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THE GREEN'S FUNCTION FOR THE INITIAL-BOUNDARY VALUE PROBLEM OF ONE-DIMENSIONAL NAVIER-STOKES EQUATION

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Huang Xiaofeng

Huang Xiaofeng
23 Jan 2014

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Contents

Summary

We study an initial-boundary value problem for the one-dimensional Navier-Stokes Equation. The point-wise structure of the fundamental solution for the initial value problem is first established. The estimate within finite Mach number area is based on the long wave-short wave decomposition. The short wave part describes the propagation of the singularity while the long wave part is shown to decay exponentially. A weighted energy estimate method is applied outside the finite Mach number area. With the Green's identity, we are able to relate the Green's function for the half space problem to the full space problem. The crucial step is to calculate the Dirichlet-Neumann map that constructs the Neumann boundary data from the known Dirichlet boundary data. Here we apply and modify the method in [23]. The full structure of the boundary data is thus determined. Thus the Green's function for the initial-boundary value problem is obtained. At last, we write the representation of the solution to the nonlinear problem which is a perturbation of a constant state by Duhamel's principle. We introduce a Picard's iteration for the representation and make an ansatz assumption according to the initial data given. We then verify our ansatz to obtain the asymptotic behavior of our solution.

The sketch of this thesis are as follows: In Chapter 2 we construct the fundamental solution to the initial value problem. In Chapter 3 we derive the Green's identity and calculate the inverse Laplace transformation to obtain the Dirichlet-Neumann map. In Chapter 4, we construct the full boundary data and get the Green's function. In Chapter 5, we make an application to the nonlinear problem.

Chapter 1

Introduction

The study of Navier-Stokes equations is an important area in fluid mechanics. The interest of studying Navier-Stokes equations rises from both practically and academically. They can be used to model the water flow in a pipe, air flow around the wing of an aeroplane, ocean currents and maybe the weather. As a result, the Navier-Stokes equations and their simplified forms are widely applied to help with the design of aircraft and cars, the analysis of water pollution, the control of blood flow and many others. They can also be used to study the magneto-hydrodynamics if been coupled with Maxwell equations. However, the existence and the smoothness of the solutions to the Navier-Stokes equations have not yet been proven by the mathematicians. This fact is somehow surprising considering the wide range of practical applications of the equations. As a result, the study of the Navier-Stokes equations becomes one of the most popular areas of modern mathematics.

In this thesis, We will focus on the one dimensional Navier-Stokes equations and consider the initial-boundary value problem. There are a lot of works on the initial value problems but the study of the problems with boundary remains open. It is known that the Navier-Stokes equations can be used to model the

compressible viscous fluid. For the one dimensional Navier-Stokes equations:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\frac{m^2}{\rho} + \rho)_x = m_{xx}. \end{cases} \tag{1.1}$$

where ρ and m stands for density and momentum respectively.

We consider the linearized form of (1.1):

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \rho_x = m_{xx}. \end{cases} \tag{1.2}$$

The reference state for the linearization is $(\rho, m) = (1, 0)$.

Let $F = \begin{pmatrix} \rho \\ m \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have the matrix form of (1.2) as follows:

$$\partial_t F + A\partial_x F = B\partial_x^2 F. \tag{1.3}$$

The fundamental solution $\mathbb{G}(x,t)$ for the initial value problem to the system (1.3) is a 2×2 matrix valued function which satisfies

$$\begin{cases} \partial_t \mathbb{G}(x,t) + A\partial_x \mathbb{G}(x,t) = B\partial_x^2 \mathbb{G}(x,t) \text{ for } x \in \mathbb{R}, t > 0, \\ \mathbb{G}(x,0) = \delta(x)I. \end{cases} \tag{1.4}$$

The Green's function $G(x,y,t)$ for the initial-boundary value problem to the

system (1.3) is also a 2×2 matrix valued function which satisfies

$$\begin{cases} \partial_t G(x,y,t) + A\partial_x G(x,y,t) = B\partial_x^2 G(x,y,t) \text{ for } x > 0, t > 0, \\ G(x,y,0) = \delta(x-y)I, \\ G(0,y,t) = 0. \end{cases} \tag{1.5}$$

In 1940s, Courant and Friedrichs systematically studied the modeling for kinds of fluid problems in their book *Supersonic Flow and Shock Waves* [4]. Many important concepts for compressible fluids were first introduced. The authors focused on wave interactions and shock reflections for ideal gas where the viscosity is neglectable. Problems introduced by this books are still hot topics in the area.

The Navier-Stokes equations are to study the viscous fluid. There are some famous books on the concepts and important problems of Naiver-Stokes equations, like [3], [13], [29]. During the past decades, there have been some breakthrough in the study on Navier-Stokes equations with constant viscosity coefficient. For the initial value satisfies some "small" conditions, the global existence, uniqueness and approximation for the solutions are well known [5], [26], [27], [28]. However, problems with large initial data are very hard. The first important result was by Lions [15]. Lions obtained the global existence of the weak solution by the weak convergence method. In [7], Feireisl, Novotny and Petzeltova consider a more general case based on Lions' work. In addition, for initial value "small" only in the energy space, Hoff [9, 10] derived the existence for the global weak solution. He and Santos also studied the propagation of the singularity in [11].

In fact, only when the density and temperature stay within certain range, the real fluid can be seen as ideal fluid where the viscosity coefficient is constant. Liu, Xin and Yang [19] studied the Cauchy problem of Navier-Stokes equations with viscosity depending on density, and proved its local well-posedness. In the other

hand, it is known that Navier-Stokes equations can be derived from Boltzmann equations by Chapman-Enskog expansion. By the expression, one can see that the viscosity also depends on temperature.

In real life, most problems we meet, as been before mentioned, like the water flow in a pipe, the air flow around the wing of an aircraft, are with boundary. As a result, the study of initial-boundary value problem seems to be much more useful practically than the initial value problem. However, so far there is not much knowledge on the initial-boundary value problem due to it's mathematical difficulty.

Our goal is to study the Navier-Stokes equations with a boundary. The traditional ways for studying well-posedness always fail with a boundary existing. In [12], Kawashima and Matsumura studied 3 types of gas dynamics equations where the second type is the one dimensional Navier-Stokes equations. In the process of proving the asymptotic stability result of traveling wave solutions, they applied an elementary energy estimate method to the integrated system of the conservation form of the original one. To make this energy method work, they supposed that the total integral of the initial disturbance to be zero. In [8], Goodman and Xin studied the zero dissipation problem for a general system of conservation laws with positive viscosity including the Navier-Stokes equations. In their proof, the authors used energy estimate method as well as a matched asymptotic analysis. However, these methods cannot be extended to problems with boundary. This is because with L^2 or L^1 estimates, local information around the boundary is not clear. Therefore, it is very difficult to combine the boundary with the internal solution structure together.

With this thought, it is inspired that the point-wise estimate for the solutions may help. In order to get point-wise estimate of the solutions, new methodology

is needed. The fundamental solution was introduced by Liu in [18]. The fundamental solution is a solution to the original equations with δ initial data. In [18], Liu studied the point-wise convergence rate of the perturbations of shock waves for viscous conservation laws. It is shown that the non-zero total integral of the perturbations gives rise to a translation of the shock front and the diffusion waves, as well as an algebraically decaying term which measures the coupling of waves pertaining to different characteristic families. The proof in [18] is based on the combination of time-asymptotic expansion, construction of approximate fundamental solution and nonlinear analysis of wave interactions. The point-wise estimate yields optimal convergence rate of the perturbations to the shock and the fundamental solution method is also useful for the studying of nonlinear wave interactions.

In [21], Liu and Yu studied the fundamental solution of one dimensional Boltzmann equation and the large time behaviors of the solutions. The proof is based on two types of decompositions: the particle-wave decomposition and the long wave-short wave decomposition. The particle component is represented by singular waves while the fluidlike wave reveals the dissipative behavior which usually can be shown by the Chapman-Enskog expansion. The long wave component is studied by the spectrum of the Fourier transform using contour integral and complex analysis while the short wave component is shown to be exponentially decay. Waves outside the finite Mach number area are estimated by a weighted energy estimate method. With combining the estimate results from the above two different angles of decompositions, the authors have constructed the full structure of the fundamental solution of the linearized Boltzmann equation according to a global Maxwellian. The point-wise description of the large time behavior then becomes an application when the initial perturbation is not necessarily smooth. The results obtained in [21] are significant and the two decompositions in constructing the fundamental solution are innovative and useful. This work paves the way of

studying the initial value problems for all kinds of nonlinear differential equations using fundamental solution. I will apply the long wave-short wave decomposition and the weighted energy estimate method in Chapter 2 in constructing the fundamental solution for the full space problem of one dimensional Navier-Stokes equations.

To achieve our main goal, it is crucial to build the relationship between the solutions of initial value problem and initial-boundary value problem. The Laplace transformation is frequently used to solve kinds of initial value problems of ordinary differential equations. It was first introduced to be applied to partial differential equations by Liu and Yu in [23]. From the first Green's identity, the representation of the difference between the solutions to the initial value problem and the initial-boundary value problem can be established. The only unknown term in this representation is the boundary Neumann data. This gives rise to the construction of the Dirichlet-Neumann map. The Dirichlet-Neumann map in the Laplace space is achieved from the Laplace transformation and the well-posedness of the original system. The discussion on the calculation of the inverse Laplace transformation of the Dirichlet-Neumann map for kinds of different PDE system remains to be the last concern for the authors in [23].

In Chapter 2, we will first construct the fundamental solution to the initial value problem of the Navier-Stokes equations (1.4). The point-wise study of the fundamental solution for a system with physical viscosity was first done by Zeng for the p-system [30]. The result was then extended to a general hyperbolic-parabolic system by Liu and Zeng [24]. Our problem can be regarded as part of the result in [24]. However, we still have to re-do the calculation to get the explicit formula of the fundamental solution for our system as the first step to obtain the Green's function of the initial-boundary value problem. The spectrum analysis in [30] is helpful and will be briefly reviewed in Chapter 2. The detailed constructions

are different and will encounter difficulties if we exactly followed [30]. Thus, we also referred to the method in [21]. In our result, within the finite Mach number area, the short wave component consists of singularity and the remaining parts are estimated by the spectrum analysis and a contour integral. Waves outside finite Mach number area are proved to decay exponentially by a weighted energy estimate method. The main theorem in this chapter is as follows:

Theorem 1 *There exists a positive constant C such that the fundamental solution $\mathbb{G}(x,t)$ of the initial value problem satisfies*

$$|\mathbb{G}(x,t) - e^{-t}\begin{pmatrix} \delta(x) & 0 \\ 0 & 0 \end{pmatrix}| \leq O(1)(\frac{e^{-\frac{|x-t|^2}{c(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{c(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/c}).$$

The norm $|\cdot|$ here stands for supnorm, that is, our estimate is point-wise.

The above result gives the point-wise estimate to the fundamental solution. It is shown that the δ-function of x variable only remains at the upper-left element of the matrix. This is different from the fundamental solution of the Boltzmann equation [21] or its simplified form, the Broadwell model [14]. This is because the variables of these equations have different meaning. The variables of the Navier-Stokes equations are thermodynamical parameters while the variables of the Boltzmann equations or the Broadwell model indicate the wave propagations. Moreover, our result is reasonable in the sense of the original system itself. The first equation with respect to variable ρ is a transport equation so the δ-function remains. The second equation has the viscosity term. From the heat equation, we can see that the solution to the parabolic equations will not maintain the singularity in the initial data for any $t > 0$.

In Chapter 3, we will first introduce some basic results on Laplace transformation and inverse Laplace transformation. We will apply the innovative method

in [23] to construct the full boundary data which is useful in the representation derived from the Green's identity.

In Chapter 4, some convolution results is proved. This can be seen as the interaction of the waves pertaining to different wave types. Finally the full structure of the Green's function to the initial-boundary value problem is derived as follows:

Theorem 2 *There exists $C > 0$ such that*

$$
\begin{aligned}
&|G(x,y,t) - e^{-t}\begin{pmatrix} \delta(x-y) & 0 \\ 0 & 0 \end{pmatrix} - j_1(y,t)\delta(x) - j_2(x,t)\delta(-y)| \\
&\leq O(1)\left(\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x-y|+t)/C}\right) \\
&\quad + O(1)\left(\frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-|y|/C}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) \\
&\quad + O(1)e^{-|x|/C}\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+|y|+t)/C},
\end{aligned}
$$

where j_1 is a matrix of the form $\begin{pmatrix} a_1(y,t) & a_2(y,t) \\ 0 & 0 \end{pmatrix}$ *satisfying*

$$
|a_1(y,t)|, |a_2(y,t)| = O(1)\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C}\right),
$$

and j_2 is a matrix of the form $\begin{pmatrix} b_1(x,t) & 0 \\ b_2(x,t) & 0 \end{pmatrix}$ *satisfying*

$$
|b_1(x,t)|, |b_2(x,t)| = O(1)\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C}\right).
$$

In Chapter 5, we make an application of the Green's function to the genuine

nonlinear problem. Let $\begin{pmatrix} \rho \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{\rho} \\ \hat{m} \end{pmatrix}$, where $\begin{pmatrix} \hat{\rho}(x,0) \\ \hat{m}(x,0) \end{pmatrix} \leq \epsilon e^{-\alpha x}$ for $\epsilon \ll 1$ and $\alpha < 1$, that is, $\begin{pmatrix} \rho(x,0) \\ m(x,0) \end{pmatrix}$ is a perturbation about the constant state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we prove the following Theorem:

Theorem 3 *The solution* $U(x,t) = \begin{pmatrix} \rho(x,t) \\ m(x,t) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *satisfies*

$$|sup_{t\to\infty} U(x,t)| = 0. \tag{1.6}$$

Moreover, we have $U(x,t) \to 0$ *by the rate* $t^{-\frac{1}{2}}$ *along the characteristic curve* $x = t$ *and away from the characteristic curve it is exponentially decaying with respect to* t.

Chapter 2

The Fundamental solution

In this chapter, we first consider the fundamental solution to the initial value problem (1.4). We apply the Fourier transformation to the equation (1.3). Our main focus is to calculate the inverse Fourier transformation. We first need the spectrum analysis as follows.

2.1 Spectrum Property

We consider the Fourier transformation of (1.3) in the x-variable

$$\hat{F}_t + i\eta A\hat{F} = -\eta^2 B\hat{F}. \tag{2.1}$$

Solve the above ODE (2.1) we have

$$\hat{F}(\eta, t) = e^{(-i\eta A - \eta^2 B)t}\hat{F}(\eta, 0) = \hat{\mathbb{G}}(\eta, t)\hat{F}(\eta, 0). \tag{2.2}$$

The operator $\hat{\mathbb{G}}(\eta, t) = e^{(-i\eta A - \eta^2 B)t}$ can be expressed as

$$\hat{\mathbb{G}}(\eta, t) = e^{(-i\eta A - \eta^2 B)t} = \sum_{j=1}^{2} e^{\lambda_j(\eta)t} l_j(\eta) \otimes r_j(\eta) = e^{\lambda_1(\eta)t} P_1(\eta) + e^{\lambda_2(\eta)t} P_2(\eta), \tag{2.3}$$

where $\lambda_1(\eta), \lambda_2(\eta)$ are the spectrum of the operator $-i\eta A - \eta^2 B = \begin{pmatrix} 0 & -i\eta \\ -i\eta & -\eta^2 \end{pmatrix}$.

They are the zeros of

$$0 = det[-i\eta A - \eta^2 B - \lambda I] \equiv \lambda^2 + \eta^2\lambda + \eta^2. \tag{2.4}$$

We have the following explicit expression:

$$\lambda_1 = -\frac{1}{2}\eta(\eta + \sqrt{\eta^2 - 4}), \lambda_2 = -\frac{1}{2}\eta(\eta - \sqrt{\eta^2 - 4}). \tag{2.5}$$

And the corresponding eigenspaces are

$$P_1 = \begin{pmatrix} \frac{1}{2} + \frac{\eta}{2\sqrt{\eta^2-4}} & \frac{i}{\sqrt{\eta^2-4}} \\ \frac{i}{\sqrt{\eta^2-4}} & \frac{1}{2} - \frac{\eta}{2\sqrt{\eta^2-4}} \end{pmatrix}, P_2 = \begin{pmatrix} \frac{1}{2} - \frac{\eta}{2\sqrt{\eta^2-4}} & -\frac{i}{\sqrt{\eta^2-4}} \\ -\frac{i}{\sqrt{\eta^2-4}} & \frac{1}{2} + \frac{\eta}{2\sqrt{\eta^2-4}} \end{pmatrix}. \tag{2.6}$$

By the inverse Fourier transform, we have the explicit formula for the fundamental solution $\mathbb{G}$:

$$\mathbb{G} = \int_{\mathbb{R}} \hat{\mathbb{G}}(\eta, t) e^{ix\eta} d\eta. \tag{2.7}$$

In the following sections, we will apply different methods, i.e., the complex analysis and weighted energy method respectively to the region inside the finite Mach number $\{|x| < Mt\}$ and outside the finite Mach number $\{|x| \geq Mt\}$. Inside the finite Mach number region, we will apply a long wave-short wave decomposition and use complex analysis.

