THREE ESSAYS ON EPISTEMIC GAME THEORY

WANG, BEN

(B.Sc.(Hons.), National University of Singapore)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ECONOMICS

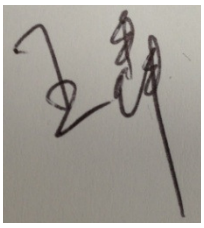
NATIONAL UNIVERSITY OF SINGAPORE

2013

Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.



Wang, Ben

21 August 2013

To my parents,

Jianjun Wang and Huirong Luo,

and my wife,

Yan Wang.

Acknowledgement

Many people played important roles in the past four years. Without their love and help, this thesis would not have been possible. On this occasion, I would like to express my gratitude toward them.

I am particularly indebted to my supervisor Prof. Xiao Luo for his support and help in the past few years. He showed great kindness and patience to me, and guided me through each step of research. My work has benefitted enormously from his comments and critique. Moreover, he is a great life mentor, and always gives invaluable suggestions on academic and non-academic matters.

Many thanks go to Prof. Yi-Chun Chen and Prof. Satoru Takahashi for their constructive comments and suggestions. Presentations and discussions in the microeconomic theory workshop they organized inspired my research, and I learnt a lot from the workshop participants, especially from Dr. Bin Miao, Dr. Xiang Sun, Dr. Yongchao Zhang, Wei He, Yunfeng Lu, Lai Yoke Mun, Xuewen Qian, Lei Qiao, Chen Qu, Yifei Sun and Guangpu Yang.

I would like to take this opportunity to thank my colleagues in Department of Economics, especially Dr. Qian Jiao, Jingping Li, Yunong Li, Ling Long, Neng Qian, Thanh Hai Vu, Peng Wang, Huihua Xie and Yingke Zhou for discussions we had and for the good time we spent together.

I am also grateful to my friends for their continuous support, especially Zhengning Liu, Wei Sun, Xuan Wang, Hui Xiao and Honghai Yu. They held me up when I was down and set the path straight for me in difficult time.

Last but not least, my deepest gratitude and thanks go to my parents, my wife and other family members. Their love and trust made me keep moving forward fearlessly.

Wang, Ben
August 21, 2013
Singapore

Contents

Summary

Epistemic game theory provides a formal language to analyze players' strategic choices, rationality, beliefs, *etc*., which enables us to formally explore the hidden assumptions behind solution concepts in the classical game theory. In this thesis, we mainly focus on epistemic conditions of three game-theoretic solution concepts, namely "mutually acceptable courses of action (MACA)" (Greenberg *et al*. (2009)), "rationalizable self-confirming equilibrium (RSCE)" (Dekel et al. (1999)), and "backward induction outcome."

(i) MACA is a unified solution concept for complex social situations where "perfectly" rational individuals with different beliefs and views of the world agree to a shared course of actions. We formulate and show, by using the notion of "lexicographic probability system (LPS)" (Blume *et al*. (1991)), that MACA is the logical consequence of common knowledge of "perfect" rationality and mutual knowledge of agreement on the underlying course of actions. (Subjective) perfect equilibrium (Selten (1975). *IJGT*), rationalizable self-confirming equilibrium (Dekel *et al*. (2002). *JET*), and (perfect version) rationalizability (Bernheim (1984), Pearce (1984). *ECTA*) are analyzed in the current epistemic approach by varying the degree of completeness of the underlying course of actions.

(ii) RSCE is a steady state where rational individuals observe the played actions and use the information about opponents' payoffs in forming the beliefs about opponents' behavior off the equilibrium path. We formulate and show, by using the notion of "conditional probability system (CPS)", that RSCE is the result of common knowledge of "sequential" rationality and mutual knowledge of the actions along the path of play. Self-confirming equilibrium (SCE) (Fudenberg and Levine (1993, *ECTA*), sequential rationalizable self-confirming equilibrium (SRSCE), and sequential rationalizability (Dekel *et al*. (2002, *JET*) are analyzed in the current epistemic framework by varying the degree of "rationality."

(iii) We suggest that conditional probability system (CPS) with the strong independence property is useful to model players' conjecture in dynamic games, and define a notion of "consistent belief" to formalize these conjectures. Subgame perfect equilibrium is shown to be the logical consequence of rationality and common consistent belief of rationality (RCCBR) in perfect information generic games.

1 Introduction

Game theory is a study of strategic thinking which provides a formal language to analyze decision makers' behavior in different interactive situations. Various solution concepts (*e.g.* iterative elimination of strictly dominated strategies, Nash equilibrium, backward induction, *etc*.) are innovated by game theorists. These concepts are mainly motivated by economic intuition. Epistemic game theory formalizes assumption about decision makers' rationality, belief and knowledge in a formal and rigorous way which allows game theorists to explore hidden assumptions behind solution concepts. This helps us have better understanding of those assumptions' behavior implications in different games. For instance, rationalizability (Bernheim (1984), Pearce (1984)) is the logical consequence of common knowledge of rationality (Tan and Werlang (1988)).

In this thesis, epistemic conditions of three game-theoretic solution concepts, namely "mutually acceptable courses of action (MACA)" (Greenberg *et al.* (2009)), "rationalizable self-confirming equilibrium (RSCE)" (Dekel *et al.* (1999)), and "backward induction outcome," will be investigated. All of these solution concepts are mainly defined for extensive games. To analyze epistemic conditions of them, one common challenge is to model players' rationality and knowledge of players' rationality in extensive games. Two non-standard probability theories are used in the analysis which will be introduced in following sections.

1.1 An Epistemic Approach to MACA

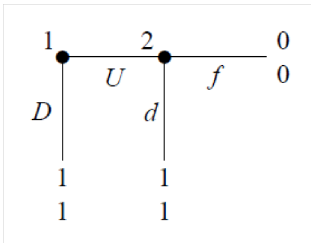
In chapter one, an epistemic approach to the notion of "mutually acceptable courses of action (MACA)" is provided. In complex social interactions, Greenberg *et al.* (Economic Theory 40 (2009) 91-112) offered a unified solution concept of "MACA" for situations where "perfectly" rational[1] individuals with different beliefs and views of the world agree to a shared course of action. In this chapter we investigate epistemic conditions for MACA by employing a non-standard probability theory.

In particular, we use the notion of "lexicographic probability system (LPS)' introduced by Blume *et al.* (Econometrica 59 (1991a) 61-79) to model players' beliefs in dynamic games.

[1]Roughly, a player is *cautious* if he/she thinks that opponents will make mistake (due to trembling hand), and hence assigns a strictly positive probability to opponents' every strategy. A player is *rational* if he/she is a utility-maximizer. A player is *"perfectly" rational* if he/she is both cautious and rational.

Blume *et al.* (1991a) presented a non-Archimedean version of subjective expected utility theory. According to the theory, an agent possesses, not a single probability distribution, but rather a vector of probability distributions that is used lexicographically in selecting an optimal action. Such a vector of probability distributions is called a lexicographic probability system (LPS).

A conditional probability system (CPS) can be viewed as a conditional-probability function which defines a probability distribution on opponents' choices at every information set, including those are not reached. The notion of "CPS" is not suitable for characterizing the epistemic condition of MACA due to the tension between "perfectly" rationality and knowledge of "perfectly" rationality. See the following example.



If player 2 is perfect rational, strategy d would be chosen. If player 1 is perfect rational, both d and f would be assigned positive probability under CPS. If player 1 thinks that player 2 is perfect rational, probability 1 should be assigned to d. under CPS. There is a conflict between the player 1's perfect rationality and player 1's belief about player 2's perfect rationality under CPS. To resolve the tension, strategy f needs to be both included and excluded in player 1's belief. The notion of LPS is designed to handle it. By using LPS, strategy f is assigned probability 0 in primary belief (the first element in the vector of probability distributions) and probability 1 in secondary belief (the second element in the vector of probability distributions).

Within a standard semantic framework, we formulate and show that, by using the notion of LPS, MACA is the logical consequence of common knowledge of "perfect" rationality and mutual knowledge of agreement on the underlying course of action. In this chapter, we also demonstrate how epistemic assumptions for various related game-theoretic solution concepts can be derived by varying the degree of completeness of the underlying course of action. This study is useful to deepen our understanding of MACA and other solution concepts in the literature, such as perfect equilibrium, (perfect) rationalizable self-confirming equilibrium, and (perfect) rationalizability.

It is worthwhile to point out that, by utilizing the notion of LPS, we will present a comprehensive and epistemic analytical framework to accommodate the tension that arises in modeling perfect rationality (that requires to include all possible strategies in a perturbed belief) and knowledge/belief about perfect rationality (that requires to exclude some strategies from the perturbed belief) in complex social interactions; *cf.*, *e.g.*, Samuelson (1992 and 2004) and Brandenburger (2007).

1.2 An Epistemic Characterization of RSCE

In chapter two, an epistemic characterization of "rationalizable self-confirming equilibrium (RSCE)" is given. Dekel *et al.* (J Econ Theory 89 (1999) 165-185) offered a solution concept of "RSCE" as a steady state where rational individuals observe the played actions and use the information about opponents' payoffs in forming the beliefs about opponents' behavior off the equilibrium path. In this chapter we investigate epistemic conditions for RSCE from a decision-theoretic point of view by employing the notion of "conditional probability system (CPS)".

Within a standard semantic framework, we formulate and show that, by using the notion of CPS, RSCE is the logical consequence of common knowledge of rationality and mutual knowledge of the actions along the path of play. We also apply this epistemic framework to other related solution concepts such that self-confirming equilibrium (SCE), sequential rationalizable self-confirming equilibrium (SRSCE), and sequential rationalizability.

1.3 Backward Induction and Consistent Belief

In chapter three, an epistemic analysis of backward induction strategy profile is offered. We suggest that conditional probability system (CPS) with strong independence property is useful to model players' conjecture in dynamic game, and define a notion of "consistent belief" to formalize these conjectures.

A CPS satisfies strong independence property if it can be generated by an independent convergent sequence of "full-support" probability distributions over the state space. Moreover, a player is said to consistently believe an event if he possesses a conditional belief system with strong independence property and believes the event at the beginning of the game.

Within a standard semantic framework, we formulate and show, by using the notion of CPS with strong independent property, backward induction strategy profile is the logical consequence of rationality and common consistent belief of rationality (RCCBR) in perfect information generic games.

2 An Epistemic Approach to MACA

2.1 Introduction

In extensive games, Greenberg *et al.* (2009) presented a unified solution concept of "mutually acceptable course of action (MACA)" which can be interpreted as an (incomplete) contract/agreement that free rational individuals would be willing to follow for their own diverse reasons. As Greenberg *et al.* (2009, p.93) put it,

> "...... a course of action is *mutually acceptable* if no player would wish, in his own world, to deviate from it. When deciding on whether or not to deviate from a course of action, every player takes into account that all players are "rational." In making their decisions, each player analyzes possible consequences of deviations from the proposed course of action. Players would be willing to conform to a proposed course of action as long as their conformity does not conflict with rational behavior. Observe that each player may rationalize his expectations in a different way, as long as this does not violate the common knowledge of rationality as perceived by each player."

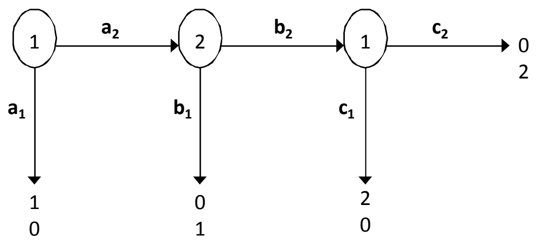
The solution concept of MACA integrates the two main forms of strategic behavior reasonings in the game theory literature: (i) players should hold consistently aligned and correct belief based on behavior specified in a contract/agreement (as in an equilibrium approach) and (ii) players might hold diverse rationalizable beliefs from introspection on the basic epistemic assumption of common knowledge of "perfectly" rationality, if there is no code of rules and behavior dictated by the (incomplete) contract/agreement (as in a non-equilibrium/rationalizability approach).[2] At a conceptual level, Greenberg *et al.* (2009) demonstrated that by varying the degree of completeness of the underlying course of action, the concept of MACA can be related to commonly used solutions, such as perfect equilibrium, rationalizable self-confirming equilibrium, and rationalizability. This approach synthesizes the contractarian and rational-choice paradigms to study extensive-form strategic behavior through the lens of a contract/agreement,

[2]MACA is related to Rubinstein and Wolinsky's (1994) notion of a "rationalizable conjectural equilibrium (RCE)" in normal-form games where players' information about opponents' play is represented by general "signal-functions." An RCE is defined as a strategy profile such that each player's chosen action maximizes his payoffs given his conjecture regarding actions of the others, and the conjectures are consistent with the player's signal and common knowledge of Bayesian rationality; see also Esponda (2012) for more discussions on the notion of RCE.

with special emphasis on the governance of contractual or informational incompleteness and asymmetries.

The purpose of this chapter is to provide expressible epistemic conditions for the solution concept of MACA. A major technical difficulty encountered in dynamic extensive-form game models is, when facing with strategic uncertainty, how to model a player's beliefs about opponents' play in every contingency, including information sets that the player thinks will not actually arise. Inspired by Selten's (1975) brilliant idea of "trembles," Greenberg *et al.* (2009, pp.95-98) offered one way to overcome this difficulty by elaborating on a player's (uncorrelated) perturbed beliefs about the behavioral strategies of opponents in extensive games; see also Dekel *et al.* (2002). In this chapter, we use the notion of "lexicographic probability system (LPS)" introduced by Blume *et al.* (1991a) to model players' beliefs and provide an epistemic characterization for the solution concept of MACA.[3] More specifically, each player is assumed to have, not a single probability distribution, but rather an "independent" vector of probability distributions, on the product of action spaces in the agent-normal form of an extensive game, that is used lexicographically in selecting an optimal strategy. Such a vector of probability distributions is called an "independent lexicographic probability system (ILPS)." The first component of LPS can be thought of as representing the player's primary theory about how the game will be played, the second component as the player's secondary theory, and so on. Within a standard semantic framework, we formulate and show that MACA is the logical consequence of common knowledge of "perfect" rationality and mutual knowledge of agreement on the underlying course of action.

The following example illustrates how to use LPS in our analysis of "perfect" strategic behavior in extensive games:



[3]Blume et al. (1991b) demonstrated how LPS can be used to provide decision-theoretic foundations for normal-form refinements of Nash equilibrium. In an interesting paper, Halpern (2009) offered an alternative and intriguing approach to sequential equilibrium, perfect equilibrium, and proper equilibrium by using "nonstandard probability;" see also Hammond (1994) and Halpern (2003) for the relationship between LPS and nonstandard probability spaces.

In this game, it is clear that there is a unique backward induction outcome: (a_1, b_1, c_1), which also satisfies the "perfect" rationality that every action chosen by a player is optimal along a trembling sequence. This "perfect" rationality can be represented by lexicographical maximization in Blume *et al.*'s (1991a) lexicographic decision theory as follows: (1a) action a_1 lexicographically maximizes player 1's expected payoff under a (full-support) LPS on $\{b_1, b_2\} \times \{c_1, c_2\}$ – namely $\rho \equiv \left(1\,(b_1, c_1)\,,\ \frac{1}{2}\,(b_1, c_2) + \frac{1}{2}\,(b_2, c_1)\,,\ 1\,(b_2, c_2)\right)$,[4] (1b) action c_1 lexicographically maximizes player 1's expected payoff under a (full-support) LPS on $\{a_1, a_2\} \times \{b_1, b_2\}$ – namely $\rho \equiv \left(1\,(a_1, b_1)\,,\ \frac{1}{2}\,(a_1, b_2) + \frac{1}{2}\,(a_2, b_1)\,,\ 1\,(a_2, b_2)\right)$, and (2) action b_1 lexicographically maximizes player 2's expected payoff under a (full-support) LPS on $\{a_1, a_2\} \times \{c_1, c_2\}$ – namely $\rho \equiv \left(1\,(a_1, c_1)\,,\ \frac{1}{2}\,(a_1, c_2) + \frac{1}{2}\,(a_2, c_1)\,,\ 1\,(a_2, c_2)\right)$. In this context, the profile (a_1, b_1, c_1) can reflect common knowledge/belief of "rationality" where rationality refers to lexicographical maximization and knowledge/belief is consistent with the primary belief determined by the first component of lexicographical probability distributions. (Intuitively, player 1 holds the primary belief that player 2 is "perfectly" rational in the sense of (2) and player 2 holds the primary belief that player 1 is "perfectly" rational in the sense of (1a) and (1b), player 1/player 2 holds the primary belief about that player 2/player 1 holds the primary belief that player 1/player 2 is "perfectly" rational, and so on.)[5]

In this chapter, we carry out the epistemic program in game theory to express formally the assumptions on players' information, knowledge and belief that lie behind the solution concept of MACA (see, *e.g.*, Dekel and Gul (1997), Battigalli and Bonanno (1999), Samuelson (2004), Brandenburger (2007), and Bonanno (2013) for surveys of the literature on epistemic game theory). In a standard semantic framework (or Aumann's model of knowledge), we offer an epistemic characterization for MACA in terms of common knowledge of "perfect" rationality

[4]Here, ρ represents player 1's LPS belief (at his first decision node) about player 2's play and player 1's play (at his second decision node). For instance, player 1 has the secondary theory that (b_1, c_2) and (b_2, c_1) are played with equal probability.

[5]By using LPS, the strategies that get infinitesimal weight can be viewed as being both included (because they do not get zero weight) and excluded (because they get only infinitesimal weight) in players' beliefs. This important feature of LPS is critical in our epistemic analysis of MACA; it is used to resolve the tension between "perfect" rationality (that requires to include all possible strategies in a perturbed belief) and knowledge/belief about "perfect" rationality (that requires to exclude some strategies from the belief). Samuelson (1992) firstly pointed out such a logical difficulty in analyzing the notion of "admissibility" in normal-form games within the conventional probability framework; cf. also Samuelson (2004, Sec. 9.1). Brandenburger (1992) and Brandenburger et al. (2008) circumvented the difficulty by using LPS; see also Asheim (2001) for an epistemic analysis of "proper rationalizability" by using LPS. (In this regard, the notion of "conditional probability system (CPS)" is not appropriate for an epistemic analysis in extensive games involving "perfectly" rational players, since there is the same kind of logical predicament in dynamic settings.)

and mutual knowledge of agreement on the underlying course of action (see Theorem 2.3.1 and Corollary 2.3.1). This result also provides a unifying epistemic approach to other related game-theoretic solution concepts such as perfect equilibrium, rationalizability, and rationalizable self-confirming equilibrium. In this chapter, we demonstrate how epistemic characterizations for various related solution concepts can be derived by varying the degree of completeness of the underlying course of action (see Propositions 2.3.1.1, 2.3.2.1 and 2.3.3.1). In the spirit of Aumann and Brandenburger's (1995) Theorems A and B, we also provide expressible epistemic assumptions for a (mixed) complete MACA when mixed strategies are interpreted as conjectures of players (see Proposition 2.3.1.2).

In an interesting paper, Asheim and Perea (2005) provided, in two-player extensive games, a unifying epistemic model for studying different "equilibrium" and "non-equilibrium" solution concepts including "sequential equilibrium/rationalizability" and "quasi-perfect equilibrium/rationalizability (where each player takes into account the possibility of the other players' mistakes, but ignores the possibility of his own mistakes)." In particular, by utilizing a more general concept of "conditional LPS" to represent a system of conditional beliefs in dynamic settings, Asheim and Perea showed that the concept of "sequential rationalizability" can be characterized by common certain belief of "sequential" rationality, and the concept of "quasi-perfect rationalizability" is the result of common certain belief of "sequential" and "cautious" rationality.[6]

Our work distinguished from Asheim and Perea (2005) in two aspects. Firstly, their work focused on two-person game which avoided the independence issue. N-person game is allowed in our work. Secondly, quasi-perfectness instead of perfectness was analyzed. In this chapter, we conduct a systematic epistemic analysis of various perfect-versions of solution concepts through MACA, by using a strong form of "perfect" rationality that reflects Selten's (1975) original idea of perfectness. This idea rested on backward induction is central to a game-theoretic analysis of rational strategic behavior in dynamic situations. Accordingly, Selten's (1975) perfectness requires that each player be "perfectly" rational based on the assumption that all the players tremble independently among all actions at each information set (including each of the player's own information sets).

[6]Asheim and Perea (2005) took a different "consistent preferences" approach to an epistemic analysis of game-theoretic solution concepts; see also Asheim (2005) for extensive discussions. In this paper, we adopt the conventional "rational choice" approach in our epistemic study of rational strategic behavior.

The rest of this chapter is organized as follows. Section 2.2 contains some preliminary notation and definitions. Section 2.3 provides an epistemic characterization for MACA and discusses its epistemic relations to other commonly used game-theoretic solution concepts. Section 2.4 offers concluding remarks.

2.2 Notation and Definitions

Since the formal description of an extensive game is by now standard (see, for instance, Kreps and Wilson (1982) and Kuhn (1954)), only the necessary notation is given below. Consider a (finite) extensive-form game with perfect recall:

$$T \equiv (N, V, H, \{A^h\}_{h\in H}, \{u^i\}_{i\in N}),$$

where $N = \{1, 2, \ldots, n\}$ is the (finite) set of players, V is the (finite) set of nodes (or vertices), H is the (finite) set of information sets, A^h is the (finite) set of pure actions available at information set h, and u^i is player i's payoff function defined on terminal nodes.

A mixed action at information set h is a probability distribution on A^h. Denote the set of mixed actions at h by $\triangle A^h$. Denote the collection of player j's information sets by H^j. A *behavioral strategy* of player j is a function, y^j, that assigns some randomization $y^j(h) \in \triangle A^h$ to every $h \in H^j$.

Let $\mathbb{Y}^j$ be the set of player j's behavioral strategies. Denote the set of behavioral strategy profiles by $\mathbb{Y}$, i.e. $\mathbb{Y} = \times_{j\in N}\mathbb{Y}^j$. For $y \in \mathbb{Y}$, we abuse notation and denote by $u^i(y)$ player i's (expected) payoff if strategy profile y is adopted from the root of the game, denote by $y(h)$ the mixed action of y at h, and denote by $y(-h)$ the profile of mixed actions of y at all information sets other than h. Write $y^j_k \rightsquigarrow y^j$ for a "trembling sequence"$\{y^j_k\}_{k=1}^{\infty}$of strictly positive behavioral strategies in $\mathbb{Y}^j$ that converges to y^j.

2.2.1 MACA: A Unifying Solution Concept

A *course of action (CA)* is defined as a mapping $x : H \to \cup_{h\in H} \triangle A^h \cup \{\emptyset\}$, with $x(h) \in \triangle A^h \cup \{\emptyset\}$ for all $h \in H$. A course of action can be interpreted as an (incomplete) contract or a (partial) agreement arising in real-life situations, which may or may not specify an action in every contingency. The interpretation of $x(h) = \emptyset$ is that the CA x does not specify which (mixed) action from $\triangle A^h$ player i would take at h, where $h \in H^i$; otherwise, $x(h)$ specifies player i's action at h. In particular, a CA x is said to be *complete* if $x(h) \neq \emptyset$ for all $h \in H$ – i.e., a complete CA is therefore a strategy profile.

