit cycle is reached. The stability of limit cycles is studied in section 2.9.

2.6.2. Existence of saddle-node bifurcation

Creation or destruction of a pair of equilibria occurs at saddle-node bifurcation (fold). As Hopf, this bifurcation is local and has codimension-one because occur when varying one parameter. The criterion that guarantee that the bifurcation take place is the existence of a single eigenvalue of a linearisation equal to zero (for details about the bifurcation theorem see [Meiss, 2007] and [Kuznetsov, 2004]) and no other with zero real and imaginary parts.

Further Hopf bifurcation, both branches of equilibria are composed of unstable equilibria which get closer as α increases. Near saddle-node bifurcation, $P6$ changes from unstable focus to unstable node, and then collide with the saddle creating a non-hyperbolic equilibrium ($\lambda_1 = 0$ and $\lambda_2 = 0{,}0272914$) as shown in Figure 2.7(a). Figures 2.7(b), 2.7(c), and 2.7(d) present the phase portraits of the system behaviour before, at, and after saddle-node bifurcation respectively. Slightly

increasing α beyond the bifurcation value internal equilibria disappear and only equilibria on the axes remain.
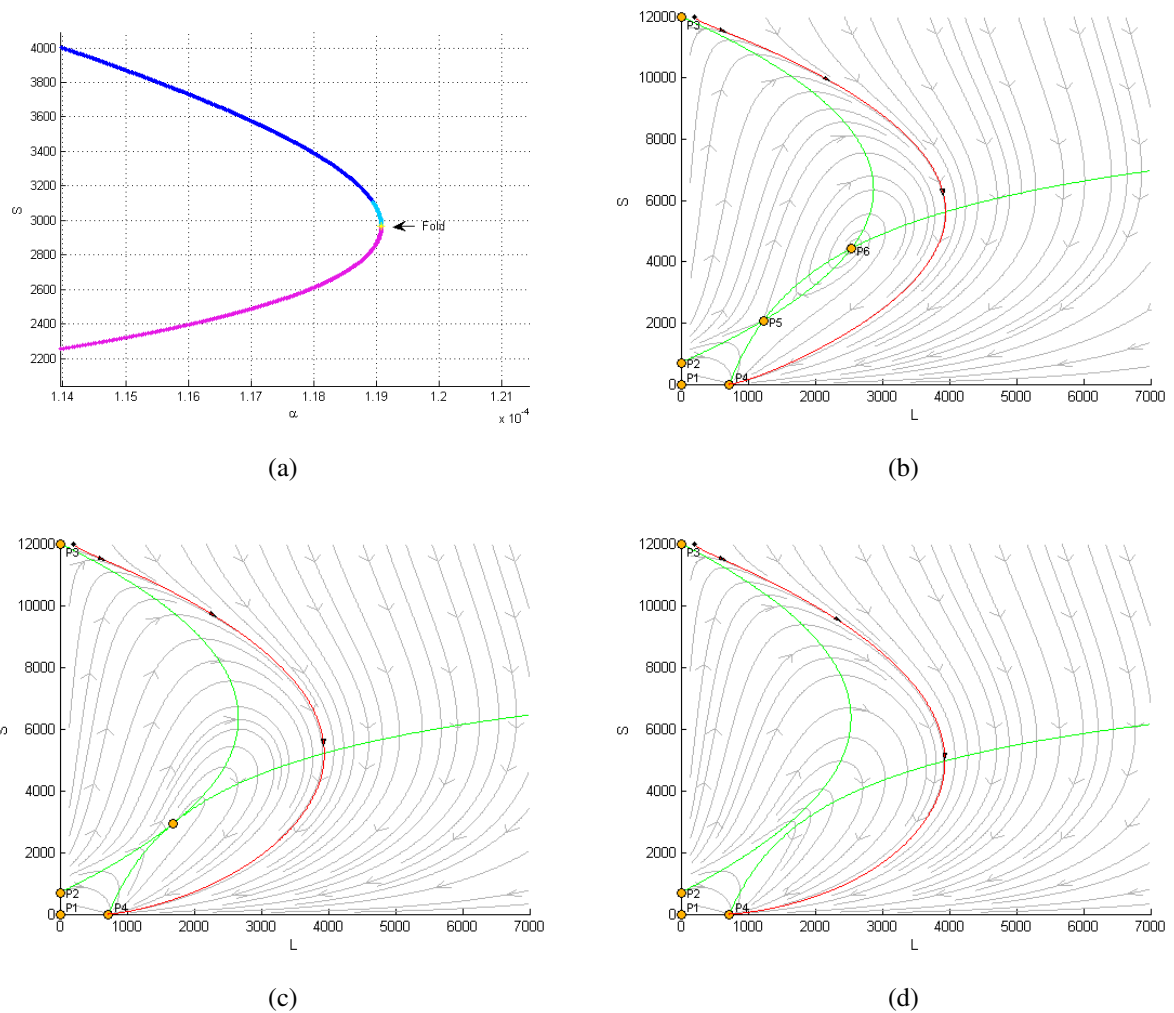

(a)

(b)

(c)

(d)

Figure 2-7: Saddle-node bifurcation for α: (a) saddle-node bifurcation on the $\alpha - S$ plane, (b) internal equilibria before bifurcation point, (c) collision of internal equilibria, and (d) disappearance of internal equilibria.

However, this bifurcation loses importance for the objectives of this work, since its occurrence does not strongly affect the dynamic of population and resources as Hopf and bifurcations of limit cycles do in section 2.7.

2.7. One-parameter bifurcation of limit cycles

As equilibria, one-parameter bifurcation also exists for limit cycles when a single parameter is varied. The main difference is that existence of these bifurcation affect the entire behaviour of the

system, therefore they are global bifurcations. Thought numerical simulations two bifurcations of limit cycles were found in this model and practically describe the route to unsustainability.

2.7.1. Existence of homoclinic bifurcation

The family of stable and unstable limit cycles associated to the Hopf bifurcation is plotted in figure 2.8(a). This cycles are commonly called feast and famine cycles where unregulated resources exploitation in societies that strongly depend on natural resources causes economy decline followed by a reduction of both population and welfare, destabilizing the society. This phenomenon occurs during a period of time T and then repeat for constant parameters. According to the figure, there is an α where the associated limit cycle and the saddle point $P5$ collide creating an homoclinic orbit[8] followed by the appearance of a unstable limit cycle[9]. This global bifurcation is commonly called *homoclinc bifurcation* and mathematical details are found in [Kuznetsov et al., 2003] and [Meiss, 2007].
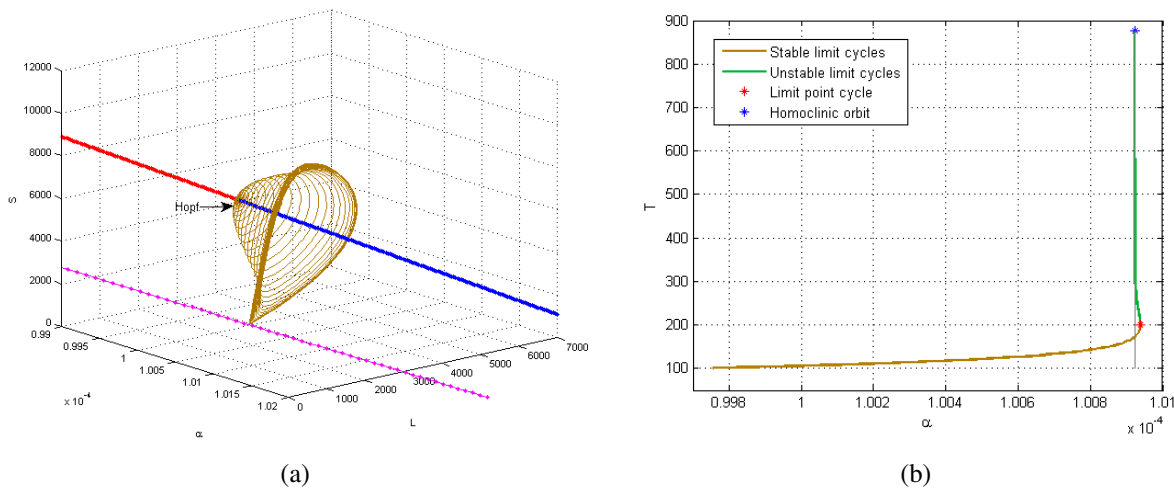

(a)	(b)

Figure 2-8: Creation and destruction of limit cycles: (a) presents the existing limit cycles associated to each unstable focus and the homoclinic orbit and (b) the cycle period near the homoclinic bifurcation when augmenting α.

2.7.2. Existence of limit point cycle

Figure 2.8(b) presents the period of limit cycles versus α. Two different regions are identified before cycles disappear. First region contains single limit cycles for each α whose period increases as α increases; that is to say that a more efficient harvesting creates longer feast and famine cycles.

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[8] An homoclinic orbit is formed for the intersection of the stable manifold and the unstable manifold of an equilibrium.
[9] As Hopf this bifurcation is associated with the birth or destruction of limit cycles.

The second region is boundaried for the homoclinic orbit and the limit point cycle (LPC) where both cycles crash. Finally, after collision of unstable and stable limit cycles no feast and famine cycles exist and the equilibrium on the axe $P4$ becomes globally stable.

Homoclinic bifurcation and limit point cycle take place in a very short interval of α and can not be perceivable. Nevertheless, they state the boundary of environment damage where development becomes unsustainable. A more detailed explanation of these bifurcation is presented in section 2.7.3.

2.7.3.   One-parameter bifurcations for preference

Below, one-parameter bifurcations of equilibria and limit cycles are describe, but this time when varying preference for resources β (($\beta - 1$) represents preference for agriculture). An increase of β involves an increased demand of resources, therefore people dedicated to the exploitation will also augment in order to produce the desired supply. Remember that $\beta \in [0, 1]$, so there are the three following supply-demand possibilities.

1. $\beta = 1$: all people demand resources, that is to say that harvesting is a profitable economic activity because of the abundant resources. Consequently, no agriculture is carried out, and system (2-2) becomes the system (2-6)

$$\begin{cases} \dot{S} = \rho \left(\frac{S}{k} - 1\right) \left(1 - \frac{S}{K}\right) S - \alpha L S; \\ \dot{L} = \gamma \left(\overline{\phi} \alpha S - \overline{\sigma}\right) L. \end{cases} \tag{2-6}$$

In this case $P4 = P1 = (0, 0)$ and $P5 = P2 = (0, k)$ (see figure **2-10**), then only the unstable focus $P6$ exist at

$$P6 = \left(\frac{-\rho \left(-\overline{\sigma} K \overline{\phi} \alpha + \overline{\sigma}^2 + kK \overline{\phi}^2 \alpha^2 - k\overline{\sigma}\overline{\phi}\alpha\right)}{\overline{\phi}^2 \alpha^3 kK}, \frac{\overline{\sigma}}{\overline{\phi}\alpha}\right);$$

2. $\beta = 0$: all people demand agricultural products, and renewable resources can growth freely without human exploitation. This condition generates the set of two disengaged differential equations (2-7)

$$\begin{cases} \dot{S} = \rho \left(\frac{S}{k} - 1\right) \left(1 - \frac{S}{K}\right) S; \\ \dot{L} = \gamma \left(\lambda L^{\delta - 1} - \overline{\sigma}\right) L, \end{cases} \tag{2-7}$$

and internal equilibria are

$$
\begin{aligned}
P5 &= \left(\left(\tfrac{\lambda}{\bar{\sigma}} \right)^{1/(1-\delta)}, k \right), \\
P6 &= \left(\left(\tfrac{\lambda}{\bar{\sigma}} \right)^{1/(1-\delta)}, K \right).
\end{aligned}
\tag{2-8}
$$

3. $0 < \beta < 1$: some people prefer resources and the rest prefer agricultural products, that is to say that human subsist from both economic activities: harvesting and agriculture, system (2-2).
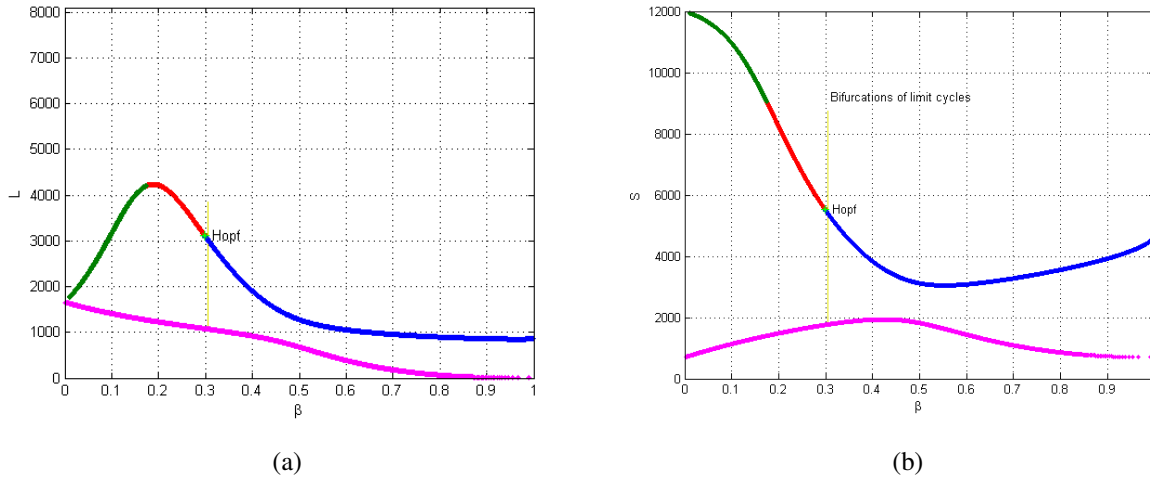

(a)   (b)

Figure 2-9: One-parameter bifurcations for β.

An increase of β means an increase of income in a first stage; nevertheless, once resources have been extensively harvested the system loses stability changing from stable node to stable focus and then, after a Hopf bifurcation, oscillatory behaviour appears and system falls into feast and famine cycles to finally becomes unsustainable through bifurcations of limit cycles. Further simulations showed that saddle-node bifurcation can also be obtained for parameter values different from Table **2-1**.