2.2 Long Wave-Short Wave decomposition

Define the long wave-short wave decomposition:

$$\mathbb{G}(x,t) = \mathbb{G}_L(x,t) + \mathbb{G}_S(x,t), \tag{2.8}$$

where

$$\mathbb{G}_L(\eta,t) = \chi(\frac{|\eta|}{\kappa})\mathbb{G}(\eta,t), \mathbb{G}_S(\eta,t) = (1-\chi(\frac{|\eta|}{\kappa}))\mathbb{G}(\eta,t). \tag{2.9}$$

Here, $\chi(y)$ is a characteristic function

$$\chi(y) = \begin{cases} 1, \text{ if } |y| \leqslant 1, \\ 0, \text{ else.} \end{cases} \tag{2.10}$$

Therefore:

$$\mathbb{G} = \int_{\mathbb{R}} \hat{\mathbb{G}}(\eta,t)e^{ix\eta}d\eta = \int_{|\eta|<\kappa} \hat{\mathbb{G}}(\eta,t)e^{ix\eta}d\eta + \int_{|\eta|\geq\kappa} \hat{\mathbb{G}}(\eta,t)e^{ix\eta}d\eta. \tag{2.11}$$

2.3 Long Wave estimate

For the long wave component, that is, the wave number η is small, we make use of the analytic property of $\hat{\mathbb{G}}$. We need the following lemma:

Lemma 2.3.1. *There exists $\kappa_0 > 0, \kappa_1 > 0$ such that for any $|\eta| > \kappa_0$,*

$$Re(\lambda_j(\eta)) < -\kappa_1 \text{ for } j = 1,2,3; \tag{2.12}$$

and for $|\eta| \leqslant \kappa_0$, the eigenvalues $\lambda_j(\eta), j = 1,2,3$ are analytic functions and satisfy

the following asymptotic representations for $|\eta| \leqslant \kappa_0$*:*

$$\begin{cases} \lambda_1(\eta) = -i\eta - \frac{1}{2}\eta^2 + O(1)\eta^3, \\ \lambda_2(\eta) = i\eta - \frac{1}{2}\eta^2 + O(1)\eta^3; \end{cases} \tag{2.13}$$

there are corresponding analytic eigenspaces $P_j(\eta)$ *satisfying the asymptotics for* $|\eta| \leq \kappa_0$*:*

$$P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + O(1)\eta, P_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + O(1)\eta. \tag{2.14}$$

Proof Similar as in [21], the first part is consequence of the spectrum gap property of the eigenvalues at the origin. We omit the proof of this part. We calculate the behavior of λ for $|\eta| \ll 1$. We make use of

$$\sqrt{\eta^2 - 4} = 2i\sqrt{1 - \frac{\eta^2}{4}} = 2i(1 - \frac{\eta^2}{8} + O(1)\eta^3). \tag{2.15}$$

Hence,

$$\lambda_1 = -\frac{1}{2}\eta(\eta + \sqrt{\eta^2 - 4}) = -\frac{1}{2}\eta(\eta + 2i(1 - \frac{\eta^2}{8} + O(1)\eta^3)) = -i\eta - \frac{1}{2}\eta^2 + O(1)\eta^3, \tag{2.16}$$

$$\lambda_2 = -\frac{1}{2}\eta(\eta - \sqrt{\eta^2 - 4}) = -\frac{1}{2}\eta(\eta - 2i(1 - \frac{\eta^2}{8} + O(1)\eta^3)) = i\eta - \frac{1}{2}\eta^2 + O(1)\eta^3 \tag{2.17}$$

The calculations for the corresponding eigenspaces are then straight forward.

Lemma 2.3.2. *For* $0 < \kappa_0 \ll 1$*, there exists* $C_0(\kappa_0) > 1$ *such that for any* $|x| \leqslant C_0(\kappa_0)(1+t)$ *we have*

$$|\int_{|\eta| \leqslant \kappa_0} e^{i\eta x + (-i\eta A - \eta^2 B)t} d\eta| \leqslant O(1)(\frac{e^{-\frac{|x-t|^2}{C_0(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C_0(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-t/C_0}. \tag{2.18}$$

Proof We prove for λ_1 only. Due to the similarity, the proof for λ_2 are omitted. We apply the complex contour integral to calculate the inverse Fourier transformation for $|\eta| \leqslant \kappa_0$:

$$\int\limits_{|\eta|\leqslant\kappa_0} e^{ix\eta+\lambda_1 t}P_1 d\eta = \int\limits_{\Gamma_1+\Gamma_2+\Gamma_3} e^{ix\eta+\lambda_1 t}P_1 d\eta, \tag{2.19}$$

where

$$\Gamma_1 = \{\eta : Re(\eta) = -\kappa_0, 0 \leqslant Im(\eta) \leqslant r\frac{x-t}{1+t}\}, \tag{2.20}$$

$$\Gamma_2 = \{\eta : -\kappa_0 \leqslant Re(\eta) \leqslant \kappa_0, Im(\eta) = r\frac{x-t}{1+t}\}, \tag{2.21}$$

$$\Gamma_3 = \{\eta : Re(\eta) = \kappa_0, 0 \leqslant Im(\eta) \leqslant r\frac{x-t}{1+t}\}. \tag{2.22}$$

Here, we choose $0 < r < \kappa_0/2(C_0+2)$. Since $|x| \leqslant C_0(\kappa_0)(1+t)$, so $\frac{x-t}{1+t} \leqslant C_0+2$. Hence, we have $r\frac{x-t}{1+t} < \kappa_0/2$. On Γ_2,

$$\begin{aligned}
&|\int_{\Gamma_2} e^{ix\eta+\lambda_1 t}P_1 d\eta| \\
&= O(1)|\int_{\Gamma_2} e^{ix\eta - i\eta t - \frac{1}{2}\eta^2 t + O(1)\eta^3 t} d\eta| \\
&= O(1)|\int_{\Gamma_2} e^{i(x-t)\eta - \frac{1}{2}\eta^2 t + O(1)\eta^3 t} d\eta| \\
&= O(1)|\int_{\Gamma_2} e^{-\frac{(x-t)}{2(1+t)} - \frac{1}{2}t(\eta - i(x-t)^2)^2 + O(1)\eta^3 t} d\eta| \\
&= O(1)|\int_{-\kappa_0}^{\kappa_0} e^{-(1-(1-r)^2)\frac{(x-t)^2}{2(1+t)} - \frac{1}{2}u^2(1+t) + 2i(1-r)u(x-t) + O(1)(u^3 + (r\frac{(x-t)}{(1+t)})^3)t} du| \\
&= O(1)e^{-(1-(1-r)^2)\frac{(x-t)^2}{4(1+t)}} \int_{-\kappa_0}^{\kappa_0} e^{-\frac{1}{4}u^2(1+t)} du \\
&= O(1)\frac{e^{-(1-(1-r)^2)\frac{(x-t)^2}{4(1+t)}}}{\sqrt{1+t}}.
\end{aligned}$$

And from the spectrum gap stated in (2.12), there exists $C_1 > 1$, such that

$$|\int_{\Gamma_1+\Gamma_3} e^{ix\eta+\lambda_1 t}P_1 d\eta| = O(1)e^{-t/C_1}. \tag{2.23}$$

The above lemma established the point-wise estimate of the fundamental so-

lution for $|\eta|$ small. For $\{\kappa < |\eta| < N\}$ inside the finite Mach number region we have the following:

Lemma 2.3.3. *For κ sufficiently small and a large number $N > 0$, we have*

$$|\int_{\kappa<|\eta|<N} \hat{\mathbb{G}}(\eta,t)e^{ix\eta}d\eta| \leq Ce^{-t/c}, \tag{2.24}$$

where positive constants C and c depend on κ and N.

Proof We observed that $Re\{-\frac{1}{2}\eta(\eta \pm \sqrt{\eta^2-4})\} < 0$ and $\hat{\mathbb{G}}$ is an entire function. In the finite region $\{\kappa < |\eta| < N\}$, we have:

$$Re\{-\frac{t}{2}\eta(\eta \pm \sqrt{\eta^2-4})\} \leq -\frac{t}{c}, \tag{2.25}$$

where c is a positive constant. Hence, we have

$$|\int_{\kappa<|\eta|<N} \hat{\mathbb{G}}(\eta,t)e^{ix\eta}d\eta| \leq Ce^{-t/c}, \tag{2.26}$$

where positive constants C and c depend on κ and N.

We have finished the point-wise estimate for the long wave component. The main theorem of this section follows:

Theorem 2.3.4. *Inside the finite Mach number region, we have the following point-wise estimate of the fundamental solution $\mathbb{G}$ for the long wave component:*

$$|\int_{|\eta|\leqslant N} e^{i\eta x}\hat{\mathbb{G}}(\eta,t)d\eta| \leqslant O(1)(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-t/C}, \tag{2.27}$$

where N is sufficiently large and C is a positive constant.

Proof The proof is straightforward derived by the above two lemmas.

Corollary 2.3.5. *For $\frac{\partial^k \mathbb{G}}{\partial x^k}$, $k \in \mathbb{N}$, we have the following point-wise estimate for the long wave component:*

$$|\int_{|\eta|\leqslant N} e^{i\eta x}(i\eta)^k \hat{\mathbb{G}}(\eta,t)d\eta| \leqslant O(1)(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-t/C}, \tag{2.28}$$

where $N > 0$ sufficiently large and C, c are positive constants.

Proof The interval $\{\kappa < |\eta| < N\}$ is precompact, so the proof of Lemma 2.3.3 is still true for $(i\eta)^k \mathbb{G}(\hat{\eta}, t)$. We can also verify the proof of Lemma 2.3.2 for $(i\eta)^k \mathbb{G}(\hat{\eta}, t)$ similarly.

2.4 Short Wave estimate

When $\eta \to \infty$, by the explicit formula (2.5) and (2.6), $\lambda_1 = -1$, $\lambda_2 = -\infty$, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore,

$$\hat{\mathbb{G}}(\infty, t) = e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.29}$$

For any complex variable $\eta, |\eta| > N$, N sufficiently large, we have

$$e^t \hat{\mathbb{G}}(\eta, t) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{i}{\eta} \\ -\frac{i}{\eta} & 0 \end{pmatrix} + O(\frac{1}{\eta^2}). \tag{2.30}$$

We will calculate the inverse Fourier Transformation for $O(\frac{1}{\eta^2})$ and $\frac{1}{\eta}$ in the following two lemmas respectively.

Lemma 2.4.1. *Let $\hat{f}(\eta)$ be the Fourier transformed function of $f(x)$ for variable $\eta = \alpha + i\beta$ with $|\beta| < \epsilon$ and $\epsilon > 0$ be any fixed number. If $\hat{f}(\eta)$ has weighted*

$L^2(\mathbb{R})-$ bound as follows:

$$\int_{\mathbb{R}}(|\eta|^2+1)|\hat{f}(\eta)|^2 d\alpha \leq K, \tag{2.31}$$

then $f(x)$ satisfies $|f(x)| \leq Ce^{-|x|/c}$ where C and c are positive constants.

Proof Denote $F(x) = f(x)e^{\beta x}$. Since $\hat{f}(\alpha+i\beta)$ is well defined with $|\beta| < \epsilon$, we have

$$\int_{\mathbb{R}} f(x)e^{-i(\alpha+i\beta)x}dx = \hat{f}(\alpha+i\beta) = \int_{\mathbb{R}} f(x)e^{\beta x}e^{-i\alpha x}dx = \hat{F}(\alpha). \tag{2.32}$$

The Parseval equality implies:

$$\int_{\mathbb{R}} |\hat{f}(\alpha+i\beta)|^2 d\eta = \int_{\mathbb{R}} |\hat{F}(\alpha)|^2 d\alpha = \int_{\mathbb{R}} |F(x)|^2 dx = \int_{\mathbb{R}} |f(x)e^{\beta x}|^2 dx, \tag{2.33}$$

and

$$\int_{\mathbb{R}} |f'(x)e^{\beta x}|^2 dx = \int_{\mathbb{R}} |(f(x)e^{\beta x})' - f(x)\beta e^{\beta x}|^2 dx \leq \int_{\mathbb{R}} |\alpha|^2 |\hat{F}(\alpha)|^2 d\alpha + \beta^2 \int_{\mathbb{R}} |F(x)|^2 dx. \tag{2.34}$$

The above two equality and the assumption (2.31) show that:

$$\int_{\mathbb{R}} |f(x)e^{\beta x}|^2 + |f'(x)e^{\beta x}|^2 dx \leq K, \tag{2.35}$$

for any β satisfying $|\beta| < \epsilon$. Hence, by the Sobolev embedding theorem, we have $|f(x)| \leq Ce^{-|x|/c}$ for some positive constants C and c.

Lemma 2.4.2. *For any real number $N > 0$,*

$$|\int_{|\eta|>N} e^{ix\eta}\frac{1}{\eta}d\eta| \leq C, \tag{2.36}$$

where C is a positive constant.

Proof The statement is true for $x = 0$. For $x \neq 0$, we have the following equality:

$$\int_0^\infty \frac{\sin x\eta}{x} d\eta = \frac{\pi}{2}. \tag{2.37}$$

Therefore,

$$\int_{|\eta|>N} e^{ix\eta} \frac{1}{\eta} d\eta = 2i \int_N^\infty \frac{\sin x\eta}{x} d\eta = \pi i - 2i \int_0^N \frac{\sin x\eta}{x} d\eta. \tag{2.38}$$

Hence, $|\int_{|\eta|>N} e^{ix\eta}\frac{1}{\eta}d\eta|$ is bounded by some constant C.

Theorem 2.4.3. *For $N > 0$ sufficiently large, we have the point-wise estimate of the fundamental solution $\mathbb{G}$ for the short wave component as follows:*

$$|\int_{|\eta|>N} (\hat{\mathbb{G}} - e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) e^{ix\eta} d\eta| \leq Ce^{-t} + Ce^{-t-|x|/c}, \tag{2.39}$$

where C and c are positive constants.

Proof For any sufficiently large real number $N > 0$, $O(1)\frac{1}{\eta^2}$ satisfies:

$$\int_{|\eta|>N} (|\eta|^2 + 1)|O(1)\frac{1}{\eta^2}|d\alpha \leq C_0 \int_{|\eta|>N} (|\eta|^2 + 1)|\frac{1}{\eta^2}|d\alpha \leq C. \tag{2.40}$$

By Lemma 2.4.1 and (2.30), we have

$$|\int_{|\eta|>N} (e^t\hat{\mathbb{G}} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) - \begin{pmatrix} 0 & \frac{i}{\eta} \\ \frac{i}{\eta} & 0 \end{pmatrix} e^{ix\eta} d\eta| \leq Ce^{-|x|/c}. \tag{2.41}$$

Therefore, by Lemma 2.4.2, we have

$$|\int_{|\eta|>N}(e^t\hat{\mathbb{G}}-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix})e^{ix\eta}d\eta| \leq |\int_{|\eta|>N}\begin{pmatrix}0 & \frac{i}{\eta}\\ \frac{i}{\eta} & 0\end{pmatrix}e^{ix\eta}d\eta|+Ce^{-|x|/c} \leq C+Ce^{-|x|/c}. \tag{2.42}$$

This shows the point-wise estimate for the short wave component.