Greenberg *et al.* (2009) offered the following solution concept of "mutually acceptable course of action (MACA)" for extensive games where "rational" individuals with different beliefs and views of the world agree to a shared course of action. Denote a subset of $\mathbb{Y}^j$ by Y^j. Denote by $y_k^j \overset{Y^j}{\rightsquigarrow} y^j$ a "trembling (belief) sequence" $\left\{y_k^j\right\}_{k=1}^{\infty}$ generated by convex combination $\sum_{t=1}^{m} \lambda_t y_{t,k}^j$ (where $y_{t,k}^j \rightsquigarrow y_t^j$ in Y^j) that converges to y^j.[7] It is easy to see that $y_k^j \rightsquigarrow y^j$ iff $y_k^j \overset{\{y^j\}}{\rightsquigarrow} y^j$.

Definition 2.2.1. A CA x is a *mutually acceptable course of action (MACA)* if there exists a set of behavioral strategy profiles $Y \equiv Y^1 \times Y^2 \cdots \times Y^n$ that supports x. That is, for every player i and every $y^i \in Y^i$, there exist $y_k^i \rightsquigarrow y^i$ and $y_k^j \overset{Y^j}{\rightsquigarrow} y^j$ for all $j \neq i$ such that

1. for all $h \in H$, $y(h) = x(h)$ whenever $x(h) \neq \emptyset$, and
2. for all $h \in H^i$ and for all $k = 1, 2, \ldots$, $u^i(y(h), y_k(-h)) \geq u^i(a^h, y_k(-h))$ for all $a^h \in A^h$.

In this chapter, we call the supporting set Y in Definition 2.2.1 a "perfectly x-rationalizable" set, and a strategy profile y in Y is said to be a "perfectly x-rationalizable" profile. For $Y^j \subseteq \mathbb{Y}^j$, let

$$\wp\left(Y^j\right) = \left\{y_k^j \rightsquigarrow y^j \,|\, y_k^j \overset{Y^j}{\rightsquigarrow} y^j\right\}.$$

That is, $\wp\left(Y^j\right)$ is the set of "trembling (belief) sequences" which can be used to represent a player's plausible "cautious" beliefs about the opponent j's behavioral strategies at all the information sets including the ones that the player thinks are impossible, given that the player knows that Y^j is a set of strategies which j might adopt.

The notion of MACA in Definition 2.2.1 provides, through the lens of a contract/agreement, a unifying game-theoretic solution concept. Greenberg *et al.* (2009) demonstrated that, by varying the degree of completeness of the underlying course of action, MACA can be related

[7] That is, there are an integer m, strategies $\{y_t^j\}_{t=1,\ldots,m}$ in Y^j, sequences of strictly positive behavioral strategies $y_{t,k}^j \rightsquigarrow y_t^j$, and a probability distribution λ on $[1, \ldots, m]$, such that the behavioral strategy y_k^j, which is outcome-equivalent to the convex combination $\sum_{t=1}^{m} \lambda_t y_{t,k}^j$, converges to y^j.

to many commonly used game-theoretic solutions, such as perfect equilibrium, rationalizable self-confirming equilibrium, and rationalizable outcomes. More specifically, there are three particular categories of MACA in extensive games:

(i) [The "Complete" MACA] A complete MACA is an MACA that specifies actions in at all information sets. The complete CA is related to the notion of perfect equilibrium.

(ii) [The "Path" MACA] A path MACA is an MACA that specifies an action at every information set that is reached with positive probability if the CA is followed. The path MACA is related to the notion of rationalizable self-confirming equilibrium.

(iii) [The "Null" MACA] The null MACA is an MACA which does not rely on a priori information regarding actions at any information set. The null MACA is associated with the notion of rationalizability.

From this perspective, the notion of MACA serves as a unifying solution concept for extensive games. The following three-person game is used to illustrate the notion of MACA. (For simplicity, we consider only pure strategies.)

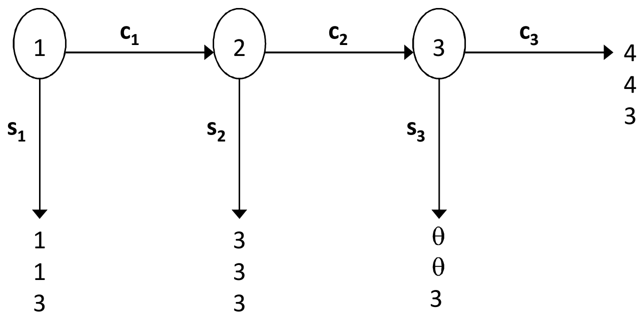


Fig. 1: A three-person game with a parameter $\theta \in [0, 1]$.

In the game depicted in Fig. 1, it is easy to see that there are two backward induction (path) outcomes: $c_1 s_2$ and $c_1 c_2 c_3$, regardless of the valuation of $\theta \in [0, 1]$. We consider two cases as follows.

Case I: $0 \leq \theta < 1$. The "complete" MACA yields the set of two strategy profiles $\{(c_1, s_2, s_3), \ (c_1, s_2, c_3)\}$, which coincides with the set of subgame perfect equilibria. The "path" MACA yields the set of three path outcomes: s_1, $c_1 s_2$ and $c_1 c_2 c_3$, which consists of (rationalizable) self-confirming equilibrium path outcomes in Fudenberg and Levine (1993) and Dekel *et al.* (1999, 2002); in particular, the "path" MACA may generate an outcome that cannot arise in the backward induction solution (because, unlike in equilibrium, players 1 and 2 need not share the same belief regarding player 3's behavior at off-path information sets). The "null" MACA yields the set of eight "perfect" rationalizable strategy profiles – i.e., the whole set of strategy profiles in this game, which coincides with the set of (subgame) rationalizable strategy profiles in the sense of Bernheim (1984) and Pearce (1984).

Case II: $\theta = 1$. Note that player 1's strategy c_1 weakly dominates s_1 and, thereby, the "perfect" rationality requires player 1 never to play strategy s_1. Thus, the "perfect-version" of rationalizability should rule out weakly dominated strategy s_1, although every strategy is still (subgame) rationalizable for $\theta = 1$. In this case, the "complete" MACA remains unchanged as in Case I, the "path" MACA yields the "refined" set of two path outcomes: $c_1 s_2$ and $c_1 c_2 c_3$, which excludes the (rationalizable) self-confirming equilibrium path outcome involving a weakly dominated strategy, and the "null" MACA yields the "refined" set of four "perfect" rationalizable strategy profiles $\{(c_1, s_2, s_3), \ (c_1, s_2, c_3), (c_1, c_2, s_3), \ (c_1, c_2, c_3)\}$.

2.2.2 LPS in Extensive Games

Blume *et al.* (1991a) presented a non-Archimedean version of subjective expected utility theory. According to the theory, an agent possesses, not a single probability distribution, but rather a vector of probability distributions that is used lexicographically in selecting an optimal action. Such a vector of probability distributions is called a "lexicographic probability system (LPS)." The first component of LPS can be thought of as representing the player's first order or primary belief about how the game will be played, the second component as the player's second order belief which is infinitely less likely than first order belief, and so on. The agent assigns to each action a vector of expected utilities calculated by LPS, and chooses an optimal action by comparing these vectors using the lexicographic ordering $\geq_{lex}$.

For the purpose of this chapter, we consider the following lexicographic preference orderings in the agent-normal game of T.[8] Let $\rho = (\rho_1, \rho_2, \ldots, \rho_L)$ be an LPS on $A = \times_{h \in H} A^h$. For $i \in N$ and $h \in H^i$, an action $a^h \in A^h$ is lexicographically preferred to another action

[8] In the agent-normal game, each agent is viewed as a separate and independent player with the same payoff as in the original game. The agent-normal game was introduced by Selten (1975) for the purpose of defining "perfect equilibrium"; cf. Kuhn's (1954) interpretation of how an extensive game is played. See also Harsanyi and Selten (1988) and van Damme (1991) for more discussions.

$b^h \in A^h$ with respect to ρ if and only if

$$[\sum_{a^{-h}\in A^{-h}} \rho_\ell^{-h}(a^{-h})u^i(a^h, a^{-h})]_{\ell=1}^L \geq_{lex} [\sum_{a^{-h}\in A^{-h}} \rho_\ell^{-h}(a^{-h})u^i(b^h, a^{-h})]_{\ell=1}^L$$

where $\rho^{-h} = (\rho_1^{-h}, \rho_2^{-h}, \ldots, \rho_L^{-h})$ is the marginal of ρ on A^{-h}.[9] The LPS ρ^{-h} represents agent h's vector-probabilistic beliefs about the other agents' actions.

Blume *et al.* (1991b) established the relationship between an LPS and a "trembling sequence" in games by using the "nested convex combination": Given an LPS $\rho = (\rho_1, \rho_2, \ldots, \rho_L)$ on A, and a vector $r = (r_1, r_2, \ldots, r_{L-1}) \in (0,1)^{L-1}$, write $r\Box\rho$ for the probability distribution on A defined by the nested convex combination

$$(1-r_1)\rho_1 + r_1(1-r_2)\rho_2 + r_1r_2(1-r_3)\rho_3 + \cdots$$
$$+ r_1r_2\cdots r_{L-2}(1-r_{L-1})\rho_{L-1} + r_1r_2\cdots r_{L-1}\rho_L.$$

This nested convex combination operator converts an LPS to a single probability measure. As $r_k \to \mathbf{0}$, an LPS ρ on A can be converted to a sequence of probability distributions $p_k = r_k\Box\rho$ on A, where ρ_ℓ is infinitely more likely than $\rho_{\ell+1}$. Blume *et al.* (1991b, Proposition 2) showed that any sequence of probability distributions $p_k \to p$ on A can also be converted to an LPS ρ on A by $p_k = r_k\Box\rho$. An LPS ρ is associated with $p_k \to p$, denoted by $\rho_{[p_k\to p]}$, if $p_k = r_k\Box\rho$ and $r_k \to \mathbf{0}$. An LPS $\rho = (\rho_1, \rho_2, \ldots, \rho_L)$ on A is *(strong) independent* if there exists $r_k \to \mathbf{0}$ such that for $k = 1, 2, ..., r_k\Box\rho$ is a product measure on A,[10] and ρ has *full support* if for each $a \in A$, $\rho_\ell(a) > 0$ for some $\ell = 1, ..., L$.

The following lemma states a relationship between the lexicographic preference ordering and the "trembling sequence" used in extensive games. That is, the standard subjective expected utility along a "trembling sequence" can be represented by a corresponding lexicographic preference over actions. This result is an immediate implication of Blume *et al.*'s (1991b) Proposition 1.[11]

[9]The marginal of ρ on A^{-h} is defined as an LPS $\rho^{-h} = (\rho_1^{-h}, \rho_2^{-h}, \ldots, \rho_L^{-h})$ on A^{-h} such that, for $\ell = 1, 2, ..., L$,

$$\forall a^{-h} \in A^{-h}, \rho_\ell^{-h}(a^{-h}) = \sum_{a^h\in A^h} \rho_\ell(a^h, a^{-h}).$$

[10]Govindan and Klumpp (2002) suggested an alternative definition of independence for LPS.

[11]It is easy to see that: Lemma 2.2.2.1 implies that a^h is a lexicographically best response with respect to

Lemma 2.2.2.1. *Let* $y_k^j \rightsquigarrow y^j \ \forall j \in N$. *For* $\forall h \in H^i$ *and* $\forall a^h, b^h \in A^h$, $u^i(a^h, y_k(-h)) > u^i(b^h, y_k(-h))$ *for* $k = 1, 2, ...$ *if, and only if,* a^h *is lexicographically preferred to* b^h *with respect to* $\rho_{[y_k \rightsquigarrow y]}$.

For $Y \subseteq \mathbb{Y}$, let

$$\wp(Y) = \times_{j \in N} \wp\left(Y^j\right),$$

where $Y^j = \{y^j | \ (y^j, y^{-j}) \in Y\}$. Define

$$ILPS^e(Y) \equiv \{\rho | \ \rho = \rho_{[y_k \rightsquigarrow y]} \text{ for some } y_k \rightsquigarrow y \text{ in } \wp(Y)\}.$$

That is, $ILPS^e(Y)$ is the set of all "independent" LPS (with full support on A) generated by $y_k^j \overset{Y^j}{\rightsquigarrow} y^j \ \forall j \in N$. Greenberg *et al.* (2009) expounded that in the context of extensive games, when faced with the subjective uncertainty about the behavioral strategies of an opponent j in Y^j, a player's plausible "cautious" belief about the opponent j's strategic behavior can be modeled as a "trembling (belief) sequence" in $\wp(Y^j)$; cf. also Dekel *et al.* (2002) for the notion of "extensive-form convex hull." By Lemma 2.2.1, such a belief can be viewed as an LPS in $ILPS^e(Y)$.

2.3 Epistemic Conditions of MACA

Following Aumann (1976, 1987, 1995 and 1999), we provide, within the standard semantic framework, an epistemic characterization of MACA by common knowledge of "rationality" and mutual knowledge of the underlying course of action. An epistemic model for game T is given by[12]

$$\mathcal{M}(T) = < \Omega, \ \{P^i\}_{i \in N}, \{\mathbf{y}^i\}_{i \in N}, \{\boldsymbol{\rho}^i\}_{i \in N} > ,$$

a full-support LPS ρ on A iff, for $y_k = r_k \Box \rho$ and $r_k \to \mathbf{0}$ (as $k \to \infty$), $u^i(a^h, y_k(-h)) \geq u^i(b^h, y_k(-h))$ $\forall b^h \in A^h$.

[12]In this paper, we are mainly concerned with the epistemic analysis of the game-theoretic solution concept of MACA. We take a point of view that an epistemic model is a pragmatic and convenient framework to be used for doing such an epistemic analysis; cf. Aumann and Brandenburger (1995, Sec. 7a) for related discussions. See also Brandenburger et al. (2008) for the epistemic model of type structure with lexicographic probabilities; cf. Brandenburger (2007) for more discussions.

where

Ω is the set of states

$P^i(\omega)$ is player i's information cell at ω

$\mathbf{y}^i(\omega)$ is player i's behavioral strategy at ω

$\boldsymbol{\rho}^i(\omega)$ is player i's vector-probabilistic belief at ω

We refer to a subset $E \subseteq \Omega$ as an *event*. For event $E \subseteq \Omega$, we take the following standard definitions in a semantic framework; see, for instance, Battigalli and Bonanno (1999), Dekel and Gul (1997), Geanakoplos (1989) and Rubinstein (1998, Chapter 3).

- $B_i E \equiv \{\omega \in \Omega | \ P^i(\omega) \subseteq E\}$ is the event that i *believes* E.
- $BE \equiv \cap_{i \in N} B_i E$ is the event that E is *mutually believed*.
- $CBE \equiv BE \cap BBE \cap BBBE \cap \cdots$ is the event that E *is commonly believed*.

Note that the information structure P^i may not be partitional; in particular, the belief operator may fail to satisfy the knowledge axiom: $E \subseteq B_i E$. Since the belief operator B satisfies the (countable) conjunction axiom: $B\left(\cap_{n=1}^{\infty} E_n\right) = \cap_{n=1}^{\infty} BE_n$, by setting

$$K_i E \equiv E \cap B_i E \text{ and } KE \equiv \cap_{i \in N} K_i E,$$

we have the following identity:

$$\begin{aligned} CKE &= KE \cap KKE \cap KKKE \cap \cdots \\ &= E \cap BE \cap BBE \cap BBBE \cap \cdots \\ &= E \cap CBE. \end{aligned}$$

In this semantic framework, we use "believe" to mean "be certain/ascribe (primary) probability 1 to" and we use "knowledge" to mean "absolute certainty/belief with no possibility for any error"; i.e., an event is said to be known when it is true and believed to be true.[13]

[13]The "belief" operator used in our semantic framework can be applied to Brandenburger et al.'s (2008) epistemic notion of "assumption" defined in a complete type structure – in this case, "i believes an event E" is interpreted as "i considers E infinitely more likely than not-E."

For $E \subseteq \Omega$, we denote by

$$\mathbf{y}(E) \equiv \{\mathbf{y}(\omega) |\, \omega \in E\}.$$

Throughout this chapter, we assume that $\mathbf{y}^i(\omega) = \mathbf{y}^i(\omega') \; \forall \omega' \in P^i(\omega)$ – i.e., each player i knows his using strategy.

We say *"agent $h \in H^i$ is perfectly rational at ω" if* we have $\boldsymbol{\rho}^i(\omega) \in ILPS^e(\mathbf{y}(P^i(\omega))$ and $\mathbf{y}^i(\omega)(h)$ is a (lexicographic) best response with respect to $\boldsymbol{\rho}^i(\omega)$ – i.e., the contingent specification $\mathbf{y}^i(\omega)(h)$ for agent h is one of lexicographically most preferred actions with respect to a vector-probabilistic belief $\boldsymbol{\rho}^i(\omega)$, where the belief that player i holds at state ω, about all the players' strategic behavior in game T, should be consistent with i's information structure at ω. (For simplicity, we use "rational" and "rationality" instead of "perfectly rational" and "perfect rationality," respectively, throughout this chapter.) Denoted by

$$R^h \equiv \{\omega |\, \text{agent } h \text{ is rational at } \omega\}\,.$$

Denoted by $R^i \equiv \cap_{h \in H^i} R^h$ the event that player i is rational and $R \equiv \cap_{i \in N} R^i$ the event that all the players are rational.

For a given course of action x, let

$$H_x = \{h \in H |\, x(h) \neq \emptyset\},$$

and let

$$R_x \equiv \cap_{h \in H_x} R^h \text{ and } R_{-x} \equiv \cap_{h \notin H_x} R^h.$$

(Define $R_x \equiv \Omega$ if $H_x = \emptyset$.) That is, R_x is the event that the players are rational at the information sets along the course of action x, and R_{-x} is the event the players are rational in all contingencies off the course of action x. Denote by $\boldsymbol{\chi}$ the restriction of $\mathbf{y}$ to H_x, i.e., $\boldsymbol{\chi}(\omega)= \mathbf{y}|_{H_x}(\omega)$ for all $\omega \in \Omega$. Let

$$[x] \equiv \{\omega \in \Omega |\, \boldsymbol{\chi}(\omega) = x\}\,.$$

We are now in a position to present the central result of this chapter which offers an epistemic characterization for the notion of MACA. Theorem 3.1 states that mutual knowledge of a course of action, "perfect" rationality along the information sets prescribed by the course of action, and common knowledge of "perfect" rationality at all other information sets, imply the underlying course of action is an MACA and, conversely, any MACA can be attained by the aforementioned epistemic assumptions.[14]

Theorem 2.3.1. *(a) Let* $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$*. Then,* $\mathbf{y}(\omega)$ *is a perfectly* x*-rationalizable profile; in particular,* $\boldsymbol{\chi}(\omega) = x$ *is an MACA. (b) Let* x *be an MACA. Then, there is an epistemic model* $\mathcal{M}(T)$ *such that* $\boldsymbol{\chi}(\omega) = x$ *for all* $\omega \in (K[x] \cap R_x) \cap CKR_{-x} \neq \emptyset$*.*

Proof. (a) For $i \in N$, define

$$Y^i \equiv \left\{ \mathbf{y}^i(\omega) |\, \omega \in (K[x] \cap R_x) \cap CKR_{-x} \right\},$$

and let $Y \equiv \times_{i \in N} Y^i$. Clearly, if $x(h) \neq \emptyset$, $y(h) = x(h)$ for all $y \in Y$. We proceed to show that Y supports x.

(i) For any $i \in N$ and $y^i \in Y^i$, there exists $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$ such that $\mathbf{y}^i(\omega) = y^i$. Since $\omega \in R_x \cap CKR_{-x}$, $\omega \in R$. Therefore, $\forall i \in N$, there is $\boldsymbol{\rho}^i(\omega) \in ILPS^e(\mathbf{y}(P^i(\omega))$ such that, for all $h \in H^i$, $\mathbf{y}^i(\omega)(h)$ is a (lexicographically) best response with respect to $\boldsymbol{\rho}^i(\omega)$.

(ii) Let $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$. Since $\omega \in K[x]$, $\forall \omega' \in P^i(\omega)$, $\mathbf{y}(\omega')(h) = x(h)$ whenever $x(h) \neq \emptyset$. That is, $\forall \omega' \in P^i(\omega)$, $\mathbf{y}(\omega')(h) = \mathbf{y}(\omega)(h)$ for all $h \in H_x$. If $x(h) = \emptyset$, then $\forall \omega' \in P^i(\omega)$,

$$\begin{aligned} \mathbf{y}(\omega')(h) &\in \{\mathbf{y}(\omega'')(h) |\, \omega'' \in CKR_{-x}\} \text{ (since } P^i(\omega) \subseteq CKR_{-x}) \\ &= \{\mathbf{y}(\omega'')(h) |\, \omega'' \in CKR_{-x} \cap (K[x] \cap R_x)\}. \end{aligned}$$

That is, $\mathbf{y}(P^i(\omega)) \subseteq Y$. But, since $\mathbf{y}^i(\omega) = \mathbf{y}^i(\omega')\ \forall \omega' \in P^i(\omega)$, we have $\mathbf{y}(P^i(\omega)) \subseteq \{y^i\} \times Y^{-i}$ for all $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$.

By (i) and (ii), it follows that for every $i \in N$ and $y^i \in Y^i$, there is $\rho \in ILPS^e(\{y^i\} \times Y^{-i})$ such that $y^i(h)$ is a (lexicographically) best response with respect to ρ for all $h \in H^i$. Thus, there exist $y^i_k \rightsquigarrow y^i$ and $y^j_k \overset{Yj}{\rightsquigarrow} y^j$ for all $j \neq i$ such that $\rho = \rho_{[y_k \rightsquigarrow y]}$ and, $\forall h \in H$, $y(h) = x(h)$

[14]Note that, in this paper, "common knowledge of rationality (CKR)" is equivalent to "rationality and common belief of rationality (RCBR)."

whenever $x(h) \neq \emptyset$. By Lemma 2.2.2.1, for every player i and every $y^i \in Y^i$, there exist $y^i_k \rightsquigarrow y^i$ and $y^j_k \overset{Y^j}{\rightsquigarrow} y^j$ for all $j \neq i$ such that (i) for all $h \in H$, $y(h) = x(h)$ whenever $x(h) \neq \emptyset$, and (ii) for all $h \in H^i$ and for all $k = 1, 2, ...,$ $u^i(y(h), y_k(-h)) \geq u^i(a^h, y_k(-h))$ for all $a^h \in A^h$. That is, Y supports x. Therefore, $\forall \omega \in (K[x] \cap R_x) \cap CKR_{-x}$, $\mathbf{y}(\omega)$ is a perfectly x-rationalizable profile and $\boldsymbol{\chi}(\omega) = x$ is an MACA.

