Author(s): Mark Gross and Sorin Popescu
Article Title: The Moduli Space of (111)-Polarized Abelian Surfaces is Unirational
Year of publication: 2001
Link to published version: http://dx.doi.org/10.1023/A:1017518027822
Publisher statement: None

Compositio Mathematica **126:** 1–23, 2001.



# The Moduli Space of (1,11)-Polarized Abelian Surfaces is Unirational

MARK GROSS [1]* and SORIN POPESCU [2]**
[1] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.*
*e-mail: mgross@maths.warwick.ac.uk*
[2] *Department of Mathematics, Columbia University, New York, NY 10027, U.S.A.*
*e-mail: psorin@math.columbia.edu*

(Received: 22 February 1999; accepted: 14 June 1999)

**Abstract.** We prove that the moduli space $\mathcal{A}_{11}^{lev}$ of $(1, 11)$-polarized Abelian surfaces with level structure of canonical type is birational to Klein's cubic hypersurface in $\mathbf{P}^4$. Therefore, $\mathcal{A}_{11}^{lev}$ is unirational but not rational, and there are no $\Gamma_{11}$-cusp forms of weight 3. The same methods also provide an easy proof of the rationality of $\mathcal{A}_9^{lev}$.

**Mathematics Subject Classification (2000).** 14K10.

**Key words.** Abelian surfaces, moduli, Kleine's cubic hypersurface.

Classical results of Tai, Freitag and Mumford and newer results of O'Grady, Gritsenko, Hulek and Sankaran say that moduli spaces of polarized Abelian varieties are almost always of general type. However, for Abelian varieties of small dimension and polarizations of small degree the situation is different and the corresponding moduli spaces usually have beautiful geometry.

In this paper we describe a projective model for the moduli of complex Abelian surfaces with a polarization of type $(1, 11)$, with level structure of canonical type. As a direct consequence we obtain the unirationality of this moduli space, which also turns out to be non-rational. However, unirationality already implies that there exist no $\Gamma_{11}$-cusp forms of weight 3.

Let $\mathcal{A}_d$ denote the moduli space of polarized Abelian surfaces of type $(1, d)$, and let $\mathcal{A}_d^{lev}$ be the moduli space of $(1, d)$-polarized Abelian surfaces with canonical level structure. The map which forgets the level structure represents $\mathcal{A}_d^{lev}$ as a finite cover of $\mathcal{A}_d$. Its general fiber is $\Gamma_d/\Gamma_d^{lev} \cong \mathrm{SL}_2(\mathbf{Z}_d)$, where $\Gamma_d$ and $\Gamma_d^{lev}$ denote the corresponding paramodular groups. In particular, if $d$ is an odd prime number, then the forgetful morphism is a ramified cover of degree $d(d^2 - 1)/2$. (See [LB], [Mum], [GP1] for the definition of canonical level structure, and basic results.)

*Supported by NSF grant DMS-9700761.
**Partially supported by NSF grant DMS-9610205 and MSRI, Berkeley. Current address: Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651, U.S.A.

Our main result is the following

THEOREM 0.1. *The moduli space* $\mathcal{A}_{11}^{lev}$ *is birational to Klein's cubic hypersurface*

$$\mathcal{K} = V\left(\sum_{i \in \mathbf{Z}_5} x_i^2 x_{i+1} = 0\right) \subset \mathbf{P}^4.$$

*In particular,* $\mathcal{A}_{11}^{lev}$ *is unirational but not rational.*

The cubic hypersurface $\mathcal{K} \subset \mathbf{P}^4$ was first studied by Klein [Kl] (see also [KlF], Band II) in connection with the $z$-embedding of the modular curve $X(11)$ of level 11, which turns out to be defined by the $4 \times 4$-minors of the Hessian of the equation of $\mathcal{K}$. In this respect, we note that $\mathcal{K}$ is the unique $\mathrm{PSL}_2(\mathbf{Z}_{11})$-invariant of degree three in $\mathbf{P}^4$, and that furthermore $\mathrm{PSL}_2(\mathbf{Z}_{11})$ is its full automorphism group [Ad1]. The Klein cubic being smooth is unirational but not rational, cf. [CG], [Mur], [Bea].

Our result should be regarded in light of the following facts: $\mathcal{A}_t$ is not unirational (and in fact $p_g(\widetilde{\mathcal{A}}_t) \geqslant 1$) if $t \geqslant 13$ and $t \neq 14,\ 15,\ 16,\ 18,\ 20,\ 24,\ 30,\ 36$ (Gritsenko [Gri1], [Gri2]), while $\widetilde{\mathcal{A}}_p^{lev}$ is a 3-fold of general type for all prime numbers $p \geqslant 37$ (Hulek and Sankaran [HS1], Gritsenko and Hulek, appendix to [Gri1]), where $\widetilde{\mathcal{A}}_p^{lev}$ is a smooth projective model of a compactification of $\mathcal{A}_p^{lev}$. See also [GH], and the survey paper [HS2] for related results, and [Bo] for a finiteness result in the same spirit. On the other hand, $\overline{\mathcal{A}_5^{lev}} \cong \mathbf{P}(H^0(F_{HM}(3)))$, where bar stands for the Igusa (=Voronoi) toroidal compactification and $F_{HM}$ is the Horrocks–Mumford bundle on $\mathbf{P}^4$ ([HM], [HKW]), while $\mathcal{A}_7^{lev}$ is rational having as birational model a smooth $V_{22}$, a prime Fano 3-fold of index 1 and genus 12 which is rational (see [MS], [Schr] and [GP2] for details). It would be interesting to know how the Klein cubic "compares" with the toroidal compactification of $\mathcal{A}_{11}^{lev}$.

In a series of forthcoming papers [GP2], [GP3], we will give details as to the structure of $\mathcal{A}_d^{lev}$, $6 \leqslant d \leqslant 12$, (excluding $d = 9$ and 11, which are covered here) and $\mathcal{A}_d$, $d = 14, 16, 18$ and 20. In particular, we will prove their rationality or unirationality.

Finally, the methods used in this paper also provide an easy proof of the rationality (over $\mathbf{Q}(\xi)$ for $\xi$ a primitive 9th root of unity) of $\mathcal{A}_9^{lev}$. The unirationality of this space also follows implicitly from O'Grady's work [O'G]. He identifies $\mathcal{A}_{p^2}^{lev}$, for $p$ prime, with the moduli space $\mathcal{A}_1(p)$ of pairs of principally polarized Abelian surfaces and rank two subspaces of the $p$-torsion points, non-isotropic for the Weil pairing. O'Grady studies the extension to (natural) toroidal compactifications of the finite natural forgetful map $\pi$ from $\mathcal{A}_1(p)$ to the moduli space $\mathcal{A}_1$ of principally polarized Abelian surfaces. Not all singularities of the toroidal compactification of $\mathcal{A}_1(p)$ are canonical, so O'Grady needs to describe carefully a partial desingularization all of whose singularities are canonical, before being able to apply Hurwitz's formula for $\pi$ to get an expression for the canonical class. For $p = 3$, our method in addition to being simpler also has the advantage of providing an explicit rational parametrization (over $\mathbf{Q}(\xi)$).

## 1. Preliminaries

We review basic properties of polarized Abelian surfaces, specializing to the case of polarizations of type $(1, 11)$. For a more detailed review of this material as used in this paper, see [GP1], §1. In general, we use the notation and definitions of [LB] and [Mum].

Let $(A, \mathcal{L})$ be a general Abelian surface with a polarization of type $(1, 11)$. Then $|\mathcal{L}|$ induces an embedding of $A \subset \mathbf{P}^{10} = \mathbf{P}(H^0(\mathcal{L})^\vee)$ as a projectively normal surface (cf. [Laz]) of degree 22 and sectional genus 12. (The projective normality of the general such Abelian surface, follows also from the proof of [GP1], Theorem 6.5.) Riemann–Roch tells us that $A$ is contained in 22 quadrics, which generate the homogeneous ideal of $A$ ([GP1], Theorem 6.5).

The line bundle $\mathcal{L}$ induces a natural map from $A$ to its dual, $\phi_{\mathcal{L}}\colon A \to \hat{A}$, given by $x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where $t_x\colon A \to A$ is the morphism given by translation by $x \in A$. Its kernel $K(\mathcal{L})$ is isomorphic to $\mathbf{Z}_{11} \times \mathbf{Z}_{11}$, and is dependent only on the polarization.

For every $x \in K(\mathcal{L})$ there is an isomorphism $t_x^*\mathcal{L} \cong \mathcal{L}$. This induces a projective representation $K(\mathcal{L}) \to \mathrm{PGL}(H^0(\mathcal{L}))$, which lifts uniquely to a linear representation of $K(\mathcal{L})$ after taking a central extension of $K(\mathcal{L})$

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{G}(\mathcal{L}) \longrightarrow K(\mathcal{L}) \longrightarrow 0,$$

whose Schur commutator map is the Weil pairing. $\mathcal{G}(\mathcal{L})$ is the *theta group* of $\mathcal{L}$ and is isomorphic to the abstract Heisenberg group $\widetilde{H}(11)$, while the above representation is isomorphic to the Schrödinger representation of $\widetilde{H}(11)$ on $V = \mathbf{C}(\mathbf{Z}_{11})$, the vector space of complex-valued functions on $\mathbf{Z}_{11}$. An isomorphism between $\mathcal{G}(\mathcal{L})$ and $\widetilde{H}(11)$, which restricts to the identity on centers induces a symplectic isomorphism between $K(\mathcal{L})$ and $\mathbf{Z}_{11} \times \mathbf{Z}_{11}$. Such an isomorphism is called a *level structure of canonical type* on $(A, c_1(\mathcal{L}))$. (See [LB], Chapter 8, §3 or [GP1], §1.)