Corollary 2.4.4. *For $\frac{\partial^k\mathbb{G}}{\partial x^k}$, $k=1,2$, we have the following point-wise estimate for the short wave component:*

$$|\int_{|\eta|>N}[(i\eta)^k\hat{\mathbb{G}}-e^{-t}(i\eta)^k\begin{pmatrix}1+\frac{1-t}{\eta^2} & -\frac{i}{\eta}+\frac{i(t-2)}{\eta^2}\\ -\frac{i}{\eta}+\frac{i(t-2)}{\eta^2} & -\frac{1}{\eta^2}\end{pmatrix}]e^{ix\eta}d\eta| \leq Ce^{-t}+Ce^{-t-|x|/c}, \tag{2.43}$$

for some constants C and c.

Proof Since we have

$$e^t(i\eta)^k\hat{\mathbb{G}}=(i\eta)^k\begin{pmatrix}1+\frac{1-t}{\eta^2} & -\frac{i}{\eta}+\frac{i(t-2)}{\eta^2}\\ -\frac{i}{\eta}+\frac{i(t-2)}{\eta^2} & -\frac{1}{\eta^2}\end{pmatrix}+O(\frac{(i\eta)^k}{\eta^4}). \tag{2.44}$$

By the same method above, we can prove this corollary.

2.5 Waves outside finite Mach number area

We will use a weighted energy estimate in this section to obtain the point-wise structure of the fundamental solution $\hat{\mathbb{G}}$ in the region outside the finite Mach number $\{|x|\geq 2t\}$.

The initial data for the fundamental solution $\hat{\mathbb{G}}$ is $\delta(x)I$ where I is the 2×2 identity matrix. Therefore, in order to obtain the point-wise estimate for $\mathbb{G}$, we

need to consider for the case with initial data $(\rho_0, m_0) = (\delta(x), 0)$ and initial data $(\rho_0, m_0) = (0, \delta(x))$ respectively. We will deal with the case $(\rho_0, m_0) = (\delta(x), 0)$ in this section. The proof for the other case is similar. Rewrite the original system as follows:

$$\rho_t + m_x = 0, \tag{2.45}$$

$$m_t + \rho_x - m_{xx} = 0. \tag{2.46}$$

Introduce new variable $\tilde{\rho} = \rho - e^{-t}\delta(x)$ and $\tilde{m} = m$, we have

$$\tilde{\rho}_t + \tilde{m}_x = -e^{-t}\delta(x), \tag{2.47}$$

$$\tilde{m}_t + \tilde{\rho}_x - \tilde{m}_{xx} = e^{-t}\delta'(x). \tag{2.48}$$

We multiply an exponential growth term $e^{\alpha(x-\frac{3}{2}t)}$ to $\rho \cdot (2.47)$ and $m \cdot (2.48)$ respectively and integrate them over $\{|x| > \frac{5}{4}t\}$ with respect to the x variable. Here the coefficient α is chosen to be positive and small, that is, $0 < \delta \ll 1$. The source terms in (2.47) and (2.48) have no effect in the integration in the interval $\{|x| > \frac{5}{4}t\}$:

$$\int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\tilde{\rho}\tilde{\rho}_t + e^{\alpha(|x|-\frac{3}{2}t)}\tilde{\rho}\tilde{m}_x dx = 0, \tag{2.49}$$

$$\int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\tilde{m}\tilde{m}_t + e^{\alpha(|x|-\frac{3}{2}t)}\tilde{m}\tilde{\rho}_x - e^{\alpha(|x|-\frac{3}{2}t)}\tilde{m}\tilde{m}_{xx} dx = 0. \tag{2.50}$$

Add up the above two equations together we have

$$\int_{|x|>\frac{5}{4}t} (\tilde{\rho}_t\tilde{\rho}+\tilde{m}_t\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dxdx + \int_{|x|>\frac{5}{4}t} (\tilde{\rho}_x\tilde{m}+\tilde{m}_x\tilde{\rho})e^{\alpha(|x|-\frac{3}{2}t)}dx - \int_{|x|>\frac{5}{4}t} \tilde{m}_{xx}\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)} = 0. \tag{2.51}$$

In order to get the weighted energy estimate, we use integration by parts for the three terms of the above equation (2.51). For notification simplicity, let

$$E(t) = \int_{|x|>\frac{5}{4}t} (\frac{\tilde{\rho}^2 + \tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}dx. \tag{2.52}$$

For the first term in (2.51), we use integration by parts for t:

$$\int_{|x|>\frac{5}{4}t}(\tilde{\rho}_t\tilde{\rho}+\tilde{m}_t\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dx=\int_{|x|>\frac{5}{4}t}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})_te^{\alpha(|x|-\frac{3}{2}t)}dx$$
$$=\int_{|x|>\frac{5}{4}t}\frac{d}{dt}[(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}]dx-\int_{|x|>\frac{5}{4}t}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})(e^{\alpha(|x|-\frac{3}{2}t)})_tdx$$
$$=\frac{d}{dt}E(t)+\frac{3\alpha}{2}E(t)+\frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=-\frac{5}{4}t}+\frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}.$$

For the second term in (2.51), we use integration by parts for x:

$$\int_{|x|>\frac{5}{4}t}(\tilde{m}_x\tilde{\rho}+\tilde{\rho}_x\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dx=\int_{|x|>\frac{5}{4}t}(\tilde{\rho}\tilde{m})_xe^{\alpha(|x|-\frac{3}{2}t)}dx$$
$$=\int_{|x|>\frac{5}{4}t}\frac{d}{dx}(\tilde{\rho}\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)})dx-\int_{|x|>\frac{5}{4}t}(\tilde{\rho}\tilde{m})(e^{\alpha(|x|-\frac{3}{2}t)})_xdx$$
$$=(\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}-\int_{|x|>\frac{5}{4}t}\frac{\alpha x}{|x|}(\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dx.$$

This term would be controlled by the arithmetic mean inequality. For the last term, we use integration by parts with respect to x variable twice:

$$-\int_{|x|>\frac{5}{4}t}\tilde{m}_{xx}\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)}$$
$$=\int_{|x|>\frac{5}{4}t}\tilde{m}_x^2e^{\alpha(|x|-\frac{3}{2}t)}+\tilde{m}_x\tilde{m}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}dx+\tilde{m}_x\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=-\frac{5}{4}t}^{x=\frac{5}{4}t}.$$

For the second term above, we have

$$\int_{|x|>\frac{5}{4}t}\tilde{m}_x\tilde{m}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}dx$$
$$=\tilde{m}^2\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}-\int_{|x|>\frac{5}{4}t}\tilde{m}\tilde{m}_x\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}+\tilde{m}^2\alpha^2e^{\alpha(|x|-\frac{3}{2}t)}dx.$$

This implies:

$$\int_{|x|>\frac{5}{4}t}\tilde{m}_x\tilde{m}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}dx=\frac{1}{2}\tilde{m}^2\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}-\frac{\alpha^2}{2}\int_{|x|>\frac{5}{4}t}\tilde{m}^2e^{\alpha(|x|-\frac{3}{2}t)}dx.$$

Finally, we have (2.51) can be written as:

$$
\begin{aligned}
0 = & \frac{d}{dt}E(t) + \frac{3\alpha}{2}E(t) - \int_{|x|>\frac{5}{4}t} \frac{\alpha x}{|x|}(\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dx \\
& + \int_{|x|>\frac{5}{4}t} \tilde{m}_x^2 e^{\alpha(|x|-\frac{3}{2}t)}dx - \frac{\alpha^2}{2}\int_{|x|>\frac{5}{4}t} \tilde{m}^2 e^{\alpha(|x|-\frac{3}{2}t)}dx \\
& + \frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=-\frac{5}{4}t} + \frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t} \\
& + (\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} + \frac{1}{2}\tilde{m}^2\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} - \tilde{m}_x\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}.
\end{aligned}
$$

For the integration part, we use arithmetic mean value inequality:

$$
\begin{aligned}
& \frac{d}{dt}E(t) + \frac{3\alpha}{2}E(t) - \int_{|x|>\frac{5}{4}t} \frac{\alpha x}{|x|}(\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}dx \\
& + \int_{|x|>\frac{5}{4}t} \tilde{m}_x^2 e^{\alpha(|x|-\frac{3}{2}t)}dx - \frac{\alpha^2}{2}\int_{|x|>\frac{5}{4}t} \tilde{m}^2 e^{\alpha(|x|-\frac{3}{2}t)}dx \\
\geq & \frac{d}{dt}E(t) + (\frac{\alpha}{2}-\alpha^2)E(t).
\end{aligned}
$$

For the boundary part

$$
\begin{aligned}
& \frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=-\frac{5}{4}t} + \frac{5}{4}(\frac{\tilde{\rho}^2+\tilde{m}^2}{2})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t} \\
& + (\tilde{\rho}\tilde{m})e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} + \frac{1}{2}\tilde{m}^2\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} \geq 0.
\end{aligned}
$$

We conclude that

$$
\frac{d}{dt}E(t) + (\frac{\alpha}{2}-\alpha^2)E(t) \leq -\tilde{m}_x\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=-\frac{5}{4}t} + \tilde{m}_x\tilde{m}e^{\alpha(|x|-\frac{3}{2}t)}|_{x=\frac{5}{4}t}. \tag{2.53}
$$

By the estimate for m_x and m at $|x| = \frac{5}{4}t$ of Corollary 2.3.5 and Corollary 2.4.4, we have

$$
\frac{d}{dt}E(t) + (\frac{\alpha}{2}-\alpha^2)E(t) \leq \frac{C}{\sqrt{1+t}}e^{-\frac{t}{4}}, \tag{2.54}
$$

where C is some positive constant. By the smallness of α, we have $\frac{\alpha}{2} - \alpha^2 > 0$.

Therefore, there exists some constant K, such that

$$\int_{|x|>\frac{5}{4}t} (\frac{\tilde{\rho}^2+\tilde{m}^2}{2}) e^{\alpha(|x|-\frac{3}{2}t)} dx \leq K. \tag{2.55}$$

Repeat the above procedure for $\tilde{\rho}_x$ and $\tilde{m}_x$, similarly we have

$$\int_{|x|>\frac{5}{4}t} (\frac{\tilde{\rho}_x^2+\tilde{m}_x^2}{2}) e^{\alpha(|x|-\frac{3}{2}t)} dx \leq K. \tag{2.56}$$

By the Sobolev embedding theorem, we have

$$e^{\alpha(|x|-\frac{3}{2}t)}|\tilde{\rho}|, e^{\alpha(|x|-\frac{3}{2}t)}|\tilde{m}| \leq K, \tag{2.57}$$

on $\{|x| > \frac{5}{4}t\}$. It is obviously true for $\{|x| > 2t\}$ also. When $|x| > 2t$,

$$|x| - \frac{3}{2}t > \frac{1}{8}|x| + \frac{7}{4}t - \frac{3}{2}t > \frac{1}{8}|x| + \frac{1}{8}t. \tag{2.58}$$

As a result, we have

$$|\tilde{\rho}|, |\tilde{m}| \leq Ce^{-(|x|+t)/c}, \tag{2.59}$$

for some constants C and c.

2.6 Conclusion

Inside the finite Mach number region $\{|x| < 2t\}$, we have

$$\begin{aligned}
&|\mathbb{G}(x,t) - e^{-t}\delta(x)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}| = |\int_{\mathbb{R}} (\hat{\mathbb{G}} - e^{-t}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}) e^{ix\eta} d\eta| \\
&\leq O(1)(\frac{e^{-\frac{|x-t|^2}{c(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{c(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-t/c} + Ce^{-t} + Ce^{-|x|/c-t} \\
&\leq O(1)(\frac{e^{-\frac{|x-t|^2}{c(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{c(1+t)}}}{\sqrt{1+t}}) + Ce^{-(|x|+t)/c},
\end{aligned}$$

where C and c are constants. For outside the finite Mach number region $\{|x| > 2t\}$, we have

$$|\mathbb{G}(x,t)| \leq Ce^{-(|x|+t)/c},$$

where C and c are constants. The above two inequalities lead to the proof of our main Theorem 1.

Chapter 3

The Dirichlet-Neumann map

From this chapter, we start to consider the problem with the presence of boundary. We will construct Green's function for the initial-boundary value problem based on the fundamental solution for the initial value problem. We make use of the property of the backward fundamental solution in our construction. We first introduce the definition of the backward fundamental solution and prove its equivalence to the normal forward fundamental solution in the next section.

3.1 The forward equation and the backward equation

We recall the definition of the forward fundamental equation first.

The fundamental solution $\mathbb{G}(x,t)$ for the initial value problem to the system (1.3) is a 2×2 matrix valued function which satisfies

$$\begin{cases} \partial_t \mathbb{G}(x,t) + A\partial_x \mathbb{G}(x,t) = B\partial_x^2 \mathbb{G}(x,t) \text{ for } x \in \mathbb{R}, t > 0, \\ \mathbb{G}(x,0) = \delta(x)I. \end{cases} \tag{3.1}$$

To differentiate from the backward fundamental solution introduced in this section, we call the above **forward fundamental solution**. The equation satisfied by the forward fundamental solution is called the **forward equation**.

We introduce the backward fundamental solution $\mathbb{G}^B$ as follows:

Definition 3.1.1. *The **backward fundamental solution** $\mathbb{G}^B(x-y,t-\tau)$ for the initial value problem to the system* (1.3) *is a 2×2 matrix valued function which satisfies the **backward equation**:*

$$\begin{cases} -\partial_\tau \mathbb{G}^B(x-y,t-\tau) - \partial_y \mathbb{G}^B(x-y,t-\tau)A - \partial_y^2 \mathbb{G}^B(x-y,t-\tau)B = 0 \text{ for } \tau \in (0,t), \\ \mathbb{G}^B(x-y,0) = \delta(x-y)I. \end{cases} \tag{3.2}$$

We will show the equivalence of the forward fundamental solution and the backward fundamental solution.

Lemma 3.1.2. *The backward fundamental solution and the forward fundamental solution are equal.*

Proof We consider the solution of the following initial value problem:

$$\begin{cases} \partial_t g(x,t) + A\partial_x g(x,t) - B\partial_x^2 g(x,t) = 0, \\ g(x,0) = g_0(x). \end{cases} \tag{3.3}$$

Left multiply it with the backward fundamental solution $\mathbb{G}^B(x-y,t-\tau)$ and integrate over $(0,t)\times(-\infty,\infty)$, we have

$$\int_0^t \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t-\tau)(\partial_\tau g(y,\tau) + A\partial_y g(y,\tau) - B\partial_y^2 g(y,\tau))dyd\tau = 0. \tag{3.4}$$

For (3.4), we do integration by parts term by term. For the first term,

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t-\tau)\partial_\tau g(y,\tau)dyd\tau \\
&= \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t-\tau)g(y,\tau)dy|_{\tau=0}^{\tau=t} - \int_0^t \int_{-\infty}^{\infty} \partial_\tau \mathbb{G}^B(x-y,t-\tau)g(y,\tau)dyd\tau \\
&= \int_{-\infty}^{\infty} \delta(x-y)g(y,t)dy - \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t)g(y,0)dy \\
&\quad - \int_0^t \int_{-\infty}^{\infty} \partial_\tau \mathbb{G}^B(x-y,t-\tau)g(y,\tau)dyd\tau \\
&= g(x,t) - \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t)g(y,0)dy - \int_0^t \int_{-\infty}^{\infty} \partial_\tau \mathbb{G}^B(x-y,t-\tau)g(y,\tau)dyd\tau.
\end{aligned}
$$

For the second term,

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t-\tau)A\partial_y g(y,\tau)dyd\tau \\
&= \int_0^t \mathbb{G}^B(x-y,t-\tau)Ag(y,\tau)d\tau|_{y=-\infty}^{y=\infty} - \int_0^t \int_{-\infty}^{\infty} \partial_y \mathbb{G}^B(x-y,t-\tau)Ag(y,\tau)dyd\tau \\
&= -\int_0^t \int_{-\infty}^{\infty} \partial_y \mathbb{G}^B(x-y,t-\tau)Ag(y,\tau)dyd\tau.
\end{aligned}
$$

For the third term,

$$
\begin{aligned}
&\quad - \int_0^t \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t-\tau)B\partial_{yy} g(y,\tau)dyd\tau \\
&= -\int_0^t \mathbb{G}^B(x-y,t-\tau)B\partial_y g(y,\tau)d\tau|_{y=-\infty}^{y=\infty} \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \partial_y \mathbb{G}^B(x-y,t-\tau)B\partial_y g(y,\tau)dyd\tau \\
&= \int_0^t \int_{-\infty}^{\infty} \partial_y \mathbb{G}^B(x-y,t-\tau)B\partial_y g(y,\tau)dyd\tau \\
&= \int_0^t \partial_y \mathbb{G}^B(x-y,t-\tau)Bg(y,\tau)d\tau|_{y=-\infty}^{y=\infty} \\
&\quad - \int_0^t \int_{-\infty}^{\infty} \partial_{yy} \mathbb{G}^B(x-y,t-\tau)Bg(y,\tau)dyd\tau \\
&= -\int_0^t \int_{-\infty}^{\infty} \partial_{yy} \mathbb{G}^B(x-y,t-\tau)Bg(y,\tau)dyd\tau.
\end{aligned}
$$

By the definition of the backward fundamental solution, we have

$$-\partial_\tau \mathbb{G}^B(x-y,t-\tau) - \partial_y \mathbb{G}^B(x-y,t-\tau)A - \partial_{yy}\mathbb{G}^B(x-y,t-\tau)B = 0.$$

Right multiply with $g(y,\tau)$ and integrate over $(0,t)\times(-\infty,\infty)$:

$$-\int_0^t \int_{-\infty}^{\infty} \partial_\tau \mathbb{G}^B(x-y,t-\tau)g(y,\tau)dyd\tau - \int_0^t \int_{-\infty}^{\infty} \partial_y \mathbb{G}^B(x-y,t-\tau)Ag(y,\tau)dyd\tau$$
$$- \int_0^t \int_{-\infty}^{\infty} \partial_{yy}\mathbb{G}^B(x-y,t-\tau)Bg(y,\tau)dyd\tau = 0.$$

Therefore, (3.4) reduced to

$$g(x,t) = \int_{-\infty}^{\infty} \mathbb{G}^B(x-y,t)g_0(y)dy. \tag{3.5}$$

Now, take $g_0(y) = \delta(y)$, from the definition of $g(x,t)$, we have

$$g(x,t) = \mathbb{G}(x,t).$$

On the other hand, since $g_0(y) = \delta(y)$, (3.5) yields:

$$g(x,t) = \mathbb{G}^B(x,t).$$

Hence, the forward fundamental solution and the backward fundamental solution are equal.