(b) Let x be an MACA which is supported by $Y \equiv Y^1 \times Y^2 \cdots \times Y^n$. We show a stronger result that there is $\mathcal{M}(T)$ such that $\boldsymbol{\chi}(\omega) = x$ for all $\omega \in CK([x] \cap R) \neq \emptyset$. For each $i \in N$ and $y^i \in Y^i$, there exist $y^i_k \rightsquigarrow y^i$ and $y^j_k \overset{Y^j}{\rightsquigarrow} y^j$ for all $j \neq i$ such that

1. for all $h \in H$, $y(h) = x(h)$ whenever $x(h) \neq \emptyset$, and
2. for all $h \in H^i$ and for all $k = 1, 2, \ldots,$ $u^i(y(h), y_k(-h)) \geq u^i(a^h, y_k(-h))$ for all $a^h \in A^h$.

Let $\rho^i(y^i) = \rho_{[y_k \rightsquigarrow y]}$ such that $y^i_k \rightsquigarrow y^i$ and $y^j_k \overset{Y^j}{\rightsquigarrow} y^j$ for all $j \neq i$. Clearly, $\rho^i(y^i) \in ILPS^e(\{y^i\} \times Y^{-i})$. Define an epistemic model for game T:

$$\mathcal{M}(T) =< \Omega,\ \{P^i\}_{i \in N}, \{\mathbf{y}^i\}_{i \in N}, \{\boldsymbol{\rho}^i\}_{i \in N} > ,$$

such that $\Omega = \left\{ (y^j, \rho^j(y^j))_{j \in N} \mid y^j \in Y^j, \forall j \in N \right\}$ and for all $i \in N$ and $\omega = (y^j, \rho^j(y^j))_{j \in N}$ in Ω,

$$\mathbf{y}^i(\omega) = y^i,\ \boldsymbol{\rho}^i(\omega) = \rho^i(y^i) \text{ and}$$
$$P^i(\omega) = \{\omega' \in \Omega |\ \mathbf{y}^i(\omega') = y^i \text{ and } \boldsymbol{\rho}^i(\omega') = \rho^i(y^i)\}.$$

Now, consider any arbitrary $\omega = (y^j, \rho^j(y^j))_{j \in N}$ in Ω. By Lemma 2.2.1, it follows that for all $i \in N$ and $h \in H^i$, $\mathbf{y}^i(\omega)(h)$ is a (lexicographically) best response with respect to $\boldsymbol{\rho}^i(\omega)$. Since $\rho^i(y^i) \in ILPS^e(\{y^i\} \times Y^{-i})$, $\boldsymbol{\rho}^i(\omega) \in ILPS^e(\mathbf{y}(P^i(\omega)))\ \forall i \in N$. Thus, $\omega \in R$. But, since $\boldsymbol{\chi}(\omega) = x$, $\omega \in [x]$. That is, $\Omega = R \cap [x]$. Therefore, $\boldsymbol{\chi}(\omega) = x$ for all $\omega \in CK([x] \cap R) = \Omega$. ■

In Theorem 2.3.1, we have identified epistemic conditions for MACA that are as spare as possible. An immediate corollary of Theorem 2.3.1 gives a readily expressible form of epistemic assumptions of MACA: The notion of MACA can be viewed as the logical consequence of common knowledge of "perfect" rationality plus mutual knowledge of agreement on the underlying course of action.

Corollary 2.3.1. *(a) Let $\omega \in K[x] \cap CKR$. Then, $\mathbf{y}(\omega)$ is a perfectly x-rationalizable profile; in particular, $\boldsymbol{\chi}(\omega) = x$ is an MACA. (b) Let x be an MACA. Then, there is an epistemic model $\mathcal{M}(T)$ such that $\boldsymbol{\chi}(\omega) = x$ for all $\omega \in CKR \cap K[x] \neq \emptyset$.*

Proof. Since $CKR \subseteq R_x \cap CKR_{-x}$, Corollary 2.3.1(a) follows directly from Theorem 2.3.1(a). Corollary 2.3.1(b) follows from the proof of Theorem 2.3.1(b). ■

At a conceptual level, Greenberg *et al.* (2009) demonstrated that by varying the degree of completeness of the underlying course of action, the notion of MACA can be related to other game-theoretic solution concepts, such as perfect equilibrium, rationalizable self-confirming equilibrium, and rationalizability. Theorem 2.3.1 provides a very general and comprehensive epistemic characterization of MACA which can be applied to a wide range of strategic environments. We go on to show how to derive the epistemic characterizations for various game-theoretic solutions from Theorem 2.3.1, by placing corresponding restrictions on the underlying course of action.

2.3.1 Complete MACA and Perfect Equilibrium

A complete CA is a course of action x where $x(h) \neq \emptyset \; \forall h \in H$ – i.e., x is a strategy profile. A complete MACA can be viewed as a "subjective" perfect equilibrium which is "self supporting" in the sense that, while all the players know that the complete MACA will be followed, it is possible for different players to have different trembling sequences that converge to this MACA. A complete MACA is a perfect equilibrium if all the players share the same trembling sequence that converges to the MACA; cf. Greenberg *et al.* (2009, Section 3.1). Analogous to Aumann and Brandenburger's (1995) preliminary epistemic observation on Nash equilibrium, the following Proposition 2.3.1.1, which is an immediate implication of Theorem 2.3.1 for a complete MACA, states a simple and straightforward epistemic characterization for (subjective) perfect equilibrium.

Proposition 2.3.1.1. *Suppose that x is a complete course of action. (a) Let $\omega \in R \cap K[x]$. Then, $\boldsymbol{\chi}(\omega) = x$ is a complete MACA – i.e., a subjective perfect equilibrium and, if all the players share a common prior LPS belief (i.e., $\boldsymbol{\rho}^i(\omega) = \boldsymbol{\rho}^j(\omega)$ for all $i, j \in N$), $\boldsymbol{\chi}(\omega) = x$ is a perfect equilibrium. (b) If x is a complete MACA, then there is an epistemic model $\mathcal{M}(T)$ such that $\boldsymbol{\chi}(\omega) = x$ for all $\omega \in (R \cap K[x]) \neq \emptyset$.*

Proof. Since x is a complete CA, $H_x = H$. Therefore, $R_x = R$, $R_{-x} = \Omega$ and $\mathbf{y}(\omega) = \boldsymbol{\chi}(\omega) = x$. Note that, if all the players share a common prior LPS belief in a subjective perfect equilibrium, this equilibrium must be a perfect equilibrium (where all the player believe in the same sequence of trembles that converges to the equilibrium). Proposition 2.3.1.1 follows directly from Theorem 2.3.1. ■

In two-person normal-form games, it is easy to see that the notion of subjective perfect equilibrium is equivalent to that of perfect equilibrium. Subsequently, in two-person simultaneous move games, $\boldsymbol{\chi}(\omega) = x$ is a perfect equilibrium for all $\omega \in R \cap K[x]$. However, the following example depicted in Fig. 2 shows that, for $\omega \in R \cap K[x]$, $\boldsymbol{\chi}(\omega) = x$ may not be a perfect equilibrium even in a two-person game with perfect information.

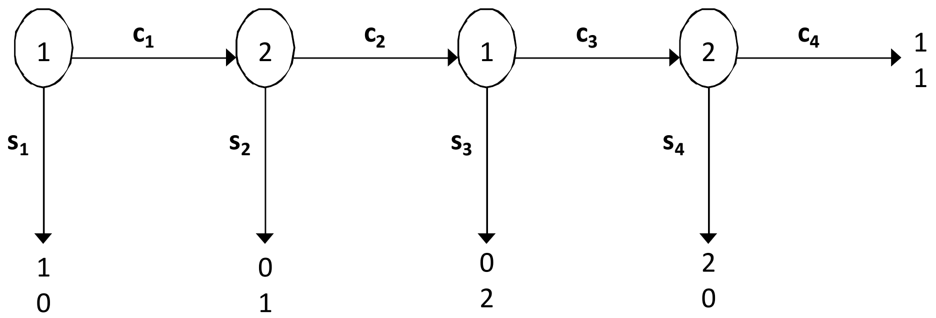


Fig. 2: A two-person game.

Example 2.3.1.1: Consider the strategy profile $x = (c_1, c_2, c_3, c_4)$ in this game. Construct a knowledge model $\mathcal{M}(T)$ such that $\Omega = \{\omega\}$, $P^1(\omega) = P^2(\omega) = \{\omega\}$, $\mathbf{y}(\omega) = x$, and

$$\boldsymbol{\rho}^1(\omega) = \begin{pmatrix} 1(c_1, c_2, c_3, c_4) \\ \frac{1}{2}(s_1, c_2, c_3, c_4) + \frac{1}{2}(c_1, c_2, c_3, s_4) \\ \frac{1}{3}(c_1, s_2, c_3, c_4) + \frac{1}{3}(c_1, c_2, s_3, c_4) + \frac{1}{3}(s_1, c_2, c_3, s_4) \\ \frac{1}{4}(s_1, s_2, c_3, c_4) + \frac{1}{4}(s_1, c_2, s_3, c_4) + \frac{1}{4}(c_1, s_2, c_3, s_4) + \frac{1}{4}(c_1, c_2, s_3, s_4) \\ \frac{1}{3}(s_1, s_2, c_3, s_4) + \frac{1}{3}(s_1, c_2, s_3, c_4) + \frac{1}{3}(c_1, s_2, s_3, c_4) \\ \frac{1}{2}(s_1, s_2, s_3, c_4) + \frac{1}{2}(c_1, s_2, s_3, s_4) \\ 1(s_1, s_2, s_3, s_4) \end{pmatrix}$$

$$\boldsymbol{\rho}^2(\omega) = \begin{pmatrix} 1(c_1, c_2, c_3, c_4) \\ \frac{1}{2}(s_1, c_2, c_3, c_4) + \frac{1}{2}(c_1, c_2, s_3, c_4) \\ \frac{1}{3}(c_1, s_2, c_3, c_4) + \frac{1}{3}(c_1, c_2, c_3, s_4) + \frac{1}{3}(s_1, c_2, s_3, c_4) \\ \frac{1}{4}(s_1, s_2, c_3, c_4) + \frac{1}{4}(s_1, c_2, c_3, s_4) + \frac{1}{4}(c_1, s_2, s_3, c_4) + \frac{1}{4}(c_1, c_2, s_3, s_4) \\ \frac{1}{3}(s_1, s_2, s_3, c_4) + \frac{1}{3}(s_1, c_2, c_3, s_4) + \frac{1}{3}(c_1, s_2, c_3, s_4) \\ \frac{1}{2}(s_1, s_2, c_3, s_4) + \frac{1}{2}(c_1, s_2, s_3, s_4) \\ 1(s_1, s_2, s_3, s_4) \end{pmatrix}.$$

It is easy to verify that, in this example, $\Omega = R \cap K[x]$ and $\boldsymbol{\chi}(\omega) = x$ is a subjective perfect equilibrium, but not a perfect equilibrium. (To see this point, assume, in negation, that x is a perfect equilibrium, supported by a trembling sequence $y_k \rightsquigarrow x$. For c_2 to be player 2's local best response to y_k, it must be the case that the probability of playing s_3 is higher than the probability of playing s_4, i.e., $y_k(s_3) \geq y_k(s_4)$. But then, as $y_k(s_2) > 0$, it follows that player 1's unique local best response to y_k at the root of the game is action s_1, but not c_1.)

In Proposition 2.3.1.1, we hold a traditional view "mixed strategies as objects of choice": players deliberately introduce randomness into their behavior. However, a mixed equilibrium strategy of a player can also be interpreted as the common conjecture of all the other players about that player's strategy choices; cf., *e.g.*, Aumann (1987) and Rubinstein (1991). We close this subsection by providing some epistemic conditions for a (mixed) complete MACA along this line of interpretation of mixed equilibrium strategies. In the spirit of Aumann and Brandenburger's (1995) Theorems A and B, we present a simple and expressible form of epistemic prerequisites for a complete MACA interpreted as beliefs: Proposition 2.3.1.2 below states that mutual belief of all players' conjectures about a complete (mixed) course of action and of "perfect" rationality implies that the complete course of action, which can be viewed as a common agreed-upon primary belief for the players, is a subjective perfect equilibrium. For the purpose of this analysis, we elaborately require each player i's strategy choice function $\mathbf{y}^i(\cdot)$ to be valued in pure strategies; mixed strategies arise only in the form of subjective beliefs about a player's strategy choices. As usual, we also assume that each player i knows his own belief –

i.e., $\boldsymbol{\rho}^i(\omega') = \boldsymbol{\rho}^i(\omega)\ \forall \omega' \in P^i(\omega)$. Consider a complete CA x and an LPS profile $\rho = (\rho^i)_{i\in N}$. Define

$$\left[\boldsymbol{\rho} \overset{x}{=} \rho\right] \equiv \{\omega \in \Omega |\ \boldsymbol{\rho}_{\geq 2}(\omega) = \rho_{\geq 2} \text{ and } \left(\boldsymbol{\rho}_1^i(\omega)\right)^{-i} = x^{-i}\ \forall i \in N\},$$

that is, $\left[\boldsymbol{\rho} \overset{x}{=} \rho\right]$ represents the event that all players hold first-order or primary beliefs agreed upon x – i.e., the marginal primary belief of each player i about the strategy choices of the opponents is given by the behavioral strategy profile x^{-i} for the opponents – and higher-order beliefs given by $\rho_{\geq 2} = \left((\rho_\ell^i)_{\ell \geq 2}\right)_{i\in N}$.[15]

Proposition 2.3.1.2. *Let $\omega \in B\left(\left[\boldsymbol{\rho} \overset{x}{=} \rho\right] \cap R\right)$. Then, there is an agreed-upon primary belief $\boldsymbol{\rho}_1^*(\omega) = x$ which is a subjective perfect equilibrium and, if all players share a common higher-order belief – i.e., $\rho_{\geq 2}^* = \rho_{\geq 2}^i\ \forall i \in N$, the primary belief $\boldsymbol{\rho}_1^*(\omega) = x$ is a perfect equilibrium.*

Proof. Let $\omega \in B\left(\left[\boldsymbol{\rho} \overset{x}{=} \rho\right] \cap R\right)$. Since each player i knows his own belief, for all $\omega' \in P^i(\omega)$, $\left(\boldsymbol{\rho}_1^i(\omega)\right)^{-i} = \left(\boldsymbol{\rho}_1^i(\omega')\right)^{-i} = x^{-i}$. That is, there is an agreed-upon primary belief $\boldsymbol{\rho}_1^*(\omega) = x$ satisfying $\left(\boldsymbol{\rho}_1^*(\omega)\right)^{-i} = \left(\boldsymbol{\rho}_1^i(\omega)\right)^{-i}\ \forall i \in N$. Now, consider any fixed player $i \in N$. Let $j \in N$ and $j \neq i$. Since $\left(\boldsymbol{\rho}_1^j(\omega)\right)^{-j} = x^{-j}$ and j knows his own using strategy, by $\boldsymbol{\rho}^j(\omega) \in ILPS^e(\mathbf{y}(P^j(\omega)))$, there exists $y_k \rightsquigarrow y$ in $\wp(\mathbf{y}(P^j(\omega)))$ such that $\boldsymbol{\rho}^j(\omega) = \rho_{[y_k \rightsquigarrow y]}$ where $y = (\mathbf{y}^j(\omega), x^{-j})$. Thus, there is $y_k^i \overset{\mathbf{y}^i\left(P^j(\omega)\right)}{\rightsquigarrow} x^i$. That is, there exist $y_t^i \in \mathbf{y}^i(P^j(\omega))$ for $t = 1, 2, ..., m$ such that $y_{t,k}^i \rightsquigarrow y_t^i$ and $y_k^i \triangleq \sum_{t=1}^m \lambda_t y_{t,k}^i \rightsquigarrow x^i$, where the notation "$\triangleq$" denotes the outcome-equivalence relation between two strategies. (For brevity, we also denote by $x^i \triangleq \sum_{t=1}^m \lambda_t y_t^i$ the limit point arising from such a situation.) Since $y_t^i \in \mathbf{y}^i(P^j(\omega))$, there exists $\omega' \in P_j(\omega)$ such that $\mathbf{y}^i(\omega') = y_t^i$. Since $\omega \in BR$, $\omega' \in P_j(\omega) \subseteq R_i$. Therefore, for all $h \in H^i$, $y_t^i(h)$ lexicographically maximizes player i's expected utilities calculated by $\boldsymbol{\rho}^i(\omega') \in ILPS^e(\mathbf{y}(P^i(\omega')))$. Since $\omega' \in P^j(\omega) \subseteq \left[\boldsymbol{\rho} \overset{x}{=} \rho\right]$ and i knows his own using strategy, $\boldsymbol{\rho}^i(\omega') = \left((y_t^i, x^{-i}), \rho_{\geq 2}^i\right)$. Therefore, for all $h \in H^i$ and $t = 1, 2, ..., m$, $y_t^i(h)$ lexicographically maximizes player i's expected utilities calculated by $\left((y_t^i, x^{-i}), \rho_{\geq 2}^i\right)$.

By Lemma 3.1.1 below, it follows that $x^i(h)$ lexicographically maximizes player i's expected utilities calculated by $\left(x, \rho_{\geq 2}^i\right)$ where $x^i \triangleq \sum_{t=1}^m \lambda_t y_t^i$. Since $\left((y_t^i, x^{-i}), \rho_{\geq 2}^i\right) \in ILPS^e(\mathbf{y}(P^i(\omega')))$

[15] We here purport to present simple and straightforward sufficient epistemic conditions for a complete (mixed) MACA, which is interpreted as a common primary belief; in particular, we do not need the epistemic assumption that all players' conjectures are commonly known as in Aumann and Brandenburger's (1995) Theorem B. The formalism of Proposition 3.1.2 is consistent with Aumann and Brandenburger's (1995) Remark 7.1 if, for each player i, ρ^i is taken to be a single (product) probability measure x on A.

for $t = 1, 2, ..., m$, there is $x_k \rightsquigarrow x$ in $\wp(\mathbb{Y})$ such that $\left(x, \rho^i_{\geq 2}\right) = \rho_{[x_k \rightsquigarrow x]}$. By Lemma 2.2.2.1, for all player $i \in N$, there exists a sequence $x_k \rightsquigarrow x$ such that, for all $h \in H^i$ and for $k = 1, 2, \ldots$,

$$u^i(x(h), x_k(-h)) \geq u^i(a^h, x_k(-h)) \text{ for all } a^h \in A^h.$$

By Greenberg *et al.*'s (2009) Claim 3.1.1, the agreed-upon primary belief $\boldsymbol{\rho}^*_1(\omega) = x$ is an complete MACA and, hence, it is a subjective perfect equilibrium. Moreover, if there is a common higher-order belief $\rho^*_{\geq 2} = \rho^i_{\geq 2}\ \forall i \in N$, then the primary belief $\boldsymbol{\rho}^*_1(\omega) = x$ is a perfect equilibrium. ■

Lemma 2.3.1.1. *If, for $t = 1, 2, ..., m$, $y^i_t(h)$ is a (lexicographically) best response with respect to $\rho^t = \left((y^i_t, x^{-i}), \rho^i_{\geq 2}\right)$, then $x^i(h)$ is a (lexicographically) best response with respect to $\rho^* = \left(x, \rho^i_{\geq 2}\right)$, where $x^i \triangleq \sum_{t=1}^m \lambda_t y^i_t$ and $h \in H^i$.*

Proof. Let

$$H^i(0) \equiv \left\{h \in H^i |\ \nexists h' \in H^i \text{ s.t. } h' \text{ can be reached from } h\right\},$$

where $h \in H^i(0)$ is interpreted as a lowest order or 0-order information set of player i from which no other information set of player i can be reached. Define, inductively, for $\kappa \geq 1$,

$$H^i(\kappa) \equiv \left\{h \in H^i \backslash \cup_{\kappa'=0}^{\kappa-1} H^i(\kappa') \mid \nexists h' \in H^i \backslash \cup_{\kappa'=0}^{\kappa-1} H^i(\kappa') \text{ s.t. } h' \text{ can be reached from } h\right\},$$

where $h \in H^i(\kappa)$ is interpreted as a κ-order information set of player i from which no higher order (i.e. κ'-order for $\kappa' \geq \kappa+1$) information set of player i can be reached. Clearly, $\{H^i(\kappa)\}_{\kappa \geq 0}$ is a (finite) partition of H^i since each player is perfect recall. We prove Lemma 2.3.1.1 by induction on the order of κ.