A decomposition $K(\mathcal{L}) = K_1(\mathcal{L}) \oplus K_2(\mathcal{L})$, with $K_1(\mathcal{L}) \cong K_2(\mathcal{L}) \cong \mathbf{Z}_{11}$ subgroups isotropic with respect to the Weil pairing, and a choice of a characteristic $c$ ([LB], Chapter 3, §1) for $\mathcal{L}$, define a unique basis $\{\vartheta_x^c \mid x \in K_1(\mathcal{L})\}$ of *canonical theta functions* for the space $H^0(\mathcal{L})$ defined in [LB], Chapter 3, §2. This basis allows an identification of $H^0(\mathcal{L})$ with $V$ via $\vartheta_\gamma^c \mapsto x_\gamma$, where $x_\gamma$ is the function on $\mathbf{Z}_{11}$ defined by

$$x_\gamma(\delta) = \begin{cases} 1 & \gamma = \delta \\ 0 & \gamma \neq \delta \end{cases} \quad \text{for } \gamma, \delta \in \mathbf{Z}_{11}.$$

The $x_0, \ldots, x_{10}$ can also be identified with coordinates on $\mathbf{P}(H^0(\mathcal{L})^\vee)$. Under this identification, the representation $\mathcal{G}(\mathcal{L}) \to \mathrm{GL}(H^0(\mathcal{L}))$ coincides with the Schrödinger representation $\widetilde{H}(11) \to \mathrm{GL}(V)$. We will only consider the action of $\mathbf{H}_{11}$, the finite subgroup of $\widetilde{H}(11) \to \mathrm{GL}(V)$ generated in the Schrödinger representation by $\sigma$ and $\tau$, where $\sigma(x_i) = x_{i-1}$, $\tau(x_i) = \xi^{-i}x_i$, for all $i \in \mathbf{Z}_{11}$ and $\xi = e^{\frac{2\pi i}{11}}$ is a primitive root of unity of order 11. Notice that $[\sigma, \tau] = \xi$, so $\mathbf{H}_{11}$ is a central extension

$$1 \longrightarrow \mu_{\mathbf{11}} \longrightarrow \mathbf{H}_{11} \longrightarrow \mathbf{Z}_{11} \times \mathbf{Z}_{11} \longrightarrow 0.$$

Thus the choice of a canonical level structure means that if $A$ is embedded in $\mathbf{P}(H^0(\mathcal{L})^\vee)$ using as coordinates $x_\gamma = \vartheta^c_\gamma$, $\gamma \in \mathbf{Z}_{11}$, then the image of $A$ will be invariant under the action of the Heisenberg group $\mathbf{H}_{11}$ via the Schrödinger representation. (See [LB], Chapter 6, §7).

If moreover the line bundle $\mathcal{L}$ is chosen to be symmetric (and there are always finitely many choices of such an $\mathcal{L}$ for a given polarization type), then the embedding via $|\mathcal{L}|$ is also invariant under the involution $\iota$, where $\iota(x_i) = x_{-i}$, $i \in \mathbf{Z}_{11}$, which restricts to $A$ as the involution $x \mapsto -x$.

Let $N(\mathbf{H}_{11})$ be the normalizer of $\mathbf{H}_{11}$ inside $\mathrm{SL}(V)$, where the inclusion $\mathbf{H}_{11} \subset \mathrm{SL}(V)$ is the Schrödinger representation. An element $\alpha \in N(\mathbf{H}_{11})$ induces an outer automorphism of $\mathbf{H}_{11}$ and, hence, an automorphism of $\mathbf{Z}_{11} \times \mathbf{Z}_{11}$ preserving the Weil pairing $e^D$, for $D = (1, 11)$. The group of such automorphisms is $\mathrm{SL}_2(\mathbf{Z}_{11})$, and thus we get a map $\psi : N(\mathbf{H}_{11}) \to \mathrm{SL}_2(\mathbf{Z}_{11})$. As in [HM] §1, one sees that the kernel of this map is $\mathbf{H}_{11}$ and that $\psi$ is surjective. This leads to extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{H}_{11} \arrow[r] \arrow[d] & N(\mathbf{H}_{11}) \arrow[r, "\psi"] \arrow[d] & \mathrm{SL}_2(\mathbf{Z}_{11}) \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \mathbf{Z}_{11} \times \mathbf{Z}_{11} \arrow[r] & N(\mathbf{H}_{11})/\mu_{\mathbf{11}} \arrow[r] & \mathrm{SL}_2(\mathbf{Z}_{11}) \arrow[r] & 1
\end{tikzcd}$$

where in the bottom row $N(\mathbf{H}_{11})/\mu_{\mathbf{11}}$ is a semidirect product by the above (symplectic) action of $\mathrm{SL}_2(\mathbf{Z}_{11})$. Since $H^2(\mathrm{SL}_2(\mathbf{Z}_{11}), \mathbf{C}^*) = 0$ it follows that $N(\mathbf{H}_{11})$ is in fact the semi-direct product $\mathbf{H}_{11} \rtimes \mathrm{SL}_2(\mathbf{Z}_{11})$ (see [HM], §1 for details in the identical case of $\mathbf{H}_5$).

Therefore the Schrödinger representation of $\mathbf{H}_{11}$ induces an 11-dimensional representation $\rho_{11}$: $\mathrm{SL}_2(\mathbf{Z}_{11}) \longrightarrow \mathrm{SL}(V)$.

In terms of generators and relations, cf. [BM], one has

$$\mathrm{PSL}_2(\mathbf{Z}_{11}) = \langle S, T \mid S^{11} = (S^2TS^6T)^3 = 1,\ (ST)^3 = T^2 = 1 \rangle,$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

while the representation $\rho_{11}$ is given projectively by

$$\rho_{11}(S) = (\xi^{ij/2}\delta_{ij})_{0 \leqslant i,j \leqslant 10} \quad \text{and} \quad \rho_{11}(T) = \frac{1}{\sqrt{11}}(\xi^{-ij})_{0 \leqslant i,j \leqslant 10},$$

where $\xi$ is the above fixed 11th root of unity. (See [Tan] and [Si] for details.)

The center of $\mathrm{SL}_2(\mathbf{Z}_{11})$ is generated by $T^2$, and $\rho_{11}(T^2) = -\iota$. Thus the representation $\rho_{11}$ is reducible. In fact, if $V_+$ and $V_-$ are the positive and negative eigenspaces, respectively, of the involution $\iota$ acting on $V$, then $V_+$ and $V_-$ are easily seen to be invariant under $\rho_{11}$ and, moreover, $\rho_{11}$ splits as $\rho_+ \oplus \rho_-$, where $\rho_\pm$ is the representation of $\mathrm{SL}_2(\mathbf{Z}_{11})$ acting on $V_\pm$. Note that $\rho_-$ is trivial on the center of $\mathrm{SL}_2(\mathbf{Z}_{11})$, so it descends to give an irreducible representation $\rho_-\colon \mathrm{PSL}_2(\mathbf{Z}_{11}) \longrightarrow \mathcal{GL}(V_-)$.

For the reader's convenience, we reproduce from the Atlas of finite groups [CNPW] the character table for $\mathrm{PSL}_2(\mathbf{Z}_{11})$:

| Size of conjugacy class | 1 | 55 | 110 | 132 | 132 | 110 | 60 | 60 |
|---|---|---|---|---|---|---|---|---|
| Conjugacy class | $I$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $\gamma_6$ | $\gamma_7$ |
| Character | | | | | | | | |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 5 | 1 | $-1$ | 0 | 0 | 1 | $\beta$ | $\bar{\beta}$ |
| $\chi_3$ | 5 | 1 | $-1$ | 0 | 0 | 1 | $\bar{\beta}$ | $\beta$ |
| $\chi_4$ | 10 | $-2$ | 1 | 0 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 10 | 2 | 1 | 0 | 0 | $-1$ | $-1$ | $-1$ |
| $\chi_6$ | 11 | $-1$ | $-1$ | 1 | 1 | $-1$ | 0 | 0 |
| $\chi_7$ | 12 | 0 | 0 | $\alpha$ | $\alpha'$ | 0 | 1 | 1 |
| $\chi_8$ | 12 | 0 | 0 | $\alpha'$ | $\alpha$ | 0 | 1 | 1 |

where $\alpha = \frac{1}{2}(-1+\sqrt{5}), \alpha' = \frac{1}{2}(-1-\sqrt{5})$, and $\beta = \frac{1}{2}(-1+\sqrt{-11})$. The conjugacy classes are represented by

$$\begin{array}{cccccccc} I & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} & \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix} & \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}$$

Similarly, the other direct summand $\rho_+\colon \mathrm{SL}_2(\mathbf{Z}_{11}) \longrightarrow \mathcal{GL}(V_+)$ is one of two mutually dual six-dimensional irreducible representations of $\mathrm{SL}_2(\mathbf{Z}_{11})$. We refer the reader to [Dor] for the character table of $\mathrm{SL}_2(\mathbf{Z}_{11})$.

## 2. Moduli of (1, 11)-Polarized Abelian Surfaces

From now on, let $(A, \mathcal{L})$ be a general Abelian surface with a polarization of type $(1, 11)$ and with canonical level structure. As seen in the previous section, $|\mathcal{L}|$ embeds $A \subset \mathbf{P}^{10} = \mathbf{P}(H^0(\mathcal{L})^\vee) = \mathbf{P}(V^\vee)$ as a projectively normal surface of degree 22 and sectional genus 12, which is invariant under the action of the Heisenberg group $\mathbf{H}_{11}$ via the Schrödinger representation. In particular, $H^0(\mathcal{I}_A(n))$ is also a representation of weight $n$ of the Heisenberg group (i.e., a central element $z \in \mathbf{C}^*$ acts by multiplication with $z^n$), and hence all its irreducible components will have dimension $11/\gcd(11, n)$. (See [LB], pg. 179, for this last fact.)

We will first determine equations for the locus of odd two-torsion points of $(1, 11)$ polarized Abelian surfaces. This is a set in $\mathbf{P}^- = \mathbf{P}(V_-{}^\vee)$. To analyze the equations which arise, we will need to make use of the $\mathrm{SL}_2(\mathbf{Z}_{11})$ symmetry present.