One parameter bifurcations of limit cycles are summarized in Figure **2-10**. Before homoclinic bifurcation and after the Hopf bifurcation there are the two basins of attraction bounded by the unstable manifold of the saddle as shown in Figure 2.10(a). $B1$ contains all initial conditions that converge to the stable limit cycle while $B2$ contains all initial conditions that approach to $P4$. Figure 2.10(b) presents the appearance of an homoclinic orbit where the stable and unstable manifolds coincide. This phenomenon reduces the basin of attraction $B1$ an only initial conditions inside the homoclinic orbit converge to the stable limit cycle. Slightly increasing β beyond the bifurcation value, a new limit cycle appears (Figure 2.10(c)). Finally limit point cycle takes place when both stable and unstable limit cycle collide, Figure 2.10(d). After collision it is not possible to obtain long-run dynamics with positive resources and all trajectories approaches to $P4$.
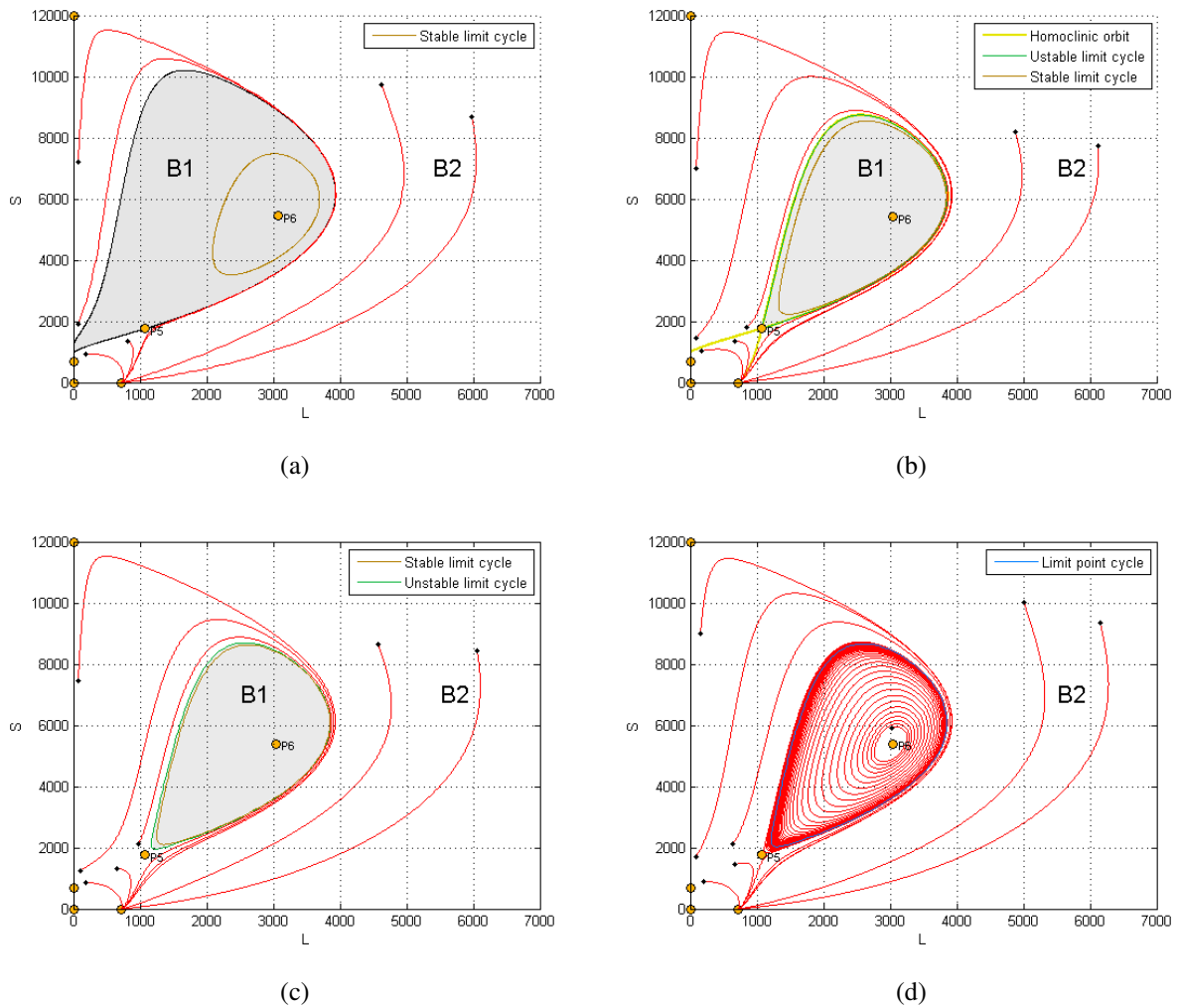
(a)

(b)

(c)

(d)

Figure 2-10: One-parameter bifurcations of limit cycles: (a) basins of attraction before the homoclinic orbit, (b) basins of attraction at homoclinic orbit, (c) coexistence of stable and unstable orbits, and (d) collision of orbits.

2.8.   One-parameter bifurcations for other important parameters

The following general results show in a stylized way sensitivity of the system when other important parameters are varied. Consider that parameter values are such in Table **2-1** and system approaches the stable limit cycle in Figure **2-3**. At certain point in time the parameter value is disturbed, changing the basins of attraction and modifying the path of convergence. Change in steady state when varying other parameters are summarized in the next graphics.

Changes in land fertility λ (for example the use of agrochemicals) affect only the dynamics of population. Therefore, population level increases when λ increases. An increasing population in-

volves that more people demand resources (βL) and agricultural products ($(1 - \beta)L$). According to Figure **2-11** the change of this parameter has the same effect as an increase in α because an increment in land fertility increases the yield of the agricultural sector and total income. Finally, an increase in population level means more potential consumers of renewable resources and again the system becomes unsustainable due to the existence of one-parameter bifurcations.
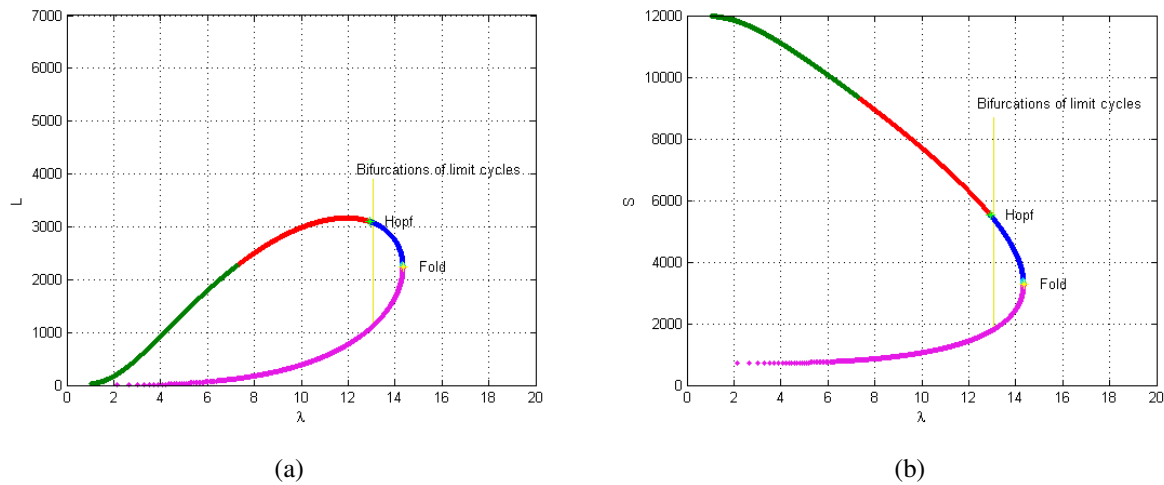

(a)   (b)

Figure 2-11: One-parameter bifurcations for λ

Regeneration rate of the resource ρ is another important parameter of the system linked to the care of land. For instance, reducing environmental damage with less external supplies replacing traditional agricultural practices for agroecological practices such as reducing single crop farming or abandon the use of agrochemical products, (see [Altieri, 1999]). This parameter is also related to the amount of land that is used for agriculture since it is not very often to recover lands for reforestation either for the interest of the owners or the loss of fertility. Figure 2.12(b) presents continuation curve when varying ρ. Positive renewable resources and population are obtained in the long by increasing ρ, but there is a minimum land regeneration rate where consumed resources are higher than natural production and resources extinction is inevitable. Different from α, β, and λ, one-parameter bifurcations occurs for decreasing parameter ρ.

2.9.   Two-parameter bifurcations

So far, this work have presented analysis of the possible long-run behaviour of population and resources by using one-parameter bifurcation analysis. This section expand results by introducing two-parameter bifurcation analysis allowing to trace the system dynamics even if not just one parameter but two parameters of the model change simultaneously. For that, examination of parameters that determine the conditions where the system becomes unsustainable is carried out.
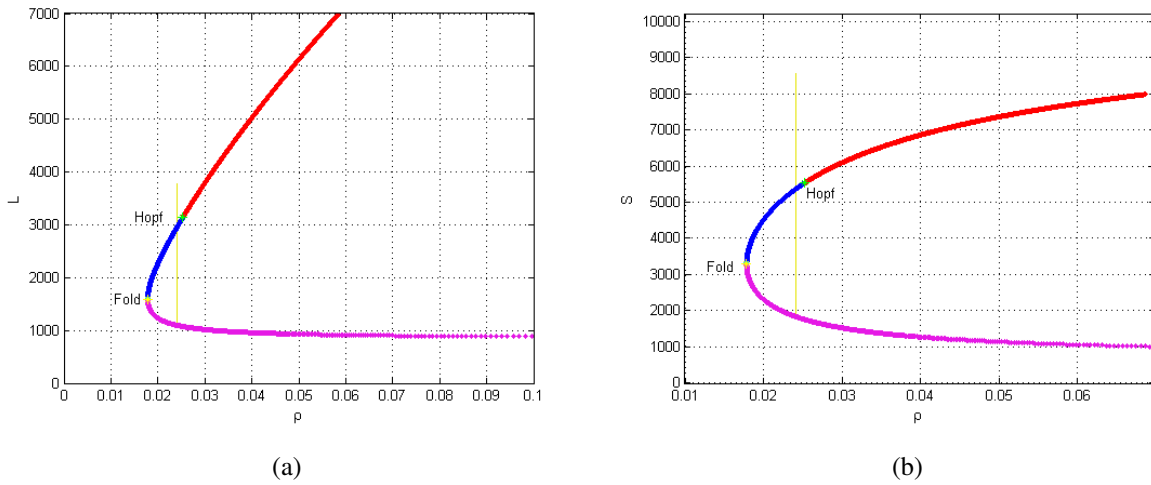
(a)   (b)

Figure 2-12: One-parameter bifurcations for ρ

2.9.1.   Existence of Bautin bifurcation

Existence of Hopf bifurcations allows to construct a Hopf bifurcation curve. For the interest of the work parameters α and β are varied simultaneously in order to obtain long-run dynamics where a combination of both parameters approach to stable limit cycles. Bautin (or generalized Hopf) bifurcation occurs when the first Lyapunov coefficient is equal to zero $l_1 = 0$, meaning that positive and negative values of l_1 may exist. If the value of l_1 is positive the bifurcation is subcritical and the associated limit cycles are unstable, but if the value is negative the bifurcation is supercritical and associated limit cycles are stable (see for example [Kuznetsov, 2004] and [Guckenheimer & Holmes, 1983]).
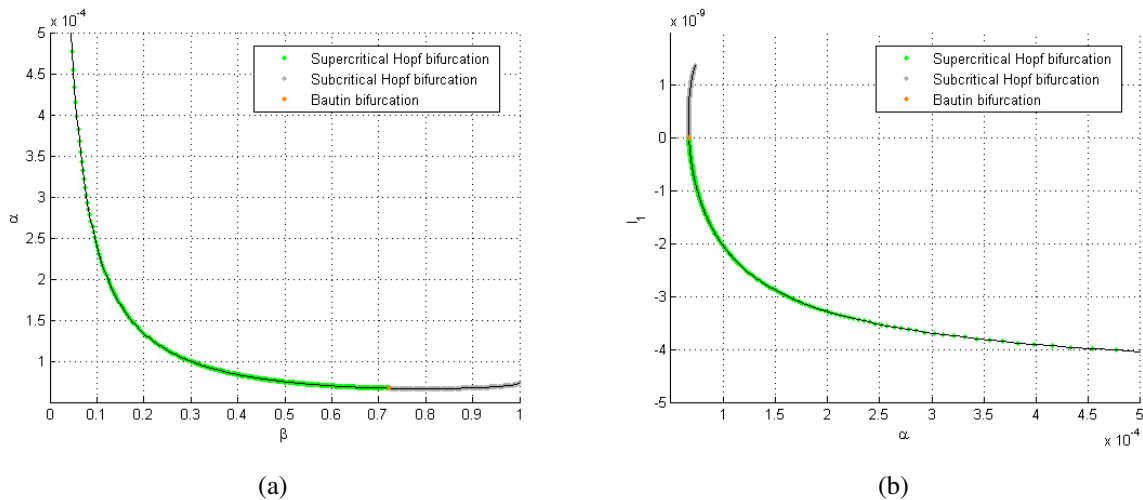

(a)   (b)

Figure 2-13: Hopf curve: (a) on the $\beta - \alpha$ plane, (b) existence of Bautin bifurcation.

Figure 2.13(a) shows the continuation of Hopf bifurcation in the parameter space $\beta - \alpha$, and Figure 2.13(b) present the transition from supercritical to subcritical Hopf bifurcations where $l_1 = 0$. If a supercritical Hopf bifurcation occurs, oscillatory behaviour is possible and feast and famine cycles exist for slight changes in parameter values. When a subcritical Hopf bifurcation occurs oscillatory behaviour is not possible and slightly increasing the parameter beyond the Hopf point the associated limit cycle is unstable and system cannot approaches to it. Consequently $P4$ becomes globally stable.

Other curve of Hopf bifurcations is that obtained when varying simultaneously $\overline{\sigma}$ and α. Results in Figure 2.14(a) suggest that as $\overline{\sigma}$ increases the allowed level of technology may increase before the feast and famine cycles appear. For instance, changes in the habits of consumption may increase the subsistence level $\overline{\sigma}$ (different countries have different misery lines according to the populations habits and economic activities), that is to say that while misery increases population decreases which is reflected in a faster regeneration of resources allowing the incorporation of new technologies. Absence of Bautin bifurcation in Figure 2.14(b) involves the existence of feast and famine cycles for any value of $\overline{\sigma}$ and α.

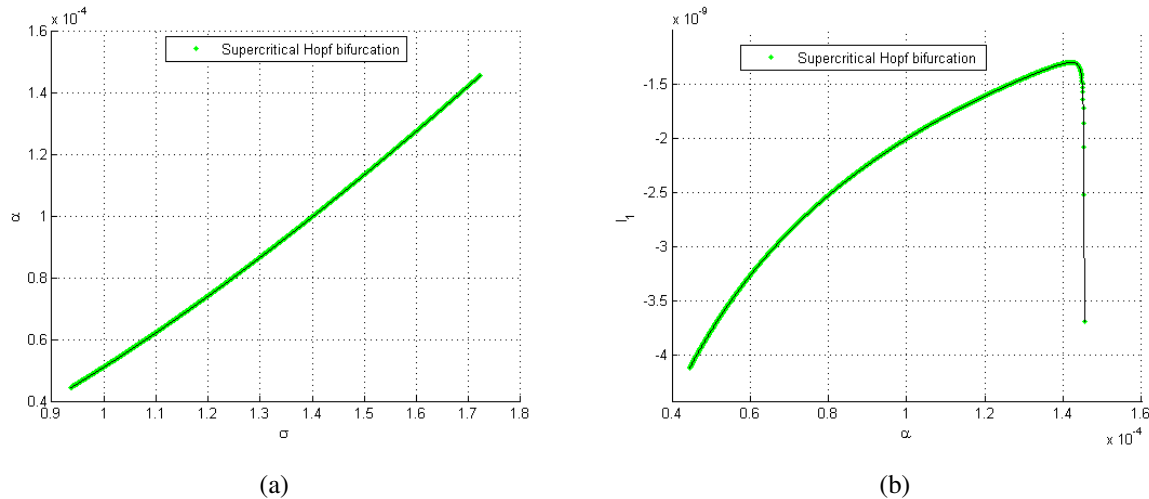(a)   (b)

Figure 2-14: Hopf curve: (a) continuation of Hopf bifurcation on the $\sigma - \alpha$-plane, (b) no existence of Bautin bifurcation.