For $\kappa = 0$, we show that the result is true for $h \in H^i(0)$. Since, for $t = 1, 2, ..., m$, $y^i_t(h)$ is a (lexicographically) best response with respect to $\rho^t = \left((y^i_t, x^{-i}), \rho^i_{\geq 2}\right)$, there exists a sequence $\left(y^i_{t,k}, x^{-i}_k\right) \rightsquigarrow (y^i_t, x^{-i})$ such that, for $k = 1, 2, \ldots$,

$$u^i(y^i_t(h), \left(y^i_{t,k}, x^{-i}_k\right)(-h)) \geq u^i(a^h, \left(y^i_{t,k}, x^{-i}_k\right)(-h)) \text{ for all } a^h \in A^h.$$

Since the game is with perfect recall and $h \in H^i(0)$ is a lowest order information set for player

i, for $t = 1, 2, ..., m$,

$$u^i(y_t^i(h), (x_k^i, x_k^{-i})(-h)) \geq u^i(a^h, (x_k^i, x_k^{-i})(-h)) \text{ for all } a^h \in A^h,$$

where $x_k^i \mathrel{\hat{=}} \sum_{t=1}^m \lambda_t y_{t,k}^i$. Therefore, for any probability distribution $\widetilde{\lambda}$ on $[1, ..., m]$,

$$u^i(\sum_{t=1}^m \widetilde{\lambda}_t y_t^i(h), x_k(-h)) \geq u^i(a^h, x_k(-h)) \text{ for all } a^h \in A^h.$$

Since $x^i \mathrel{\hat{=}} \sum_{t=1}^m \lambda_t y_t^i$, it follows that $x^i(h)(a^h) = \lim_{k\to\infty} x_k^i(h)(a^h) = 0$ if $a^h \neq y_t^i(h)$ for $t = 1, 2, ..., m$. Therefore, $x^i(h)$ can be viewed as a convex combination $\sum_{t=1}^m \widetilde{\lambda}_t y_t^i(h)$ and, hence,

$$u^i(x^i(h), x_k(-h)) \geq u^i(a^h, x_k(-h)) \text{ for all } a^h \in A^h.$$

Now, consider $\kappa = 1$. We proceed to show that the result is true for $h \in H^i(1)$. Since, for $t = 1, 2, ..., m$, $y_t^i(h)$ is a (lexicographically) best response with respect to $\rho^t = ((y_t^i, x^{-i}), \rho_{\geq 2}^i)$, there exists a sequence $(y_{t,k}^i, x_k^{-i}) \rightsquigarrow (y_t^i, x^{-i})$ such that, for $k = 1, 2, \ldots$,

$$u^i(y_t^i(h), (y_{t,k}^i, x_k^{-i})(-h)) \geq u^i(a^h, (y_{t,k}^i, x_k^{-i})(-h)) \text{ for all } a^h \in A^h.$$

By the proof for $\kappa = 0$, player i's expected payoff conditional on $h' \in H^i(0)$ satisfies:[16]

$$\begin{aligned} u^i(y_t^i(h'), (y_{t,k}^i, x_k^{-i})(-h') | h') &= u^i(y_t^i(h'), x_k(-h') | h') \\ &= u^i(x^i(h'), x_k(-h') | h') \\ &= \max_{a^{h'} \in A^{h'}} u^i(a^{h'}, x_k(-h') | h'). \end{aligned}$$

Since $\rho^* = (x, \rho_{\geq 2}^i)$ and $\rho^t = ((y_t^i, x^{-i}), \rho_{\geq 2}^i)$ for $t = 1, 2, \ldots m$, we can have

$$x_k = (1 - r_{1,k})\, x + r_{1,k} (r_{\geq 2,k} \square \rho_{\geq 2}^i) \text{ and } (y_{t,k}^i, x_k^{-i}) = (1 - r_{1,k}) (y_t^i, x^{-i}) + r_{1,k} (r_{\geq 2,k} \square \rho_{\geq 2}^i),$$

[16]Note that the conditional expected payoff at an information set from a behavioral strategy profile is well defined, given that the information set is reached with positive probability when the game is played according to the specified strategy profile.

where $(r_{1,k}, r_{\geq 2,k}) \to \mathbf{0}$. Therefore, for all $h' \in H^i(0)$,

$$\begin{aligned} & u^i(y^i_{t,k}(h'), (y^i_{t,k}, x^{-i}_k)(-h')\,|h') \\ = \;& (1-r_1)\, u^i(y^i_t(h'), (y^i_{t,k}, x^{-i}_k)(-h')\,|h') + r_1 u^i((r_{\geq 2}\Box\rho^i_{\geq 2})(h'), (y^i_{t,k}, x^{-i}_k)(-h')\,|h') \\ = \;& (1-r_1)\, u^i(x^i(h'), (x^i_k, x^{-i}_k)(-h')\,|h') + r_1 u^i((r_{\geq 2}\Box\rho^i_{\geq 2})(h'), (x^i_k, x^{-i}_k)(-h')\,|h') \\ = \;& u^i(x^i_k(h'), x_k(-h')\,|h'). \end{aligned}$$

Since the game is with perfect recall, for $t = 1, 2, ..., m$, it follows

$$\begin{aligned} u^i(y^i_t(h), x_k(-h)\,|h) &= u^i(y^i_t(h), (y^i_{t,k}, x^{-i}_k)(-h)\,|h) \\ &\geq u^i(a^h, (y^i_{t,k}, x^{-i}_k)(-h)\,|h) \\ &= u^i(a^h, x_k(-h)\,|h),\ \forall a^h \in A^h. \end{aligned}$$

Therefore, $u^i(y^i_t(h), x_k(-h)) \geq u^i(a^h, x_k(-h)),\ \forall a^h \in A^h$. Again, by the similar argument above, we have

$$u^i(x^i(h), x_k(-h)) \geq u^i(a^h, x_k(-h)) \text{ for all } a^h \in A^h.$$

Repeating the argument for $\kappa \geq 2$, we conclude that the result is true for all $h \in H^i$. ■

2.3.2 Path MACA and Self-Confirming Equilibrium

A path CA is a course of action, x, that specifies a (mixed) action at the root of the game and at every information set that is reached with positive probability if x is followed. The path MACA is related to the notions of "self-confirming equilibrium (SCE)" (see Fudenberg and Levine (1993)) and "rationalizable self-confirming equilibrium (RSCE)" (see Dekel *et al.* (1999, 2002)),[17] since they are based on the same idea that the requirement of "commonality of beliefs" about the actions, which would have been taken in contingencies that were not realized during the play, cannot be justified and, therefore, should not be required for a solution concept.

The notion of path MACA indeed refines the notion of "sequential RSCE" in which each

[17] See also Fudenberg and Kreps (1995) and Kalai and Lehrer (1993a, 1993b).

player is assumed to be sequentially rational at all his information sets (see Dekel *et al.* (1999, 2002)). Intuitively, the path MACA adopts more stringent "perfect" rationality restrictions in place of "sequential" rationality used in the definition of sequential RSCE; a path MACA requires not only that players be "perfectly" rational at information sets along the path of the play, but also that players commonly know they are "perfect" rational in contingencies off the equilibrium path.

By restricting attention to a path course of action, Theorem 2.3.1 delivers an epistemic characterization for the path MACA, a perfect-version of rationalizable self-confirming equilibrium.

Proposition 2.3.2.1. *Suppose that x is a path course of action. (a) Let $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$. Then, $\boldsymbol{\chi}(\omega) = x$ is a path MACA and, hence, it is supported by a sequential RSCE. (b) If x is a path MACA, then there is an epistemic model $\mathcal{M}(T)$ such that $\boldsymbol{\chi}(\omega) = x$ for all $\omega \in (K[x] \cap R_x) \cap CKR_{-x}$.*

Proof. Proposition 2.3.2.1 follows immediately from Theorem 2.3.1. ■

2.3.3 Null MACA and Rationalizability

A course of action, x, is the null CA if $x(h) = \emptyset$ for all information sets h. The concept of null MACA, which is related to Bernheim's (1984) and Pearce's (1984) notion of rationalizability, is applicable to situations where players have no common background agreement (based on, say, past observations or social norms) concerning the actions to be taken at some decision moments. The null MACA suggests an interesting notion of "perfect rationalizability" with independent perturbed beliefs:

Definition 2.3.3.1. A set of strategy profiles $Y \equiv Y^1 \times Y^2 \cdots \times Y^n$ is *perfectly rationalizable* if, for every player i and every $y^i \in Y^i$ there exist $y_k^i \rightsquigarrow y^i$ and $y_k^j \overset{Y^j}{\rightsquigarrow} y^j$ for all $j \neq i$ such that, for all $h \in H^i$ and for all $k = 1, 2, \ldots$,

$$u^i(y(h), y_k(-h)) \geq u^i(a^h, y_k(-h)) \text{ for all } a^h \in A^h.$$

In particular, $y \in Y$ is said to be a perfectly rationalizable strategy profile.

Bernheim (1984, Section 6(b)) defined the concept of "subgame rationalizability" by using Selten's (1965) subgame perfectness criterion, and Definition 2.3.3.1 can be viewed as a natural "perfect" extension of subgame rationalizability suitable for general extensive games. In normal-form games, the null MACA yields a rather intuitive refinement of rationalizability, since perfect rationalizability never uses weakly dominated strategies. Definition 2.3.3.1 is indeed Herings and Vannetelbosch's (1999) definition of "weakly perfect rationalizability" in the class of simultaneous-move games. As Herings and Vannetelbosch (2000) showed, this notion is equivalent to the DF procedure (Dekel-Fudenberg 1990) if allowed for correlated perturbed beliefs.

By applying Theorem 2.3.1 to the case of the null CA, we can obtain the following Proposition 2.3.3.1 which provides an epistemic characterization for the "perfect" version of rationalizability. In normal-form games, Proposition 2.3.3.1 is consistent with Brandenburger's (1992) characterization for the DF procedure by using LPS.[18]

Proposition 2.3.3.1. *(a) Let* $\omega \in CKR$*. Then,* $\mathbf{y}(\omega)$ *is a perfectly rationalizable strategy profile and, hence, a rationalizable strategy profile. (b) Let* y *be a perfectly rationalizable strategy profile. Then, there is an epistemic model* $\mathcal{M}(T)$ *such that* $\mathbf{y}(\omega) = y$ *for* $\omega \in CKR$*.*

Proof. (a) Let $\boxed{R}$ denote a "self-evident event in R" – i.e., $\boxed{R} \subseteq R$ and $\boxed{R} \subseteq B_i \boxed{R}$ $\forall i \in N$. It is easy to see that $\omega \in CKR$ iff there is $\boxed{R} \ni \omega$; cf., *e.g.*, Aumann (1976). For each $i \in N$ define

$$Y_i \equiv \left\{ \mathbf{y}_i(\omega') \mid \omega' \in \boxed{R} \right\}.$$

Therefore, for every $\omega' \in \boxed{R}$, $\mathbf{y}_{-i}(P_i(\omega')) \subseteq Y_{-i}$. Since $\omega' \in R_i$, $\mathbf{y}_i(\omega')$ is a (lexicographically) best response of player i to $\boldsymbol{\rho}^i(\omega') \in ILPS^e(Y)$. But, since $\omega \in \boxed{R}$, $y(\omega) \in Y$ is a perfectly rationalizable strategy profile.

(b) Since x is the null CA, $H_x = \emptyset$. Therefore, $[x] = \Omega$, $R_x = \Omega$ and $R_{-x} = R$. Proposition 2.3.3.1(b) follows directly from the proof of Theorem 2.3.1(b). ■

In a "generic" PI game (i.e. perfect-information game) where no two different terminal

[18] In normal-form games, Borgers (1994) provided an alternative characterization for the DF procedure by common p-belief of "rationality" (where $p \to 1$) – i.e. it is approximate common belief that players maximize expected utility using full-support conjectures; see also Hu (2007) for more discussions. Gul (1996) demonstrated that the DF procedure can be viewed as a weakest perfect version of τ-theory. Barelli and Galanis (2013) also offered an alternative and interesting approach to providing epistemic conditions for admissible behavior in (two-person) normal-form games, including the DF procedure and iterated admissibility.

nodes generate the same payoff for any of the players, Proposition 2.3.3.1 yields the following corollary that re-states Aumann's (1995) central result: In a generic PI game, common knowledge of "rationality" implies the backward induction outcome and, moreover, the backward induction outcome can be attained in terms of common knowledge of "rationality."[19]

Corollary 2.3.3.1. *Suppose T is a generic PI game. Let $\omega \in CKR$. Then $\mathbf{y}(\omega)$ is the backward induction outcome. Moreover, there exists an epistemic model $\mathcal{M}(T)$ such that $\mathbf{y}(\omega)$ is the backward induction outcome for $\omega \in CKR \neq \emptyset$.*

Proof. Note that, in a generic PI game, the backward induction outcome is the unique perfectly rationalizable strategy profile. The result of Corollary 2.3.3.1 follows directly from Proposition 2.3.3.1. ■

2.4 Concluding Remarks

In the conventional framework of extensive-form games, Greenberg *et al.* (2009) presented a unified game-theoretic solution concept of "mutually acceptable course of action (MACA)" for situations where "perfectly" rational individuals with different beliefs and views of the world agree to a shared course of action. In this chapter, we have carried out the epistemic program in game theory to explore epistemic conditions for MACA.

We have established an expressible epistemic characterization for MACA. More specifically, by using the notion of "lexicographic probability system (LPS)" introduced by Blume *et al.* (1991a), we have defined "rationality" as lexicographic maximization through LPS beliefs and, within Aumann's semantic framework, we have formulated and shown that MACA is the logical consequence of common knowledge of "perfect" rationality and mutual knowledge of agreement on the underlying course of action (see Theorem 2.3.1 and Corollary 2.3.1).[20]

[19] Aumann (1995) used the conventional semantic model of knowledge with standard "partitional" information structures. Halpern (2001) provided a nice synthesis of the knowledge-based approach to different theories for PI games in light of different kinds of counterfactual reasonings; see also Halpern (1999). From this perspective, players in our framework can be viewed as if they used (full-support) LPS beliefs to revise their beliefs about other players' strategic behavior when doing such hypothetical reasoning. In a different framework (i.e., a finite extensive-form type model), Ben-Porath (1997) defined a "weak" extensive-form notion of common (initial) belief of "sequential rationality" and showed that, for a "generic" PI game, this notion leads to an extensive-form analog of the DF procedure which does not necessarily imply the backward induction outcome; cf. Dekel and Gul (1997, Sec. 5.4) for more discussion.

[20] Feinberg (2005a) presented a "subjective" epistemic framework in a syntactic fashion for describing and

This chapter therefore provides an epistemic counterpart of MACA in terms of what players know and believe about "rationality," actions, information, and knowledge in complex social environments with emerging a shared course of action.

One important feature of this work is that we take a strong form of "perfect" and "cautious" rationality that reflects Selten's (1975) idea of "trembles" in our analysis. The chapter thus can provide a simple and useful analytical framework for comparing various perfect-versions of solution concepts, from an epistemic perspective, in game situations where players are "perfectly" rational individuals; cf. Table 1. In this chapter, we have shown how epistemic characterizations for various related solution concepts can be obtained, in a direct and simple way, by varying the degree of completeness of the underlying course of action, as well as assuming different epistemic conditions to players in the game (see Propositions 2.3.1.1, 2.3.2.1 and 2.3.3.1). In the spirit of Aumann and Brandenburger (1995), we have also offered an additional epistemic characterization for a (mixed) complete MACA – i.e., a (subjective) perfect equilibrium – if mixed strategies are interpreted as conjectures of players (see Proposition 2.3.1.2). It is worthwhile to point out that, by utilizing the notion of LPS, we have presented a comprehensive and epistemic analytical framework to accommodate the tension that arises in modeling "perfect" rationality (that requires to include all possible strategies in a perturbed belief) and knowledge/belief about "perfect" rationality (that requires to exclude some strategies from the perturbed belief) in complex social interactions; cf., *e.g.*, Samuelson (1992 and 2004) and Brandenburger (2007). The study of this chapter is useful to deepen our epistemic understanding of MACA and related game-theoretic solution concepts in the literature.

Finally, we would like to point out that, in this chapter, we define "rationality" as lexicographic maximization by "independent" (cautious) LPS beliefs. This formalism is used for capturing Selten's original idea of "trembles" in analyzing dynamic strategic behavior. It is natural and interesting to extend the epistemic analysis of this chapter to MACA by allowing for "correlated" LPS beliefs. We leave this issue for future research.

---

analyzing dynamic strategic behavior. The framework is particularly useful for accounting for the subjective reasoning of players in hypothetical situations. At a conceptual level, this subjective framework models players' beliefs from an a posteriori viewpoint, while our approach in this paper models players' cautious lexicographic beliefs, which can be used in all possible contingencies, from an a priori viewpoint; cf. Feinberg (2005a, Sec. 1) for more discussions. Feinberg (2005b) also studied, in such an epistemic framework, various solution concepts based on the subjective reasoning of players about hypothetical events in dynamic games.

3 An Epistemic Characterization of RSCE

3.1 Introduction

In extensive games, Fudenberg and Levine (1993) presented a solution concept of "self-confirming equilibrium (SCE)" which arises as a steady state where players correctly predict the moves their opponents actually make, but may have misconceptions about what their opponents would do at information sets that are never reached when the equilibrium is played. That is, the notion of SCE is designed to model situations where players have no *a priori* information about opponents' play or payoffs and, when each time the game is played, they observe only the actions actually played by their opponents along the equilibrium path; cf. also Fudenberg and Kreps (1995) and Fudenberg and Levine (2006, 2009). A particular and noteworthy feature of SCE is that beliefs about off-path play are completely arbitrary so that players may hold false and inconsistent belief about off-path play; in particular, the notion of SCE allows players to use a "noncredible" threats in beliefs about off-path play (see Dekel *et al.* (1999, Fig. 2.1)). If, however, players can use information about opponents' payoffs and think strategically, players should be able to deduce and make use of information about opponents' payoff functions and, thus, can alleviate inconsistency in players' beliefs about off-path play. To fulfil this purpose, by using Bernheim's (1986) and Pearce's (1984) idea of rationalizability, Dekel *et al.* (1999, 2001) provided a solution concept of "rationalizable self-confirming equilibrium (RSCE)" which refines SCE by requiring a player's rationality at the player's information sets that are not precluded by his own strategy. Dekel *et al.* (1999) showed that RSCE is robust to payoff uncertainty in the sense of Fudenberg *et al.* (1988). Dekel *et al.* (1999) also defined a stronger concept of "sequentially rationalizable self-confirming equilibrium (SRSCE)" by requiring a player's rationality at all of the player's information sets, so that the sequential rationalizability notion implies backward induction in finite games of perfect information with generic payoffs; SRSCE is related to Greenberg *et al.*'s (2009) notion of "mutually acceptable course of action (MACA)". Sequential rationalizability is also introduced as a byproduct of SRSCE which refines rationalizability by imposing optimality at every information set.

The purpose of this chapter is to offer a simple epistemic characterization for RSCE. This line of study can help to deepen our understanding of RSCE and other related solution concepts from an epistemic perspective. In doing so, a technical difficulty encountered in dynamic extensive-form game models is, when facing with strategic uncertainty, how to model a player's beliefs about opponents' play in every contingency, including information sets that the player thinks will not actually arise. Inspired by Selten's (1975) idea of "trembles," Dekel *et al.* (2002) defined the "extensive-form convex hull" of a set of behavior strategies to model a player's beliefs about the play of an opponent's strategic behavior in extensive games; cf. also Greenberg *et al.* (2009, pp.95-98) for related discussions. In this chapter, we use the notion of "conditional probability system (CPS)" introduced by Myerson (1986) to represent players' beliefs and provide an epistemic characterization for the solution concept of RSCE. More specifically, each player is assumed to hold an "independent" CPS over on the product of action spaces in the agent-normal form of an extensive game, which is based on the information along the path of play.

Within a standard semantic framework or Aumann's model of knowledge, we formulate and show that RSCE is the logical consequence of mutual knowledge of actions and rationality along the path of play and common knowledge of rationality off the path of play (Theorem 3.3.1.1 and Corollary 3.3.1.1). This result provides a unifying epistemic approach to other related solution concepts such as SCE, SRSCE, sequential rationalizability and sequential equilibrium; we demonstrate, in this chapter, how various epistemic characterizations for related solution concepts can be derived by varying the restrictions of rationality (Corollaries 3.3.2.1, 3.3.3.1, 3.3.3.2, and 3.3.4.1).

The rest of this chapter is organized as follows. Section 3.2 contains some preliminary notation and definitions. Section 3.3 presents a simple epistemic characterization for RSCE and discusses its epistemic relations to other related solution concepts such as SCE, SRSCE, sequential rationalizability and sequential equilibrium.. Section 3.4 offers concluding remarks.

3.2 Notation and Definitions

Since the formal description of an extensive game is by now standard (see, for instance, Kreps and Wilson (1982) and Kuhn (1954)), only the necessary notation is given below. Consider a (finite) extensive-form game with perfect recall:

$$T \equiv (N, W, H, \{A^h\}_{h\in H}, \{u_i\}_{i\in N}),$$

where $N = \{1, 2, \ldots, n\}$ is the (finite) set of players, W is the (finite) set of nodes (or vertices), H is the set of information sets (which is a partition of nonterminal nodes), A^h is the (finite) set of pure actions available at information set h, and u_i is player i's payoff function defined on terminal nodes. A mixed action at information set h is a probability measure on A^h. Denote the set of mixed actions at h by $\triangle A^h$. Denote the collection of player i's information sets by H^i. Denote by $A \equiv \times_{h \in H} A^h$ the set of actions.

A *behavior strategy* of player i is a function, π_i, that assigns some randomization $\pi_i(h) \in \triangle A^h$ to every $h \in H^i$. Let $\mathbf{\Pi}_i$ be the set of player i's behavior strategies. Denote the set of behavior strategy profiles by $\mathbf{\Pi}$, i.e. $\mathbf{\Pi} = \times_{j \in N} \mathbf{\Pi}_j$. For $\pi \in \mathbf{\Pi}$, we denote by $u_i(\pi)$ player i's (expected) payoff if strategy profile π is adopted from the root of the game. For $\pi \in \mathbf{\Pi}$, we denote by $\pi(h)$ the mixed action of π at h, and denote by $\pi(-h)$ the profile of mixed actions of π at all information sets other than h. Given $\pi \in \mathbf{\Pi}$, let H_π be the set of information sets reached with positive probability under π. Denote by $H^i_\pi = H_\pi \cap H^i$ the set of player i's information sets reached by π and $H^i_{\pi_i} = \cup_{\pi_{-i} \in \mathbf{\Pi}_{-i}} H^i_{(\pi_i, \pi_{-i})}$ the set of player i's information sets that are reachable under π_i.

Write $\pi^k_i \rightsquigarrow \pi_i$ for the "trembling" sequence $\left\{\pi^k_i\right\}^\infty_{k=1}$ of strictly positive behavior strategies in $\mathbf{\Pi}_i$ that converges to π_i.

3.2.1 RSCE: A Definition

Dekel *et al.* (1999) proposed a solution concept of "rationalizable self-confirming equilibrium (RSCE)" for extensive games where players learn the path of the play and incorporate the information of opponents' payoffs into the original notion of SCE. Following Dekel *et al.* (1999), an *assessment* η_i for player i is a function that assigns a probability measure over the nodes at each of his own information sets. A *belief* of player i is a pair (η_i, π^i_{-i}) where η_i is player i's assessment and $\pi^i_{-i} = (\pi^i_j)_{j \neq i}$ represents player i's conjecture about opponents' strategies. A *version* of player i is a strategy-belief pair $v_i = (\pi_i, (\eta_i, \pi^i_{-i}))$. Given a version $v_i = (\pi_i, (\eta_i, \pi^i_{-i}))$, $\pi_i(h)$ is a *best response with respect to* $(\pi_i, (\eta_i, \pi^i_{-i}))$ *at* $h \in H^i$ if

$$u_i\left(\pi_i, \pi^i_{-i} | h, \eta_i(h)\right) \geq u_i\left(a^h, \left(\pi_i, \pi^i_{-i}\right)(-h) | h, \eta_i(h)\right) \ \forall a^h \in A^h$$

where $u_i\left(\pi|h, \eta_i\left(h\right)\right)$ represents player i's conditional expected payoff given that information set h is reached, that player i's assessment is given by $\eta_i(h)$, and that the strategy profile is π.

A version $v_i = (\pi_i,\, (\eta_i,\, \pi^i_{-i}))$ is *consistent* (Kreps and Wilson (1982)) if $\eta_{i,k} \to \eta_i$ where $\eta_{i,k}$ is obtained using Bayes rule from a trembling sequence $\pi^i_{-i,k} \rightsquigarrow \pi^i_{-i}$. A *belief model* $V = (V_1,\, V_2,\, ..., V_n)$ where V_i is the set of consistent versions for player i.

