BOUNDARY WAVE AND INTERIOR WAVE PROPAGATIONS

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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Contents

Abstract

This thesis is concerned with the mathematical study of the boundary wave and interior wave propagation. The models we considered are the Broadwell model of the Boltzmann equation in the kinetic theory and a simplified model from magnetohydrodynamics (MHD).

In the first part, the initial boundary value problem for the Broadwell model in the half space is studied to understand the interaction of boundary waves and interior fluid waves. The Green's function for the linearized system in the half space is constructed. Moreover, the optimal rate of convergence of the solution to a global Maxwellian is obtained by combining this Green's function for the half space with nonlinear terms.

In the second part we study the interaction of interior nonlinear waves. We consider two models, one is a conservative system from the MHD, the other is the Broadwell model from the kinetic theory. We seek a unified approach to solve the linearized problem around the general shock profile with general amplitude, which is a variable coefficient PDE(system). With the explicit structure of solution for the linearized problem, we study the nonlinear wave propagation and to conclude the convergence with an optimal convergent rate around the shock front.

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Chapter 1

Introduction

1.1 Background

Gas is a basic state of matter in nature without a definite shape or volume, which contain a collection of particles, e.g., molecules, atoms, ions, electrons. The gas motions are described by the different mathematical models at different physical scales: statistical mechanics (Newton equation) at the microscopic scale; hydrodynamics (Euler and Navier-Stokes equations) at the macroscopic scale; kinetic theory (Boltzmann equation) at the mesoscopic scale, which connects of macroscopic and microscopic theories. There are close relations among these models in the sense that some of them generally can be seen as the approximations from others after taking limits or truncations for the special physical parameters. This has raised many challenging mathematical problems in the theories of asymptotic analysis and singular perturbations.

In the theories of gas motion, there are two basic components: the boundary layers and the interior nonlinear waves. Systematic asymptotic expansions have been developed to study the inter-relations of these two components, see Grad [13]. In this asymptotic analysis based on the

Hilbert expansion for the hydrodynamics limits problem, three singular slips, the initial layer, boundary layer, and shock layer, were excluded from the hydrodynamic regimes. Grad proposed to study the time asymptotic behaviors of these slips on the level of the original kinetic equation for the purpose to develop a general asymptotic expansion theory with singular non-hydrodynamic structures.

With the knowledge of the interaction of the fluid waves with boundary layer, the purely kinetic phenomenon such as thermal creep, some bifurcations due to curvature effect, ghost effects in the rarefied gas has been established, see the works lead by Kyoto group Sone and Aoki [31, 32]. Further analytical study requires more quantitative, pointwise estimates on the wave interaction. It is still hardly reachable for current available mathematical analytical tools.

Shock waves are interior nonlinear waves. For the compressible Navier-Stokes equations, there are also the interior nonlinear waves of contact layers and rarefaction waves. The micro-parts in the contact layers and rarefaction waves of the Boltzmann equation, will converge to zero. Thus on the level of contact layers and rarefaction waves, the Navier-stokes equation and the Boltzmann equation are time asymptotically equivalent. As a consequence, the time-asymptotic analysis for these two waves on the level of the Navier-Stokes equation can be generalized to the Boltzmann equation. However, the micro-part of the Boltzmann shock profile is time-invariant and the analysis designed for the Navier-Stokes equation is not sufficient. The study of the positivity of shock waves for Boltzmann equation, [22], established more direction connection of the kinetic theory and shock wave theory for conservation laws. The study is based on the energy method, which is motivated by the energy method in [12] for the stability of a viscous shock profile, The energy method is a basic techniques in the study of differential equations and works for the stability analysis of contact and rarefaction waves. Although it is not sufficient for shock wave studies in general, It has helped to initiate the studies of nonlinear waves, e.g., [23, 24, 25, 28, 33, 34]. In [23, 24], a Green's function pointwise estimate approach was initiated for the purpose of better understanding of the qualitative and quantitative behavior of solutions. In [34], the author generalized and refined this approach to better handling the local wave interactions. He analysed the interaction of initial layer and shock layer and also the time-asymptotic stability of the shock waves for the kinetic equation. In the initial time, the kinetic particle-like behavior dominates; in the intermediate time, the Burgers nonlinearity dominates; in the large-time, the fluid type behavior dominates. The situation differs totally from the Navier-Stokes equation [14].

As we have already seen above, quantitative and qualitative analysis of wave interaction plays an important role in understanding of physical phenomena. The Green's function approach is indispensable, and allows for the quantitative description of the rich phenomena resulting from the interaction. However, the derivation of explicit formula for the Green's function is still a difficult task in general. Since there are many open problems relating to the study of the interaction of the interior fluid wave and boundary layers, initial layers, new analytical ideas are needed to give explicit expression of the Greens function.

1.2 Main goals of dissertation

In this section, we state the main goals of this dissertation. We devote ourselves to studying the following two issues: boundary waves and interior waves propagation. We construct Green's function for the initial-boundary problem and the Green's function for the shock profile. With the explicit expression of Green's function, we obtain the sharp pointwise nonlinear wave propagation structure respectively. We focus our effort on the following two models: one is the Broadwell model of the Boltzmann equation from the kinetic theory, the other is a simplified model from the magneto-hydrodynamics. It should be emphasized that the streamlined approaches to deal with these two issues are general and unified, and can be applied to other models. Our approaches rely on **master relationship** which is a useful tool introduced recently by [27].

In considering the spatial domain with boundary, the derivation of explicit formula for the Green's function for an initial boundary value problem for constant coefficient PDE is a task of fundamental importance. However, the explicit construction of the Green's function is in general a difficult task, as there are very rich and hidden wave structures along and around the boundary. With the presence of a physical boundary, a precise pointwise structure of the Green's function is even more important in the sense of relevance and richness to both physics and mathematics. There have been many essential progress on the boundary value problems for rarefied gas, numerical computations by Sone et.al. in[31]; analytical studies, particularly structure of Green's function in [5], [17], [18], [25].

For a better understanding the pointwise wave structure around the boundary, a new aspect of initial-boundary problem aroused, by deriving the master-relationship [26, 27]. One could construct the full boundary data in terms of the imposed boundary conditions. With such boundary relation, a well-posed boundary problem, with partial information of boundary data, would have a solution formula. In other words, the Green's function together with the boundary relation, yield the explicit expression for the Green's function for the initial-boundary value problem. Our first work is one of a series of studies followed with the methodology developed in that paper.

The second work in my dissertation is to study the wave behavior of wave perturbation around the shock profile. The shock profile can be classified into two cases: the classical Lax shocks; the nonclassical shocks which contain overcompressive and undercompressive shocks. The shock becomes overcompressive if more characteristics impinge into the shock area, and undercompressive if less characteristics impinge into the shock area, as compared to the Lax shock. The study of the stability of shock profile can be traced back to [12], where the energy method was used. However, this method has not been shown to be sufficient for the shock wave in general. The study of the large time coupling of nonlinear waves was initiated by [20] to obtain the time-asympototic convergence to a shock profile for a system of viscous conservation laws with an artificial viscosity matrix. [28] established a completion on the viscous shock stability along the framework of [20]. All the above mentioned studies are about the Lax shock profile with a small assumption on the shock strength. For the nonclassical case study, see [11, 28]. We seek a unified approach to solve the wave perturbation around the general shock profile with general amplitude.

Our works of the above problem were summarized in some research papers [8, 9, 10].

1.3 Summary

The contents of this dissertation are described as follows. We shall use the master relationship tool to study two wave patterns. The fist application is into the study of the boundary wave interaction, which is covered by Chapter 2. The second is about the shock wave interactions, covered the rest two chapters, from Chapter 3 to Chapter 4. The Appendix is given at the end of dissertation including some useful lemmas for the wave interaction.

In Chapter 2, we study the half space problem for the Broadwell model of Boltzmann equation. This problem is very interesting because of the characteristic boundary, where the speed of the boundary coincides with one speed of the transport matrix. The Green's function for the initial boundary value problem is decomposed into two parts: one is the Green's function for the initial value problem, we call it the fundamental solution for the whole space; the other is the convolution of this fundamental solution with full boundary data. The first part has been already established in [17]. To get the second part, we derive the master relationship: incoming-outgoing map to get the full boundary data. Once the Green's function is obtained, we can prove the nonlinear time-asymptotic stability of a given equilibrium state.

In Chapter 3, we consider the overcompressive shock wave propagation for a simple rotationally invariant system, which is originated from the study of MHD and nonlinear elasticity. We derive the master relationship: Dirichlet-Neumann map for the preparation. To handle the linearized system around the large amplitude profile, we initiated a method in our research. The structure of the linear wave propagation around the profile for Cauchy problem could be obtained by solving a variable coefficient PDE system. Firstly, we obtain a non-decaying structure which is caused by initial data through a standard procedure. With this observation, we extract the non-decaying part precisely. Otherwise, one would fail to get the nonlinear stability. The remainder satisfies an error equation. Then, we construct a function r to approximate the remainder, which satisfies a modified error equation, here we only modify the values of the shock profile at far fields. Due to this modification, one could separate the whole physical domain into three parts: two far fields, one finite domain region. This splitting method is similar to the work by Kreiss [15, 16]. In the left and right far field domains, we only need to consider the constant-coefficient initial boundary problem. Structures of solution in the finite domain could be obtained through the standard PDE method. So all the difficulties are shifted to how to give the boundary data in each part. It is very necessary to emphasize that Dirichlet and Neumann data at two inside boundaries are connected through profiles. Therefore, one could solve all the boundary information by setting up several equations, not just giving arbitrarily. Once all the boundary information is obtained, the structure in each part is clear. Hence we get the pointwise structure of the approximate solution. The truncation error produced in the approximate procedure satisfies a similar variable coefficient PDE system. Therefore, based on this approximate procedure, we define an iteration scheme to estimate the truncation error of each approximation. The smallness and pointwise localization property of r will assure that the series of errors $\sum_{j=0}^{\infty} r^j$ obtained in each iteration step converges. Due the overcompressive property, the Green's function of the linearized problem can get a sharp exponential decaying structure, excluding the non-decaying term. This sharp yields the global pointwise nonlinear wave structure for the full system.

In Chapter 4, we shall analyze the Lax shock wave propagation for the Broadwell model of Boltzmann equation. The approach is similar to the overcompressive case. However, since there exist transversal waves which propagate away from the shock regions, the nonlinear stability requires more detailed analysis on the nonlinear wave coupling with the estimate of the linearized problem.

In the Appendix, we list some lemmas on the wave interaction without proof. Some of them are from exiting results, while others could be proved by the hints given.

Chapter 2

Characteristic Half Space Problem for the Broadwell Model

2.1 Introduction

The most basic initial-boundary value problem in the rarefied gas is the one-dimensional Broadwell model given incoming boundary condition $b_+(t)$:

$$\begin{cases} \partial_t \tilde{F} + V\partial_x \tilde{F} = Q(\tilde{F}), \quad x > 0, \quad t > 0, \\ \tilde{F}(x,0) = \tilde{I}_0(x), \\ (1,0,0)\tilde{F}(0,t) = \tilde{b}_+(t), \end{cases} \tag{2.1}$$

where

$$\tilde{F}(x,t) = \begin{pmatrix} \tilde{f}_+(x,t) \\ \tilde{f}_0(x,t) \\ \tilde{f}_-(x,t) \end{pmatrix}, V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$Q(\tilde{F}) = \begin{pmatrix} \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_- \\ -(\frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_-) \\ \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_- \end{pmatrix}.$$

Here the unknown functions $\tilde{f}_+, \tilde{f}_0, \tilde{f}_-$ represent the mass densities for the gas particles moving in $x-$ direction with constant speed 1, 0 and -1 respectively. The part $\partial_t\tilde{F}+V\partial_x\tilde{F}$ represents the free transport mechanism in the gas flow, $Q(\tilde{F})$ models the collision mechanism.

Even for smooth, compatible initial and boundary data, there usually exists singularity in the solution around the boundary. The classical energy method is not enough to study the nonlinearity due to the boundary singularity. So we need to consider the pointwise estimates of the Green's function for the linearized problem and then close the nonlinearity.

For the collision operator Q, the equilibrium states $\tilde{F}$ are the positive-valued vector solutions of $Q(\tilde{F}) = 0$. Furthermore, an absolute equilibrium state M satisfies

$$M = (1/6, 1/3, 1/6)^t.$$

We are interested in the structure of solutions close to the absolute equilibrium state M.

The linearized equation around the absolute equilibrium state M is

$$\partial_t F + V\partial_x F = LF, \tag{2.2}$$

with the linearized collision operator L:

$$L = -\frac{1}{6}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

The macro-micro decomposition $(\mathbf{P_0}, \mathbf{P_1})$ is based on the kernel and co-kernel of the linearized collision operator L:

$$\begin{cases} \mathbf{P_0} : \mathbb{R}^3 \longmapsto kel(L), & \mathbf{P_0}|_{ker(L)} = I_{ker(L)}; \\ \mathbf{P_1} : \mathbb{R}^3 \longmapsto cokel(L), & \mathbf{P_1} \equiv I - \mathbf{P_0}. \end{cases}$$

Moreover, the decomposition $(\mathbf{P_0}, \mathbf{P_1})$ satisfies

$$\begin{cases} \mathbf{P_0}L = L\mathbf{P_0} = 0, \\ \mathbf{P_1}L = L\mathbf{P_1} = L, \\ \mathbf{P_0}Q = 0. \end{cases}$$

The explicit forms of the operators P_0 and P_1 are [17]:

$$\mathbf{P_0} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad \mathbf{P_1} = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \tag{2.3}$$

The Green's function $\mathbb{G}_b(x, y, t, \tau) = \mathbb{G}_b(x, y, t-\tau)$ for the linearized initial boundary value problem of Broadwell model is a 3×3 matrix-valued function which satisfies:

$$\begin{cases} \partial_t \mathbb{G}_b + V\partial_x \mathbb{G}_b = L\mathbb{G}_b, \quad x, \quad y > 0, \quad t > \tau, \\ \mathbb{G}_b(x, y, 0) = \delta(x-y)I, \\ (1, 0, 0)\mathbb{G}_b(0, y, t, \tau) = (0, 0, 0). \end{cases} \tag{2.4}$$

To construct $\mathbb{G}_b(x, y, t)$, we define a fundamental pair$(\mathbb{G}, \mathbb{H})$:

$$\begin{cases} \mathbb{G} = \mathbb{G}(x-y, t), \\ \mathbb{H}(x, y, t) = \mathbb{G}_b(x, y, t) - \mathbb{G}(x-y, t). \end{cases} \tag{2.5}$$

Here the first part is the fundamental solution of the linearized initial problem of Broadwell model for the whole space, which is also a 3×3 matrix valued function satisfying:

$$\begin{cases} \partial_t \mathbb{G} + V\partial_x \mathbb{G} = L\mathbb{G}, \quad x \in \mathbb{R}, \quad t > 0, \\ \mathbb{G}(x, 0) = \delta(x)I. \end{cases}$$

The second part $\mathbb{H}(x, y, t)$ satisfies the following system:

$$\begin{cases} \partial_t \mathbb{H} + V\partial_x \mathbb{H} = L\mathbb{H}, \quad x > 0, \quad t > 0, \\ \mathbb{H}(x, y, 0) = 0, \\ (1, 0, 0)\mathbb{H}(0, y, t) = -(1, 0, 0)\mathbb{G}(-y, t). \end{cases} \tag{2.6}$$

By applying the first Green's identity to (2.6), we have the representation for the second part:

$$\mathbb{H}(x, y, t) = \int_0^t \mathbb{G}(x, t-\tau) V \mathbb{H}(0, y, \tau) d\tau. \tag{2.7}$$

Because of the pointwise description of Green's function $\mathbb{G}(x,t)$ obtained in [17], the representation (2.7) yields a pointwise description of the interaction of the boundary data and the propagation of the interior fluid waves. However, the representation demands the full boundary data $\mathbb{H}(0, y, \tau)$, while physically the boundary data are given only for particle moving in $x-$ direction with speed 1. The global boundary data $\mathbb{H}(0, y, \tau)$ can be obtained through Fourier transformation and wellposedness of the half space problem. One can apply complex analysis to yield the exponential sharp global estimates of the boundary data.

Once the Green's function for the initial-boundary problem (2.4) is obtained, we can get the representation for the solution of initial-boundary problem (2.1). As is usually done, the solution $\tilde{F}$ is written as

$$\tilde{F} = M + W.$$

The boundary value $b_+(t)$, for simplicity, is assumed to be part of the absolute equilibrium state M. To illustrate the wave propagation properties of the solution, we assume the initial perturbation $W_0(x) \equiv \tilde{I}_0(x) - M$ satisfying $\|W_0(x)\| \leq \epsilon e^{-\sigma|x|}$, with $\epsilon \ll 1$. Then the perturbation satisfies

$$\begin{cases} \partial_t W + V \partial_x W - LW = Q(W), \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ W(x,0) = W_0(x), \\ (1,0,0) W(0,t) = 0. \end{cases} \tag{2.8}$$

Now we state the main theorem in this chapter:

Theorem 2.1.1. *There exists a constant C, such that the solution $W(x,t)$*

for (2.8) *satisfies*

$$\begin{aligned}\|W(x,t)\| &\le C\epsilon\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right)\\ &+C\epsilon\begin{cases}\frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}, & \textit{for}\ \ x\in(0,\frac{1}{\sqrt{3}}t-\sqrt{t}),\\ 0, & \textit{for}\ \ x\in(\frac{1}{\sqrt{3}}t-\sqrt{t},\infty).\end{cases}\end{aligned}$$

The rest of this chapter is as follows. In section 2.2, we prepare the full boundary data through Fourier transformation and wellposedness of the half space problem. In section 2.3, we will review the Green's function for the linearized initial value problem, then construct the Green's function for the initial-boundary problem by using the aforementioned fundamental pair $(\mathbb{G},\mathbb{H})$. In the section 2.4, we prove the main nonlinear stability theorem. This problem is very interesting due to a particular fact that the speed of the boundary coincides with one speed of the transport matrix. The resonance between particles and boundary can be clearly realized by the Green's function we constructed. The analysis in this chapter also provides a unified tool for studying the initial-boundary value problem for this kinetic equation with differential physical characteristics, as compared to the previous works in [5], [17], [18].