Since the forward and backward fundamental solution are equivalent, in the following we will denote them uniquely as $\mathbb{G}(x,t)$.

3.2 The Green's Identity

The main objective of this and the next chapter is to study the behavior of the difference between the solutions to the initial value problem and the initial-boundary value problem. We denote the Green's function to the initial-boundary value problem $G(x, y, t)$ as: $G(x, y, t) = \mathbb{G}(x - y, t) + H(x, y, t)$, where $\mathbb{G}(x, t)$ is the fundamental solution to the initial value problem which is obtained in the previous chapter. Therefore, we have the following equations for H:

$$\begin{cases} \partial_t H(x, y, t) + A\partial_x H(x, y, t) = B\partial_x^2 H(x, y, t), x, t > 0 \\ H(x, y, 0) = 0, \\ H(0, y, t) = -\mathbb{G}(-y, t). \end{cases} \tag{3.6}$$

Left multiply $\mathbb{G}(x - z, t - \tau)$ with the first equation of (3.6) and integrate over $(0, t) \times (0, \infty)$,

$$\int_0^t \int_0^\infty \mathbb{G}(x - z, t - \tau)(\partial_\tau H(z, y, \tau) + A\partial_z H(z, y, \tau) - B\partial_z^2 H(z, y, \tau))dzd\tau = 0. \tag{3.7}$$

We do integration by parts for (3.7) term by term. For the first term

$$\begin{aligned} &\int_0^t \int_0^\infty \mathbb{G}(x - z, t - \tau)\partial_\tau H(z, y, \tau)dzd\tau \\ &= \int_0^\infty \mathbb{G}(x - z, t - \tau)H(z, y, \tau)dz|_{t=0}^{t=\tau} - \int_0^t \int_0^\infty \partial_\tau \mathbb{G}(x - z, t - \tau)H(z, y, \tau)dzd\tau. \\ &= \int_0^\infty \delta(x - z)H(z, y, t)dz - \int_0^\infty \mathbb{G}(x - z, t)H(z, y, 0)dz \\ &\quad - \int_0^t \int_0^\infty \partial_\tau \mathbb{G}(x - z, t - \tau)H(z, y, \tau)dzd\tau \\ &= H(x, y, t) - \int_0^t \int_0^\infty \partial_\tau \mathbb{G}(x - z, t - \tau)H(z, y, \tau)dzd\tau. \end{aligned}$$

For the second term

$$
\begin{aligned}
&\int_0^t \int_0^\infty \mathbb{G}(x-z,t-\tau)A\partial_z H(z,y,\tau)dzd\tau \\
&= \int_0^t \mathbb{G}(x-z,t-\tau)AH(z,y,\tau)d\tau|_{z=0}^{z=\infty} - \int_0^t \int_0^\infty \partial_z \mathbb{G}(x-z,t-\tau)AH(z,y,\tau)dzd\tau \\
&= -\int_0^t \mathbb{G}(x,t-\tau)AH(0,y,\tau)d\tau - \int_0^t \int_0^\infty \partial_z \mathbb{G}(x-z,t-\tau)AH(z,y,\tau)dzd\tau.
\end{aligned}
$$

For the third term

$$
\begin{aligned}
&-\int_0^t \int_0^\infty \mathbb{G}(x-z,t-\tau)B\partial_{zz} H(z,y,\tau)dzd\tau \\
&= -\int_0^t \mathbb{G}(x-z,t-\tau)B\partial_z H(z,y,\tau)d\tau|_{z=0}^{z=\infty} \\
&\quad + \int_0^t \int_0^\infty \partial_z \mathbb{G}(x-z,t-\tau)B\partial_z H(z,y,\tau)dzd\tau \\
&= \int_0^t \mathbb{G}(x,t-\tau)B\partial_z H(0,y,\tau)d\tau + \int_0^t \int_0^\infty \partial_z \mathbb{G}(x-z,t-\tau)B\partial_z H(z,y,\tau)dzd\tau \\
&= \int_0^t \mathbb{G}(x,t-\tau)B\partial_z H(0,y,\tau)d\tau + \int_0^t \partial_z \mathbb{G}(x-z,t-\tau)BH(z,y,\tau)d\tau|_{z=0}^{z=\infty} \\
&\quad - \int_0^t \int_0^\infty \partial_{zz} \mathbb{G}(x-z,t-\tau)BH(z,y,\tau)dzd\tau \\
&= \int_0^t \mathbb{G}(x,t-\tau)B\partial_x H(0,y,\tau)d\tau - \int_0^t \partial_x \mathbb{G}(x-z,t-\tau)BH(z,y,\tau)d\tau|_{z=0}^{z=\infty} \\
&\quad - \int_0^t \int_0^\infty \partial_{zz} \mathbb{G}(x-z,t-\tau)BH(z,y,\tau)dzd\tau \\
&= \int_0^t \mathbb{G}(x,t-\tau)B\partial_x H(0,y,\tau)d\tau + \int_0^t \partial_x \mathbb{G}(x,t-\tau)BH(0,y,\tau)d\tau \\
&\quad - \int_0^t \int_0^\infty \partial_{zz} \mathbb{G}(x-z,t-\tau)BH(z,y,\tau)dzd\tau.
\end{aligned}
$$

Since $\mathbb{G}(x-z,t-\tau)$ satisfies the backward equation (3.2),

$$
-\partial_\tau \mathbb{G}(x-z,t-\tau) - \partial_z \mathbb{G}(x-z,t-\tau)A - \partial_{zz} \mathbb{G}(x-z,t-\tau)B = 0. \tag{3.8}
$$

Therefore, we have the following representation for $H(x, y, t)$

$$H(x,y,t) = \int_0^t \mathbb{G}(x,t-\tau)AH(0,y,\tau) - \mathbb{G}(x,t-\tau)B\partial_x H(0,y,\tau) - \partial_x \mathbb{G}(x,t-\tau)BH(0,y,\tau)d\tau. \tag{3.9}$$

In the above representation of $H(x, y, t)$, the only term unknown is the Neumann boundary data $\partial_x H(0, y, \tau)$. As a result, it would be great if we can construct the Neumann boundary data from the given Dirichlet boundary data $H(0, y, \tau)$.

We will apply Laplace transformation and inverse Laplace transformation to construct a Dirichlet-Neumann map in the following sections. We will first introduce some basic properties for the Laplace transformation and inverse Laplace transformation in the next section.

3.3 Laplace transformation and inverse Laplace transformation

In order to calculate the Dirichlet-Neumann map, we first introduce the definition and some properties of the Laplace transformation and inverse Laplace transformation.

Definition 3.3.1. *For any function $f(t), t > 0$, the **Laplace transformation** of $f(t)$ is defined to be a function $F(s)$, by*

$$F(s) = \mathbb{L}[f](s) = \int_0^\infty e^{-st} f(t)dt. \tag{3.10}$$

Definition 3.3.2. *For function $V(x, t), t > 0, x > 0$, the **Laplace transforma-***

tion *of* $V(x,t)$ *over variable* t *is defined by*

$$\mathbb{L}[V](x,s) = \int_0^\infty e^{-st} V(x,t)dt, \tag{3.11}$$

and the ***Laplace transformation*** *of* $V(x,t)$ *over variable* t *and* x *is defined by*

$$\mathbb{J}[V](\xi,s) = \int_0^\infty e^{-\xi x} \mathbb{L}[V](x,s)dx. \tag{3.12}$$

Lemma 3.3.3. *Let* $\mathbb{L}[f](s) = F(s)$*, then*

$$\mathbb{L}[\frac{df}{dt}] = sF(s) - f(0). \tag{3.13}$$

Proof By definition and integration by parts,

$$\int_0^\infty e^{-st}\frac{df(t)}{dt}dt = [e^{-st}f(t)]_0^\infty - \int_0^\infty \frac{de^{-st}}{dt}f(t)dt = -f(0) + sF(s). \tag{3.14}$$

Lemma 3.3.4. *Let* $\mathbb{L}[f](s) = F(s)$*, then*

$$\mathbb{L}[\frac{d^2f}{dt^2}] = s^2F(s) - sf(0) - f'(0). \tag{3.15}$$

Proof By definition and integration by parts,

$$\int_0^\infty e^{-st}\frac{d^2f(t)}{dt^2}dt = [e^{-st}\frac{df(t)}{dt}]_0^\infty - \int_0^\infty \frac{de^{-st}}{dt}\frac{df(t)}{dt}dt = -f'(0) + s\mathbb{L}[\frac{df}{dt}]. \tag{3.16}$$

Substitute in the result for $\mathbb{L}[\frac{df}{dt}]$ from the previous Lemma, we proved (3.15).

Definition 3.3.5. *For function* $F(s)$*, the* ***inverse Laplace transformation*** *or* ***Bromwich integral*** *of* $F(s)$ *is defined to be a function* $f(t)$*, by the following*

complex integral

$$f(t) = \mathbb{L}^{-1}[F](t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s) ds, \tag{3.17}$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$.

Lemma 3.3.6. *Let $F(s) = \frac{1}{\sqrt{s}}$, then*

$$f(t) = \mathbb{L}^{-1}[F](t) = \frac{1}{\sqrt{\pi t}}. \tag{3.18}$$

Proof $\frac{1}{\sqrt{s}}$ is convergent in the region $\{Re(s) > 0\}$. Hence, we choose $\gamma = 0$, and by Bromwich integral

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{-iT}^{+iT} e^{st} \frac{1}{\sqrt{s}} ds. \tag{3.19}$$

Let

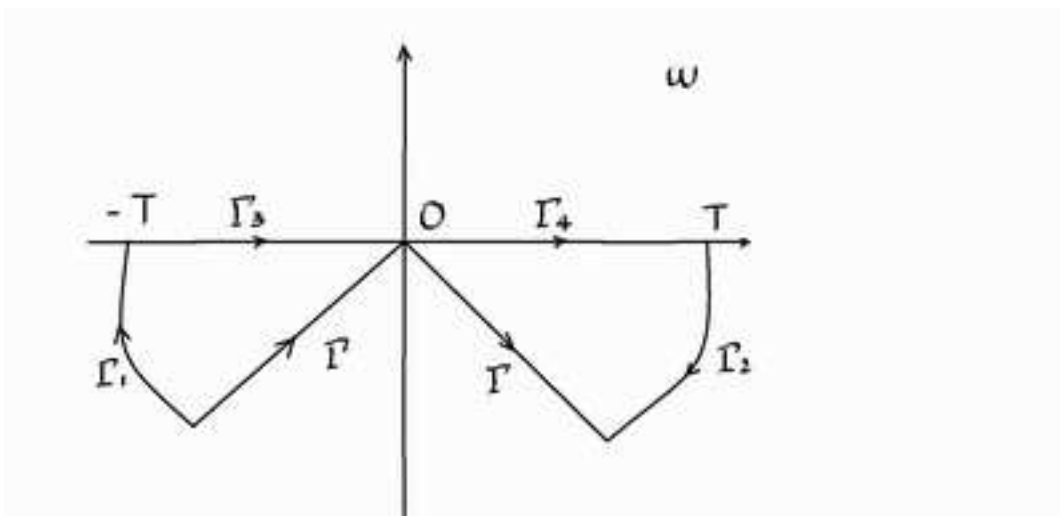
$$i\omega = \sqrt{s},$$

therefore,

$$s = -\omega^2.$$

By complex contour integral,

$$
\begin{aligned}
&\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-iT}^{+iT} e^{st}\frac{1}{\sqrt{s}}ds \\
=&\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\Gamma} e^{-\omega^2 t}\frac{-2\omega}{i\omega}d\omega \\
=&\frac{1}{\pi}\lim_{T\to\infty}\int_{\Gamma} e^{-\omega^2 t}d\omega \\
=&\frac{1}{\pi}\lim_{T\to\infty}\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4} e^{-\omega^2 t}d\omega \\
=&\frac{1}{\pi}\lim_{T\to\infty}\int_{\Gamma_3+\Gamma_4} e^{-\omega^2 t}d\omega \\
=&\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-\omega^2 t}d\omega = \frac{1}{\sqrt{\pi t}}. \\
=&\frac{1}{\sqrt{\pi t}}.
\end{aligned}
$$



*Pic.*1

Lemma 3.3.7. *Let $F(s)$ be convergent in the region $\{Re(s) > 0\}$, and the Bromwich integral $f(t)$ exists, then the inverse Laplace transformation of $F(s+K), K > 0$*

satisfies

$$\mathbb{L}^{-1}[F(s+K)](t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{-K-iT}^{K+iT} e^{st}F(s)ds = e^{-Kt}f(t). \tag{3.20}$$

Proof By direct calculation. Moreover, combine this and the above Lemma, we have $\mathbb{L}^{-1}[\frac{1}{\sqrt{s+K}}] = \frac{1}{\sqrt{\pi t}}e^{-Kt}$.