A strategy π_i of player i is in the *extensive-form convex hull of a subset* $\Pi_i \subseteq \mathbf{\Pi}_i$ (Dekel *et al.* (2002)), denote by $co^e\left(\Pi_i\right)$, if there is an integer m, strategies $\{\pi_{i,t}\}_{t=1,\,...,\,m}$ in Π_i, sequences of strictly positive behavior strategies $\pi_{i,\,t,\,k} \rightsquigarrow \pi_{i,t}$, and a sequence $\alpha_k \to \alpha$ of probability distributions on $[1, ..., \, m]$, such that the behavior strategies $\pi_{i,\,k}$, which is outcome-equivalent to convex combination $\sum_{t=1}^{m} \alpha_{t,k}\pi_{i,\,t,\,k}$, converges to π_i (in this situation we denote by $\pi_{i,\,k} \rightsquigarrow \pi_i \in co^e\left(\Pi_i\right)$).

Dekel *et al.* (1999, 2002) defined SCE, RSCE and SRSCE as strategy profiles. Since only the path of play is essential in these notions, we give the following alternative definition in terms of paths of play.

Definition 3.2.1. (Dekel *et al.* 1999, 2002). Let $\widehat{\pi}$ be a path of play. Given a belief model $V = (V_1,\, V_2,\, ..., V_n)$, for every player $i \in N$ and every $\left(\pi_i, \left(\eta_i, \pi^i_{-i}\right)\right) \in V_i$, we consider the following conditions for V:

(1) $\forall h \in H^i_{\pi_i}$, $\pi_i\left(h\right)$ is a best response with respect to (η_i, π^i_{-i}).

(1') $\forall h \in H^i_{(\pi_i, \pi^i_{-i})}$, $\pi_i\left(h\right)$ is a best response with respect to (η_i, π^i_{-i}).

(1") $\forall h \in H^i$, $\pi_i\left(h\right)$ is a best response with respect to (η_i, π^i_{-i}).

(2) the path of play resulting from (π_i, π^i_{-i}) is $\widehat{\pi}$.

(3) $\forall j \neq i$, $\pi^i_j \in co^e\left(\Pi^V_j\right)$ where $\Pi^V_j = \{\pi_j : (\pi_j, (\eta_j, \pi^j_{-j})) \in V_j$ for some belief $(\eta_j, \pi^j_{-j})\}$.

The path $\widehat{\pi}$ is a *rationalizable self-confirming equilibrium (RSCE)* if there is a belief model V satisfying (1), (2) and (3), $\widehat{\pi}$ is a *self-confirming equilibrium (SCE)* if there is a belief model V satisfying (1'), (2) and (3), and $\widehat{\pi}$ is a *sequential rationalizable self-confirming equilibrium (SRSCE)* if there is a belief model V satisfying (1"), (2) and (3).

Dekel *et al.* (1999, p.173) demonstrated, through the example of Selten's Horse, that arbitrary and heterogeneous false beliefs about off-path play can lead to non-Nash outcomes: SCE, RSCE and SRSCE all can arrive at a steady state that cannot arise in Nash equilibrium. Dekel *et al.* (1999, Sec. 4) showed that the notion of RSCE is not robust to the presence of a small amount of payoff uncertainty in the sense of Fudenberg *et al.* (1988). The following example illustrates the differences in SCE, RSCE and SRSCE.

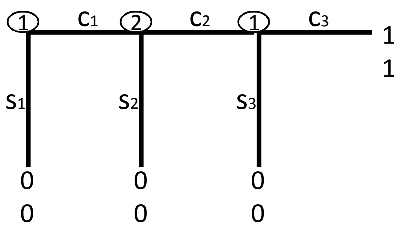


Fig. 1

In this game depicted in Fig. 1, it is easy to verify that the path outcomes of SCE, RSCE and SRSCE are as follows:

SCE: s_1; c_1s_2; $c_1c_2c_3$

RSCE: s_1; $c_1c_2c_3$

SRSCE: $c_1c_2c_3$

For instance, while the path s_1 can arise in RSCE by using a "rationalizable" belief that player 2 will play s_2 with probability 1 (since player 1's second decision node is precluded by his strategy and, thus, there is no rationality requirement for player 1 at this decision node), the path c_1s_2 cannot arise in RSCE (since player 1's second decision node is not precluded by his strategy in this case and, thus, the rationality at this decision node requires player 1's choice to be c_3). In particular, SRSCE yields the unique backward induction outcome: $c_1c_2c_3$.

3.2.2 CPS in Extensive Games

In this chapter, we consider the "conditional probability system (CPS)" on the space, $A = \times_{h\in H}A^h$, of action profiles in the agent-normal form of T. Accordingly, a CPS on A can be viewed as a conditional-probability function which define a probability distribution on agents' actions at every information set, including those are not reached. Formally, a CPS $\mu|$ on A is a function that specifies, for every nonempty subset $B \subseteq A$, a conditional probability distribution $\mu|_B$ given B and satisfies the property:

$$\mu|_B(D) = \mu|_C(D)\mu|_B(C) \text{ for } D \subseteq C \subseteq B \text{ and } C \neq \emptyset.$$

See, *e.g.*, Myerson (1991, Sec. 1.6). Denote by

$$A(h) \equiv \{a \in A : a \text{ reaches } h\}$$

the set of action profiles by each of which h can be reached. For $i \in N$ and $h \in H^i$, $a^h \in A^h$ is a *best response with respect to a CPS* $\mu|$ *on* A if

$$\sum_{a^{-h} \in A^{-h}} \mu|_{A(h)}^{-h}(a^{-h}) u_i(a^h, a^{-h}) \geq \sum_{a^{-h} \in A^{-h}} \mu|_{A(h)}^{-h}(a^{-h}) u_i(b^h, a^{-h}) \ \ \forall b^h \in A^h$$

where $\mu|_{A(h)}^{-h}$ is the marginal of $\mu\,|_{A(h)}$ on A^{-h},[21] which specifies the agent h's belief about opponents' choices given that information set h is reached.

By Myerson's (1986) Theorem 1, a CPS on a (finite) state space can be expressed by a convergent sequence of "full-support" probability distributions over the state space. A CPS $\mu|$ on A is *associated with a probability distribution* p (on A), denoted by $\mu|_{[p_k \rightsquigarrow p]}$, if there exists a sequence of probability distributions $p_k \to p$ such that:

(i) For $k = 1, 2, ...$ and every $a \in A$, $p_k(a) > 0$;

(ii) For any $B,\ C \subseteq A$ with $B \neq \emptyset$, $\mu|_B(C) = \lim_{k \to \infty} \frac{p_k(B \cap C)}{p_k(B)}$.

For the purpose of this chapter, we say "a CPS $\mu|$ on A is independent" if $\mu| = \mu|_{[p_k \rightsquigarrow p]}$ where p_k are product measures on the (product) space A; cf., *e.g.*, McLennan (1989) for more discussions.

The following lemma is an immediate implication of Myerson's (1986) Theorem 1, which states a relationship between "sequential rationality" and "conditionally preference ordering by CPS."

[21] The marginal of $\mu|_{A(h)}$ on A^{-h} is defined as probability measure on A^{-h} such that

$$\forall a^{-h} \in A^{-h},\ \mu|_{A(h)}^{-h}\left(a^{-h}\right) \equiv \sum_{a^h \in A^h} \mu|_{A(h)}\left(a^h,\, a^{-h}\right).$$

Lemma 3.2.1. *Let $\pi_{j,k} \rightsquigarrow \pi_j$ $\forall j \in N$. For all $h \in H^i$, $\pi_i(h)$ is a best response with respect to a consistent version $(\pi_i, (\eta_i, \pi_{-i}))$ with $\pi_{j,k} \rightsquigarrow \pi_j$ $\forall j \neq i$ if, and only if, $\pi_i(h)$ is preferred to a^h with respect to $\mu|_{[\pi_k \rightsquigarrow \pi]}$ for all $a^h \in A^h$.*

For any subset $\Pi \subseteq \mathbf{\Pi}$, let

$$co^e(\Pi) = \times_{j \in N} co^e(\Pi_j),$$

where $\Pi_j = \{\pi_j : (\pi_j, \pi_{-j}) \in \Pi\}$. Written $\pi_k \rightsquigarrow \pi \in co^e(\Pi)$ for "$\pi_{j,k} \rightsquigarrow \pi_j \in co^e(\Pi_j)$ $\forall j \in N$." Define

$$ICPS^e(\Pi) \equiv \left\{\mu| \; : \; \mu| = \mu|_{[\pi_k \rightsquigarrow \pi]} \text{ for some } \pi_k \rightsquigarrow \pi \text{ in } co^e(\Pi)\right\}.$$

That is, $ICPS^e(\Pi)$ is the set of all independent CPS on A that can be generated by $\pi \in co^e(\Pi)$.

3.3 Epistemic Characterization of RSCE

Following Aumann (1976, 1987, 1995, and 1999), we provide, within the standard partition model, epistemic conditions for RSCE by common knowledge of "rationality" and mutual knowledge of the equilibrium path. A model of knowledge for game T is given by

$$\mathcal{M}(T) = < \Omega, \{P_i\}_{i \in N}, \{\boldsymbol{\pi}_i\}_{i \in N}, \{\boldsymbol{\mu}_i|\}_{i \in N} >,$$

where

Ω is the set of states

$P_i(\omega)$ is player i's information partition at ω

$\boldsymbol{\pi}_i(\omega)$ is player i's behavior strategy at ω

$\boldsymbol{\mu}_i|(\omega)$ is player i's conditional belief systems at ω

We refer to a subset $E \subseteq \Omega$ as an *event*. For an event $E \subseteq \Omega$, we take the following standard definitions.

- $K_i E \equiv \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$ is the event that i *knows* E.

- $KE \equiv \cap_{i\in N} K_i E$ is the event that E is *mutually known*.
- $CKE \equiv KE \cap KKE \cap KKKE \cap \cdots$ is the event that E *is commonly known*.

For $E \subseteq \Omega$, we denote by

$$\boldsymbol{\pi}(E) \equiv \{\boldsymbol{\pi}(\omega) :\ \omega \in E\}.$$

Throughout this chapter, we assume $\boldsymbol{\pi}_i(\cdot)$ is measurable w.r.t. information partition P_i – i.e. $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega')\ \forall \omega' \in P_i(\omega)$.

Agent $h \in H^i$ is *rational at* ω if we have $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P_i(\omega)))$ and $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|(\omega)$. For every $i \in N$ and every $h \in H^i$, denote by

$$\mathring{R}^h \equiv \left\{\omega : \text{agent } h \text{ is rational at } \omega \text{ if } h \in H^i_{\boldsymbol{\pi}_i(\omega)}\right\},$$

i.e., $\mathring{R}^h$ represents the event that agent h is robust-rational whenever information set h is not excluded by his strategy choice. (Apparently, $\omega \in \mathring{R}^h$ if $h \notin H^i_{\boldsymbol{\pi}_i(\omega)}$) For any given path of play $\hat{\pi}$, let

$$\mathring{R}^{\hat{\pi}} \equiv \cap_{h\in H_{\hat{\pi}}} \mathring{R}^h \text{ and } \mathring{R}^{-\hat{\pi}} \equiv \cap_{h\notin H_{\hat{\pi}}} \mathring{R}^h,$$

where $H_{\hat{\pi}}$ is the information sets reached by $\hat{\pi}$. That is, $\mathring{R}^{\hat{\pi}}$ is the event that players are robust-rational at the information sets along the path $\hat{\pi}$ and $\mathring{R}^{-\hat{\pi}}$ is the event that players are robust-rational at the off-path information sets.

The *path of play* under π can be viewed as the restriction of π to reached information sets:

$$\widehat{\pi} = \times_{h\in H_\pi} \pi(h).$$

Denote by $\hat{\boldsymbol{\pi}}$ the restriction of $\boldsymbol{\pi}$ to $H_{\hat{\pi}}$, i.e., $\hat{\boldsymbol{\pi}}(\omega)= \boldsymbol{\pi}|_{H_{\hat{\pi}}}(\omega)\ \forall \omega \in \Omega$. Let

$$[\hat{\pi}] \equiv \{\omega : \hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}\},$$

i.e., $[\hat{\pi}]$ is the event that the path of play is $\hat{\pi}$.

3.3.1 Rationalizable Self-Confirming Equilibrium

We are now in a position to present the central result of this chapter which provides a simple epistemic characterization for the notion of RSCE. Theorem 3.3.1.1 states that mutual knowledge of a path of play, robust-rationality along the information sets prescribed by the path, and common knowledge of robust-rationality at off-path information sets, imply an RSCE. Conversely, any RSCE can be attained by the aforementioned epistemic assumptions.

Theorem 3.3.1.1. *(a) Let* $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$*. Then,* $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ *is an RSCE. (b) Let* $\hat{\pi}$ *be an RSCE. Then, there is a knowledge model* $\mathcal{M}(T)$ *such that* $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ *for all* $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$.

Proof. (a) For any $i \in N$, define

$$\Pi_i^V = \{\boldsymbol{\pi}_i(\omega) : \omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}\},$$

and let $\Pi^V \equiv \times_{i \in N} \Pi_i^V$.

Clearly, if $h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$ for all $\pi \in \Pi^V$. That is, for all $\pi \in \Pi^V$, π induces the same distribution over outcomes as $\hat{\pi}$.

(i) For any $i \in N$, $\pi_i \in \Pi_i^V$, there exists $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$ such that $\boldsymbol{\pi}_i(\omega) = \pi_i$. Since $\omega \in \mathring{R}^{\hat{\pi}} \cap CK\mathring{R}^{-\hat{\pi}}$, $\omega \in \mathring{R}$. Therefore, $\forall i \in N$ there is $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P^i(\omega))$ such that $\forall h \in H_{\boldsymbol{\pi}_i(\omega)}$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$.

(ii) Since $\omega \in K_i[\hat{\pi}]$, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \hat{\pi}(h)$ for all $h \in H_{\hat{\pi}}$. That is, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \boldsymbol{\pi}(\omega)(h)$ for all $h \in H^{\hat{\pi}}$.

If $h \notin H_{\hat{\pi}}$, then $\forall \omega' \in P_i(\omega)$,

$$\begin{aligned} \boldsymbol{\pi}(\omega')(h) &\in \{\boldsymbol{\pi}(\omega'')(h) : \omega'' \in CK\mathring{R}^{-\hat{\pi}}\} \text{ (since } P_i(\omega) \subseteq CK\mathring{R}^{-\hat{\pi}}) \\ &= \{\boldsymbol{\pi}(\omega'')(h) : \omega'' \in CK\mathring{R}^{-\hat{\pi}} \cap (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}})\}. \end{aligned}$$

Therefore, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \Pi^V$. Since $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega')\ \forall \omega' \in P_i(\omega)$, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \{\pi_i\} \times \Pi_{-i}^V$ for all $\omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$.

By (i) and (ii), it follows that for every $i \in N$ and $\pi_i \in \Pi_i^V$, there is a $\mu_i| \in ICPS^e(\Pi^V)$ such that for every $h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to $\mu_i|_{A(h)}$. Thus, there exists $\pi_k \rightsquigarrow \pi \in co^e\left(\Pi^V\right)$ such that $\mu_i| = \mu|_{[\pi_k \rightsquigarrow \pi]}$, and $\forall h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$. By Lemma 3.2.1,

$\forall i \in N$ and $\pi_i \in \Pi_i^V$, there exists (η_i, π_{-i}^i), which is consistent with $\pi_k \rightsquigarrow \pi$, such that $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is best response with respect to (η_i, π_{-i}^i). As $\forall j \neq i$, $\pi_{j,k} \rightsquigarrow \pi_j \in co^e\left(\Pi_j^V\right)$ and $\pi_j = \pi_j^i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$. That is, $\forall h \in H_{\hat{\pi}}$ $\left(\pi_i, \pi_{-i}^i\right)(h) = \hat{\pi}(h)$.

For all $\forall i \in N$, let

$$V_i \equiv \left\{ (\boldsymbol{\pi}_i(\omega), (\eta_i, \pi_{-i}^i)) \; : \; \begin{array}{c} (\eta_i, \pi_{-i}^i) \text{ is consistent with } \pi_k \rightsquigarrow \pi \\ \text{where } \mu|_{[\pi_k \rightsquigarrow \pi]} = \boldsymbol{\mu}_i|(\omega) \\ \text{and } \omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}} \end{array} \right\},$$

and $V \equiv (V_1, V_2, ..., V_n)$. Then, for all $i \in N$ and $\left(\pi_i, \left(\eta_i, \pi_{-i}^i\right)\right) \in V_i$, we have

(1) $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to (η_i, π_{-i}^i).

(2) $\left(\pi_i, \pi_{-i}^i\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi_j^i \in co^e\left(\Pi_j^V\right)$ where $\Pi_j^V = \{\pi_j' : (\pi_j', (\eta_i, \pi_{-j}^j)) \in V_j$ for some belief $(\eta_i, \pi_{-j}^j)\}$.

That is, $\forall \omega \in (K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \cap CK\mathring{R}^{-\hat{\pi}}$, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ and $\hat{\pi}$ is an RSCE.

(b)Let $\hat{\pi}$ be an RSCE that is supported by $V = (V_1, V_2, ..., V_n)$.

We proceed to show a stronger result that there is $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in CK([\hat{\pi}] \cap \mathring{R}) \neq \emptyset$. For all $i \in N$, for every $(\pi_i, (\eta_i, \pi_{-i}^i)) \in V_i$,

(1) $\forall h \in H_{\pi_i}$, $\pi_i(h)$ is a best response with respect to (η_i, π_{-i}^i).

(2) $\left(\pi_i, \pi_{-i}^i\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi_j^i \in co^e\left(\Pi_j^V\right)$ where $\Pi_j^V = \{\pi_j' : (\pi_j', (\eta_i, \pi_{-j}^j)) \in V_j$ for some belief $(\eta_i, \pi_{-j}^j)\}$.

Let $\mu_i|(\pi_i) = \mu|_{[\pi_k \rightsquigarrow \pi]}$ such that $\pi_{i,k} \rightsquigarrow \pi_i \in co^e(\{\pi_i\})$, and $\pi_{j,k}^i \rightsquigarrow \pi_j^i \in co^e\left(\Pi_j^V\right)$ $\forall j \neq i$. Clearly, $\mu_i|(\pi_i) \in ICPS^e\left(\{\pi_i\} \times \Pi_{-i}^V\right)$. Define a knowledge model for game T:

$$\mathcal{M}(T) =< \Omega, \{P_i\}_{i \in N}, \{\boldsymbol{\pi}_i\}_{i \in N}, \{\boldsymbol{\mu}_i|\}_{i \in N} >,$$

such that $\Omega = \left\{\left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N} : \pi_j \in \Pi_j^V, \forall j \in N\right\}$ and for all $i \in N$ and $\omega = \left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N}$ in Ω,

$$\boldsymbol{\pi}_i(\omega) = \pi_i, \; \boldsymbol{\mu}_i|(\omega) = \mu_i|(\pi_i) \text{ and}$$

$$P_i(\omega) = \{\omega' \in \Omega : \boldsymbol{\pi}_i(\omega') = \pi_i \text{ and } \boldsymbol{\mu}_i|(\omega') = \mu_i|(\pi_i)\}.$$

Now, consider any arbitrary $\omega = \left(\pi_j, \mu_j|\left(\pi_j\right)\right)_{j\in N} \in \Omega$. By Lemma 3.2.1, it follows that for all $i \in N$ and $h \in H_{\pi_i}$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$. Since $\mu_i|\left(\pi_i\right) \in ICPS^e\left(\{\pi_i\} \times \Pi^V_{-i}\right)$, $\boldsymbol{\mu}_i|(\omega) \in ICPS^e\left(\boldsymbol{\pi}\left(P_i(\omega)\right)\right) \forall i \in N$. Therefore, $\omega \in \mathring{R}$. But, since $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$, $\omega \in [\hat{\pi}]$. Therefore, $\Omega = \mathring{R} \cap [\hat{\pi}]$ and, hence, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in CK([\hat{\pi}] \cap \mathring{R}) = \Omega$. ■

An immediate corollary of Theorem 3.3.1.1 gives a more readily expressible and readable form of epistemic assumptions of RSCE: The notion of RSCE can be viewed as the logical consequence of common knowledge of robust-rationality plus mutual knowledge of a path of play.

Corollary 3.3.1.1. *Let $\mathring{R}^i \equiv \cap_{h\in H^i}\mathring{R}^h$ and $\mathring{R} \equiv \cap_{i\in N}\mathring{R}^i$. (a) Let $\omega \in K[\hat{\pi}] \cap CK\mathring{R}$. Then, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ is an RSCE. (b) Let $\hat{\pi}$ be an RSCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in K[\hat{\pi}] \cap CK\mathring{R}$.*

Proof. Since $CK\mathring{R} \subseteq \mathring{R}^{\hat{\pi}} \cap CK\mathring{R}^{-\hat{\pi}}$, Corollary 3.3.1.1(a) follows directly from Theorem 3.3.1.1(a). Corollary 3.3.1.1(b) follows from the proof of Theorem 3.3.1.1(b). ■

This theorem says that mutual knowledge of the on-path actions, robust-rationality along on-path information sets, and common knowledge of robust-rationality at off-path information sets lead to "rationalizable self-confirming equilibrium (RSCE)." The "robust-rationality" is defined only at reachable information sets, rather than at all information sets. In particular, this "rationality" at off-path information sets does require that each player be optimal at all these information sets, but it requires only that each player be optimal at the information sets that are not precluded by the player's strategy at the state. The epistemic assumption of "common knowledge of robust-rationality at off-path information sets" can be justified by using the prior payoff information (cf. Dekel *et al.* (1999)).

3.3.2 Self-Confirming Equilibrium

In Fudenberg and Levine (1993) and Fudenberg and Kreps (1995), players are assumed to have no a priori information about each others' payoffs, and only observe the actions chosen by their opponents. In such an environment, Fudenberg and Kreps (1995) proposed the solution concept of "self-confirming equilibrium (SCE)" in which players' behavior is required to be optimal only at the observed information sets and players' behavior at off the equilibrium path information sets imposes no requirement of rationality. Without imposing any "rationality" restriction on the off-path behavior, we obtain an epistemic characterization for SCE as a corollary of Theorem 3.3.1.1.

Corollary 3.3.2.1. *(a) Let $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$. Then, $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ is an SCE. (b) Let $\hat{\pi}$ be an SCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$.*

Proof. (a) For any $i \in N$, define

$$\Pi_i^V = \{\boldsymbol{\pi}_i(\omega) : \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}\}$$

and let $\Pi^V \equiv \times_{i \in N} \Pi_i^V$.

Clearly, if $h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$ for all $\pi \in \Pi^V$. That is, for all $\pi \in \Pi^V$, π has the same distribution over outcomes as induced by $\hat{\pi}$.

