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Uniform topologies on types

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We study the robustness of interim correlated rationalizability to perturbations of higher-order beliefs. We introduce a new metric topology on the universal type space, called *uniform-weak topology*, under which two types are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. This topology generalizes the now classic notion of proximity to common knowledge based on *common p-beliefs* (Monderer and Samet 1989). We show that convergence in the uniform-weak topology implies convergence in the *uniform-strategic topology* (Dekel et al. 2006). Moreover, when the limit is a finite type, uniform-weak convergence is also a necessary condition for convergence in the strategic topology. Finally, we show that the set of finite types is nowhere dense under the uniform strategic topology. Thus, our results shed light on the connection between similarity of beliefs and similarity of behaviors in games.

Keywords. Rationalizability, incomplete information, higher-order beliefs, strategic topology, Electronic Mail game.

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1. Introduction

The Bayesian analysis of incomplete information games requires the specification of a *type space*, which is a representation of the players' uncertainty about fundamentals, their uncertainty about the other players' uncertainty about fundamentals, and so on, ad infinitum. Thus the strategic outcomes of a Bayesian game may depend on entire infinite hierarchies of beliefs. Critically, in some games this dependence can be very sensitive at the tails of the hierarchies, so that a mispecification of higher-order beliefs, even at arbitrarily high orders, can have a large impact on the predictions of strategic behavior, as shown by the Electronic Mail game of Rubinstein (1989). As a matter of fact, this phenomenon is not special to the E-Mail game. Recently, Weinstein and Yildiz (2007) have shown that in any game satisfying a certain payoff richness condition, if a player has multiple actions that are consistent with *interim correlated rationalizability*—the solution concept that embodies common knowledge of rationality[1]—then any of these actions can be made *uniquely* rationalizable by suitably perturbing the player's higher-order beliefs at any arbitrarily high order. This phenomenon raises a conceptual issue: if predictions of strategic behavior are not robust to mispecification of higher-order beliefs, then the common practice in applied analysis of modeling uncertainty using small type spaces—often finite—may give rise to spurious predictions.

A natural approach to study this robustness problem is topological. Consider the correspondence that maps each type of player into his set of interim correlated rationalizable (ICR) actions. The fragility of strategic behavior identified by Rubinstein (1989) and Weinstein and Yildiz (2007) can be recast as a certain kind of discontinuity of the ICR correspondence in the *product topology* over hierarchies of beliefs, i.e., the topology of weak convergence of k-order beliefs, for each $k \geq 1$. While in every game the ICR correspondence is upper hemicontinuous in the product topology, lower hemicontinuity can fail even for the *strict* ICR correspondence—a refinement of ICR that requires the incentive constraints to hold with strict inequality.[2] Strictness rules out incentives that hinge on a "knife edge," which can always be destroyed by suitably perturbing the payoffs of the game. Indeed, nonstrict solution concepts are known to fail lower hemicontinuity in other contexts, e.g., in complete information games, Nash equilibrium, and, in fact, even best-reply correspondences fail to be lower hemicontinuous with respect to payoff perturbations. By contrast, the strict Nash equilibrium and the strict best-reply correspondences *are* lower hemicontinuous. It is, therefore, surprising that this form of continuity breaks down when it comes to perturbations of higher-order beliefs.

There exist, of course, finer topologies under which the ICR correspondence is upper hemicontinuous and the strict ICR correspondence is lower hemicontinuous in all games. The coarsest such topology is the *strategic topology* introduced by Dekel et al. (2006); it embodies the minimum restrictions on the class of admissible perturbations of higher-order beliefs necessary to render rationalizable behavior continuous. Thus

[1] See Dekel et al. (2007, Proposition 2) and Battigalli et al. (2008, Theorem 4).

[2] Here, the notion of strictness is actually quite strong: the slack in the incentive constraints is required to be bounded away from zero uniformly on a best-reply set. Despite this, the strict ICR correspondence fails to be lower hemicontinuous in the product topology.

the strategic topology gives a tight measure of the robustness of strategic behavior: if the analyst considers any larger set of perturbations, he is bound to make a nonrobust prediction in some game. Given this significance, we believe the strategic topology deserves closer examination. Indeed, Dekel et al. (2006) only define it *implicitly* in terms of proximity of behavior in games, as opposed to *explicitly* using some notion of proximity of probability measures. This leaves open the important question as to what proximity in the strategic topology means in terms of the beliefs of the players.

To address this question, we introduce a new metric topology on types, called *uniform-weak topology*, under which a sequence of types $(t_n)_{n\geq 1}$ converges to a type t if the k-order belief of t_n weakly converges to that of t and the rate of convergence is uniform over $k \geq 1$. More precisely, for each $k \geq 1$, we consider the *Prohorov* metric, d^k, over k-order beliefs—a standard metric that metrizes the topology of weak convergence of probability measures—and then define the uniform-weak topology as the topology of convergence in the metric $d^{\mathrm{UW}} \equiv \sup_{k\geq 1} d^k$. Our first main result, Theorem 1, is that convergence in the uniform-weak topology implies convergence in the *uniform-strategic topology*. The latter, also introduced by Dekel et al. (2006), is the coarsest topology on types under which the ICR correspondence is upper hemicontinuous and the strict ICR correspondence is lower hemicontinuous, where the continuity is now required to hold uniformly across all games.[3] In particular, Theorem 1 implies that convergence in the uniform-weak topology is a sufficient condition for convergence in the strategic topology.

The uniform-weak topology is interesting in its own right, as it generalizes the classic notion of approximate common knowledge due to Monderer and Samet (1989). Given a payoff-relevant parameter θ, say that a type of a player has *common p-belief in θ* if he assigns probability no smaller than p to θ, assigns probability no smaller than p to the event that θ obtains and the other players assign probability no smaller than p to θ, and so forth, ad infinitum. A sequence of types $(t_n)_{n\geq 1}$ has *asymptotic common certainty of θ* if for every $p < 1$, t_n has common p-belief in θ for all n large enough. Monderer and Samet (1989) use this notion of proximity to common knowledge to study the robustness of Nash equilibrium to small amounts of incomplete information. Although they focus on an ex ante notion of robustness and consider only common prior perturbations, their main result has the following counterpart in our interim, noncommon prior, nonequilibrium framework.

> *If a sequence of types $(t_n)_{n\geq 1}$ has asymptotic common certainty of θ, then, for every game, every action that is strictly interim correlated rationalizable when θ is common certainty remains interim correlated rationalizable for type t_n, for all n large enough.*

It turns out that asymptotic common certainty of θ is *equivalent* to uniform-weak convergence to the type that has common certainty of θ (i.e., common 1-belief). Thus, our Theorem 1 is a generalization of Monderer and Samet's (1989) main result to environments where the limit game has incomplete information.

[3]See Section 3 for the precise definition of the modulus of continuity on which the uniformity is based.

An important corollary of Theorem 1 is that the strategic, uniform-strategic, and product topologies generate the same σ-algebra.[4] Indeed, a fundamental result of Mertens and Zamir (1985), which is the Bayesian foundation of Harsanyi's (1967–1968) model of types, is that the space of hierarchies of beliefs, called the *universal type space*, exhausts all the relevant uncertainty of the players when endowed with the product σ-algebra. It is reassuring to know that this universality property remains valid when the players can reason about any strategic event.[5]

Our second main result, Theorem 2, is that uniform-weak convergence is also a necessary condition for strategic convergence when the limit is a *finite type*, i.e., a type belonging to a finite type space. Indeed, for any finite type t and for any sequence of (possibly infinite) types $(t_n)_{n\geq 1}$ that fails to converge to t uniform-weakly, we construct a game in which an action is strictly interim correlated rationalizable for t, but not interim correlated rationalizable for t_n, infinitely often along the sequence.[6] Thus, the uniform-weak topology fully characterizes the strategic topology around finite types. Moreover, the assumption that the limit is a finite type cannot be dispensed with. Under the uniform-weak topology, the universal type space is not *separable*, i.e., it does not contain a countable dense subset; by contrast, Dekel et al. (2006) show that a countable set of finite types is dense under the strategic topology.[7] This implies the existence of infinite types to which uniform-weak convergence is not a necessary condition for strategic convergence. (We explicitly construct such an example in Section 4.) While this fact imposes a natural limit to our analysis, finite type spaces play a prominent role in both applied and theoretical work, so it is important to know that our sufficient condition for strategic convergence is also necessary in this case.

Finite types are also the focus of our third main result, Theorem 3. We show that, under the uniform-strategic topology, the set of finite types is *nowhere dense*, i.e., its closure has an empty interior. To understand the conceptual implications of this result, recall that Dekel et al. (2006) demonstrate the denseness of finite types under the nonuniform version of the strategic topology.[8] Arguably, this result provides a compelling justification for why it might be without loss of generality to model uncertainty with finite type spaces: Irrespective of how large the "true" type space T is, for any given game there is always a finite type space T' with the property that the predictions of strategic behavior

[4] This is because uniform-weak balls are countable intersections of finite-order cylinders and the strategic topologies are sandwiched between the uniform-weak and the product topologies, by Theorem 1.

[5] Morris (2002, Section 4.2) raises the question of whether the Mertens–Zamir construction is still meaningful when strategic topologies are assumed.

[6] This complements the main result of Weinstein and Yildiz (2007), who *fix a game* (satisfying a payoff-richness assumption) and a finite type t, and then *construct a sequence of types* converging to t in the product topology such that the behavior of t is bounded away from the behavior of all types in the sequence. By way of contrast, we *fix a sequence of types* that fails to converge to a finite type t in the uniform-weak topology and then *construct a game* for which the behavior of t is bounded away from the behavior of the types in the sequence infinitely often.

[7] While Dekel et al. (2006) state only the weaker result that the set of *all* finite types is dense in the strategic topology, their proof actually establishes the stronger result above.

[8] Mertens and Zamir (1985) prove the denseness of finite types under the product topology. Dekel et al. (2006) argue that this result does not provide a sound justification for restricting attention to finite types, for strategic behavior is not continuous in the product topology.

based on T' are arbitrarily close to those based on T. Our nowhere denseness result implies that such finite type space T' cannot be chosen independently of the game. This is particularly relevant for environments such as those of mechanism design, where the game—both payoffs and action sets—is not a priori fixed. More generally, our result implies that the uniform-strategic topology is strictly finer than the strategic topology. Thus, while a priori these two notions of strategic continuity seem equally compelling, assuming one or the other can have a large impact on the ensuing theory.

The exercise in this paper is similar in spirit to that of Monderer and Samet (1996) and Kajii and Morris (1998), who, like us, consider perturbations of incomplete information games. These papers provide belief-based characterizations of strategic topologies for Bayesian Nash equilibrium in countable partition models à la Aumann (1976). However, since both of these papers assume a common prior and adopt an ex ante approach, while we adopt an interim approach without imposing a common prior, it is difficult to establish a precise connection.[9] Another important difference between their approach and ours is in the distinct *payoff-relevance constraints* adopted: we fix the set of payoff-relevant states, so our games cannot have payoffs that depend directly on players' higher-order beliefs; Monderer and Samet (1996) and Kajii and Morris (1998) have no such payoff-relevance constraint.

The connection between uniform and strategic topologies first appears in Morris (2002), who studies a special class of games, called *higher-order expectation* (HOE), games, and shows that the topology of uniform convergence of higher-order *iterated expectations* is equivalent to the coarsest topology under which a certain notion of strict ICR correspondence—different from the one we consider—is lower hemicontinuous in every game of the HOE class.[10] Compared to the uniform-weak topology, the topology of uniform convergence of iterated expectations is neither finer nor coarser, even around finite types. We further elaborate on this relationship in Section 5.

This paper is also related to contemporaneous work by Ely and Pęski (2008). Following their terminology, a type t is *critical* if, under the product topology, the strict ICR correspondence is discontinuous at t in some game. Ely and Pęski (2008) provide an insightful characterization of critical types in terms of a common belief property: a type is critical if and only if, for some $p > 0$, it has common p-belief in some closed (in product topology) proper subset of the universal type space.[11] Conceptually, this result shows that the usual type spaces that appear in applications consist almost entirely of critical types, as these type spaces typically embody nontrivial common belief assumptions. For instance, all finite types are critical and so are almost all types belonging to a common

[9] Monderer and Samet (1996) fix the common prior and consider proximity of information partitions, whereas Kajii and Morris (1998) vary the common prior on a fixed information structure. For this reason, the precise connection between these papers is already unclear.

[10] Morris (2002) defines his strategic topology for HOE games using a distance that makes no reference to ICR. But, as we claimed above, it can be shown that his strategic topology coincides with the coarsest topology under which a certain notion of strict ICR correspondence is continuous in every HOE game. The notion of strictness implicit in Morris (2002) analysis, unlike ours, does not require the slack in the incentive constraints to be uniform.

[11] Moreover, they show that under the product topology the *regular* types, i.e., those types which are not critical, form a *residual* subset of the universal type space—a standard topological notion of a "generic" set.

prior type space. Thus Ely and Pęski's (2008) result tells us when—based on the common beliefs of the players—there will be some game and some product-convergent sequence along which strategic behavior is discontinuous, whereas we identify a condition for an arbitrary sequence to display continuous strategic behavior in all games.

The rest of the paper is organized as follows. Section 2 introduces the standard model of hierarchies of beliefs and type spaces, and reviews the solution concept of ICR. Section 3 reviews the strategic and uniform-strategic topologies of Dekel et al. (2006), introduces the uniform-weak topology, and presents our two main results concerning the relationship between these topologies (Theorems 1 and 2). Section 4 examines the nongenericity of finite types under the uniform-strategic and uniform-weak topologies, and presents the nowhere denseness result (Theorem 3). Section 5 discusses the relation with some other topologies. Section 6 concludes with some open questions for future research.

2. Preliminaries

Throughout the paper, we fix a two-player set I and a finite set Θ of payoff-relevant states with at least two elements.[12] Given a player $i \in I$, we write $-i$ to designate the other player in I. All topological spaces, when viewed as measurable spaces, are endowed with their Borel σ-algebra. For a topological space S, we write $\Delta(S)$ to designate the space of probability measures over S equipped with the topology of weak convergence. Unless explicitly noted, all product spaces are endowed with the product topology and subspaces are endowed with the relative topology.

2.1 *Hierarchies of beliefs and types*

Our formulation of incomplete information follows Mertens and Zamir (1985).[13] Define $X^0 = \Theta$, and $X^1 = X^0 \times \Delta(X^0)$, and, for each $k \geq 2$, define recursively

$$X^k = \left\{ (\theta, \mu^1, \dots, \mu^k) \in X^0 \times \bigtimes_{\ell=1}^{k} \Delta(X^{\ell-1}) : \operatorname{marg}_{X^{\ell-2}} \mu^\ell = \mu^{\ell-1} \, \forall \ell = 2, \dots, k \right\}.$$

By virtue of the above coherency condition on marginal distributions, each element of X^k is determined by its first and last coordinates, so we can identify X^k with $\Theta \times \Delta(X^{k-1})$. For each $i \in I$ and $k \geq 1$, we let $\mathcal{T}_i^k = \Delta(X^{k-1})$ designate the space of *k-order beliefs* of player i, so that $\mathcal{T}_i^k = \Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$. The space $\mathcal{T}_i$ of *hierarchies of beliefs* of player i is

$$\mathcal{T}_i = \left\{ (\mu^k)_{k \geq 1} \in \bigtimes_{k \geq 1} \Delta(X^k) : \operatorname{marg}_{X^{k-2}} \mu^k = \mu^{k-1} \, \forall k \geq 2 \right\}.$$

[12] We restrict attention to two-player games for ease of notation. Our results remain valid with any finite number of players.