Notice that $S^2(V) = H^0(\mathcal{O}_{\mathbf{P}(H^0(\mathcal{L})^\vee)}(2))$ is 66-dimensional, and as a representation of weight 2, splits into six isomorphic 11-dimensional representations of $\mathbf{H}_{11}$, each isomorphic to a twist $V'$ of the Schrödinger representation. On the other hand, by (2.2) of [HM], $\mathrm{Hom}_{\mathbf{H}_{11}}(V', V \otimes V)$ is an $N(\mathbf{H}_{11})/\mathbf{H}_{11} \cong \mathrm{SL}_2(\mathbf{Z}_{11})$-module and coincides with the representation $\rho_{11}$ described in Section 1. Decomposing into the positive and negative eigenspaces of the involution $\iota$ we deduce that

$$V_+ = \mathrm{Hom}_{\mathbf{H}_{11}}(V', S^2(V)) \quad \text{and} \quad V_- = \mathrm{Hom}_{\mathbf{H}_{11}}(V', \wedge^2(V)),$$

as $\mathrm{SL}_2(\mathbf{Z}_{11})$-modules. By (2.2) of [HM], $\varphi : V' \otimes \mathrm{Hom}_{\mathbf{H}_{11}}(V', S^2(V)) \to S^2(V)$ is an isomorphism of $N(\mathbf{H}_{11})$-modules, and since $S^2(V)$ is an irreducible $N(\mathbf{H}_{11})$-module it follows that $V_+$ is a six-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{Z}_{11})$.

The $\mathrm{SL}_2(\mathbf{Z}_{11})$-isomorphism $\varphi : V_+ \otimes V' \to S^2(V)$ can be represented as follows. We use the usual basis $x_0, \dots, x_{10}$ for $V$ (identified with the basis of canonical theta functions of $H^0(\mathcal{L})$), and the basis $f_0, \dots, f_{10}$ of $V'$ such that $\sigma(f_i) = f_{i-1}$ and $\tau(f_i) = \xi^{-2i} f_i$, $i \in \mathbf{Z}_{11}$. Then there is a basis $e_0, \dots, e_5$ of $V_+$ (in fact the projection of $x_0, \dots, x_5$ onto $V_+$ coming from the decomposition $V = V_+ \oplus V_-$) such that the map $\varphi$ takes $e_i \otimes f_j$ to the $(i, j)$th entry of the $6 \times 11$ matrix $R_5$, whose entries are

$$(R_5)_{ij} = x_{j+i} x_{j-i}, \quad 0 \leqslant i \leqslant 5, 0 \leqslant j \leqslant 10,$$

where the indices of the variables are mod 11. Thus the span of the entries in any row of $R_5$ are a $\mathbf{H}_{11}$-subrepresentation of $S^2(V)$. Also, any $\mathbf{H}_{11}$-subrepresentation of $S^2(V)$ can be obtained by taking a linear combination of the rows, and taking the span in $S^2(V)$ of the resulting 11 quadratic polynomials. In addition, if $P \in \mathbf{P}^{10}$ and $v \in V_+$, then $v \cdot R_5(P) = 0$ if and only if $P$ is contained in the scheme cut out by the $\mathbf{H}_{11}$-subrepresentation of quadrics determined by $v$.

Abusing notation, we will also denote by $V_\pm$ the eigenspaces of the involution $\iota$ acting on $V'$ (by $\iota(f_i) = f_{-i}$). Then the restriction of $\varphi$ to $V_+ \otimes V_+$ induces a $\mathrm{SL}_2(\mathbf{Z}_{11})$-isomorphism $\Phi : \wedge^2(V_+) \longrightarrow S^2(V_-)$, usually called the *intertwining operator* (see [We], [AR] pp. 62–63, 74, or [Ad5] for details). We may regard the intertwining isomorphism $\Phi$ as being induced by a skew-symmetric matrix with entries quadratic polynomials in the coordinates of $V_-$, namely: $\Phi$ takes the element $e_i \wedge e_j$ of $\wedge^2(V_+)$ to the $(i, j)$th entry of the matrix

$$S = \begin{pmatrix} 0 & x_1{}^2 & x_2{}^2 & x_3{}^2 & x_4{}^2 & x_5{}^2 \\ -x_1{}^2 & 0 & x_1x_3 & x_2x_4 & x_3x_5 & -x_4x_5 \\ -x_2{}^2 & -x_1x_3 & 0 & x_1x_5 & -x_2x_5 & -x_3x_4 \\ -x_3{}^2 & -x_2x_4 & -x_1x_5 & 0 & -x_1x_4 & -x_2x_3 \\ -x_4{}^2 & -x_3x_5 & x_2x_5 & x_1x_4 & 0 & -x_1x_2 \\ -x_5{}^2 & x_4x_5 & x_3x_4 & x_2x_3 & x_1x_2 & 0 \end{pmatrix}.$$

Here we are abusing notation by identifying $x_i \in V$ with the projection of $x_i$ onto $V_-$ under the decomposition $V = V_+ \oplus V_-$. Note that $S$ can also be viewed as the restriction of the first $6 \times 6$-block of $R_5$ to $\mathbf{P}^-$, where we use $x_1, \ldots, x_5$ as coordinates on $\mathbf{P}^-$ (where $x_i = x_{-i}$).

We will only need the fact that $S$ arises from the intertwining operator in Lemma 2.1 (1) below. The key fact we will need later about $S$ is the simple observation that if $P \in \mathbf{P}^-$, then $v \cdot R_5(P) = 0$ if and only if $v \cdot S(P) = 0$.

Following [GP1], §6, we define $D_i \subseteq \mathbf{P}^-$ to be the locus where the matrix $S$ has rank $\leqslant 2i$, for $i = 1, 2$. By the previous observation, we may interpret $D_i$ as the locus of points in $\mathbf{P}^- \subset \mathbf{P}^{10} = \mathbf{P}(V^\vee)$ which are contained in at least a $(6 - 2i)$-dimensional family of $\mathbf{H}_{11}$-representations of quadrics. Remark also that all the loci $D_i$ are invariant under the action of $\mathrm{PSL}_2(\mathbf{Z}_{11})$ via the representation $\rho_-$ defined in Section 1.

LEMMA 2.1. (1) $D_1 \subset \mathbf{P}^-$ *is a smooth curve of degree 20 and genus 26 isomorphic to the modular curve* $X(11)$. *(This is Klein's* $z$*-model of the modular curve* $X(11)$. *It is the "trace" of the origins in the Shioda compactification of elliptic normal curves with level structure in* $\mathbf{P}^{10}$*, and its embedding is induced by* $\lambda^4$*, where* $\lambda$*, a* 10*th root of the canonical bundle on* $X(11)$*, is the generator of the group of* $\mathrm{PSL}_2(\mathbf{Z}_{11})$*-invariant line bundles on* $X(11)$*. See [Kl], pp. 153-156, [AR] and [Dol] for details.)*

(2) $D_2 \subset \mathbf{P}^-$ *is an irreducible sextic hypersurface, defined by the* $6 \times 6$ *Pfaffian of* $S$*. It contains as an open subset the locus in* $\mathbf{P}^{10}$ *of odd* 2*-torsion points of* $(1, 11)$*-polarized Abelian surfaces with canonical level structure.*

*Proof.* (1) Consider the composition

$$\Psi : \mathbf{P}^- = \mathbf{P}(V_-{}^\vee) \longrightarrow \mathbf{P}(S^2(V_-{}^\vee)) \xrightarrow{\cong} \mathbf{P}(\wedge^2(V_+{}^\vee)),$$

where the first map is the Veronese embedding and the second isomorphism is induced by the intertwining operator. Then for $P \in \mathbf{P}^-$, one can write $\Psi(P) = S(P)$, the latter being a skew-symmetric matrix which should be interpreted as element of $\wedge^2(V_+{}^\vee)$. It is then clear that $D_1$ is the pull-back under $\Psi$ of the locus in $\mathbf{P}(\wedge^2(V_+{}^\vee))$ of rank 2 skew-symmetric matrices. This latter locus can be identified with $\mathrm{Gr}(2, V_+{}^\vee)$. Thus $D_2$ is isomorphic to the pull-back of $\mathrm{Gr}(2, V_+{}^\vee)$ under $\Psi$.

Adler-Ramanan [AR], Theorem 19.17, study this pull-back. In particular, they show that this pull-back gives the same variety in $\mathbf{P}^-$ as that defined by Klein's equations studied by Vélu in [Ve]. It is proven in [Ve], (summarized in Théorème 10.6), that Klein's equations give the so-called the $z$-model of $X(11)$, and this model is always nonsingular. (See [Dol], Theorem 5.1 for the explicit statement.) Thus $D_1$ is isomorphic to $X(11)$.

The genus of $X(11)$ is well known: see for example [Hu], pg. 59, for the genus of the modular curve of level $n$. The degree of the $z$-model is calculated in [Ve]. (See also [AR], Corollary 23.28.)