Lemma 3.3.8. *Let $F(s) = s$, then*

$$f(t) = \mathbb{L}^{-1}[F](t) = \frac{d}{dt}\delta(t). \tag{3.21}$$

Lemma 3.3.9. *Let $F(s) = \mathbb{L}[f]$, $G(s) = \mathbb{L}[g]$, then*

$$\mathbb{L}^{-1}[F(s)G(s)](t) = \int_0^t g(t-\tau)f(\tau)d\tau \equiv f(t) * g(t). \tag{3.22}$$

3.4 Dirichlet-Neumann map

In this section, we will construct the Dirichlet-Neumann map, that is, to write the representation of the Neumann boundary data in terms of the Dirichlet boundary data. For notation simplicity in calculation, in this section, we denote $\rho(0,y,t) \equiv \rho_b$, $\rho_x(0,y,t) \equiv \rho_x$, $m(0,y,t) \equiv m_b$, $m_x(0,y,t) \equiv m_x$. We have the following representation for the Neumann data m_x in terms of m_b:

Theorem 3.4.1. *The Nuemann boundary data $m_x(0,y,t)$ can be represented in terms of the Dirichelt boundary data $m(0,y,t)$ as follows:*

$$m_x(0,y,t) = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-(t-\tau)}}{\sqrt{t-\tau}}\partial_\tau m(0,y,\tau)d\tau. \tag{3.23}$$

Proof We calculate the Laplace transformation for each equation of system (1.2) for both x and t variable, apply Lemma 3.3.3, Lemma 3.3.4 and denote $\tilde{V} \equiv \mathbb{J}[V]$:

$$s\tilde{\rho} + \xi\tilde{m} = \tilde{m}_b, \tag{3.24}$$

$$s\tilde{m} + \xi\tilde{\rho} = \tilde{\rho}_b + \xi^2\tilde{m} - \tilde{m}_x - \xi\tilde{m}_b, \tag{3.25}$$

where $\tilde{\rho}_b = \tilde{\rho}(0, y, s)$, $\tilde{m}_b = \tilde{m}(0, y, s)$, and $\tilde{m}_x = \tilde{m}_x(0, y, s)$. Moreover, calculate the Laplace transformation of the first equation of system (1.2) for t variable, and take value at $x = 0$:

$$s\mathbb{L}[\rho_b] + \mathbb{L}[m_x] = 0, \tag{3.26}$$

Apply Laplace transformation to (3.26) for x variable:

$$s\tilde{\rho}_b = -\tilde{m}_x. \tag{3.27}$$

Combine (3.24), (3.25) and (3.27) together, we have:

$$[(s+1)\xi^2 - s^2]\tilde{m} = (s+1)(\tilde{m}_x + \xi\tilde{m}_b). \tag{3.28}$$

Hence,

$$\tilde{m} = \frac{(s+1)(\tilde{m}_x + \xi\tilde{m}_b)}{(s+1)\xi^2 - s^2} = \frac{\tilde{m}_x + \xi\tilde{m}_b}{\xi^2 - \frac{s^2}{(s+1)}}. \tag{3.29}$$

Solve $\xi^2 - \frac{s^2}{(s+1)} = 0$, we have:

$$\xi_1 = \frac{s}{\sqrt{s+1}}, \tag{3.30}$$

$$\xi_2 = -\frac{s}{\sqrt{s+1}}, \tag{3.31}$$

Hence, we can write down the representation formula for m:

$$\mathbb{L}[m] = e^{\xi_1 x}F_1 + e^{\xi_2 x}F_2 = 0, \tag{3.32}$$

where

$$F_1 = Res_{\xi=\xi_1}\tilde{m} = \frac{\tilde{m}_x + \xi_1\tilde{m}_b}{\xi_1 - \xi_2}, \tag{3.33}$$

$$F_2 = Res_{\xi=\xi_2}\tilde{m} = \frac{\tilde{m}_x + \xi_2\tilde{m}_b}{\xi_2 - \xi_1}, \tag{3.34}$$

Therefore, by the well-posedness of P.D.E., for the positive solution ξ_1, we have

$$F_1 = 0, \tag{3.35}$$

that is

$$\tilde{m}_x + \xi_1\tilde{m}_b = 0. \tag{3.36}$$

Therefore,

$$\tilde{m}_x = -\frac{s}{\sqrt{s+1}}\tilde{m}_b. \tag{3.37}$$

Apply the inverse Laplace transformation with respect to x, we have

$$\mathbb{L}(m_x) = -\frac{s}{\sqrt{s+1}}\mathbb{L}(m_b). \tag{3.38}$$

Hence, it would be sufficient if we are able to obtain the inverse Laplace transformation for $-\frac{s}{\sqrt{s+1}}$. By Lemma 3.3.6 and Lemma 3.3.7, we have

$$\mathbb{L}^{-1}[\frac{1}{\sqrt{s+1}}] = \frac{1}{\sqrt{\pi t}}e^{-t}. \tag{3.39}$$

By Lemma 3.3.8,

$$\mathbb{L}^{-1}[F](t) = \frac{d}{dt}\delta(t). \tag{3.40}$$

Therefore, by Lemma 3.3.9,

$$m_x = -\mathbb{L}^{-1}(\frac{1}{\sqrt{s+1}}) * \mathbb{L}^{-1}(s) * m_b = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-(t-\tau)}}{\sqrt{t-\tau}}m_b'(\tau)d\tau. \tag{3.41}$$

The above relationship is called Dirichlet-Neumann map. We will construct the Green's function for the initial-boundary value problem based on this relationship in the next coming Chapter.

Chapter 4

The Green's function

In this chapter, we will derive the exact estimate of the Green's function $G(x, y, t)$ for the initial-boundary value problem to the linearized Navier-Stokes equations (1.5). We recall the representation of $H(x, y, t)$ which is the difference of the Green's function for the initial-boundary value problem $G(x, y, t)$ and the fundamental solution for the initial value problem $\mathbb{G}(x - y, t)$:

$$H(x, y, t) = \int_0^t \mathbb{G}(x, t-\tau)AH(0, y, \tau) - \mathbb{G}(x, t-\tau)BH_x(0, y, \tau) - \mathbb{G}_x(x, t-\tau)BH(0, y, \tau)d\tau. \tag{4.1}$$

From the above representation, to construct the full boundary data from the known boundary data becomes our first step.

4.1 A Priori Estimate on the Neumann boundary data $H_x(0, y, t)$

Lemma 4.1.1. *For any variable $x > 0$ and constant $c_1 > 0$, there exists constants $c_2, c_3 > 0$, such that*

$$xe^{-c_1 x} \leqslant c_2 e^{-c_3 x}. \tag{4.2}$$

Proof There exists constant $c_2 > 0$, such that

$$x \leqslant c_2 e^{c_1 x}, \text{ for any } x > 0 \text{ and fixed } c_1 > 0. \tag{4.3}$$

Choose $c_3 = \frac{1}{2}c_1$, we have

$$xe^{-c_1 x} \leqslant c_2 e^{-\frac{1}{2}c_1 x} = c_2 e^{-c_3 x}. \tag{4.4}$$

The following two lemmas are on the estimate of the derivative of heat kernel and exponential decay term.

Lemma 4.1.2. *There exists $C_1 > 0$, such that*

$$\frac{d}{dt}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}}\right) \leqslant O(1)\frac{1}{\sqrt{1+t}}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C_1(1+t)}}}{\sqrt{1+t}}\right). \tag{4.5}$$

Proof By direct calculation,

$$\frac{d}{dt}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}}\right) = \left(\frac{2\lambda(x-\lambda t)}{C(1+t)} + \frac{(x-\lambda t)^2}{C(1+t)^2} - \frac{1}{2}\frac{1}{1+t}\right)\left(\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}}\right). \tag{4.6}$$

By Lemma 4.1.1,

$$\frac{2\lambda(x-\lambda t)}{C(1+t)}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}}\right) \leqslant O(1)\frac{1}{\sqrt{1+t}}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C_1(1+t)}}}{\sqrt{1+t}}\right), \tag{4.7}$$

$$\frac{(x-\lambda t)^2}{C(1+t)^2}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}}\right) \leqslant O(1)\frac{1}{1+t}\left(\frac{e^{-\frac{(x-\lambda t)^2}{C_1(1+t)}}}{\sqrt{1+t}}\right). \tag{4.8}$$

Combine the above together, we proved (4.5).

Lemma 4.1.3.

$$\frac{d}{dt}e^{-(|x|+t)/C} \leqslant O(1)e^{-(|x|+t)/C}. \tag{4.9}$$

Lemma 4.1.4.

$$|m_b(y,t)| \leqslant O(1)(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-(|y|+t)/C}. \tag{4.10}$$

Proof This can be derived directly from Theorem 1 and replace x by $-y$ from the definition of $H_b(y,t) = H(0,y,t) = -\mathbb{G}(-y,t)$.

The following two lemmas are on the convolution of exponential decay term with heat kernel and exponential decay term.

Lemma 4.1.5. *There exists $C_2 > 0$, such that*

$$e^{-(|x|+t)/C} * e^{-t/C_1} \equiv \int_0^t e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds \leqslant O(1)e^{-(|x|+t)/C_2}. \tag{4.11}$$

Proof Separate the integration scales:

$$\int_0^t e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds = \int_0^{\frac{t}{2}} e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds + \int_{\frac{t}{2}}^t e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds, \tag{4.12}$$

and

$$\int_0^{\frac{t}{2}} e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds \leqslant e^{-|x|/C} \cdot \int_0^{\frac{t}{2}} 1 \cdot e^{-(t-s)/C_1} ds \leqslant e^{-|x|/C} \cdot e^{-t/2C_1}, \tag{4.13}$$

$$\int_{\frac{t}{2}}^t e^{-(|x|+s)/C} \cdot e^{-(t-s)/C_1} ds \leqslant e^{-|x|/C} \cdot \int_{\frac{t}{2}}^t e^{-(|x|+s)/C} \cdot 1 ds \leqslant e^{-|x|/C} \cdot e^{-t/2C}. \tag{4.14}$$

Lemma 4.1.6. *There exists $C_2 > 0$, such that*

$$\frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}} * e^{-t/C_1} \equiv \int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \leqslant O(1)\frac{e^{-\frac{(x-\lambda t)^2}{C_2(1+t)}}}{\sqrt{1+t}} + O(1)e^{-(|x|+t)/C_2}. \tag{4.15}$$

Proof We consider $\lambda > 0$ first. Without loss of generality, we suppose $C > \lambda^2 C_1$. For $x < 2\lambda t$,

$$\begin{aligned}
&\int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \\
&= \int_0^{\frac{t}{2}} \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds + \int_{\frac{t}{2}}^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \\
&\leqslant O(1) \int_0^{\frac{t}{2}} \frac{e^{-(\frac{x^2}{C(1+t-s)} - 2\lambda x + \lambda^2(t-s))}}{\sqrt{1+t-s}} \cdot e^{-\lambda^2 s/C} \cdot e^{-(s/C_1 - \lambda^2 s/C)} ds \\
&\quad + O(1) e^{-t/2C_1} \cdot \int_{\frac{t}{2}}^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} ds \\
&\leqslant O(1) \int_0^{\frac{t}{2}} \frac{e^{-(\frac{x^2}{C(1+t)} - 2\lambda x + \lambda^2 t)}}{\sqrt{1+t-s}} \cdot e^{-(s/C_1 - \lambda^2 s/C)} ds + O(1) e^{-(|x|+t)/C_2} \\
&\leqslant O(1) \int_0^{\frac{t}{2}} \frac{e^{-\frac{(x-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t-s}} e^{-(s/C_1 - \lambda^2 s/C)} ds + O(1) e^{-(|x|+t)/C_2} \\
&= O(1) e^{-\frac{(x-\lambda t)^2}{C(1+t)}} \int_0^{\frac{t}{2}} \frac{1}{\sqrt{1+t-s}} e^{-(s/C_1 - \lambda^2 s/C)} ds + O(1) e^{-(|x|+t)/C_2} \\
&\leqslant O(1) \frac{e^{-\frac{(x-\lambda t)^2}{C_2(1+t)}}}{\sqrt{1+t}} + O(1) e^{-(|x|+t)/C_2}.
\end{aligned}$$

For $x > 2\lambda t$,

$$\int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \leqslant O(1) \int_0^t \frac{e^{-(|x|+t-s)/C}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \leqslant O(1) e^{-(|x|+t)/C_2}. \tag{4.16}$$

The proof for $\lambda = 0$ is similar. For $\lambda < 0$,

$$\frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \leqslant O(1) e^{-(|x|+t-s)/C}, \tag{4.17}$$

therefore

$$\int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C(1+t-s)}}}{\sqrt{1+t-s}} \cdot e^{-s/C_1} ds \leqslant O(1) e^{-(|x|+t)/C_2}. \tag{4.18}$$

Combining the above lemmas together, we have the following a priori estimate theorem on Neumann data:

Theorem 4.1.7. *There exists $C > 0$, such that*

$$|m_x(0,y,t)| \leqslant O(1)\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|y|+t)/C}. \tag{4.19}$$

Proof The representation for $|m_x(0,y,t)|$ is from the Dirichlet-Neumann map Theorem 3.4.1. Combining with the above Lemma 4.1.2 to Lemma 4.1.6 which estimate the derivative, initial data and convolution, we proved this theorem.

From next section, we are able to see that only $|m_x(0,y,t)|$ is useful in the estimate of $H(x,y,t)$. Hence we finished the estimate of $H_x(0,y,t)$.

4.2 Estimate on $H(x,y,t)$

In this section, we will calculate the point-wise estimate on $H(x,y,t)$ using the representation (4.1). We need the following lemma for the convolution of the heat kernel.

Lemma 4.2.1. *For given constants $C_0 > 0$ and $C_1 > 0$, there exists $C_2 > 0$ such that*

$$\int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C_0(t-s)}}}{\sqrt{1+t-s}} \cdot \frac{e^{-\frac{(y-\lambda s)^2}{C_1(1+s)}}}{\sqrt{1+s}} ds = O(1)\frac{e^{-\frac{(x+y-\lambda t)^2}{C_2(1+t)}}}{\sqrt{1+t}} \tag{4.20}$$

Proof Without loss of generality, we suppose $C_0 > C_1$,

$$
\begin{aligned}
&\int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C_0(t-s)}}}{\sqrt{1+t-s}} \cdot \frac{e^{-\frac{(y-\lambda s)^2}{C_1(1+s)}}}{\sqrt{1+s}} ds \\
&= O(1) \int_0^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{C_0(t-s)}}}{\sqrt{1+t-s}} \cdot \frac{e^{-\frac{(y-\lambda s)^2}{C_0(1+s)}}}{\sqrt{1+s}} ds \\
&= O(1) e^{-\frac{(x+y-\lambda t)^2}{2C_0 t}} \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) \frac{e^{-\frac{(x-\lambda(t-s))^2}{2C_0(t-s)}}}{\sqrt{1+t-s}} \cdot \frac{e^{-\frac{(y-\lambda s)^2}{2C_0(1+s)}}}{\sqrt{1+s}} ds \\
&= O(1) e^{-\frac{(x+y-\lambda t)^2}{2C_0 t}} \left(\int_0^{\frac{t}{2}} \frac{e^{-\frac{(y-\lambda s)^2}{2C_0(1+s)}}}{\sqrt{1+t}\sqrt{1+s}} ds + \int_{\frac{t}{2}}^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{2C_0(t-s)}}}{\sqrt{1+t-s}\sqrt{1+t}} ds\right) \\
&= O(1) \frac{e^{-\frac{(x+y-\lambda t)^2}{2C_0 t}}}{\sqrt{1+t}} \left(\int_0^{\frac{t}{2}} \frac{e^{-\frac{(y-\lambda s)^2}{2C_0(1+s)}}}{\sqrt{1+s}} ds + \int_{\frac{t}{2}}^t \frac{e^{-\frac{(x-\lambda(t-s))^2}{2C_0(t-s)}}}{\sqrt{1+t-s}} ds\right) \\
&= O(1) \frac{e^{-\frac{(x+y-\lambda t)^2}{C_2(1+t)}}}{\sqrt{1+t}}.
\end{aligned}
$$

For the second equal we make use of the following:

$$
\begin{aligned}
&\frac{(x-\lambda(t-s))^2}{2C_0(t-s)} + \frac{(y-\lambda s)^2}{2C_0(1+s)} \\
&= \frac{1}{2C_0}\left(\frac{x^2}{t-s} - 2\lambda x + \lambda^2(t-s) + \frac{y^2}{1+s} - 2\lambda y + \lambda(1+s)\right) \\
&= \frac{1}{2C_0}\left(\frac{x^2+y^2}{1+t} - 2\lambda(x+y) + \lambda^2(1+t) + \frac{(t-s)^2x^2 + (1+s)^2y^2}{(1+s)(t-s)(1+t)}\right) \\
&\geqslant \frac{1}{2C_0}\left(\frac{x^2+y^2}{1+t} - 2\lambda(x+y) + \lambda^2(1+t) + \frac{2xy}{1+t}\right) \\
&= \frac{(x+y-\lambda t)^2}{2C_0 t}.
\end{aligned}
$$

Lemma 4.2.2. *For given constants $C_0 > 0$ and $C_1 > 0$, there exists $D > C_0, C_1$ such that*

$$
\begin{aligned}
&\left(\frac{e^{-\frac{|x-t|^2}{C_0(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C_0(1+t)}}}{\sqrt{1+t}}\right) * \left(\frac{e^{-\frac{|y-t|^2}{C_1(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C_1(1+t)}}}{\sqrt{1+t}}\right) \\
\leqslant\ & O(1)\left(\frac{e^{-\frac{|x-y-t|^2}{D(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{D(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y-t|^2}{D(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{D(1+t)}}}{\sqrt{1+t}}\right).
\end{aligned}
$$

Proof By similar calculation as in the proof of Lemma 4.2.1, we have

$$\frac{e^{-\frac{|x-t|^2}{C_0(1+t)}}}{\sqrt{1+t}} * \frac{e^{-\frac{|y-t|^2}{C_1(1+t)}}}{\sqrt{1+t}} = O(1)\frac{e^{-\frac{|x+y-t|^2}{C_2(1+t)}}}{\sqrt{1+t}}, \tag{4.21}$$

and

$$\frac{e^{-\frac{|x-t|^2}{C_0(1+t)}}}{\sqrt{1+t}} * \frac{e^{-\frac{|y+t|^2}{C_1(1+t)}}}{\sqrt{1+t}} = \frac{e^{-\frac{|x-t|^2}{C_0(1+t)}}}{\sqrt{1+t}} * \frac{e^{-\frac{|-y-t|^2}{C_1(1+t)}}}{\sqrt{1+t}} = O(1)\frac{e^{-\frac{|x-y-t|^2}{C_2(1+t)}}}{\sqrt{1+t}}, \tag{4.22}$$

Other calculations for the convolution are similar. Hence we proved this lemma.