2.2 Master relationship: incoming-outgoing map

Consider the solution (2.7). To obtain full boundary $\mathbb{H}(0,y,\tau)$, we make use of the Laplace transform to construct a map

$$(1,0,0)\mathbb{H}(0,y,\tau)\to(0,0,1)\mathbb{H}(0,y,\tau).$$

For convenience, we denote

$$\mathbb{H}_+(x,y,t)\equiv(1,0,0)\mathbb{H}(x,y,t),$$

$$\mathbb{H}_0(x,y,t) \equiv (0,1,0)\mathbb{H}(x,y,t),$$
$$\mathbb{H}_-(x,y,t) \equiv (0,0,1)\mathbb{H}(x,y,t).$$

Then

$$\mathbb{H}(x,y,t) = \begin{pmatrix} \mathbb{H}_+(x,y,t) \\ \mathbb{H}_0(x,y,t) \\ \mathbb{H}_-(x,y,t) \end{pmatrix}.$$

For a function $y(t)$ defined for $t \geq 0$, its Laplace transform and inverse Laplace transform are defined as follows:

Definition 2.2.1. *For a function $y(t)$ defined for $t \geq 0$, its Laplace transform and inverse Laplace transform are defined as follows:*

$$\begin{aligned} Y(s) &= \mathbb{L}[y](s) \equiv \int_0^\infty e^{-st} y(t)dt, \\ y(t) &= \mathbb{L}^{-1}[Y](t) \equiv \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st}Y(s)ds, Re(s) = \gamma, \quad s.t. \\ &\quad \gamma \quad \text{is greater than the real part of all singularities of} \quad Y(s). \end{aligned}$$

Let $\mathbb{L}_s$ and $\mathbb{L}_\xi$ denote the Laplace transform with respect to time variable t and space variable x respectively. Take the Laplace transform of the first equation of (2.6) in the x and t variables:

$$\begin{cases} s\mathbb{J}[\mathbb{H}_+](\xi,y,s) + \xi\mathbb{J}[\mathbb{H}_+](\xi,y,s) = \mathbb{L}_s[\mathbb{H}_+](0,y,s) \\ -\frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s) - \mathbb{J}[\mathbb{H}_0](\xi,y,s) + \mathbb{J}[\mathbb{H}_-](\xi,y,s)), \\ s\mathbb{J}[\mathbb{H}_0](\xi,y,s) = \frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s) - \mathbb{J}[\mathbb{H}_0](\xi,y,s) + \mathbb{J}[\mathbb{H}_-](\xi,y,s)), \\ s\mathbb{J}[\mathbb{H}_-](\xi,y,s) - \xi\mathbb{J}[\mathbb{H}_-](\xi,y,s) = -\mathbb{L}_s[\mathbb{H}_-](0,y,s) \\ -\frac{1}{6}(\mathbb{J}[\mathbb{H}_+](\xi,y,s) - \mathbb{J}[\mathbb{H}_0](\xi,y,s) + \mathbb{J}[\mathbb{H}_-](\xi,y,s)), \end{cases} \tag{2.9}$$

here $\mathbb{J}[\mathbb{H}] \equiv \mathbb{L}_s[\mathbb{L}_\xi[\mathbb{H}]], s > 0, \xi > 0$.

From (2.9), we have

$$\begin{cases} \mathbb{J}[\mathbb{H}_+](\xi,y,s) = \frac{\mathbb{L}_s[\mathbb{H}_+](0,y,s) - s\mathbb{J}[\mathbb{H}_0](\xi,y,s)}{\xi+s}, \\ \mathbb{J}[\mathbb{H}_-](\xi,y,s) = \frac{\mathbb{L}_s[\mathbb{H}_-](0,y,s) + s\mathbb{J}[\mathbb{H}_0](\xi,y,s)}{\xi-s}. \end{cases}$$

Substitute these two representations into the second equation of (2.9), we get

$$\mathbb{J}[\mathbb{H}_0](\xi, y, s) = \frac{1}{6s+1} \frac{(\xi - s)\mathbb{L}_s[\mathbb{H}_+](0, y, s) + (\xi + s)\mathbb{L}_s[\mathbb{H}_-](0, y, s)}{\xi^2 - \frac{6s^3+3s^2}{6s+1}}.$$

Denote $\lambda_1(s) \equiv \sqrt{\frac{6s^3+3s^2}{6s+1}}$, then

$$\mathbb{J}[\mathbb{H}_0](\xi, y, s) = \frac{Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)}{\xi - \lambda_1} + \frac{Res_{\xi=-\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)}{\xi + \lambda_1}, \quad (2.10)$$

here $Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s)$ means the residue of function $\mathbb{J}[\mathbb{H}_0](\xi, y, s)$ at λ_1. Take the inverse Laplace transform of (2.10) with respect to space variable ξ:

$$\mathbb{L}_s[\mathbb{H}_0](x, y, s) = e^{\lambda_1 x} Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s) + e^{-\lambda_1 x} Res_{\xi=-\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s).$$

For the wellposedness of a differential equation imposes the solution $\mathbb{L}_S[\mathbb{H}_0](x, y, s)$ decays to zero as $x \to \infty$. This implies that

$$Res_{\xi=\lambda_1}\mathbb{J}[\mathbb{H}_0](\xi, y, s) = 0,$$

which yields the following relationship:

$$\begin{aligned}\mathbb{L}_s[\mathbb{H}_-](0, y, s) &= -\frac{\lambda_1 - s}{\lambda_1 + s}\mathbb{L}_s[\mathbb{H}]_+ y(0, y, s) \\ &= -(6s + 2 - \sqrt{(6s+3)(6s+1)})\mathbb{L}_s[\mathbb{H}_+](0, y, s) \quad (2.11)\end{aligned}$$

By the inverse Laplace transform of (2.11), we finally get the incoming-outgoing map formula:

$$\begin{aligned}\mathbb{H}_-(0, y, t) &= -6\partial_t \mathbb{H}_+(0, y, t) - 2\mathbb{H}_+(0, y, t) \\ &+6\partial_t * \frac{e^{-1/2t}}{2\sqrt{\pi t}} * \partial_t * \frac{e^{-1/6t}}{2\sqrt{\pi t}} * \mathbb{H}_+(0, y, t). \quad (2.12)\end{aligned}$$

Here, instead of studying the inverse Laplace transform of $\sqrt{s}$, we consider $\frac{\sqrt{s}}{s} = \frac{1}{\sqrt{s}}$ by the usual inversion formula of the Laplace transform:

$$L(t) \equiv \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{1}{\sqrt{s}} ds = \frac{1}{2\pi\sqrt{s}} \int_{-\infty}^{\infty} e^{is} \frac{1}{\sqrt{is}} ds = \frac{1}{\sqrt{\pi t}}. \quad (2.13)$$

The inversion of the Laplace transform has the property that division in s corresponds to differentiation in t.

2.3 Construct the Green's function $\mathbb{G}_b(x, y, t)$

The fundamental pair (2.5) yields the decomposition $\mathbb{G}_b(x, y, t) = \mathbb{G}(x - y, t) + \mathbb{H}(x, y, t)$. Firstly, we recall the following theorem in [17] on Green's function $\mathbb{G}(x, t)$ for the initial value problem.

Theorem 2.3.1. *There exists a positive constant C such that*

$$\left\| \mathbb{G}(x,t) - e^{-t/6} \begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t) \end{pmatrix} \right\| \leq C \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} \right) + Ce^{-(|x|+t)/C}, \tag{2.14}$$

for all $x \in \mathbb{R}, \quad t > 0.$

To get the pointwise estimate for the function $\mathbb{H}(x, y, t)$, we also need another lemma in [25].

Lemma 2.3.2. *Suppose that λ, $\mu > 0$. Then for given positive constants D_0 and D_1, there exists $D_2 > 0$ such that for any x, z, $t \geq 0$, and $\alpha \geq 1$,*

$$\int_0^t \frac{e^{-\frac{(x-\lambda(t-\tau))^2}{D_1(t-\tau)}}}{\sqrt{t-\tau+1}} \frac{e^{-\frac{(z-\mu\tau)^2}{D_0(\tau+1)}}}{(\tau+1)^{\frac{\alpha}{2}}} d\tau = O(1) \left(\frac{1}{(z+1)^{\frac{\alpha-1}{2}}} + \frac{1}{(t+1)^{\frac{\alpha-1}{2}}} \right) \frac{e^{-\frac{(x-\lambda t+\frac{\lambda}{\mu}z)^2}{D_2(t+1)}}}{\sqrt{t+1}}.$$

Lemma 2.3.3.

$$\|\mathbb{G}_+(-y,\tau)\| \leq O(1) \left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}} + \frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}} + e^{-(|y|+\tau)/C} \right),$$

$$\begin{aligned} &\|\partial_\tau \mathbb{G}_+(-y,\tau)\| \\ \leq\ & O(1) \left(\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{(1+\tau)} + \frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{(1+\tau)} \right) (1 + \frac{1}{\sqrt{1+\tau}}) + e^{-(|y|+\tau)/C} \right), \end{aligned}$$

$$
\begin{aligned}
&\left\|\partial_\tau * \frac{e^{-1/2\tau}}{2\sqrt{\pi\tau}} * \partial_\tau * \frac{e^{-1/6\tau}}{2\sqrt{\pi\tau}} * \mathbb{G}_+(-y,\tau)\right\| \\
\leq\ & O(1)\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}(1+\tau)} + \frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}(1+\tau)} + e^{-(|y|+\tau)/C}\right).
\end{aligned}
$$

Proof. The fist two inequalities are straightforward. For the third one, just apply Lemma 2.3.2 when $z = 0$. □

Now we have the following theorem:

Theorem 2.3.4.

$$
\begin{aligned}
&\left\|\mathbb{G}_b(x,y,t) - e^{-t/6}\begin{pmatrix} \delta(x-y-t) & 0 & 0 \\ 0 & \delta(x-y) & 0 \\ 0 & 0 & \delta(x-y+t)\end{pmatrix}\right\| \\
\leq\ & O(1)\left[e^{-(|x|+|y|+t)/C} + \frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}\right]
\end{aligned}
$$

Remark 1. *The last term represents the reflections at the boundary of waves with negative speed* $-\frac{1}{\sqrt{3}}$ *to waves with positive speed* $\frac{1}{\sqrt{3}}$.

Proof. Denote the 0th order Particle-Wave decomposition for $\mathbb{G}(x,t)$ as

$$
\mathbb{G}(x,t) = e^{-t/6}\begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t)\end{pmatrix} + \mathbb{W}_0(x,t).
$$

From (2.7), (2.12) and Theorem 2.3.1 we have

$$
\begin{aligned}
\mathbb{H}(x,y,t) = &\int_0^t \mathbb{W}_0(x,t-\tau)V\mathbb{H}(0,y,\tau)d\tau \\
+ &\int_0^t e^{-(t-\tau)/6}\begin{pmatrix} \delta(x-(t-\tau)) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t-\tau)\end{pmatrix} V\mathbb{H}(0,y,\tau)d\tau.
\end{aligned}
$$

For the first term, using Lemma 2.3.3 we have the following estimate

$$
\begin{aligned}
&\left\|\int_0^t \mathbb{W}_0(x,t-\tau)V\mathbb{H}(0,y,\tau)d\tau\right\| \\
\le\ & O(1)\int_0^t \|\mathbb{W}_0(x,t-\tau)\|\left(\|\mathbb{H}_+(0,y,\tau)\|+\|\mathbb{H}_-(0,y,\tau)\|\right)d\tau \\
\le\ & O(1)\int_0^t \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+(t-\tau))}}}{\sqrt{1+t-\tau}}+\frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}+e^{-(|x|+t-\tau)/C}\right)\left(\|\mathbb{G}_+(-y,\tau)\|\right. \\
&\left.+\left\|\partial_t\mathbb{G}_+(-y,\tau)+\partial_\tau * \frac{e^{-1/2\tau}}{2\sqrt{\pi\tau}} * \partial_\tau * \frac{e^{-1/6\tau}}{2\sqrt{\pi\tau}} * \mathbb{G}_+(-y,\tau)\right\|\right)d\tau \\
\le\ & O(1)\int_0^t \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+(t-\tau))}}}{\sqrt{1+t-\tau}}+\frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}+e^{-(|x|+t-\tau)/C}\right) \\
&\cdot\left(\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}\right)+e^{-(|y|+\tau)/C}\right)d\tau \\
\le\ & O(1)\left[e^{-(|x|+|y|+t)/C}+\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}}\right]
\end{aligned}
\tag{2.15}
$$

For the second term, we have

$$
\begin{aligned}
&\left\|\int_0^t e^{-\frac{t-\tau}{6}}\begin{pmatrix}\delta(x-(t-\tau)) & 0 & 0\\ 0 & \delta(x) & 0\\ 0 & 0 & \delta(x+t-\tau)\end{pmatrix}V\mathbb{H}(0,y,\tau)d\tau\right\| \\
&\le O(1)\int_0^t e^{-\frac{t-\tau}{6}}\left[(\delta(x-(t-\tau))\,\|\mathbb{G}_+(-y,\tau)\|\right. \\
&\qquad\left.+\delta(x+(t-\tau))\,\|\mathbb{G}_-(-y,\tau)\|\right]d\tau \\
&\le O(1)\int_0^t e^{-\frac{t-\tau}{6}}\delta(x-(t-\tau)) \\
&\qquad\left(\frac{e^{-\frac{|-y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{\sqrt{1+\tau}}+e^{-(|y|+\tau)/C}\right)d\tau \\
&\le O(1)e^{-x/C}\left(\frac{e^{-\frac{|-y+\frac{1}{\sqrt{3}}(t-x)|^2}{C(1+(t-x))}}}{\sqrt{1+(t-x)}}+e^{-(|y|+(t-x))/C}\right)
\end{aligned}
$$

$$\leq O(1)e^{-(x+y+t)/C}. \tag{2.16}$$

From (2.15), (2.16) and (2.5) we get the pointwise estimate for $\mathbb{G}_b(x, y, t)$.

□

2.4 Nonlinear stability of an absolute equilibrium state

In this section, we prove the main Theorem 2.1.1 of this chapter. For such a problem (2.8), the solution can be represented by the Green's function $\mathbb{G}_b(x, y, t)$:

$$\begin{aligned}
&W(x,t)\\
= &\int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy + \int_0^t \int_0^\infty \mathbb{G}_b(x,y,t-\tau)Q(W)(y,\tau)dyd\tau\\
= &\int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy + \int_0^t \int_0^\infty \mathbb{G}_b(x,y,t-\tau)\mathbf{P_1}Q(W)(y,\tau)dyd\tau,
\end{aligned}$$

here $\mathbf{P_1}$ is the micro part of macro-micro decomposition (2.3).

$W(x, t)$ can be solved by a Picard's iteration:

$$\begin{cases}
W^{(0)}(x,t) = \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy,\\
W^{(l)}(x,t) = W^{(0)}(x,t) + \int_0^t \int_0^\infty \mathbb{G}_b(x,y,t-\tau)Q(W^{(l-1)})(y,\tau)dyd\tau,\\
\quad \text{for} \ \ l \geq 1.
\end{cases}$$

By Theorem 2.3.4, there exists some constant K_0 such that

$$\left\| \int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy \right\| \leq K_0\epsilon \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right).$$

We make an ansatz assumption:

$$\begin{aligned}
&\|W(x,t)\|\\
\leq\ &2K_0\epsilon \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right)
\end{aligned}$$

$$+2K_0\epsilon \begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}, & \text{for } x \in (0, \frac{1}{\sqrt{3}}t - \sqrt{t}), \\ 0, & \text{for } x \in (\frac{1}{\sqrt{3}}t - \sqrt{t}, \infty). \end{cases} \tag{2.17}$$

Now, we proof the ansatz.

Note that for $\mathbb{G}(x,t)$:

$$\|\mathbb{G}(x,t)\mathbf{P_1}\| = O(1)\left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{1+t} + \frac{e^{-\frac{(x+\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{1+t} + e^{-(|x|+t)/C} \right).$$

However, the algebraic decaying rate $(1+t)^{-1}$ of $\mathbb{G}(x,y,t)\mathbf{P_1}$ is not necessarily true for $\mathbb{G}_b(x,y,t)\mathbf{P_1}$ globally due to the presence of the boundary.

Case 1: $\{0 < x < \frac{1}{\sqrt{3}}t/2\}$.