(i) For any $i \in N$, $\pi_i \in \Pi_i^V$, there exists $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$ such that $\boldsymbol{\pi}_i(\omega) = \pi_i$. Since $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\omega \in \mathring{R}^{\hat{\pi}}$. Therefore, $\forall i \in N$ there is $\boldsymbol{\mu}_i|(\omega) \in ICPS^e(\boldsymbol{\pi}(P^i(\omega))$ such that $\forall h \in H_{\hat{\pi}} \cap H^i$, $\boldsymbol{\pi}_i(\omega)(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}(\omega)$.

(ii) Since $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\omega \in K_i[\hat{\pi}] \subseteq [\hat{\pi}]$. Then, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \hat{\pi}(h)$ for all $h \in H_{\hat{\pi}}$. That is, for all $\omega' \in P_i(\omega)$, $\boldsymbol{\pi}(\omega')(h) = \boldsymbol{\pi}(\omega)(h)$ for all $h \in H_{\hat{\pi}}$. Therefore, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \Pi^V$. Since $\boldsymbol{\pi}_i(\omega) = \boldsymbol{\pi}_i(\omega')$ $\forall \omega' \in P_i(\omega)$, $\boldsymbol{\pi}(P_i(\omega)) \subseteq \{\pi_i\} \times \Pi_{-i}^V$ for all $\omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$.

By (i) and (ii), it follows that for every $i \in N$ and $\pi_i \in \Pi_i^V$, there is a $\mu_i \in ICPS^e(\Pi^V)$ such that for all $h \in H_{\hat{\pi}} \cap H^i$, $\pi_i(h)$ is best response with respect to μ_i. Thus, there exists $\pi_k \rightsquigarrow \pi \in co^e\left(\Pi^V\right)$ such that $\mu_i| = \mu|_{[\pi_k \rightsquigarrow \pi]}$, and $\forall h \in H_{\hat{\pi}}$, $\pi(h) = \hat{\pi}(h)$. By Lemma 3.2.1, $\forall i \in N$ and $\pi_i \in \Pi_i^V$, there exists (η_i, π_{-i}^i), which is consistent with $\pi_k \rightsquigarrow \pi$, such that $\forall h \in H_{(\pi_i, \pi_{-i}^i)} \cap H^i$, $\pi_i(h)$ is best response with respect to (η_i, π_{-i}^i). As $\forall j \neq i$, $\pi_{j,k} \rightsquigarrow \pi_j \in co^e\left(\Pi_j^V\right)$ and $\pi_j = \pi_j^i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$. That is, $\forall h \in H_{\hat{\pi}}$ $\left(\pi_i, \pi_{-i}^i\right)(h) = \hat{\pi}(h)$.

For all $i \in N$, let

$$V_i \equiv \left\{ \left(\boldsymbol{\pi}_i(\omega), (\eta_i, \pi_{-i}^i)\right) \; : \; \begin{array}{c} (\eta_i, \pi_{-i}^i) \text{ is consistent with } \pi_k \rightsquigarrow \pi \\ \text{where } \mu|_{[\pi_k \rightsquigarrow \pi]} = \boldsymbol{\mu}_i|(\omega) \\ \text{and } \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}} \end{array} \right\},$$

and $V \equiv (V_1,\, V_2,\, ..., V_n)$. Then, for all $i \in N$ and $\left(\pi_i, \left(\eta_i, \pi^i_{-i}\right)\right) \in V_i$, we have

(1') $\forall h \in H_{(\pi_i, \pi^i_{-i})} \cap H^i$, $\pi_i\,(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e\left(\Pi^V_j\right)$ where $\Pi^V_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some belief $(\eta_i, \pi^j_{-j})\}$.

That is, $\forall \omega \in K[\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}$, $\boldsymbol{\pi}\,(\omega) = \hat{\pi}$ and $\hat{\pi}$ is an SCE.

(b) Let $\hat{\pi}$ be an SCE that is supported by $V = (V_1,\, V_2,\, ..., V_n)$.

We proceed to show a stronger result that there is $\mathcal{M}\,(T)$ such that $\hat{\boldsymbol{\pi}}\,(\omega) = \hat{\pi}$ for all $\omega \in K([\hat{\pi}] \cap \mathring{R}^{\hat{\pi}}) \neq \emptyset$. For all $i \in N$, for every $(\pi_i, (\eta_i, \pi^i_{-i})) \in V_i$,

(1') $\forall h \in H_{(\pi_i, \pi^i_{-i})} \cap H^i$, $\pi_i\,(h)$ is a best response with respect to (η_i, π^i_{-i}).

(2) $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$.

(3) $\forall j \neq i$, there exists $\pi^i_j \in co^e\left(\Pi^V_j\right)$ where $\Pi^V_j = \{\pi'_j : (\pi'_j, (\eta_i, \pi^j_{-j})) \in V_j$ for some belief $(\eta_i, \pi^j_{-j})\}$.

Let $\mu_i|\,(\pi_i) = \mu|_{[\pi_k \rightsquigarrow \pi]}$ such that $\pi_{i,k} \rightsquigarrow \pi_i \in co^e\,(\{\pi_i\})$, and $\pi^i_{j,k} \rightsquigarrow \pi^i_j \in co^e\left(\Pi^V_j\right)$ $\forall j \neq i$. Clearly, $\mu_i|\,(\pi_i) \in ICPS^e\left(\{\pi_i\} \times \Pi^V_{-i}\right)$. Define a knowledge model for game T:

$$\mathcal{M}\,(T) =< \Omega,\, \{P_i\}_{i\in N}, \{\boldsymbol{\pi}_i\}_{i\in N}, \{\boldsymbol{\mu}_i|\}_{i\in N} > ,$$

such that $\Omega = \left\{\left(\pi_j, \mu_j|\,(\pi_j)\right)_{j\in N} :\ \pi_j \in \Pi^V_j, \forall j \in N\right\}$ and for all $i \in N$ and $\omega = \left(\pi_j, \mu_j|\,(\pi_j)\right)_{j\in N}$ in Ω,

$$\boldsymbol{\pi}_i\,(\omega) = \pi_i,\ \mu_i|\,(\omega) = \mu_i|\,(\pi_i) \text{ and}$$

$$P_i\,(\omega) = \{\omega' \in \Omega : \boldsymbol{\pi}_i\,(\omega') = \pi_i \text{ and } \boldsymbol{\mu}_i|\,(\omega') = \mu_i|\,(\pi_i)\}.$$

Since $\left(\pi_i, \pi^i_{-i}\right)$ has the distribution over outcomes induced by $\hat{\pi}$ and perfect recall, $\forall h \in H_{\hat{\pi}}$, $\left(\pi_i, \pi^i_{-i}\right)(h) = \hat{\pi}(h)$. Now, consider any arbitrary $\omega = \left(\pi_j, \mu_j|\,(\pi_j)\right)_{j\in N} \in \Omega$. By Lemma 3.2.1, it follows that for all $i \in N$ and $h \in H_{\hat{\pi}} \cap H^i$, $\boldsymbol{\pi}_i\,(\omega)\,(h)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}\,(\omega)$. Since $\mu_i|\,(\pi_i) \in ICPS^e\left(\{\pi_i\} \times \Pi^V_{-i}\right)$, $\boldsymbol{\mu}_i|\,(\omega) \in ICPS^e\,(\boldsymbol{\pi}\,(P_i\,(\omega)))$ $\forall i \in N$. Therefore, $\omega \in \mathring{R}^{\hat{\pi}}$. But, since $\hat{\boldsymbol{\pi}}\,(\omega) = \hat{\pi}$, $\omega \in [\hat{\pi}]$. Therefore, $\Omega = \mathring{R}^{\hat{\pi}} \cap [\hat{\pi}]$ and, hence, $\hat{\boldsymbol{\pi}}\,(\omega) = \hat{\pi}$ for all $\omega \in K([\hat{\pi}] \cap \mathring{R}^{\hat{\pi}})$. ■

3.3.3 Sequential Rationalizable Self-Confirming Equilibrium

As pointed out, Dekel *et al.* (1999) defined RSCE by using robust-rationality. If "rationality" is defined as the conventional (sequential) rationality in the sense of Kreps and Wilson (1982) – i.e., it requires to be sequentially rational at every information set, including those unreachable information sets, we can obtain a stronger version of "sequentially rationalizable self-confirming equilibrium (SRSCE)"; see Dekel *et al.* (1999, Sec. 4). Denoted by

$$R^h \equiv \{\omega : \text{ agent } h \text{ is rational at } \omega\},$$

i.e., player i is (sequential) rational at information set h where $h \in H^i$. For any given path of play $\hat{\pi}$, let

$$R^{\hat{\pi}} \equiv \cap_{h \in H_{\hat{\pi}}} R^h \text{ and } R^{-\hat{\pi}} \equiv \cap_{h \notin H_{\hat{\pi}}} R^h.$$

That is, $R^{\hat{\pi}}$ is the event that players are (sequential) rational along on-path information sets specified by strategy profile $\hat{\pi}$, and $R^{-\hat{\pi}}$ is the event that players are (sequential) rational at off-path information sets.

Corollary 3.3.3.1. *(a) Let $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$. Then, $\boldsymbol{\pi}(\omega) = \hat{\pi}$ is an SRSCE. (b) Let $\hat{\pi}$ be an SRSCE. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\boldsymbol{\pi}(\omega) = \hat{\pi}$ for all $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$.*

Proof. $\mathring{R}^h$ (where $h \notin H_i$) has no restrictions on the behavior at the information set h if it is excluded by player i's strategy choice. Then, $R^h \subseteq \mathring{R}^h$. Corollary 3.3.3.1 follows immediately from Theorem 3.3.1.1. ■

If there is a unique backward induction outcome in a game with perfect information – i.e. a PI game (which is true for "generic" assignments of payoffs to terminal nodes), then SRSCE coincides with the backward induction solution since the "rationality" requirement is strengthened so that each player's strategy is (sequentially) optimal at each of his information sets, including those precluded by the player's own strategy.

Corollary 3.3.3.2. *Suppose a PI game T has a unique backward induction outcome. (a) Let $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$. Then, $\boldsymbol{\pi}(\omega) = \hat{\pi}$ is the backward induction (path) outcome. (b)*

Let $\hat{\pi}$ be the backward induction (path) outcome. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\hat{\boldsymbol{\pi}}(\omega) = \hat{\pi}$ for all $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$.

Proof. Clearly, $\boldsymbol{\pi}(\omega)$ is the unique backwards induction outcome if $\omega \in (K[\hat{\pi}] \cap R^{\hat{\pi}}) \cap CKR^{-\hat{\pi}}$. The result of Corollary 3.3.3.2 follows directly from Corollary 3.3.3.1. ■

3.3.4 Sequential Rationalizability

Sequential rationalizability is also introduced in Dekel *et al.* (1999) which refines rationalizability (Bernheim (1984) and Pearce (1984)) by imposing that players behave rationally at every information set. This is a variant of Definition 4.1 in Dekel *et al.* (1999).

Definition 3.3.4.1. Given a belief model $V = (V_1, V_2, ..., V_n)$, let $\Pi^V = \times_j \Pi_j^V$ where $\Pi_j^V = \{\pi_j : (\pi_j, (\eta_j, \pi_{-j}^j)) \in V_j$ for some belief $(\eta_j, \pi_{-j}^j)\}$. Π^V is sequential rationalizable if for every player $i \in N$ and every $\left(\pi_i, \left(\eta_i, \pi_{-i}^i\right)\right) \in V_i$, we have

(1") $\forall h \in H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π_{-i}^i).

(3) $\forall j \neq i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$.

The following result is in align with Tan and Werlang (1988), and sequential rationalizability is shown to be a logical consequence of common knowledge of (sequential) rationality.

Corollary 3.3.4.1. *(a) Let $\omega \in CKR$. Then, $\boldsymbol{\pi}(\omega)$ is a sequential rationalizable strategy profile. (b) Let π be sequential rationalizable. Then, there is a knowledge model $\mathcal{M}(T)$ such that $\boldsymbol{\pi}(\omega) = \pi$ for all $\omega \in CKR$.*

Proof. (a) Let $\boxed{R}$ denote a "self-evident event in R" – i.e., $\boxed{R} \subseteq R$ and $\boxed{R} \subseteq K_i \boxed{R}\ \forall i \in N$. It is easy to see that $\omega \in CKR$ iff there is $\boxed{R} \ni \omega$; cf., *e.g.*, Aumann (1976). For each $i \in N$ define

$$\Pi_i^V \equiv \left\{\boldsymbol{\pi}_i(\omega') \mid \omega' \in \boxed{R}\right\}.$$

Therefore, for every $\omega' \in \boxed{R}$, $\boldsymbol{\pi}_{-i}(P_i(\omega')) \subseteq \Pi_{-i}^V$. Since $\omega' \in R_i$, $\forall h \in H_i\ \boldsymbol{\pi}_i(\omega')(h)$ is a best response of player i to $\boldsymbol{\mu}_i|(\omega') \in ICPS^e\left(\Pi^V\right)$.

Thus, there exists $\pi_k \rightsquigarrow \pi \in co^e\left(\Pi^V\right)$ such that $\mu_i| = \mu|_{[\pi_k \rightsquigarrow \pi]}$. By Lemma 3.2.1, $\forall i \in N$ and $\pi_i \in \Pi_i^V$, there exists $(\eta_i,\, \pi_{-i}^i)$, which is consistent with $\pi_k \rightsquigarrow \pi$, such that $\forall h \in H_i$, $\pi_i(h)$ is best response with respect to $(\eta_i,\, \pi_{-i}^i)$. As $\forall j \neq i$, $\pi_{j,k} \rightsquigarrow \pi_j \in co^e\left(\Pi_j^V\right)$ and $\pi_j = \pi_j^i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$.

For all $i \in N$, let

$$V_i \equiv \left\{ \left(\boldsymbol{\pi}_i(\omega), (\eta_i,\, \pi_{-i}^i)\right) \; : \; \begin{array}{c} (\eta_i,\, \pi_{-i}^i) \text{ is consistent with } \pi_k \rightsquigarrow \pi \\ \text{where } \mu|_{[\pi_k \rightsquigarrow \pi]} = \boldsymbol{\mu}_i|(\omega) \\ \text{and } \omega \in \boxed{R} \end{array} \right\},$$

and $V \equiv (V_1,\, V_2,\, ...,V_n)$. Then, for all $i \in N$ and $\left(\pi_i, \left(\eta_i, \pi_{-i}^i\right)\right) \in V_i$, we have

(1”) $\forall h \in H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π_{-i}^i).

(3) $\forall j \neq i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$.

That is, $\forall \omega \in CKR$, $\boldsymbol{\pi}(\omega)$ is a sequential rationalizable strategy profile.

(b) Let π is a sequential rationalizable strategy profile that is supported by $V = (V_1,\, V_2,\, ...,V_n)$.

We proceed to show a stronger result that there is $\mathcal{M}(T)$ such that $\boldsymbol{\pi}(\omega) = \pi$ for all $\omega \in CKR \neq \emptyset$. For all $i \in N$, for every $(\pi_i, (\eta_i, \pi_{-i}^i)) \in V_i$,

(1”) $\forall h \in H^i$, $\pi_i(h)$ is a best response with respect to (η_i, π_{-i}^i).

(3) $\forall j \neq i$, $\pi_j^i \in co^e\left(\Pi_j^V\right)$.

Let $\mu_i|(\pi_i) = \mu|_{[\pi_k \rightsquigarrow \pi]}$ such that $\pi_{i,k} \rightsquigarrow \pi_i \in co^e(\{\pi_i\})$, and $\pi_{j,k}^i \rightsquigarrow \pi_j^i \in co^e\left(\Pi_j^V\right)$ $\forall j \neq i$. Clearly, $\mu_i|(\pi_i) \in ICPS^e\left(\{\pi_i\} \times \Pi_{-i}^V\right)$. Define a knowledge model for game T:

$$\mathcal{M}(T) = < \Omega,\, \{P_i\}_{i \in N}, \{\boldsymbol{\pi}_i\}_{i \in N}, \{\boldsymbol{\mu}_i|\}_{i \in N} >,$$

such that $\Omega = \left\{ \left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N} : \; \pi_j \in \Pi_j^V, \forall j \in N \right\}$ and for all $i \in N$ and $\omega = \left(\pi_j, \mu_j|(\pi_j)\right)_{j \in N}$ in Ω,

$$\boldsymbol{\pi}_i(\omega) = \pi_i,\ \mu_i|(\omega) = \mu_i|(\pi_i) \text{ and}$$

$$P_i(\omega) = \{\omega' \in \Omega : \boldsymbol{\pi}_i(\omega') = \pi_i \text{ and } \boldsymbol{\mu}_i|(\omega') = \mu_i|(\pi_i)\}.$$

Now, consider any arbitrary $\omega = \left(\pi_j, \mu_j| \left(\pi_j\right)\right)_{j\in N} \in \Omega$. By Lemma 3.2.1, it follows that for all $i \in N$ and $h \in H_i$, $\boldsymbol{\pi}_i\left(\omega\right)\left(h\right)$ is a best response with respect to $\boldsymbol{\mu}_i|_{A(h)}\left(\omega\right)$. Since $\mu_i| \left(\pi_i\right) \in ICPS^e\left(\{\pi_i\} \times \Pi^V_{-i}\right)$, $\boldsymbol{\mu}_i| \left(\omega\right) \in ICPS^e\left(\boldsymbol{\pi}\left(P_i\left(\omega\right)\right)\right) \forall i \in N$. Therefore, $\omega \in R$. Therefore, $\Omega = R$ and, hence, $\boldsymbol{\pi}\left(\omega\right) = \pi$ for all $\omega \in CKR = \Omega$. ■

3.4 Concluding Remarks

In extensive-form games, Fudenberg and Levine (1993) and Fudenberg and Kreps (1995) presented a solution concept of "self-confirming equilibrium (SCE)" which arise as a steady state where players have no prior information about opponents' payoff functions or strategies, and each player observes only the actions played by opponents at each round of the game. Dekel *et al.* (1999) offered a solution concept of "rationalizable self-confirming equilibrium (RSCE)," where each player observes only the actions played by opponents at each round of the game and behaves rationally at all of his information sets that are not precluded by his own strategy, as a refinement of SCE. In this chapter, we have carried out the epistemic program in game theory to explore epistemic conditions for RSCE.

We have presented a simple epistemic characterization of RSCE. More specifically, by using the notion of "conditional probability system (CPS)" introduced by Myerson (1986), we have defined "rationality" as conditional maximization through CPS beliefs and, within a standard semantic framework, we have formulated and shown that RSCE is the logical consequence of common knowledge of "robust-rationality" and mutual knowledge of actions along the path. This chapter therefore provides an epistemic counterpart of RSCE in terms of what players know and believe about "rationality," actions, information, and knowledge in complex social environments with emerging a commonly observed path.

This chapter provides a unifying epistemic approach to other related game-theoretic solution concepts such as SCE, "sequential rationalizable self-confirming equilibrium (SRSCE).", and sequential rationalizability. In this chapter, we have shown how epistemic characterizations for various related solution concepts can be obtained, in a direct and simple way, by varying the requirements of "rationality," as well as assuming different epistemic conditions to players in the game. For instance, SCE can be formally represented as the result of mutual knowledge of actions along the path and rationality along the path; it coincides with the motivation of SCE where each player's strategy is a best response to his beliefs about the play of his opponents, and each player's beliefs are correct along the equilibrium path of play. The study of this chapter is useful to deepen our understanding of RSCE and related solution concepts in the literature.

We would like to point out that, in this chapter, we define "rationality" as conditional expected maximization by "independent" CPS beliefs. This formalism is used to capture the conventional notion of sequential rationality in Kreps and Wilson (1982). Greenberg *et al.* (2009) presented a unified game-theoretic solution concept of "mutually acceptable course of action (MACA)" suitable for situations where "perfectly" and "cautiously" rational individuals with different beliefs and views of the world agree to a shared course of action. When the underlying course of action is taken as the form of "path of play," MACA delivers a strong perfect-version of SRSCE which can rule out weakly dominated strategies. On the other hand, the "null MACA" can be viewed as a perfect version of sequential rationalizability, called "perfect" rationalizability. Chapter one provided expressible epistemic characterization for MACA by using "lexicographic probability system (Blume *et al.* (1991a,b))."

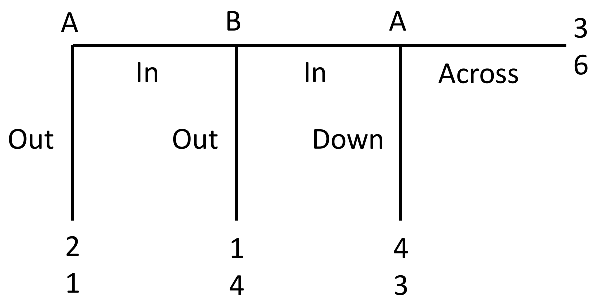
In an interesting and related paper, Asheim and Perea (2005) provided, in two-player extensive games, a different epistemic model for studying both "equilibrium" and "non-equilibrium" solution concepts including "sequential equilibrium/rationalizability" and "quasi-perfect equilibrium/rationalizability (where each player takes into account the possibility of the other players' mistakes, but ignores the possibility of his own mistakes)." By utilizing a more general concept of "conditional lexicographic probability system" to represent a system of conditional beliefs in dynamic settings, Asheim and Perea showed that the concept of "sequential rationalizability" can be characterized by common certain belief of "sequential" rationality, and the concept of "quasi-perfect rationalizability" is the result of common certain belief of "sequential" and "cautious" rationality. As we have emphasized before, the main focus of this chapter is concentrated on presenting a simple epistemic characterization for the notion of RSCE.

4 Backward Induction and Consistent Belief

4.1 Introduction

Backward induction (BI), one of the most classical solution concepts in dynamic games, is still in the center of theoretical analysis today (*e.g.*, Aumann (1995); Ben-Porath (1997); Battigalli and Siniscalchi (2002); Battigalli and Friedenberg (2012); Samet (2013); Bonanno (2013)). Dated back to Zermelo (1913), the existence of optimal pure strategies in chess game was proved. In a perfect information game (*e.g.* chess), the backward induction procedure is simple. At the last decision node, players choose an action that maximizes their payoff. At the second last decision node, players take this as given and choose an action that maximizes their payoff. This procedure continues until the root of the game is reached. The backward induction inspired the development of subgame perfect equilibrium (Selten (1965)) and perfect equilibrium (Selten (1975)).

With the emergence of epistemic game theory, theorists are able to formally analyze players' knowledge, belief, rationality, *etc*. Through the analysis, the hidden assumptions behind solution concepts are uncovered. The backward induction, which seems natural and intuitive, was found to have logical difficulties when theorists try to epistemically characterize it.(see Brandenburger (2007)) To implement the BI procedure, players are required to believe in BI procedure even if there was a clearly observed contradiction of the procedure. Consider the following centipede game (see example 8.1 in Brandenburger (2007)).