Now we can prove the main result of this Chapter as follow:

Theorem 4.2.3. *There exists $C > 0$ such that*

$$H(x,y,t) = j_1(y,t)\delta(x) + j_2(x,t)\delta(-y) + j_3(x,y,t), \tag{4.23}$$

where j_1 is a matrix of the form $\begin{pmatrix} a_1(y,t) & a_2(y,t) \\ 0 & 0 \end{pmatrix}$ *satisfying*

$$|a_1(y,t)|, |a_2(y,t)| = O(1)(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C}), \tag{4.24}$$

j_2 is a matrix of the form $\begin{pmatrix} b_1(x,t) & 0 \\ b_2(x,t) & 0 \end{pmatrix}$ *satisfying*

$$|b_1(x,t)|, |b_2(x,t)| = O(1)(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C}), \tag{4.25}$$

and $j_3(x,y,t)$ is a matrix satisfy

$$
\begin{aligned}
|j_3(x,y,t)| = O(1)(&\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}) \\
&+O(1)e^{-|y|/C}(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}) \\
&+O(1)e^{-|x|/C}(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}) + O(1)e^{-(|x|+|y|+t)/C}.
\end{aligned}
$$

Proof We first calculate

$$
\int_0^t \mathbb{G}(x,t-\tau)AH(0,y,\tau)d\tau = -\int_0^t \mathbb{G}(x,t-\tau)A\mathbb{G}(-y,\tau)d\tau. \tag{4.26}
$$

By the structure of the fundamental solution, we have

$$
\mathbb{G}(x,t) = e^{-t}\delta(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(1)(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C})\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = I_1 + I_2, \tag{4.27}
$$

$$
\mathbb{G}(-y,t) = e^{-t}\delta(-y)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(1)(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C})\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = I_3 + I_4, \tag{4.28}
$$

where I_1 and I_3 denote the part include the δ-function.

To estimate the convolutions in (4.26), we need the following results for matrix calculation,

$$
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.29}
$$

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.30}
$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{4.31}$$

Therefore,

$$\int_0^t I_1(x, t-\tau) A I_3(-y, \tau) d\tau = 0, \tag{4.32}$$

that is, there is no term of the form $\delta(x)\delta(-y)$ in the result. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ 0 & 0 \end{pmatrix} \equiv M_1, \tag{4.33}$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ a_{22} & 0 \end{pmatrix} \equiv M_2, \tag{4.34}$$

we have,

$$\int_0^t I_1(x, t-\tau) A I_4(-y, \tau) d\tau = h_1(y, t)\delta(x) M_1, \tag{4.35}$$

and

$$\int_0^t I_2(x, t-\tau) A I_3(-y, \tau) d\tau = h_2(y, t)\delta(-y) M_2, \tag{4.36}$$

where for some constant $C > 0$

$$|h_1(y,t)| = O(1)\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C}\right), \tag{4.37}$$

$$|h_2(x,t)| = O(1)\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C}\right). \tag{4.38}$$

For the remaining term, we have

$$\begin{aligned}
&\int_0^t I_2(x,t-\tau)AI_4(-y,\tau)d\tau \equiv h_3(x,y,t)\\
&= O(1)\left(\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right)\\
&\quad +O(1)e^{-|y|/C}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right)\\
&\quad +O(1)e^{-|x|/C}\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+|y|+t)/C}.
\end{aligned}$$

We then calculate the second term

$$\int_0^t \mathbb{G}(x,t-\tau)BH_x(0,y,\tau)d\tau, \tag{4.39}$$

where

$$\mathbb{G}(x,t) = e^{-t}\delta(x)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + O(1)\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C}\right)\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = I_1 + I_2, \tag{4.40}$$

$$B = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{4.41}$$

and

$$H_x(0,y,t) = \begin{pmatrix}(\rho_1)_x & (\rho_2)_x\\ (m_1)_x & (m_2)_x\end{pmatrix}. \tag{4.42}$$

Since

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0, \tag{4.43}$$

we have

$$\int_0^t I_1(x,t-\tau)BH_x(0,y,\tau)d\tau = 0. \tag{4.44}$$

Since

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (\rho_1)_x & (\rho_2)_x \\ (m_1)_x & (m_2)_x \end{pmatrix} = \begin{pmatrix} a_{12}(m_1)_x & a_{12}(m_2)_x \\ a_{22}(m_1)_x & a_{22}(m_2)_x \end{pmatrix}, \tag{4.45}$$

we have

$$\int_0^t I_2(x,t-\tau)BH_x(0,y,\tau)d\tau = O(1)(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C})*\begin{pmatrix} a_{12}(m_1)_x & a_{12}(m_2)_x \\ a_{22}(m_1)_x & a_{22}(m_2)_x \end{pmatrix}. \tag{4.46}$$

We note that there is no singularity $\delta(x)$, $\delta(-y)$ or $\delta(x)\delta(-y)$ in the above representation. Moreover, the only required Neumann boundary data in this estimate is the derivative of the momentum m_x which is calculated in the previous Chapter by the Dirichlet-Neumann map. Hence, combined with the estimate of m_x, we have

$$\begin{aligned} &|\int_0^t \mathbb{G}(x,t-\tau)BH_x(0,y,\tau)d\tau| \\ &= O(1)(\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}) \\ &+O(1)e^{-|y|/C}(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}) \\ &+O(1)e^{-|x|/C}(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}})+O(1)e^{-(|x|+|y|+t)/C}. \end{aligned}$$

At last, we estimate the third term

$$\int_0^t \mathbb{G}_x(x,t-\tau)BH(0,y,\tau)d\tau = -\int_0^t \mathbb{G}_x(x,t-\tau)B\mathbb{G}(-y,\tau)d\tau \tag{4.47}$$

Similar as in the proof of Lemma 4.1.2 and Lemma 4.1.3, we have the following

estimate for $\mathbb{G}_x(x,t)$,

$$\mathbb{G}_x(x,t) = e^{-t}\delta'(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(1)\frac{1}{\sqrt{1+t}}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+t)/C}, \tag{4.48}$$

Due to the presence of $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in the representation, there will be no singular terms after the matrix multiplication as in before. Therefore,

$$\begin{aligned}
&\left|\int_0^t \mathbb{G}_x(x,t-\tau)BH(0,y,\tau)d\tau\right| \\
&= [O(1)\frac{1}{\sqrt{1+t}}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+t)/C}] \\
&\quad *[O(1)\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C}\right)] \\
&= O(1)\left(\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) \\
&\quad + O(1)e^{-|y|/C}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) \\
&\quad + O(1)e^{-|x|/C}\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+|y|+t)/C}.
\end{aligned}$$

Combined the above estimates together, we proved this Theorem. It is then natural to write down the representation of the Green's function $G(x,y,t)$ of the initial-boundary value problem as in Theorem 2 based on the above Theorem and the fundamental solution $\mathbb{G}(x,t)$ derived in Chapter 2.

In the next coming Chapter, we will make use of the structure of the Green's $G(x,y,t)$ of the initial-boundary value problem to study the asymptotic behavior of the general nonlinear problem.

Chapter 5

The nonlinear problem

In this Chapter, we will study the following initial-boundary value problem:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\frac{m^2}{\rho} + \rho)_x = m_{xx}, \\ \rho(0,t) = 1, m(0,t) = 0, \\ \rho(x,0) = 1 + \epsilon e^{-\alpha x}, m(x,0) = \epsilon e^{-\alpha x} \end{cases} \tag{5.1}$$

where ρ and m stands for density and momentum respectively, $x, t \geq 0$, ϵ and α are positive constants, $\epsilon \ll 1$ sufficiently small and $\alpha < 1$. We note that the boundary data is fixed at the reference state $(\rho, m) = (1, 0)$ while the initial data is a small perturbation of the reference state $(\rho, m) = (1, 0)$. Therefore, its linearization at the reference state $(\rho, m) = (1, 0)$ is (1.2). Let $\hat{\rho} = \rho - 1$, $\hat{m} = m$, then (5.1) becomes

$$\begin{cases} \hat{\rho}_t + \hat{m}_x = 0, \\ \hat{m}_t + (\frac{\hat{m}^2}{1+\hat{\rho}} + \hat{\rho})_x = \hat{m}_{xx}, \\ \hat{\rho}(0,t) = \hat{m}(0,t) = 0, \\ \hat{\rho}(x,0) = \epsilon e^{-\alpha x}, \hat{m}(x,0) = \epsilon e^{-\alpha x} \end{cases} \tag{5.2}$$

Let $U = \begin{pmatrix} \hat{\rho} \\ \hat{m} \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 0 \\ (\frac{\hat{m}^2}{1+\hat{\rho}})_x \end{pmatrix}$, we have the matrix form of (5.2) as follows:

$$\begin{cases} \partial_t U + A\partial_x U - B\partial_x^2 U = -N, \\ U(0,t) = 0, \\ U(x,0) = U_0(x) = \epsilon e^{-\alpha x}. \end{cases} \tag{5.3}$$

With the help of the Green's function derived in the previous Chapter, we are able to obtain the point-wise estimate and the asymptotic behavior for the solution U. Before we write out the explicit representation of the solution U with respect to the Green's function, initial data and the nonlinear term, we first need to introduce the backward Green's function.

5.1 Green's function: backward and forward, and their equivalence

We first recall the definition of the (forward) Green's function. The Green's function $G(x, y, t)$ for the initial-boundary value problem to the system (1.3) is a 2×2 matrix valued function which satisfies

$$\begin{cases} \partial_t G(x,y,t) + A\partial_x G(x,y,t) = B\partial_x^2 G(x,y,t) \text{ for } x > 0, t > 0, \\ G(0,y,t) = 0, \\ G(x,y,0) = \delta(x-y)I. \end{cases} \tag{5.4}$$

Correspondingly, we introduce the backward Green's function G^B as follows:

Definition 5.1.1. *The* ***backward Green's function*** $G^B(x, y, t-\tau)$ *for the initial-boundary value problem to the system* (1.3) *is a* 2×2 *matrix valued function which satisfies the* ***backward equation****:*

$$\begin{cases} -\partial_\tau G^B(x,y,t-\tau) - \partial_y G^B(x,y,t-\tau)A - \partial_y^2 G^B(x,y,t-\tau)B = 0 \text{ for } \tau \in (0,t), \\ G^B(x,0,t) = 0, \\ G^B(x,y,0) = \delta(x-y)I. \end{cases} \tag{5.5}$$

We will then show the equivalence of the forward Green's function and the backward Green's function.

Lemma 5.1.2. *The backward Green's function and the forward Green's function are equal.*

Proof For the first equation of (5.4), we left multiply it with the backward Green's function $G^B(x, y, t-\tau)$ and integrate over $(0,t)\times(0,\infty)$, we have

$$\int_0^t \int_0^\infty G^B(x,z,t-\tau)(\partial_\tau G(z,y,\tau) + A\partial_z G(z,y,\tau) - B\partial_z^2 G(z,y,\tau))dzd\tau = 0. \tag{5.6}$$

For (5.6), we calculate the integration term by term. For the first term,

$$\begin{aligned} &\int_0^t \int_0^\infty G^B(x,z,t-\tau)\partial_\tau G(z,y,\tau)dzd\tau \\ =& \int_0^\infty G^B(x,z,t-\tau)G(z,y,\tau)dz|_{\tau=0}^{\tau=t} - \int_0^t \int_0^\infty \partial_\tau G^B(x,z,t-\tau)G(z,y,\tau)dzd\tau \\ =& \int_0^\infty \delta(x-z)G(z,y,t)dz - \int_{-\infty}^\infty G^B(x,z,t)\delta(z-y)dz \\ &- \int_0^t \int_0^\infty \partial_\tau G^B(x,z,t-\tau)G(z,y,\tau)dzd\tau \\ =& G(x,y,t) - G^B(x,y,t) - \int_0^t \int_0^\infty \partial_\tau G^B(x,z,t-\tau)G(z,y,\tau)dzd\tau. \end{aligned}$$

For the second term,

$$
\begin{aligned}
&\int_0^t \int_0^\infty G^B(x,z,t-\tau)A\partial_z G(z,y,\tau)dzd\tau \\
&= \int_0^t G^B(x,z,t-\tau)AG(z,y,\tau)d\tau|_{z=0}^{z=\infty} - \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)AG(z,y,\tau)dzd\tau \\
&= -\int_0^t G^B(x,0,t-\tau)AG(0,y,\tau)d\tau - \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)AG(z,y,\tau)dzd\tau \\
&= -\int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)AG(z,y,\tau)dzd\tau,
\end{aligned}
$$

where we make use of the boundary data $G^B(x,0,t-\tau) = G(0,y,\tau) = 0$. For the third term,

$$
\begin{aligned}
&-\int_0^t \int_0^\infty G^B(x,z,t-\tau)B\partial_z^2 G(z,y,\tau)dzd\tau \\
&= -\int_0^t G^B(x,z,t-\tau)B\partial_z G(z,y,\tau)d\tau|_{z=0}^{z=\infty} \\
&\quad + \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)B\partial_z G(z,y,\tau)dzd\tau \\
&= \int_0^t G^B(x,0,t-\tau)B\partial_z G(0,y,\tau)d\tau + \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)B\partial_z G(z,y,\tau)dzd\tau \\
&= \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)B\partial_z G(z,y,\tau)dzd\tau \\
&= \int_0^t \partial_z G^B(x,z,t-\tau)BG(z,y,\tau)d\tau|_{z=0}^{z=\infty} - \int_0^t \int_0^\infty \partial_z^2 G^B(x,z,t-\tau)BG(z,y,\tau)dzd\tau \\
&= -\int_0^t \partial_z G^B(x,0,t-\tau)BG(0,y,\tau)d\tau - \int_0^t \int_0^\infty \partial_z^2 G^B(x,z,t-\tau)BG(z,y,\tau)dzd\tau \\
&= -\int_0^t \int_0^\infty \partial_z^2 G^B(x,z,t-\tau)BG(z,y,\tau)dzd\tau.
\end{aligned}
$$

In the above calculation, we make use of the boundary data of the backward Green's function $G^B(x,0,t-\tau) = 0$ in the third equal and the boundary data of the forward Green's function $G(0,y,\tau) = 0$ in the sixth equal. By the definition of the backward Green's function, we have

$$
-\partial_\tau G^B(x,z,t-\tau) - \partial_z G^B(x,z,t-\tau)A - \partial_z^2 G^B(x,z,t-\tau)B = 0.
$$

Therefore,

$$-\int_0^t \int_0^\infty \partial_\tau G^B(x,z,t-\tau)G(z,y,\tau)dzd\tau - \int_0^t \int_0^\infty \partial_z G^B(x,z,t-\tau)AG(z,y,\tau)dzd\tau$$
$$-\int_0^t \int_0^\infty \partial_z^2 G^B(x,z,t-\tau)BG(z,y,\tau)dzd\tau = 0.$$

Hence, (5.6) reduced to

$$G(x,y,t) - G^B(x,y,t) = 0. \tag{5.7}$$

So we have that the forward Green's function and the backward Green's function are equal.

Since the forward and backward Green's function are equivalent, in the following we will denote them uniquely as $G(x,y,t)$.