(2) For a general point $P \in \mathbf{P}^- = \mathbf{P}(V_-)$, $S$ has rank 6, the intertwining operator being an isomorphism, thus $\mathbf{P}^- = D_3 \neq D_2$. In fact $D_2 \subset \mathbf{P}^-$ is the sextic

hypersurface given by the Pfaffian of $S$. For the record, the equation $f_6$ of $D_2$ is

$$f_6 = -x_1^2x_2x_3^3 + x_1^3x_3x_4^2 - x_2^3x_3^2x_5 + x_1x_4^3x_5^2 + x_2^2x_4x_5^3 + \\ + x_1x_2^4x_4 - x_2x_3x_4^4 - x_1^4x_2x_5 + x_3^4x_4x_5 + x_1x_3x_5^4 + \\ + x_1x_2x_3^2x_4^2 - x_1^2x_2^2x_3x_5 - x_1x_2^2x_4^2x_5 - x_1^2x_3x_4x_5^2 + x_2x_3^2x_4x_5^2,$$

though we will not make use of its explicit form. To prove its irreducibility we will use the fact that it is a $\mathrm{PSL}_2(\mathbf{Z}_{11})$-invariant in $S^6(V_-)$. Without loss of generality we may assume that as a $\mathrm{PSL}_2(\mathbf{Z}_{11})$ representation $V_-$ has character $\chi_3$. It is easy to see that $\chi_{S^2(V_-)} = \chi_3 + \chi_5$, so in particular there are no $\mathrm{PSL}_2(\mathbf{Z}_{11})$ invariants in $S^2(V_-)$. On the other hand, $\chi_{S^3(V_-)} = \chi_1 + \chi_5 + \chi_6 + \chi_7$, so in $S^3(V_-)$ there is precisely one $\mathrm{PSL}_2(\mathbf{Z}_{11})$-invariant, which we will denote by $f_3$. Thus the only way that $f_6$ could fail to be irreducible is if $f_6 = f_3^2$. But $f_6$ is not a square. To see this, set, say $x_4 = x_5 = 0$ in the matrix $S$ and take its Pfaffian, or just set $x_4 = x_5 = 0$ in the above equation for $f_6$. We get only one term, $-x_1^2x_2x_3^3$, which is not a square, so $f_6$ itself cannot be a square, and thus $f_6$ is irreducible. (Alternatively, the irreducibility of $D_2$ follows from [Ad3].)

For any point $P \in D_2 \setminus D_1$, $S(P)$ is of rank 4, thus $P$ is contained in a pencil of $\mathbf{H}_{11}$-subrepresentations of quadrics in $\mathbf{P}^{10}$. The last claim follows now from the fact that $\mathbf{H}_{11} \rtimes \langle \iota \rangle$-invariant Abelian surfaces in $\mathbf{P}^{10}$ lie on a pencil of $\mathbf{H}_{11}$-representations of quadrics, and that odd 2-torsion points of a general Abelian surface get mapped to $D_2 \setminus D_1 \subset \mathbf{P}^-$, and uniquely determine the surface (cf. [GP1], Lemma 6.3 and Lemma 6.4). $\square$

By Lemma 2.1, we may define now the morphism

$$\Theta : D_2 \setminus D_1 \longrightarrow \mathrm{Gr}(2, V_+) = \mathrm{Gr}(2, 6),$$

$$D_2 \setminus D_1 \ni P \mapsto \ker(S(P)) = \{v \in V_+ \mid v \cdot S(P) = 0\},$$

which sends a point $P$ to the pencil of $\mathbf{H}_{11}$-subrepresentations of quadrics containing it. The general (1, 11)-polarized Abelian surface embedded with level structure via a symmetric line bundle meets $\mathbf{P}^-$ in the (images of the) six odd 2-torsion points. By [GP1], Lemma 6.4 they are mapped via $\Theta$ to a single point in $\mathrm{Gr}(2, 6)$. Thus $\Theta$ factorizes as a rational map

$$\Theta_{11} : \mathcal{A}_{11}^{lev} \dashrightarrow \mathrm{Gr}(2, 6) = \mathrm{Gr}(2, V_+),$$

which essentially takes an Abelian surface $A$ to the point in $\mathrm{Gr}(2, 6)$ corresponding to the $\mathbf{H}_{11}$-subrepresentation $H^0(\mathcal{I}_A(2)) \subset H^0(\mathcal{O}_{\mathbf{P}(V)}(2))$.

THEOREM 2.2. *The map* $\Theta_{11}$: $\mathcal{A}_{11}^{lev} \dashrightarrow \mathrm{Gr}(2, 6)$ *yields a birational map between* $\mathcal{A}_{11}^{lev}$ *and* $\mathrm{im}(\Theta)$.

*Proof.* This is an immediate consequence of the degeneration arguments in [GP1], Theorem 6.5, and the above results. $\square$

In order to prove Theorem 0.1 we will need to determine the precise structure of $\mathrm{im}(\Theta)$. To do so, we will use the following representation of the Plücker embedding of the Grassmannian $\mathrm{Gr}(2, 2m) = \mathrm{Gr}(2, W)$, where $W \cong \mathbf{C}^{2m}$, $m \geqslant 2$.

$\mathrm{Gr}(2, 2m)$ is embedded in $\mathbf{P}^{\binom{2m}{2}-1} = \mathbf{P}(\wedge^2(W))$ via the Plücker embedding, as the variety of those 2-vectors which are totally decomposable. Thus hyperplanes in the Plücker embedding can be identified with (projectivized) skew symmetric forms $H \in \mathbf{P}(\wedge^2(W)^*)$, and thus with $2m \times 2m$ skew-symmetric matrices. In this setting the Grassmannian $\mathrm{Gr}(2, 2m)$ can be also identified (as an embedded variety) with the subvariety $R_1$ of $2m \times 2m$ skew-symmetric matrices of rank two. The hyperplane sections corresponding to points of $R_1$ are the Schubert cycles $\sigma_1$, the sets of lines intersecting a given subspace of codimension 2 in $\mathbf{P}(W)$.

More precisely, if $P \in \mathrm{Gr}(2, 2m)$ corresponds to a subspace of $W \cong \mathbf{C}^{2m}$ spanned by the rows of a $2 \times 2m$ matrix

$$L = \begin{pmatrix} a_1 & \cdots & a_{2m} \\ b_1 & \cdots & b_{2m} \end{pmatrix},$$

then the corresponding $2m \times 2m$ skew-symmetric matrix $H_P = (p_{ij})$ has as entries the Plücker coordinates of $L$

$$p_{ij} = a_i b_j - a_j b_i.$$

This matrix is rank 2, all the rows being linear combinations of the rows of $L$. Conversely, given any $2m \times 2m$ skew-symmetric matrix of rank 2, the span of the rows yields a two-dimensional subspace of $\mathbf{C}^{2m} = W$, and hence a point in $\mathrm{Gr}(2, 2m)$. Furthermore, with obvious abuse of notation, the following correspondence holds:

LEMMA 2.3. *For $H \in \mathbf{P}(\wedge^2(W)^*)$ and $k \in \{1, \ldots, m\}$ the following are equivalent*

(1) $H \in R_k := \{H \in \mathbf{P}(\wedge^2(W)^*) \mid \mathrm{rank}(H) \leqslant 2k\}$
(2) *$H$ lies in the $k$-chordal locus of $R_1 \cong \mathrm{Gr}(2, 2m)$ (i.e., lies in the union of linear subspaces $\mathbf{P}^{k-1} \subset \mathbf{P}(\wedge^2(W)^*)$ which is $k$-secant to $R_1$).*

*Proof.* All these facts are classical, and easy to prove. See, for instance, [SR]. □

An easy computation shows that the codimension of $R_{m-k}$ in $\mathbf{P}(\wedge^2(W)^*)$ is $\binom{2k}{2}$. In particular, $R_{m-1}$ is a hypersurface in $\mathbf{P}(\wedge^2(W)^*)$ of degree $m$, defined by the Pfaffian of the generic skew-symmetric matrix.

*Remark* 2.4. The Plücker embedding is compatible with the natural action of $\mathrm{PGL}(W)$, and the orbits under this action are exactly $R_k \setminus R_{k-1}$, for $k = \overline{1, m}$, where $R_0 = \emptyset$.

The following lemma describes the map $\Theta$ in the above setting:

LEMMA 2.5. *Let $M$ be a $2m \times 2m$ skew-symmetric matrix of forms of rank $2m-2$ on a variety $X$. Then the map $\Theta : X \to \mathrm{Gr}(2, 2m)$ induced by $x \in X \mapsto \ker M(x) \subseteq \mathbf{C}^{2m}$ is given in (dual) Plücker coordinates by*

$$x \longmapsto M^*(x) \in R_1 = \mathrm{Gr}(2, 2m),$$

*where $M^*$ is the $2m \times 2m$ skew-symmetric matrix defined by*

$$M^*_{ij} = \begin{cases} (-1)^{i+j}\mathrm{Pf}^{ij}(M), & i < j, \\ 0, & i = j, \\ (-1)^{i+j+1}\mathrm{Pf}^{ij}(M), & i > j, \end{cases}$$

*and where $\mathrm{Pf}^{ij}(M)$ is the Pfaffian of the matrix obtained by deleting the $i$th and $j$th rows and columns from $M$.*

*Proof.* We will make use of the following standard facts concerning Pfaffian identities (see [BE] and [Re] for more details). Let $F$ be a free module of rank $n = 2m$ over the ring $R$, let $F^*$ denote the dual module, and let $f\colon F^* \to F$ be a skew-symmetric morphism (that is the matrix of $f$ corresponding to the choice of a basis in $F$ and the dual basis in $F^*$ is a skew-symmetric $n \times n$-matrix $M$). Now, $\wedge F^*$ and $\wedge F$ are modules over each other and we adopt here [BE]'s notation in writing $a(b)$ for the result of an operation of $a \in \wedge F$ on $b \in \wedge F^*$ and vice-versa; thus $a(b) \in \wedge F^*$ and $b(a) \in \wedge F$. The skew-symmetric map $f$ corresponds to an element $\varphi \in \wedge^2 F$, such that for all $a^* \in F^*$ we have $f(a^*) = -a^*(\varphi)$. In terms of a basis $e_1, \dots, e_{2m}$ of $F$, if $(f_{ij})$ is the matrix of $f$ with $f_{ij} = -f_{ji}$, then

$$\varphi = \sum_{i<j} f_{ij} e_i \wedge e_j.$$

Now fix an orientation $e^* \in \wedge^{2m} F^*$ (that is a generator of this module). This yields a correspondence between skew-symmetric maps $g : F \to F^*$ and elements $\psi \in \bigwedge^{2m-2} F$, via $g(a) = \psi(a(e^*))$. In coordinates, write

$$\psi = \sum_{i<j} g_{ij} e_{\{i,j\}^*},$$

where $\{i, j\}^*$ denotes the complement of the set $\{i, j\}$ in $\{1, \dots, 2m\}$, and for a subset $H = \{i_1, \dots, i_n\} \subseteq \{1, \dots, 2m\}$ with $i_1 < \dots < i_n$, $e_H = e_{i_1} \wedge \dots \wedge e_{i_n}$. Notice that the matrix corresponding to $g$ is $((-1)^{i+j} g_{ij})$. We then have, for $a \in F$,

$$(f \circ g)(a) = f(\psi(a(e^*))) = -\psi(a(e^*))(\varphi).$$

Recall that divided powers are related to Pfaffians by the formula

$$\varphi^{(p)} = \sum_{|H|=2p} \mathrm{Pf}(M_H) e_H,$$

where $M$ denotes the skew-symmetric matrix of $f$ with respect to $\{e_i\}$ and the dual basis $\{e_i^*\}$, while $M_H$ denotes the principal submatrix of $M$ determined by rows and columns indexed by $H$. Thus, if $M$ has rank $2m-2$, then $\varphi^{(m)}=0$. Now Lemma 2.4 of [BE] tells us that for $a \in F$,

$$\varphi^{(m)}(a(e^*)) = \varphi^{(m-1)}(a(e^*))(\varphi),$$

so if $g : F \to F^*$ is the morphism corresponding to $\varphi^{(m-1)}$ and $\varphi^{(m)}=0$, then $f \circ g = 0$.