[13] An alternative, equivalent formulation is found in Brandenburger and Dekel (1993).

Since Θ is finite, $\mathcal{T}_i$ is a compact metrizable space. Moreover, there is a unique mapping $\mu_i : \mathcal{T}_i \to \Delta(\Theta \times \mathcal{T}_{-i})$ that is *belief preserving*, i.e., for all $t_i = (t_i^1, t_i^2, \ldots) \in \mathcal{T}_i$ and $k \geq 1$,

$$\mu_i(t_i)[\theta \times (\pi_{-i}^k)^{-1}(E)] = t_i^{k+1}[\theta \times E] \quad \text{for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^k,$$

where π_i^k is the natural projection of $\mathcal{T}_i$ onto $\mathcal{T}_i^k$. Furthermore, the mapping μ_i is a homeomorphism, so to save on notation, we identify each hierarchy of belief $t_i \in \mathcal{T}_i$ with its corresponding belief $\mu_i(t_i)$ over $\Theta \times \mathcal{T}_{-i}$. Similarly, for each $t_i \in \mathcal{T}_i$, we write $t_i^k \in \mathcal{T}_i^k$ instead of the more cumbersome $\pi_i^k(t_i)$.

Hierarchies of beliefs can be implicitly represented using a *type space*, i.e., a tuple $(T_i, \phi_i)_{i \in I}$, where each T_i is a Polish space of *types* and each $\phi_i : T_i \to \Delta(\Theta \times T_{-i})$ is a measurable function. Indeed, every type $t_i \in T_i$ is mapped into a hierarchy of beliefs $\nu_i(t_i) = (\nu_i^k(t_i))_{k \geq 1}$ in a natural way: $\nu_i^1(t_i) = \text{marg}_\Theta \, \phi_i(t_i)$ and, for $k \geq 2$,

$$\nu_i^k(t_i)[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i}^{k-1})^{-1}(E)] \quad \text{for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^{k-1}.$$

The type space $(\mathcal{T}_i, \mu_i)_{i \in I}$ is called the *universal type space*, since for every type space $(T_i, \phi_i)_{i \in I}$ there is a unique belief-preserving mapping from T_i into $\mathcal{T}_i$, namely the mapping ν_i above.[14] When the mappings $(\nu_i)_{i \in I}$ are injective, the type space $(T_i, \phi_i)_{i \in I}$ is called *nonredundant*. In this case, $(\nu_i)_{i \in I}$ are measurable embeddings onto their images $(\nu_i(T_i))_{i \in I}$, which are measurable and can be viewed as a nonredundant type space, since we have $\mu_i(\nu_i(t_i))[\Theta \times \nu_{-i}(T_{-i})] = 1$ for all $i \in I$ and $t_i \in T_i$. Conversely, any $(T_i)_{i \in I}$ such that $T_i \subseteq \mathcal{T}_i$ and $\mu_i(t_i)[\Theta \times T_{-i}] = 1$ for all $i \in I$ and $t_i \in T_i$ can be viewed as a nonredundant type space.

2.2 *Bayesian games and interim correlated rationalizability*

A *game* is a tuple $G = (A_i, g_i)_{i \in I}$, where A_i is a finite set of *actions* for player i and $g_i : A_i \times A_{-i} \times \Theta \to [-M, M]$ is his *payoff* function, with $M > 0$ an arbitrary bound on payoffs that we fix throughout.[15] We write $\mathcal{G}$ to denote the set of all games and, for each integer $m \geq 1$, we write $\mathcal{G}^m$ for the set of games with $|A_i| \leq m$ for all $i \in I$.

The solution concept of *interim correlated rationalizability* (ICR) was introduced in Dekel et al. (2007). Given a $\gamma \in \mathbb{R}$, a type space $(T_i, \phi_i)_{i \in I}$, and a game G, for each player $i \in I$, integer $k \geq 0$, and type $t_i \in T_i$, we let $R_i^k(t_i, G, \gamma) \subseteq A_i$ designate the set of *k-order γ-rationalizable actions* of t_i. These sets are defined as:

$$R_i^0(t_i, G, \gamma) = A_i,$$

and recursively for each integer $k \geq 1$, $R_i^k(t_i, G, \gamma)$ is the set of all actions $a_i \in A_i$ for which there is a *conjecture*, i.e., a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that[16]

$$\text{supp}\, \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \gamma) \quad \forall (\theta, t_{-i}) \in \Theta \times T_{-i} \tag{1}$$

[14]To say that ν_i is belief-preserving means that $\mu_i(\nu_i(t_i))[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i})^{-1}(E)]$ for all $\theta \in \Theta$ and measurable $E \subseteq \mathcal{T}_{-i}$.

[15]We will also denote by g_i the payoff function in the mixed extension of G, writing $g_i(\alpha_i, \alpha_{-i}, \theta)$ with the obvious meaning for any $\alpha_i \in \Delta(A_i)$ and $\alpha_{-i} \in \Delta(A_{-i})$.

[16]Relaxing condition (1) by requiring it to hold only for $\phi_i(t_i)$-almost every (θ, t_{-i}) would not alter the definition of rationalizability. Indeed, any conjecture that has a $(k-1)$-order rationalizable support $\phi_i(t_i)$-almost everywhere can be changed into one that yields the same expected payoff and satisfies the condition

and for all $a_i' \in A_i$,

$$\int_{\Theta\times T_{-i}} \big[g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(a_i', \sigma_{-i}(\theta, t_{-i}), \theta)\big]\phi_i(t_i)(d\theta \times dt_{-i}) \geq -\gamma. \qquad (2)$$

For future reference, a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ that satisfies the former condition will be called a $(k-1)$*-order* γ*-rationalizable conjecture*. The set of γ*-rationalizable* actions of type t_i is then defined as

$$R_i(t_i, G, \gamma) = \bigcap_{k\geq 1} R_i^k(t_i, G, \gamma).$$

Finally, following Ely and Pęski (2008), an action $a_i \in A_i$ is *strictly interim correlated* γ*-rationalizable* for type t_i and we write $a_i \in \overset{\circ}{R}_i (t_i, G, \gamma)$ if $a_i \in R_i(t_i, G, \gamma')$ for some $\gamma' < \gamma$.

As shown in Dekel et al. (2007), $R_i(t_i, G, \gamma)$ is nonempty for every game G, type t_i and $\gamma \geq 0$.[17]

Interim correlated rationalizability has a characterization in terms of *best-reply sets*. A pair of measurable functions $\varsigma_i : T_i \to 2^{A_i}$, $i \in I$, has the γ*-best-reply property* if for each $i \in I$ and $t_i \in T_i$, each action $a_i \in \varsigma_i(t_i)$ is a γ-best reply for t_i to a conjecture σ_{-i} : $\Theta \times T_{-i} \to \Delta(A_{-i})$ with

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq \varsigma_{-i}(t_{-i}) \quad \forall(\theta, t_{-i}) \in \Theta \times T_{-i}.$$

If $(\varsigma_i)_{i\in I}$ has the γ-best-reply property, then $\varsigma_i(t_i) \subseteq R_i(t_i, G, \gamma)$ for all $i \in I$ and $t_i \in T_i$. As shown in Dekel et al. (2007), the pair $(R_i(\cdot, G, \gamma))_{i\in I}$ is the *maximal* pair of correspondences with the γ-best-reply property. This means there is no other pair $(\varsigma_i)_{i\in I}$ with the γ-best-reply property such that $R_i(t_i, G, \gamma) \subseteq \varsigma_i(t_i)$ for each $i \in I$ and $t_i \in T_i$, with strict inclusion for some $i \in I$ and $t_i \in T_i$. Therefore, an action is γ-rationalizable for a type t_i if and only if it is a γ-best reply to a γ*-rationalizable conjecture*, i.e., a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \gamma) \quad \forall(\theta, t_{-i}) \in \Theta \times T_{-i}.$$

Dekel et al. (2007) also show that the set of γ-rationalizable actions of a type is determined by the induced hierarchy of beliefs. Indeed, for any $k \geq 1$, any two types (possibly belonging to different type spaces) mapping into the same k-order belief must have the same set of k-order γ-rationalizable actions. This has two implications. First, for interim correlated rationalizability, it is without loss of generality to identify types with their corresponding hierarchies. Thus, in what follows we restrict attention to type spaces $(T_i)_{i\in I}$ with $T_i \subseteq \mathcal{T}_i$ and $t_i[\Theta \times T_{-i}] = 1$ for all $i \in I$ and $t_i \in T_i$.[18] Accordingly, we take the universal type space $\mathcal{T}_i$ to be the domain of the correspondence $R_i(\cdot, G, \gamma) : \mathcal{T}_i \rightrightarrows A_i$. Second,

everywhere. This is possible because the correspondence R_{-i}^{k-1} is upper hemicontinuous, and hence it admits a measurable selection by the Kuratowski–Ryll–Nardzewski selection theorem (see, e.g., Aliprantis and Border 1999).

[17] Note that for $\gamma < -2M$, we have $R_i(t_i, G, \gamma) = \varnothing$, and for $\gamma > 2M$ we have $R_i(t_i, G, \gamma) = A_i$.

[18] Recall that we identify each type $t_i \in \mathcal{T}_i$ with his belief $\mu_i(t_i) \in \Delta(\Theta \times \mathcal{T}_{-i})$.

to establish whether an action is k-order γ-rationalizable for a type t_i, we can restrict attention to $(k-1)$-order γ-rationalizable conjectures σ_{-i}, which are measurable with respect to $(k-1)$-order beliefs.[19]

Finally, the following result shows that, similar to rationalizability in complete information games, interim correlated rationalizability has a characterization in terms of iterated dominance, where the notion of dominance now becomes an interim one.

Proposition 1. *Fix γ and a game $G = (A_i, g_i)_{i\in I}$. For each $k \geq 1$, player $i \in I$, type $t_i \in \mathcal{T}_i$, and action $a_i \in A_i$, we have $a_i \in R_i^k(t_i, G, \gamma)$ if and only if, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, there exists a measurable $\sigma_{-i}: \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$ with*

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \in R_{-i}^{k-1}(t_{-i}, G, \gamma) \quad \forall (\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i} \tag{3}$$

such that

$$\int_{\Theta\times\mathcal{T}_{-i}} \big[g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}), \theta)\big] t_i(d\theta \times dt_{-i}) \geq -\gamma.$$

The proof of this proposition, relegated to the Appendix, uses a separation argument analogous to that which establishes the equivalence between strictly dominated and never best-reply strategies in complete information games. Here, too, the usefulness of the result comes from the fact that to check whether an action is rationalizable for a type, we are able to reverse the order of quantifiers and seek a possibly different conjecture for each possible (mixed) deviation.

3. Topologies on types

The *strategic* (or simply S) *topology* introduced in Dekel et al. (2006) is the coarsest topology on the universal type space $\mathcal{T}_i$ under which the ICR correspondence is upper hemicontinuous and the strict ICR correspondence is lower hemicontinuous in all games. More explicitly, following a formulation due to Ely and Pęski (2008), the S topology is the topology generated by the collection of all sets of the form

$$\{t_i \in \mathcal{T}_i : a_i \notin R_i(t_i, G, \gamma)\} \quad \text{and} \quad \{t_i \in \mathcal{T}_i : a_i \in \overset{\circ}{R}_i(t_i, G, \gamma)\},$$

where $G = (A_i, g_i)_{i\in I}$, $a_i \in A_i$, and $\gamma \in \mathbb{R}$.[20]

The S topology on $\mathcal{T}_i$ is metrizable by the distance d_i^{S}, defined as follows.[21] For each game $G = (A_i, g_i)_{i\in I}$, action $a_i \in A_i$, and type $t_i \in \mathcal{T}_i$, let

$$h_i(t_i|a_i, G) = \inf\{\gamma : a_i \in R_i(t_i, G, \gamma)\}.$$

[19]This means that $\sigma_{-i}(\theta, s_{-i}) = \sigma_{-i}(\theta, t_{-i})$ for all θ and all types s_{-i}, t_{-i} with the same $(k-1)$-order beliefs.

[20]The strategic topology can be given an equivalent definition that makes no direct reference to γ-rationalizability for $\gamma \neq 0$. Indeed, by Ely and Pęski (2008, Lemma 4), a subbasis of the strategic topology is the collection of all sets of the form $\{t_i : a_i \notin R(t_i, G, 0)\}$ and $\{t_i : a_i \in \overset{\circ}{R}_i(t_i, G, 0)\}$.

[21]Dekel et al. (2006) define the S topology directly using the distance d_i^{S}, rather than using the topological definition above.

Then, for each s_i and $t_i \in \mathcal{T}_i$,

$$d_i^{\mathrm{S}}(s_i, t_i) = \sum_{m\geq 1} 2^{-m} \sup_{G=(A_i,g_i)_{i\in I}\in\mathcal{G}^m} \max_{a_i\in A_i} \big|h_i(s_i|a_i, G) - h_i(t_i|a_i, G)\big|.$$

In terms of convergence of sequences, Dekel et al. (2006) show that for every $t_i \in \mathcal{T}_i$ and every sequence $(t_{i,n})_{n\geq 1}$ in $\mathcal{T}_i$, we have $d_i^{\mathrm{S}}(t_{i,n}, t_i) \to 0$ if, and only if, for every game $G = (A_i, g_i)_{i\in I}$, action $a_i \in A_i$, and $\gamma \in \mathbb{R}$, the following upper hemicontinuity (u.h.c.) and lower hemicontinuity (l.h.c.) properties hold: For every sequence $\gamma_n \to \gamma$,

$$a_i \in R_i(t_{i,n}, G, \gamma_n) \quad \forall n \geq 1 \quad \Longrightarrow \quad a_i \in R_i(t_i, G, \gamma), \tag{u.h.c.}$$

and for some sequence $\gamma_n \searrow \gamma$,

$$a_i \in R_i(t_i, G, \gamma) \quad \Longrightarrow \quad a_i \in R_i(t_{i,n}, G, \gamma_n) \quad \forall n \geq 1. \tag{l.h.c.}$$

Dekel et al. (2006) also introduce the *uniform-strategic (US) topology*, which strengthens the definition of the strategic topology by requiring the convergence to be uniform over all games. More precisely, the US topology is the topology of convergence under the metric d_i^{US}, which is defined as

$$d_i^{\mathrm{US}}(t_i, s_i) = \sup_{G=(A_i,g_i)_{i\in I}\in\mathcal{G}} \max_{a_i\in A_i} \big|h_i(t_i|a_i, G) - h_i(s_i|a_i, G)\big|.$$

This uniformity renders the US topology particularly relevant for environments where the game—both payoffs and action sets—is not fixed a priori, such as in a mechanism design environment.

We now introduce a metric topology on types, which we call *uniform-weak* (UW) *topology*, under which two types of player are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. Thus, unlike the S and US topologies, which are *behavior-based*, the UW topology is a *belief-based* topology, i.e., a metric topology defined explicitly in terms of proximity of hierarchies of beliefs. The two main results of this section, Theorems 1 and 2 below, establish a connection between these behavior- and belief-based topologies.