Example 4.1

According to backward induction procedure, Ann chooses *Down* at the last stage. Taking this as given, Bob will choose *Out* at the second stage. Based on these analysis, Ann chooses *Out* at the beginning of the game. Let us look at the logic behind it carefully. Ann chooses *Out* at the root of game because she believes that Bob would choose *Out* at the second stage. Ann's belief in Bob's choice, *Out*, is deduced from her belief that Bob thinks Ann would choose the optimal choice, *Down*, at the last stage. Note that there would be a violation of backward induction when it is Bob's turn to make decision as the game *should* stop at the first stage under the BI procedure. In other words, Ann believes that Bob believes that she would follow backward induction at the last stage given his observation of her violation of backward induction at the first stage.

The purpose of this chapter is to provide the epistemic characterization of backward induction strategy profiles with Bayesian updating belief in a perfect information generic game. It provides an explanation of Bob's ignorance of observed violation in the previous example. The major difficulty that we encounter is how to model players' subjective uncertainty at every information set, particularly at those unreached. In this chapter, we use the notion of "conditional probability system (CPS)" (Myerson (1986)) to model players' beliefs and provide an epistemic characterization for the solution concept of the BI strategy profile. More specifically, we define the notion of "consistent belief" by using CPS with strong independence property. Within a standard semantic framework, we formulate and show that BI strategy profile is the logical consequence of rationality and common consistent belief of rationality (RCCBR) in perfect information generic game. It is important to obtain the backwards induction strategy profile instead of only the outcome as strategy profiles are the focus of game theory analysis.

In the pioneer work, Aumann (1995, 1996), the epistemic condition of backward induction was investigated under the knowledge model, where the backward induction outcome is shown to be the logical consequence of common knowledge of rationality. In Aumann's framework, players take the "from that point on" view, *e.g.* player acts as if that information set is reached when deciding what to do at an information set. Common knowledge of rationality implies that at each decision node players ignore the irrationality at previous stage, and believe that opponents would behave rationally at the subsequent game.

The major critic of Aumann's result is that there is no belief revision (Samet (1996), Halpern (2001)). The concept of "initial belief"[22] (Ben-Porath (1997)) was developed to model players' belief and revision process. In Ben-Porath's model, players have a conditional belief system that specifies players' belief at every information set. If a player initially believes an event, he would assign probability one to this event at the beginning of game, under the conditional

[22]The original term is common certainty of rationality. We adopt the terminology in Brandenburg (2007) for convenience of comparison with the notion of "strong belief" (Battigalli and Siniscalchi (1999,2002)).

belief system. Moreover, if there is no contradiction of the initial belief at the subsequent information sets, players update their original belief at these information sets based on Bayes' rule. Otherwise, players may revise their belief arbitrarily. In perfect information generic games, DF procedure (Dekel and Fudenberg (1990)), one round of elimination of weakly dominated strategies followed by iterative elimination of strictly dominated strategies, was shown to be led by rationality and common initial belief of rationality (RCIBR) *e.g.* each player initially believes that each player is rational, each player initially believes that each player initially believes that each player is rational, and so on. Clearly, DF procedure may result in more than backward induction outcomes, *e.g.*, (ID, I) in the Example 4.1.

Battigalli and Siniscalchi (1999, 2002) strengthened the definition of "initial belief" and introduced the concept of "strong belief". At those information sets contradicting with initial belief, players employ forward reasoning to revise their belief. They try to explain the status quo (contradiction in the previous stage) by players' alternatively rational behavior. Rationality and common strong belief of rationality (RCSBR) is shown to be equivalent to extensive-form rationalizability (EFR), due to Pearce (1984) and Battigalli (1997), which coincides with the backward induction outcome in perfect information generic games.

However, RCSBR may result in non-SPE strategy profile. The difficulty rises at those information sets that totally falsify the rationality assumption. In this chapter, we provide an alternative notion of belief operator "consistent belief" and investigate the properties of it (Lemma 4.3.2.1, Corollary 4.3.2.1 and Proposition 4.3.2.1). Moreover, we show that backward induction strategy profile is the logical consequence of rationality and common consistent of rationality in perfect information generic games (Theorem 4.4.2).

The rest of this chapter is organized as follows. An example is demonstrated in Section 4.2. The framework is introduced in Section 4.3. Section 4.4 provides the formal definition of consistent belief and an epistemic characterization for backward induction strategy profile. Discussion and comments on the related literature (Aumann's model, initial belief, strong belief, *etc.*) are presented in Section 4.5. Section 4.6 concludes.

4.2 Example and CPS

In this section, we use an example to illustrate how to use CPS in our epistemic analysis. Consider the extensive game in Example 4.1. Name the first information set as h_1, the second one as h_2, and the last one as h_3. Let $T_a = \{t_1^a, t_2^a\}$ and $T_b = \left\{t_1^b, t_2^b\right\}$ be type spaces for Ann and Bob respectively. Each type of Ann/Bob induces a conditional probability system like the following:

$$
\begin{aligned}
\lambda_a\left(t_1^a\right)|_{A\times T} &= 1\circ\left(OD,t_1^a,O,t_1^b\right) \text{ and } \lambda_a\left(t_1^a\right)|_{A(h_3)\times T}=1\circ\left(ID,t_1^a,I,t_2^b\right);\\
\lambda_a\left(t_2^a\right)|_{A\times T} &= 1\circ\left(ID,t_2^a,I,t_2^b\right) \text{ and } \lambda_a\left(t_2^a\right)|_{A(h_3)\times T}=1\circ\left(ID,t_2^a,I,t_2^b\right);\\
\lambda_b\left(t_1^b\right)|_{A\times T} &= 1\circ\left(OD,t_1^a,O,t_1^b\right) \text{ and } \lambda_b\left(t_1^b\right)|_{A(h_2)\times T}=1\circ\left(ID,t_2^a,O,t_1^b\right);\\
\lambda_b\left(t_2^b\right)|_{A\times T} &= 1\circ\left(OD,t_1^a,I,t_2^b\right) \text{ and } \lambda_b\left(t_2^b\right)|_{A(h_2)\times T}=1\circ\left(IA,t_2^a,I,t_2^b\right).
\end{aligned}
$$

$\lambda_a\left(t_2^a\right)|_{A(h_3)\times T}=1\circ\left(ID,t_2^a,I,t_2^b\right)$ represents that Ann with type t_2^a possesses the conditional belief $1\circ\left(ID,t_2^a,I,t_2^b\right)$ at information set h_3. A state of the world is a 4-tuple (s_a,t_a,s_b,t_b) where $s_a\,(\in S_a)$ and $t_a\,(\in T_a)$ are Ann's actual choice and type respectively. In the extensive form game, a strategy-type pair, $(s_i,t_i)\,(\in S_i\times T_i)$ is rational if at every his information set, the action prescribed by s_i is a best response to his conditional belief induced by his type, $\lambda_i\,(t_i)$. The rational strategy-type pairs in the above example are $R_a=\{(OD,t_1^a),(ID,t_2^a)\}$ and $R_b=\left\{\left(O,t_1^b\right),\left(I,t_2^b\right)\right\}$. At state $\left(ID,t_2^a,I,t_2^b\right)$, Ann is rational, and initially believes that [Bob is rational and initially believes that [she is rational]], and so on. Likewise for Bob. In other words, there is rationality and common initial belief of rationality (RCIBR) at $\left(ID,t_2^a,I,t_2^b\right)$. However, (ID,I) does not induce a backward induction outcome.

We strengthen the notion of "initial belief" operator by imposing the strong independence property. A CPS satisfies strong independence property if it can be generated by a convergent sequence of "full-support" product measures on $\times_h A^h$. It is not difficult to verify that $\lambda_a\left(t_2^a\right)$ satisfies strong independence property but not $\lambda_b\left(t_2^b\right)$. A player is said to consistently believe (CB) an event if he possesses a conditional belief system with strong independence property and initially believes the event. In the above example, the only state satisfying "rationality and common consistent belief of rationality (RCCBR)" is $\left(OD,t_1^a,O,t_1^b\right)$ where (OD,O) is the unique subgame perfect equilibrium.

4.3 Notation and Definitions

Since the formal description of an extensive game is by now standard (see, for instance, Kreps and Wilson (1982) and Kuhn (1954)), only the necessary notation is given below. Consider a (finite) perfect information generic extensive-form game:

$$T \equiv (N, V, H, \left\{A^h\right\}_{h\in H}, \{u_i\}_{i\in N}),$$

where $N = \{1, 2, \ldots, n\}$ is the (finite) set of players, V is the (finite) set of nodes (or vertices), H is the set of information sets (which is a partition of non-terminal nodes), A^h is the (finite) set of pure actions available at information set h, and u_i is player i's payoff function defined on terminal nodes. Throughout this chapter, we only consider generic game where the payoffs of each player at terminal nodes of the game are different from each other. Denote the collection of player i's information sets by H_i. Denote by $A \equiv \times_{h\in H} A^h$ the set of actions, and $A^{-h} \equiv \times_{h'\neq h} A^{h'}$.

A *strategy* of player i is a function, s_i, that assigns an action $a^h \in A^h$ to every $h \in H_i$. Let S_i be the set of player i's strategies. Denote the set of strategy profiles by $S = \times_{j\in N} S_j$. For $s \in S$, we denote by $u_i(s)$ player i's payoff if strategy profile s is adopted from the root of the game.

4.3.1 CPS in Extensive Games

In this chapter, we consider the "conditional probability system (CPS)" on the space, $A = \times_{h\in H} A^h$, of action profiles in the agent-normal form of T. A CPS $\mu|$ on A is a function that specifies, for every nonempty subset $B \subseteq A$ a conditional probability distribution given B over A, denoted by $\mu|_B$, and satisfies the property:

$$\mu|_B(D) = \mu|_C(D)\mu|_B(C) \text{ for } D \subseteq C \subseteq B \subseteq A \text{ and } C \neq \emptyset.$$

See, *e.g.*, Myerson (1991, Section 1.6).

Denote by

$$A(h) \equiv \{a \in A : a \text{ reaches } h\}$$

the set of action profiles that reach h. For $i \in N$ and $h \in H_i$, $a^h \in A^h$ is a *best response with respect to a CPS* $\mu_i|$ on A if

$$\sum_{a^{-h}\in A^{-h}} \mu|_{A(h)}^{-h}(a^{-h})u_i(a^h, a^{-h}) \geq \sum_{a^{-h}\in A^{-h}} \mu|_{A(h)}^{-h}(a^{-h})u_i(b^h, a^{-h}) \ \ \forall b^h \in A^h$$

where $\mu|_{A(h)}^{-h}$ is the marginal of $\mu|_{A(h)}$ on A^{-h},[23] which specifies the agent h's belief about opponents' choices given that information set h is reached.

4.3.2 Strong Independence Property

By Theorem 1 in Myerson (1986), a CPS on a (finite) state space can be expressed by a convergent sequence of "full-support" probability distributions over the state space. A CPS $\mu|$ on A is *associated with a probability distribution* p (on A), denoted by $\mu|_{[p_k \rightsquigarrow p]}$, if there exists a sequence of probability distributions $p_k \rightsquigarrow p$ such that:

(i) For $k = 1, 2, ...$ and every $a \in A$, $p_k(a) > 0$;

(ii) For any B, $C \subseteq A$ with $B \neq \emptyset$, $\mu|_B(C) = \lim_{k\to\infty} \frac{p_k(B\cap C)}{p_k(B)}$.

For the purpose of this chapter, we say "a CPS $\mu_i|$ on A satisfies *strong independence property*" if $\mu_i| = \mu_i|_{[p_k \rightsquigarrow p]}$ where each p_k is a product measure on space $A^h \times A^{-h}$ for any $h \in H$, *e.g.* for any $a \in A$, $p_k(a) = p_k^h(a^h) \cdot p_k^{-h}(a^{-h})$ where p_k^h $\left(p_k^{-h}\right)$ is the marginal of p on A^h $\left(A^{-h}\right)$.[24]

For any two distinct h, $h' \in H$, we say that h is a *precedent* of h', denoted as $h \prec h'$, if $A(h') \subseteq A(h)$. Define $h \not\prec h'$ as h is not a precedent of h'. Then, we have the following lemma.

Lemma 4.3.2.1. If $\mu_i|$ is strongly independent on A. $\forall h \in H_i$, $\mu_i|_A^{h'} = \mu_i|_{A(h)}^{h'}$ for $h' \in H$ where $h' \not\prec h$.

[23]The marginal of $\mu|_{A(h)}$ on A^{-h} is defined as probability measure on A^{-h} such that

$$\forall a^{-h} \in A^{-h},\ \mu|_{A(h)}^{-h}\left(a^{-h}\right) \equiv \sum_{a^h\in A^h} \mu|_{A(h)}\left(a^h,\, a^{-h}\right).$$

[24]It should be noticed that the strong independence property is different from independence property in Definition 2.1 Battigalli (1996). Under their definition, the conditional probability measure specified by CPS at each information set is a product measure on strategy space, $\times_j S_j$.

Proposition 3.3 shows that CPS with strong independence property specifies a product measure on action space, $\times_h A^h$, at each information set as a conditional belief.

Proof. Let $p_k \rightsquigarrow p$ be product measures on space $A^h \times A^{-h}$ for any $h \in H$ such that $\mu_i| = \mu_i|_{[p_k \rightsquigarrow p]}$. Let $h \in H_i$ and $h' \in H$ such that $h' \nprec h$.

Pick $a \in A$ randomly, and we have

$$\begin{aligned} \mu_i|_A(a) &= \lim_{k\to\infty} \frac{p_k(a)}{\sum_{b\in A} p_k(b)} \\ &= \lim_{k\to\infty} p_k(a) \\ &= p(a) \end{aligned}$$

Denote by $p^{h'}$ the marginal of p on $A^{h'}$. Therefore,

$$\begin{aligned} \mu_i|_A^{h'}\left(a^{h'}\right) &= \sum_{c^{-h'}\in A^{-h'}} \mu_i|_A\left(a^{h'}, c^{-h'}\right) \\ &= \sum_{c^{-h'}\in A^{-h'}} p\left(a^{h'}, c^{-h'}\right) \\ &= p^{h'}\left(a^{h'}\right). \end{aligned}$$

Note that $A(h) = \times_{h''} A^{h''}(h)$ in PI games. Since $\mu_i|_{A(h)}(a) = \lim_{k\to\infty} \frac{p_k(\{a\}\cap A(h))}{\sum_{b\in A(h)} p_k(b)}$, we have

$$
\begin{aligned}
\mu_i|_{A(h)}^{h'}\left(a^{h'}\right) &= \sum_{c^{-h'}\in A^{-h'}} \mu_i|_{A(h)}\left(a^{h'},c^{-h'}\right) \\
&= \sum_{c^{-h'}\in A^{-h'}} \lim_{k\to\infty} \frac{p_k\left(\left\{\left(a^{h'},c^{-h'}\right)\right\}\cap A(h)\right)}{\sum_{\left(b^{h'},d^{-h'}\right)\in A(h)} p_k\left(b^{h'},d^{-h'}\right)} \\
&= \lim_{k\to\infty} \sum_{c^{-h'}\in A^{-h'}} \frac{p_k\left(\left\{a^{h'}\right\}\cap A^{h'}(h),\left\{c^{-h'}\right\}\cap A^{-h'}(h)\right)}{\sum_{\left(b^{h'},d^{-h'}\right)\in A(h)} p_k\left(b^{h'},d^{-h'}\right)} \\
&= \lim_{k\to\infty} \frac{p_k^{h'}\left(\left\{a^{h'}\right\}\cap A^{h'}(h)\right)\cdot\left(\sum_{c^{-h'}\in A^{-h'}} p_k^{-h'}\left(\left\{c^{-h'}\right\}\cap A^{-h'}(h)\right)\right)}{\sum_{\left(b^{h'},d^{-h'}\right)\in A(h)} p_k^{h'}\left(b^{h'}\right)\cdot p_k^{-h'}\left(d^{-h'}\right)} \\
&= \lim_{k\to\infty} \frac{p_k^{h'}\left(\left\{a^{h'}\right\}\cap A^{h'}(h)\right)\cdot p_k^{-h'}\left(A^{-h'}(h)\right)}{\sum_{b^{h'}\in A^{h'}(h)} p_k^{h'}\left(b^{h'}\right)\cdot\left(\sum_{d^{-h'}\in A^{-h'}(h)} p_k^{-h'}\left(d^{-h'}\right)\right)} \\
&= \lim_{k\to\infty} \frac{p_k^{h'}\left(a^{h'}\right)}{\sum_{b^{h'}\in A^{h'}} p_k^{h'}\left(b^{h'}\right)} \text{since } A^{h'}(h)=A^{h'} \\
&= \lim_{k\to\infty} p_k^{h'}\left(a^{h'}\right) \\
&= p^{h'}\left(a^{h'}\right). \blacksquare
\end{aligned}
$$

With Lemma 4.3.2.1, we have the following corollary immediately.

Corollary 4.3.2.1. If $\mu_i|$ is strongly independent on A, $\forall h,h'\in H_i$ $\mu_i|_{A(h)}^{h''}=\mu_i|_{A(h')}^{h''}$ for $h''\in H$ with $h''\not\prec h$ and $h''\not\prec h'$.

The following proposition says that if $\mu|$ satisfies strong independence property, the conditional belief at every information set specified under $\mu|$ is a product measure on A.

Proposition 4.3.2.1. If $\mu_i|$ is strongly independent on A, $\mu_i|_{A(h)}(a)=\Pi_{h'\in H}\mu_i|_{A(h)}^{h'}\left(a^{h'}\right)$ for any $h\in H_i$ and $a\in A$.

Proof. It is trivial if $a\notin A(h)$. Hence, we only consider the case that $a\in A(h)$.

Let m be the number of information set in the game. Index all the information as $h_1,h_2,\ldots,h_m$.

($\Rightarrow$) Let $H_p \subseteq H$ such that $\forall h'' \in H_p$, $h'' \prec h$.

$$\begin{aligned}
\mu_i|_{A(h)}(a) &= \lim_{k\to\infty} \frac{p_k(a)}{\sum_{b\in A(h)} p_k(b)} \\
&= \lim_{k\to\infty} \frac{\Pi_{h'\in H} p_k^{h'}\left(a^{h'}\right)}{\sum_{b\in A(h)} p_k(b)} \text{by lemma 4.3.2.2} \\
&= \lim_{k\to\infty} \frac{p_k^{H_p}\left(a^{H_p}\right)\cdot p_k^{-H_p}\left(a^{-H_p}\right)}{\sum_{b\in A(h)} p_k^{H_p}\left(b^{H_p}\right)\cdot p_k^{-H_p}\left(b^{-H_p}\right)}
\end{aligned}$$

Since it is a PI game, there is a unique path to h. That is,

$$\begin{aligned}
\mu_i|_{A(h)}(a) &= \lim_{k\to\infty} \frac{p_k^{H_p}\left(a^{H_p}\right)\cdot p_k^{-H_p}\left(a^{-H_p}\right)}{p_k^{H_p}\left(b^{H_p}\right)\left(\sum_{b^{-H_p}\in A^{-H_p}(h)} p_k^{-H_p}\left(b^{-H_p}\right)\right)} \\
&= \lim_{k\to\infty} \frac{p_k^{-H_p}\left(a^{-H_p}\right)}{\sum_{b^{-H_p}\in A^{-H_p}} p_k^{-H_p}\left(b^{-H_p}\right)} \text{ since } A^{-H_p} = A^{-H_p}(h) \\
&= \lim_{k\to\infty} p_k^{-H_p}\left(a^{-H_p}\right) \\
&= \Pi_{h'\in H\setminus H_p} p^{h'}\left(a^{h'}\right)
\end{aligned}$$

Since there is a unique path to h, $\mu_i|_{A(h)}^{h''}\left(a^{h''}\right) = 1$ for all $h'' \prec h$. Then,

$$\begin{aligned}
\mu_i|_{A(h)}(a) &= \Pi_{h'\in H\setminus H_p} \mu_i|_{A(h)}^{h'}\left(a^{h'}\right) \text{ by lemma 4.3.2.1} \\
&= \Pi_{h'\in H} \mu_i|_{A(h)}^{h'}\left(a^{h'}\right). \blacksquare
\end{aligned}$$

Lemma 4.3.2.2. For any $a \in A$, if $p_k(a) = p_k^h\left(a^h\right)\cdot p_k^{-h}\left(a^{-h}\right)$, $p_k(a) = \Pi_{h'} p_k^{h'}\left(a^{h'}\right)$.

Proof. Let $a \in A$. Let m be the number of information set in the game. Index all the information as $h_1, h_2, \ldots, h_m$.

$$\begin{aligned}
p_k(a) &= p_k^{h_1}\left(a^{h_1}\right) \cdot p_k^{-h_1}\left(a^{-h_1}\right) \\
&= p_k^{h_1}\left(a^{h_1}\right) \cdot \left(\sum_{b^{h_1} \in A^{h_1}} p_k\left(b^{h_1}, a^{h_2}, a^{h_3}, \ldots, a^{h_m}\right)\right) \\
&= p_k^{h_1}\left(a^{h_1}\right) p_k^{h_2}\left(a^{h_2}\right) \cdot \left(\sum_{b^{h_1} \in A^{h_1}} p_k^{-h_2}\left(b^{h_1}, a^{h_3}, \ldots, a^{h_m}\right)\right) \\
&= p_k^{h_1}\left(a^{h_1}\right) p_k^{h_2}\left(a^{h_2}\right) \cdot \left(\sum_{b^{h_1} \in A^{h_1}} \sum_{b^{h_2} \in A^{h_2}} p_k\left(b^{h_1}, b^{h_2}, a^{h_3}, \ldots, a^{h_m}\right)\right) \\
&\cdots \\
&= p_k^{h_1}\left(a^{h_1}\right) p_k^{h_2}\left(a^{h_2}\right) \ldots p_k^{h_m}\left(a^{h_m}\right) \left(\sum_{b^{h_1} \in A^{h_1}} \sum_{b^{h_2} \in A^{h_2}} \cdots \sum_{b^{h_m} \in A^{h_m}} p_k\left(b^{h_1}, b^{h_2}, \ldots, b^{h_m}\right)\right) \\
&= p_k^{h_1}\left(a^{h_1}\right) p_k^{h_2}\left(a^{h_2}\right) \ldots p_k^{h_m}\left(a^{h_m}\right) \ \blacksquare
\end{aligned}$$

Remark. The reverse of Proposition 4.3.2.1 is not true. In the example 4.1, Bob's second type t_2^b induces the CPS $\lambda_b\left(t_2^b\right)|$ as following,

$$\begin{aligned}
\lambda_b\left(t_2^b\right)|_{A \times T} &= 1 \circ \left(OD, t_1^a, I, t_2^b\right) \\
\text{and } \lambda_b\left(t_2^b\right)|_{A(h_2) \times T} &= 1 \circ \left(IA, t_2^a, I, t_2^b\right).
\end{aligned}$$

Clearly, $\lambda_b\left(t_2^b\right)|_{A(h)}(a) = \Pi_{h' \in H} \lambda_b\left(t_2^b\right)|_{A(h)}^{h'}\left(a^{h'}\right)$ for any $h \in H_b$ and $a \in A$. However, $\lambda_b\left(t_2^b\right)|$ does not satisfy strong independence property.