In coordinates, let $M$ be the skew symmetric $n \times n$-matrix corresponding to $f$ (and the above choice of bases), denote by $\mathrm{Pf}(M)$ its Pfaffian and by $\mathrm{Pf}^{ij}(M)$, for $i<j$, the Pfaffian of the skew-symmetric matrix obtained from $M$ by ommiting the $i$th and $j$th rows and columns, and set $\mathrm{Pf}^{ij}(M) = -\mathrm{Pf}^{ji}(M)$, if $i>j$.

Thus if the matrix $M^* = (m^*_{ij})$ is the skew-symmetric matrix with entries

$$m^*_{ij} = \begin{cases} (-1)^{i+j}\mathrm{Pf}^{ij}(M) & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

then $M^*$ is the matrix of $g$ associated with the bases $\{e_i\}$ of $F$ and $\{e^*_i\}$ of $F^*$. The above compositions read as $M \cdot M^* = 0$. Thus also $M^* \cdot M = (M \cdot M^*)^t = 0$, as required. $\square$

We show next that $\overline{\mathrm{im}(\Theta)}$ is a smooth Fano 3-fold of genus 8 and index one. As a corollary of Theorem 2.2 and Theorem 2.6 below we obtain then Theorem 0.1.

THEOREM 2.6. *The Zariski closure of* $\mathrm{im}(\Theta) \subset \mathrm{Gr}(2,6) = \mathrm{Gr}(2, V_+)$ *in the Plücker embedding has equations given in Plücker coordinates by*

$$p_{23} = -p_{15}, \qquad p_{26} = p_{13}, \qquad p_{14} = -p_{35}, \qquad p_{16} = p_{45}, \qquad p_{46} = -p_{12}.$$

*Furthermore, this linear section of* $\mathrm{Gr}(2,6)$ *is three-dimensional, smooth and, hence, a Fano 3-fold of type* $V_{14}$. *Furthermore it is birational to the Klein cubic hypersurface*

$$\mathcal{K} = V\left(\sum_{i \in \mathbf{Z}_5} x_i^2 x_{i+1} = 0\right) \subset \mathbf{P}^4.$$

*Proof.* The first thing to check is that $\mathrm{im}(\Theta) \subset \mathbf{P}^9$ satisfies the five linear relations given above. This can easily be checked by hand using Lemma 2.5 simply by computing the corresponding Pfaffians of the matrix $S$ and showing they satisfy the given relations. Observe also that the $\mathrm{SL}_2(\mathbf{Z}_{11})$ representation on $\wedge^2(V_+)$ induced by $\rho_+$ decomposes as the sum of a 5 and a 10-dimensional irreducible representations. The equations in the statement of Theorem 2.6 define this 10-dimensional representation as a subspace of $\wedge^2(V_+)$.

To conclude that the closure of the image of $\Theta$ is actually given by these equations, we will show that the subscheme $X$ of $\mathrm{Gr}(2,6)$ defined by these equations is three-dimensional and nonsingular, and thus a Fano 3-fold $V_{14}$ of genus 8 and index 1. To this end we will use a classical construction due to G. Fano [Fa], and recast in modern language by Iskovskih [Is1], [Is2] (see also [Pu]), which shows that any $V_{14}$ is birationally equivalent to a (smooth) cubic threefold in $\mathbf{P}^4$.

Let $H_1, \ldots, H_5$ be five (linearly independent) hyperplanes in $\mathbf{P}^{14} = \mathbf{P}(H^0(\mathcal{O}_{\mathrm{Gr}(2,6)}(1)))$, and let

$$X := \mathrm{Gr}(2,6) \cap H_1 \cap \cdots \cap H_5.$$

By Lemma 2.3 and the discussion preceding it, we may identify $(\mathbf{P}^{14})^\vee$ with the space of $6 \times 6$ skew-symmetric matrices, and so $\mathrm{Gr}(2,6)^\vee$ can be naturally identified with the locus $R_2$ of skew-symmetric matrices of rank $\leqslant 4$. As seen above $\mathrm{Gr}(2,6)^\vee$ is then a cubic hypersurface in $(\mathbf{P}^{14})^\vee$ defined by the $6 \times 6$ Pfaffian of the generic skew-symmetric matrix. Its singular locus is the locus $R_1$ of $6 \times 6$ skew-symmetric matrices of rank $\leqslant 2$, which is isomorphic (as an embedded variety) to $\mathrm{Gr}(2,6)$. By Lemma 2.3, the Pfaffian cubic is also the secant variety to $R_1 \cong \mathrm{Gr}(2,6)$.

Let $\mathbf{P}^4 := \langle H_1, \ldots, H_5 \rangle \subseteq (\mathbf{P}^{14})^\vee$ denote the span of the above five hyperplanes as points in $(\mathbf{P}^{14})^\vee$, and let

$$B := \mathrm{Gr}(2,6)^\vee \cap \langle H_1, \ldots, H_5 \rangle.$$

Define now a (possibly rational) map

$$\Psi : B \longrightarrow \mathrm{Gr}(2,6)$$

$$B \ni b \longmapsto \ker(b) \in \mathrm{Gr}(2,6),$$

where we think of each element $b$ of $B$ as a $6 \times 6$ skew-symmetric matrix of rank $\leqslant 4$. Therefore $\Psi$ is defined on all of $B$ iff $B$ is disjoint from the singular locus of $\mathrm{Gr}(2,6)^\vee$.

LEMMA 2.7. *If the cubic hypersurface $B$ is smooth and* $\mathrm{im}(\Psi) \cap X = \emptyset$*, then $X$ is a nonsingular threefold (Fano of genus 8, index 1).*

*Proof.* First, let $x \in X$ be a point where the Zariski tangent space of $X$ at $x$, $T_{X,x}$, has $\dim T_{X,x} > 3$. If $l_1, \ldots, l_5$ are the Zariski tangent spaces to $H_1, \ldots, H_5$ at $x$, then

$$T_{X,x} = T_{\mathrm{Gr}(2,6),x} \cap l_1 \cap \ldots \cap l_5 \subseteq T_{\mathbf{P}^{14},x}.$$

Suppose now that $H_1, \ldots, H_5$ are given by the linear equations $h_1 = 0, \ldots, h_5 = 0$, respectively. The only way $T_{X,x}$ could fail to be three-dimensional is if there exist a hyperplane section $H$ whose equation is $\sum_{i=1}^5 a_i h_i = 0$ for some $a_i$, with $T_{H,x} \supseteq T_{\mathrm{Gr}(2,6),x}$. Thus $H$ must be tangent to $\mathrm{Gr}(2,6)$, and so $H \in B$. If the cubic $B$ is smooth, then $B$ is disjoint from the singular locus of $\mathrm{Gr}(2,6)^\vee$, and the map $\Psi$ above is defined everywhere on $B$. Now $\Psi(H)$ is the point of the Grassmannian $\mathrm{Gr}(2,6)$ which $H$ is tangent to; thus, in particular, if $H$ is tangent to $\mathrm{Gr}(2,6)$ at a point of $X$, we must have $\Psi(H) \in X$ and $\Psi(B) \cap X \neq \emptyset$. In conclusion, if $\Psi(B) \cap X = \emptyset$, we must have $\dim T_{X,x} = 3$ and so $X$ is a non-singular Fano threefold. $\square$

*Proof of Theorem 2.6 continued.* We use Lemma 2.7 to check that the subscheme $X$ of $\mathrm{Gr}(2,6)$ defined by the equations in the statement of Theorem 2.6 is three-dimensional and nonsingular. We can now compute $B$ directly in our case:

Let $x_{ij}$, $1 \leqslant i < j \leqslant 6$, be coordinates on $(\mathbf{P}^{14})^\vee$ dual to the Plücker coordinates $p_{ij}$ on $\mathbf{P}^{14}$. Then, in our particular case, the $\mathbf{P}^4$ spanned by $H_1, \dots, H_5$ is cut out by the equations

$$\begin{aligned} &x_{12} - x_{46} = 0, && x_{13} + x_{26} = 0, \\ &x_{14} - x_{35} = 0, && x_{15} - x_{23} = 0, \\ &x_{16} + x_{45} = 0, && x_{24} = x_{25} = x_{34} = x_{36} = x_{56} = 0. \end{aligned}$$

Making now the substitutions $x_0 = x_{12}, x_2 = x_{13}, x_1 = x_{14}, x_4 = x_{15}$, and $x_3 = x_{16}$, we see that the equation of $B \subseteq \mathbf{P}^4$, with coordinates $x_0, \dots, x_4$, is given by the $6 \times 6$ Pfaffian of the skew-symmetric matrix

$$M = \begin{pmatrix} 0 & x_0 & x_2 & x_1 & x_4 & x_3 \\ -x_0 & 0 & x_4 & 0 & 0 & -x_2 \\ -x_2 & -x_4 & 0 & 0 & x_1 & 0 \\ -x_1 & 0 & 0 & 0 & -x_3 & x_0 \\ -x_4 & 0 & -x_1 & x_3 & 0 & 0 \\ -x_3 & x_2 & 0 & -x_0 & 0 & 0 \end{pmatrix}$$

which is

$$B = \{x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0 = 0\}.$$

Thus $B$ is Klein's cubic $\mathcal{K} = \left\{ \sum_{i=0}^4 x_i^2 x_{i+1} = 0 \right\}$, the only $\mathrm{PSL}_2(\mathbf{Z}_{11})$-invariant cubic in $\mathbf{P}^4$. This cubic is known to be smooth. (See also [Ad4], Lemma 47.2 for the Pfaffian description of the Klein cubic.)