We compute $W(x,t)$ by the forward representation:

$$\begin{aligned} & \|W(x,t)\| \\ = & \|\int_0^\infty \mathbb{G}_b(x,y,t)W_0(y)dy\| + \int_0^t\int_0^\infty \|\mathbb{G}_b(x,y,t-\tau)\mathbf{P_1}Q(W)(y,\tau)\|\,dyd\tau \\ & \le K_0\epsilon\left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right) \\ & + \int_0^t\int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \|W(y,\tau)\|^2 dyd\tau \\ & + \int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)} \frac{e^{-\frac{[x+y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\|W(y,\tau)\|^2 dyd\tau \end{aligned} \tag{2.18}$$

Under the ansatz assumption (2.17), we have:

$$\begin{aligned} & \int_0^t\int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \|W(y,\tau)\|^2 dyd\tau \\ \le \; & O(1)K_0^2\epsilon^2 \left[\int_0^t\int_0^\infty \left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \right. \end{aligned}$$

$$\left(\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}\tau)^2}{C(1+\tau)}}}{1+\tau}+e^{-(|y|+\tau)/C}\right)dyd\tau$$

$$+\int_0^t\int_0^{\frac{1}{\sqrt{3}}\tau-\sqrt{\tau}}\left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}\right)$$

$$\left.\frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)}dyd\tau\right].$$

By Lemma A.0.2 and Lemma A.0.3 in the Appendix,

$$\int_0^t\int_0^\infty\left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}\right)$$

$$\left(\frac{e^{-\frac{(y-\frac{1}{\sqrt{3}}\tau)^2}{C(1+\tau)}}}{1+\tau}+e^{-(|y|+\tau)/C}\right)dyd\tau$$

$$\leq I^{2,2}(x,t;0,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)+I^{2,2}(x,t;0,t;\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},D)$$

$$\leq O(1)\left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right)$$

$$+O(1)\begin{cases}\frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}, & \text{for } x\in(0,\frac{1}{\sqrt{3}}t-\sqrt{t}),\\ 0, & \text{for } x\in(\frac{1}{\sqrt{3}}t-\sqrt{t},\infty).\end{cases}$$

By Lemma A.0.4 in the Appendix,

$$\int_0^t\int_0^{\frac{1}{\sqrt{3}}\tau-\sqrt{\tau}}\left(\frac{e^{-\frac{[x-y+\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{[x-y-\frac{1}{\sqrt{3}}(t-\tau)]^2}{C(1+t-\tau)}}}{1+t-\tau}\right)$$

$$\frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)}dyd\tau$$

$$\leq H(x,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}};C)+H(x,t;-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};C)$$

$$=O(1)\left(\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right)$$

$$+O(1)\begin{cases} \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}}, & \text{for } x\in(0,\frac{1}{\sqrt{3}}t-\sqrt{t}), \\ 0, & \text{for } x\in(\frac{1}{\sqrt{3}}t-\sqrt{t},\infty). \end{cases}$$

$$\begin{aligned}
&\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\|W(y,\tau)\|^2dyd\tau\\
&\le O(1)K_0^2\epsilon^2\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\\
&\quad\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\left(\frac{e^{-\frac{|y-\frac{1}{\sqrt{3}}\tau|^2}{C(1+\tau)}}}{1+\tau}+e^{-(|y|+\tau)/C}\right)dyd\tau\\
&+O(1)K_0^2\epsilon^2\int_0^t\int_0^{\frac{1}{\sqrt{3}}(t-\tau)}\\
&\quad\frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+y)}}\frac{1}{(|y+\frac{1}{\sqrt{3}}\tau|+1)(|y-\frac{1}{\sqrt{3}}\tau|+1)}dyd\tau\\
&\le O(1)K_0^2\epsilon^2(t+1)^{-1}.
\end{aligned}\tag{2.19}$$

Here the decaying rate -1 in (2.19) is obtained by the similar calculation in the proof of Theorem 1.2 in [25].

Case 2: $\{\frac{1}{\sqrt{3}}t/2<x<\frac{1}{\sqrt{3}}t\}$.

Fix (x,t) and treat $\mathbb{G}_b(x,y,t,\tau)$ as an operator-valued function of (y,τ). From a symmetry consideration, we have

$$\begin{aligned}
&\|\mathbb{G}_b(x,y,t,\tau)\mathbf{P_1}\|\\
=\;& O(1)\left(\frac{1}{\sqrt{t-\tau+1}}+\frac{1}{\sqrt{x+1}}\right)\left(\frac{e^{-\frac{|x+y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{1+t-\tau}}\right)\\
&+O(1)\left(\frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}+\frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau}+e^{-(x+y+t-\tau)}\right).
\end{aligned}$$

Using the fact that, in the region $x\in(\frac{1}{\sqrt{3}}t/2,\frac{1}{\sqrt{3}}t)$,

$$\|\mathbb{G}_b(x,y,t,\tau)\mathbf{P_1}\|=\|\mathbb{G}(x,y,t,\tau)\mathbf{P_1}\|,$$

by similar calculation, we can prove that

$$\begin{aligned}
&\int_0^t \int_0^\infty \mathbb{G}_b(x, y, t-\tau) Q(W)(y, \tau) dy d\tau \\
\leq\ & O(1) K_0^2 \epsilon^2 \left(\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right) \right. \\
&\left. + \frac{1}{\sqrt{(|x+\frac{1}{\sqrt{3}}t|+1)(|x-\frac{1}{\sqrt{3}}t|+1)}} \right).
\end{aligned}$$

Case 3: $\{x > \frac{1}{\sqrt{3}}t\}$.

$$\begin{aligned}
&\left\| \int_0^t \int_0^\infty \mathbb{G}_b(x, y, t-\tau) Q(W)(y, \tau) dy d\tau \right\| \\
\leq\ & \int_0^t \int_0^\infty \left(\frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} + \frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} \right) \|W(y,\tau)\|^2 dy d\tau \\
& + \int_0^t \int_0^{\frac{1}{\sqrt{3}}(t-\tau)} \frac{e^{-\frac{|x+y-\frac{1}{\sqrt{3}}(t-\tau)|^2}{C(1+t-\tau)}}}{\sqrt{(1+t-\tau)(1+x)}} \|W(y,\tau)\|^2 dy d\tau
\end{aligned}$$

The calculation for the first term is similar to case 1. To deal with the second term, note that

$$\frac{1}{\sqrt{1+x}} < O(1) \frac{1}{\sqrt{1+t}},$$

we bring out the decaying factor $\frac{e^{-\frac{(x-\frac{1}{\sqrt{3}}t)^2}{C(1+t)}}}{\sqrt{1+t}}$ and the left is $O(1)$, so

$$\begin{aligned}
&\left\| \int_0^t \int_0^\infty \mathbb{G}_b(x, y, t-\tau) Q(W)(y, \tau) dy d\tau \right\| \\
\leq\ & O(1) K_0^2 \epsilon^2 \left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C} \right),
\end{aligned}$$

and the last case is proved.

Therefore, we verify the ansatz and get the nonlinear stability theorem.

Chapter 3

Over-compressive Shock Profile for a Simplified Model of MHD

3.1 Introduction

Consider the following simple rotationally invariant system originated from the study of MHD and nonlinear elasticity by Freistuler [11],

$$\begin{cases} \tilde{u}_t + (\tilde{u}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{u}_{xx}, \\ \tilde{v}_t + (\tilde{v}(\tilde{u}^2 + \tilde{v}^2))_x = \mu \tilde{v}_{xx}. \end{cases} \tag{3.1}$$

The characteristics are

$$\begin{aligned} &r_1(\tilde{u}, \tilde{v}) = (\tilde{v}, -\tilde{u}), \quad r_2(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}), \\ &\lambda_1(\tilde{u}, \tilde{v}) = \tilde{u}^2 + \tilde{v}^2, \quad \lambda_2(\tilde{u}, \tilde{v}) = 3\lambda_1(\tilde{u}, \tilde{v}). \end{aligned} \tag{3.2}$$

The 1-characteristic is linearly degenerate and the 2-characteristic is genuinely nonlinear except at the origin

$$\nabla\lambda_1 \cdot r_1(\tilde{u}, \tilde{v}) = 0, \quad \nabla\lambda_2 \cdot r_2(\tilde{u}, \tilde{v}) = 6(\tilde{u}^2 + \tilde{v}^2). \tag{3.3}$$

A viscous shock wave has end states along the same radial direction through the origin, i.e., in the direction of $r_2(\tilde{u}, \tilde{v})$. The system is rotational invariant and so, without loss of generality, consider the ends states to have the second component zeros $(\tilde{u}_\pm, 0)$. When $\tilde{u}_\pm$ are of the same sign, the shock is classical, and there is only one connecting orbit along the $\tilde{u}_-$ axis. When $\tilde{u}_+ < 0$, the shock may cross the point of non-strictly hyperbolic point $(\tilde{u}, \tilde{v}) = 0$ and becomes over-compressive.

Here we are interested in over-compressive shock, which is characterized by

$$\lambda_2(\tilde{u}_-, \tilde{v}_-) > \lambda_1(\tilde{u}_-, \tilde{v}_-) > b > \lambda_2(\tilde{u}_+, \tilde{v}_+) > \lambda_1(\tilde{u}_+, \tilde{v}_+), \tag{3.4}$$

b is the speed of over-compressive shock. By (3.4), the over-compressive shock is a node-node connection. Thus when exists, there is a 1-parameter family of viscous profiles.

The goal of this chapter is to prove that over-compressive shock profiles for (3.1) can be stable against small perturbations: given the profile $\Phi = (\phi, \psi)$ of an (appropriate) over-compressive shock profile, and a function $(\tilde{u}_0(x), \tilde{v}_0(x))$ (of appropriate type) such that

$$\begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} \equiv \begin{pmatrix} \tilde{u}_0(x) \\ \tilde{v}_0(x) \end{pmatrix} - \Phi$$

is small(in an appropriate sense), then the solution $(\tilde{u}, \tilde{v})$ of (3.1) with initial data $(\tilde{u}_0(x), \tilde{v}_0(x))$ converges time-asymptotically to another profile $\Phi^* = (\phi^*, \psi^*)$. We give the pointwise convergence rate to the new profile.

Since there is a 1-parameter family of viscous shocks with given end states, the stability of a shock would have to be understood in the following way: The perturbation of a stable shock profile would convergence to another profile in the 1-parameter family. Thus, in addition of the phase shift, the perturbation also change the time-asymptotic profile of the solution. Therefore, instead of using the conservation laws to identify the phase shift and diffusion waves as for the Laxian shocks, we should use the two conservation laws to identify the phase shift and the new profile. This should make the situation well-posed as we have two conservation laws and the same number of parameters to determine.

Theorem 3.1.1. *(Main Theorem) Assume that $\mu = 1$. Given an over-compressive shock profile $\Phi = (\phi, 0)$, $\Phi(\pm\infty) = (\tilde{u}_\pm, \tilde{v}_\pm)$, with $(\tilde{u}_-, \tilde{v}_-) = (1, 0)$, $(\tilde{u}_+, \tilde{v}_+) = (-\frac{1}{4}, 0)$. Let C be a universal positive constant, and let*

$$\left\{\begin{array}{c} \tilde{u}_0(x) \\ \\ \tilde{v}_0(x) \end{array}\right\} = \Phi(x) + \left\{\begin{array}{c} u_0(x) \\ \\ v_0(x) \end{array}\right\}$$

with $\|u_0(x)\|, \|v_0(x)\| \le O(1)\varepsilon e^{-|x|/C}$. Then for ε sufficiently small, there exists a unique profile Φ^* with $\Phi(\pm\infty) = \Phi^*(\pm\infty)$, such that the solution of (3.1) satisfies:

$$\left\| \left\{\begin{array}{c} \tilde{u}(x,t) \\ \\ \tilde{v}(x,t) \end{array}\right\} - \Phi^*(x - bt - x_0) \right\| \le O(1)\varepsilon e^{-(|x-bt|+t)/C}.$$

Here the constant x_0 and profile Φ^* are determined by

$$\int_R (\tilde{u}_0, \tilde{v}_0)(x, 0)dx = \int_R ((\phi^*, \psi^*)(x - bt - x_0) - (\phi, \psi)(x - bt))dx.$$

This nonlinear stability will be proved in Section 3.5, after preliminary preparation for later use in Section 3.2, brief framework introduction on solving an simplified variable coefficient PDE system related to the linearized system in Section 3.3 and rigorous analysis on linearized problem in Section 3.4.

To handle the linearized system around the general amplitude profile, we initiated a method in our research. The structure of the linear wave propagation around the profile for Cauchy problem could be obtained by solving a variable coefficient PDE system. Firstly, we obtain a non-decaying structure which is caused by initial data through a standard procedure.

With this observation, we extract the non-decaying part precisely. Otherwise, one would fail to get the nonlinear stability. The remainder satisfies an error equation. Then, we construct a function r to approximate the remainder, which satisfies a modified error equation, here we only modify the values of the shock profile at far fields. Due to this modification, one could separate the whole physical domain into three parts: two far fields, one finite domain region. This splitting method is similar to the work by Kreiss [15, 16]. In the left and right far field domains, we only need to consider the constant-coefficient initial boundary problem. Structures of solution in the finite domain could be obtained through the standard PDE method. So all the difficulties are shifted to how to give the boundary data in each part. It is very necessary to emphasize that Dirichlet and Neumann data at two inside boundaries are connected through profiles. Therefore, one could solve all the boundary information by setting up several equations, not just giving arbitrarily. Once all the boundary information is obtained, the structure in each part is clear. Hence we get the pointwise structure of the approximate solution. The truncation error produced in the approximate procedure satisfies a similar variable coefficient PDE system. Therefore, based on this approximate procedure, we define an iteration scheme to estimate the truncation error of each approximation. The smallness and pointwise localization property of approximate function will assure that the series of errors obtained in each iteration step converges.

There are many other works on the stability problem around different profiles, using different approaches. See[20, 28, 34] for the case of small amplitude Laxian shock profile. For the large-amplitude profile, see series of results by Zumbrun and various collaborators, most of which are based on the framework of the Evans-function, e.g., [29].

In the rest of introduction, we want to comment on the system (3.1) and the non-classical shock. In physical systems of conservation laws, rotational invariance typically arises due to natural isotropy. Unlike the classical conservation law theory, in which the isotropy is superposed with Galilean invariance in a geometrically non-generic way, the MHD or elastic plane waves display rotational symmetry in generic form, which induces (3.2), (3.3) and (3.4). The over-compressive shock wave we study in this chapter are good descriptions of certain non-classical shock waves that arise in physical systems, especially in magnetohydrodynamics. Analysis shows that viscous profiles for the over-compressive shocks are nonlinearly stable, but not uniformly with respect to the strength of the viscosity, see [11] and references therein. The over-compressive shocks for (3.1) are physical provided that dissipations are not small. They are not admissible inviscid shocks.

3.2 Preliminary

3.2.1 Profiles of over-compressive shocks

Profile (ϕ, ψ) of any viscous shock wave is a heteroclinic orbit of the O.D.E. system

$$
\begin{aligned}
\phi' &= (\phi^2 + \psi^2 - b)\phi - b_1, \\
\psi' &= (\phi^2 + \psi^2 - b)\psi - b_2,
\end{aligned} \tag{3.5}
$$

in which the speed b and the relative flux b_1, b_2 are given by the Rankine-Hugoniot conditions:

$$
\begin{aligned}
(\tilde{u}_+^2 + \tilde{v}_+^2 - b)\tilde{u}_+ = (\tilde{u}_-^2 + \tilde{v}_-^2 - b)\tilde{u}_- = b_1, \\
(\tilde{u}_+^2 + \tilde{v}_+^2 - b)\tilde{v}_+ = (\tilde{u}_-^2 + \tilde{v}_-^2 - b)\tilde{v}_- = b_2,
\end{aligned} \tag{3.6}
$$

with $\tilde{u}_\pm = \phi(\pm\infty), \tilde{v}_\pm = \psi(\pm\infty)$.

Quote a lemma in [11]:

Lemma 3.2.1. (i): *$\tilde{u}_\pm$, $\tilde{v}_\pm$ satisfy the inequality* (3.4) *with b_1 and b_2 from* (3.6) *if and only if* $(\tilde{u}_-, \tilde{v}_-) \neq (0,0)$ *and* $(\tilde{u}_+, \tilde{v}_+) = \alpha(\tilde{u}_-, \tilde{v}_-)$, $\alpha \in (-\frac{1}{2}, 0)$.

(ii): *In this case, there is a 1-parameter family of viscous profiles satisfy* (3.5) *and* (3.6).

In the following Figure 3.1: the left is the 1-parameter family of profiles, the right is 1-component plot for the over-compressive shock profile we give in our main theorem: $\Phi = (\phi, 0) = (\frac{4-49e^x+7\sqrt{16-24e^x+49e^{2x}}}{32}, 0)$. The end states are $(\tilde{u}_-, \tilde{v}_-) = (1,0)$, $(\tilde{u}_+, \tilde{v}_+) = (-\frac{1}{4}, 0)$.

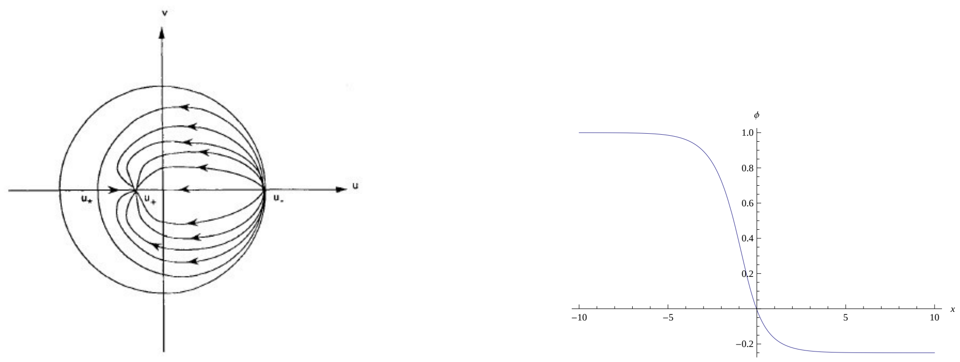

Figure 3.1: Overcompressive shock profiles

3.2.2 Master relationship: Dirichlet-Neumann map

We prepare the basic materials for constructing Dirichlet-Neumann map we will need in following sections, see [26] for more details.

Consider the following half space problem:

$$\begin{cases} V_t + AV_x = V_{xx}, \quad \text{for} \quad x > 0, \quad t > 0, \\ V(0,t) = V_b(0,t), \\ V(x,0) = V_0(x), \end{cases} \tag{3.7}$$

A is a constant. A functional property on given boundary data V_b is imposed for the consistency with zero initial data and for an application condition for the Bromwich integral:

$$V_b \in \{g|\mathbb{L}[g](s) \text{ exists for } Re(s) > 0, g^{[n]}(0) = 0 \text{ for } n \in \mathbb{N} \cup \{0\}\}.$$

Under the definition of (2.2.1), we define $\mathbb{L}_s$ as the Laplace transform with respect to time variable t, $\mathbb{L}_\xi$ as the Laplace transform with respect to space variable x, $\mathbb{L}^{-1}$ as the inverse Laplace transform.