2.9.2.   Existence of Bodganov-Takens bifurcation

This phenomenon occurs when a saddle-node curve and a Hopf curve collide. The equilibrium at that point satisfy conditions either of saddle-node or Hopf bifurcations (see [Kuznetsov, 2004]). This bifurcation is characterized by the appearance in two-parameter families and two eigenvalues equal to zero $\lambda_1 = \lambda_2 = 0$. Results show that this collision take place for negative values of α and β when parameter values are those in Table **2-1**, as shown in Figure **2-15**. After the collision, Hopf curve ends at the singularity which is another characteristic of the bifurcation. Further simulations

could not find out this bifurcation for the parameter values of the system, losing importance for the objectives of this work.
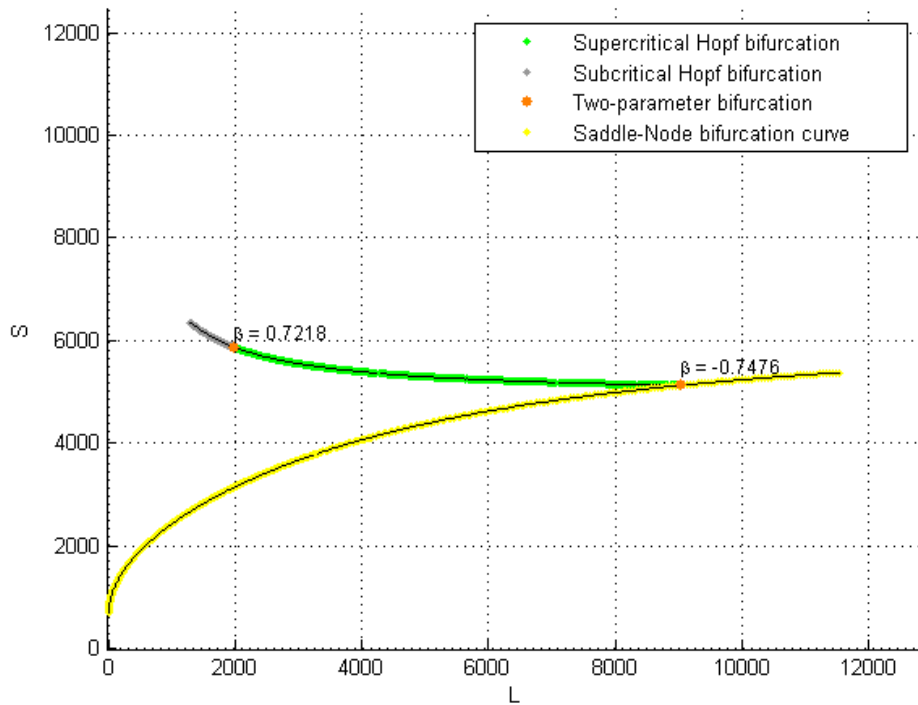

Figure 2-15: Two-parameter bifurcations.

2.10. **Mathematical model for preference as a function of resources.**

Suppose that high level of resources increases the supply and therefore the income spent in wood (demand). Now, consider a threshold of resources T where β begins to decrease because of resource reduction. Section 2.5 describes sensitivity to initial conditions in a stationary environment in which all parameters of the model are constant. This suggest that over time preference parameter β is constant and so does the portion of income spent on consuming natural resources. The current section introduces the fact that preference decline when renewable resources level is reduced. For this purpose and algebraic function of β that depends on the resource level is presented. Consider the dynamic interaction between population and natural resources in (2-2) but now β is a function

of S

$$\begin{cases} \dot{S} = \rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right)S - \alpha\beta\left(S\right)LS; \\ \dot{L} = \gamma\left(\lambda(1 - \beta\left(S\right))^{\delta}L^{\delta-1} + \overline{\phi}\alpha\beta\left(S\right)S - \overline{\sigma}\right)L; \end{cases} \qquad (2\text{-}9)$$

where $\beta(S)$ is the algebraic expression

$$\beta\left(S\right) = \beta_m\left(\frac{S}{K}\right)\left[\left(1 - \frac{S}{K}\right) - \left(\frac{S - K}{K} - 1\right)\right]; \qquad (2\text{-}10)$$

and $\beta_m = 0{,}3$ is the β average. According to Figure **2-16** there is a value of S where preference begins to decrease. i.e. $T = \frac{3K}{4}$. That is to say that if resources level is below $3/4$ of their carrying capacity the supply begins to decrease.
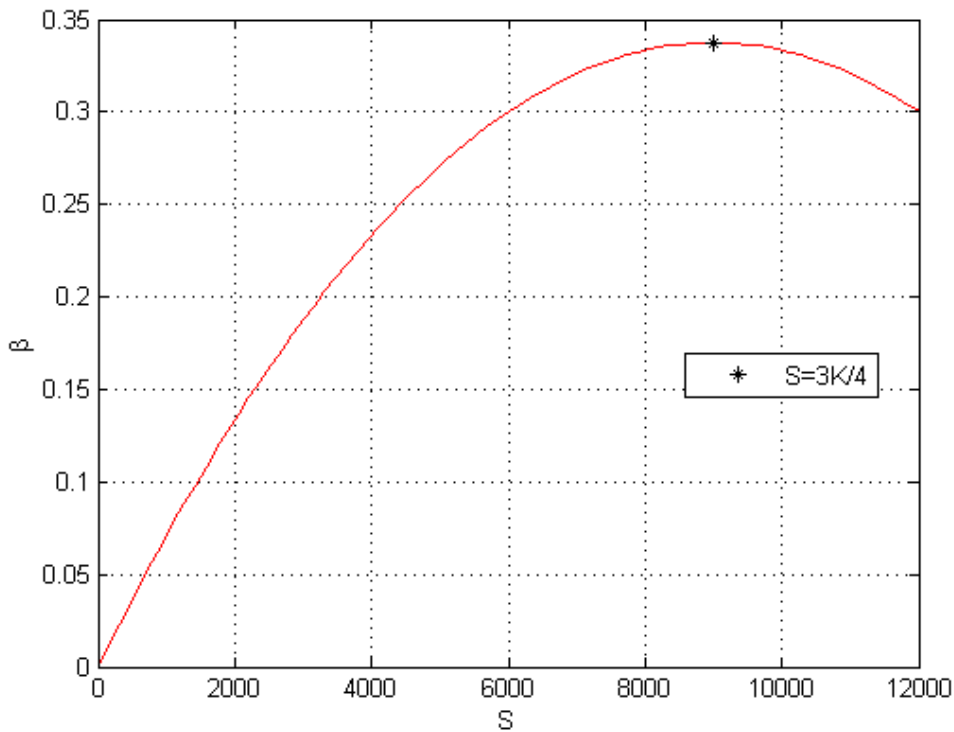

Figure 2-16: β as a function of resources.

$S = K$ involves that $\beta = \beta_m$. As soon as resources begin to be exploited, there are two stages of consumption according to Figure **2-16**. The first one presents increasing preference due to the resource abundance, and the second one presents reduction of preference due to the reduction of resources level, meaning that a reduced level of resources is not the better source of income and preference for agriculture $(1 - \beta)$ tends to increases.

Phase portrait of this system is presented in Figure 2.17(a) and again nullclines help to find out internal equilibria. Comparing the result with that in Figure **2-3** there is a significant difference, that is the disappearance of the limit cycle. This means that auto-regulation of preference for resources reduces the impact on the environment by means of a reduction in the loss of stability. Temporal evolution of the system is shown in Figure 2.17(b) in order to check the relation between β and S: at carrying capacity resources begins to decrease while β augment, once the the threshold T is crossed both preference and resources behave the same way.
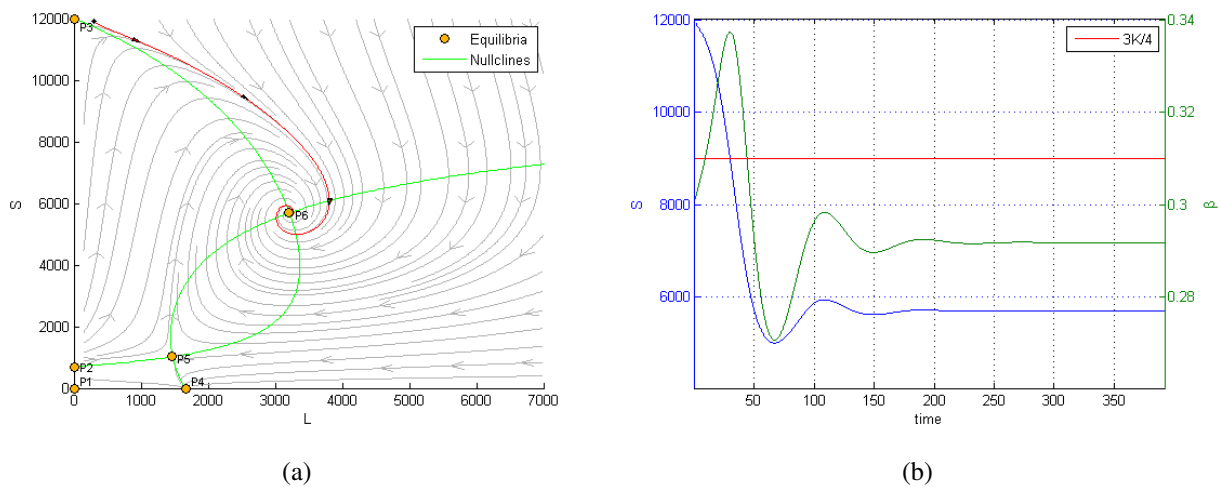

(a)

(b)

Figure 2-17: Evolution of the system when β depends on resources level: (a) phase portrait and (b) temporal evolution.

2.10.1. Sensitivity to initial conditions

The introduction of β as a function of resources level affects the basins of attraction of the system. In section 2.5 two attractors were found and depending on the initial conditions the system approaches to one or the other in the long-run. When $\beta(S)$ is introduced, these basins of attraction change as shown in Figure **2-18**. The main characteristic of this result is the expansion of basin of attraction $B1$, meaning that it is possible to obtain long-run behaviour with positive population and no resources exhaustion for high initial population level.

2.10.2. Continuation of equilibrium points

In section 2.6 dynamical analysis was developed by using bifurcation theory. Results suggest that an increasing technological progress α in the harvesting sector may cause the resources extinction through a set of one-parameter bifurcations of equilibria and limit cycles. Following results confirm the fact that auto-regulations of preference have positive effect on long-run dynamics. The equilibrium curve in Figure 2.19(a) presents the changes in stability of internal equilibria. The
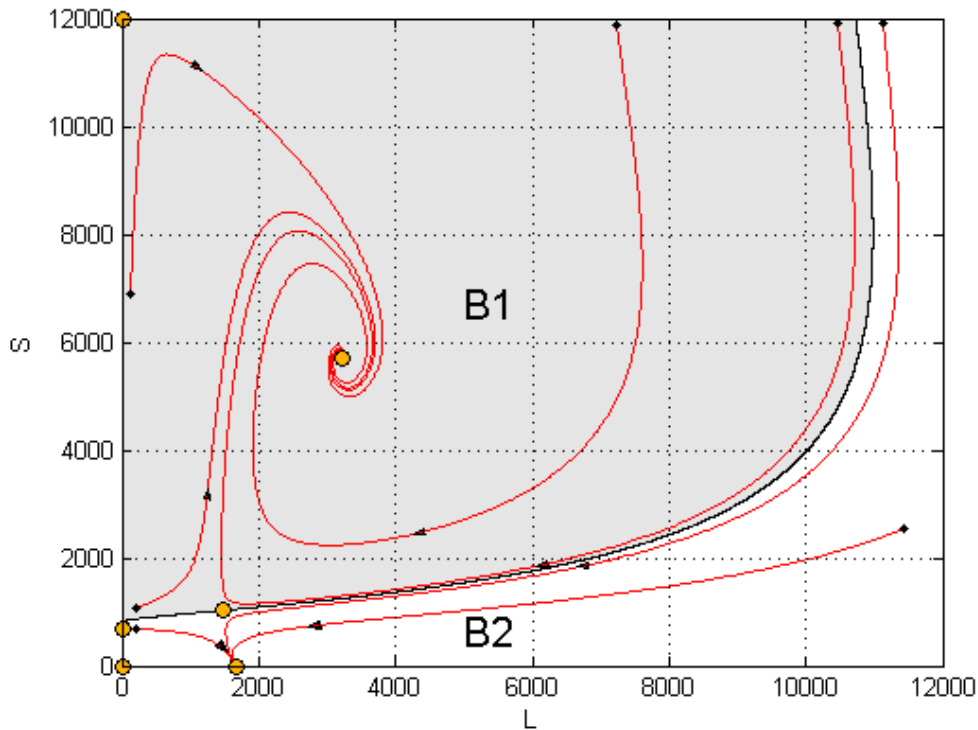
Figure 2-18: Basins of attraction for $\beta(S)$.

inferior branch which correspond to $P5$ is always a saddle while the superior one $P6$ changes stability, but never becomes unstable. Implications of this behaviour falls in the absence of limit cycles, consequently Hopf bifurcation does not occur. At the end both equilibria collide and disappear through a saddle-node bifurcation. The disappearance of internal equilibria implies that $P4$ becomes globally stable and only basin of attraction $B2$ exists. On the other hand, when varying regeneration rate ρ (Figure 2.19(b)) the result confirms the non existence of limit cycles and again unsustainability is reached at saddle-node bifurcation.
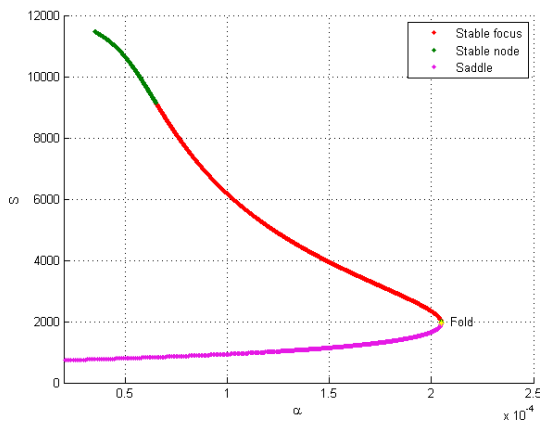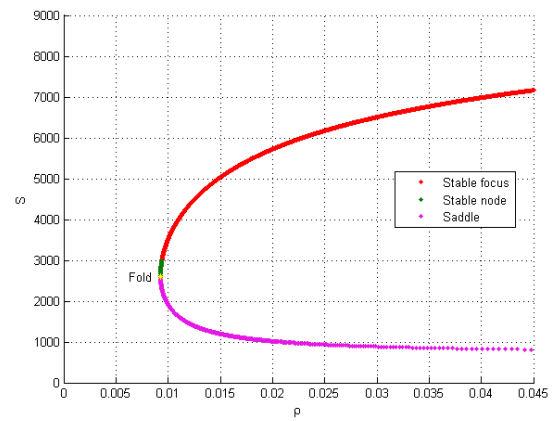
(a)   (b)

Figure 2-19: Continuation of equilibrium points: (a) when varying α, (b) when varying ρ.