To show that $X$ is nonsingular, we need now to check the second hypothesis of Lemma 2.7. By Lemma 2.5, the map $\Psi\colon B \to \mathrm{Gr}(2, 6)$ is given in Plücker coordinates by the matrix

$$(M)^* =$$

$$\begin{pmatrix} 0 & x_0x_1 & x_2x_3 & x_1x_2 & x_0x_4 & x_3x_4 \\ -x_0x_1 & 0 & x_3^2 + x_0x_4 & x_1x_3 & -x_0x_2 & -x_1^2 - x_2x_3 \\ -x_2x_3 & -x_3^2 - x_0x_4 & 0 & -x_2x_4 & x_0^2 + x_1x_2 & x_0x_3 \\ -x_1x_2 & -x_1x_3 & x_2x_4 & 0 & -x_2^2 - x_3x_4 & x_0x_1 + x_4^2 \\ -x_0x_4 & x_0x_2 & -x_0^2 - x_1x_2 & x_2^2 + x_3x_4 & 0 & -x_1x_4 \\ -x_3x_4 & x_1^2 + x_2x_3 & -x_0x_3 & -x_0x_1 - x_4^2 & x_1x_4 & 0 \end{pmatrix}.$$

In order for a point $P = (x_0 : \ldots : x_4) \in B$ to satisfy $\Psi(B) \in X$, the Plücker coordinates of $\Psi(P)$ must satisfy the five linear equations defining $X$, which yields that $P$ must satisfy the equations

$$x_i^2 + 2x_{i+1}x_{i+2} = 0, \quad \leqslant i \leqslant 4, \qquad \sum_{i=0}^{4} x_i^2 x_{i+1} = 0.$$

The first set of equations is precisely the Jacobian of the Klein cubic $B$, and since $B$ is smooth, there are no points $P \in B$ satisfying these equations. Hence, by Lemma 2.7, $X$ is non-singular.

We are left now to construct a birational map between $B(= \mathcal{K}) \subset \mathbf{P}^4$ and $X \subset \mathbf{P}^9$. As mentioned above, a classical construction due to Fano and Iskovskih [Fa], [Is2] (see also [Pu] for details) provides such a (rather indirect) birational transformation. For the reader's convenience we sketch it in the sequel.

The lines $L_p$ in $\mathbf{P}^5 = \mathbf{P}(V_+{}^\vee)$ represented by points $p \in X$ sweep out an irreducible quartic hypersurface $\Gamma \subset \mathbf{P}^5$ (called "da Palatini" by Fano). Through the generic point $x \in \Gamma$, passes exactly one $L_p$, with $p \in X$.

On the other hand, since $B$ is smooth, each point $q \in B = \mathrm{Gr}(2,6)^\vee \cap \mathbf{P}^4$ corresponds to a hyperplane $H_q$ tangent to the Grassmannian in exactly one point $n_q$, called the "centre" of $H_q$. The lines $N_q$ in $\mathbf{P}^5$ represented by centres $n_q$ of points $q \in B$ sweep out an irreducible variety $\Sigma \subset \mathbf{P}^5$.

It is easy to see that $\Sigma \subset \Gamma$. (See, for instance, [Pu], pp. 83–84, where the given argument holds whenever $B$ is a smooth 3-fold.) Now through the generic point of $\Sigma$ passes exactly one line $N_q$, with $q \in B$. Otherwise, if $q, q' \in B$ and $N_q \cap N_{q'} \neq \emptyset$, then the whole pencil spanned by $q$ and $q'$ lies in $B$, which means that we found a line in $B$. But this contradicts the fact that the Fano variety of the Klein cubic is 2-dimensional. Therefore $\Sigma \subset \mathbf{P}^5$ is an irreducible hypersurface, and so we must have $\Sigma = \Gamma$. (The quartic equation defining $\Gamma = \Sigma$ is the unique quartic invariant for the action of $\mathrm{SL}_2(\mathbf{Z}_{11})$ on $\mathbf{P}^+$ via $\rho_+$; see [Ad4], Corollary 50.2 for the explicit equation.)

Choose now a generic hyperplane $\Pi \subset \mathbf{P}^5$, and let $\bar{\Gamma} := \Gamma \cap \Pi$. We may define birational maps

$$\eta : X \dashrightarrow \bar{\Gamma}, \qquad \eta(p) := \Pi \cap L_p,$$

$$\gamma : B \dashrightarrow \bar{\Gamma}, \qquad \gamma(q) := \Pi \cap N_q.$$

The composition $\chi := \gamma^{-1} \circ \eta$ defines now a birational isomorphism between $X$ and $B$, as required. (See [Pu] and [Is2] for a detailed analysis of this mapping.) $\square$

*Remark* 2.8. (1) The birational isomorphism $\chi$ provided in the proof of Theorem 2.6 and Theorem 0.1 depends on the choice of a hyperplane $\Pi \subset \mathbf{P}^5$, and thus is not compatible with the action of $\mathrm{PSL}_2(\mathbf{Z}_{11})$. The indeterminacy locus of the isomorphism $\chi$ (as well as of its inverse) turns out to be the union of an elliptic quintic curve $E$ and 25 mutually disjoint secant lines to it (which are flopped by $\chi$). In terms of linear systems, $\chi$ is defined by $|5H - 3E|$, where $H$ is the hyperplane class on $X$. Similarly, $\chi^{-1}$ is induced by $|7H' - 4E'|$, where $H'$ is the hyperplane class on $B$ and $E'$ is the base locus of $\gamma$.

(2) Takeuki [Ta] and Tregub [Tr] have constructed a different birational isomorphism of a smooth Fano 3-fold $V_{14}$ of genus 8, index one onto a smooth cubic hypersurface $B \subset \mathbf{P}^4$, which can be briefly described as follows. Let $C$ be a

(general) rational normal curve $C$ on $B \subset \mathbf{P}^4$. There are exactly 16 chords $l_i$, $1 \leqslant i \leqslant 16$ to $C$ on $B$. Let $\widetilde{B}$ be the blowing-up of $B$ along $C$ and the $l_i$'s. Then the linear system $L = |8H - 5C - \sum_{i=1}^{16} 2L_i|$ provides a birational morphism from $\widetilde{B}$ onto the intersection $X$ of $\mathrm{Gr}(2, 6)$ with a codimension 5 linear subspace. Under this morphism, the unique divisor $D \in |3H - 2C - \sum_{i=1}^{16} L_i|$ is contracted to a point $p$. The inverse birational morphism is then induced by the linear system $|2H' - 3p|$, where $H'$ is the hyperplane class on $X$.

(3) Notice also that every (abstract) smooth Fano 3-fold of genus 8, index one is isomorphic to a codimension 5 linear section of $\mathrm{Gr}(2, 6)$, cf. [Gu].

*Question* 2.9. It seems plausible that the lines on the codimension 5 linear section $Y$ of $\mathrm{Gr}(2, 6)$, which is the Zariski closure of $\mathrm{im}(\Theta)$ in Theorem 2.6, are parametrized by the modular curve $X(11)$, and that the intermediate Jacobian of $Y$ is isomorphic to the generalized Prym variety corresponding to the (symmetric) Hecke correspondence $T_3$ on $X(11)$. See, for instance, [Ad2] and [Ed] for a geometric description of this Hecke correspondence.

*Remark* 2.10. Let $(A, \mathcal{L})$ be a general $(1, 11)$-polarized Abelian surface, where $\mathcal{L}$ is assumed to be symmetric. One can show that the linear system $|2\mathcal{L} - 2\sum_{i=1}^{16} e_i|^+$, of even divisors of $2\mathcal{L}$ having multiplicity two in the half periods, descends to a (complete) very ample linear system on the (desingularized) Kummer surface $X$ associated to $A$, and embeds it as a codimension 8 linear section of the spinor variety $\mathcal{S} \subset \mathbf{P}^{15}$, which parametrizes isotropic $\mathbf{P}^4$'s in an eight-dimensional smooth quadric in $\mathbf{P}^9$. It would be interesting, in the light of [Muk], to determine exactly which codimension 8 linear sections correspond to such Kummer surfaces.

## 3. Moduli of $(1, 9)$-Polarized Abelian Surfaces

As mentioned in the introduction, an argument similar to the one used in Section 2 allows us to prove the rationality of $\mathcal{A}_9^{lev}$.

For the remaining of the paper, let $(A, \mathcal{L})$ be a general Abelian surface with a polarization of type $(1, 9)$ and with canonical level structure. Most of the facts concerning theta groups from Section 1 can be adapted to this case, but we will make little use of them in the sequel. We will also assume that $\mathcal{L}$ is chosen to be a symmetric line bundle.