Case 1: $V_0(x) = 0$.

Apply the Laplace-Laplace transform to the equation of V in (3.7), one has:

$$(s + A\xi - \xi^2)\mathbb{J}[V] = (A - \xi)\mathbb{L}_s[V_b](s) - \partial_x \mathbb{L}_s[V](0, s),$$

$\mathbb{J}[V] \equiv \mathbb{L}_\xi[\mathbb{L}_s[V]$.

By the purely algebraic manipulations one can have the solution in terms of variable ξ, s, $\mathbb{L}_s[V_b]$ and $\partial_x \mathbb{L}_s[V]$:

$$\begin{aligned} \mathbb{J}[V] &= \frac{-\partial_x \mathbb{L}_s[V](0,s) + \frac{1}{2}\mathbb{L}_s[V_b](s)(A - \sqrt{A^2+4s})}{\sqrt{A^2+4s}(\xi - \frac{1}{2}(A + \sqrt{A^2+4s}))} \\ &\quad + \frac{-\partial_x \mathbb{L}_s[V](0,s) + \frac{1}{2}\mathbb{L}_s[V_b](s)(A + \sqrt{A^2+4s})}{\sqrt{A^2+4s}(\xi - \frac{1}{2}(A - \sqrt{A^2+4s}))}. \end{aligned} \tag{3.8}$$

Note that the characteristic polynomial $s + A\xi - \xi^2$ has only one positive root $\xi = \frac{1}{2}(A + \sqrt{A^2+4s})$ for $s > 0$, the well-posedness of V requires that $|V| < \infty$ as $x \to \infty$. This results in:

$$\frac{-\partial_x \mathbb{L}[V](0,s) + \frac{1}{2}\mathbb{L}[V_b](s)(A - \sqrt{A^2+4s})}{\sqrt{A^2+4s}} = 0,$$

which gives the Dirichlet-Neumann map:

$$\partial_x \mathbb{L}_s[V](0,s) = -\frac{2s}{2A + \sqrt{A^2+4s}}\mathbb{L}_s[V_b](s).$$

Case 2: $V_0(x) \neq 0$.

One needs to shift the initial datum to the boundary datum for the new variable:

$$\begin{cases} \bar{V} = V - I(x,t), \\ I(x,t) = \int_0^\infty \mathbb{G}_1(x-y,t)V_0(y)dy. \end{cases} \tag{3.9}$$

$\mathbb{G}_1$ is the fundamental solution for

$$\begin{cases} \mathbb{G}_{1t} + A\mathbb{G}_{1x} = \mathbb{G}_{1xx}, \quad \text{for} \quad x \in R, \quad t > 0, \\ \mathbb{G}_1(x,0) = \delta(x). \end{cases} \tag{3.10}$$

Therefore, $\bar{V}$ satisfies:

$$\begin{cases} \bar{V}_t + A\bar{V}_x = \bar{V}_{xx}, \quad \text{for} \quad x > 0, \quad t > 0, \\ \bar{V}(0,t) = V_b(0,t) - I(0,t), \\ \bar{V}(x,0) = 0, \end{cases} \tag{3.11}$$

which is the case 1.

3.3 A general framework to solve a variable coefficient PDE system

In this section, we present a framework containing several steps to solve a simplified problem:

$$\begin{cases} w_t + (f(\phi)w)_x = w_{xx}, \\ w(x,0) = w_0(x), \end{cases} \tag{3.12}$$

where

$$\begin{cases} f \in C^1, \\ \phi(\pm\infty) = M_\pm, M_+ < 0, M_- > 0, \\ \|\phi'(x)\| = O(1)e^{-|x|/C}, \\ \|w_0(x)\| = O(1)\varepsilon e^{-|x|/C}. \end{cases}$$

Conservation law for w implies that there exists a non-decaying structure due to initial data. The first step in our framework is to extract this non-decaying part precisely. The non-decaying component is defined to be:

$\beta(t)\varphi(x)$, with the property that

$$\begin{cases} \|\beta'(t)\| = O(1)\varepsilon\frac{e^{-C_1 t}}{\sqrt{t+1}}, \\ \|\varphi(x)\| = O(1)e^{-|x|/C}. \end{cases}$$

$\varphi(x)$ is the stationary solution of (3.12) with $\varphi(\pm\infty) = 0$.

The remainder $z \equiv w - \beta(t)\varphi(x)$ satisfies

$$\begin{cases} z_t + (f(\phi)z)_x = z_{xx} + S(x,t), \\ \|z(x,0)\| = O(1)\varepsilon e^{-|x|/C}, \int_R z(x,0)dx = 0, \\ \|S(x,t)\| = O(1)\varepsilon e^{-(|x|+t)/C}. \end{cases} \tag{3.13}$$

Since the profile we considered here is strong, the approach of solving the equation of z is based on an approximate problem:

$$\begin{cases} r_t + (f(\phi_L)r)_x = r_{xx} + S(x,t), \\ r(x,0) = z(x,0), \end{cases} \tag{3.14}$$

where

$$\phi_L(x) = \begin{cases} \phi(\infty), & \text{if } \ x \geq L, \\ \phi(x), & \text{if } \ |x| < L, \\ \phi(-\infty), & \text{if } \ x \leq -L, \end{cases} \tag{3.15}$$

choosing L sufficiently large, $L = O(1)|\ln\varepsilon|$.

In the second step, we consider the problem (3.14). With the help of approximating function $\phi_L(x)$, one could split the whole space domain into three parts: the far field $(-\infty, -L]$, $[L, \infty)$, and finite domain region $(-L, L)$. Convergence rate for $r(x,t)$ at each boundary can be obtained through the wave interactions in these three domains. Once all the boundary information is clear, one could solve the problem (3.14) in each domain separately. Therefore, all the difficulties in our second step are shifted to how to solve the boundary data for r at $x = \pm L$.

When $|x| > L$, (3.14) turns to be a constant coefficient problem. The solvability condition for the constant coefficient problem in each far field gives a Dirichlet-Neumann map:

$$\mathbb{L}_s[r]_x(-L-, s) = \mathbf{D}_1 \mathbb{L}_s[r](-L-, s) + I_1(-L-, s), \qquad (3.16)$$

$$\mathbb{L}_s[r]_x(L+, s) = \mathbf{D}_2 \mathbb{L}_s[r](L+, s) + I_2(L+, s). \qquad (3.17)$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are Dirichlet-Neumann kernels, I_1 and I_2 are terms due to initial datum and source term $S(x, t)$.

Define a transition function $G(x, s)$ in the Laplace space with the boundary data given on purpose:

$$\begin{cases} sG - f(\phi_L)G_x - G_{xx} = 0, \\ G(-L, s) = 1, \\ G(-L, s) = -\mathbf{D}_1 - f(\phi_L)(-\infty). \end{cases} \qquad (3.18)$$

Apply the Laplace transformation to the first equation in (3.14) with respect to time variable, multiply it with $G(x, s)$, integrate in the domain $[-L, L]$ yield

$$\int_{-L}^{L} G(x, s) \left(s\mathbb{L}_s[r] + (f(\phi_L)\mathbb{L}_s[r])_x - \mathbb{L}_s[r]_{xx} - z(x, 0) - \mathbb{L}_s[S](x, s) \right) dx = 0.$$

Integrate by parts, with the help of C^1 continuity for r at $x = -L$, one has

$$\begin{aligned} & I_1(-L, s) + \int_{-L}^{L} G(x, s)(z_0(x) + \mathbb{L}_s[S](x, s))dx \\ = \;& (Gf(\phi_L)\mathbb{L}_s[r] - G\mathbb{L}_s[r]_x + G_x \mathbb{L}_s[r])|_L, \end{aligned} \qquad (3.19)$$

which gives another relationship for $\mathbb{L}_s[r]_x(L, s)$ and $\mathbb{L}_s[r](L, s)$.

Combing (3.17), (3.19) and C^1 continuity for r at $x = L$ together, we solve $\mathbb{L}_s[r]_x(L, s)$ and $\mathbb{L}_s[r](L, s)$. Similarly one could get all the boundary information at $x = -L$.

The truncation error of the above approximation $e(x, t) \equiv z(x, t) -$

$r(x,t)$ satisfies:

$$\begin{cases} e_t + (f(\phi)e)_x = e_{xx} + ((f(\phi_L) - f(\phi))r)_x, \\ e(x,0) = 0, \end{cases} \tag{3.20}$$

which is similar to system (3.13).

In the third step, based on the approximate procedure given in the second step and the truncation error system produced by the approximation (3.20), we introduce an iteration scheme:

$$\begin{cases} r^0 = r, \\ e^0 = z, \\ e^k = e^{k-1} - r^{k-1}, k \geq 1. \end{cases}$$

r^k is the approximate solution of e^k. we prove the convergence of scheme and establish the wave propagation patterns for system (3.13).

3.4 Pointwise estimate of solution for the linearized system

Linearizing system (3.1) around the given profile $\Phi = (\phi, 0)$ gives the decoupled system:

$$\begin{cases} u_t + ((3\phi^2 - b)u)_{\bar{x}} = u_{\bar{x}\bar{x}}, \\ v_t + ((\phi^2 - b)v)_{\bar{x}} = v_{\bar{x}\bar{x}}, \\ u_0(\bar{x}), v_0(\bar{x}) = O(1)\varepsilon e^{-|\bar{x}|/C}, \end{cases} \tag{3.21}$$

here we have changed the coordinate system to make the presentation of this problem convenient:

$$\begin{cases} \bar{x} = x - bt, \\ t = t. \end{cases}$$

Set $f_1(\phi) \equiv 3\phi^2 - b$, $f_2(\phi) \equiv \phi^2 - b$. We use the framework given in Section 3.3 to solve the system (3.21).

3.4.1 Extract the non-decaying structure

Before extracting the non-decaying structure, we lay down a standard procedure to catch the pointwise estimate non-decaying structure.

Approximate the over-compressive shock profile with the discontinuous function, which takes the form:

$$H(\bar{x},t) \equiv \begin{cases} \phi(-\infty) = 1, & \bar{x} < 0, \\ \phi(\infty) = \alpha, & \bar{x} > 0, \end{cases}$$

$\alpha \in (-\frac{1}{2}, 0)$. We choose $\alpha = -\frac{1}{4}$.

Consider the approximate linearized equations:

$$\begin{cases} \bar{u}_t + (f_1(H)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \\ \bar{v}_t + (f_2(H)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-C|\bar{x}|}. \end{cases} \tag{3.22}$$

Due to the discontinuous property of $H(x,t)$, we separate the whole space into two half-spaces:

$$\begin{cases} \bar{u}_t + ((3-b)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \quad \text{for} \quad \bar{x} < 0, \quad t > 0, \\ \bar{v}_t + ((1-b)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \quad \text{for} \quad \bar{x} < 0, \quad t > 0, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-C|\bar{x}|}, \\ \bar{u}(0-,t), \quad \bar{v}(0-,t) \quad \text{to be determined.} \end{cases} \tag{3.23}$$

$$\begin{cases} \bar{u}_t + ((3\alpha^2-b)\bar{u})_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}}, \quad \text{for} \quad \bar{x} > 0, \quad t > 0, \\ \bar{v}_t + ((\alpha^2-b)\bar{v})_{\bar{x}} = \bar{v}_{\bar{x}\bar{x}}, \quad \text{for} \quad \bar{x} > 0, \quad t > 0, \\ \bar{u}_0(\bar{x}), \bar{v}_0(\bar{x}) = O(1)\varepsilon e^{-C|\bar{x}|}, \\ \bar{u}(0+,t), \quad \bar{v}(0+,t) \quad \text{to be determined.} \end{cases} \tag{3.24}$$

$\bar{u}(0\pm,t), \bar{v}(0\pm,t)$ will be determined by C^1 continuity of $(\bar{u},\bar{v})$ and two Dirichlet-Neumann maps.

Apply the Laplace-Laplace transformation to the above two systems, omitting the initial data temporarily, one could get Dirichlet-Neumann maps, for example for $\bar{u}$, we have:

$$\begin{cases} \mathbb{L}_s[\bar{u}]_x(0-,s) &= \mathbf{D}_1\mathbb{L}_s[\bar{u}](0-,s) \\ & \equiv \frac{3-b+\sqrt{(3-b)^2+4s}}{2}\mathbb{L}_s[\bar{u}](0-,s), \\ \mathbb{L}_s[\bar{u}]_x(0+,s) &= \mathbf{D}_2\mathbb{L}_s[\bar{u}](0+,s) \\ & \equiv \frac{3\alpha^2-b-\sqrt{(3\alpha^2-b)^2+4s}}{2}\mathbb{L}_s[\bar{u}](0+,s). \end{cases} \tag{3.25}$$

Combine (3.25) with C^1 continuity of $(\bar{u},\bar{v})$ together, we have:

$$\begin{cases} \mathbb{L}_s[\bar{u}](0-,s) = \mathbb{L}_s[\bar{u}](0+,s), \\ (3-b)\mathbb{L}_s[\bar{u}](0-,s) - \mathbb{L}_s[\bar{u}]_x(0-,s) \\ = (3\alpha^2-b)\mathbb{L}_s[\bar{u}](0+,s) - \mathbb{L}_s[\bar{u}]_x(0+,s), \\ \mathbb{L}_s[\bar{u}]_x(0-,s) - \mathbb{L}_s[I]_x(0-,s) = \mathbf{D}_1(\mathbb{L}_s[\bar{u}](0-,s) - \mathbb{L}_s[I](0-,s)), \\ \mathbb{L}_s[\bar{u}]_x(0+,s) - \mathbb{L}_s[I]_x(0+,s) = \mathbf{D}_2(\mathbb{L}_s[\bar{u}](0+,s) - \mathbb{L}_s[I](0+,s)), \end{cases} \tag{3.26}$$

taking the initial data into account. $I(0,t)$ is defined in the way of (3.9).

Solving (3.26),

$$\begin{aligned} & (3-3\alpha^2-\mathbf{D}_1+\mathbf{D}_2)\mathbb{L}_s[\bar{u}](0-,s) \\ = \;& -\mathbf{D}_1\mathbb{L}_s[I](0-,s) + \mathbf{D}_2\mathbb{L}_s[I](0+,s) + \mathbb{L}_s[I]_x(0-,s) - \mathbb{L}_s[I]_x(0+,s) \\ \equiv \;& \mathbb{L}_s[I_0](s). \end{aligned}$$

The operator which converts $\mathbb{L}_s[\bar{u}](0-,s)$ into $\mathbb{L}_s[I_0](s)$ has the estimates:

$$\begin{aligned} & (3-3\alpha^2-\mathbf{D}_1+\mathbf{D}_2) \\ = \;& \frac{3-3\alpha^2-\sqrt{(3-b)^2+4s}-\sqrt{(3\alpha^2-b)^2+4s}}{2} \\ = \;& 2s\left(\frac{1}{3\alpha^2-b-\sqrt{(3\alpha^2-b)^2+4s}} - \frac{1}{3-b+\sqrt{(3-b)^2+4s}}\right). \end{aligned}$$

Hence,

$$\mathbb{L}_s[\bar{u}](0-,s) = -\frac{C_1+\sqrt{s+C_1}}{s}\mathbb{L}_s[I_0](s),$$

C_1 is a positive constant.

With the same reason we have explained in (2.13), taking the inverse Laplace transformation, we have the following estimates:

$$\|\bar{u}(0\pm,t)\| \equiv \|\beta_1(t)\| = \| \int_0^t C_1 I_0(\tau)d\tau + \int_0^t \frac{e^{-C_1\tau}}{2\sqrt{\pi\tau}} * (\partial_\tau + C_1)I_0(\tau)d\tau\|$$

$$\leq \ O(1)\varepsilon \int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau. \tag{3.27}$$

Similarly, one could get the same estimate for $\bar{v}(0\pm,t) \equiv \beta_2(t)$. For large times, the boundary information obtained in (3.27) implies that there is a non-decaying component caused by initial data.

Moreover, if extracting the boundary information, one could get the pointwise estimate for (3.22) with the help of first Green's identity:

Lemma 3.4.1.

$$\left\|(\bar{u}(\bar{x},t),\bar{v}(\bar{x},t)) - \beta(t)\bar{\Psi}(\bar{x})\right\| = O(1)\varepsilon e^{-(\bar{x}+t)/C},$$

$\bar{\Psi}$ is the stationary solution of (3.23) *with* $\bar{\Psi}(\pm\infty) = 0$, $\beta(t) \equiv diag(\beta_1(t), \beta_2(t))$. *One could have the following estimates:* $\|\bar{\Psi}(\bar{x})\| \leq O(1)e^{-\frac{\bar{x}}{C}}$, $\|\beta(t)\| \leq O(1)\varepsilon \int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau$. *For large times,* $\int_R \beta(t)\bar{\Psi}(\bar{x})d\bar{x} = \int_R(\bar{u}_0(\bar{x}),\bar{v}_0(\bar{x}))d\bar{x}$ *due to conservation laws.*

If the initial data $\int_R(\bar{u}_0(\bar{x}),\bar{v}_0(\bar{x}))d\bar{x} = 0$, *one has*

$$\|(\bar{u}(\bar{x},t),\bar{v}(\bar{x},t))\| = O(1)\varepsilon e^{-(\bar{x}+t)/C}.$$

Remark 2. *This lemma implies that the stationary part doesn't exist if the initial data carry zero mass. The proof is not hard if using the Cancelation and Recombination Lemmas in [7], and we omit the details.*

With the above observation, we introduce a non-decaying component stacked in the shock region: $\beta(t)\Psi(\bar{x})$, $\Psi(\bar{x}) \equiv (\varphi_1,\varphi_2)$ is a stationary solution of (3.21) with $\Psi(\pm\infty) = 0$. Extract it from (3.21), set

$$(z_1,z_2) \equiv (u,v) - \beta(t)\Psi,$$

then

$$\begin{cases} z_{1t} + (f_1(\phi)z_1)_{\bar{x}} = z_{1\bar{x}\bar{x}} + \beta'(t)\varphi_1, \\ z_{2t} + (f_2(\phi)z_2)_{\bar{x}} = z_{2\bar{x}\bar{x}} + \beta'(t)\varphi_2, \\ \|z_i(\bar{x},0)\| = O(1)\varepsilon e^{-C|\bar{x}|}, \int_R z_i(\bar{x},0)dx = 0, i = 1,2. \end{cases} \tag{3.28}$$

3.4.2 Pointwise estimate of the approximate problem

The approximate problem for (3.28) is given as follows:

$$\begin{cases} r_{1t} + (f_1(\phi_L)r_1)_{\bar{x}} = r_{1\bar{x}\bar{x}} + \beta'(t)\varphi_1, \\ r_{2t} + (f_2(\phi_L)r_2)_{\bar{x}} = r_{2\bar{x}\bar{x}} + \beta'(t)\varphi_2, \\ r_i(\bar{x},0) = z_i(\bar{x},0), i = 1,2. \end{cases} \tag{3.29}$$

where ϕ_L is defined by (3.15). We only consider the equation for r_1.