Before we present the formal definition of the UW topology, recall that for a complete separable metric space (S, d), the topology of weak convergence on $\Delta(S)$ is metrizable by the *Prohorov* distance ρ, defined as

$$\rho(\mu, \mu') = \inf\{\delta > 0 : \mu(E) \leq \mu'(E^\delta) + \delta \text{ for each measurable } E \subseteq S\} \quad \forall \mu, \mu' \in \Delta(S),$$

where $E^\delta = \{s \in S : \inf_{s'\in S} d(s, s') < \delta\}$. The UW *topology* is the metric topology on $\mathcal{T}_i$ generated by the distance

$$d_i^{\mathrm{UW}}(s_i, t_i) = \sup_{k\geq 1} d_i^k(s_i, t_i) \quad \forall s_i, t_i \in \mathcal{T}_i,$$

where d^0 is the discrete metric on Θ and recursively for $k \geq 1$, d_i^k is the Prohorov distance on $\Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$ induced by the metric $\max\{d^0, d_{-i}^{k-1}\}$ on $\Theta \times \mathcal{T}_{-i}^{k-1}$.

In the remainder of Section 3 we explore the relationship between the UW topology and the S and US topologies. First, we show that the UW topology is finer than the US topology (Theorem 1). Second, we prove a partial converse, namely that around *finite types*, i.e., types belonging to a finite type space, the S topology (and hence also the US topology) is finer than the UW topology (Theorem 2).

3.1 *UW convergence implies US convergence*

THEOREM 1. *For each player $i \in I$ and for all types $s_i, t_i \in \mathcal{T}_i$,*

$$d_i^{\mathrm{US}}(s_i, t_i) \le 4M d_i^{\mathrm{UW}}(s_i, t_i).$$

Thus the UW topology is finer than the US topology.

This theorem is a direct implication of the following proposition.

PROPOSITION 2. *Fix a game G, $\gamma \ge 0$ and $\delta > 0$. For each integer $k \ge 1$,*

$$d_i^k(s_i, t_i) < \delta \quad \Longrightarrow \quad R_i^k(t_i, G, \gamma) \subseteq R_i^k(s_i, G, \gamma + 4M\delta) \quad \forall i \in I, \forall s_i, t_i \in \mathcal{T}_i.$$

The main challenge in proving this result is due to the fact that $(k-1)$-order rationalizable conjectures $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$ need not be continuous under the topology of weak convergence of $(k-1)$-order beliefs. This implies that, keeping the conjecture fixed, the incentive constraints of player i for k-order γ-rationalizability (cf. (2)) may be discontinuous in his type under the topology of weak convergence of k-order beliefs. Our proof overcomes this issue by endowing close-by types with similar, but not identical, conjectures. Indeed, the characterization of ICR from Proposition 1 implies that for a given action $a_i \in A_i$ and a given mixed deviation $\alpha_i \in \Delta(A_i)$, there always exists a $(k-1)$-order rationalizable conjecture that is *optimal* to γ-rationalize a_i against α_i at order k.[22] Following this observation, in our proof we endow type t_i with an optimal conjecture for γ-rationalizability and endow type s_i with an optimal conjecture for $(\gamma + 4M\delta)$-rationalizability. Using these optimal conjectures, we then prove, using an integration-by-parts type argument, that every action that is k-order γ-rationalizable for t_i remains k-order $(\gamma + 4M\delta)$-rationalizable for s_i.

PROOF OF PROPOSITION 2. Fix a game $G = (A_i, g_i)_{i \in I}$, $\gamma \ge 0$ and $\delta > 0$. The proof is by induction on k. For $k = 1$, let s_i and $t_i \in \mathcal{T}_i$ be such that $d_i^1(s_i, t_i) < \delta$. Fix an arbitrary $a_i \in R_i^1(t_i, G, \gamma)$ and let us show that $a_i \in R_i^1(s_i, G, \gamma + 4M\delta)$ using Proposition 1. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$ and let $\sigma_{-i} : \Theta \to \Delta(A_{-i})$ be a conjecture such that[23]

$$\sum_{\theta \in \Theta} \big(g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta)\big) t_i^1[\theta] \ge -\gamma. \tag{4}$$

[22] To be precise, when we say that σ_{-i} is an *optimal conjecture to γ-rationalize a_i against α_{-i} at order k*, we mean that σ_{-i} is a $(k-1)$-order γ-rationalizable conjecture that satisfies the following property: for any type t_i, the expected payoff difference between a_i and α_i for type t_i is at least $-\gamma$ under *some* $(k-1)$-order γ-rationalizable conjecture if and only if this expected payoff difference is at least $-\gamma$ under σ_{-i}.

[23] Recall that t_i^1 designates the first-order belief of type t_i.

(Note that condition (3) is trivial for $k=1$.) Pick any function $\mathrm{a}_{-i}: \Theta \to A_{-i}$ such that

$$\mathrm{a}_{-i}(\theta) \in \arg\max_{a_{-i}\in A_{-i}} [g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)] \quad \forall \theta \in \Theta$$

and define

$$h(\theta) = g_i(a_i, \mathrm{a}_{-i}(\theta), \theta) - g_i(\alpha_i, \mathrm{a}_{-i}(\theta), \theta) \quad \forall \theta \in \Theta,$$

so that

$$h(\theta) \geq g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta) \quad \forall \theta \in \Theta. \tag{5}$$

To conclude the proof for $k=1$, we now show that $\sum_{\theta\in\Theta} h(\theta) s_i^1[\theta] \geq -\gamma - 4M\delta$. Indeed, let $\{\theta_n\}_{n=1}^N$ be an enumeration of Θ such that $h(\theta_n) \geq h(\theta_{n+1})$ for all $1 \leq n \leq N-1$. Thus, it follows from $d_i^1(s_i, t_i) < \delta$ and $|h(\theta)| \leq 2M$ for all θ that

$$\begin{aligned}
\sum_{\theta\in\Theta} h(\theta)(s_i^1[\theta] - t_i^1[\theta]) &= \sum_{n=1}^{N-1} (h(\theta_n) - h(\theta_{n+1})) \sum_{m=1}^{n} (s_i^1[\theta_m] - t_i^1[\theta_n]) \\
&= \sum_{n=1}^{N-1} \underbrace{(h(\theta_n) - h(\theta_{n+1}))}_{\geq 0} \underbrace{\left(s_i^1[\{\theta_m\}_{m=1}^n] - t_i^1[\{\theta_m\}_{m=1}^n]\right)}_{\geq -\delta} \\
&\geq -\delta \sum_{n=1}^{N-1} h(\theta_n) - h(\theta_{n+1}) \\
&= -\delta(h(\theta_1) - h(\theta_N)) \\
&\geq -4M\delta,
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{\theta\in\Theta} h(\theta) s_i^1[\theta] &= \sum_{\theta\in\Theta} h(\theta)(s_i^1[\theta] - t_i^1[\theta]) + \sum_{\theta\in\Theta} h(\theta) t_i^1[\theta] \geq -4M\delta + \sum_{\theta\in\Theta} h(\theta) t_i^1[\theta] \\
&\geq -4M\delta + \sum_{\theta\in\Theta} \big(g_i(a_i, \sigma_{-i}(\theta), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta), \theta)\big) t_i^1[\theta] \geq -\gamma - 4M\delta,
\end{aligned}$$

where the penultimate inequality follows from (5) and the last inequality follows from (4). Thus, $a_i \in R_i^1(s_i, G, \gamma + 4M\delta)$ by Proposition 1, which proves the desired result for $k=1$.

Proceeding by induction, we now suppose the result is valid for some $k \geq 1$ and show that it remains valid for $k+1$. Let $s_i, t_i \in \mathcal{T}_i$ be such that $d_i^{k+1}(s_i, t_i) < \delta$. Fix an arbitrary $a_i \in R_i^{k+1}(t_i, G, \gamma)$ and let us show that $a_i \in R_i^{k+1}(s_i, G, \gamma + 4M\delta)$. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$ and let $\sigma_{-i}: \Theta \times \mathcal{T}_{-i}^k \to \Delta(A_{-i})$ be a k-order γ-rationalizable conjecture such that[24]

$$\int_{\Theta\times\mathcal{T}_{-i}^k} \big(g_i(a_i, \sigma_{-i}(\theta, t_{-i}^k), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}^k), \theta)\big) t_i^{k+1}(d\theta \times dt_{-i}^k) \geq -\gamma. \tag{6}$$

[24]Recall that t_i^k designates the k-order belief of type t_i.

Pick any measurable function $\mathrm{a}_{-i}:\Theta\times\mathcal{T}^k_{-i}\to A_{-i}$ such that

$$\mathrm{a}_{-i}(\theta,t^k_{-i})\in\operatorname*{arg\,max}_{a_{-i}\in R^k_{-i}(t^k_{-i},G,\gamma+4M\delta)}(g_i(a_i,a_{-i},\theta)-g_i(\alpha_i,a_{-i},\theta))\quad\forall(\theta,t^k_{-i})\in\Theta\times\mathcal{T}^k_{-i}.$$

By construction, a_{-i} is a k-order $(\gamma+4M\delta)$-rationalizable conjecture. Thus, by Proposition 1, to conclude that $a_i\in R^{k+1}_i(s_i,G,\gamma+4M\delta)$, we need show only that

$$\int_{\Theta\times\mathcal{T}^k_{-i}}\big(g_i(a_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)\big)s^{k+1}_i(d\theta\times dt^k_{-i})\geq-\gamma-4M\delta. \tag{7}$$

Let $\bar{A}_1,\dots,\bar{A}_L$ be an enumeration of the nonempty subsets of A_{-i} and define

$$h_\ell(\theta)=\max_{a_{-i}\in\bar{A}_\ell}[g_i(a_i,a_{-i},\theta)-g_i(\alpha_i,a_{-i},\theta)]\quad\forall\theta\in\Theta,\forall 1\leq\ell\leq L.$$

Next, define a partition $\{P_1,\dots,P_L\}$ of $\mathcal{T}^k_{-i}$ as

$$P_\ell=\{t^k_{-i}\in\mathcal{T}^k_{-i}:R^k_{-i}(t^k_{-i},G,\gamma)=\bar{A}_\ell\}\quad\forall 1\leq\ell\leq L.$$

Since σ_{-i} is a k-order γ-rationalizable conjecture, we have

$$h_\ell(\theta)\geq g_i(a_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)\quad\forall(\theta,t^k_{-i})\in\Theta\times P_\ell$$

and, therefore,

$$\sum_{\theta\in\Theta}\sum_{\ell=1}^{L}h_\ell(\theta)t^{k+1}_i[\theta\times P_\ell]\geq\int_{\Theta\times\mathcal{T}^k_{-i}}\big[g_i(a_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)\big]t^{k+1}_i(d\theta\times dt^k_{-i}). \tag{8}$$

Likewise, define a partition $\{Q_1,\dots,Q_L\}$ as

$$Q_\ell=\{t^k_{-i}\in\mathcal{T}^k_{-i}:R^k_{-i}(t^k_{-i},G,\gamma+4M\delta)=\bar{A}_\ell\}\quad\forall 1\leq\ell\leq L.$$

Thus we have

$$\int_{\Theta\times\mathcal{T}^k_{-i}}[g_i(a_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)]s^{k+1}_i(d\theta\times dt^k_{-i})=\sum_{\theta\in\Theta}\sum_{\ell=1}^{L}h_\ell(\theta)s^{k+1}_i[\theta\times Q_\ell],$$

which, together with (6) and (8), implies

$$\int_{\Theta\times\mathcal{T}^k_{-i}}[g_i(a_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\mathrm{a}_{-i}(\theta,t^k_{-i}),\theta)]s^{k+1}_i(d\theta\times dt^k_{-i})$$
$$\geq\int_{\Theta\times\mathcal{T}^k_{-i}}\big[g_i(a_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)-g_i(\alpha_i,\sigma_{-i}(\theta,t^k_{-i}),\theta)\big]t^{k+1}_i(d\theta\times dt^k_{-i})$$

$$+\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta\times Q_\ell]-t_i^{k+1}[\theta\times P_\ell])$$
$$\geq -\gamma+\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta\times Q_\ell]-t_i^{k+1}[\theta\times P_\ell]).$$

Therefore, to prove (7) and conclude that $a_i\in R_i^{k+1}(s_i,G,\gamma+4M\delta)$, we need only show that

$$\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta\times Q_\ell]-t_i^{k+1}[\theta\times P_\ell])\geq -4M\delta.$$

To prove this inequality first note that the induction hypothesis implies

$$P_\ell^\delta\subseteq\bigcup_{n:\bar{A}_n\supseteq\bar{A}_\ell} Q_n\quad \forall 1\leq\ell\leq L. \tag{9}$$

Next, let $N=|\Theta|L$ and consider an enumeration $\{(\theta_n,\ell_n)\}_{n=1}^N$ of $\Theta\times\{1,\ldots,L\}$ such that for all n,

$$h_{\ell_n}(\theta_n)\geq h_{\ell_{n+1}}(\theta_{n+1}),$$

and for all m, n,

$$(\theta_m=\theta_n \text{ and } \bar{A}_{\ell_m}\supseteq\bar{A}_{\ell_n})\quad\Longrightarrow\quad m\leq n.\text{[25]} \tag{10}$$

Thus, for each $n=1,\ldots,N$,

$$\begin{aligned} s_i^{k+1}\left[\bigcup_{m=1}^{n}\theta_m\times Q_{\ell_m}\right] &\geq s_i^{k+1}\left[\bigcup_{m=1}^{n}\theta_m\times P_{\ell_m}^\delta\right]\quad\text{(by (9) and (10))}\\ &= s_i^{k+1}\left[\left(\bigcup_{m=1}^{n}\theta_m\times P_{\ell_m}\right)^\delta\right]\\ &\geq t_i^{k+1}\left[\bigcup_{m=1}^{n}\theta_m\times P_{\ell_m}\right]-\delta\quad(\text{by } d_i^{k+1}(s_i,t_i)<\delta)\end{aligned}$$

and, therefore,

$$\begin{aligned}&\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)(s_i^{k+1}[\theta\times Q_\ell]-t_i^{k+1}[\theta\times P_\ell])\\ &\quad=\sum_{n=1}^{N} h_{\ell_n}(\theta_n)(s_i^{k+1}[\theta_n\times Q_{\ell_n}]-t_i^{k+1}[\theta_n\times P_{\ell_n}])\end{aligned}$$

[25]To see why an enumeration of $\Theta\times\{1,\ldots,L\}$ that satisfies these two properties exists, note that it follows directly from the definition of $h_\ell(\theta)$ that $\bar{A}_\ell\supseteq\bar{A}_m$ implies $h_\ell(\theta)\geq h_m(\theta)$.

$$= \sum_{n=1}^{N-1} (h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})) \sum_{m=1}^{n} (s_i^{k+1}[\theta_m \times Q_{\ell_m}] - t_i^{k+1}[\theta_m \times P_{\ell_m}])$$

$$= \sum_{n=1}^{N-1} \underbrace{(h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1}))}_{\geq 0} \underbrace{\left(s_i^{k+1}\left[\bigcup_{m=1}^{n} \theta_m \times Q_{\ell_m}\right] - t_i^{k+1}\left[\bigcup_{m=1}^{n} \theta_m \times P_{\ell_m}\right]\right)}_{\geq -\delta}$$

$$\geq -\delta \sum_{n=1}^{N-1} (h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})) = -\delta[h_{\ell_1}(\theta_1) - h_{\ell_N}(\theta_N)] \geq -4M\delta$$

as required. □

Corollary 1. *The Borel σ-algebras of the UW, US, S, and product topologies coincide.*

Proof. Theorem 1 implies that the Borel σ-algebra of the US topology is contained in the Borel σ-algebra of the UW topology. Moreover, Lemma 4 in Dekel et al. (2006) implies that the Borel σ-algebra of the strategic topology contains the product σ-algebra. Hence, it suffices to show that the product σ-algebra contains the UW σ-algebra. In effect, every uniform-weak ball is a countable intersection of cylinders, therefore, every uniform-weak ball is product-measurable, which implies that every UW-measurable set is product measurable. □

An important implication of this corollary is that the Mertens–Zamir universal type space $(\mathcal{T}_i, \mu_i)_{i\in I}$ remains a universal type space when equipped with any of the topologies S, US, or UW instead of the product topology, a fact that was not known prior to this paper. Indeed these topologies leave the measurable structure unchanged, so $\mu_i : \mathcal{T}_i \to \Delta(\Theta \times \mathcal{T}_{-i})$ remains the unique belief-preserving mapping and a Borel isomorphism, albeit no longer a homeomorphism.