3 Technological progress dynamics

In chapter 2 long-term dynamic interaction between the exploitation of natural resource and population growth was studied. In that model technology used in harvesting (α) was a system parameter and results showed the negative effect of improving this kind of technology. The aim of this section is to introduce a new differential equation which represents the dynamic behaviour of α.
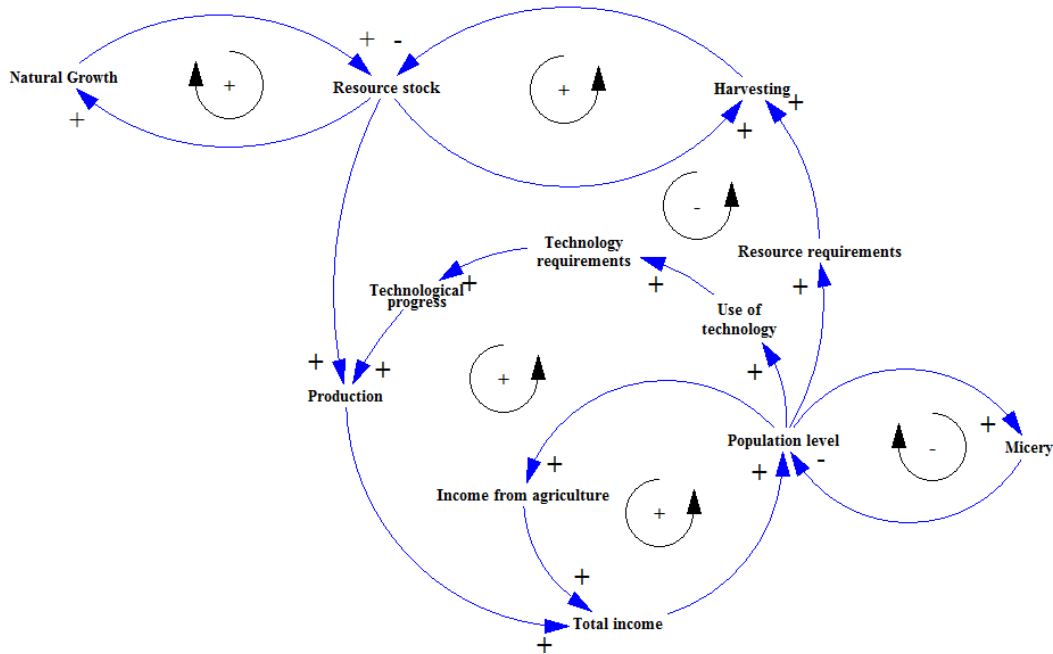
Figure 3-1: Causal diagram of renewable resources and population with endogenous technological progress.

According to the endogenous growth theory, demand induces technological progress, then technological progress depends on population level. An increase in population leads an increase in technology since a higher population means more potential inventors, [Kremer, 1993]. Introducing this assumption a new causal diagram is obtained. Figure **3-1** contains a new positive feedback cycle representing the dynamics of technological progress. An increase in population level involves an increasing use of the current technologies. Then, technology requirements may also augment. Finally, when technological progress is carried out and an increase in production is obtained. Endogenous technological progress as is described in causal diagram **3-1** can be modelled in a general

way for the set of differential equations (3-1)

$$\begin{cases} \dot{S} = G\left(S\right) - H\left(L, S, \alpha\right), \\ \dot{L} = E\left(L\right) + F\left(L, S, \alpha\right), \\ \dot{\alpha} = I\left(L, \alpha\right). \end{cases} \tag{3-1}$$

3.1. Mathematical model

[Angulo et al., 2009] proposed a production function for harvesting technology. This model depends on current technological state α, the minimum amount of people required to develop this technology $L_{\text{mín}}$, total population L, and production factors δ_2 and k_L. The final set of three coordinated differential equations is system (3-2).

$$\begin{cases} \dot{S} = \left[\rho\left(\frac{S}{k} - 1\right)\left(1 - \frac{S}{K}\right) - \alpha\beta L\right] S; \\ \dot{L} = \gamma\left(\lambda\left(1 - \beta\right)^{\delta} L^{\delta-1} + \overline{\phi}\alpha\beta S - \overline{\sigma}\right) L \quad ; \\ \dot{\alpha} = k_L \alpha L^{\delta_2}\left[\frac{L - L_{\text{mín}}}{L_{\text{mín}}^2 + (L - L_{\text{mín}})^2}\right]. \end{cases} \tag{3-2}$$

3.2. Equilibrium points

This system presents five equilibria inside the positive octant. Last two: $P4$ and $P5$ are the coordinates $(L_{\text{mín}}, S_{5,6}, \alpha_{5,6})$ where $S_{5,6}$ and $\alpha_{5,6}$ have to be obtained numerically by solving system (3-2) equalled to zero; other three equilibria are trivial equilibrium and correspond to the next solutions (L, S, α)

$$P1 = (0, 0, 0).$$
$$P2 = \left(\left(\frac{\lambda(1-\beta)^{\delta}}{\overline{\sigma}}\right)^{1/(1-\delta)}, 0, 0\right),$$
$$P3 = \left(\left(\frac{\lambda(1-\beta)^{\delta}}{\overline{\sigma}}\right)^{1/(1-\delta)}, k, 0\right),$$
$$P4 = \left(\left(\frac{\lambda(1-\beta)^{\delta}}{\overline{\sigma}}\right)^{1/(1-\delta)}, K, 0\right).$$

Three unstable subspace also exist:

$$S1 = (0, 0, \alpha);$$
$$S2 = (0, k, \alpha);$$
$$S3 = (0, K, \alpha).$$

3.3. Stability of equilibrium points

Subspaces affect the basins of attraction and so the system dynamics. However, their unchangeable stability allows a simpler analysis of the non-linear phenomena and the interest is focused in stability of equilibria. According to the trivial equilibria, system never approaches to a long-run with population extinction because as in the planar system (2-2), in absence of renewable resource population survives from agriculture.

Trivial equilibria have invariable stability, then $P2$ is always unstable while $P1$ and $P3$ are stable. On the other hand internal equilibrium $P5$ is always unstable while $P4$ have a variable stability depending on parameter values. Therefore, the system evolution may approach to one of the attractors: if resources has been completely extinguished $S = 0$, a low population may survive from agricultural activities, but if people decide not to use renewable resources as an economic activity, they can also survive from agriculture preventing resources disappearance $S = K$. If system falls into the attractor associated $P4$, both economic activities are carried out simultaneously and long-run dynamics with positive population, resources, and technological progress if obtained.

Considering a stationary environment with constant parameters values reported in Table **2-1** and Table **3-1**, and a initial condition $(200, 12000, 0,0001)$. The phase portrait of system (3-2) is that in Figure **3-2**.
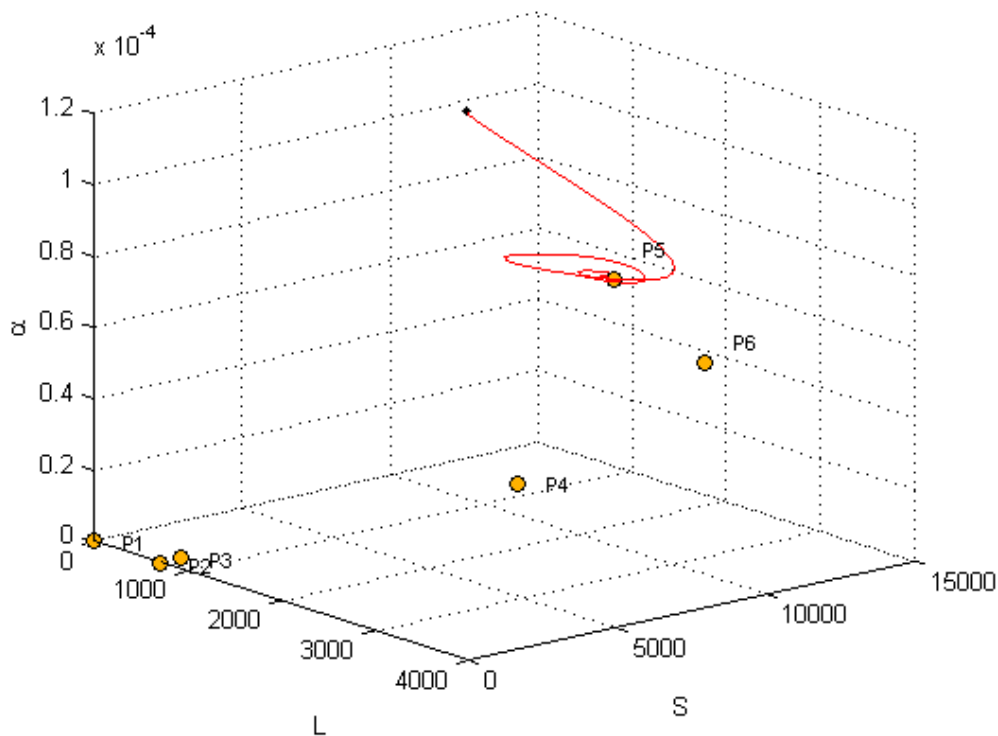

Figure 3-2: Phase portrait of system (3-2) with initial condition at $(200, 12000, 0,0001)$.

3.4.   Sensitivity to initial conditions

All trajectories of the system are associated to one of the three attractors $P1$, $P3$, and $P4$. Basins of attraction in this model are not too easy to calculate as in the planar system; nevertheless, three 2D-projections are drawn and slices of the global basins of attraction are calculated for a constant third initial condition as shown in Figure **3-3**. For the initial value of $\alpha = 0{,}0001$ Figure 3.3(a) shows a slight difference with regard to Figure **2-4**. $B1$ is the region containing all initial conditions of population and renewable resources that converge to $P4$ which is the internal attractor where population survives from both economic activities. As in the planar system there is a maximum initial population that renewable resources can support in the long-run, if initial population is beyond this separatrix, resources tend to the extinction.

Parameter	Value
$L_{\text{mín}}$	3500
δ_2	0.01
kl	15

Table 3-1: Parameters for technology dynamics.

Given an initial condition $L = 200$ Figure 3.3(b) explores how the initial value of resources and technology affect the long-run behaviour. Results suggest that resources extinction is a consequence either of low initial resources level (below the Allee threshold) or high initial technology ($B2$). On the contrary, if initial resources level is higher than Allee threshold, and initial technology is as low that resources harvesting is not a profitable economic activity ($B3$), the behaviour in the long-run suggests that people finally abandon harvesting and survive from agriculture while resources tend to their carrying capacity (equilibrium point $P3$). For moderate initial technology and initial resources above the Allee threshold ($B1$), it is possible to approaches the internal attractor $P4$ where long-run with positive variables are found. Finally, Figure 3.3(c) presents initial conditions of population and technology for $S = K$. Results involves that regardless of the level of population, increasing initial technology affects dramatically the long run dynamic of resources.

3.5.   Steady state solutions

This section studies the steady-state dynamics of system (3-2) for different values of parameters. First, three different phase portraits are used to present some interesting non-linear phenomena that take place when parameters are varied.

According to results in section 3.4 for constant parameter values system approaches to an attractor or another. Since basins of attraction $B2$ and $B3$ have trivial steady states (the system converges to the trivial equilibria $P1$ and $P4$ respectively) the analysis is restricted to the study of the long-term dynamics with positive values of variables. For this purpose, only initial conditions in $B1$
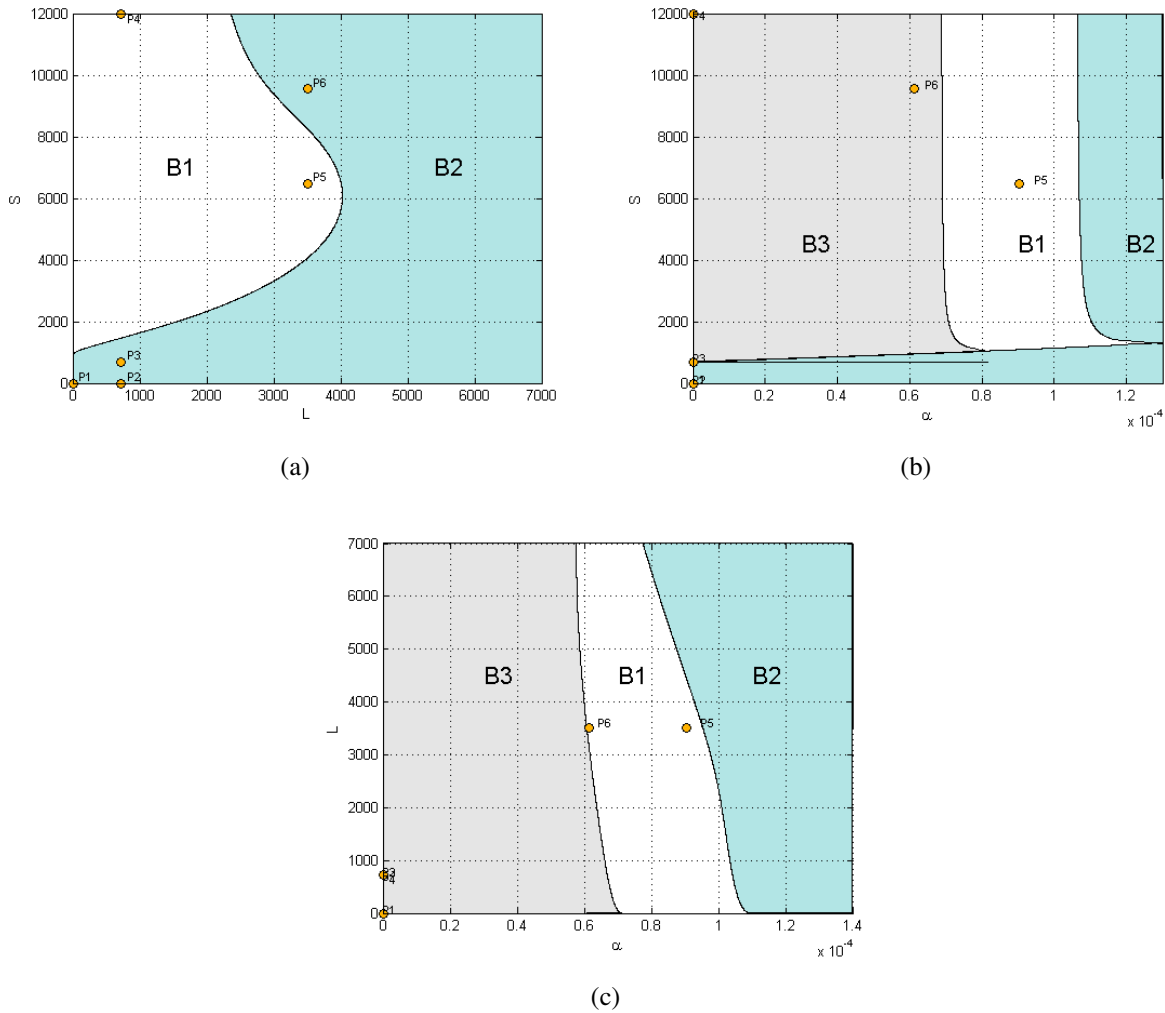
(a)

(b)

(c)

Figure 3-3: Basins of attraction: (a) initial condition $\alpha = 0,0001$, (b) initial condition $L = 200$, (c) initial condition of $S = 12000$

are taken into account. Figure 3.4(a) is the typical case where the attractor is a stable equilibrium while Figures 3.4(b) and 3.4(b) presents periodic solutions either a limit cycle or quasiperiodic orbits. An orbit is said to be N-quasiperiodic if N is the number of frequencies, in this case an orbit 8-quasiperiodic is obtained. Chaotic behaviour of the system also occurs as shown in Figure 3.4(d) and no long-term prediction is possible as in periodic and quasiperiodic solutions.