Following the framework, we split the whole space domain into three parts. The solvability of problem in the left and right far fields gives two Dirichlet-Neumann maps:

$$\mathbb{L}_s[r_1]_x(-L-,s) = \mathbf{D}_1\mathbb{L}_s[r_1](-L-,s) + I_1(-L-,s), \tag{3.30}$$

$$\mathbb{L}_s[r_1]_x(L+,s) = \mathbf{D}_2\mathbb{L}_s[r_1](L+,s) + I_2(L+,s), \tag{3.31}$$

where

$$I_1(-L-,s) = \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2} + C)},$$

$$I_2(L+,s) = \frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2} + C)},$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are defined in (3.25).

Apply the Laplace transformation to the equation of r_1 with respect to time variable t, multiply it with $G(x,s)$ defined by (3.18), integrate in the domain $[-L,L]$ by parts, with the help of C^1 continuity of r_1 at $\bar{x} = -L$, we get

$$\int_{-L}^{L} G(\bar{x},s)(z_1(\bar{x},0)+\mathbb{L}_s[S](\bar{x},s))d\bar{x} + \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)}$$
$$= (Gf_1(\phi_L)\mathbb{L}_s[r_1] - G\mathbb{L}_s[r_1]_{\bar{x}} + G_{\bar{x}}\mathbb{L}_s[r_1])|_{L-}, \tag{3.32}$$

which is the second relationship for Dirichlet and Neumann data at $\bar{x}=L$. Here $S(\bar{x},t)\equiv\beta'(t)\varphi_1$.

Combine (3.32) and (3.31) together, with the help of C^1 continuity of r_1 at $\bar{x}=L$, one could solve:

$$\mathbb{L}_s[r_1](L-,s)$$
$$= \frac{\frac{\int_{-L}^{L} G(\mathbb{L}_s[S]+z_1(\bar{x},0))d\bar{x}}{G(L-,s)} + \frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)G(L-,s)}}{f_1(\phi_L(\infty))+\mathbf{D}_2+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}$$
$$+\frac{\frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2}+C)}}{f_1(\phi_L(\infty))+\mathbf{D}_2+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}$$
$$= \frac{\frac{\int_{-L}^{L} G(\mathbb{L}_s[S]+z_1(\bar{x},0))d\bar{x}}{G(L-,s)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}} + \frac{\frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)G(L-,s)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}$$
$$+\frac{\frac{O(1)\varepsilon}{(s+C)(\frac{3\alpha^2-b+\sqrt{(3\alpha^2-b)^2+4s}}{2}+C)}}{\frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2}+\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}}. \tag{3.33}$$

To invert $\mathbb{L}_s[r_1]$, we need to find the locations of all possible poles. Firstly, we give some observation on the transition function $G(\bar{x},s)$:

$$\begin{cases} sG - f_1(\phi_L)G_x - G_{xx} = 0, \\ G(-L,s)=1, \\ G_{\bar{x}}(-L,s) = -\mathbf{D}_1 - f_1(\phi_L)(-\infty) = -\frac{3(3-b)+\sqrt{(3-b)^2+4s}}{2} < 0. \end{cases} \tag{3.34}$$

Lemma 3.4.2. *There exists a positive constant C such that when $Re[s] > -C$, $\frac{G_{\bar{x}}(L-,s)}{G(L-,s)}$ and $G(L-,s)$ are analytic functions having the following properties:*

$$\begin{cases} G(L-,s) < 0, \\ \frac{3(3\alpha^2-b)-\sqrt{(3\alpha^2-b)^2+4s}}{2} + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)} < 0. \end{cases} \tag{3.35}$$

Proof. Set $P = -\frac{G_{\bar{x}}(x,s)}{G(x,s)}$, which satisfies:

$$\begin{cases} P_{\bar{x}} = P^2 - f_1(\phi_L)P - s, \\ P(-L,s) = \frac{3(3-b)+\sqrt{(3-b)^2+4s}}{2}. \end{cases}$$

When s large enough,

$$P(L,s) > \frac{f_1(\phi_L(+\infty)) - \sqrt{{f_1}^2(\phi_L(\infty)) + 4s}}{2}.$$

Hence,

$$\frac{3(3\alpha^2 - b) - \sqrt{(3\alpha^2 - b)^2 + 4s}}{2} + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)} < 0, \quad \text{as} \quad s \to \infty.$$

Moreover, $\frac{G_{\bar{x}}(L,s)}{G(L,s)} = O(1)\sqrt{s}$, $G(L,s) < 0$, $G_{\bar{x}}(L,s) < 0$. Therefore, one could choose a suitable large constant $D = O(1)L$ such that (3.35) is true when $Re[s] > D$.

When $-\frac{(3\alpha^2-b)^2}{4} < Re[s] < D$, by the computation(for example generate a program in the Mathematcia 8.0) one can get a positive number $C_1 = 0.000104$ such that when $Re[s] > -C_1$,

$$\frac{3(3\alpha^2 - b) - \sqrt{(3\alpha^2 - b)^2 + 4s}}{2} + \frac{G_{\bar{x}}(L-,s)}{G(L-,s)} < 0.$$

There also exists a positive number $C_2 = 0.00012$ such that when $Re[s] > -C_2$, $G(L-,s) < 0$. See Figure 3.2 for the plots of two functions at $s = -C_1$ and $s = -C_2$ respectively. □

One can also conclude that $\int_{-L}^{L} G(\mathbb{L}_s[S] + z(\bar{x},0))d\bar{x}$ has no pole when $Re[s] > -C_2$. For other terms in (3.33), the locations of poles are obvious since C is a positive number.

Choosing a suitable positive C_3, we conclude that $\mathbb{L}_s[r_1](L-,s)$ is a analytic function of the variable s for $Re[s] > -C_3$. Note that $\mathbb{L}_s[r_1](L-,s)$ decays to zero when $s \to \pm i\infty$. Therefore, by the complex analysis we have

$$\|r_1(L-,t)\| = O(1)\varepsilon e^{-t/C_3} \left\| \int_R e^{i\eta t}\mathbb{L}_s[r_1](L-, i\eta - C_3)d\eta \right\| = O(1)\varepsilon e^{-t/C_4}.$$

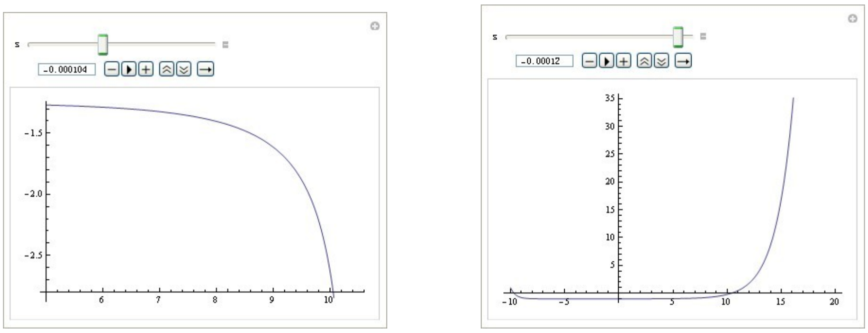

Figure 3.2: Plot of two functions with $\varepsilon = 5 \times 10^{-5}$, $L \simeq 10$

Similar ways to estimate $r_1(-L+, t)$.

Once all the boundary data are clear, with the help of the spectrum gap, one could solve the problem in finite domain and get the following estimate:

$$\sup_{x \in [-L, L]} |r_1(\bar{x}, t)| = O(1)\varepsilon e^{-t/C_4}. \tag{3.36}$$

This exponential decaying property of solution for the viscous conservation law in the finite domain has been investigated by many authors and we omit the proof.

Problem restricted to the left or right far field is just a constant coefficient problem in half space. Using the Green's function for the linearized viscous Burger's equation around the constant state, one could obtain the estimate easily. Consider the problem in the left far field:

$$\begin{cases} r_{1t} + (f_1(\phi_L(-\infty))r_1)_{\bar{x}} = r_{1\bar{x}\bar{x}} + S(\bar{x}, t), \bar{x} < -L, t > 0, \\ r_1(\bar{x}, 0) = z_1(\bar{x}, 0), \\ r_1(-L, t) = O(1)\varepsilon e^{-t/C_4}. \end{cases} \tag{3.37}$$

The Green's function for the linearized viscous Burger's equation around the constant state is defined as follows:

$$\begin{cases} \bar{\mathbb{G}}_t + \lambda \bar{\mathbb{G}}_{\bar{x}} = k \bar{\mathbb{G}}_{\bar{x}\bar{x}}, \\ \bar{\mathbb{G}}(\bar{x}, 0) = \delta(\bar{x}), \\ \bar{\mathbb{G}}(\bar{x}, t, \lambda) = \frac{e^{-\frac{(\bar{x}-\lambda t)^2}{4kt}}}{\sqrt{4\pi kt}}. \end{cases}$$

With the Green's identity, we have the solution for (3.37):

$$\begin{aligned} & r_1(\bar{x}, t) \\ = & \int_{-\infty}^{-L} \bar{\mathbb{G}}(\bar{x} - y, t, f_1(\phi_L(-\infty))) r_1(y, 0) dy \\ & + \int_0^t \bar{\mathbb{G}}(\bar{x}, t - \tau, f_1(\phi_L(-\infty))) r_{1\bar{x}}(0, \tau) d\tau \\ & + \int_0^t \int_{-\infty}^{-L} \bar{\mathbb{G}}(\bar{x} - y, t - \tau, f_1(\phi_L(-\infty))) S(y, \tau) dy d\tau, \\ & \bar{x} < -L, t > 0. \end{aligned} \tag{3.38}$$

The Neumann datum $r_{1\bar{x}}(0, \tau)$ can be obtained by the Dirichlet-Neumann map (3.30). Take the inverse Laplace transform to (3.30), with the help of complex analysis, we have the following estimates:

$$\begin{cases} \mathbb{L}_t^{-1}[\mathbf{D}_1] = \frac{(3-b)\delta(t) + (3-b)^2 \frac{e^{-(3-b)^2/4t}}{\sqrt{\pi t}} + 4 \frac{e^{-(3-b)^2/4t}}{\sqrt{\pi t}} * \partial_t}{2}, \\ \mathbb{L}_t^{-1}[\frac{O(1)\varepsilon}{(s+C)(\frac{-(3-b)+\sqrt{(3-b)^2+4s}}{2}+C)}] = O(1)\varepsilon e^{-Ct} * e^{-(3-b)^2 t/4}. \end{cases}$$

Substitute these into (3.38), one has:

$$\begin{aligned} & \|r_1(\bar{x}, t)\| \\ = & \left\| O(1)\varepsilon \int_{-\infty}^{-L} \bar{\mathbb{G}}(\bar{x} - y, t, f_1(\phi_L(-\infty))) e^{-y/C} dy \right. \\ & + O(1)\varepsilon \int_0^t \bar{\mathbb{G}}(\bar{x}, t - \tau, f_1(\phi_L(-\infty)))(\mathbb{L}_\tau^{-1}[\mathbf{D}_1] * e^{-\tau/C} + e^{-\tau/C}) d\tau \\ & \left. + O(1)\varepsilon \int_0^t \int_{-\infty}^{-L} \bar{\mathbb{G}}(\bar{x} - y, t - \tau, f_1(\phi_L(-\infty))) e^{(-y-\tau)/C} dy d\tau \right\| \\ = & O(1)\varepsilon e^{(-\bar{x}-t)/C}, \bar{x} < -L, t > 0. \end{aligned} \tag{3.39}$$

Similar computations in the right far field.

Combining (3.36) with (3.39), we get the estimate for the whole space:

$$\|r_1(\bar{x},t)\| = O(1)\varepsilon e^{-(\bar{x}+t)/C}, \text{for } \ \bar{x} \in R, t > 0, \tag{3.40}$$

C is a universal positive constant. Similar procedure for r_2.

3.4.3 Iterated scheme

The error function $e(\bar{x},t) = (e_1, e_2) \equiv (z_1 - r_1, z_2 - r_2)$ of approximate problem (3.29) to (3.28) satisfy the initial value problem:

$$\begin{cases} e_{1t} + (f_1(\phi)e_1)_{\bar{x}} = e_{1\bar{x}\bar{x}} + ((f_1(\phi_L) - f_1(\phi))r_1)_{\bar{x}}, \\ e_{2t} + (f_2(\phi)e_2)_{\bar{x}} = e_{2\bar{x}\bar{x}} + ((f_2(\phi_L) - f_2(\phi))r_2)_{\bar{x}}, \\ e_i(\bar{x},0) = 0, i = 1,2. \end{cases} \tag{3.41}$$

With the help of the estimate of (3.40) and the choice of $L = O(1)|\ln\varepsilon|$, we have

$$\begin{cases} \|r_i\| = O(1)\varepsilon e^{-(\bar{x}+t)/C}, i = 1,2, \\ \|f_i(\phi_L) - f_i(\phi)\| = O(1)\varepsilon, i = 1,2. \end{cases}$$

With the same method of solving (3.28), we consider the following approximate problem:

$$\begin{cases} r^1_{1t} + (f_1(\phi_L)r^1_1)_{\bar{x}} = r^1_{1\bar{x}\bar{x}} + ((f_1(\phi_L) - f_1(\phi))r_1)_{\bar{x}}, \\ r^1_{2t} + (f_2(\phi_L)r^1_2)_{\bar{x}} = r^1_{2\bar{x}\bar{x}} + ((f_2(\phi_L) - f_2(\phi))r_2)_{\bar{x}}, \\ r^1_i(\bar{x},0) = 0, i = 1,2. \end{cases}$$

Similarly, one could obtain the following estimate:

$$\|r^1_i(\bar{x},t)\| = O(1)\varepsilon^2 e^{-(\bar{x}+t)/C}, \text{for } \ \bar{x} \in R, \quad t > 0.$$

Now we introduce an iteration scheme to construct the solution of (3.28):

$$\begin{cases} r^0_i = r_i, \\ e^0_i = z_i, \\ e^1_i \equiv e^0_i - r^0_i, \end{cases} \tag{3.42}$$

for $k \geq 1$,

$$\begin{cases} e_{1t}^k + (f_1(\phi)e_1^k)_{\bar{x}} = e_{1\bar{x}\bar{x}}^k + ((f_1(\phi_L) - f_1(\phi))r_1^{k-1})_{\bar{x}}, \\ e_{2t}^k + (f_2(\phi)e_2^k)_{\bar{x}} = e_{2\bar{x}\bar{x}}^k + ((f_2(\phi_L) - f_2(\phi))r_2^{k-1})_{\bar{x}}, \\ e_i^k(\bar{x}, 0) = 0, i = 1, 2, \end{cases} \tag{3.43}$$

$$\begin{cases} r_{1t}^k + (f_1(\phi_L)r_1^k)_{\bar{x}} = r_{1\bar{x}\bar{x}}^k + ((f_1(\phi_L) - f_1(\phi))r_1^{k-1})_{\bar{x}}, \\ r_{2t}^k + (f_2(\phi_L)r_2^k)_{\bar{x}} = r_{2\bar{x}\bar{x}}^k + ((f_2(\phi_L) - f_2(\phi))r_2^{k-1})_{\bar{x}}, \\ r_i^k(\bar{x}, 0) = 0, i = 1, 2, \end{cases} \tag{3.44}$$

$$e_i^{k+1} = e_i^k - r_i^k.$$

For each r_i^k, we have the estimate:

$$\|r_i^k(\bar{x}, t)\| = O(1)\varepsilon^{k+1}e^{-(\bar{x}+t)/C}, \text{for} \quad \bar{x} \in R, t > 0. \tag{3.45}$$

The solution of (3.28) can be written formally in terms of the iteration scheme:

$$z_i = \sum_{k=0}^{\infty} r_i^k.$$

From (3.45), the series is convergent, and

$$\|z_i(\bar{x}, t)\| = O(1)\varepsilon e^{-(\bar{x}+t)/C}, \quad \text{for} \quad \bar{x} \in R, \quad t > 0.$$

To summarize, we have the follow theorem:

Theorem 3.4.3. *The solution of* (3.21) *defines a semi-group* $\mathbb{G}_\Phi^t$ *as follows:*

$$\mathbb{G}_\Phi^t[u_0, v_0](\bar{x}) \equiv (u(\bar{x}, t), v(\bar{x}, t))$$

which satisfies:

$$\|(u(\bar{x}, t), v(\bar{x}, t)) - \beta(t)\Psi(\bar{x})\| = O(1)\varepsilon e^{-(\bar{x}+t)/C},$$

where: $\|\Psi(\bar{x})\| = O(1)e^{-\frac{\bar{x}}{C}}$,$\beta(t) \le O(1)\varepsilon \int_0^t \frac{e^{-C_1\tau}}{\sqrt{\tau+1}}d\tau$, *$C$ is a universal constant. For large times,* $\int_R \beta(t)\Psi(\bar{x})d\bar{x} = \int_R(\bar{u}_0(\bar{x}), \bar{v}_0)(\bar{x})d\bar{x}$ *due to conservation laws.*

If the initial data $\int_R(u(\bar{x},0), v(\bar{x},0)d\bar{x} = 0,$ *one has*

$$\|u(\bar{x},t), v(\bar{x},t)\| = O(1)\varepsilon e^{-(\bar{x}+t)/C}.$$

3.5 Nonlinear stability of over-compressive shock waves

The estimate obtained in Theorem 3.4.3 tells us that there is a non-decaying structure $O(1)\varepsilon e^{-\bar{x}/C}$ determined by the initial data. To prove the nonlinear stability Theorem 3.1.1, instead of extracting the over-compressive shock profile, the shock profile with a proper shift and a proper shape changed is the real final state (extracting the non-decaying term). Consider the perturbation of new profile

$$\Phi^* \equiv (\phi^*(\bar{x}-x_0), \psi^*(\bar{x}-x_0)),$$

which satisfies:

$$\begin{cases} \int_R(\tilde{u}_0, \tilde{v}_0)(x,0)dx = \int_R((\phi^*, \psi^*) - (\phi, \psi))dx, \\ \|\Phi^* - \Phi\| = O(1)\varepsilon e^{-\bar{x}/C}. \end{cases} \tag{3.46}$$

Note that the phase shift x_0 contributes to the integral only for the first component, because the end states $(\tilde{u}_\pm, \tilde{v}_\pm)$ have zero second component. For simplicity, assume that $x_0 = 0$.