4.4 Epistemic Characterization of Backward Induction

4.4.1 Type Structure and Consistent Belief Operator

Fix a finite extensive game

$$\Gamma \equiv (N, V, H, \{A^h\}_{h\in H}, \{u_i\}_{i\in N}).$$

Let T_i be the finite set of types of player i. Members of T_i are called player i's types. Denote by $T \equiv \times_{j\in N} T_j$ the set of type profiles. Denote by $\Delta^* (S \times T)$ the set of conditional probability systems defined over $A\times T$.[25] Denote by $\Delta^{*\circ} (S \times T)$ the set of conditional probability systems where the marginal of CPS on A satisfies strong independence property[26].

Definition 4.4.1. A $\{S_j\}_{j\in N}$*-based type structure* is a structure

$$\langle S_1, ..., S_n, T_1, ..., T_n, \lambda_1, ..., \lambda_n\rangle$$

where for all $i \in N$, $\lambda_i : T_i \to \Delta^* (S \times T)$. Members of $S \times T$, are called *states of the world*.

Fix $i \in N$, and an event $E \subseteq S \times T$. We say player i *consistently believes* E at t_i if $\lambda_i (t_i) | \in \Delta^{*\circ} (S \times T)$ and $\lambda_i (t_i) |_{A\times T} (E) = 1$, and write

$$CB_i (E) \equiv \{t_i \in T_i : \lambda_i (t_i) | \in \Delta^{*\circ} (S \times T) \text{ and } \lambda_i (t_i) |_{A\times T} (E) = 1\}.$$

Throughout this chapter, we assume that player i knows his own type, i.e., $CB_i (E) \subseteq \text{proj}_{T_i} E$. For any $h \in H_i$, denote by $s_i (h)$ ($\in A^h$) the action of player i at information set h prescribed by strategy s_i. Player i is *rational* at (s_i, t_i) if for any $h \in H_i$, $s_i (h)$ is a best response with respect to $\text{marg}_A (\lambda_i (t_i) |)$. Denoted by

$$R_i \equiv \{(s_i, t_i) \in S_i \times T_i : \text{Player } i \text{ is rational at } (s_i, t_i)\}.$$

[25] We abuse the notation, and define the CPS on $S \times T$ as a CPS on $A \times T$. For any $E \subseteq S \times T$, the $\mu|_{A\times T} (E)$ is defined in the usual sense.

[26] For any CPS $\mu|$ in $\Delta^{*\circ} (S \times T)$, denote the marginal of it on A as a new CPS $\mu'|$ defined over A. $\mu'|$ satisfies the strong independence property.

Let $R = \times_{j \in N} R_j$ and $R_i^1 \equiv R_i$. For finite $m \geq 1$, define R_i^m and R^m inductively by

$$\begin{aligned} R^m &= \times_{j \in N} R_j^m \\ R_i^{m+1} &= R_i^m \cap [S_i \times CB_i(R^m)] . \end{aligned}$$

If $(s_1, t_1, ..., s_n, t_n) \in \times_{j \in N} \left(\cap_{m=1}^{\infty} R_j^m \right)$, say there is *rationality and common consistent belief of rationality (RCCBR)* at this state.

4.4.2 Characterization of BI

We are now in a position to present the central result of this chapter which offers an epistemic characterization for the notion of backward induction. Recall that in perfect information generic games, there is only one subgame perfect equilibrium which is the unique backward induction strategy profile. Theorem 4.4.2 states that rationality and common consistent belief of rationality (RCCBR) implies the underlying strategy profile is the unique subgame perfect equilibrium (SPE) in the perfect information generic game, and conversely, any SPE can be attained by the aforementioned epistemic assumptions.

Theorem 4.4.2. *In a perfect information generic game,*
(a) Fix a type structure

$$\langle S_1, ..., S_n, T_1, ..., T_n, \lambda_1, ..., \lambda_n \rangle .$$

Let $s \in \times_{j \in N} proj_{S_j} \left(\cap_{m=1}^{\infty} R_j^m \right)$. Then, s is the unique subgame perfect equilibrium.
(b) Let s^* be the subgame perfect equilibrium. There is a type structure model $\mathcal{M}(\Gamma)$ such that $s^* \in \times_{j \in N} proj_{S_j} \left(\cap_{m=1}^{\infty} R_j^m \right)$.

Proof. (a) Let s^* be the unique SPE in the PI generic game. Let $s \in \times_{j \in N} proj_{S_j} \left(\cap_{m=1}^{\infty} R_j^m \right)$. We want to show $s = s^*$.

Let

$$H_i(0) \equiv \{ h \in H_i : \nexists h' \in H_i \text{ s.t. } h' \text{ can be reached from } h \} ,$$

where $h \in H_i(0)$ is interpreted as a lowest order or 0-order information set of player i from which no other information set of player i can be reached. Define, inductively, for $\kappa \geq 1$,

$$H_i(\kappa) \equiv \left\{h \in H_i \backslash \cup_{\kappa'=0}^{\kappa-1} H_i(\kappa') : \nexists h' \in H_i \backslash \cup_{\kappa'=0}^{\kappa-1} H_i(\kappa') \text{ s.t. } h' \text{ can be reached from } h\right\},$$

where $h \in H_i(\kappa)$ is interpreted as an κ-order information set of player i from which no higher order (i.e. κ'-order for $\kappa' \geq \kappa + 1$) information set of player i can be reached. Clearly, $\{H_i(\kappa)\}_{\kappa \geq 0}$ is a (finite) partition of H_i since each player is perfect recall. We prove $s = s^*$ by induction on the order of κ.

For $\kappa = 0$, we show that for every $i \in N$, $s_i(h) = s_i^*(h)$ for $h \in H_i(0)$. Since $s \in \times_{j \in N} proj_{S_j}\left(\cap_{m=1}^{\infty} R_j^m\right)$, $(s_i, t_i) \in \cap_{m=1}^{\infty} R_i^m$ for some $t_i \in T_i$. That is, $(s_i, t_i) \in R_i$. Then, $s_i(h)$ is a best response to $\text{marg}_A\left(\lambda_i(t_i)|_{A(h)\times T}\right)$. Since $h \in H_i(0)$ and it is a generic game, $s_i(h) = s_i^*(h)$.

Now, consider $\kappa = 1$. We proceed to show that for every $i \in N$, $s_i(h) = s_i^*(h)$ for $h \in H_i(1)$. As shown in the last step, for all $j \in N$ and $h' \in H_j(0)$, $s_j(h') = s_j^*(h')$. Since $(s_i, t_i) \in \cap_{m=1}^{\infty} R_i^m$, $t_i \in CB_iR$. That is, $\text{marg}_{A^{h'}}(\lambda_i(t_i)|_{A\times T}) = 1 \circ s^*(h')$ for all $h' \in H(0)$. By Corollary 4.3.2.1, for all $h' \in H(0)$

$$\begin{aligned} \text{marg}_{A^{h'}}\left(\lambda_i(t_i)|_{A(h)\times T}\right) &= \text{marg}_{A^{h'}}(\lambda_i(t_i)|_{A\times T}) \\ &= 1 \circ s^*(h'). \end{aligned}$$

Since $(s_i, t_i) \in R_i$, $s_i(h)$ is a best response to $\text{marg}_A\left(\lambda_i(t_i)|_{A(h)\times T}\right)$. Since it is a generic game, $s_i(h) = s_i^*(h)$.

Repeating the argument for $\kappa \geq 2$, we conclude that $s(h) = s^*(h)$ for all h.

(b)Fix a subgame perfect equilibrium s^*. For each $i \in N$, let $T_i \equiv \{t_i\}$ where $\lambda_i(t_i) \in \Delta^{*\circ}(S \times T)$ and $\lambda_i(t_i)|_{A\times T} = 1 \circ (s^*, t)$. We have constructed the type structure

$$\langle S_1, ..., S_n, T_1, ..., T_n, \lambda_1, ..., \lambda_n \rangle.$$

Clearly, $R_i = \{s_i^*\} \times T_i$ for all $i \in N$. Then, $CB_i(R) = T_i$ and

$$\begin{aligned} R_i^2 &= R_i \cap \left[S_i \times CB_i\left(R^1\right)\right] \\ &= \{s_i^*\} \times T_i \cap [S_i \times T_i] \\ &= \{s_i^*\} \times T_i. \end{aligned}$$

By induction, $(s_i^*, t_i) \in R_i^m$ for $m \geq 1$. That is, $s^* \in \times_{j \in N} \text{proj}_{S_j}\left(\cap_{m=1}^{\infty} R_j^m\right)$. ■

4.5 Discussion

In this section, we are going to discuss and comment on the related literature. The current framework will be compared with Aumann's model (1995). Moreover, the relationship among the notion of "consistent belief", "initial belief" (Ben-Porath (1997)) and "strong belief" (Battigalli and Siniscalchi (2002)) will be analyzed. Lastly, the rationality and common consistent belief of rationality is related to sequential rationalizability, SRSCE and "null MACA".

4.5.1 Aumann's Framework

Aumann (1995) provided the first epistemic characterization of backward induction. It shows that backward induction outcome is a logical consequence of common knowledge of rationality. In his model, every player initially believes that all players will choose behave rationally at every information set. Moreover, every player will stick to their belief about opponents' rationality at all information set, particularly at those information sets reached by opponents' suboptimal actions. In other words, even if there was an observed contradiction of opponent's rationality, players will ignore this contradiction and still assume the rationality of opponents.

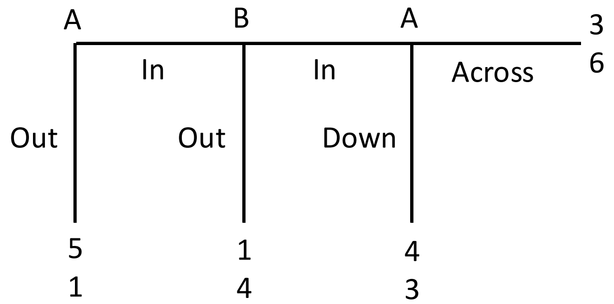
Aumann (1996) section 9 enriched his original model by adding belief system explicitly. Each player has a belief system which specifies player's belief at his every information set. A player is said to be Bayesian rational if he maximizes his (expected) payoff given the belief at all his information sets. Moreover, players take the "from that point on" view. It says if players know something at the beginning of the game, they will believe it at all subsequent information sets. With the belief system, Aumann (1995) showed that backward induction is the result of common knowledge of Bayesian rationality.

The major critic of these two treatments is that there is no belief revision process (Halpern 1999), to justify why they would maintain their belief even if there is a contradiction. However, our work give a possible explanation of Aumann's result with Bayesian updating. Particularly, the belief system in Aumann (1996) can be viewed as a independent conditional probability system in our model, which is used to characterize BI strategy profile.

4.5.2 Initial Belief and Strong Belief

Ben-Porath (1997) introduced the notion of "initial belief", and showed that the rationality and common initial belief of rationality (RCIBR) is characterized by DF procedure (Dekel and Fudenberg (1990)), *e.g.* one round of elimination of weakly dominated strategies followed by iterative elimination of strictly dominated strategies. As shown in the example 4.1 and section 4.2, backward induction may not be the only result led by RCIBR. This is because the initial belief operator only puts the restriction on belief revision at information sets reached with positive probability. It means that at information sets that contradict with the initial belief, players may arbitrarily revise their beliefs.

In the spirit of forward reasoning, Battigalli and Siniscalchi (2002) strengthened the definition by providing the strong belief operator. Strong belief of rationality says when there is a contraction between the initial belief and the observation at some subsequent information sets, players will have a second thought at these information sets and try to find out the reason to rationalize why such a behavior, *e.g.* why these information sets are reached before revising the beliefs. In other words, only at informations sets that totally falsify the rationality assumption, players will arbitrarily revise their beliefs. With the strong belief operator, backward induction outcomes is shown to be the logical consequence of rationality and common strong belief of rationality (RCSBR) in generic PI game. Although strong belief provides the desired belief revision process, RCSBR may not lead to backward induction profile.



Example 4.2

In example 4.2, if Bob strongly believes that Ann is rational, he cannot find out a belief about her belief to justify her choice of *In* at the first node. In this case, Bob may believe that she would choose *Across* at the third stage, and hence choose *In* at the second stage. $((Out, Down), In)$, which leads to BI outcome, is not a BI strategy profile.

4.5.3 Sequential Rationalizability, SRSCE and MACA

The path mutually acceptable courses of action (MACA) is a perfect version of sequential rationalizable self-confirming equilibrium (SRSCE), and the latter one is the logical consequence of common knowledge of rationality and mutual knowledge of players' action along the path. Players' belief at SRSCE is a CPS generated by independent trembling sequence of players' behavior strategies. Together with the Theorem 4.4.2, we conclude that SRSCE is a backward induction strategy profile in PI generic game.

Moreover, "null MACA", which is the result of common knowledge of (perfect) rationality (Chapter 2 Proposition 2.3.3.1. and Corollary 2.3.3.1), can be viewed as a "perfect" rationalizability which is a refinement of sequential rationalizability (Dekel et. al (1999)). Meanwhile, the notion of "sequential rationalizability" is the logical consequence of common knowledge of "sequential" rationality (Chapter 3 Corollary 3.3.4.1). Both of them lead to backward induction in PI generic game.

4.6 Concluding Remarks

Throughout this chapter, we develop "strong independence property" for a conditional probability system. Based on this concept, we define the notion of "consistent belief" which strengthens the notion of "initial belief" (Ben-Porath (1997)). Within a standard semantic framework, we formulate and show backward induction strategy profile is the logical consequence of rationality and common consistent belief of rationality in a perfect information generic game.

References

[1] G.B. Asheim, Proper rationalizability in lexicographic beliefs, Int J Game Theory 30 (2001), 453-478.

[2] G.B. Asheim, The Consistent Preferences Approach to Deductive Reasoning in Games. Berlin: Springer Verlag (2006).

[3] G.B. Asheim and A. Perea, Sequential and quasi-perfect rationalizability in extensive games, Games Econ Behav 53 (2005), 15-42.

[4] R.J. Aumann, Agreeing to disagree, Ann Statist 4 (1976), 1236-1239

[5] R.J. Aumann, Correlated equilibrium as an expression of Bayesian rationality, Econometrica 55 (1987), 1-18.

[6] R.J. Aumann, Backward induction and common knowledge of rationality, Games Econ Behav 8 (1995), 6-19.

[7] R.J. Aumann, Interactive epistemology I: knowledge, Int J Game Theory 28 (1999), 263-300.

[8] R.J. Aumann, Subjectivity and correlation in randomized strategies, J Math Econ 1 (1974), 67-96.

[9] R.J. Aumann and A. Brandenburger, Epistemic conditions for Nash equilibrium, Econometrica 63 (1995), 1161-1180.

[10] P. Barelli and S. Galanis, Admissibility and event-rationality, Games Econ Behav 77 (2013), 21-40.

[11] P. Battigalli, Strategic Independence and Perfect Bayesian Equilibria, J Econ Theory 70 (1996), 201-234

[12] P. Battigalli and G. Bonanno, Recent results on belief, knowledge and the epistemic foundations of game theory, Research in Economics 53 (1999), 149-225.

[13] P. Battigalli and A. Friedenberg, Forward induction reasoning revisted, Theoretical Economics 7 (2012), 57-98

[14] P. Battigalli and M. Siniscalchi, Hierarchies of conditional beliefs and interactive epistemology in dynamic games, J Econ Theory 88 (1999), 188–230.

[15] P. Battigalli and M. Siniscalchi, Strong belief and forward induction reasoning, J Econ Theory 106 (2002), 356-391.

[16] E. Ben-Porath, Rationality, Nash equilibrium, and backward induction in perfect information game, Rev Econ Stud 64 (1997), 23-46.

[17] B.D. Bernheim, Rationalizable strategic behavior, Econometrica 52 (1984), 1007-1028.

[18] L. Blume, A. Brandenburger and E. Dekel, Lexicographic probabilities and choice under uncertainty, Econometrica 59 (1991a), 61-79.

[19] L. Blume, A. Brandenburger and E. Dekel, Lexicographic probabilities and equilibrium refinements, Econometrica 59 (1991b), 81-98.

[20] G. Bonanno, A dynamic epistemic characterization of backward induction without counterfactuals, Games Econ Behav 78 (2013), 31-43

[21] G. Bonanno, Epistemic foundations of game theory. Forthcoming in: H. van Ditmarsch *et al.* (eds.), Handbook of Epistemic Logic, London: College Publications (2013).

[22] T. Borgers, Weak dominance and approximate common knowledge, J Econ Theory 64 (1994), 265-276.

[23] A. Brandenburger, Lexicographic probabilities and iterated admissibility. In: P. Dasgupta, D. Gale, O. Hart and E. Maskin (eds.), Economic Analysis of Markets and Games. MA: MIT Press (1992), pp. 282-290.

[24] A. Brandenburger, The power of paradox: some recent developments in interactive epistemology, Int J Game Theory 35 (2007), 465-492.

[25] A. Brandenburger, A. Friedenberg and H.J. Keisler, Admissibility in games, Econometrica 76 (2008), 307-352.

[26] E. Dekel and D. Fudenberg, Rational behavior with payoff uncertainty, J Econ Theory 52 (1990), 243-267.

[27] E. Dekel, D. Fudenberg and D.K. Levine, Payoff information and self-confirming equilibrium, J Econ Theory 89 (1999), 165-185.

[28] E. Dekel, D. Fudenberg and D.K. Levine, Subjective uncertainty over behavior strategies: a correction, J Econ Theory 104 (2002), 473-478.

[29] E. Dekel and F. Gul, Rationality and knowledge. In: D.M. Kreps and K.F. Wallis (eds.), Advances in Economics and Econometrics: Theory and Applications, Cambridge: Cambridge University Press (1997), pp. 87-172.

[30] I. Esponda, Rationalizable conjectural equilibrium: a framework for robust predictions, forthcoming in: Theoretical Econ (2013).

[31] Y. Feinberg, Subjective reasoning – dynamic games, Games Econ Behav 52 (2005a), 54-93.

[32] Y. Feinberg, Subjective reasoning – solutions, Games Econ Behav 52 (2005b), 94-132.

[33] D. Fudenberg and D.M. Kreps, Learning in extensive games I: self-confirming equilibria, Games Econ Behav 8 (1995), 20-55.

[34] D. Fudenberg and D.M. Kreps, and D.K. Levine, On the robustness of equilibrium refinements, J Econ Theory 44 (1988), 354-380.

[35] D. Fudenberg and D.K. Levine, Self-confirming equilibrium, Econometrica 61 (1993), 523-546.

[36] D. Fudenberg and D.K. Levine, Superstition and rational learning, Amer Econ Rev 96 (2006), 630-651.

[37] D. Fudenberg and D.K. Levine, Self-confirming equilibrium and the Lucas critique, J Econ Theory 144 (2009) , 2354-2371.

[38] J. Geanakoplos, Game theory without partitions and applications to speculation and consensus, Cowles Foundation Discussion Paper No. 914 (1989).

[39] S. Govindan and T. Klumpp, Perfect equilibrium and lexicographic beliefs, Int J Game Theory 31 (2002), 229-243.

[40] J. Greenberg, S. Gupta and X. Luo, Mutually acceptable courses of action, Econ Theory 40 (2009), 91-112.

[41] F. Gul, Rationality and coherent theories of strategic behavior, J Econ Theory 70 (1996), 1-31.

[42] J.Y. Halpern, Hypothetical knowledge and counterfactual reasoning, Int J Game Theory 28 (1999), 315-330.

[43] J.Y. Halpern, Substantive rationality and backward induction, Games Econ Behav 37 (2001), 425-435.

[44] J.Y. Halpern, A nonstandard characterization of sequential equilibrium, perfect equilibrium, and proper equilibrium, Int J Game Theory (2009) 38, 37-49.

[45] J.Y. Halpern, Lexicographic probability, conditional probability, and nonstandard probability, Games Econ Behav 68 (2010),155-179.

[46] P.J. Hammond, Elementary non-Archimedean representations of probability for decision theory and games. In: P. Humphreys (ed.), Patrick Suppes: Scientific Philosopher, Vol. I: Probability and Probabilistic Causality. Dordrecht: Kluwer Academic Publishers (1994), pp. 25-49.

[47] J.C. Harsanyi and R. Selten, A General Theory of Equilibrium Selection in Games. MA: MIT Press (1988).

[48] P.J.J. Herings and V.J. Vannetelbosch, Refinements of rationalizability for normal-form games, Int J Game Theory 28 (1999), 53-68.

[49] P.J.J. Herings and V.J. Vannetelbosch, The equivalence of the Dekel-Fudenberg iterative procedure and weakly perfect rationalizability?, Econ Theory 15 (2000), 677-687.

[50] T.W. Hu, On p-rationalizability and approximate common certainty of rationality, J Econ Theory 136 (2007), 379-391.

[51] E. Kalai and E. Lehrer, Rational learning leads to Nash equilibrium, Econometrica 61 (1993a), 1019-1045.

[52] E. Kalai and E. Lehrer, Subjective equilibrium, Econometrica 61 (1993b), 1231-1240.

[53] D.M. Kreps and R. Wilson, Sequential equilibria, Econometrica 50 (1982), 863-894.

[54] H.W. Kuhn, Extensive games and the problem of information. In: H.W. Kuhn and A.W. Tucker (eds.), Contributions to the Theory of Games, Vol. II. NJ: Princeton University Press (1953), pp. 193-216.

[55] A. McLennan, Consistent conditional systems in noncooperative game theory, Inter J Game Theory 18 (1989), 141-174.

[56] R. Myerson, Multistage games with communication, Econometrica 54 (1986), 323-358.

[57] R. Myerson, Game Theory: Analysis of Conflict. Harvard Univ. Press, MA, 1991.

[58] D. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52 (1984), 1029-1050.

[59] A. Rubinstein, Modeling Bounded Rationality. MA: MIT Press (1998).

[60] A. Rubinstein and A. Wolinsky, Rationalizable conjectural equilibrium: between Nash and rationalizability, Games Econ Behav 6 (1994), 299-311.

[61] D. Samet, Common beleif of rationality in games of perfect information, Games Econ Behav 79 (2013), 192-200

[62] L. Samuelson, Dominated strategies and common knowledge, Games Econ Behav 4 (1992), 284-313.

[63] L. Samuelson, Modeling knowledge in economic analysis, J Econ Lit 57 (2004), 367-403.

[64] R. Selten, Spieltheoretische behandlung eines oligopolmodells mit nachfragetragheit, Zeitschrift fur die gesamte Staatswissenschaft 121 (1965), 301-324 and 667-689.

[65] R. Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, Int J Game Theory 4 (1975), 25-55.

[66] E. van Damme, Stability and Perfection of Nash Equilibria. Berlin: Springer Verlag (1991).