3.5.1. Bifurcation diagrams

Results in previous section 3.5 can be presented in a stylized way by constructing bifurcations diagrams. These diagrams are constructed through *Poincaré maps* which are return maps of a vector field (see for example [Lee et al., 2008]) and are mainly used to study attractors. The main
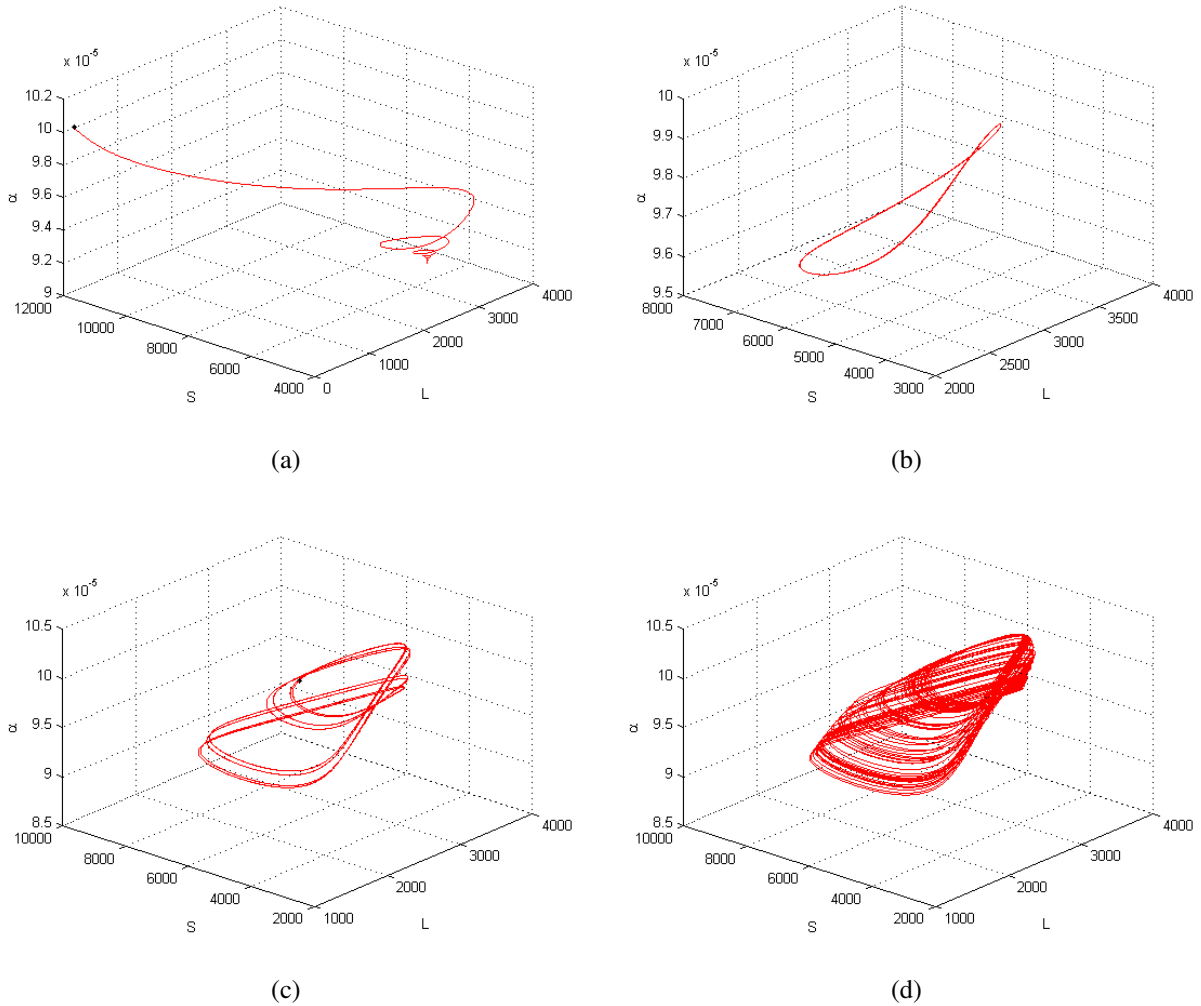
Figure 3-4: Phase portraits of system (3-2): (a) $\beta = 0,3$ and $L_{min} = 3500$, (b) $\beta = 0,3$ and $L_{min} = 3000$, (c) $\beta = 0,3027$ $L_{min} = 2800$, and (d) $\beta = 0,2997$ and $L_{min} = 2800$.

problem is the choice of the adequate section. The section used in the current work is a plane parallel to the $\alpha - S$ plane and pass through L_{min}. This chose guarantee that trajectories in basin of attraction $B1$ cross the plane since solutions are associated to internal equilibria $P4$. An illustration of the section used for the steady state analysis is presented in Figure **3-5**. Following the flow of the vector field in steady state, the Poincaré map is constructed by all the intersections of the flow in the same direction. Therefore, bifurcation diagram can be constructed by concatenating all the solutions when varying one parameter.

Parameter values affect the long-term dynamics of the system by modifying the basins of attraction. Then, it is necessary to take into account the initial conditions when a bifurcation diagram is constructed. There are two options: to take the same initial condition for each parameter value which no guarantee the fall into the desired attractor, or to take an adequate initial condition in
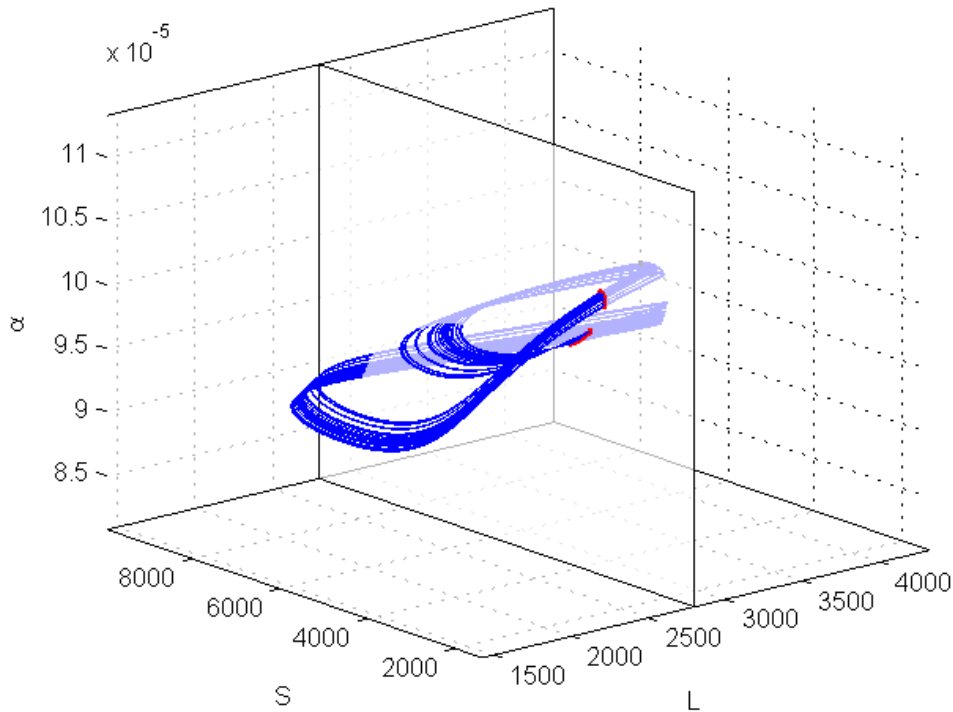
Figure 3-5: Phase portrait of system (3-2) with a Poincaré section.

order to follow the attractor and avoid the falling into other basin of attraction. For the objective of
the work, bifurcations diagrams were constructed by following the attractor.

Steady state of two parameters are studied: the minimum amount of people required to develop
technology $L_{\text{mín}}$ and preference parameter β. Bifurcation diagrams in Figure **3-6** show the evolu-
tion toward chaos with decreasing L_{min}. Three regions with different steady state are identified.
Region C is characterized for the convergence to stable equilibria, namely $P4$ is stable in that
subinterval. An increase in L_{min} means more potential inventors, then α increases in a first stage.
However, an increasing technology involves faster and efficient harvesting reducing resources le-
vel and per-capita income. Finally population level declines and so does technological progress.
In conclusion, decreasing L_{min} implies an increase of resources level (Figure 3.6(c)) and a reduc-
tion in technological progress (Figure 3.6(a)) when the steady-state is reached. The single line in
region B correspond to the fixed points of periodic orbits of the Poincaré map. This means that $P4$
becomes unstable and the system becomes oscillatory. When L_{min} is at the level of region A some
interesting non-linear phenomena occurs while L_{min} keeps decreasing. Figures 3.6(b) and 3.6(d)
shows the existence of N-periodic orbits obtained through doubling-period bifurcations. At the end
the system becomes very sensible to slight variations of the parameter and initial conditions and
falls into a chaotic attractor.

Analogous results are obtained when varying preference parameter β. Since N-periodic orbits are
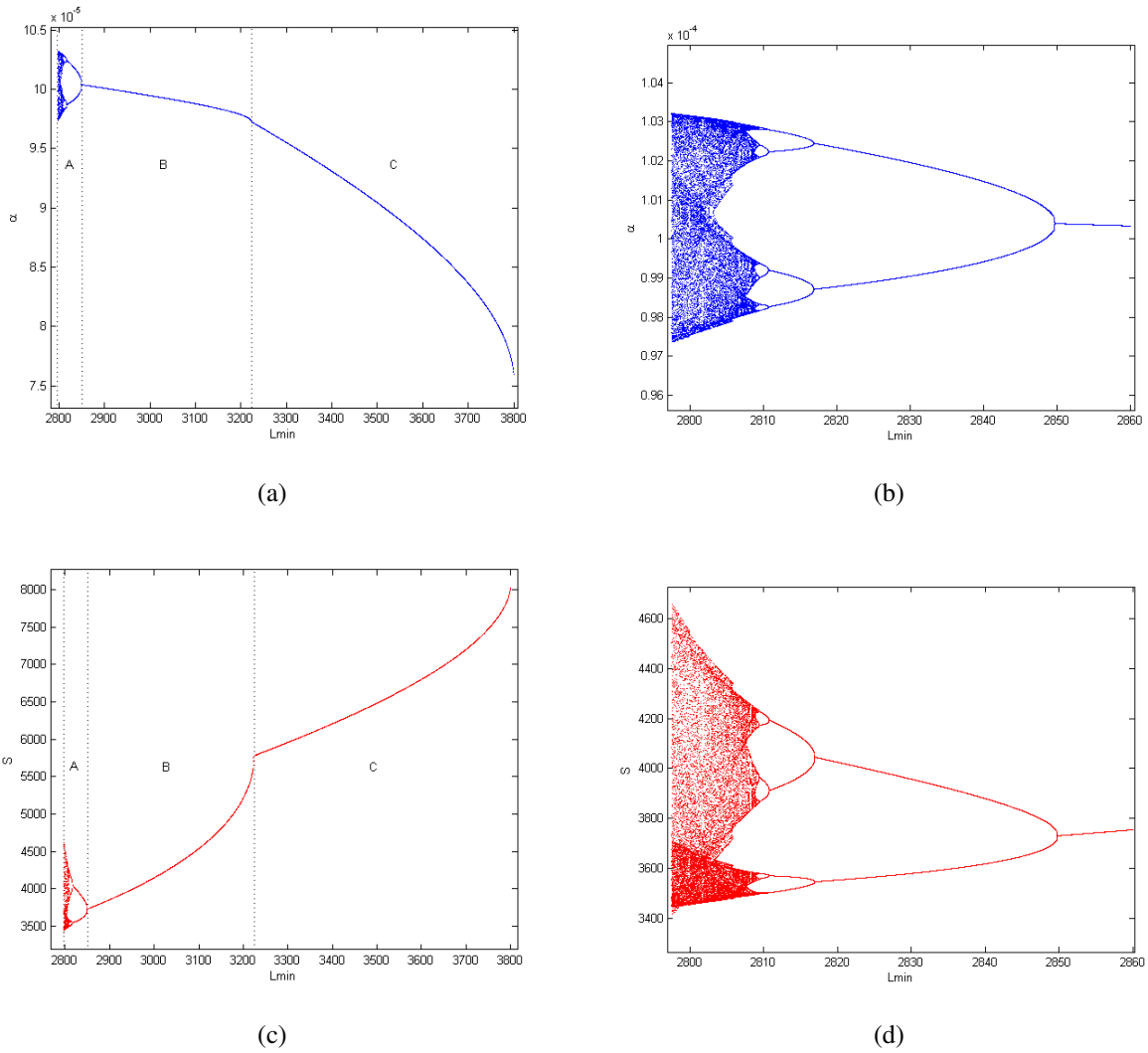
(a)

(b)

(c)

(d)

Figure 3-6: Bifurcation diagrams for L_{min}.

generated through doubling-period bifurcations to finally fall into the chaotic attractor as shown in Figure **3-7**.
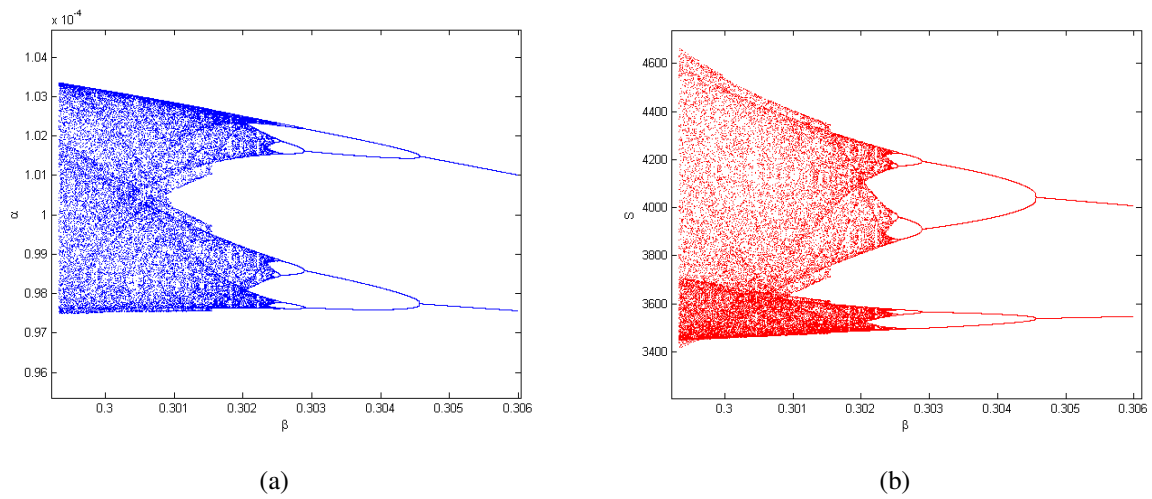
<center>(a)   (b)</center>

Figure 3-7: Bifurcation diagrams for β.