The perturbation $\begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \equiv \begin{pmatrix} \tilde{u}(x,t) \\ \tilde{v}(x,t) \end{pmatrix} - \Phi^*$ satisfies:

$$\begin{cases} \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix}_t + \left((A(\Phi^*(\bar{x})) - bI) \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \right)_x \\ = \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix}_{\bar{x}\bar{x}} - Q\left(\Phi^*, \begin{pmatrix} u^*(\bar{x},t) \\ v^*(\bar{x},t) \end{pmatrix} \right)_{\bar{x}}, \\ \left\| \begin{pmatrix} u^*(\bar{x},0) \\ v^*(\bar{x},0) \end{pmatrix} \right\| = O(1)\varepsilon e^{-\bar{x}/C}, \\ \int_R \begin{pmatrix} u^*(\bar{x},0) \\ v^*(\bar{x},0) \end{pmatrix} d\bar{x} = 0, \end{cases}$$

where $A(\Phi^*) \equiv |\Phi^*|^2 I + 2\Phi^*\Phi^{*T}$, Q is the quadratic remainder which is easily checked to obey the estimate:

$$\|Q(\Phi^*, Z)\| \leq 3(\|\Phi^*\| + \|Z\|)\|Z\|^2.$$

Due to the second estimate in (3.46), we rewrite the first equation system of (3.46) as follows:

$$\begin{cases} {u^*}_t + (f_1(\phi,\psi)u^*)_{\bar{x}} = {u^*}_{\bar{x}\bar{x}} + O(1)\varepsilon(e^{-\bar{x}/C}(u^* + v^*))_x + Q_1(\bar{x},t), \\ {v^*}_t + (f_2(\phi,\psi)v^*)_{\bar{x}} = {v^*}_{\bar{x}\bar{x}} + O(1)\varepsilon(e^{-\bar{x}/C}(u^* + v^*))_x + Q_2(\bar{x},t). \end{cases} \tag{3.47}$$

Now make an ansatz assumption on the solution:

$$\|u^*(\bar{x},t), v^*(\bar{x},t)\| \leq O(1)\varepsilon e^{-(\bar{x}+t)/C}, \tag{3.48}$$

to deal with the nonlinear term.

To verify the ansatz assumption, we only need to show that under the ansatz assumption, the following holds:

$$\left\| \int_R \mathbb{G}_\Phi(\bar{x} - y, t)[u^*(y,0), v^*(y,0)]dy \right.$$

$$
\begin{aligned}
+ & \quad \int_0^t \int_R \mathbb{G}_\Phi(\bar{x}-y,t-s)[O(1)\varepsilon(e^{-y/C}(u^*+v^*))_y \\
& \quad +Q_1(y,s), O(1)\varepsilon(e^{-y/C}(u^*+v^*))_y + Q_2(y,s)]dyds\| \\
\leq & \quad O(1)\varepsilon e^{-(\bar{x}+t)/C}, \quad \text{for} \quad \bar{x}\in R, \quad t>0. \qquad (3.49)
\end{aligned}
$$

By Theorem 3.4.3 $\mathbb{G}_\phi^t[u_0,v_0](\bar{x}) = (u(\bar{x},t), v(\bar{x},t))$,

$$\|\mathbb{G}_\phi^t[u_0,v_0](\bar{x})\| \leq O(1)\varepsilon e^{-(\bar{x}+t)/C}.$$

From the ansatz assumption (3.48), we have:

$$\|O(1)\varepsilon(e^{-\bar{x}/C}(u^*+v^*))_x\| \leq O(1)\varepsilon^2 e^{-(\bar{x}+t)/D},$$

and

$$\|Q_i(\bar{x},t)\| \leq O(1)\varepsilon^2 e^{-(\bar{x}+t)/D}.$$

Therefore (3.49) is true and we prove the Main Theorem 3.1.1.

Chapter 4

A Strong Shock Profile for the Broadwell Model

4.1 Introduction

In this chapter, we are interested in the wave behavior of planar wave perturbation around a Broadwell shock profile. Consider the Broadwell model in the whole space

$$\begin{cases} \partial_t \tilde{f}_+ + \partial_x \tilde{f}_+ = \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_-, \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \\ \partial_t \tilde{f}_0 = -(\frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_-), \\ \partial_t \tilde{f}_- - \partial_x \tilde{f}_- = \frac{1}{4}\tilde{f}_0^2 - \tilde{f}_+\tilde{f}_-. \end{cases} \tag{4.1}$$

The details about the Broadwell model are already introduced in the Chapter 2.

A shock profile for the Broadwell model is a traveling wave $\tilde{F}(x,t) = \varphi(x - bt)$ satisfies:

$$\begin{cases} (V - bI)\varphi' = Q(\varphi), \\ \varphi(\pm\infty) = M^{\pm}, \end{cases} \tag{4.2}$$

the end states M^-, M^+ are equilibrium states defined above, b is the shock profile speed and we assume the shock is forward ($b > 0$).

The Broadwell model is a simplified kinetic description for gas dynamics. The simplest related macroscopic equations are Euler equations in gas dynamics:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \hat{z}(\rho, m) = 0, \end{cases} \tag{4.3}$$

$\hat{z}(\rho, m) \equiv \frac{\rho(2\sqrt{1+3(\frac{m}{\rho})^2}-1)}{3}$.

Euler equations can be obtained from the Broadwell model by assuming that the gas is a local Maxwellian at every (x, t). Though the fluid dynamic equations do not model the shock profile, the end states of a shock profile are governed by the Rankine-Hugoniot condition for the fluid equations:

$$\begin{cases} m^- - b\rho^- = m^+ - b\rho^+, \\ \hat{z}^- - bm^- = \hat{z}^+ - bm^+. \end{cases}$$

Hence, the end states $M^{\pm}$ in (4.2) should satisfy:

$$\begin{cases} (1-b)M_+^- - 2bM_0^- - (1+b)M_-^- = (1-b)M_+^+ - 2bM_0^+ - (1+b)M_-^+, \\ (1-b)M_+^- + (1+b)M_-^- = (1-b)M_+^+ + (1+b)M_-^+, \end{cases}$$

and additional entropy condition:

$$b(M_+^- + 2M_0^- + M_-^-) > b(M_+^+ + 2M_0^+ + M_-^+).$$

In this chapter, without loss of generality, we set

$$M^- = \begin{pmatrix} 5/14 \\ 3/7 \\ 9/70 \end{pmatrix}, \quad M^+ = \begin{pmatrix} 1/6 \\ 1/3 \\ 1/6 \end{pmatrix}, \quad b = 2/3. \tag{4.4}$$

Solving the system (4.2) with (4.4) , we get the explicit form of the shock wave profile connecting these two end states [2]:

$$\begin{pmatrix} \varphi_+ \\ \varphi_0 \\ \varphi_- \end{pmatrix} = \begin{pmatrix} -\frac{2}{21} \\ -\frac{1}{21} \\ \frac{2}{105} \end{pmatrix} \tanh(\frac{3}{20}(x-bt)) + \begin{pmatrix} \frac{11}{42} \\ \frac{8}{21} \\ \frac{31}{210} \end{pmatrix}.$$

Setting $\vec{\mathbf{u}} = (\rho, m)$, $\mathbf{F}(\vec{\mathbf{u}}) = (m, \hat{z}(\rho, m))$, The Euler eqs. (2.7) can be rewritten as a 2×2 strictly hyperbolic and genuinely nonlinear system, [3],

$$\vec{\mathbf{u}}_t + \mathbf{F}(\vec{\mathbf{u}})_x = 0, \quad \vec{\mathbf{u}} \in \mathbb{R}^2.$$

The eigenvalues of $\mathbf{F}'(\vec{\mathbf{u}})$,

$$\mathbf{F}'(\vec{\mathbf{u}}) = \begin{pmatrix} 0 & 1 \\ (\rho - \hat{z})/(\rho + 3\hat{z}) & 4m/(\rho + 3\hat{z}) \end{pmatrix},$$

define two Euler characteristics

$$\lambda_1(\vec{\mathbf{u}}) = \frac{2m - \sqrt{4m^2 + (\rho - \hat{z})(\rho + 3\hat{z})}}{\rho + 3\hat{z}},$$
$$\lambda_2(\vec{\mathbf{u}}) = \frac{2m + \sqrt{4m^2 + (\rho - \hat{z})(\rho + 3\hat{z})}}{\rho + 3\hat{z}},$$

with the corresponding right eigenvector fields $r_1(\vec{\mathbf{u}})$, $r_2(\vec{\mathbf{u}})$, left eigenvectors $l_1(\vec{\mathbf{u}})$, $l_2(\vec{\mathbf{u}})$,

$$\begin{aligned}
&\{\mathbf{r}_1(\vec{\mathbf{u}}), \mathbf{r}_2(\vec{\mathbf{u}})\} \\
= \;& \left\{ \begin{pmatrix} -\frac{\rho+3\hat{z}}{2\sqrt{4m^2+(\rho-\hat{z})(\rho+3\hat{z})}} \\ -\frac{m}{\sqrt{4m^2+(\rho-\hat{z})(\rho+3\hat{z})}} + \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\rho+3\hat{z}}{2\sqrt{4m^2+(\rho-\hat{z})(\rho+3\hat{z})}} \\ \frac{m}{\sqrt{4m^2+(\rho-\hat{z})(\rho+3\hat{z})}} + \frac{1}{2} \end{pmatrix} \right\}, \\
&\{\mathbf{l}_1(\vec{\mathbf{u}}), \mathbf{l}_2(\vec{\mathbf{u}})\} \\
= \;& \left\{ \left(-\frac{2m + \sqrt{4m^2 + (\rho - \hat{z})(\rho + 3\hat{z})}}{\rho + 3\hat{z}}, 1 \right), \right. \\
& \left. \left(\frac{-2m + \sqrt{4m^2 + (\rho - \hat{z})(\rho + 3\hat{z})}}{\rho + 3\hat{z}}, 1 \right) \right\},
\end{aligned} \tag{4.5}$$

where

$$\mathbf{F}'(\vec{\mathbf{u}})\mathbf{r}_i(\vec{\mathbf{u}}) = \lambda_i(\vec{\mathbf{u}})\mathbf{r}_i(\vec{\mathbf{u}}),$$
$$\mathbf{l}_i(\vec{\mathbf{u}})\mathbf{F}'(\vec{\mathbf{u}}) = \lambda_i(\vec{\mathbf{u}})\mathbf{l}_i(\vec{\mathbf{u}}),$$
$$\mathbf{l}_j(\vec{\mathbf{u}}) \cdot \mathbf{r}_i(\vec{\mathbf{u}}) = \delta_i^j, \quad i, j = 1, 2.$$

If the components $\tilde{f}_+, \tilde{f}_0, \tilde{f}_-$ of the solution are initially nonnegative, they remain nonnegative [1]. For such a solution $|\frac{m}{\rho}| \leq 1$ (i.e., the average velocity is no larger than the molecular speed) and $(\rho+3\hat{z}) > 0$, $(\rho-\hat{z}) > 0$. Thus $\lambda_1(\vec{\mathbf{u}}) < 0 < \lambda_2(\vec{\mathbf{u}})$.

The main interest in this chapter is to give the time asymptotic stability of the Broadwell shock profile $\varphi(x,t)$ with respect to small perturbation in initial data, in the pointwise L^∞-norm:

$$\begin{cases} \partial_t \tilde{F} + V\partial_x \tilde{F} = Q(\tilde{F}), \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \\ \|\tilde{F}(x,0) - \varphi(x,0)\| \le O(1)\epsilon e^{-|x|/C}, \epsilon \ll 1. \end{cases} \tag{4.6}$$

Since the shock is forward, there is a basic compressibility condition:

$$\begin{cases} \lambda_1^- < b < \lambda_2^-, \\ \lambda_1^+ < \lambda_2^+ < b, \end{cases} \tag{4.7}$$

where we use the abbreviations $\lambda_1^- \equiv \lambda_1(M^-)$, $\lambda_2^- \equiv \lambda_2(M^-)$, $\lambda_1^+ \equiv \lambda_1(M^+)$, $\lambda_2^+ \equiv \lambda_2(M^+)$.

The nonlinear stability of shock profile for the Broadwell model was first investigated in [3]. With the small-amplitude assumption, the solution was consisted as the sum of a shock wave, a diffusion wave, a linear hyperbolic wave and an error term. The stability analysis was obtained through the energy estimates. However, this energy estimates were too global to understand the wave interactions. For the purpose of better understanding of the qualitative and quantitative behavior of these wave interactions, a Green's function pointwise estimate approach was initiated for the Boltzmann equation in [22]. Such an idea was adopted by [17] to study the structures of Green's function for the Broadwell model linearized around a global Maxwellian. They showed that the primary fluid wave structure is a sum of transported heat kernels along each characteristic fields. With this detailed structure, one could obtain the pointwise nonlinear wave coupling structure around an absolute Maxwellian state. In a subsequent study, the half space problem was resolved through the Green's function with the global energy estimates [26]. Recently, following these previously studies, [34] devised a Transverse-Compressive scheme (T-C scheme) to study the problem such as the initial value problem of a linearized system around the weak shock profile. However, this T-C scheme cannot be adopted to study the nonlinear stability of strong shock profile. The aim of this chapter is to fill this gap.

There are other attempts in studying the strong shock profiles. Paper [29] and [30] established $L^1 \cap H^2 \to L^p$ nonlinear orbital stability, $1 \leq p \leq \infty$, with sharp rates of decay, of large-amplitude Lax-type shock profiles for a general class of hyperbolic-parabolic systems and relaxation systems under the necessary conditions of strong spectral stability, i.e., stable point spectrum of the linearized operator about the wave, transversality of the profile, and hyperbolic stability of the associated ideal shock. Our philosophy here is different from them.

In the last chapter, we developed an effective way to solve a viscous conservation law. We continue to develop the program to fit into the a simple kinetic model and hope that this could pave a way towards the Boltzmann equation eventually. In the rest of introduction, we present the framework to solve a linearized equation around the given shock profile φ, which satisfies:

$$\begin{cases} \partial_t h + (V - bI)\partial_{\bar{x}} h = L_{\varphi(\bar{x})} h, \\ \|h(\bar{x}, 0)\| \leq O(1)\epsilon e^{-|\bar{x}|/C}. \end{cases} \tag{4.8}$$

In the first step, we introduce a local wave front tracing and a transversal component caused by initial data. Due to the compressibility of the shock wave, there is a non-decaying component stacked at the shock front. We need to extract this non-decaying part from the linearzied equation precisely. Otherwise, one would fail to get the nonlinear stability. The construction of this component requires the essential information on the profile, and it preserves the localized structure around the shock front as well as the macroscopic conservation laws. To fit these criteria, we introduce a standard procedure to catch the non-decaying part, which is defined to be $diag(\ell(t))\psi(\bar{x})$. The extraction of the transverse components towards the far field depends on the Green's function of the constant coefficient problem at the far fields.

The reminder $z(\bar{x}, t)$ satisfies:

$$\begin{cases} \partial_t z + (V - bI)\partial_{\bar{x}} z = L_{\varphi(\bar{x})} z + S(\bar{x}, t), \\ z(\bar{x}, 0) = 0, \end{cases} \tag{4.9}$$

with $S(\bar{x}, t) = (L_{\varphi(\bar{x})} - L_{H(\bar{x})})v(\bar{x}, t)$ purely microscopic.

To handle (4.9), we consider an approximate problem:

$$\begin{cases} \partial_t r + (V - bI)\partial_{\bar{x}} r = L_{\varphi_L(\bar{x})} r + S(x, t), \\ r(\bar{x}, 0) = z(\bar{x}, 0). \end{cases} \tag{4.10}$$

where

$$\varphi_L(x) = \begin{cases} \varphi(\infty), & \text{if } \ x \geq L, \\ \varphi(x), & \text{if } \ |x| < L, \\ \varphi(-\infty), & \text{if } \ x \leq -L, \end{cases} \tag{4.11}$$

choosing L sufficiently large, $L = O(1)|\ln \varepsilon|$. This is the second step.