3.2 *S convergence to finite types implies UW convergence*

Here we provide a partial converse to Theorem 1. We show that, as far as convergence to finite types is concerned, convergence in the S topology implies convergence in the UW topology (and hence also in the US topology).

Theorem 2. *Around finite types the S topology is finer than the UW topology, i.e., for each player $i \in I$, finite type $t_i \in \mathcal{T}_i$ and $\delta > 0$ there exists $\varepsilon > 0$ such that for each $s_i \in \mathcal{T}_i$,*

$$d_i^{\mathrm{S}}(s_i, t_i) \leq \varepsilon \quad \Longrightarrow \quad d_i^{\mathrm{UW}}(s_i, t_i) \leq \delta.$$

This theorem is a direct implication of Proposition 3 below, which in turn relies on the following result.

LEMMA 1. *Let $(T_i)_{i\in I}$ be a finite type space. For every $\delta > 0$, there exist $\varepsilon > 0$ and a game $G = (A_i, g_i)_{i\in I}$ with $A_i \supseteq T_i$ for all $i \in I$, such that for every $i \in I$ and $t_i \in T_i$,*

$$t_i \in \arg\max_{a_i \in A_i} \sum_{\theta\in\Theta} \sum_{t_{-i}\in T_{-i}} g_i(a_i, t_{-i}, \theta) t_i[\theta, t_{-i}], \tag{11}$$

and for every $\psi \in \Delta(\Theta \times A_{-i})$ such that $\psi[D] \le t_i[D] - \delta$ for some $D \subseteq \Theta \times T_{-i}$,

$$\min_{a_i\in A_i} \sum_{\theta\in\Theta} \sum_{a_{-i}\in A_{-i}} (g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta))\psi[\theta, a_{-i}] < -\varepsilon. \tag{12}$$

The proof of this lemma, given in the Appendix, uses a "report-your-beliefs" game embedded in a "coordination" game. More precisely, we construct a game where each player i chooses a point in a finite grid $A_i \subseteq \Delta(\Theta \times T_{-i})$ that includes all types in T_i (viewed as probability distributions over $\Theta \times T_{-i}$). If player $-i$ chooses an action in T_{-i} the payoff to player i is given by a *proper scoring rule*,[26,27] which guarantees that coordinating on truthful reporting has the best-reply property, as shown in (11). If, instead, player $-i$ chooses an action in $A_{-i} \setminus T_{-i}$, then the payoff to player i is no greater than the minimum payoff under the scoring rule and strictly less when choosing an action in T_i. Thus, if the grid $A_i \subseteq \Delta(\Theta \times T_{-i})$ is sufficiently fine, no action $t_i \in T_i$ can be an ε-best reply to a conjecture $\psi \in \Delta(\Theta \times A_{-i})$ that is far from t_i (viewed as a probability distribution over $\Theta \times A_{-i}$), as shown in (12). Indeed, either ψ assigns large probability to $-i$ choosing an action in $A_{-i} \setminus T_{-i}$, which makes any $a_i \in A_i \setminus T_i$ a profitable deviation, or it assigns enough probability to $\Theta \times T_{-i}$ so that the conditional $\bar{\psi} = \psi(\cdot|\Theta \times T_{-i})$ is close to ψ and hence far from t_i. Thus, in both cases, any grid point $a_i \in A_i \setminus T_i$ sufficiently close to $\bar{\psi}$ is a profitable deviation.

PROPOSITION 3. *Let $(T_i)_{i\in I}$ be a finite type space. For each $\delta > 0$, there exist $\varepsilon > 0$ and a game G such that for each integer $k \ge 1$, each player $i \in I$, and each $(t_i, s_i) \in T_i \times \mathcal{T}_i$,*

$$d_i^k(s_i, t_i) > \delta \quad\Longrightarrow\quad R_i(t_i, G, 0) \nsubseteq R_i^k(s_i, G, \varepsilon).$$

[26] A *proper scoring rule* on a measurable space Ω is a measurable function $f : \Omega \times \Delta(\Omega) \to \mathbb{R}$ such that $\int f(\omega, \mu)\mu(d\omega) \ge \int f(\omega, \mu')\mu(d\omega)$ for all μ, $\mu' \in \Delta(\Omega)$, with strict inequality whenever $\mu' \ne \mu$. In the proof of the lemma, we use the scoring rule $f_i : \Theta \times T_{-i} \times \Delta(\Theta \times T_{-i}) \to [-1, 1]$ such that $(\theta, t_{-i}, \psi) \mapsto 2\psi[\theta, t_{-i}] - \|\psi\|_i^2$.

[27] Dekel et al. (2006) use a report-your-beliefs game to prove their Lemma 4, which states that for every $k \ge 1$ and $\delta > 0$, there exists $\varepsilon > 0$ such that for all $t_i, s_i \in \mathcal{T}_i$, $d_i^k(s_i, t_i) \ge \delta$ implies $d_i^{\mathrm{S}}(s_i, t_i) \ge \varepsilon$. However, it can be shown that, as $k \to \infty$, the number of actions in their game grows without bound and ε shrinks to 0. Thus, we cannot use a similar construction to prove our result. The game we construct differs from theirs in two respects: First, in our game the players report infinite hierarchies of beliefs, albeit in a *finite* type space, whereas in their game players report only finitely many orders; second, Dekel et al. (2006) use a pure report-your-beliefs game, while we embed a report-your-beliefs game in a coordination game. The coordination feature ensures that the rationalizable outcomes of our game hinge on infinitely many levels of the hierarchy. This is important because when types fail to be close under d_i^{UW}, there is no upper bound on the lowest order at which the failure of proximity occurs.

Proof. Fix a finite type space $(T_i)_{i\in I}$ and $\delta>0$. Choose $0<\eta<\delta$ such that for all $k\geq 1$, $i\in I$ and $t_i, u_i\in T_i$,[28]

$$t_i^k \neq u_i^k \quad \Longrightarrow \quad d_i^k(t_i,u_i)>2\eta. \tag{13}$$

By Lemma 1, there exist $\varepsilon>0$ and a game $G=(A_i,g_i)_{i\in I}$ with $A_i\supseteq T_i$ such that (11) and (12) hold for every $t_i\in T_i$ and every $\psi\in\Delta(\Theta\times A_{-i})$ such that $\psi[D]\leq t_i[D]-\eta$ for some $D\subseteq\Theta\times T_{-i}$. Thus, for each $(t_i,s_i)\in T_i\times\mathcal{T}_i$ and each measurable function $\sigma_{-i}:\Theta\times\mathcal{T}_{-i}\to\Delta(A_{-i})$, if for some $D\subseteq\Theta\times T_{-i}$,

$$\sum_{(\theta,a_{-i})\in D}\underbrace{\int_{\mathcal{T}_{-i}}\sigma_{-i}(\theta,s_{-i})[a_{-i}]s_i(\theta\times ds_{-i})}_{\psi(\theta,a_{-i})}\leq t_i[D]-\eta,$$

then for some $a_i\in A_i$,

$$\int_{\Theta\times\mathcal{T}_{-i}}\big[g_i(t_i,\sigma_{-i}(\theta,s_{-i}),\theta)-g_i(a_i,\sigma_{-i}(\theta,s_{-i}),\theta)\big]s_i(d\theta\times ds_{-i})<-\varepsilon. \tag{14}$$

We now show that for each $i\in I$,

$$t_i\in R_i(t_i,G,0)\quad \forall t_i\in T_i \tag{15}$$

$$d_i^k(s_i,t_i)\geq\eta \quad\Longrightarrow\quad t_i\notin R_i^k(s_i,G,\varepsilon)\quad \forall k\geq 1, \forall(t_i,s_i)\in T_i\times\mathcal{T}_i. \tag{16}$$

For $i\in I$ and $t_i\in T_i$ consider the conjecture $\sigma_{-i}:\Theta\times T_{-i}\to\Delta(A_{-i})$ with $\sigma_{-i}(\theta,t_{-i})[t_{-i}]=1$ for all $(\theta,t_{-i})\in\Theta\times T_{-i}$. Then action t_i is a best reply for type t_i to conjecture σ_{-i} by (11), hence $t_i\in R_i(t_i,G,0)$ by the characterization of ICR in terms of best-reply sets, thus proving (15).

To prove (16) for $k=1$, pick $s_i\in\mathcal{T}_i$ with $d_i^1(s_i,t_i)\geq\eta$. Then there exists $E\subseteq\Theta$ such that $s_i^1[E]\leq t_i^1[E]-\eta$ and, hence, for every $\sigma_{-i}:\Theta\times\mathcal{T}_{-i}\to\Delta(A_{-i})$, letting $D=E\times T_{-i}$,

$$\sum_{(\theta,a_{-i})\in D}\int_{\mathcal{T}_{-i}}\sigma_{-i}(\theta,s_{-i})[a_{-i}]s_i(\theta\times ds_{-i})=\sum_{\theta\in E}\int_{\mathcal{T}_{-i}}\sigma_{-i}(\theta,s_{-i})[T_{-i}]s_i(\theta\times ds_{-i})$$
$$\leq s_i^1[E]\leq t_i^1[E]-\eta=t_i[D]-\eta.$$

It follows from (14) that $t_i\notin R_i^1(s_i,G,\varepsilon)$.

Proceeding by induction, let $k\geq 2$ and assume that (16) holds for $k-1$. Fix $i\in I$ and $t_i\in T_i$, and pick $s_i\in\mathcal{T}_i$ with $d_i^k(t_i,s_i)\geq\eta$. Then there exists some $E\subseteq\Theta\times\pi_{-i}^{k-1}(T_{-i})$ with

$$s_i^k[E^\eta]\leq t_i^k[E]-\eta. \tag{17}$$

Define $D=\{(\theta,t_{-i})\in\Theta\times T_{-i}:(\theta,t_{-i}^{k-1})\in E\}$, so that $t_i[D]=t_i^k[E]$. Consider an arbitrary $(k-1)$-order ε-rationalizable conjecture $\sigma_{-i}:\Theta\times\mathcal{T}_{-i}\to\Delta(A_{-i})$, i.e.,

$$\operatorname{supp}\sigma_{-i}(\theta,s_{-i})\subseteq R_{-i}^{k-1}(s_{-i},G,\varepsilon)\quad\forall(\theta,s_{-i})\in\Theta\times\mathcal{T}_{-i}.$$

[28]Such positive η exists because, given any finite type space $(T_i)_{i\in I}$, there exists $K\geq 1$ such that $d_i^k(t_i,u_i)=d_i^K(t_i,u_i)$ for all $k\geq K$ and $t_i,u_i\in T_i$.

By the induction hypothesis and the condition above,

$$d_{-i}^{k-1}(s_{-i},t_{-i}) \geq \eta \quad \Longrightarrow \quad \sigma_{-i}(\theta,s_{-i})[t_{-i}]=0 \quad \forall(\theta,s_{-i},t_{-i}) \in \Theta \times \mathcal{T}_{-i} \times T_{-i}. \tag{18}$$

Thus,

$$\begin{aligned}
&\sum_{(\theta,a_{-i})\in D} \int_{\mathcal{T}_{-i}} \sigma_{-i}(\theta,s_{-i})[a_{-i}] s_i(\theta \times ds_{-i}) \\
&\quad = \sum_{(\theta,t_{-i}^{k-1})\in E} \int_{\mathcal{T}_{-i}} \sigma_{-i}(\theta,s_{-i})[T_{-i} \cap (\pi_{-i}^{k-1})^{-1}(t_{-i}^{k-1})] s_i(\theta \times ds_{-i}) \\
&\quad \leq \sum_{(\theta,t_{-i}^{k-1})\in E} \int_{(\pi_{-i}^{k-1})^{-1}(\{t_{-i}^{k-1}\}^{\eta})} \sigma_{-i}(\theta,s_{-i})[T_{-i} \cap (\pi_{-i}^{k-1})^{-1}(t_{-i}^{k-1})] s_i(\theta \times ds_{-i}) \\
&\quad \leq \sum_{(\theta,t_{-i}^{k-1})\in E} s_i^k[\theta \times \{t_{-i}^{k-1}\}^{\eta}] = s_i^k[E^{\eta}] \leq t_i^k[E] - \eta = t_i[D] - \eta,
\end{aligned}$$

where the first inequality follows from (18), the second equality follows from (13), and the last inequality follows from (17). By (14), this implies $t_i \notin R_i^k(s_i, G, \varepsilon)$. □

Theorems 1 and 2 combined yield the following corollary.

Corollary 2. *The UW, US, and S topologies are all equivalent around finite types.*

To end this section, we remark that in Theorem 2 we cannot dispense with the assumption that t_i is a finite type. Indeed, in the next section we prove that the US topology is strictly finer than the S topology. Thus, the UW topology cannot be equivalent to the S topology, for we have shown that the UW topology is finer than the US topology (Theorem 1).

A more direct way to argue that the UW topology is strictly finer than the S topology is to note that the universal type space is not separable under the UW topology (a result that is interesting in its own right), whereas Dekel et al. (2006) show that a countable set of finite types is dense under the strategic topology. To see why the uniform-weak topology is not separable, fix two states θ_0 and θ_1 in Θ, and consider the nonredundant type space $(X_i)_{i\in I}$, where $X_i = \{0,1\}^{\mathbb{N}}$ and each type $x_i = (x_{i,n})_{n\in\mathbb{N}}$ assigns probability 1 to the pair $(\theta_{x_{i,1}}, L_i(x_i))$, where $L_i : X_i \to X_{-i}$ is the shift operator, i.e., $L((x_{i,1}, x_{i,2}, \ldots)) = (x_{i,2}, x_{i,3}, \ldots)$ for each $x_i = (x_{i,n})_{n\in\mathbb{N}}$. Clearly, the UW distance between any two different types in X_i is 1 and, hence, under the UW metric, X_i is a discrete subset of the universal type space. Since X_i is uncountable, it follows that the universal type space is not separable under the UW topology.