4 A piecewise smooth model

The interest for models in which evolution of the variables is matching by sudden changes in behaviour has been increasing during the last years. Some examples of these systems are focussed to study plankton blooms, insets pests in forest, and populations with selective switching between different habitats. There are different classes of discontinuous system (see Chapter 1) whose analysis required different methods. The system discussed in this work belongs to the *piecewise smooth* class which are often called Filippov systems and are describe for standard differential equations.

In the last 15 years problems in ecology and populations have been approached through Filippov systems ([Krivan, 1996], [Krivan, 1998], [Krivan & Sikder, 1999], [Krivan & Eisner, 2003], [Dercole et al., 2007]). Populations in these models switch between alternative habitats or diets depending on their size in order to control the density of some population. Filippov systems have also been used to simulate the exploitation of natural resources if harvesting is prohibited when a critical threshold is reached ([Dercole et al., 2003], [Costa & Meza, 2006]). These result suggest that stocks could be preserved under strict political regulations.

Non-linear systems in Chapters 2 and 3 are defined by sets of smooth ordinary differential equations whose vector field are continuous. This chapter study the long-term behaviour of the interaction between population and renewable resources when smooth systems are replaced by Filippov systems.

4.1. Mathematical model

Consider the planar system of smooth differential equations (2-2) and suppose that there is an amount of renewable resource R such that below that threshold it is not possible to continue with the excessive harvesting of the renewable resource. Once the R value is crossed, the ecological complex begins to recovery by means of β reduction. The boundary R in this case is a constant, so the discontinuity occurs in a straight line parallel to the population axes. This boundary divides the state space into two regions 1 and 2 which have their own set of smooth differential equations.

In section 2.10 a proposed function of β depending on level of resources was presented. Nevertheless, since the objective is to study the effect of excessive harvesting, a simplest form of $\beta(S)$ can be used obtaining similar results.

According to the restriction (4-1) while $S > R$, preference for resources stay at the average value

$\beta_m = 0,3$; and if $S < R$, preference id reduced in a linear form depending on resources level.

$$\beta = \begin{cases} \beta_m & if \quad S > R \\ \beta_m \left(\frac{S}{K}\right) & if \quad S < R \end{cases} \tag{4-1}$$

The system (4-2) is the Filippov system where the single boundary Σ and the regions F_1 and F_2 are

$$\begin{aligned} \Sigma &= \{(L,S) : S = R\}, \\ F_1 &= \{(L,S) : S{>}R\}, \\ F_2 &= \{(L,S) : S{<}R\}, \end{aligned} \tag{4-2}$$

then, the Filippov system is described for the standard ordinary differential equations

$$\dot{x} = \begin{cases} f^1(x) & if \quad x \in F_1, \\ f^2(x) & if \quad x \in F_2, \end{cases} \tag{4-3}$$

The analysis of this kind of systems is non-trivial as system (2-2) since $\dot{x}$ are not defined on the discontinuity Σ. First, if components of f^1 and f^2 are transversal to Σ they crosses the boundary if have the same sign. Second, if f^1 and f^2 are transversal to Σ but have different sign the system remains on the boundary and slice on it.

To analyse the behaviour of system (4-3) for different possible values of parameters it is necessary to find out if sliding is possible. For this purpose, the *Utkin's equivalent control method* ([Utkin, 1992]) is used to construct the state portrait of the Filippov system.

4.1.1. Equations for flows that slide

Solutions for system (4-3) are generally constructed by linking together the solutions of $F_{1,2}$ and the sliding solutions on Σ. The discontinuity boundary Σ is totally composed by the crossing set Σ_c and the sliding set Σ_s. The crossing set is the set of all points $x \in \Sigma$, where the two vectors $f^i(x)$ have no transversal components to Σ of the same sign. The sliding region is defined by the left sliding limit T_1 and the right sliding limit T_2 where both vectors f^1 and f^2 are tangent to Σ (see for example [Kuznetsov et al., 2003], [Dercole & Kuznetsov, 2005], and [di Bernardo et al., 2002]).

Σ is described by H which is a smooth scalar function with non-vanishing gradient $H(x) = 0$ on Σ. Following the same notation, population is the abscissa and resources the ordinate, then, $H = [0\,1]$ meets that conditions. The sliding region of the discontinuity set of system (4-3) is given by the portion of boundary of $H(x)$ for which $(H_x f^1) \cdot (H_x f^2) {<}0$. That is to say that $(H_x f^1)$ has opposite sign to $(H_x f^2)$. Thus, the boundary is simultaneously attracting or repelling from f^1 and f^2.

To formulate the equations for flows that slide, the Utkin's method is used. This method supposes that the system flows according to the sliding vector field f^s which is considered the average of the two vector field f^1 and f^2 in regions F_1 and F_2 respectively plus a control $\theta(x) \in [-1, 1]$ in the direction of the difference between the vector fields

$$f^s(x) = \frac{f^1(x) + f^2(x)}{2} + \frac{f^2(x) - f^1(x)}{2}\theta_s(x), \tag{4-4}$$

and f^s must be tangential to the switching manifold, i.e. $H_x f^s = 0$, then

$$\theta_s(x) = -\frac{H_x(f^1 + f^2)}{H_x(f^2 - f^1)}.$$

4.1.2. Sliding region

The sliding region may be defined as:

$$\Sigma_s = \{x \in \Sigma : |\theta_s(x)| \leq 1\}, \tag{4-5}$$

and its boundaries as:

$$T_1 = \{x \in \Sigma : \theta_s(x) = -1\},$$
$$T_2 = \{x \in \Sigma : \theta_s(x) = 1\}. \tag{4-6}$$

Because this boundaries exist on $S = R$, solutions can be obtained by solving for L the equations: $\theta_s(x) + 1 = 0$ and $\theta_s(x) - 1 = 0$. The obtained sliding limits are:

$$T_1 = \frac{\rho\left(RK - R^2 - kK + kR\right)}{\alpha\beta kK},$$
$$T_2 = \frac{\rho\left(RK - R^2 - kK + kR\right)}{\alpha\beta RK}. \tag{4-7}$$

Note that the sliding set exist and depends on values of some parameter, and the value of the prescribed reserve of resources R.

4.1.3. Pseudo-equilibria and singular sliding points

Pseudo-equilibria are internal points of the sliding segment where the vectors f^i are transversal to Σ and anti-collinear. They can be found when $f^s = 0$. Points where $f^s = 0$ and either $f^1 = 0$ or $f^2 = 0$ are called boundary equilibria. Then, sliding limits T_1 and T_2 can be either at a boundary equilibrium, or at a tangent point where the vector $f^i \neq 0$, but one of them is tangent to Σ.
The sliding set Σ_s may also contain *singular sliding points*. These points occurs under three conditions: both vectors f^1 and f^2 are tangent to Σ, one of the vectors vanishes while the other is tangent to Σ (boundary equilibria), or both vectors vanish.

4.1.4. System evolution

To construct the phase portrait of the Filippov system it is necessary to include equilibria in F_1 and F_2. Figure **4-3** shows phase portraits for different values of R. Suppose that the resource reserve is at level of the Allee threshold k, then, $T_1 = T_2$ located at $(0, k)$ and no sliding set is possible. For this condition, the concatenation of vectors f^1, f^2, and f^s is that in Figure 4.1(a). Note that both internal equilibria are in F_1 and qualitative behaviour is similar to that in which no discontinuity exists (Chapter 2). For a higher desired reserve, i.e. $R = 5000$, population and resource coexist on a limit cycle (the existence of periodic solutions confirm the existence of Hopf bifurcation) that contains a sliding phase during which population in harvesting sector decrease allowing the regeneration of resources (Figure 4.1(b)). For a further increase, namely $R = 6000$, a stable pseudo-equilibrium P is present, and therefore long-run behaviour in which resources regenerates at the same rate they are harvested (Figure 4.1(c)) is obtained. Finally, Figure 4.1(d) shows the phase portrait in the case of high desired reserve in a society that present a strong economic dependence from resources. The result involves that even if harvesting is necessary for population to survive, a good distribution of the two economical sectors (harvesting and agriculture) is reflected in high levels of both population and resources. The boundary case where $R = K$ implies that no sliding is possible because again $T_1 = T_2 = 0$. In this case, the Filippov system becomes a smooth system in which f^2 describe the entire behaviour of the system. On the other hand, note that Allee effect exists in all cases, then, resources extinction is always a possibility.

4.1.5. Influence of the reserve level

Previous results are summarised in Figure **4-2**. There, tangent points T_1 and T_2 are calculated for $R \in [k, K]$ and plotted. The region between the T_1 and T_2 curves is the sliding set for the interval of R. According to the figure, pseudo-equilibria exists for different subintervals of R and always are stable. Since the reserve R is a new parameter in the model, small perturbations of it, imply small perturbations of the sliding limits T_1, T_2, and the pseudo-equilibrium P, so that two bifurcations may occur: the collision of P with T_1 and the collision of P with T_2. At these bifurcations, the pseudo-equilibrium becomes a standard equilibrium because $\dot{S} = \dot{L} = 0$.

4.1.6. Influence of technological progress

According to the sliding limits in (4-7), the sliding set Σ_s depends on values of some parameters, i.e. ρ, α, β, k, K, and R. Following results present the influence of small perturbations of technological progress α in the global behaviour. First, suppose that the desired level of resources is $R = 5000$ and standard parameter values are those in Table **2-1**. Under these assumptions the system behaves as in Figure 4.1(b). Now, to check the way α influences the entire behaviour, sliding limits for different values of α are calculated and plotted. Figure 4.3(a) shows that, there is an interval of α where pseudo-equilibria exist and two bifurcation points where sliding limits collide with pseudo-equilibria. Figure 4.3(b) shows the sliding segment $T_1 T_2$ on Σ and the nullclines

(a)

(b)

(c)

(d)

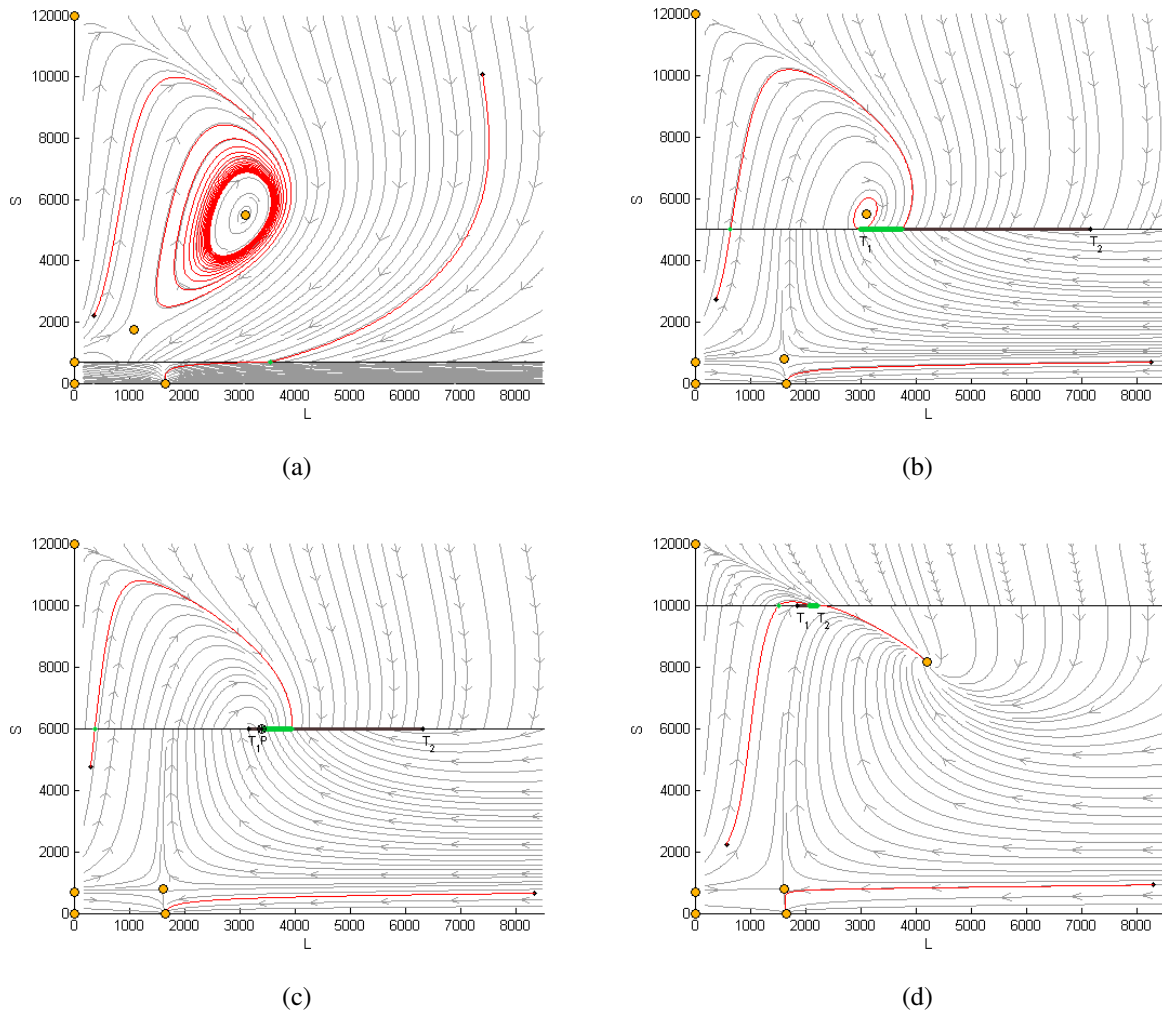**Figure 4-1**: Phase portraits for different values of reserve capacity: (a) R=700, (b) R=5000, (c) R=6000, and (d) R=10000