With the help of approximate function $\varphi_L(x)$, one could split the whole space domain into three parts: the far field $(-\infty, -L]$, $[L, \infty)$, and finite domain region $(-L, L)$. Convergence rate for $r(\bar{x}, t)$ at each boundary can be obtained through the wave interactions in these three domains. Once all the boundary information is clear, one could solve the problem (4.10) in each domain separately. Therefore, all the difficulties in this step are shifted to how to solve the boundary data for r at $\bar{x} = \pm L$.

When $|\bar{x}| > L$, (4.10) turns to be a constant coefficient problem. The solvability condition for the constant coefficient problem in each far field gives a incoming-outgoing map:

$$\begin{cases} \mathbb{L}_s[r_0](L+,s) = \mathbf{M}_2(\mathbb{L}_s[r_+](L+,s)) + I_1(L+,s), \\ \mathbb{L}_s[r_-](L+,s) = \mathbf{M}_3(\mathbb{L}_s[r_+](L+,s)) + I_2(L+,s), \\ \mathbb{L}_s[r_+](-L-,s) = \mathbf{M}_1(\mathbb{L}_s[r_0](-L-,s), \mathbb{L}_s[r_-](-L-,s) + I_3(-L-,s). \end{cases} \tag{4.12}$$

$\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are incoming-outgoing kernels, I_1,I_2 and I_3 are terms due to initial data and source term $S(\bar{x},t)$.

Define a transition function $G(\bar{x},s)$ in the Laplace space with the boundary data given on purpose:

$$\begin{cases} sG^T(\bar{x},s) - (V-bI)G^T(\bar{x},s)_{\bar{x}} = L^T_{\varphi(\bar{x})}G^T(\bar{x},s), \\ G^T(-L,s) = I. \end{cases} \tag{4.13}$$

Apply the Laplace transformation to the equation in (4.10) with respect to time variable, multiply it with $G(\bar{x},s)$, integrate by parts in the domain $[-L,L]$, yields

$$\begin{aligned} & G(L,s)(V-bI)\mathbb{L}_s[r](L,s) - G(-L,s)(V-bI)\mathbb{L}_s[r](-L,s) \\ = & \int_{-L}^{L} G(\bar{x},s)\mathbb{L}_s[S](\bar{x},s)d\bar{x}. \end{aligned} \tag{4.14}$$

With the help of C^1 continuity for r at $x=\pm L$, one could solve $\mathbb{L}_s[r]_0(-L,s)$ $\mathbb{L}_s[r]_-(-L,s)$ and $\mathbb{L}_s[r]_+(L,s)$.

The truncation error of the above approximation $e(\bar{x},t) \equiv z(\bar{x},t) - r(\bar{x},t)$ satisfies:

$$\begin{cases} \partial_t e + (V-bI)\partial_{\bar{x}} e = L_{\varphi(\bar{x})}e + S^0(\bar{x},t), \\ e(\bar{x},0) = 0, \end{cases} \tag{4.15}$$

where $S^0(\bar{x},t) = (L_{\varphi(\bar{x})} - L_{\varphi_L(\bar{x})})r(\bar{x},t)$.

In the third step, based on the approximate procedure given in the second step and the truncation error system produced by the approximation

(4.10), we introduce an iterated scheme:

$$\begin{cases} r^0 = r, \\ e^0 = z, \\ e^k = e^{k-1} - r^{k-1}, k \geq 1. \end{cases}$$

r^k is the approximate solution of e^k. we prove the convergence of scheme and establish the wave propagation patterns for system (4.9).

Once the linearized problem is solved, we use the Duhamel's principle to close the nonlinear stability of the perturbation, which satisfies:

$$\begin{cases} \partial_t h + (V - bI)\partial_{\bar{x}} h = L_{\varphi(\bar{x})} h + Q(h), \\ \|h(\bar{x}, 0)\| \leq O(1)\epsilon e^{-|\bar{x}|/C}. \end{cases} \tag{4.16}$$

Now we state the main theorem of this chapter:

Theorem 4.1.1. *(Main Theorem) Suppose the strength of the forward shock wave profile is $O(1)$, then there exists $x_0 > 0$ such that the solution $\tilde{F}(x,t)$ of (4.6) satisfies:*

$$\begin{aligned} \|\tilde{F}(x - bt, t) - \varphi(x - bt - x_0)\| \leq \; & O(1)\epsilon \frac{\chi_{[(\lambda_1^- - \frac{2}{3})t + \sqrt{t}, 0]}(x - bt)}{\sqrt{(|x - \lambda_1^- t| + 1)(|x - \lambda_2^- t| + 1)}} \\ & + O(1)\epsilon \frac{e^{-\frac{|x - \lambda_1^- t|^2}{C(1+t)}}}{\sqrt{1+t}} + O(1)\epsilon e^{-(|x - bt| + t)/C}. \end{aligned}$$

Here the constant x_0 is determined by the algebraic relationship: $\int_R (\tilde{F}(x,0) - \varphi(x - x_0, 0))dx \in Span\{\boldsymbol{r}_1(\vec{\boldsymbol{u}}^-)\}$, and the vector $\boldsymbol{r}_1(\vec{\boldsymbol{u}}^-)$ is defined in (4.5). When the shock is backward, one replaces $Span\{\boldsymbol{r}_1(\vec{\boldsymbol{u}}^-)\}$ with $Span\{\boldsymbol{r}_2(\vec{\boldsymbol{u}}^-)\}$.

The arrangement of the rest of the chapter is as follows: in Section 4.2, we review the necessary materials for this chapter. In Section 4.3, pointwise estimate of solution of linearized problem is obtained. In section 4.4, we make an ansatz assumption of the nonlinear wave propagation and use the estimates obtained to verify the ansatz assumption and prove the main theorem.

4.2 Preliminaries

4.2.1 Green's function of the linearized Broadwell model around a global maxwellian

For the linearized equation around a global Maxwellian state M,

$$\partial_t F + V\partial_x F = L_M F,$$

its Green's function $\mathbb{G}(x,t)$ is a solution of a particular initial condition (completely particle-like initial data): $\mathbb{G}(x,0) = \delta(x)$. We use the long wave-short wave decomposition and particle-wave like decomposition in [26] to yield the following Green's function linearized around M^+, which is the general form of Theorem 2.3.1:

Theorem 4.2.1. [17] There exists a positive constant C such that

$$\begin{aligned} &\left\| \mathbb{G}(x,t) - e^{-t/6} \begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t) \end{pmatrix} \right\| \\ &\leq C \left(\frac{e^{-\frac{|x-\lambda_1^+ t|^2}{C(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x-\lambda_2^+ t|^2}{C(1+t)}}}{\sqrt{1+t}} \right) + Ce^{-(|x|+t)/C} \\ &\text{for all} \quad x \in \mathbb{R}, \quad t > 0. \end{aligned} \tag{4.17}$$

Similar result for M^-.

4.2.2 Master relationship: Incoming-outgoing map

We prepare the basic materials for constructing incoming-outgoing map we will need in next sections. Consider the following linearized problem around a constant equilibrium state $M = (M_+, M_0, M_-)$ in half space:

$$\begin{cases} (\partial_t + (V - bI)\partial_{\bar{x}} - L_M)F(\bar{x}, t) = 0, \text{for } \ \bar{x} > 0, t > 0, \\ F(\bar{x}, 0) = I_0(\bar{x}), \\ f_+(0, t) = f_+(t). \end{cases} \tag{4.18}$$

We first consider the case $I_0(\bar{x}) = 0$. We will convert the differential equations into an algebraic system to obtain the full boundary data. Take the Laplace-Laplace transform of (4.18):

$$\mathbb{L}[F](\bar{x}, s) = \int_0^\infty e^{-st} F(\bar{x}, t) dt,$$
$$\mathbb{J}[F](\xi, s) = \int_0^\infty e^{-\xi\bar{x}} \mathbb{L}[F](\bar{x}, s) d\bar{x}.$$

and it becomes

$$\mathbb{J}[F](\xi, s) = \frac{adj(s + \xi(V - bI) - L_M)}{p(\xi, s)}(V - bI)\mathbb{L}[F](\bar{x}, s)|_{\bar{x}=0}, \tag{4.19}$$

where $p(\xi, s) = \det(s + \xi(V - bI) - L_M)$.

The characteristic polynomial $p(\xi, s)$ is a degree 3 polynomial in ξ. One denotes the three roots $\xi = \xi_i(s, b)$, $i = 1, 2, 3$, of $p(\xi, s)$ in ξ:

$$Re(\xi_3(s, b)) \le Re(\xi_2(s, b)) \le Re(\xi_1(s, b)).$$

One can prove that

$$Re(\xi_3(s, b)) < 0 < Re(\xi_2(s, b)) \le Re(\xi_1(s, b)), \text{for } \ Re(s) > 0.$$

The expression for $\mathbb{J}[F](\xi, s)$ in (4.19) is a rational function in ξ so that one can apply the inverse Laplace transform with respect to space variable ξ. The wellposedness of a differential equation imposed the solution $\mathbb{L}[F](\bar{x}, s)$ decaying to zero as $\bar{x} \to \infty$, which implies that

$$Res_{\xi=\xi_i, Re(\xi_i)>0} \frac{adj(s + \xi(V - bI) - L_M)}{p(\xi, s)}(V - bI)\mathbb{L}[F](\bar{x}, s)|_{\bar{x}=0} = 0. \tag{4.20}$$

This and the boundary condition in (4.18) give a linear system on the full boundary $\mathbb{L}[F](0,s)$. One can solve the linear system to get the incoming-outgoing map.

Take the inverse Laplace transform with respect to s variable, one could convert $\mathbb{L}[F](0,s)$ into $F(0,t)$ by complex analysis.

For the case that $I_0(\bar{x}) \neq 0$. We need to shift the initial data to the boundary for the new variable:

$$\begin{cases} \bar{F} = F - I(\bar{x},t), \\ I(\bar{x},t) = \int_0^\infty \mathbb{G}(\bar{x}-y,t)F_0(y)dy, \end{cases} \tag{4.21}$$

$\mathbb{G}(\bar{x}-y,t)$ is the Green's function defined in Section 4.2.1. Therefore, $\bar{F}$ satisfies:

$$\begin{cases} (\partial_t + (V-bI)\partial_{\bar{x}} - L_M)\bar{F}(\bar{x},t) = 0, \text{for} \quad \bar{x} > 0, t > 0, \\ \bar{F}(\bar{x},0) = 0, \\ \bar{f}_+(0,t) = f_+(t) - I_+(0,t), \end{cases}$$

which is the first case.

4.2.3 Wave decomposition

(4.7) implies that there are two characteristic curves entering the shock from the right of the shock: one characteristic curve catching up with the shock from the left of the shock; and one characteristic curve leaving to the left of the shock. Combine this property with our initial condition that there're essentially no waves in front of the shock in the initial data, we can conclude that time-asymptotically there are no waves in front of the shock. So we only need to take care the waves behind the shock.

Definition 4.2.2. *T-C decomposition is defined as follows:*

$$\begin{cases} F = F^t + F^c + F^m, \\ \textit{Transverse Component}: \quad F^t \parallel \boldsymbol{r}_1(\vec{\boldsymbol{u}}^-), \\ \textit{Compressive Component}: \quad F^c \parallel \boldsymbol{r}_2(\vec{\boldsymbol{u}}^-), \\ \textit{Microscopic Component}: \quad F^m \equiv F - F^t - F^c. \end{cases}$$

4.2.4 Shock profile of any strength for the Broadwell model

The shock profile for the Broadwell model is found to satisfies (4.2). The equation has two conservation laws which are obtained by multiplying $\chi_1 = (1,2,1)$ and $\chi_2 = (1,0,-1)$. The constants can be identified using the limiting conditions and the Rankine-Hugoniot conditions. We have

$$(\varphi_+ - \varphi_-) - b(\varphi_+ + 2\varphi_0 + \varphi_-) = a_1,$$
$$(\varphi_+ + \varphi_-) - b(\varphi_+ - \varphi_-) = a_2.$$

From these conservation laws we find φ_+, φ_- in terms of φ_0, a_1 and a_2, as

$$\varphi = \Lambda\varphi_0 + a, \tag{4.22}$$

where

$$\Lambda = \begin{pmatrix} \frac{b}{1-b} \\ 1 \\ -\frac{b}{1+b} \end{pmatrix},$$
$$a = \begin{pmatrix} \frac{a_1+a_2}{2(1-b)} \\ 0 \\ -\frac{a_1-a_2}{2(1+b)} \end{pmatrix}.$$

Define

$$\bar{Q}(M,N) = \frac{1}{4}M_0N_0 - \frac{1}{2}(M_+N_- + M_-N_+).$$

Then $Q(\varphi)$ has the simple expression

$$
\begin{aligned}
Q(\varphi) &= \{\bar{Q}(\Lambda,\Lambda)\varphi_0^2 + 2\bar{Q}(\Lambda,a)\varphi_0 + \bar{Q}(a,a)\} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
&\equiv q(\varphi_0) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Since $M^{\pm}$ are Maxwellian, and they satisfy the conservation laws, so $M^{\pm}$ are two distinct roots of the quadratic q. Thus

$$q(\varphi_0) = \bar{Q}(\Lambda,\Lambda)(\varphi_0 - M_0^+)(\varphi_0 - M_0^-).$$

Now the shock profile equation in (4.2) reduced to a single autonomous ordinary differential equation for φ_0:

$$\varphi_0' = \frac{1}{b}\bar{Q}(\Lambda,\Lambda)(\varphi_0 - M_0^+)(\varphi_0 - M_0^-).$$

This has the solution

$$\varphi_0(\bar{x}) = -\frac{1}{2}(M_0^- + M_0^+)\tanh\left(\frac{1+3b^2}{4b(1-b^2)}(M_0^- - M_0^+)\bar{x}\right) + \frac{1}{2}(M_0^+ - M_0^+).$$

Combine this with (4.22), we get the explicit form for the shock profile.

4.3 The linearized problem

Linearizing system (4.6) around the profile φ gives the following system:

$$
\begin{cases}
\partial_t h + (V - bI)\partial_{\bar{x}} h = L_{\varphi(\bar{x})} h, \\
h(\bar{x},0) = a(\bar{x},y_0)\mathbf{r}_1(\vec{\mathbf{u}}^-) + b(\bar{x},y_0)\mathbf{r}_2(\vec{\mathbf{u}}^-),
\end{cases}
\tag{4.23}
$$

$$\|a(\bar{x},y_0), b(\bar{x},y_0)\| \le O(1)\epsilon e^{-|\bar{x}-y_0|/C}.$$

y_0 is a parameter to model the location of the initial data. Firstly consider the case that $y_0 = 0$. Here we changed the coordinate system to make the presentation of this problem convenient:

$$\begin{cases} \bar{x} = x - bt, \\ t = t. \end{cases}$$

We use the framework introduced in the Section 4.1 to solve this system.

4.3.1 Non-decaying structure stacked around the wave front

Firstly, we lay down a standard procedure to catch the non-decaying component which stacked around the shock front. Approximate the shock profile with a discontinuous function, which takes the form:

$$H(x,t) \equiv \begin{cases} \varphi(-\infty), & x < bt, \\ \varphi(\infty), & x > bt. \end{cases}$$

Set

$$F(x,t) \equiv (f_+(x,t), f_0(x,t), f_-(x,t))^T,$$

satisfies:

$$\begin{cases} \partial_t F + (V - bI)\partial_{\bar{x}} F = L_{H(\bar{x})} F, \\ F(\bar{x}, 0) = h(\bar{x}, 0). \end{cases} \tag{4.24}$$

Separate the whole space into two half-spaces:

$$\begin{cases} \partial_t f_+ + (1 - \frac{2}{3})\partial_{\bar{x}} f_+ = -(\frac{9}{70} f_+ - \frac{3}{14} f_0 + \frac{5}{14} f_-), \quad \text{for} \quad \bar{x} < 0, \quad t > 0, \\ \partial_t f_0 - \frac{2}{3}\partial_{\bar{x}} f_0 = \frac{9}{70} f_+ - \frac{3}{14} f_0 + \frac{5}{14} f_-, \\ \partial_t f_- - (1 + \frac{2}{3})\partial_{\bar{x}} f_- = -(\frac{9}{70} f_+ - \frac{3}{14} f_0 + \frac{5}{14} f_-), \\ F(\bar{x}, 0) = I_0(\bar{x}), \\ f_0(0,t) = \alpha(t), f_-(0,t) = \beta(t), \end{cases} \tag{4.25}$$

$$\begin{cases} \partial_t f_+ + (1-\frac{2}{3})\partial_{\bar{x}} f_+ = -\frac{1}{6}(f_+ - f_0 + f_-), \quad \text{for} \quad \bar{x} > 0, \quad t > 0, \\ \partial_t f_0 - \frac{2}{3}\partial_{\bar{x}} f_0 = \frac{1}{6}(f_+ - f_0 + f_-), \\ \partial_t f_- - (1+\frac{2}{3})\partial_{\bar{x}} f_- = -\frac{1}{6}(f_+ - f_0 + f_-), \\ F(\bar{x},0) = I_0(\bar{x}), \\ f_+(0,t) = \gamma(t), \end{cases} \tag{4.26}$$

with $\alpha(t)$,$\beta(t)$ and $\gamma(t)$ to be determined later.

A compatibility property must be satisfied on the surfaces of discontinuity: when the perturbations pass through the shock, the mass, momentum (mass flux) and momentum flux should be the same on both sides. With the help of three algebraic relationships (incoming-outgoing map)we will obtain below, one can represent the mass, momentum and momentum flux in terms of $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and thus solve them easily.

Apply the Laplace-Laplace transform to the above two systems (4.25) and (4.26), temporarily omit the initial data to make the statements more economic. Firstly, we state the algebraic properties of the roots of two characteristic polynomials with the proof, the algebraic manipulation is standard.