4. Nongenericity of finite types

Dekel et al. (2006) show that finite types are dense under the S topology, thus strengthening an early result of Mertens and Zamir (1985) that finite types are dense under the product topology. In contrast, in Theorem 3 below we show that under the US topology, finite types are *nowhere dense*, i.e., the closure of finite types has an empty interior.[29] An implication of this result and Theorem 1 is that the US topology is *strictly* finer than the S topology.[30]

The proof of Theorem 3 relies on Lemmas 2 and 3 below. Lemma 2 states that finite types are not dense under the UW topology. To prove this, we consider an instance of the countably infinite common-prior type space from Rubinstein's (1989) E-Mail game and show that none of its types can be UW-approximated by a sequence of finite types. In Lemma 3 we show that any sequence of types that fails to converge to a type in the E-Mail type space under the UW topology must also fail to converge under the US topology. Together, these lemmas imply that finite types are bounded away from the E-mail type space in US distance, which we state as Proposition 4 below. This implies that the set of finite types is not dense under the US topology. Using this result, the proof of Theorem 3 shows that every finite type can be US-approximated by a sequence of infinite types, none of which is the US limit of a sequence of finite types, thereby establishing nowhere denseness.

In effect, consider the following instance of the E-Mail type space. Let $\Theta = \{\theta_0, \theta_1\}$ and let the type space (U_1, U_2) be[31]

$$U_1 = \{u_{1,0}, u_{1,1}, u_{1,2}, \dots\}, \qquad U_2 = \{u_{2,0}, u_{2,1}, u_{2,2}, \dots\},$$

where $u_{1,0}[\theta_0, u_{2,0}] = 1$, $u_{2,0}[\theta_0, u_{1,0}] = 2/3$, $u_{2,0}[\theta_1, u_{1,1}] = 1/3$,

$$u_{1,n}[\theta_1, u_{2,n-1}] = 2/3, \qquad u_{1,n}[\theta_1, u_{2,n}] = 1/3 \quad \forall n \geq 1$$
$$u_{2,n}[\theta_1, u_{1,n}] = 2/3, \qquad u_{2,n}[\theta_1, u_{1,n+1}] = 1/3 \quad \forall n \geq 1.$$

We have the following result.

Proposition 4. *For every $i \in I$, finite type $t_i \in \mathcal{T}_i$, and $n \geq 0$, $d_i^{\mathrm{US}}(t_i, u_{i,n}) \geq M/6$.*

The proposition is a direct consequence of the following two lemmas.

Lemma 2. *For every $i \in I$, finite type $t_i \in \mathcal{T}_i$, and $n \geq 0$, $d_i^{\mathrm{UW}}(t_i, u_{i,n}) \geq 1/3$.*

Lemma 3. *For every $i \in I$, $t_i \in \mathcal{T}_i$, and $n \geq 0$, $d_i^{\mathrm{US}}(t_i, u_{i,n}) \geq (M/2) d_i^{\mathrm{UW}}(t_i, u_{i,n})$.*

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[29]This is equivalent to saying that the complement of the set of finite types contains an open and dense set under the US topology.

[30]Dekel et al. (2006) state the result that the US topology is strictly finer than the S topology. However, as reported in Chen and Xiong (2008), the proof in that paper contains a mistake.

[31]This type space is an instance of the E-Mail type space where the more informed player 1 who received k messages attaches probability $p = 2/3$ (resp. $1 - p = 1/3$) to player 2 having received $k - 1$ (resp. k) messages, and the less informed player 2 who received k messages attaches probability p (resp. $1 - p$) to player 1 having received k (resp. $k + 1$) messages. Our choice that $p = 2/3$ is immaterial; our results hold true if we assume any other value for p.

θ_0	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	s_2
$a_{1,0}$	$0, 0$	$-4, 1$	$-4, -1$	$-4, -4$	$-4, 0$
$a_{1,1}$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, 0$
$b_{1,1}$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, 0$
$c_{1,1}$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, 0$
s_1	$0, -4$	$0, -4$	$0, -4$	$0, -4$	$0, 0$

θ_1	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	s_2
$a_{1,0}$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, -4$	$-4, 0$
$a_{1,1}$	$0, 0$	$-4, -2$	$-4, 2$	$0, 0$	$-4, 0$
$b_{1,1}$	$1, -4$	$-4, -4$	$-4, -4$	$-2, -4$	$-4, 0$
$c_{1,1}$	$-1, -4$	$-4, -4$	$-4, -4$	$2, -4$	$-4, 0$
s_1	$0, -4$	$0, -4$	$0, -4$	$0, 0$	$0, 0$

FIGURE 1. The game from Lemma 3 for $N = 1$ and $M = 4$.

In the proof of Lemma 2, given in the Appendix, we first show that the UW distance between any two distinct types of any player in the E-Mail type space above is at least 2/3.[32] Second, we show that any finite type $t_{2,n}$ whose UW distance from $u_{2,n}$ is less than 1/3 must attach positive probability to (and hence implies the existence, in the same finite type space, of) a type $t_{1,n+1}$ whose UW distance from $u_{1,n+1}$ is less than 1/3, which in turn implies the existence in the same finite type space of some type $t_{2,n+1}$ whose UW distance from $u_{2,n+1}$ is less than 1/3 and so on. These two facts together imply the contradiction that the types $t_{i,1}, t_{i,2}, \ldots$ are all different but belong to the same finite type space, whence the result follows.

Turning to Lemma 3, the proof, also in the Appendix, constructs, for each $\delta \geq 0$ and $N \geq 0$, a game such that for each $0 \leq n \leq N$, a certain action $a_{i,n}$ is rationalizable for $u_{i,n}$ but is not δ-rationalizable for any type t_i with $d_i^k(t_i, u_{i,n}) > 2\delta/M$, where the order k grows with the difference $N - n$. To provide intuition, we sketch the argument for the case $N = 1$. The game corresponding to this case is depicted in Figure 1, with the payoff bound normalized to $M = 4$.

It is clear that in this game, for all $i = 1, 2$ and $n = 0, 1$, action $a_{i,n}$ is rationalizable for $u_{i,n}$.[33] However, $a_{i,n}$ is weakly dominated by s_i, and the payoffs from $b_{i,n}$ and $c_{i,n}$ are such that whenever the beliefs of a type t_i are sufficiently far from those of $u_{i,n}$, then any δ-rationalizable conjecture about player $-i$ that δ-rationalizes $a_{i,n}$ against s_i cannot do so against both $b_{i,n}$ and $c_{i,n}$ as well. Indeed, we have

$$d_i^k(t_i, u_{i,n}) > 2\delta/M \quad \Longrightarrow \quad a_{i,n} \notin R_i^k(t_i, \delta) \quad \forall 1 \leq k \leq 2 - n. \tag{19}$$

To see this for $k = 1$, first note that $a_{1,0}$ is weakly dominated by s_1, hence $a_{1,0} \notin R_1^1(t_i, \delta)$ for any type t_1 with $d_1^1(t_1, u_{1,0}) > \delta/2$. Indeed, $u_{1,0}^1[\theta_0] = 1$ and hence $d_1^1(t_1, u_{1,0}) > \delta/2$ implies $t_1^1[\theta_0] < 1 - \delta/2$, so the highest possible expected payoff for t_1 under $a_{1,0}$ is -2δ, whereas s_1 yields 0. By the same token, $a_{1,1} \notin R_1^1(t_1, \delta)$ for any t_1 with $d_1^1(t_1, u_{1,1}) > \delta/2$ and $a_{2,1} \notin R_2^1(t_2, \delta)$ for any t_2 with $d_2^1(t_2, u_{2,1}) > \delta/2$. Consider action $a_{2,0}$ now and pick any t_2 such that $d_2^1(t_2, u_{2,0}) > \delta/2$. Since $u_{2,0}^1[\theta_1] = 1/3$, we must have either $t_2^1[\theta_1] <$

[32] The type $u_{1,k}$ of player 1 who received k messages assigns probability 2/3 to the other player having received $k - 1$ messages, while $u_{1,k+1}$ attaches probability 0 to that event, and similarly for player 2.

[33] The pair $(\varsigma_1, \varsigma_2)$ with $\varsigma_i(u_{i,n}) = a_{i,n}$ if $n \leq 1$ and $\varsigma_i(u_{i,n}) = s_i$ if $n \geq 2$ has the best-reply property.

$1/3-\delta/2$ or $t_2^1[\theta_0] < 2/3+\delta/2$. Pick any conjecture σ_1 that δ-rationalizes $a_{2,0}$, so that the difference in expected payoff between s_1 and $a_{2,0}$ is at most δ. This requires the induced distribution over $\Theta \times A_1$ to satisfy

$$\Pr[\theta_0, a_{1,0}|t_2, \sigma_1] + \Pr[\theta_1, a_{1,1}|t_2, \sigma_1] \geq 1-\delta/4,$$

hence the difference in expected payoffs between $b_{2,0}$ and $a_{2,0}$ is

$$\Pr[\theta_0, a_{1,0}|t_2, \sigma_1] - 2\Pr[\theta_1, a_{1,1}|t_2, \sigma_1] \geq -3\Pr[\theta_1, a_{1,1}|t_2, \sigma_1] + 1 - \delta/4,$$

which is greater than δ when $t_2^1[\theta_1] < 1/3-\delta/2$. Likewise, the difference in expected payoffs between $c_{2,0}$ and $a_{2,0}$ is

$$-\Pr[\theta_0, a_{1,0}|t_2, \sigma_1] + 2\Pr[\theta_1, a_{1,1}|t_2, \sigma_1] \geq -3\Pr[\theta_0, a_{1,0}|t_2, \sigma_1] + 2 - \delta/2,$$

which is greater than δ when $t_2^1[\theta_0] < 2/3+\delta/2$. Thus, in any case, $a_{2,0} \notin R_2^1(t_2, \delta)$ and the proof of (19) for $k=1$ is complete. The proof for $n=0$ and $k=2$ uses the arguments just given for the case $k=1$ and is completely analogous—for instance, those arguments show that if σ_2 is a first-order δ-rationalizable conjecture that δ-rationalizes $a_{1,0}$ for a type t_1, then we must have $1-\delta/4 \leq \Pr[\theta_0, a_{2,0}|t_1, \sigma_2] \leq t_1^2[\theta_0 \times \{u_{2,0}^1\}^{2\delta/M}]$ and hence the distance between the second-order beliefs of t_1 and $u_{1,0}$ is at most δ.

We are now ready to prove the main result of this section.

Theorem 3. *Finite types are nowhere dense under the US and the UW topology.*

Proof. It suffices to prove that every finite type can be UW-approximated by a sequence of infinite types, none of which is the US limit of a sequence of finite types.[34] Fix a finite type space (T_1, T_2) and a type $t_2 \in T_2$. For each $n \geq 1$, let $\delta_n = 1/(n+1)$ and define the infinite type $t_{2,n}$ by the requirement that, for every $k \geq 1$ and every measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$,

$$t_{2,n}^k[E] = (1-\delta_n)t_2^k[E] + \delta_n u_{2,0}^k[E].$$

Note that for all $n \geq 1$, $k \geq 1$, and measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$, we have

$$t_{2,n}^k[E] = (1-\delta_n)t_2^k[E] + \delta_n u_{2,0}^k[E] \leq t_{2,n}^k[E^{\delta_n}] + \delta_n,$$

hence $d_2^{\mathrm{UW}}(t_{2,n}, t_2) \leq \delta_n \longrightarrow 0$.

It remains to prove that none of the types in the sequence $(t_{2,n})_{n\geq 1}$ is in the US closure of the set of finite types, i.e., for every $n \geq 1$, there exists $\varepsilon_n > 0$ such that the US distance between $t_{2,n}$ and every finite type in $\mathcal{T}_2$ is at least ε_n. Thus, fix $n \geq 1$, pick any $0 < \varepsilon_n < \min\{M/6, M/(3n+1)\}$, any finite type space (S_1, S_2), and any type $s_2 \in S_2$, and let us show that $d_2^{\mathrm{US}}(t_{2,n}, s_2) \geq \varepsilon_n$. Using Lemma 2, choose $N \geq 1$ large enough so that

$$d_1^{2(N+1)}(t_1, u_{1,0}) \geq 1/3 \quad \forall t_1 \in T_1 \cup S_1 \tag{20}$$

[34] Indeed, by Theorem 1, the sequence also US-approximates the finite type, hence nowhere denseness in the US topology follows. By the same theorem, none of the types in the sequence will be the UW limit of a sequence of finite types, thus nowhere denseness in the UW topology also follows.

and let $G_N = (A_{i,N}, g_{i,N})_{i=1,2}$ be the game defined in the proof of Lemma 3. Now define another game $G'_N = (A'_{i,N}, g'_{i,N})_{i=1,2}$ as

$$A'_{1,N} = A_{1,N}, \qquad A'_{2,N} = A_{2,N} \times \{0,1\},$$

and for all $a_1 \in A_{1,N}$, $a_2 \in A_{2,N}$, $x \in \{0,1\}$ and $\theta \in \Theta$,

$$g'_{1,N}(a_1, a_2, x, \theta) = \tfrac{1}{2} g_{1,N}(a_1, a_2, \theta)$$

$$g'_{2,N}(a_1, a_2, x, \theta) = \tfrac{1}{2} g_{2,N}(a_1, a_2, \theta) + \begin{cases} M/2 & \text{if } x = 1 \text{ and } a_1 = a_{1,0} \\ -M/(3n+1) & \text{if } x = 1 \text{ and } a_1 \neq a_{1,0} \\ 0 & \text{otherwise.} \end{cases}$$

Note that since all payoffs in G_N are between $-M$ and M, the same is true for all payoffs in G'_N. Moreover, we have the following lemma, which is proved in the Appendix.

Lemma 4. *For all $k \geq 0$ and all $\varepsilon \geq 0$,*

$$R_1^k(t_1, G_N, 2\varepsilon) = R_1^k(t_1, G'_N, \varepsilon) \quad \forall t_1 \in \mathcal{T}_1 \tag{21}$$

$$R_2^k(t_2, G_N, 2\varepsilon) = \operatorname{proj}_{A_{2,N}} R_2^k(t_2, G'_N, \varepsilon) \quad \forall t_2 \in \mathcal{T}_2. \tag{22}$$

We now prove that $(a_2, 1) \in R_2(t_{2,n}, G'_N, 0)$ for some $a_2 \in A_{2,N}$, but $(a_2, 1) \notin R_2(s_2, G'_N, \varepsilon_n)$ for all $a_2 \in A_{2,N}$, reaching the desired conclusion that $d_2^{\mathrm{US}}(t_{2,n}, s_2) \geq \varepsilon_n$.

To show that $(a_2, 1) \in R_2(t_{2,n}, G'_N, 0)$ for some $a_2 \in A_{2,N}$, it suffices to construct a rationalizable conjecture σ'_1 in game G'_N under which, for all $a_2 \in A_{2,N}$, actions $(a_2, 0)$ and $(a_2, 1)$ give $t_{2,n}$ the same expected payoff. Let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be an arbitrary rationalizable conjecture in G_N and define $\sigma'_1 : \Theta \times \mathcal{T}_1 \to \Delta(A'_{1,N})$ as

$$\sigma'_1(\theta, t_1)[a_1] = \sigma_1(\theta, t_1)[a_1] \quad \forall t_1 \in \mathcal{T}_1 \setminus U_1, \forall a_1 \in A'_{1,N}$$

$$\sigma'_1(\theta, u_{1,k})[a_{1,k}] = 1 \quad \forall k \geq 0.$$

From the proof of Lemma 3, it follows, using (21) with $\varepsilon = 0$, that σ'_1 is a rationalizable conjecture in G'_N and also, using (20) and the fact that $\varepsilon_n < M/6$, it follows that

$$a_{1,0} \notin R_1(t_1, G_N, \varepsilon_n) \quad \forall t_1 \in T_1 \cup S_1. \tag{23}$$

Thus, $\sigma'_1(\theta, t_1)[a_{1,0}] = 0$ for all $\theta \in \Theta$ and $t_1 \in T_1$, hence for all $a_2 \in A_{2,N}$ we have

$$\int_{\Theta \times \mathcal{T}_1} \left[g'_{2,N}(\sigma'_1(\theta, t_1), a_2, 1, \theta) - g'_{2,N}(\sigma'_1(\theta, t_1), a_2, 0, \theta) \right] t_{2,n}(d\theta \times dt_1)$$

$$= \frac{2\delta_n}{3} \frac{M}{2} - \left(1 - \frac{2\delta_n}{3}\right) \frac{M}{3n+1} = 0.$$

This proves that $(a_2, 0)$ and $(a_2, 1)$ give type $t_{2,n}$ the same expected payoff under σ'_1 for all $a_2 \in A_{2,N}$, as was to be shown.