$\dot{S} = 0$ and $\dot{L} = 0$ for $\alpha = 0{,}00010488$. For that value, system slides and approaches to the stable pseudo-equilibrium at the terminal point T_1. Slightly increasing the parameter beyond the bifurcation value ($\alpha = 0{,}00020367$) the system slides and approaches to pseudo-equilibria that moves on the sliding set. Once the right sliding limit is reached by the pseudo-equilibrium, i.e. $P = T_2$, the other bifurcation takes place and the phase portrait is that in Figure 4.3(c). Finally, with a further increase of α no pseudo-equilibria exists an two standard equilibrium are in F_2 as shown in Figure 4.3(d). These results suggest that protect a reserve of renewable resources has positive effects on population and environment dynamics, since even with an effective harvesting positive levels of resource exist in the long run.
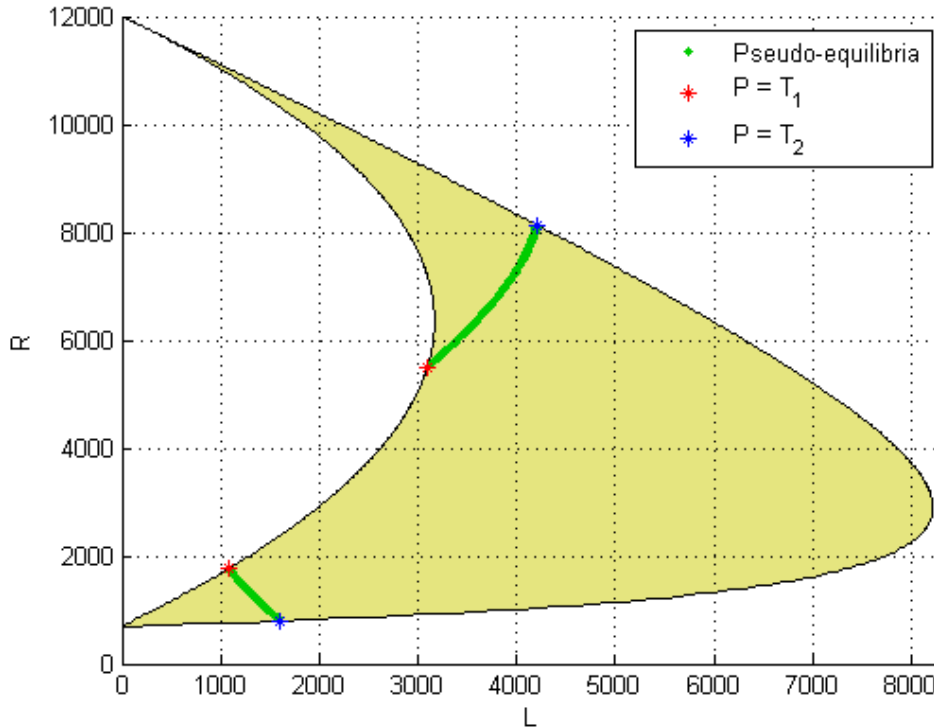
Figure 4-2: Sliding set for $R \in [k, K]$

4.2. Third-order Filippov system

Analogous results can be obtained when restriction (4-1) is applied to the system (3-2). The final set of differential equations is a third-order Filippov system whose sliding boundaries T_1 and T_2 are those in equation (4-7), but this time they describe a sliding surface since α is another state. Results in Figure **4-4** present behaviour for different reserve levels. Given an initial condition on F_1, the system has a normal evolution following vector field f^1; once the discontinuity Σ is reached on its sliding set Σ_s the system slide according to f^s. Finally, the sliding limit T_2 is reached and the trajectory returs to F_2 as shown in Figure 4.4(a). Temporal evolution in Figure 4.4(b) shows that the system quickly approaches the sliding set and finally evolves periodically following an periodic orbit with N different frequencies. Now, suppose that a higher level of resources is desired to preserve, i.e. $R = 5000$. According to Figure 4.4(c) the trajectory followed by the system implies that it is not possible to preserve that amount of resources since once the sliding limit T_2 is reached, the trajectory crosses Σ and falls to a stable equilibrium in F_2. As in the planar system this result suggest that a society that strongly depends on natural resources can find a better distribution of the economic sectors in order to preserve the maximum resource stock.
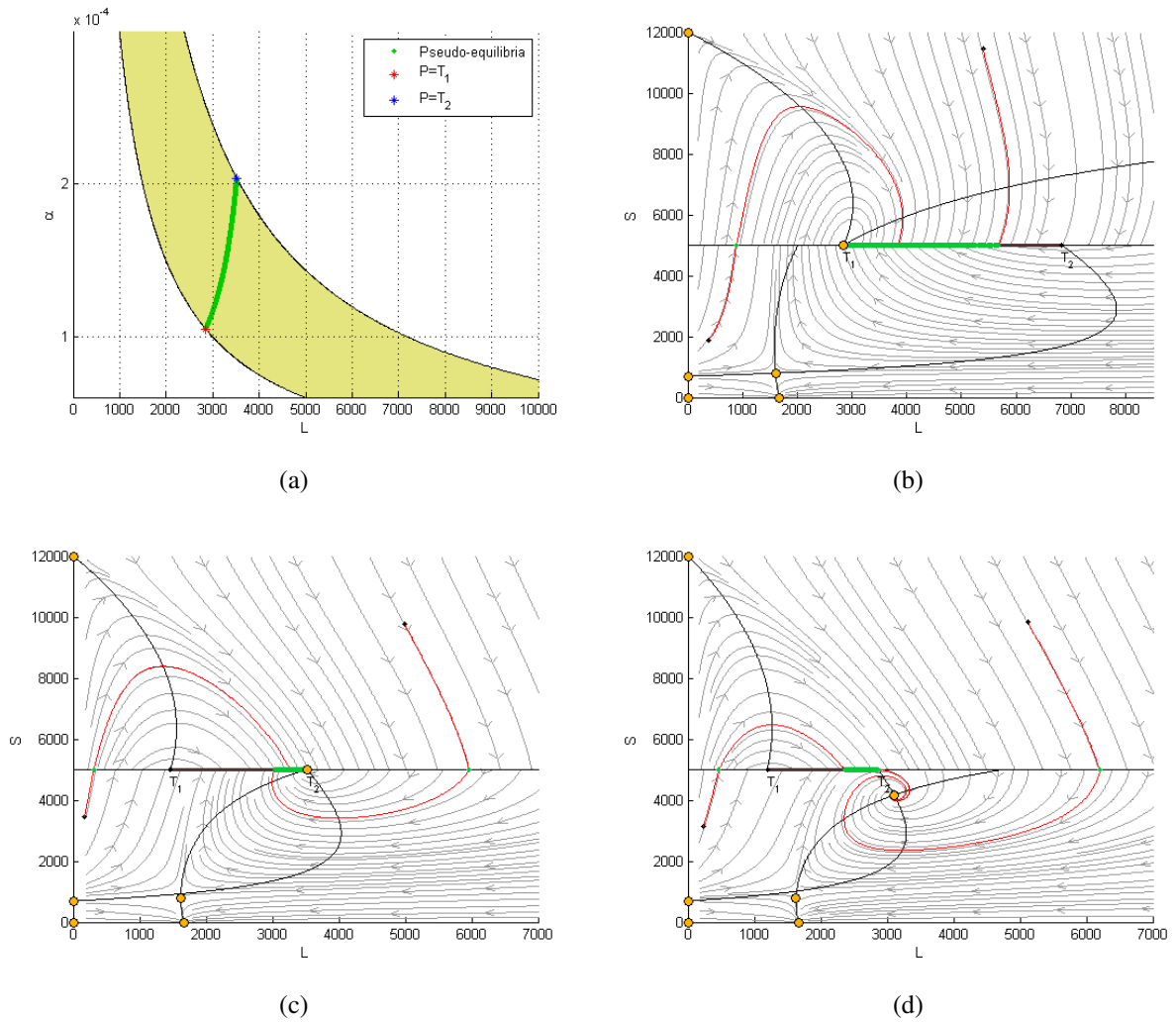
(a)   (b)

(c)   (d)

Figure 4-3: Phase portraits for different values of α including standard equilibria and sliding
segment.

4.3.   Filippov systems with two discontinuities

In section 2.10 an algebraic equation for β was introduced in order to self-regulate preference for resources. The proposed equation implies that high levels of resources increases the supply, but once they are reduced below a certain value the supply begins to decrease. This behaviour can also be obtained by introducing abrupt changes in preference. Suppose a new threshold of renewable resource R_2 such that above it the fraction of income spent on wood β increases due to the high supply. Once this value has been crossed β becomes constant because of resource reduction. Finally, the prescribed reserve of renewable resource R_1 acts in order to avoid resources extinction. According to the restriction (4-8) β has different functions depending on renewable
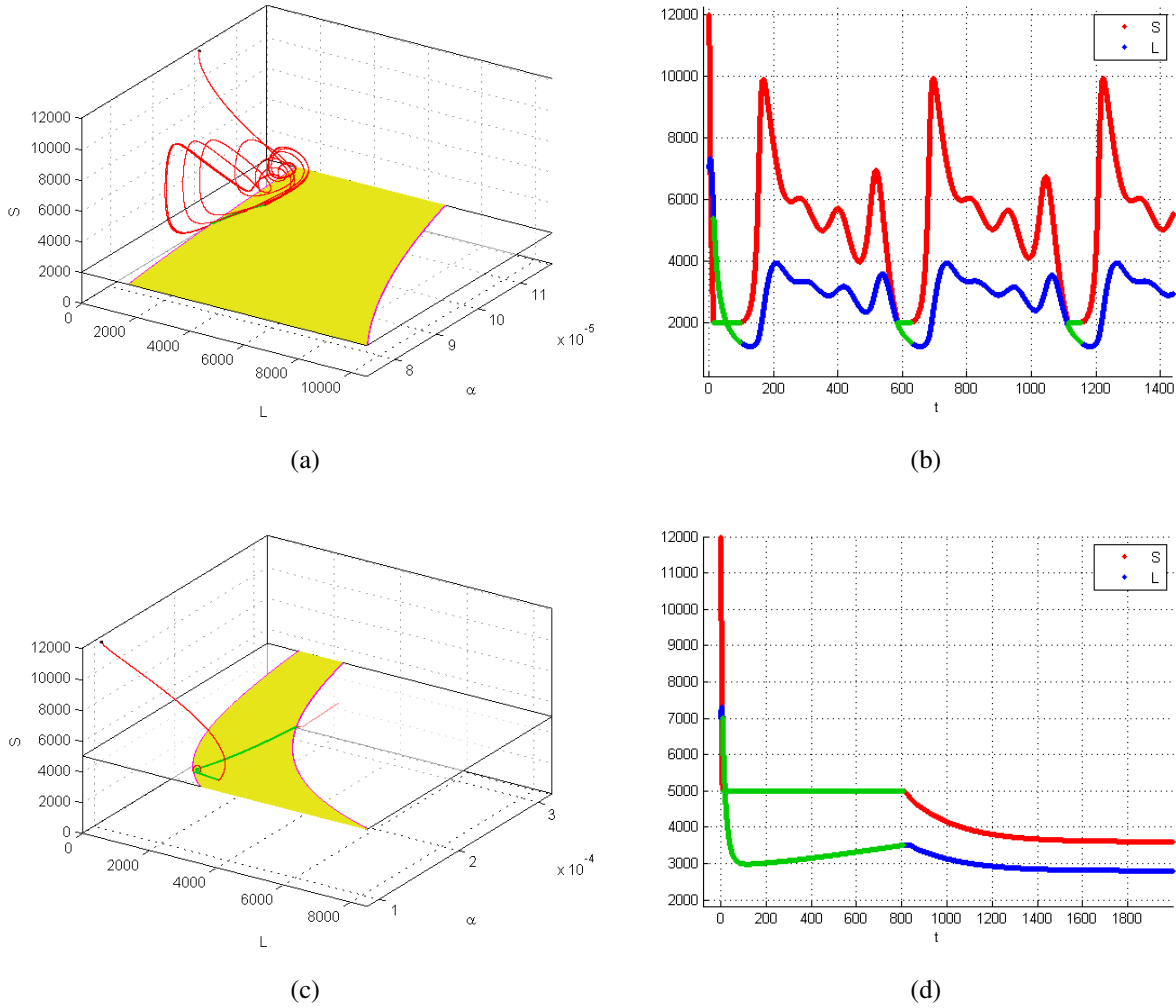
(a)

(b)

(c)

(d)

Figure 4-4: Phase portraits for different values of reserve: (a) phase portrait for $R = 2000$, (b) temporal evolution for $R = 2000$, (c) phase portrait for $R = 5000$, (d) temporal evolution for $R = 5000$.

resource stock.

$$\beta = \begin{cases} \beta_{\mathrm{m}}\left(\frac{K}{S}\right) & if & S > R_2 \\ \beta_{\mathrm{m}} & if & R_1 < S < R_2 \\ \beta_{\mathrm{m}}\left(\frac{S}{K}\right) & if & S < R \end{cases} \tag{4-8}$$

Again $\beta_m = 0{,}3$ is the average preference. Note that $\beta = 1$ if $S = 4000$, then $R_2 \in [4000, 12000]$. The system (4-9) is the Filippov system where the boundaries Σ_1 and Σ_2 and the regions F_1, F_2,

and F_3 are

$$
\begin{aligned}
\Sigma_1 &= \{(L, S) : S = R_1\}, \\
\Sigma_2 &= \{(L, S) : S = R_2\}, \\
F_1 &= \{(L, S) : S > R_2\}, \\
F_2 &= \{(L, S) : R_1 < S < R_2\}, \\
F_3 &= \{(L, S) : S < R_1\},
\end{aligned}
\tag{4-9}
$$

and is described by set of standard differential equations

$$
\dot{x} = \begin{cases}
f^1(x) & if \quad x \in F_1, \\
f^2(x) & if \quad x \in F_2, \\
f^3(x) & if \quad x \in F_3,
\end{cases}
\tag{4-10}
$$

For the new discontinuity there are two new sliding limits

$$
\begin{aligned}
T_3 &= \frac{R_2 \rho \left(R_2 K - R_2^2 - kK + R_2 k \right)}{\alpha \beta K^2 k}, \\
T_4 &= \frac{\rho \left(R_2 K - R_2^2 - kK + R_2 k \right)}{\alpha \beta k K}.
\end{aligned}
\tag{4-11}
$$

4.3.1. Planar system

Results about single R_1 has already been obtained, therefore the following results describe the behaviour when R_2 is also introduced. Figure **4-5** presents three particular cases where pseudo-equilibria exists on one, two, or no discontinuity. Figure 4.5(a) is the phase portrait where no discontinuity has pseudo-equilibria. For some initial conditions the system slides on R_2 then falls to R_1 where a second sliding occurs to finally converge to the stable limit cycle. The no existence of pseudo-equilibria implies that necessarily system converge to an attractor (stable equilibrium or stable limit cycle) in regions F_i. The reduction of R_2 from 8000 to 5000 and R_1 from 4000 to 2000 permit the appearance of a pseudo-equilibrium on R_2 as shown in figure 4.5(b). Figure 4.5(c) suggests that even for low desired reserve the system approaches to a steady state with positive population and resources, but depending on initial conditions convergence to a pseudo-equilibrium on R_2 is also a possibility. General results suggest that the middle region between discontinuities R_1 and R_2 may be considered as a control region where not only excessive preference is mitigated but also a reserve or renewable resources is preserved. Consequently, long-run dynamics falls either region F_2 or a pseudo-equilibria on sliding sets Σ_{S_1} or Σ_{S_2}.