As seen in Section 1 $|\mathcal{L}|$ embeds $A \subset \mathbf{P}^8 = \mathbf{P}(H^0(\mathcal{L})^\vee) = \mathbf{P}(V^\vee)$ invariantly under the action of the Heisenberg group $\mathbf{H}_9$ via the Schrödinger representation, and the involution $\iota$. In particular, $H^0(\mathcal{I}_A(n))$ is a representation of weight $n$ of the Heisenberg group, whose irreducible components will have dimension $9/\gcd(9, n)$. Via $|\mathcal{L}|$, $A$ is embedded as a projectively normal surface of degree 18 which is contained in nine quadrics (cf. [Laz], or [GP1], Theorem 6.5). However, in contrast with Section 2, we are in a boundary case, in that these quadrics do not generate the homogeneous ideal of $A$. Moreover, in general the quadrics containing the degenerations used in the proof of Theorem 2.2 and [GP1], Theorem 6.5, (b), cut out only a threefold.

As in Section 2, we will investigate the locus of odd 2-torsion points, which in this simpler case turns out to be the whole of $\mathbf{P}^- = \mathbf{P}(V_-{}^\vee) \cong \mathbf{P}^3$.

The space of quadrics $H^0(\mathcal{O}_{\mathbf{P}^8}(2))$ decomposes into five nine-dimensional representations of the Heisenberg group, each one isomorphic to the Schrödinger representation. As above, one such decomposition is given by the spans of the rows of the matrix defined in [GP], §6:

$$R_4 = \begin{pmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 & x_8^2 \\ x_1x_8 & x_2x_0 & x_3x_1 & x_4x_2 & x_5x_3 & x_6x_4 & x_7x_5 & x_8x_6 & x_0x_7 \\ x_2x_7 & x_3x_8 & x_4x_0 & x_5x_1 & x_6x_2 & x_7x_3 & x_8x_4 & x_0x_5 & x_1x_6 \\ x_3x_6 & x_4x_7 & x_5x_8 & x_6x_0 & x_7x_1 & x_8x_2 & x_0x_3 & x_1x_4 & x_2x_5 \\ x_4x_5 & x_5x_6 & x_6x_7 & x_7x_8 & x_8x_0 & x_0x_1 & x_1x_2 & x_2x_3 & x_3x_4 \end{pmatrix}.$$

Thus every nine-dimensional $\mathbf{H}_9$-subrepresentation of quadrics is spanned by $v \cdot R_4$ for some $v \in \mathbf{C}^5 = V_+$, and thus these representations are parametrized by $\mathbf{P}^+ := \mathbf{P}(V_+{}^\vee)$. If we restrict $R_4$ to $\mathbf{P}^- = \mathbf{P}(V_-{}^\vee)$, the $(-1)$-eigenspace of the involution $\iota$, and consider as before the first $5 \times 5$ block, we obtain the matrix

$$S = \begin{pmatrix} 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ -x_1^2 & 0 & x_3x_1 & x_4x_2 & -x_4x_3 \\ -x_2^2 & -x_3x_1 & 0 & -x_4x_1 & -x_3x_2 \\ -x_3^2 & -x_4x_2 & x_4x_1 & 0 & -x_2x_1 \\ -x_4^2 & x_3x_4 & x_2x_3 & x_1x_2 & 0 \end{pmatrix},$$

representing the intertwining operator $\Phi : \wedge^2(V_+) \to S^2(V_-)$.

As in Section 2, or [GP1], §6, it follows that for a point $P \in \mathbf{P}^-$, $v \cdot R_4(P) = 0$ if and only if $v \cdot S(P) = 0$.

LEMMA 3.1

(1) $\operatorname{rank}(S(P)) \geqslant 2$, *for all* $P \in \mathbf{P}^-$.
(2) *The locus* $D_1 \subseteq \mathbf{P}^-$ *where matrix* $S$ *has rank* 2 *is the disjoint union of a smooth curve* $C \subset \mathbf{P}^-$ *of degree 9, which is the complete intersection*

$$\{x_1^2x_2 - x_2^2x_4 - x_1x_4^2 = x_1x_2^2 - x_3^3 + x_1^2x_4 - x_2x_4^2 = 0\} \subset \mathbf{P}^-,$$

*and the four points*

$$P_1 = (0:0:0:1:0:0:-1:0:0),$$
$$P_2 = (0:-1:1:0:-1:1:0:-1:1),$$
$$P_3 = (0:-1:\xi^3:0:-\xi^6:\xi^6:0:-\xi^3:1),$$
$$P_4 = (0:-1:\xi^6:0:-\xi^3:\xi^3:0:-\xi^6:1),$$

*where* $\xi = e^{\frac{2\pi i}{9}}$ *is a primitive root of order nine of the unity. The curve C is isomorphic to the modular curve* $X(9)$.

*Proof.* Direct computation and arguments similar to those used in the proof of Lemma 2.1. □

As in Section 2, we may interpret $D_1$ as the locus of points in $\mathbf{P}^- \subset \mathbf{P}^8 = \mathbf{P}(V^\vee)$ which are contained in a net of $\mathbf{H}_9$-subrepresentations of quadrics. On the other hand, $S$ is a $5 \times 5$ skew-symmetric matrix, so $S$ drops rank on all of $\mathbf{P}^-$. Therefore we can define again a map

$$\Theta : \mathbf{P}^- \setminus D_1 \longrightarrow \mathbf{P}^+$$
$$\mathbf{P}^- \setminus D_1 \ni P \longmapsto \mathbf{P}(\ker(S(P))) = \mathbf{P}(\{v \in V_+ \mid v \cdot S(P) = 0\}),$$

which sends a point $P$ to the unique $\mathbf{H}_9$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^8}(2))$ of quadrics containing it.

By an argument similar to the one used in the proof of Lemma 2.5, the morphism $\Theta$ is easily seen to map a point $P \in \mathbf{P}^-$ to the point of $\mathbf{P}^+$ whose coordinates are given by the $4 \times 4$-Pfaffians of $S(P)$, taken with suitable signs. In coordinates, if $P = (x_1 : \ldots : x_4) \in \mathbf{P}^-$ this yields $\Theta(x_1 : \ldots : x_4) = (v_0 : \ldots : v_4)$, where

$$\begin{aligned}
v_0 &= -x_1^2x_2x_3 + x_2^2x_3x_4 + x_1x_3x_4^2,\\
v_1 &= x_1x_2^3 - x_2x_3^3 + x_1x_4^3,\\
v_2 &= -x_1^3x_2 + x_3^3x_4 + x_2x_4^3,\\
v_3 &= x_1^2x_2x_3 - x_2^2x_3x_4 - x_1x_3x_4^2,\\
v_4 &= x_1x_3^3 - x_1^3x_4 - x_2^3x_4.
\end{aligned}$$

Since $v_0 = -v_3$, we deduce that the image of $\Theta$ is contained in the linear subspace $\Pi$ of $\mathbf{P}^+$ defined by $v_0 = -v_3$. As in Section 2, by [GP1], Lemma 6.4, $\Theta$ induces a rational map

$$\Theta_9 : \mathcal{A}_9^{lev} \dashrightarrow \Pi,$$

which essentially is defined by taking an Abelian surface $A \subseteq \mathbf{P}^8$ to $\Theta((A \cap \mathbf{P}^-) \setminus D_1)$, that is to the point corresponding to the unique $\mathbf{H}_9$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^8}(2))$ of quadrics containing the Abelian surface.

*Remark* 3.2. It is easy to see that a $(1, 9)$-polarized Abelian surface $A \subseteq \mathbf{P}^8$ is not cut out by quadrics. Indeed, if $v = (v_0 : \ldots : v_4) = \Theta_9(A) \in \mathrm{im}(\Theta)$, then $v_0 = -v_3$ and each quadric entry of $v \cdot R_4$ vanishes at the (fixed) point

$$P = (1 : 0 : 0 : 1 : 0 : 0 : 1 : 0 : 0) \in \mathbf{P}^8.$$

However, since $\sigma^3(P) = P$ and $\tau^3(P) = P$, where $\sigma$ and $\tau$ acting by translation by 9-torsion points on $A$ are the usual generators of $\mathbf{H}_9$ in the Schrödinger representation, $P$ cannot be contained in $A$. Thus $A$ is not cut out by quadrics since the only quadrics containing $A$ are linear combinations of the entries of $v \cdot R_4$. In fact one may show that for the general Abelian surface $A$, the quadrics defined by $v \cdot R_4$ cut out the union of $A$ and the set of nine points which form the $\mathbf{H}_9$ orbit of $P$. A degeneration argument, in the spirit of [GP1], §6, shows that the homogeneous ideal of $A$ is in fact generated by the 9 quadrics and 6 extra cubics (use Lemma 3.4 below).

We can now prove the rationality of $\mathcal{A}_9^{lev}$:

THEOREM 3.3. $\Theta_9 : \mathcal{A}_9^{lev} \dashrightarrow \Pi \cong \mathbf{P}^3$ *is a birational map.*

*Proof.* We follow much the same strategy as the proof of [GP1], Theorem 6.5, however since, by Remark 3.2, quadrics do not cut out an Abelian surface, we will need to involve cubic equations, and the process is a bit more difficult computationally.

We will also make use of the ubiquitous (Moore) $9 \times 9$-matrices

$$M_4'(x, y) = (x_{5(i+j)} y_{5(i-j)})_{i,j \in \mathbf{Z}_9},$$

where we think of $x = (x_i)_{i \in \mathbf{Z}_9}$ as a point in the ambient $\mathbf{P}^8$ and $y = (y_i)_{i \in \mathbf{Z}_9}$ as a parameter point. We refer the reader to [GP1], §§2 and 6 for a detailed discussion of their properties. Note also that the matrix $R_4$ above, up to transpose and permutations of rows and columns, is a submatrix of $M_4'(x, x)$.