Turning to the proof that $(a_2, 1) \notin R_2(s_2, G'_N, \varepsilon_n)$ for all $a_2 \in A_{2,N}$, consider an arbitrary ε_n-rationalizable conjecture σ'_1 in game G'_N. By (21) and (23), for all $\theta \in \Theta$ and $s_1 \in S_1$, we must have $\sigma'_1(\theta, s_1)[a_{1,0}] = 0$. Thus, for all $a_2 \in A_{2,N}$,

$$\sum_{(\theta,s_1)\in\Theta\times S_1} s_2[\theta, s_1]\big[g'_{2,N}(\sigma_1(\theta, s_1), a_2, 1, \theta) - g'_{2,N}(\sigma_1(\theta, s_1), a_2, 0, \theta)\big] = -\frac{M}{3n+1} < -\varepsilon_n,$$

which proves that $(a_2, 1)$ is not ε_n-rationalizable for s_2 in game G'_N. □

5. Discussion

5.1 *Relation with common p-beliefs*

As we mentioned in the Introduction, the uniform-weak topology is related to the notion of *common p-belief* due to Monderer and Samet (1989). Fix a state $\theta \in \Theta$ and $p \in [0, 1]$. For each player $i \in I$, define

$$B_i^{1,p}(\theta) = \{t_i^1 \in \mathcal{T}_i^1 : t_i^1[\theta] \geq p\} \quad \text{and} \quad B_i^{k,p}(\theta) = \big\{t_i^k \in \mathcal{T}_i^k : t_i^k[\theta \times B_{-i}^{k-1,p}(\theta)] \geq p\big\}$$

recursively for all $k \geq 2$. A type t_i has *common p-belief in* θ, and we write $t_i \in C_i^p(\theta)$, if $t_i^k \in B_i^{k,p}(\theta)$ for all $k \geq 1$. A sequence of types $(t_{i,n})_{n\geq 1}$ has *asymptotic common certainty of* θ if for every $p < 1$, we have $t_{i,n} \in C_i^p(\theta)$ for n large enough.

Monderer and Samet (1989) use this notion of proximity to common certainty, i.e., common 1-belief, to study the robustness of Nash equilibrium to small amounts of incomplete information. Their main result states that for any game and any sequence of common-prior type spaces, a sufficient condition for Nash equilibrium to be robust to incomplete information (relative to the given sequence of type spaces) is that for some sequence $p_n \nearrow 1$, the prior probability of the event that the players have common p_n-belief on the payoffs from the complete information game converges to 1 as $n \to \infty$. A related paper, Kajii and Morris (1997), shows that asymptotic common certainty is actually a necessary condition for robustness in all games. Since both results are formulated for Bayesian Nash equilibrium in common-prior type spaces, to facilitate comparison with our results, we report (without proof) an analogue of their results for interim correlated rationalizability without imposing common priors.

Proposition 5. *A sequence of types $(t_{i,n})_{n\geq 1}$ has asymptotic common certainty of θ if and only if for every game and every $\varepsilon > 0$, every action that is rationalizable for player i when θ is common certainty remains interim correlated ε-rationalizable for type $t_{i,n}$ for all n large enough.*

Thus the "only if" part is an interim version of Monderer and Samet (1989, Theorem B*) and the "if" part is an interim version of Kajii and Morris (1997, Proposition 10).

As it turns out, the uniform-weak topology can be viewed as an extension of the concept of asymptotic common certainty: these two notions of convergence coincide when the limit type has common certainty of some state. Indeed, letting $t_{i,\theta}$ designate the type of player i who has common certainty of θ, we can make the following proposal.

Proposition 6. *A sequence* $(t_{i,n})_{n\geq 0}$ *has asymptotic common certainty of* θ *if and only if* $d_i^{\mathrm{UW}}(t_{i,n}, t_{i,\theta}) \to 0$ *as* $n \to \infty$.

Proof. It suffices to show that for each $i \in I$, $p \in [0,1]$, and $k \geq 1$, we have $B_i^{k,p}(\theta) = \{t_{i,\theta}^k\}^{1-p}$. For $k = 1$, this follows directly from $t_{i,\theta}^1[\theta] = 1$. Now suppose this holds for $k-1$ and let us show that it also holds for k. Indeed,

$$\begin{aligned} B_i^{k,p}(\theta) &= \{t_i^k \in \mathcal{T}_i^k : t_i^k[\theta \times B_{-i}^{k-1,p}(\theta)] \geq p\} \\ &= \{t_i^k \in \mathcal{T}_i^k : t_i^k[\theta \times \{t_{-i,\theta}^{k-1}\}^{1-p}] \geq p\} = \{t_{i,\theta}^k\}^{1-p}, \end{aligned}$$

where the second equality follows from the induction hypothesis and the third equality follows from the fact that $t_{i,\theta}^k[\theta, t_{-i,\theta}^{k-1}] = 1$. □

Thus, taken together, Theorems 1 and 2 extend Proposition 5 to environments where the limit type has nondegenerate incomplete information.[35]

5.2 *Other uniform metrics*

The Prohorov metric, on which the uniform-weak topology is based, is but one of many equivalent distances that metrize the topology of weak convergence of probability measures. For any such distance, one can consider the associated uniform distance over hierarchies of beliefs. Interestingly, these metrics can generate different topologies over infinite hierarchies, even though the induced topologies over k-order beliefs coincide for each $k \geq 1$. Below we provide such an example.

Given a metric space (S, d), let $\mathrm{BL}(S, d)$ designate the vector space of real-valued, bounded, Lipschitz continuous functions over S, endowed with the norm

$$\|f\|_{\mathrm{BL}} = \max\left\{\sup_x |f(x)|, \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)}\right\} \quad \forall f \in \mathrm{BL}(S,d).$$

Recall that the *bounded Lipschitz* distance over $\Delta(S, d)$ is

$$\beta(\mu, \mu') = \sup\left\{\left|\int f\,d\mu - \int f\,d\mu'\right| : f \in \mathrm{BL}(S,d) \text{ with } \|f\|_{\mathrm{BL}} \leq 1\right\} \quad \forall \mu, \mu' \in \Delta(S,d).$$

This distance metrizes the topology of weak convergence and it relates to the Prohorov metric ρ as[36]

$$(2/3)\rho^2 \leq \beta \leq 2\rho.$$

Now define a uniform metric β_i^{UW} over hierarchies of beliefs as follows. Let β^0 denote the discrete metric over Θ and, recursively, for $k \geq 1$, let β_i^k denote the bounded Lipschitz metric on $\Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$ when $\Theta \times \mathcal{T}_{-i}^{k-1}$ is equipped with the metric $\max\{\beta^0, \beta_{-i}^{k-1}\}$. Then

$$\beta_i^{\mathrm{UW}} = \sup_{k\geq 1} \beta_i^k.$$

[35] Note that $t_{i,\theta}$ is a finite type.

[36] See Dudley (2002, pp. 398 and 411).

For each $k \geq 1$, the metric β_i^k is equivalent to d_i^k, as they both induce the weak topology on k-order beliefs. However, as we now show, β_i^{UW} is *not* equivalent to d_i^{UW}.[37] Suppose that $\Theta = \{\theta_0, \theta_1\}$ and for each $n \geq 1$, consider the type space $(T_{i,n})_{i \in I}$, where

$$T_{i,n} = \{u_{i,0}, u_{i,1}, t_{i,n}\} \quad \forall i \in I$$

and beliefs are

$$u_{i,0}[\theta_0, u_{-i,0}] = 1, \qquad u_{i,1}[\theta_1, u_{-i,1}] = 1 \quad \forall i \in I$$

and

$$t_{i,n}[\theta_0, u_{-i,0}] = 1/n, \qquad t_{i,n}[\theta_1, t_{-i,n}] = 1 - 1/n \quad \forall i \in I.$$

Thus $d_i^k(t_{i,n}, u_{i,1}) = 1/n$ for all $k \geq 1$ and, therefore, $d_i^{\mathrm{UW}}(t_{i,n}, u_{i,1}) \to 0$ as $n \to \infty$. We now show that $\beta_i^{\mathrm{UW}}(t_{i,n}, u_{i,1}) \not\to 0$. Let f be the indicator function of $\{\theta_1\}$, i.e., $f(\theta_m) = m$ for $m \in \{0, 1\}$. Then define the k-order *iterated expectation* of f for each $k \geq 1$ and each player i, denoted $f_i^k : \mathcal{T}_i^k \to \mathbb{R}$, as

$$f_i^1(t_i^1) = \int f \, dt_i^1 = t_i^1[\theta_1] \quad \text{and} \quad f_i^k(t_i^k) = \int f_{-i}^{k-1} \, dt_i^k \quad \text{for } k \geq 2.$$

Thus, we have

$$\int f_{-i}^{k-1} \, du_{i,1}^k = 1 \quad \text{and} \quad \int f_{-i}^{k-1} \, dt_{i,n}^k = (1 - 1/n)^k.$$

Since it can be shown that $f_i^k \in \mathrm{BL}(\mathcal{T}_i^k, \beta_i^k)$ and $\|f_i^k\|_{\mathrm{BL}} \leq 1$, we have $\beta_i^k(t_{i,n}, u_{i,1}) \geq 1 - (1 - 1/n)^k$ and hence $\beta_i^{\mathrm{UW}}(t_{i,n}, u_{i,1}) \geq 1$ for every $n \geq 1$.

This example is also relevant for the comparison between our work and Morris (2002), who shows that the topology of uniform convergence of iterated expectations is equivalent to the strategic topology associated with a restricted class of games, called *higher-order expectations* (HOE) games. By this result and the example above, uniform-weak convergence is not sufficient for convergence in the strategic topology for HOE games. This might seem puzzling at first, given that uniform-weak convergence has been shown to imply convergence in Dekel et al. (2006) strategic topology, which is defined by requiring lower hemicontinuity of the strict ICR correspondence in *all* games, not just HOE games. To reconcile these facts, we note that the notion of strict ICR correspondence implicitly used in Morris (2002) is different from the one we use, in that it does not require the slack in the incentive constraints to hold uniformly in a best-reply set. Thus, for a given game, continuity of Morris (2002) notion of strict ICR is more demanding than ours.

[37] The example below actually shows that the two metrics are not equivalent even around complete information types. In particular, asymptotic common certainty does not guarantee convergence under β_i^{UW}.

6. Conclusion

Our results shed light on the connection between similarity of beliefs and similarity of behaviors in games, but leave open a number of interesting questions for future research. One question is whether uniform-weak convergence is also a necessary condition for uniform-strategic convergence. We believe the answer is in the affirmative and are pursuing this conjecture in ongoing research. This question is of particular interest because of the tension between Theorem 2 and Theorem 3, which imply that the uniform-weak and the uniform-strategic topologies are equivalent around types in a nowhere dense set. Another important avenue of research is to characterize the (nonuniform) strategic topology in terms of proximity of beliefs; we are also exploring this question in ongoing work. Finally, it would be interesting to examine strategic topologies for solution concepts that refine ICR, such as Bayesian equilibrium, incomplete information versions of correlated equilibrium, or interim independent rationalizability.

Appendix: Omitted Proofs

Proof of Proposition 1. Fix $k \geq 1$, $t_i \in \mathcal{T}_i$, and $a_i \in A_i$. Let Σ_{-i} denote the set of equivalence classes of measurable functions $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$ such that

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R^{k-1}_{-i}(t_{-i}, G, \gamma) \quad \text{for } t_i\text{-almost every } (\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i},$$

where we identify pairs of functions that are equal t_i-almost surely. The set Σ_{-i} can be viewed as a compact convex subset of the topological vector space L of (equivalence classes of) $\mathbb{R}^{|A_{-i}|}$-valued measurable functions over $\Theta \times \mathcal{T}_{-i}$.[38]

Consider the function $F : \Delta(A_i \setminus \{a_i\}) \times \Sigma_{-i} \to \mathbb{R}$ such that

$$F(\alpha_i, \sigma_{-i}) = \int_{\Theta \times \mathcal{T}_{-i}} \big[g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \sigma_{-i}(\theta, t_{-i}), \theta)\big] t_i(d\theta \times dt_{-i}).$$

Thus, F is the restriction of a continuous bilinear functional on $\mathbb{R}^{(|A_i|-1)} \times L$ to the Cartesian product of compact, convex sets. By a minmax theorem of Fan (1953),

$$\min_{\alpha_i \in \Delta(A_i \setminus \{a_i\})} \max_{\sigma_{-i} \in \Sigma_{-i}} F(\alpha_i, \sigma_{-i}) = \max_{\sigma_{-i} \in \Sigma_{-i}} \min_{\alpha_i \in \Delta(A_i \setminus \{a_i\})} F(\alpha_i, \sigma_{-i}).$$

[38]The space L is equipped with the weak* topology induced by the probability measure $t_i \in \Delta(\Theta \times \mathcal{T}_{-i})$. Under this topology, a sequence $(f_n)_{n \in \mathbb{N}}$ in L converges to $f \in L$ if and only if for each continuous and bounded function $h : \Theta \times \mathcal{T}_{-i} \to \mathbb{R}^{|A_{-i}|}$,

$$\int \langle h(\theta, t_{-i}), f_n(\theta, t_{-i}) \rangle t_i(d\theta \times dt_{-i}) \to \int \langle h(\theta, t_{-i}), f(\theta, t_{-i}) \rangle t_i(d\theta \times dt_{-i}) \quad \text{as } n \to \infty,$$

where $\langle \cdot, \cdot \rangle$ designates the Euclidean inner product in $\mathbb{R}^{|A_{-i}|}$. To see why Σ_{-i} is compact, note that the disintegration property of probability measures yields a natural homeomorphism between Σ_{-i} and

$$\big\{\nu \in \Delta(\Theta \times \mathcal{T}_{-i} \times A_{-i}) : \nu\big(\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in R^{k-1}_{-i}(t_{-i}, G, \gamma)\}\big) = 1, \operatorname{marg}_{\Theta \times \mathcal{T}_{-i}} \nu = t_i\big\},$$

which is a closed subset of the compact space $\Delta(\Theta \times \mathcal{T}_{-i} \times A_{-i})$.