(a)   (b)
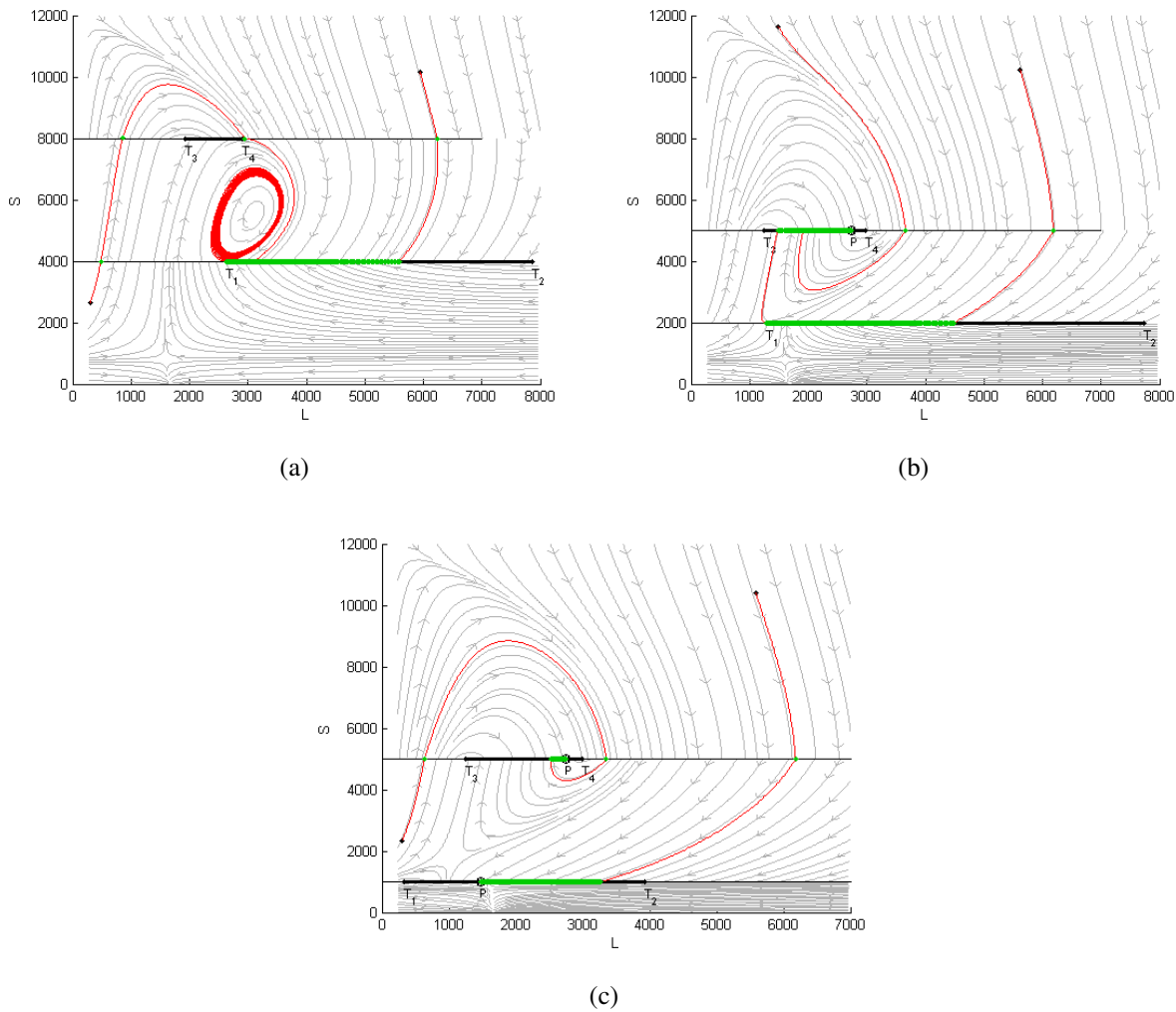

(c)

Figure 4-5: Planar Filippov system with two discontinuities

4.3.2.   3D system

Results are analogous to the ones obtained in the previous section. Figure **4-6** presents phase portrait and temporal evolution of the system for two different values of R_1 and R_2. First two figures 4.6(a) and 4.6(b) present the case where the system slides only on the sliding set Σ_{s_1} and finally converges to a steady state with positive population and resources. The oscillatory behaviour creates feast and famine cycles with sliding segments: when Σ_{s_1} is reached, population decreases allowing the regeneration of resources and then returns to F_2 to slide again. Another possibility is the existence of feast and famine cycles with sliding on both discontinuities as shown in figures 4.6(c) and 4.6(d). Fist, the trajectory slides on Σ_{s_1} allowing the regeneration of the resource, then returns to region F_2 and finally on Σ_{s_2} to repeat the cycle.
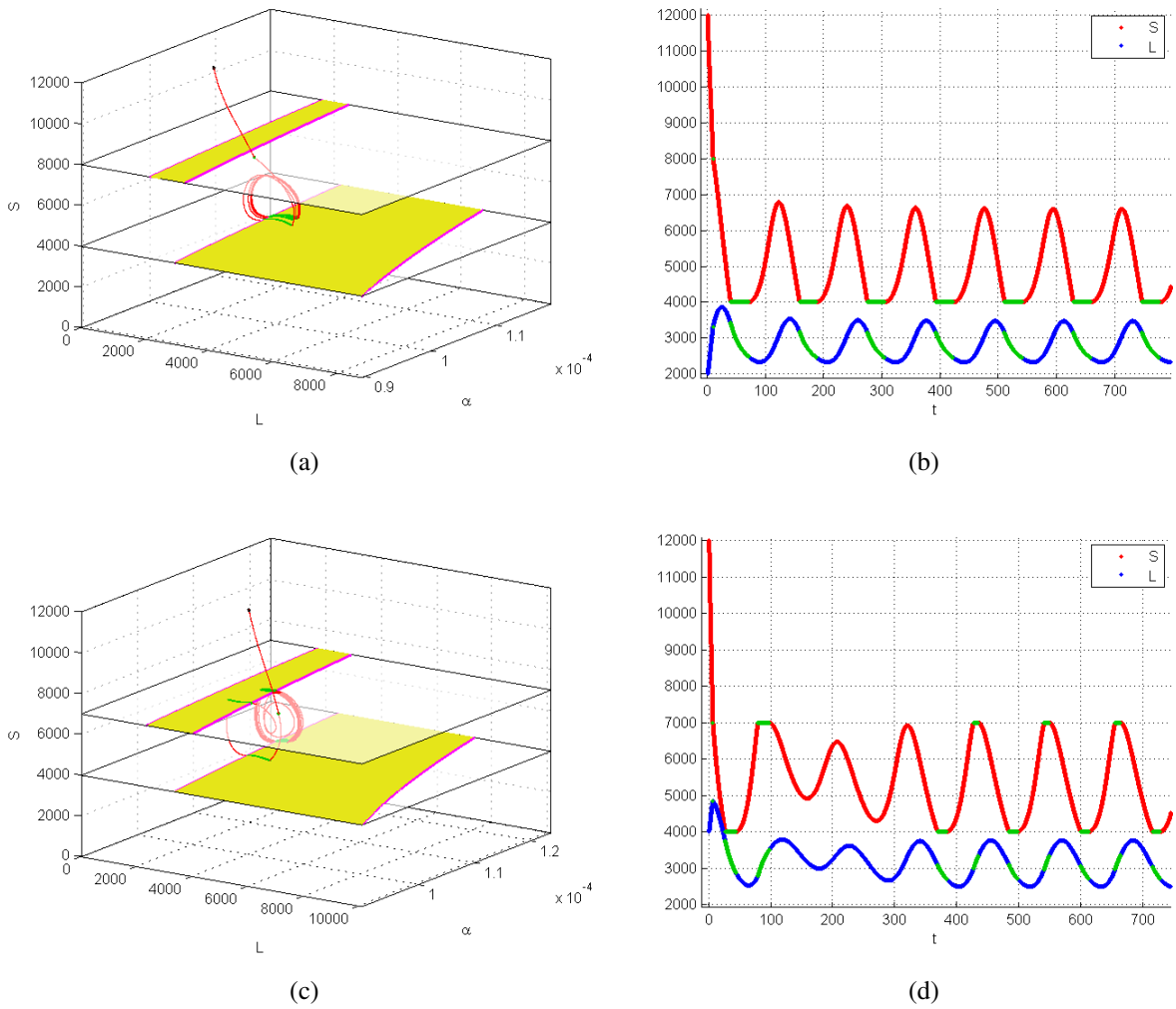
(a)

(b)

(c)

(d)

Figure 4-6: Third-order Filippov system with two discontinuities

5 Final remarks, summary, and future work

The paper evaluates the dynamic of an endogenous process for population growth in a economy based on resources consumption. The variation in population is assumed to be equal to the difference between calories consumed per-capita and the per-capita natural level of calories needed to survive. This assumption is a representation of a Ricardo-Malthus model where the increasing population is a consequence of the food capacity expansion and only hunger and misery can control it. In this sense and according to Ricardo's theory about the wage, an increase in per-capita income above a particular equilibrium level of consumption leads to increasing population size, but this increase of population reduces per-capita resources and consequently falls back to the equilibrium level. This relation is summarized in causal diagram **2-2**.

The income in the Ricardo-Malthus model studied in chapter 2 is assumed to come from two economic activities: the harvest of a natural resource and agricultural production. This assumption allows the introduction of non-linearities into the system and the possibility of multiple equilibria. The obtained results shows the economic convergence to different steady states: one with positive population and resources and the other with a minimum population surviving strictly from agriculture because of the resources extinction.

[D'Alessandro, 2007] adjusted parameters taking into account the values reported by [Brander & Taylor, 1998] which are consistent with Easter Island according to archaeological findings and historical data. Although D'Alessandro's model includes agriculture as an extra economical sector, it also represents the economic interaction in a primitive isolated society since no consider the possibility of emigration and immigration. For instance, the Mayan culture overexploited their resources to finally depend on agricultural production which could no longer support population and emigration to other regions was unavoidable. On the contrary, migration on Easter island was not possible since the island is really isolated and civilization falls due to the resources scarcity, then models like [Brander & Taylor, 1998] and system (2-2) can provide insights that should be considering in the resources management.

The inputs and outputs of population in a region leads to changes in total population, therefore it is necessary to include birth and death rates in order to obtain the existing population at any time. This expansion of the model involves to deal with more complex structures that could be applied to modern resources system.

In chapter 2 details of possible long-run dynamics of population and resources based in a recent model by [D'Alessandro, 2007] have been presented. In a stationary environment with constant

parameter values the system may converge to an attractor or the other depending on initial conditions. Nonetheless, it is also possible to obtain sustainable future paths with only one attractor when parameter values change. By using local bifurcation analysis various different long-run system dynamics can be distinguished since they depend on parameter values or a combination of them. Expanding bifurcation theory to two-dimensional bifurcation diagrams when no just one but two parameters of the model changes simultaneously, regions where resources scarcity followed by population decline are identified.

Results were referred specifically to find out conditions where resources scarcity causes populations death. In particular long-run dynamics of population and resources changes when parameter such as preference for resources β, regeneration rate of resource ρ, technology in harvesting α, and land fertility λ are changed. An increase of β involves and increase on harvesting rate since population dedicated to exploit resources also increases. Results indicate that as β increases population equilibrium increase while resources equilibrium decreases, during this process the system loses stability through bifurcations and finally becomes unsustainable. Analogous results were found for increasing α since also affects the harvesting rate.

Agricultural practices and land management are represented by λ and ρ. The use of traditional agricultural practices in places where land fertility depends strongly on the use of agrochemicals have severe environmental consequences since over time the vegetative mantle could be severely damaged causing erosion and groundwater seepage. These improper practices cause in a first stage an economic rise, but as soon as population begins to increase the necessity of resources and agricultural products also increases, then sustainable equilibrium falls to finally becomes unsustainable. Moreover, the increasing use of soil to produce enough food for the increasing population reduces resources stock by affecting the environment. For example in absence of humans, resources grow naturally and neither harvesting nor agriculture is carried out. Agriculture production involves the use of a portion of fertile land reducing moderately forest and other associated natural resources (for example water sources). Overexploitation of land by means of traditional agricultural practices leads to use of large areas of land and agrochemicals. Then, the regeneration rate of the resource ρ is associated to the recovery of lands for reforestation and the lost of fertility due to the overexploitation of land. An alternative for this trouble in order to obtain more production without lost of fertility is the implementation of agro-ecological practices such as reducing single crop farming, increasing crop rotation, and abandon the use of agrochemicals.

According to endogenous technological progress theory an increase in population would stimulate technologist to increase food production. Under pressure of increasing population, humans put more labour and effort in production and find new ways to increase their income. According to the system (3-2) the effort is put in harvesting improvement $\alpha(L, S)$. They exploit forest more intensively without any care reducing the resources until they are no longer sufficient to feed the population, after that people begins to decrease, and so does the effort to harvest. finally, the system falls down to the equilibrium. Results suggest that long-run dynamics could present oscillatory behaviour or falls in a chaotic attractor depending on parameter values and initial conditions.

In chapter 4 the dynamics of population and renewable resources has been modelled when a reser-

ve is created to protect a certain amount of resources from harvesting. Results help to derive some intuitive principles to conserve resources and understand the behaviour and dependence of population on the reserve capacity. The study shows that the reserve capacity parameter R plays the important role of controlling the state of the vector field when other parameters remain constant. Namely, the existence of long-run dynamics conserving the desired amount of resources, sliding sets, and pseudo-equilibria depends on the resources capacity.

When a second discontinuity is introduced the density of resources can described by a control system consisting in preference for harvesting resources. This control system is bounded by the two discontinuities: the first is a population effort to mitigate the resources exhaustion by reducing the demand, and the second is the prescribed reserve of resources to be preserved. This control system is chosen in order to avoid resources extinction by considering cultural and institutional changes in order to maximize population without over-harvesting resources. Results enhance the importance of preference reduction either with abrupt changes or with a constant effort (for example $\beta(S)$). Abrupt changes are related to institutional changes and political decisions while the constant effort of population for protecting resources is related with human habits, reproductive decisions, culture, and education. Nowadays conservation efforts like reforestation or forest preservation are difficult to carry out at least in areas with higher population density because these areas are in constant conflict with agricultural intensification and only strict political decisions can protect the remaining resources.

Malthus wrote his essay before industrial revolution therefore he excluded technology from the theory. At that time, population grows slowly with low per-capita income which is consistent with the theory. After industrial revolution new economic sectors appeared and technology plaid an important role in production. Nowadays the panorama is very different because the relation between population and economy turns negative, meaning that population has begun to decrease while economy grows rapidly. This is the case of richest countries which have the lower population growth rates and the largest economic growth. This phenomenon can also be observed in cities since poorer families have the highest birth rates. The main limitation of Malthus's theory lies in its ability to predict the reduced population in countries that develop economically. However, study economical models described by Malthus and Buserup theories allows the possibility of developing complex models for modern resources societies.

Future research about modelling sustainable development through non-linear and non-smooth dynamical systems may include not only migration in and out but also other economic sectors in order to scape from the Malthusian trap where population is limited by the existing resources. Innovative models should also include human capital, welfare, and economic growth to fit completely with the structure of sustainable development. For this purpose it is necessary to take into account social aspects of development such as education, health, and violence. An important aid to building these models is the use of system dynamics. Once the problem is fully known in all its social, environmental, and economic aspects the causal diagram can be constructed and the variables and parameters may be identified. The overall obtained framework could be complex and may be necessary to work with sub-models. For instance migration between rural and urban areas

or production in the primary and secondary sectors.

For future work is suggested to validate the obtained models. For this purpose it is necessary to calibrate parameters according to available information. For example [Brander & Taylor, 1998] obtained values for Easter Island from archaeological findings and historical data. Nowadays, statistical and historical data, and indicator systems can provide enough information for calibration of models. For example [Prskawetz et al., 1999] calculated production parameters for 23 low-income countries in Africa and used them to prove a three-sector model; or [Capello & Faggian, 2002] studied the importance of urban size for urban sustainability based in real data for 95 Italian cities using a predator-prey model.

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