5.2 Duhamel's Principle: The representation of the solution

In order to study the original nonlinear problem, we first transfer the original differential equation to an integral equation. The solution is then represented by an integration in terms of the Green's function, initial data and the nonlinear term. This process is so called the Duhamel's Principle. We will show the derivation of the representation as follows.

The solution $U(x,t)$ of the nonlinear initial-boundary value problem (5.3) satisfies

$$\partial_t U + A\partial_x U - B\partial_x^2 U = -N. \tag{5.8}$$

Left multiply the above equation with the Green's function $G(x, y, t)$ and integrate over $(0, t) \times (0, \infty)$,

$$\int_0^t \int_0^\infty G(x, y, t-\tau)(\partial_\tau U(y, \tau) + A\partial_y U(y, \tau) - B\partial_z^2 U(y, \tau) + N)dyd\tau = 0. \quad (5.9)$$

We do integration by parts for (5.9) term by term. For the first term

$$\begin{aligned}
&\int_0^t \int_0^\infty G(x, y, t-\tau)\partial_\tau U(y, \tau)dyd\tau \\
&= \int_0^\infty G(x, y, t-\tau)U(y, \tau)dy|_{t=0}^{t=\tau} - \int_0^t \int_0^\infty \partial_\tau G(x, y, t-\tau)U(y, \tau)dyd\tau \\
&= \int_0^\infty \delta(x-y)U(y, t)dy - \int_0^\infty G(x, y, t)U(y, 0)dy \\
&\quad - \int_0^t \int_0^\infty \partial_\tau G(x, y, t-\tau)U(y, \tau)dyd\tau \\
&= U(x, t) - \int_0^\infty G(x, y, t)U_0(y)dy - \int_0^t \int_0^\infty \partial_\tau G(x, y, t-\tau)U(y, \tau)dyd\tau.
\end{aligned}$$

For the second term

$$\begin{aligned}
&\int_0^t \int_0^\infty G(x, y, t-\tau)A\partial_y U(y, \tau)dyd\tau \\
&= \int_0^t G(x, y, t-\tau)AU(y, \tau)d\tau|_{y=0}^{y=\infty} - \int_0^t \int_0^\infty \partial_y G(x, y, t-\tau)AU(y, \tau)dyd\tau \\
&= \int_0^t G(x, 0, t-\tau)AU(0, \tau)d\tau - \int_0^t \int_0^\infty \partial_y G(x, y, t-\tau)AU(y, \tau)dyd\tau \\
&= -\int_0^t \int_0^\infty \partial_y G(x, y, t-\tau)AU(y, \tau)dyd\tau.
\end{aligned}$$

In the above estimate, we make use of the boundary condition $U(0, \tau) = 0$. For

the third term

$$
\begin{aligned}
&-\int_0^t\int_0^\infty G(x,y,t-\tau)B\partial_y^2U(y,\tau)dyd\tau\\
&=-\int_0^t G(x,y,t-\tau)B\partial_yU(y,\tau)d\tau|_{y=0}^{y=\infty}+\int_0^t\int_0^\infty \partial_yG(x,y,t-\tau)B\partial_yU(y,\tau)dyd\tau\\
&=\int_0^t G(x,0,t-\tau)B\partial_yU(0,\tau)d\tau+\int_0^t\int_0^\infty \partial_yG(x,y,t-\tau)B\partial_yU(y,\tau)dyd\tau\\
&=\int_0^t\int_0^\infty \partial_yG(x,y,t-\tau)B\partial_yU(y,\tau)dyd\tau\\
&=\int_0^t \partial_yG(x,y,t-\tau)BU(y,\tau)d\tau|_{y=0}^{y=\infty}-\int_0^t\int_0^\infty \partial_y^2G(x,y,t-\tau)BU(y,\tau)dyd\tau\\
&=-\int_0^t \partial_yG(x,0,t-\tau)BU(0,\tau)d\tau-\int_0^t\int_0^\infty \partial_y^2G(x,y,t-\tau)BU(y,\tau)dyd\tau\\
&=-\int_0^t\int_0^\infty \partial_y^2G(x,y,t-\tau)BU(y,\tau)dyd\tau.
\end{aligned}
$$

In the above estimate, we make use of the boundary condition $G(x,0,t-\tau)$ of the backward Green's function in the third equal and the boundary condition $U(0,\tau)=0$ of the nonlinear equation in the sixth equal. Since $G(x,y,t-\tau)$ satisfies the backward equation (5.5),

$$-\partial_\tau G^B(x,y,t-\tau)-\partial_yG^B(x,y,t-\tau)A-\partial_y^2G^B(x,y,t-\tau)B=0.$$

Hence, we have the following representation for $U(x,t)$

$$U(x,t)=\int_0^\infty G(x,y,t)U_0(y)dy-\int_0^t\int_0^\infty G(x,y,t-\tau)N(y,\tau)dyd\tau. \tag{5.10}$$

This representation is an integral equation because the nonlinear term N is a function on the solution U. We will first estimate the first integration in the next section. Based on this estimate result, we make an ansatz assumption for the solution. We then prove our ansatz assumption by calculating the second double integration.

5.3 Estimate regarding to the initial data

We estimate the integration of the Green's function with the initial data $\int_0^\infty G(x,y,t)U_0(y)dy$ in this section, where $U_0(y) = \epsilon e^{-\alpha y}$. Recall the representation of the Green's function $G(x,y,t)$:

$$
\begin{aligned}
G(x,y,t) = e^{-t}\begin{pmatrix} \delta(x-y) & 0 \\ 0 & 0 \end{pmatrix} &+ j_1(y,t)\delta(x) + j_2(x,t)\delta(-y) \\
&+O(1)\left(\frac{e^{-\frac{|x-y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x-y|+t)/C}\right) \\
&+O(1)\left(\frac{e^{-\frac{|x+y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-|y|/C}\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) \\
&+O(1)e^{-|x|/C}\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}}\right) + O(1)e^{-(|x|+|y|+t)/C} \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8,
\end{aligned}
$$

where j_1 is a matrix of the form $\begin{pmatrix} a_1(y,t) & a_2(y,t) \\ 0 & 0 \end{pmatrix}$ satisfying

$$
|a_1(y,t)|, |a_2(y,t)| = O(1)\left(\frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C}\right),
$$

and j_2 is a matrix of the form $\begin{pmatrix} b_1(x,t) & 0 \\ b_2(x,t) & 0 \end{pmatrix}$ satisfying

$$
|b_1(x,t)|, |b_2(x,t)| = O(1)\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C}\right).
$$

We will estimate for $J_i, i = 1, \cdots, 8$ term by term. We will use some constants

D_i instead of $O(1)$ in the calculation to make the structure more clear. For J_1,

$$\int_0^\infty J_1(x,y,t)U_0(y)dy = \int_0^\infty e^{-t}\begin{pmatrix}\delta(x-y) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}D_1\epsilon e^{-\alpha y}\\ D_2\epsilon e^{-\alpha y}\end{pmatrix}dy = \begin{pmatrix}D_1\epsilon e^{-\alpha x-t}\\ D_1\epsilon e^{-\alpha x-t}\end{pmatrix}.$$

For J_2,

$$\int_0^\infty J_2(x,y,t)U_0(y)dy = \delta(x)\int_0^\infty j_1(y,t)U_0(y)dy.$$

For $x,t>0$, $\int_0^\infty j_1(y,t)U_0(y)dy$ is a function on t and $\delta(x)=0$. Therefore,

$$\int_0^\infty J_2(x,y,t)U_0(y)dy = 0.$$

For J_3,

$$\int_0^\infty J_3(x,y,t)U_0(y)dy = j_2(x,t)\int_0^\infty \delta(-y)U_0(y)dy,$$

where

$$\int_0^\infty \delta(-y)U_0(y)dy = \begin{pmatrix}D_1\epsilon\\ D_2\epsilon\end{pmatrix}.$$

Therefore,

$$\int_0^\infty J_3(x,y,t)U_0(y)dy = \begin{pmatrix}D_1\epsilon b_1(x,t)\\ D_1\epsilon b_2 x,t\end{pmatrix},$$

where the estimate of $b_1(x,t)$ and $b_2(x,t)$ are given in the above representation of $G(x,y,t)$. We note for $x,t>0$, the term $O(1)\frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}$ is dominated by $O(1)\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}}$ and $O(1)e^{-(|x|+t)/C}$. Therefore,

$$\int_0^\infty J_3(x,y,t)U_0(y)dy = D_3\epsilon\left(\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(x+t)/C}\right),$$

where $D_3>0$ is some constant.

For the estimate on J_4 and J_5, we need the following Lemma:

Lemma 5.3.1. *There exists $C_1 > 0$, such that*

$$\int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy \leq O(1) \frac{e^{-\frac{(x-t)^2}{C_1(1+t)}}}{\sqrt{1+t}} + O(1) e^{-\alpha(x+t)/C_1}. \tag{5.11}$$

Proof In order to do the above estimate, we consider 3 cases: $0 < x < t$, $t \leq x < 2t$ and $x \geq 2t$. For $0 < x < t$, we have

$$|x - y - t| > |x - t|.$$

Therefore,

$$\int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy \leq \frac{e^{-\frac{(x-t)^2}{C(1+t)}}}{\sqrt{1+t}} \int_0^\infty e^{-\alpha y} dy = O(1) \frac{e^{-\frac{(x-t)^2}{C(1+t)}}}{\sqrt{1+t}}.$$

For $t \leq x < 2t$, $x - t \geq 0$,

$$\int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy = \int_0^{\frac{x-t}{2}} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy + \int_{\frac{x-t}{2}}^\infty \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy$$

For the first term of above,

$$|x - y - t| > \frac{1}{2}|x - t|,$$

for $y < \frac{x-t}{2}$. Hence,

$$\int_0^{\frac{x-t}{2}} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy \leq \frac{e^{-\frac{(x-t)^2}{4C(1+t)}}}{\sqrt{1+t}} \int_0^\infty e^{-\alpha y} dy = O(1) \frac{e^{-\frac{(x-t)^2}{4C(1+t)}}}{\sqrt{1+t}}.$$

For the second term,

$$\begin{aligned}
&\int_{\frac{x-t}{2}}^{\infty} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy \\
\leq & \frac{1}{\sqrt{1+t}} \int_{\frac{x-t}{2}}^{\infty} e^{-\frac{(x-y-t)^2}{C(1+t)}} \cdot e^{-\frac{\alpha}{2}y} \cdot e^{-\frac{\alpha}{2}y} dy \\
\leq & \frac{1}{\sqrt{1+t}} e^{-\frac{\alpha}{4}(x-t)} \int_{\frac{x-t}{2}}^{\infty} e^{-\frac{(x-y-t)^2}{C(1+t)}} \cdot e^{-\frac{\alpha}{2}y} dy \\
\leq & \frac{e^{-\frac{\alpha}{4}\frac{(x-t)^2}{x-t}}}{\sqrt{1+t}} \int_{\frac{x-t}{2}}^{\infty} e^{-\frac{\alpha}{2}y} dy \\
\leq & O(1) \frac{e^{-\frac{\alpha}{4}\frac{(x-t)^2}{1+t}}}{\sqrt{1+t}}.
\end{aligned}$$

For $x \geq 2t$, we still have

$$\int_0^{\infty} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy = \int_0^{\frac{x-t}{2}} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy + \int_{\frac{x-t}{2}}^{\infty} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy.$$

For the first term, the calculation for the case $t \leq x < 2t$ still works. For the second term, we have

$$\int_{\frac{x-t}{2}}^{\infty} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} \cdot e^{-\alpha y} dy \leq e^{-\frac{\alpha(x-t)}{2}} \int_{\frac{x-t}{2}}^{\infty} \frac{e^{-\frac{(x-y-t)^2}{C(1+t)}}}{\sqrt{1+t}} dy \leq O(1) e^{-\frac{\alpha(x-t)}{2}} \leq O(1) e^{-\frac{\alpha(x+t)}{6}}.$$

Combined the above 3 cases together, we proved this lemma.

The integration for the other heat kernel terms in J_4 and J_5 are similar as shown in Lemma 5.3.1.

For the rest term $e^{-(|x-y|+t)/C}$, we have

$$\int_0^{\infty} e^{-|x-y|/C} \cdot e^{-\alpha y} dy = \int_0^{x} e^{-(x-y)/C} \cdot e^{-\alpha y} dy + \int_x^{\infty} e^{-(y-x)/C} \cdot e^{-\alpha y} dy.$$

Without loss of generality, we let $\alpha > 1/C$. Therefore,

$$\int_0^\infty e^{-|x-y|/C} \cdot e^{-\alpha y} dy \leq x e^{-x/C} + O(1) e^{-\alpha x} \leq O(1) e^{-x/C_1}, \tag{5.12}$$

for some $C_1 > 0$.

By Lemma 5.3.1 and (5.12), we conclude

$$\int_0^\infty (J_4(x,y,t) + J_5(x,y,t)) U_0(y) dy = D_4 \epsilon (\frac{e^{-\frac{|x-t|^2}{C_1(1+t)}}}{\sqrt{1+t}} + e^{-(x+t)/C_1}),$$

where $C_1, D_4 > 0$ are some constants.

For J_6, we have

$$\begin{aligned}\int_0^\infty J_6(x,y,t) U_0(y) dy &= O(1)\epsilon (\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}) \int_0^\infty e^{-y/C - \alpha y} dy \\ &= O(1)\epsilon (\frac{e^{-\frac{|x-t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+t|^2}{C(1+t)}}}{\sqrt{1+t}}),\end{aligned}$$

For J_7, we have

$$\int_0^\infty \frac{e^{-\frac{|y-t|^2}{C(1+t)}}}{\sqrt{1+t}} e^{-\alpha y} dy = O(1) \int_0^\infty \frac{e^{-\frac{|y-(1+\frac{\alpha}{2})t|^2}{C(1+t)}}}{\sqrt{1+t}} e^{-\alpha(1+\frac{\alpha}{4})t/C} dy = O(1) e^{-\alpha t/C}.$$

The remaining part $\int_0^\infty \frac{e^{-\frac{|y+t|^2}{C(1+t)}}}{\sqrt{1+t}} e^{-\alpha y} dy$ is dominated by the above integration. Therefore,

$$\int_0^\infty J_7(x,y,t) U_0(y) dy = O(1) \epsilon e^{-x/C - \alpha t}.$$

The calculation for J_8 is straightforward,

$$\int_0^\infty J_8(x,y,t)U_0(y)dy = O(1)e^{-x/C-t/C}\int_0^\infty e^{-y/C-\alpha y}dy = O(1)\epsilon e^{-x/C-t/C}.$$

Now, we conclude the main Theorem of this section as follows:

Theorem 5.3.2. *The solution to the following initial-boundary value problem*

$$\begin{cases} \partial_t U + A\partial_x U - B\partial_x^2 U = 0, x, t > 0 \\ U(0,t) = 0, \\ U(x,0) = U_0(x) = O(1)\epsilon e^{-\alpha x} \end{cases} \tag{5.13}$$

satisfies

$$|U(x,t)| = D\epsilon\frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}, \tag{5.14}$$

where $D > 0$ is a universal constant.

We finished the estimate of the first integration of (5.10). In the next section, we will introduce a Picard's iteration for the integral equation (5.10). We first make an ansatz assumption based on the above estimate (5.14), then we verify our ansatz assumption by the Picard's iteration.

5.4 Proof of the main result

We introduce the following Picard's iteration to solve the nonlinear problem (5.3):

$$\begin{cases} U_{(0)}(x,t) = \int_0^\infty G(x,y,t)U_0(y)dy, \\ U_{(l)}(x,t) = U_{(0)}(x,t) + \int_0^t \int_0^\infty G(x,y,t-\tau)(-N(U_{(l-1)}))dyd\tau, \text{ for } l \geq 1. \end{cases} \tag{5.15}$$

From last section, we have

$$|U_{(0)}(x,t)| = D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D},$$

where $D > 0$ is a universal constant. We make our ansatz assumption to be

$$|U(x,t)| \leq 2(D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}).$$

That is, for all $l \geq 1$, if

$$|U_{(l-1)}(x,t)| \leq 2(D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}),$$

we are going to verify from the second equation of (5.15) that

$$|U_{(l)}(x,t)| \leq 2(D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}).$$

This is equivalent to show that

$$|\int_0^t \int_0^\infty G(x,y,t-\tau)N(U_{(l-1)})dyd\tau| \leq D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}. \tag{5.16}$$

For the Green's function $G(x,y,t-\tau)$, we still apply the separation as in the

last section that $G(x, y, t-\tau) = \sum J_i, i = 1, \cdots, 8$. We calculate the double integration for J_i term by term. For notation simplicity, we write $N = \begin{pmatrix} 0 \\ N^* \end{pmatrix}$, where $N^* = (\frac{\hat{m}^2}{1+\hat{\rho}})_x$.