Now, $a_i \in R_i^k(t_i, G, \gamma)$ if and only if the right-hand side is greater than or equal to $-\gamma$. Thus, $a_i \in R_i^k(t_i, G, \gamma)$ if and only if for every $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, there exists $\sigma_{-i} \in \Sigma_{-i}$ such that $F(\alpha_i, \sigma_{-i}) \geq -\gamma$, which is the desired result. □

Proof of Lemma 1. For each $i \in I$, let ρ_i and $\|\cdot\|_i$ denote the Prohorov distance on $\Delta(\Theta \times T_{-i})$ and the Euclidean norm on $\mathbb{R}^{|\Theta||T_{-i}|}$, respectively. Also, let $f_i : \Theta \times T_{-i} \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$f_i(\theta, t_{-i}, \psi) = 2\psi[\theta, t_{-i}] - \|\psi\|_i^2$$

and let $F_i : \Delta(\Theta \times T_{-i}) \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$F_i(\psi', \psi) = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} f_i(\theta, t_{-i}, \psi')\psi[\theta, t_{-i}].$$

Note that $F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|_i^2$ for all $\psi, \psi' \in \Delta(\Theta \times T_{-i})$, hence

$$\eta \equiv \tfrac{1}{2} \min\left\{F_i(\psi, \psi) - F_i(\psi', \psi) : \psi', \psi \in \Delta(\Theta \times T_{-i}), \rho_i(\psi, \psi') \geq \tfrac{1}{2}\delta\right\} > 0,$$

and also[39]

$$\rho_i(\psi, \psi') < \eta/2 \quad \Longrightarrow \quad F_i(\psi, \psi) - F_i(\psi', \psi) < \eta \quad \forall \psi, \psi' \in \Delta(\Theta \times T_{-i}).$$

The compact set $\Delta(\Theta \times T_{-i})$ can be covered by a finite union of open balls of radius $\eta/2$. (These balls are taken according to the metric ρ_i.) Choose one point in each of these balls and let $A_i \subseteq \Delta(\Theta \times T_{-i})$ denote the finite set of selected points. Enlarge A_i, if necessary, to ensure $A_i \supseteq T_i$. (Recall that we identify each $t_i \in T_i$ with $\mu_i(t_i)$.) Thus, for every $\psi \in \Delta(\Theta \times T_{-i})$, there exists $a_i \in A_i \setminus T_i$ such that $F_i(\psi, \psi) - F_i(a_i, \psi) < \eta$.

Now define the payoff function $g_i : \Theta \times A_i \times A_{-i} \to \mathbb{R}$, as

$$g_i(\theta, a_i, a_{-i}) = \begin{cases} f_i(\theta, a_{-i}, a_i) & \text{if } a_{-i} \in T_{-i} \\ -4/\delta & \text{if } a_i \in T_i \text{ and } a_{-i} \notin T_{-i} \\ -1 & \text{if } a_i \notin T_i \text{ and } a_{-i} \notin T_{-i}. \end{cases}$$

It follows directly from the definition of g_i and the fact that $t_i[\Theta \times T_{-i}] = 1$ that each $a_i \in A_i$ yields an expected payoff of $F_i(a_i, t_i)$ to type t_i under the conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that $\sigma_{-i}(\theta, t_{-i})[t_{-i}] = 1$ for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Since $F_i(t_i, t_i) \geq F_i(a_i, t_i)$ for all $a_i \in A_i$, (11) follows.

Fix any $0 < \varepsilon < \min\{\eta(1 - \delta/2), \delta/2\}$. We shall prove (12) now. Fix $t_i \in T_i$ and $\psi \in \Delta(\Theta \times A_{-i})$, and assume that there exists $D \subseteq \Theta \times T_{-i}$ such that $\psi[D] \leq t_i[D] - \delta$. First

[39] Letting $h : \Theta \times T_{-i} \to [-1, 1]$ denote the mapping $(\theta, t_{-i}) \mapsto h(\theta, t_{-i}) = \psi[\theta, t_{-i}] - \psi'[\theta, t_{-i}]$, for each $\zeta \geq 0$, we have

$$F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|^2 = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi[\theta, t_{-i}]h(\theta, t_{-i}) - \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi'[\theta, t_{-i}]h(\theta, t_{-i}) \leq 2\zeta$$

whenever $\rho_i(\psi, \psi') \leq \zeta$.

suppose $\psi[\Theta \times T_{-i}] < 1 - \delta/2$. Pick any $a_i \in A_i \setminus T_i$. Since f_i maps into $[-1, 1]$,

$$\sum_{\theta\in\Theta}\sum_{a_{-i}\in A_{-i}}(g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta))\psi[\theta, a_{-i}] \le 2(1-\delta/2) + (\delta/2)(-4/\delta + 1)$$
$$= -\delta/2 < -\varepsilon,$$

which proves (12) for the case $\psi[\Theta \times T_{-i}] < 1 - \delta/2$. Now suppose that $\psi[\Theta \times T_{-i}] \ge 1 - \delta/2$. Consider the conditional probability $\bar\psi(\cdot) \equiv \psi(\cdot|\Theta \times T_{-i})$. Then

$$\bar\psi[D] \ge \psi[D] = \bar\psi[D]\psi[\Theta \times T_{-i}] \ge \bar\psi[D] - \delta/2,$$

hence

$$|\bar\psi[D] - t_i[D]| \ge |\psi[D] - t_i[D]| - |\psi[D] - \bar\psi[D]| \ge \delta - \delta/2 = \delta/2,$$

which implies $F_i(\bar\psi, \bar\psi) - F_i(t_i, \bar\psi) \ge 2\eta$ by the definition of η. Now pick any $a_i \in A_i \setminus T_i$ with $\rho_i(\bar\psi, a_i) < \eta/2$, so that $F_i(a_i, \bar\psi) - F_i(\bar\psi, \bar\psi) > -\eta$. Then $F_i(a_i, \bar\psi) - F_i(t_i, \bar\psi) > \eta$ and hence

$$\sum_{\theta\in\Theta}\sum_{a_{-i}\in A_{-i}}(g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta))\psi[\theta, a_{-i}]$$
$$= (F_i(t_i, \bar\psi) - F_i(a_i, \bar\psi))\psi[\Theta \times T_{-i}] + (-4/\delta + 1)(1 - \psi[\Theta \times T_{-i}])$$
$$\le (F_i(t_i, \bar\psi) - F_i(a_i, \bar\psi))\psi[\Theta \times T_{-i}] < (1-\delta/2)(-\eta) < -\varepsilon,$$

which proves (12) also for the case $\psi[\Theta \times T_{-i}] \ge 1 - \delta/2$.[40] □

Proof of Lemma 2. First we prove by induction that

$$d_i^{\text{UW}}(u_{i,n}, u_{i,m}) \ge 2/3 \quad \forall i = 1, 2, \forall n \ge 0, \forall m \ge 0 \text{ s.t. } m \ne n. \tag{24}$$

For all $n \ge 1$ we have $u^1_{1,0}[\theta_0] = 1$ and $u^1_{1,n}[\theta_0] = 0$, hence $d^1_1(u_{1,0}, u_{1,n}) = 1 > 2/3$; moreover, $u^1_{2,0}[\theta_0] = 2/3$ and $u^1_{2,n}[\theta_0] = 0$, hence $d^1_2(u_{2,0}, u_{2,n}) \ge 2/3$. Assume that we have proved $d^n_i(u_{i,n-1}, u_{i,m}) \ge 2/3$ for all $i = 1, 2$, some $N \ge 1$, all $1 \le n \le N$, and all $m \ge n$. Then, for all $m > n$, since $u_{1,n}[\theta_1 \times u_{2,n-1}] = 2/3$ and $u_{1,m}[\theta_1 \times u_{2,\ell}] = 0$ for all $\ell < n$, we obtain $u^{n+1}_{1,n}[\theta_1 \times u^n_{2,n-1}] = 2/3$ and $u^{n+1}_{1,m}[\theta_1 \times \{u^n_{2,n-1}\}^{2/3}] = 0$, hence $d^{n+1}_1(u_{1,n}, u_{1,m}) \ge 2/3$. Since $u_{2,n}[\theta_1 \times u_{1,n}] = 2/3$ and $u_{2,m}[\theta_1 \times u_{1,\ell}] = 0$ for all $\ell \le n$, we also get $u^{n+1}_{2,n}[\theta_1 \times u^n_{1,n}] = 2/3$ and $u^{n+1}_{2,m}[\theta_1 \times \{u^n_{1,n}\}^{2/3}] = 0$, hence $d^{n+1}_2(u_{2,n}, u_{2,m}) \ge 2/3$. The proof of (24) is complete.

Now let (T_1, T_2) be a finite type space, and for every $i = 1, 2$ and every $n \ge 0$, define

$$T_{i,n} = \{t_i \in T_i : d_i^{\text{UW}}(t_i, u_{i,n}) < 1/3\}.$$

We must show that each $T_{i,n}$ is empty. Note that (24) implies $T_{i,n} \cap T_{i,m} = \varnothing$ for each player i, and all $n \ge 0$ and $m \ge 0$ such that $m \ne n$. Thus, it suffices to show that if $T_{i,n} \ne \varnothing$

[40] To ensure that the payoffs are bounded by M, we can multiply g_i and ε by a factor of $M\delta/4$, if necessary. This normalization does not affect the validity of (12).

for some player i and some $n \geq 0$, then $T_{1,m} \neq \varnothing$ and $T_{2,m} \neq \varnothing$ for all $m > n$, as this contradicts the finiteness of T_1 and T_2.

Assume that $T_{1,0} \neq \varnothing$. Pick any $t_{1,0} \in T_{1,0}$ and $1/3 > \delta > d_1^{\mathrm{UW}}(t_{1,0}, u_{1,0})$. Then

$$t_{1,0}^k[\theta_0 \times \{u_{2,0}^{k-1}\}^\delta] \geq u_{1,0}^k[\theta_0 \times u_{2,0}^{k-1}] - \delta = 1 - \delta \quad \forall k \geq 1,$$

and hence, using the fact that $\delta < 1/3$ and $t_{1,0}[\theta_0 \times T_2] = t_{1,0}[\theta_0 \times \mathcal{T}_2]$, also

$$t_{1,0}[\theta_0 \times T_{2,0}] \geq t_{1,0}\big[\theta_0 \times \{t_2 \in \mathcal{T}_2 : d_2^{\mathrm{UW}}(t_2, u_{2,0}) < \delta\}\big] \geq 1 - \delta > 0,$$

implying that $T_{2,0} \neq \varnothing$ as well. Now let $n \geq 0$ and assume $T_{2,n} \neq \varnothing$. Pick any $t_{2,n} \in T_{2,n}$ and $1/3 > \delta > d_2^{\mathrm{UW}}(t_{2,n}, u_{2,n})$. Then

$$t_{2,n}^k[\theta_1 \times \{u_{1,n+1}^{k-1}\}^\delta] \geq u_{2,n}^k[\theta_1 \times u_{1,n+1}^{k-1}] - \delta = 1/3 - \delta \quad \forall k \geq 1$$

and hence, as before,

$$t_{2,n}[\theta_1 \times T_{1,n+1}] \geq t_{2,n}\big[\theta_1 \times \{t_1 \in \mathcal{T}_1 : d_1^{\mathrm{UW}}(t_1, u_{1,n+1}) < \delta\}\big] \geq 1/3 - \delta > 0,$$

so $T_{1,n+1} \neq \varnothing$. Similarly, we can show that $T_{1,n} \neq \varnothing$ implies $T_{2,n} \neq \varnothing$ for all $n \geq 1$. □

Proof of Lemma 3. For any given $N \geq 1$ we construct a game G_N with action sets

$$A_{1,N} = \{a_{1,0}, a_{1,1}, b_{1,1}, c_{1,1}, \ldots, a_{1,N}, b_{1,N}, c_{1,N}, s_1\}$$
$$A_{2,N} = \{a_{2,0}, b_{2,0}, c_{2,0}, \ldots, a_{2,N-1}, b_{2,N-1}, c_{2,N-1}, a_{2,N}, s_2\}$$

such that

$$a_{i,n} \in R_i(u_{i,n}, G_N, 0) \quad \forall i \in I, \forall 0 \leq n \leq N \tag{25}$$

and, moreover, for every $\delta \geq 0$ and $0 \leq k \leq N$,

$$a_{1,n} \in R_1^{2(k+1)}(t_1, G_N, \delta) \implies d_1^{2(k+1)}(t_1, u_{1,n}) \leq 2\delta/M \quad \forall n \leq N-k, \forall t_1 \in \mathcal{T}_1 \tag{26}$$
$$a_{2,n} \in R_2^{2k+1}(t_2, G_N, \delta) \implies d_2^{2k+1}(t_2, u_{2,n}) \leq 2\delta/M \quad \forall n \leq N-k, \forall t_2 \in \mathcal{T}_2. \tag{27}$$

Indeed, this implies the statement of the lemma.

Fix $N \geq 1$. For convenience, throughout the proof let $a_{1,N+1} = s_1$ and $\theta_n = \theta_1$ for every $n \geq 2$. The payoffs in G_N are as follows. Actions s_1 and s_2 give constant payoffs

$$g_{1,N}(\theta, s_1, a_2) = g_{2,N}(\theta, a_1, s_2) = 0 \quad \text{for every } \theta \in \Theta, a_1 \in A_{1,N}, \text{ and } a_2 \in A_{2,N}.$$

Actions $a_{1,0}, \ldots, a_{1,N}$ and $a_{2,0}, \ldots, a_{2,N}$ are weakly dominated by s_1 and s_2, respectively:

$$g_{1,N}(\theta, a_{1,n}, a_2) = \begin{cases} 0 & \text{if } n = 0 \text{ and } (\theta, a_2) = (\theta_0, a_{2,0}) \\ 0 & \text{if } n > 0 \text{ and } (\theta, a_2) \in \{(\theta_1, a_{2,n-1}), (\theta_1, a_{2,n})\} \\ -M & \text{otherwise} \end{cases}$$

$$g_{2,N}(\theta, a_1, a_{2,n}) = \begin{cases} 0 & \text{if } (\theta, a_1) \in \{(\theta_n, a_{1,n}), (\theta_1, a_{1,n+1})\} \\ -M & \text{otherwise.} \end{cases}$$

The payoffs for actions $b_{1,1}, c_{1,1}, \dots, b_{1,N}, c_{1,N}$ are

$$g_{1,N}(\theta, b_{1,n}, a_2) = -g_{1,N}(\theta, c_{1,n}, a_2) = \begin{cases} M/4 & \text{if } (\theta, a_2) = (\theta_1, a_{2,n-1}) \\ -M/2 & \text{if } (\theta, a_2) = (\theta_1, a_{2,n}) \end{cases}$$
$$g_{1,N}(\theta, b_{1,n}, a_2) = g_{1,N}(\theta, c_{1,n}, a_2) = -M \quad \text{otherwise.}$$

Finally, the payoffs for $b_{2,0}, c_{2,0}, \dots, b_{2,N-1}, c_{2,N-1}$ are

$$g_{2,N}(\theta, a_1, b_{2,n}) = -g_{2,N}(\theta, a_1, c_{2,n}) = \begin{cases} M/4 & \text{if } (\theta, a_1) = (\theta_n, a_{1,n}) \\ -M/2 & \text{if } (\theta, a_1) = (\theta_1, a_{1,n+1}) \end{cases}$$
$$g_{2,N}(\theta, a_1, b_{2,n}) = g_{2,N}(\theta, a_1, c_{2,n}) = -M \quad \text{otherwise.}$$

It is immediate to verify that (25) holds. To see this, just note that the mappings $\varsigma_i : U_i \to 2^{A_{i,N}}$ such that $\varsigma_i(u_{i,n}) = a_{i,n}$ for $0 \le n \le N$ and $\varsigma_i(u_{i,n}) = s_i$ for $n > N$ have the best reply property.