Let $Z := \Theta^{-1}(\mathrm{im}(\Theta_9)) \subseteq \mathbf{P}^- \setminus D_1$, and let $\bar{Z}$ denote the closure of $Z$ in $\mathbf{P}^- \setminus D_1$. Let $\mathcal{A} \subseteq \mathbf{P}^8_{\bar{Z}}$ be the family defined by the condition that the ideal of a fibre $\mathcal{A}_z$, $z \in \bar{Z}$, is generated by the 9 quadrics, which are entries of $\Theta(z) \cdot R_4$ (i.e., the $\mathbf{H}_9$-subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^8}(2))$ vanishing at $z$), along with the cubics which are the $6 \times 6$-Pfaffians of the skew-symmetric $9 \times 9$-matrix $M_4'(x, z)$. By [GP1], Corollary 2.8 and Lemma 6.4, $\mathcal{A}_z$ contains all Abelian surfaces whose odd 2-torsion points map to $z$.

We need to show that there exists an open set $U \subset \bar{Z}$, such that the restricted family $\mathcal{A}_U \longrightarrow U$ is flat, and every smooth fiber is an $\mathbf{H}_9$-invariant (and thus $(1, 9)$-polarized) Abelian surface.

The degeneration argument in [GP1], Theorem 3.1 and Theorem 6.2 shows that if $E \subset \mathbf{P}^8$ is a Heisenberg invariant elliptic normal curve of degree 9, then $\mathrm{Sec}(E) \cap (\mathbf{P}^- \setminus D_1) \subseteq \bar{Z}$. The same is also true if we take $E$ to be the "standard 9-gon" $X(\Gamma_9)$, and $\mathrm{Sec}(E)$ to be its "secant variety", that is, with notation as in [GP1], §4:

$$X(\Gamma_9) = \cup_{i \in \mathbf{Z}_9} l_{i,i+1} \subseteq \mathbf{P}^8,$$

where $l_{i,i+1} = \langle e_i, e_{i+1} \rangle$ is the line joining the vertices $e_i$ and $e_{i+1}$ of the standard simplex in $\mathbf{P}^8$. In particular, for $E = X(\Gamma_9)$, the set $\mathrm{Sec}(E) \cap (\mathbf{P}^- \setminus D_1)$ and thus also

$\bar{Z}$ contain the point

$$z_0 = (0 : 0 : -1 : -1 : 0 : 0 : 1 : 1 : 0).$$

Let $I_0$ be the homogeneous ideal of the fibre $\mathcal{A}_{z_0}$. To conclude the result, it will be enough to show that $\mathcal{A}_{z_0}$ is contained in a surface of degree 18.

Now $\Theta(z_0) = (0 : 1 : 0 : 0 : 0) \in \Pi \subset \mathbf{P}^+$, so $I_0$ contains the quadrics $\Theta(z) \cdot R_4$, namely: $\{x_i x_{i+2},\ i \in \mathbf{Z}_9\}$. On the other hand, the matrix $M_9'(x, z_0)$ is

$$\begin{pmatrix} 0 & 0 & 0 & -x_6 & x_2 & -x_7 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_7 & x_3 & -x_8 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_8 & x_4 & -x_0 & x_5 \\ x_6 & 0 & 0 & 0 & 0 & 0 & -x_0 & x_5 & -x_1 \\ -x_2 & x_7 & 0 & 0 & 0 & 0 & 0 & -x_1 & x_6 \\ x_7 & -x_3 & x_8 & 0 & 0 & 0 & 0 & 0 & -x_2 \\ -x_3 & x_8 & -x_4 & x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_4 & x_0 & -x_5 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_5 & x_1 & -x_6 & x_2 & 0 & 0 & 0 \end{pmatrix}$$

We consider first two of its $6 \times 6$-Pfaffians: The skew-symmetric $6 \times 6$-minor coming from taking rows (and columns) 1,2,3,5,6 and 7 has Pfaffian

$$-x_2x_3x_4 + x_4x_7^2 - x_3x_7x_8 + x_2x_8^2 \in I_0,$$

and similarly taking rows (and columns) 1,2,3,4,6, and 8, we get another cubic Pfaffian

$$-x_0x_3x_6 + x_4x_6x_8 \in I_0.$$

Taking into account the quadrics in $I_0$, we observe that $I_0$ also contains the polynomials $x_4x_7^2 - x_3x_7x_8 + x_2x_8^2$ and $x_0x_3x_6$. Since the matrix $M_9'$ is Heisenberg invariant (in the $x$-coordinate) up to permutations of rows and columns, it follows that $I_0$ is $\mathbf{H}_9$-invariant, and hence contains

$$\begin{gathered} x_i x_{i+2}, \quad i \in \mathbf{Z}_9 \\ x_0x_3x_6, \quad x_1x_4x_7, \quad x_2x_5x_8, \\ x_{i+4}x_{i+7}^2 - x_{i+3}x_{i+7}x_{i+8} + x_{i+2}x_{i+8}^2, \quad i \in \mathbf{Z} \end{gathered}$$

The claim of Theorem 3.3 follows now from the following combinatorial lemma, which determines the Hilbert polynomial of $I_0$:

LEMMA 3.4.

(1) *The ideal $J_1$ generated by the quadric and cubic monomials*

$$\{x_i x_{i+2}, \quad x_i x_{i+3} x_{i+6}, \quad x_{i+3} x_{i+7} x_{i+8} \mid i \in \mathbf{Z}_9\}$$

*is the Stanley–Reisner face ideal $I_{X(\Delta_9)}$ corresponding to the triangulation $\Delta_9$ of the* *torus $T_1$ in [GP1], Proposition 4.4. In particular, $J_1$ has the same Hilbert polynomial as a $(1,9)$-polarized Abelian surface.*

(2) *The ideal $J_2$ generated by the 12 quadric and cubic monomials*

$$\{x_i x_{i+2}, \quad x_i x_{i+3} x_{i+6} \mid i \in \mathbf{Z}_9\},$$

*cuts out the threefold*

$$\Sigma = \bigcup_{i=0}^{8} L_i, \qquad L_i = \sigma^i(L_0),$$

*where $L_0$ is the $\mathbf{P^3}$ determined by $\{x_0 = x_1 = x_4 = x_5 = x_8 = 0\}$. $J_2$ is the face ideal of the "solid" torus whose triangulation $\Delta_9$ is described in* (1). *This is the complex whose two-simplices are those of $\Delta_9$ but which has in addition three-simplices with vertices $(x_i, x_{i+1}, x_{i+3}, x_{i+4})$.*

(3) *The ideals*

$$J_{(\lambda:\mu)} = J_2 + \langle \lambda x_{i+4} x_{i+7}^2 - \mu x_{i+3} x_{i+7} x_{i+8} + \lambda x_{i+2} x_{i+8}^2, \ i \in \mathbf{Z}_9 \rangle,$$

*for $(\lambda : \mu) \in \mathbf{P}^1$, define a flat family of surfaces $X_{(\lambda:\mu)} \subset \mathbf{P}^8$ with the same Hilbert polynomial as a $(1,9)$-polarized Abelian surface. In particular, $I_0 = J_{(1:1)}$ defines a surface of degree 18 as desired.*

*Proof.* The proof is easy and left to the reader. Observe that $X_{(\lambda:\mu)}$ is defined by $J_2$ and 9 trinomials, from which it can be shown that set theoretically $X_{(\lambda:\mu)}$ is the union of 9 distinct (smooth) quadric surfaces

$$X_{(\lambda:\mu)} = \bigcup_{i=0}^{8} Q_i, \qquad Q_i = \sigma^i(Q_0),$$

where $Q_0$ is defined by

$$Q_0 = L_0 \cap \{\lambda x_3 x_6 - \mu x_2 x_7 = 0\}.$$

On the other hand, $J_{(0:1)} = J_1$ is the Stanley–Reisner ideal of the triangulation $\Delta_9$ of the torus, and thus has the required Hilbert function. (See also [GP1], Proposition 4.4, for details.) $\square$

*Remark* 3.5. The linear projection $\pi_-\colon \mathbf{P}(V^\vee) = \mathbf{P}^8 \dashrightarrow \mathbf{P}^-$ commutes with the involution $\iota$ and thus maps a general $\mathbf{H}_9 \rtimes \langle \iota \rangle$ Abelian surface $A \subset \mathbf{P}^8$ to a 6-nodal Kummer quartic surface $K \subset \mathbf{P}^3 = \mathbf{P}^-$ (whose nodes are the odd 2-torsion points of $A$). The linear system of quadrics through the set $S$ of nodes of $K$ maps the Kummer surface to a smooth quartic $K' \subset \mathbf{P}^3$. Such a smooth quartic surface has 16 skew conics, and in fact any smooth quartic surface in $\mathbf{P}^3$ containing 16 skew conics is a Kummer surface of an Abelian surface $(A, \mathcal{L})$ with a polarization of type $(1,9)$, via the linear system $|2\mathcal{L} - 2\sum_{i=1}^{16} e_i|^+$ of even divisors of the totally symmetric line bundle $2\mathcal{L}$, having multiplicity two in the half periods (see [BB], Claims 2–4, and [Bau], Theorem 2.1 for a detailed discussion).

*Remark* 3.6. There is a second family of (minimal) Abelian surfaces of degree 18, and sectional genus 10 embedded in $\mathbf{P}^8$, namely those embedded via a polarization of type (3, 3). These are also contained in nine independent quadrics, that, in contrast with the (1, 9) case, cut out scheme theoretically the Abelian surface. The homogeneous ideal of a (3, 3) polarized Abelian surface is generated by (quadrics and three independent) cubics (cf. [Se]). See [Co], [Gra], [vdG], and [Ba] for explicit equations and their relation to the Burchardt quartic.

## Acknowledgements

We thank Igor Dolgachev, David Eisenbud, Klaus Hulek, and Kristian Ranestad for many useful discussions, and Allan Adler and Gregory Sankaran for a careful reading of a preliminary version of this paper. We are also grateful to Dave Bayer, Dan Grayson and Mike Stillman for *Macaulay* [BS], and *Macaulay2* [GS] which helped us tremendously to understand the shape of the equations described in this paper. The second author also thanks the Mathematical Sciences Research Institute, Berkeley for its hospitality while part of this paper was being written.

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