For J_1,

$$\int_0^t \int_0^\infty J_1(x, y, t-\tau) N dy d\tau = \int_0^t \int_0^\infty e^{-(t-\tau)} \begin{pmatrix} \delta(x-y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ N^* \end{pmatrix} dy d\tau = 0.$$

For J_2,

$$\int_0^t \int_0^\infty J_2(x, y, t-\tau) N dy d\tau = \delta(x) \int_0^t \int_0^\infty j_1(y, t-\tau) N dy d\tau.$$

For $x, t > 0$, $\int_0^t \int_0^\infty j_1(y, t-\tau) N dy d\tau$ is a function on t and $\delta(x) = 0$. Therefore,

$$\int_0^t \int_0^\infty J_2(x, y, t-\tau) N dy d\tau = 0.$$

For J_3,

$$\int_0^t \int_0^\infty J_3(x, y, t-\tau) N dy d\tau = \int_0^t \int_0^\infty \delta(-y) \begin{pmatrix} b_1(x,t) & 0 \\ b_2(x,t) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ N^* \end{pmatrix} dy d\tau = 0.$$

For the remaining terms, since there is no singularity in them, we apply an

integration by parts before we do the estimate, which is as follows:

$$\begin{aligned}
&\int_0^t\int_0^\infty J_i(x,y,t-\tau)N^*dyd\tau = \int_0^t\int_0^\infty J_i(x,y,t-\tau)(\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)})_y dyd\tau\\
&= \int_0^t J_i(x,y,t-\tau)\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}d\tau|_{y=0}^{y=\infty} - \int_0^t\int_0^\infty (J_i)_y(x,y,t-\tau)\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}dyd\tau\\
&= -\int_0^t J_i(x,0,t-\tau)\frac{\hat{m}^2(0,\tau)}{1+\hat{\rho}(0,\tau)}d\tau - \int_0^t\int_0^\infty (J_i)_y(x,y,t-\tau)\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}dyd\tau\\
&= -\int_0^t\int_0^\infty (J_i)_y(x,y,t-\tau)\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}dyd\tau.
\end{aligned}$$

We make use of the boundary condition $\hat{m}(0,\tau)=0$ in the above integration.

For $(J_i)_y$, similar as Lemma 4.1.2, we have

$$|\frac{\partial}{\partial y}(\frac{e^{-\frac{(y-\lambda t)^2}{C(1+t)}}}{\sqrt{1+t}})| \le O(1)\frac{1}{\sqrt{1+t}}(\frac{e^{-\frac{(y-\lambda t)^2}{C_1(1+t)}}}{\sqrt{1+t}}),$$

where $C_1 > C$ is a constant and can be taken slightly larger than C, that is, we can choose $C_1 = \frac{5}{4}C$. And similar as in Lemma 4.1.3, we have

$$|\frac{\partial}{\partial y}e^{-(y+t)/C}| \le O(1)e^{-(y+t)/C}.$$

For $\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}$, we have

$$\begin{aligned}
|\frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)}| \le \frac{(D\epsilon\frac{e^{-\frac{(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} + D\epsilon e^{-\alpha(y+\tau)/D})^2}{1+D\epsilon} &\le \frac{3}{2}(D\epsilon\frac{e^{-\frac{(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} + D\epsilon e^{-\alpha(y+\tau)/D})^2\\
&\le 3D^2\epsilon^2\frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{1+\tau} + 3D^2\epsilon^2 e^{-2\alpha(y+\tau)/D},
\end{aligned}$$

for ϵ sufficiently small.

For the universal constant C in the Green's function, we can choose $D > 2C$ in our ansatz assumption. For the estimate on J_4 and J_5, we need the following Lemmas:

Lemma 5.4.1. *There exists $C_1 > 0$, such that*

$$\int_0^t \int_0^\infty \frac{e^{-\frac{(x-y-(t-\tau))^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot e^{-2\alpha(y+\tau)/D} dy d\tau \leq C_1 (\frac{e^{-\frac{(x-t)^2}{D(1+t)}}}{\sqrt{1+t}} + e^{-\alpha(x+t)/D}). \tag{5.17}$$

Proof For $D > 2C$, we have

$$\begin{aligned} &\int_0^t \int_0^\infty \frac{e^{-\frac{(x-y-(t-\tau))^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot e^{-2\alpha(y+\tau)/D} dy d\tau \\ \leq\ & O(1) \int_0^t \int_0^\infty \frac{e^{-\frac{2(x-y-(t-\tau))^2}{D(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot e^{-2\alpha(y+\tau)/D} dy d\tau. \end{aligned}$$

The proof is then similar with the proof of Lemma 5.3.1.

Lemma 5.4.2. *There exists $C_2 > 0$, such that*

$$\int_0^t \int_0^\infty \frac{e^{-\frac{(x-y-(t-\tau))^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} dy d\tau \leq C_2 \frac{e^{-\frac{(x-t)^2}{D(1+t)}}}{\sqrt{1+t}}. \tag{5.18}$$

Proof For$D > 2C$, we have

$$\int_0^t \int_0^\infty \frac{e^{-\frac{(x-y-(t-\tau))^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} dy d\tau \leq O(1) \int_0^t \int_0^\infty \frac{e^{-\frac{2(x-y-(t-\tau))^2}{D(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} dy d\tau. \tag{5.19}$$

We make use of the inequality:

$$\frac{(x-y-(t-s))^2}{C(1+t-s)} + \frac{(y-s)^2}{C(1+s)} \leq \frac{(x-t)^2}{C(1+t)}.$$

$$
\begin{aligned}
&\int_0^t \int_0^\infty \frac{e^{-\frac{2(x-y-(t-\tau))^2}{D(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} dy d\tau \\
&\leq O(1) e^{-\frac{(x-t)^2}{D(1+t)}} (\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t) \int_0^\infty \frac{e^{-\frac{2(x-y-(t-\tau))^2}{D(1+t-\tau)}}}{\sqrt{1+t-\tau}} \cdot \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}} dy d\tau \\
&\leq O(1) e^{-\frac{(x-t)^2}{D(1+t)}} (\int_0^{\frac{t}{2}} \int_0^\infty \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{\sqrt{1+\tau}\sqrt{1+t}} dy d\tau + \int_{\frac{t}{2}}^t \int_0^\infty \frac{e^{-\frac{2(x-y-(t-\tau))^2}{D(1+t-\tau)}}}{\sqrt{1+t-\tau}\sqrt{1+t}} dy d\tau) \\
&\leq O(1) \frac{e^{-\frac{(x-t)^2}{D(1+t)}}}{\sqrt{1+t}}.
\end{aligned}
$$

The integration for the other heat kernel terms in J_4 and J_5 are dominated by the integrations in Lemma 5.4.1 and 5.4.2.

For the rest term $e^{-(|x-y|+t)/C}$, we have

$$
\int_0^t \int_0^\infty e^{-(|x-y|+t-\tau)/C} \cdot e^{-2\alpha(y+\tau)/D} dy d\tau \leq O(1) e^{-\alpha(x+t)/D},
$$

for $D > 2C$.

Hence, we conclude

$$
|\int_0^t \int_0^\infty (J_4(x,y,t) + J_5(x,y,t)) N^* dy d\tau| \leq D_1 \epsilon^2 (\frac{e^{-\frac{(x-t)^2}{D(1+t)}}}{\sqrt{1+t}} + e^{-\alpha(x+t)/D}),
$$

where $D_1 > 0$ is some constant.

For J_6, we have

$$
\begin{aligned}
&|\int_0^t \int_0^\infty (J_6)_y(x,y,t) \frac{\hat{m}^2(y,\tau)}{1+\hat{\rho}(y,\tau)} dy d\tau| \\
&\leq O(1)\epsilon^2 (\frac{e^{-\frac{(x-t)^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{(x+t)^2}{C(1+t)}}}{\sqrt{1+t}}) \int_0^t \int_0^\infty e^{-y/C} \cdot (3D^2\epsilon^2 \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{1+\tau} + 3D^2\epsilon^2 e^{-2\alpha(y+\tau)/D}) dy d\tau \\
&\leq O(1)\epsilon^2 (\frac{e^{-\frac{(x-t)^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{(x+t)^2}{C(1+t)}}}{\sqrt{1+t}}) \\
&\leq D_2 \epsilon^2 \frac{e^{-\frac{(x-t)^2}{D(1+t)}}}{\sqrt{1+t}},
\end{aligned}
$$

where $D_2 > 0$ is a constant and we choose $D > 2C > C$ as in the above case.

For J_7, we make use of

$$\begin{aligned}
&\int_0^t \int_0^\infty \frac{e^{-\frac{|y-(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} e^{-2\alpha(y+\tau)/D} dy d\tau \\
&\leq O(1) \int_0^t \int_0^\infty \frac{e^{-\frac{2|y-(1+\frac{\alpha}{2})(t-\tau)|^2}{D(1+t-\tau)}}}{1+t-\tau} e^{-2\alpha(1+\frac{\alpha}{4})(t-\tau)} e^{-2\alpha\tau/D} dy d\tau \\
&\leq O(1) \int_0^t e^{-2\alpha(1+\frac{\alpha}{4})(t-\tau)} e^{-2\alpha\tau/D} d\tau \\
&\leq O(1) e^{-\alpha t/D},
\end{aligned}$$

for $D > 2C$. The integration for the remaining part $\frac{e^{-\frac{|y+(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}$ is dominated by the above integration. Therefore,

$$|\int_0^t \int_0^\infty J_7(x,y,t) N^* dy d\tau| \leq D_3 \epsilon^2 e^{-\alpha(x+t)/D},$$

where $D_3 > 0$ is a constant.

For J_8, we have

$$\begin{aligned}
&|\int_0^t \int_0^\infty J_8(x,y,t) N^* dy d\tau| \\
&\leq O(1) e^{-x/C} \int_0^t \int_0^\infty e^{-(t-\tau)/C} (3D^2\epsilon^2 \frac{e^{-\frac{2(y-\tau)^2}{D(1+\tau)}}}{1+\tau} + 3D^2\epsilon^2 e^{-2\alpha(y+\tau)/D}) dy d\tau \\
&\leq D_4 \epsilon^2 e^{-\alpha(x+t)/D},
\end{aligned}$$

where $D_4 > 0$ is a constant.

Conclude the above calculations, we have

$$|\int_0^t \int_0^\infty G(x,y,t-\tau) N dy d\tau| \leq (D_1 + D_2 + D_3 + D_4)\epsilon^2 (\frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + e^{-\alpha(x+t)/D}).$$

Choose our ϵ sufficiently small such that $(D_1 + D_2 + D_3 + D_4)\epsilon^2 \leq D\epsilon$, then we

verified our ansatz assumption. We conclude the whole section as the following Theorem.

Theorem 5.4.3. *The solution to the initial-boundary value problem* (5.3) *satisfies*

$$|U(x,t)| \leq D\epsilon \frac{e^{-\frac{|x-t|^2}{D(1+t)}}}{\sqrt{1+t}} + D\epsilon e^{-\alpha(x+t)/D}, \tag{5.20}$$

where $D > 0$ is a universal constant.

From this Theorem, we conclude that the perturbation $U(x,t)$ tends to zero when the time t tends to infinity. Since our estimate is point-wise, we also obtained the decay rate. For the wave component along the characteristic curve $x = t$, the decay rate is $t^{-\frac{1}{2}}$. Other wave components are exponentially decaying with respect to t.

Bibliography

[1] Chern, I. L.; Liu, T. P. Convergence to diffusion waves of solutions for viscous conservation laws. *Comm. Math. Phys* **110**(1987), 503-517

[2] Chern, I. L.; Liu, T. P. Convergence to diffusion waves of solutions for viscous conservation laws, Erratum. *Comm. Math. Phys* **120**(1989), 525-527

[3] Constantin, P.; Foias, C. *Navier-Stokes Equations.* Chicago Lectures in Mathematics, The University of Chicago Press, 1988

[4] Courant, R.; Friedriches, K. O. *Supersonic Flow and Shock Waves.* New York: Interscience Publishers, 1948

[5] Danchin, R. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.* **141**(2000), 579-614

[6] Ellis, R.; Pinsky, M. The first and second fluid approximations to the linearized Boltzmann equation. *CJ. Math. Pures Appl.* **54**(1975), no.9, 125-156

[7] Feireisl, E.; Novotny, A.; Petzelotova, H. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.* **3**(2001), 358-392

[8] Goodman, J.;Xin, Z. Viscous limits for piecewise smooth solutions to systems of conservation laws *Arch. Rati. Mech. Anal.* **121**(1994), 235-265

[9] Hoff, D. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations* **120**(1995), 215-254

[10] Hoff, D. Strong convergenc to global solutions for multidimensional compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.* **132**(1995), 1-14

[11] Hoff, D.; Santos, M. M. Lagrangean structure and propagation of singularities in multidimensional compressible flow. *Arch. Rational Mech. Anal.* **188**(2008), 509-543

[12] Kawashima, S.; Matsumura, A. Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. *Comm. Math. Phys.* **101**(1985), 97-127

[13] Ladyzhensakaya, O. A. *The Mathematical Theory of Viscous Imcompressible Flow,* Second Edition. New York: Gordon and Breach, 1969

[14] Lan, C. Y.; Lin. H. E.;Yu, S. H. The Green's functions for the Broadwell model in a half space problem. *Netw. Heterog. Media* **1**(2006), no.1, 167-183

[15] Lions, P. L. *Mathematical topics in fluid mechanics.* vol.2. New York: Oxford University Press, 1998

[16] Liu, T. P. Nonlinear stability of shock waves for viscous conservation laws. *Mem. Amer. Math. Soc.* **56**(1985), no.328, v+108 pp

[17] Liu, T. P. Interactions of nonlinear hyperbolic waves, in: Nonlinear Analysis, F. C. Liu and T. P. Liu, eds. *World Scientific* (1991), 171-184

[18] Liu, T. P. Pointwise convergence to shock waves for viscous conservation laws. *Comm. Pure Appl. Math.* **50**(1997), no.11, 1113-1182

[19] Liu, T. P.; Xin, Z. P.; Yang, T. Vacuum states of compressible flow. *Discrete Contin. Dynam. Systems* **4**(1998), 1-32

[20] Liu, T. P.; Yu, S. H. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. *Comm. Math. Phys* **246**(2004), no.1, 133-179

[21] Liu, T. P.; Yu, S. H. The Green's function and large-time behavior of solutions for the one-dimensional Boltzmann equation. *Comm. Pure Appl. Math.* **57**(2004), no.12, 1543-1608

[22] Liu, T. P.; Yu, S. H. Initial-boundary value problem for one-dimensional wave solutions of the Boltzmann equation. *Comm. Pure Appl. Math.* **60**(2007), no.3, 295-356

[23] Liu, T. P.; Yu, S. H. Dirichlet-Neumann kernel for dissipative system in half-space. (to appear)

[24] Liu, T. P.; Zeng, Y. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. *Mem. Amer. Math. Soc.* **125**(1997), no.599, viii+120 pp

[25] Liu, T. P.; Zeng, Y. On Green's function for hyperbolic-parabolic systems . *Acta Math. Sci. Ser. B Engl. Ed.* **29**(2009), no.6, 1556-1572

[26] Matsumura, A.; Nishida, T. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acad. Ser. A Math. Sci.* **55**(1979), 337-342

[27] Matsumura, A.; Nishida, T. The initial value problem for the equations of motion of viscous and heat-conductive fluids. *J. Math. Kyoto Univ.* **20**(1980), 67-104

[28] Matsumura, A.; Nishida, T. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.* **89**(1983), 445-464

[29] Teman, R. *Navier-Stokes Equations, Theory and Numeriacl Anaylsis.* North-Holland, 1984

[30] Zeng, Y. L^1 Asymptotic behavior of compressible, isentropic, viscous 1-D flow. *Comm. Pure Appl. Math.* **47**(1994), no. 8, 1053-1082