It remains to prove that (26) and (27) hold for every $0 \le k \le N$. To do this, we now fix $\delta \ge 0$ and establish the following three claims. First, we show that (27) holds for $k = 0$. Second, we prove that (27) implies (26) for all $0 \le k \le N$. Third, we show that if (26) holds for some $0 \le k < N$, then (27) holds with $k+1$ substituted for k, thus concluding the proof. To ease notation, for every player i, type $t_i \in \mathcal{T}_i$, and conjecture $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i,N})$, in what follows we write $\Pr[\cdot | t_i, \sigma_{-i}]$ for the probability distribution over $\Theta \times A_{-i,N}$ induced by t_i and σ_{-i}, i.e.,

$$\Pr[\theta, a_{-i} | t_i, \sigma_{-i}] = \int_{\mathcal{T}_{-i}} \sigma_{-i}(\theta, t_{-i})[a_{-i}] t_i(\theta \times dt_{-i}) \quad \forall (\theta, a_{-i}) \in \Theta \times A_{-i,N}.$$

To prove our first claim, namely that (27) is valid for $k = 0$, fix any $t_2 \in \mathcal{T}_2$ and $0 \le n \le N$, assume that $a_{2,n} \in R_2^1(t_2, G_N, \delta)$, and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be a corresponding 0-order δ-rationalizable conjecture. Since $a_{2,n}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 instead of $a_{2,n}$ under σ_1 must be at most δ, hence

$$\Pr[\theta_n, a_{1,n} | t_2, \sigma_1] + \Pr[\theta_1, a_{1,n+1} | t_2, \sigma_1] \ge 1 - \delta/M. \tag{28}$$

Similarly, the difference in expected payoff when choosing $b_{2,n}$ or $c_{2,n}$ instead of $a_{2,n}$ under σ_1 must be at most δ, hence

$$-\delta \le \tfrac{1}{4} M \Pr[\theta_n, a_{1,n} | t_2, \sigma_1] - \tfrac{1}{2} M \Pr[\theta_1, a_{1,n+1} | t_2, \sigma_1] \le \delta.$$

The latter inequalities together with (28) imply

$$\Pr[\theta_n, a_{1,n} | t_2, \sigma_1] \ge 2/3 - 2\delta/M, \qquad \Pr[\theta_1, a_{1,n+1} | t_2, \sigma_1] \ge 1/3 - 2\delta/M, \tag{29}$$

hence $t_2^1[\theta_n] \ge 2/3 - 2\delta/M$ and $t_2^1[\theta_1] \ge 1/3 - 2\delta/M$. Moreover, if $n > 0$, then (28) implies $t_2^1[\theta_1] \ge 1 - 2\delta/M$. Thus, $d_2^1(t_2, u_{2,n}) \le 2\delta/M$, as (27) requires for $k = 0$.

To prove our second claim, namely that (27) implies (26) for all $0 \le k \le N$, fix any such k, any $t_1 \in \mathcal{T}_1$, and any $0 \le n \le N$, assume that $a_{1,n} \in R_1^{2(k+1)}(t_1, G_N, \delta)$, and let $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ be a corresponding $(2k+1)$-order δ-rationalizable conjecture.

First consider the case $n=0$. Since $a_{1,0}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, hence

$$\Pr[\theta_0, a_{2,0}|t_1, \sigma_2] \geq 1 - \delta/M \geq 1 - 2\delta/M.$$

Since σ_2 is $(2k+1)$-order δ-rationalizable, from (27) we thus obtain

$$t_1^{2(k+1)}[\theta_0 \times \{u_{2,0}^{2k+1}\}^{2\delta/M}] \geq 1 - 2\delta/M,$$

as required by (26) when $n=0$. Next consider the case $n>0$. Since $a_{1,n}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, hence

$$\Pr[\theta_1, a_{2,n-1}|t_1, \sigma_2] + \Pr[\theta_1, a_{2,n}|t_1, \sigma_2] \geq 1 - \delta/M.$$

Similarly, comparing $a_{1,n}$ to $b_{1,n}$ and $c_{1,n}$, we must have

$$-\delta \leq \tfrac{1}{4}M \Pr[\theta_1, a_{2,n-1}|t_1, \sigma_2] - \tfrac{1}{2}M \Pr[\theta_1, a_{2,n}|t_1, \sigma_2] \leq \delta.$$

The latter three inequalities together imply

$$\Pr[\theta_1, a_{2,n-1}|t_1, \sigma_2] + \Pr[\theta_1, a_{2,n}|t_1, \sigma_2] \geq 1 - 2\delta/M \tag{30}$$

$$\Pr[\theta_1, a_{2,n-1}|t_1, \sigma_2] \geq 2/3 - 2\delta/M \tag{31}$$

$$\Pr[\theta_1, a_{2,n}|t_1, \sigma_2] \geq 1/3 - 2\delta/M. \tag{32}$$

Since σ_2 is $(2k+1)$-order δ-rationalizable, by (27) we have $\sigma_2(\theta_1, t_2)[a_{2,n-1}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2, u_{2,n-1}) > 2\delta/M$ and $\sigma_2(\theta_1, t_2)[a_{2,n}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2, u_{2,n}) > 2\delta/M$. By (30), (31), and (32) this implies

$$t_1^{2(k+1)}[\theta_1 \times \{u_{2,n-1}^{2k+1}, u_{2,n}^{2k+1}\}^{2\delta/M}] \geq 1 - 2\delta/M$$

$$t_1^{2(k+1)}[\theta_1 \times \{u_{2,n-1}^{2k+1}\}^{2\delta/M}] \geq 2/3 - 2\delta/M$$

$$t_1^{2(k+1)}[\theta_1 \times \{u_{2,n}^{2k+1}\}^{2\delta/M}] \geq 1/3 - 2\delta/M,$$

as required by (26) when $n>0$.

It remains to prove our third claim. Assuming (26) for some $0 \leq k < N$, we must show that (27) remains valid when k is replaced by $k+1$. Pick any $t_2 \in \mathcal{T}_2$ and $0 \leq n \leq N-k-1$, assume that $a_{2,n} \in R_2^{2(k+1)+1}(t_2, G_N, \delta)$, and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be a corresponding $2(k+1)$-order δ-rationalizable conjecture. Since $a_{2,n}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 or $b_{2,n}$ or $c_{2,n}$ instead of $a_{2,n}$ under σ_1 must be at most δ. Thus, as before, (28) and (29) must hold. Moreover, since σ_1 is $2(k+1)$-order δ-rationalizable, by (26) we have $\sigma_1(\theta_n, t_1)[a_{1,n}] = 0$ for all $t_1 \in \mathcal{T}_1$ with $d_1^{2(k+1)}(t_1, u_{1,n}) > 2\delta/M$ and $\sigma_1(\theta_1, t_1)[a_{1,n+1}] = 0$ for all $t_1 \in \mathcal{T}_1$ with $d_1^{2(k+1)}(t_1, u_{1,n+1}) > 2\delta/M$. This implies

$$t_2^{2(k+1)+1}[\theta_n \times \{u_{1,n}^{2(k+1)}\}^{2\delta/M}] \geq 2/3 - 2\delta/M$$

$$t_2^{2(k+1)+1}[\theta_1 \times \{u_{1,n+1}^{2(k+1)}\}^{2\delta/M}] \geq 1/3 - 2\delta/M,$$

and, if $n > 0$, also

$$t_2^{2(k+1)+1}[\theta_1 \times \{u_{1,n}^{2(k+1)}, u_{1,n+1}^{2(k+1)}\}^{2\delta/M}] \geq 1 - 2\delta/M,$$

as required by (27) when k is replaced by $k+1$. □

Proof of Lemma 4. Fix $\varepsilon \geq 0$ and note that (21) and (22) are trivially true for $k = 0$. Now we assume they are true for some $k \geq 0$ and prove that they hold for $k+1$. Note that since (22) holds for k, there exists a mapping $\xi : \mathcal{T}_2 \times A_{2,N} \to \{0,1\}$ that satisfies

$$(a_2, \xi(t_2, a_2)) \in R_2^k(t_2, G'_N, \varepsilon) \quad \forall t_2 \in \mathcal{T}_2, \forall a_2 \in R_2^k(t_2, G_N, 2\varepsilon). \tag{33}$$

Let us prove (21) for $k+1$ now. Fix any $t_1 \in \mathcal{T}_1$ and $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$, and let $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ be a corresponding k-order 2ε-rationalizable conjecture. Define the conjecture $\sigma'_2 : \Theta \times \mathcal{T}_2 \to \Delta(A'_{2,N})$ for game G'_N as

$$\sigma'_2(\theta, t_2)[a_2, \xi(t_2, a_2)] = \sigma_2(\theta, t_2)[a_2] \quad \forall \theta \in \Theta, \forall t_2 \in \mathcal{T}_2, \forall a_2 \in A_{2,N}.$$

By (33), σ'_2 is a k-order ε-rationalizable conjecture. Moreover, the difference in expected payoff for t_1 between any $a'_1 \in A'_{1,N}$ and a_1 under σ'_2 in game G'_N is

$$\begin{aligned}&\int_{\Theta \times \mathcal{T}_2^k} [g'_{1,N}(a'_1, \sigma'_2(\theta, t_2), \theta) - g'_{1,N}(a_1, \sigma'_2(\theta, t_2), \theta)] t_1^{k+1}(d\theta \times dt_2^k) \\ &\quad = \tfrac{1}{2} \int_{\Theta \times \mathcal{T}_2^k} [g_{1,N}(a'_1, \sigma_2(\theta, t_2), \theta) - g_{1,N}(a_1, \sigma_2(\theta, t_2), \theta)] t_1^{k+1}(d\theta \times dt_2^k) \leq \tfrac{1}{2} 2\varepsilon = \varepsilon,\end{aligned}$$

where the inequality follows from the fact that $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$. This proves that $a_1 \in R_1^{k+1}(t_1, G'_N, \varepsilon)$, and we have thus shown that $R_1^{k+1}(t_1, G_N, 2\varepsilon) \subseteq R_1^{k+1}(t_1, G'_N, 2\varepsilon)$. Conversely, pick any $a_1 \in R_1^{k+1}(t_1, G'_N, \varepsilon)$ and let $\sigma'_2 : \Theta \times \mathcal{T}_2 \to \Delta(A'_{2,N})$ be a corresponding k-order ε-rationalizable conjecture. Define $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,N})$ as

$$\sigma_2(\theta, t_2) = \operatorname{marg}_{A_{2,N}} \sigma'_2(\theta, t_2) \quad \forall \theta \in \Theta, \forall t_2 \in \mathcal{T}_2.$$

Since (22) holds for k, this is a k-order 2ε-rationalizable conjecture in G_N. Moreover, the difference in expected payoff for t_1 between any $a'_1 \in A_{1,N}$ and a_1 under σ_2 in game G_N is

$$\begin{aligned}&\int_{\Theta \times \mathcal{T}_2^k} [g_{1,N}(a'_1, \sigma_2(\theta, t_2), \theta) - g_{1,N}(a_1, \sigma_2(\theta, t_2), \theta)] t_1^{k+1}(d\theta \times dt_2^k) \\ &\qquad = 2 \int_{\Theta \times \mathcal{T}_2^k} [g'_{1,N}(a'_1, \sigma'_2(\theta, t_2), \theta) - g'_{1,N}(a_1, \sigma'_2(\theta, t_2), \theta)] t_1^{k+1}(d\theta \times dt_2^k) \leq 2\varepsilon,\end{aligned}$$

hence $a_1 \in R_1^{k+1}(t_1, G_N, 2\varepsilon)$. This shows that $R_1^{k+1}(t_1, G'_N, 2\varepsilon) \subseteq R_1^{k+1}(t_1, G_N, 2\varepsilon)$, so the proof of (21) for $k+1$ is complete.

Now we show that (22) also remains true for $k+1$, thus concluding the proof. Fix $t_2 \in \mathcal{T}_2$, let $a_2 \in R_2^{k+1}(t_2, G_N, 2\varepsilon)$, and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,N})$ be a corresponding k-order 2ε-rationalizable conjecture. Choose any

$$x^* \in \arg\max_{x\in\{0,1\}} \int_{\Theta\times\mathcal{T}_1^k} g'_{2,N}(\sigma_1(\theta,t_1), a_2, x, \theta) t_2^{k+1}(d\theta \times dt_1^k).$$

Then the difference in expected payoff for t_2 between any $(a'_2, x) \in A'_{2,N}$ and (a_2, x^*) under σ_1 in game G'_N is

$$\begin{aligned}
&\int_{\Theta\times\mathcal{T}_1^k} [g'_{2,N}(\sigma_1(\theta,t_1), a'_2, x, \theta) - g'_{2,N}(\sigma_1(\theta,t_1), a_2, x^*, \theta)] t_2^{k+1}(d\theta \times dt_1^k)\\
&\quad \le \int_{\Theta\times\mathcal{T}_1^k} [g'_{2,N}(\sigma_1(\theta,t_1), a'_2, x, \theta) - g'_{2,N}(\sigma_1(\theta,t_1), a_2, x, \theta)] t_2^{k+1}(d\theta \times dt_1^k)\\
&\quad = \tfrac{1}{2}\int_{\Theta\times\mathcal{T}_1^k} [g_{2,N}(\sigma_1(\theta,t_1), a'_2, \theta) - g_{2,N}(\sigma_1(\theta,t_1), a_2, \theta)] t_2^{k+1}(d\theta \times dt_1^k) \le \tfrac{1}{2}2\varepsilon = \varepsilon,
\end{aligned}$$

hence $(a_2, x^*) \in R_2^{k+1}(t_2, G'_N, \varepsilon)$. This proves $R_2^{k+1}(t_2, G_N, 2\varepsilon) \subseteq \operatorname{proj}_{A_{2,N}} R_2^k(t_2, G'_N, \varepsilon)$. Conversely, let $(a_2, x) \in R_2^{k+1}(t_2, G'_N, \varepsilon)$ and let $\sigma'_1 : \Theta \times \mathcal{T}_1 \to \Delta(A'_{1,N})$ be a corresponding k-order ε-rationalizable conjecture. Then the difference in expected payoff for t_2 between any $a'_2 \in A_{2,N}$ and a_2 under σ'_1 in game G_N is

$$\begin{aligned}
&\int_{\Theta\times\mathcal{T}_1^k} [g_{2,N}(\sigma'_1(\theta,t_1), a'_2, \theta) - g_{2,N}(\sigma'_1(\theta,t_1), a_2, \theta)] t_2^{k+1}(d\theta \times dt_1^k)\\
&\quad = 2\int_{\Theta\times\mathcal{T}_1^k} [g'_{2,N}(\sigma'_1(\theta,t_1), a'_2, x, \theta) - g'_{2,N}(\sigma'_1(\theta,t_1), a_2, x, \theta)] t_2^{k+1}(d\theta \times dt_1^k) \le 2\varepsilon,
\end{aligned}$$

hence $a_2 \in R_2^{k+1}(t_2, G_N, 2\varepsilon)$. This proves $\operatorname{proj}_{A_{2,N}} R_2^k(t_2, G'_N, \varepsilon) \subseteq R_2^{k+1}(t_2, G_N, 2\varepsilon)$, so the proof of (22) for $k+1$ is complete. □

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