The characteristic polynomial of $p(\xi,s)$ and $q(\xi,s)$ for the two systems are defined:

$$\begin{aligned} p(\xi,s) &= 49s^2 + 70s^3 + \frac{148s\xi}{3} + 140s^2\xi - \frac{35\xi^2}{9} + \frac{70s\xi^2}{3} - \frac{700\xi^3}{27}, \\ q(\xi,s) &= 3s^2 + 6s^3 - (4s+12s^2)\xi + (\frac{1}{3}+2s)\xi^2 + \frac{20}{9}\xi^3. \end{aligned} \tag{4.27}$$

Only consider the first characteristic polynomial $p(\xi,s)$, there is a root $\xi \sim O(1)s$ as $s \to 0$. Thus for the purpose to approximate this root, one could consider the following polynomial ignoring some higher order terms:

$$p_0(\xi,s) \equiv 49s^2 + \frac{148s\xi}{3} - \frac{35\xi^2}{9}.$$

Two roots of this polynomial are $\xi = (\frac{222}{35} + \frac{9\sqrt{799}}{35})s$ and $\xi = (\frac{222}{35} - \frac{9\sqrt{799}}{35})s$. These two roots give the asymptotic behavior of the roots of $p(\xi,s) = 0$ as

$s \to 0$. Since the production of three roots of $p(\xi, s) = 0$ in ξ is $\frac{189s^2+270s^3}{100}$, $\xi = -\frac{3}{20}$ is an asymptotic root of $p(\xi, s) = 0$ as $s \to 0$.

Now we consider a root ξ of $p(\xi, s)$ of the form: $\xi = -\frac{3}{20} + \alpha(s)$, and by using the implicit function theory, we can define an analytic root $\xi(s) = -\frac{3}{20} + \alpha(s)$ of $p(\xi, s) = 0$ around $s = 0$ with $\xi(0) = -\frac{3}{20}$.

Next, use the Euclid's algorithm to find equation of the branch point of $p(\xi, s) = 0$. The necessary condition for the branch point (ξ, s) is

$$s^2(5593 + 1899396s + 6351408s^2 + 9843120s^3 + 11113200s^4) = 0.$$

Since $\xi(s)$ is analytic around $s = 0$, we get the possible branch point for $\xi(s)$ are the roots of

$$(5593 + 1899396s + 6351408s^2 + 9843120s^3 + 11113200s^4) = 0. \quad (4.28)$$

Lemma 4.3.1. *There exists $C > 0$ such that the root s of the polynomial in* (4.28) satisfies

$$Re(s) < -C.$$

Lemma 4.3.2. *There are three analytic roots $\xi = \mu(s)$ of characteristic polynomial $p(\xi, s) = 0$ for $s \in \{Re(s) > -C\}$ and one has the following asymptotic expansion for $|s| \ll 1$:*

$$\begin{cases} \mu_1(s) = (\frac{222}{35} + \frac{9\sqrt{799}}{35})s + O(1)s^2, \\ \mu_2(s) = (\frac{222}{35} - \frac{9\sqrt{799}}{35})s + O(1)s^2, \\ \mu_3(s) = -\frac{3}{20} - \frac{165}{14}s + O(1)s^2. \end{cases}$$

Here, the function $O(1)$ is an analytic function in the region $Re(s) > -C$. We can also expand the roots as follows:

$$\begin{aligned} \lim_{s\to\infty} \frac{\mu_1(s)}{s} &= 3, \\ \lim_{s\to\infty} \frac{\mu_2(s)}{s} &= -3/5, \\ \lim_{s\to\infty} \frac{\mu_3(s)}{s} &= -3/2. \end{aligned}$$

Similarly, we have the following:

Lemma 4.3.3. *There exists some positive constant C such that there are three analytic roots $\xi = \nu(s)$ of $q(\xi, s) = 0$ for $s \in \{Re(s) > -C\}$ and one has the following asymptotic expansion for $|s| \ll 1$:*

$$\begin{cases} \nu_1(s) = 3(2+\sqrt{3})s + O(1)s^2, \\ \nu_2(s) = 3(2-\sqrt{3})s + O(1)s^2, \\ \nu_3(s) = -\frac{3}{20} - \frac{129}{10}s + O(1)s^2. \end{cases}$$

Here, the function $O(1)$ is an analytic function in the region $Re(s) > -C$.

One can also expand the roots as follows:

$$\begin{aligned} \lim_{s\to\infty} \frac{\nu_1(s)}{s} &= 3/2, \\ \lim_{s\to\infty} \frac{\nu_2(s)}{s} &= 3/5, \\ \lim_{s\to\infty} \frac{\nu_3(s)}{s} &= -3. \end{aligned}$$

Now we can derive the incoming-outgoing relationship for the full boundary data and construct the full boundary date in terms of the imposed boundary conditions:

$$\begin{cases} \mathbb{L}_s[f_+](0-,s) &= \mathbf{M}_1(\mathbb{L}_s[f_0](0-,s), \mathbb{L}_s[f_-](0-,s)) \\ &\equiv -\frac{p_2(\mu_1,s)}{p_3(\mu_1,s)}\mathbb{L}_s[f_0](0-,s) - \frac{p_1(\mu_1,s)}{p_3(\mu_1,s)}\mathbb{L}_s[f_-](0-,s), \\ \mathbb{L}_s[f_0](0+,s) &= \mathbf{M}_2(\mathbb{L}_s[f_+](0+,s)) \\ &\equiv -\frac{q_3(\nu_2,s)q_1(\nu_1,s)-q_3(\nu_1,s)q_1(\nu_2,s)}{q_2(\nu_1,s)q_3(\nu_2,s)-q_2(\nu_2,s)q_3(\nu_1,s)}\mathbb{L}_s[f_+](0+,s) \\ &= \frac{1}{2+60s}\mathbb{L}_s[f_+](0+,s), \\ \mathbb{L}_s[f_-](0+,s) &= \mathbf{M}_3(\mathbb{L}_s[f_+](0+,s)) \\ &\equiv -\frac{q_1(\nu_2,s)q_2(\nu_1,s)-q_1(\nu_1,s)q_2(\nu_2,s)}{q_2(\nu_1,s)q_3(\nu_2,s)-q_2(\nu_2,s)q_3(\nu_1,s)}\mathbb{L}_s[f_+](0+,s) \\ &= -\frac{1+6s}{5+150s}\mathbb{L}_s[f_+](0+,s), \end{cases} \tag{4.29}$$

where

$$
\begin{cases}
p_1(\xi, s) = \frac{125s}{3} - \frac{125\xi}{9}, \\
p_2(\xi, s) = \frac{68s}{3} + \frac{140s^2}{3} + \frac{40\xi}{9} + \frac{560s\xi}{9} - \frac{700\xi^2}{27}, \\
p_3(\xi, s) = -3s - 5\xi, \\
q_1(\xi, s) = \frac{s}{3} - \frac{5}{9}\xi, \\
q_2(\xi, s) = -\frac{4}{3}s - 4s^2 + \frac{8}{9}\xi + \frac{16}{3}s\xi + \frac{20}{9}\xi^2, \\
q_3(\xi, s) = -\frac{5}{3}s - \frac{5}{9}\xi.
\end{cases}
$$

With the help of (4.29), three continuities can be given in terms of Laplace transformed functions $\alpha(s)$, $\beta(s)$, $\gamma(s)$, taking the initial data into account:

$$
\begin{cases}
\begin{aligned}
& \mathbf{M}_1(\mathbb{L}_s[\alpha](s) - \mathbb{L}_s[A_0](0, s), \mathbb{L}_s[\beta](s) - \mathbb{L}_s[A_-](0, s)) + \mathbb{L}_s[A_+](0, s) \\
& + 2\mathbb{L}_s[\alpha](s) + \mathbb{L}_s[\beta](s) \\
= \; & \mathbb{L}_s[\gamma](s) + 2\mathbf{M}_2(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) + 2\mathbb{L}_s[B_0](0, s) \\
& + \mathbf{M}_3(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) + \mathbb{L}_s[B_-](0, s), \\
& \mathbf{M}_1(\mathbb{L}_s[\alpha](s) - \mathbb{L}_s[A_0](0, s), \mathbb{L}_s[\beta](s) - \mathbb{L}_s[A_-](0, s)) \\
& + \mathbb{L}_s[A_+](0, s) - \mathbb{L}_s[\beta](s) \\
= \; & \mathbb{L}_s[\gamma](s) - \mathbf{M}_3(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) - \mathbb{L}_s[B_-](0, s), \\
& (1 - b)\mathbf{M}_1(\mathbb{L}_s[\alpha](s) - \mathbb{L}_s[A_0](0, s), \mathbb{L}_s[\beta](s) - \mathbb{L}_s[A_-](0, s)) \\
& + (1 - b)\mathbb{L}_s[A_+](0, s) + (1 + b)\mathbb{L}_s[\beta](s) \\
= \; & (1 - b)\mathbb{L}_s[\gamma](s) + (1 + b)\mathbf{M}_3(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) \\
& + (1 + b)\mathbb{L}_s[B_-](0, s).
\end{aligned}
\end{cases}
$$

$A(\bar{x}, t)$ and $B(\bar{x}, t)$ are terms similar to $I(\bar{x}, t)$ defined in (4.21), taking care the initial information.

From (4.30), we get the following relationships:

$$
\begin{cases}
\begin{aligned}
& \mathbf{M}_1(\mathbb{L}_s[\alpha](s) - \mathbb{L}_s[A_0](0, s), \mathbb{L}_s[\beta](s) - \mathbb{L}_s[A_-](0, s)) \\
& + \mathbb{L}_s[A_+](0, s) = \mathbb{L}_s[\gamma](s), \\
& \mathbb{L}_s[\alpha](s) = \mathbf{M}_2(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) + \mathbb{L}_s[B_0](0, s), \\
& \mathbb{L}_s[\beta](s) = \mathbf{M}_3(\mathbb{L}_s[\gamma](s) - \mathbb{L}_s[B_+](0, s)) + \mathbb{L}_s[B_-](0, s).
\end{aligned}
\end{cases}
\tag{4.30}
$$

Note that:

$$\begin{aligned}
&\mathbf{M}_1(a+b, c+d) = \mathbf{M}_1(a,c) + \mathbf{M}_1(b,d),\\
&\mathbf{M}_2(a+b) = \mathbf{M}_2(a) + \mathbf{M}_2(b),\\
&\mathbf{M}_3(a+b) = \mathbf{M}_3(a) + \mathbf{M}_3(b).
\end{aligned}$$

From (4.30), we have

$$\mathbb{L}_s[\gamma](s) = \mathbf{M}_1(\mathbf{M}_2(\mathbb{L}_s[\gamma](s)), \mathbf{M}_3(\mathbb{L}_s[\gamma](s))) + I_1(0,s) \tag{4.31}$$

where

$$\begin{aligned}
I_1(0,s) \quad = \quad & \mathbf{M}_1\left(-\mathbf{M}_2(\mathbb{L}_s[B_+](0,s)) + \mathbb{L}_s[B_0](0,s) - \mathbb{L}_s[A_+](0,s),\right.\\
&\qquad \left.-\mathbf{M}_3(\mathbb{L}_s[B_+](0,s)) + \mathbb{L}_s[B_-](0,s) - \mathbb{L}_s[A_-](0,s)\right)\\
&+\mathbb{L}_s[A_+](0,s).
\end{aligned}$$

The operator which converts $\mathbb{L}_s[\gamma](s)$ into $I_1(0,s)$ satisfies:

$$\begin{aligned}
& 1 - \frac{p_2(\mu_1,s)}{p_3(\mu_1,s)} \frac{q_3(\nu_2,s)q_2(\nu_1,s) - q_3(\nu_1,s)q_2(\nu_2,s)}{q_2(\nu_1,s)q_3(\nu_2,s) - q_2(\nu_2,s)q_3(\nu_1,s)}\\
& -\frac{p_1(\mu_1,s)}{p_3(\mu_1,s)} \frac{q_1(\nu_2,s)q_2(\nu_1,s) - q_1(\nu_1,s)q_2(\nu_2,s)}{q_2(\nu_1,s)q_3(\nu_2,s) - q_2(\nu_2,s)q_3(\nu_1,s)}\\
= \quad & \frac{O(1)s}{s+O(1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{L}_s[\gamma](s) \quad &= \quad O(1)\frac{s+O(1)}{s} I_1(0,s),\\
\mathbb{L}_s[\alpha](s) \quad &= \quad O(1)\frac{s+O(1)}{s} \mathbf{M}_2(I_1(0,s)) + I_2(0,s),\\
\mathbb{L}_s[\beta](s) \quad &= \quad O(1)\frac{s+O(1)}{s} \mathbf{M}_3(I_1(0,s)) + I_3(0,s),
\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}
&I_2(0,s) = -\mathbf{M}_2(\mathbb{L}_s[B_+](0,s) + \mathbb{L}_s[B_0](0,s),\\
&I_3(0,s) = -\mathbf{M}_3(\mathbb{L}_s[B_+](0,s) + \mathbb{L}_s[B_-](0,s),
\end{aligned}$$

$\mathbf{M}_2, \mathbf{M}_3$ are bounded operators. Take the inverse Laplace transformation, we finally get the estimate of each boundary data:

Lemma 4.3.4. *The boundary data $\gamma(t)$, $\alpha(t)$ and $\beta(t)$ satisfy the following estimates:*

$$\left\| \begin{pmatrix} \gamma(t) \\ \alpha(t) \\ \beta(t) \end{pmatrix} \right\| \equiv \|\ell(t)\| = O(1)\epsilon \int_0^t \frac{e^{-C\tau}}{\sqrt{\tau}} d\tau. \tag{4.33}$$

According to the boundary information (4.33), we introduce a wave front component parallelling to $\psi(\bar{x})$, which satisfies the following steady equation: $(V - bI)\partial_{\bar{x}}\psi = L_{\varphi(\bar{x})}\psi$, with $\psi(\pm\infty) = 0$.

This system contains two conservation laws:

$$(1,2,1)(V - bI)\partial_{\bar{x}}\psi = 0, (1,0,-1)(V - bI)\partial_{\bar{x}}\psi = 0.$$

With these two conservation laws, one can solve the system and get the following estimate on the local wave front component:

$$|\psi(\bar{x})| \leq O(1)e^{-C|\bar{x}|}.$$

Now we can extract the non-decaying component $diag(\ell(t))\psi(\bar{x})$, the remainder satisfies:

$$\begin{cases} \partial_t w + (V - bI)\partial_{\bar{x}} w = L_{\varphi(\bar{x})} w + diag(\ell'(t))\psi(\bar{x}), \\ w(\bar{x}, 0) = (l_1(\vec{\mathbf{u}}^-), F(\bar{x}, 0)), \|w(\bar{x}, 0)\| = O(1)\varepsilon e^{-C|\bar{x}|}. \end{cases} \tag{4.34}$$

4.3.2 Transverse waves

The transverse wave is devised for the purely transverse initial data in (4.34). Consider the following approximate problem:

$$\begin{cases} \partial_t v + (V - bI)\partial_{\bar{x}} v = L_{H(\bar{x})} v + diag(\ell'(t))\psi(\bar{x}), \\ v(\bar{x}, 0) = w(\bar{x}, 0), \end{cases} \tag{4.35}$$

which could be separated into two systems:

$$\begin{cases} \partial_t v(\bar{x},t) + (1-\frac{2}{3})\partial_{\bar{x}} v(\bar{x},t) - L_{M^-} v(\bar{x},t) = S_1(\bar{x},t), \\ \quad \text{for } \ \bar{x}<0, \quad t>0, \\ v(\bar{x},0) = O(1)\epsilon e^{-C|\bar{x}|}, \\ v_0(0,t) = O(1)\epsilon e^{-Ct}, v_-(0,t) = O(1)\epsilon e^{-Ct}, \end{cases} \tag{4.36}$$

$$\begin{cases} \partial_t v(\bar{x},t) + (1-\frac{2}{3})\partial_{\bar{x}} v(\bar{x},t) - L_{M^+} v(\bar{x},t) = S_1(\bar{x},t), \\ \quad \text{for } \ \bar{x}>0, \quad t>0, \\ v(\bar{x},0) = O(1)\epsilon e^{-C|\bar{x}|}, \\ v_+(0,t) = O(1)\epsilon e^{-Ct}, \end{cases} \tag{4.37}$$

the source term $S_1(\bar{x},t) = diag(\ell'(t))\psi(\bar{x})$ is bounded by $O(1)\epsilon\frac{e^{-C|\bar{x}|-Ct}}{\sqrt{t}}$.

Applying the first Green's identity to (4.36) and (4.37), we have

$$\begin{aligned} v(\bar{x},t) &= \int_{-\infty}^{0} \mathbb{G}^-(\bar{x}-y+\frac{2}{3}t,t)v(y,0)dy \\ &\quad + \int_0^t \mathbb{G}^-(\bar{x}+\frac{2}{3}(t-\tau),t-\tau)v(0,\tau)d\tau \\ &\quad + \int_0^t \int_{-\infty}^{0} \mathbb{G}^-(\bar{x}-y+\frac{2}{3}(t-\tau),t-\tau)S_1(y,\tau)dyd\tau, \bar{x}<0, \\ v(\bar{x},t) &= \int_0^{\infty} \mathbb{G}^+(\bar{x}-y+\frac{2}{3}t,t)v(y,0)dy \\ &\quad + \int_0^t \mathbb{G}^+(\bar{x}+\frac{2}{3}(t-\tau),t-\tau)v(0,\tau)d\tau \\ &\quad + \int_0^t \int_0^{\infty} \mathbb{G}^+(\bar{x}-y+\frac{2}{3}(t-\tau),t-\tau)S_1(y,\tau)dyd\tau, \bar{x}>0. \end{aligned}$$

$\mathbb{G}^-$ and $\mathbb{G}^+$ are the global Green's functions linearized around M^- and M^+ respectively.

Since $\lambda_1^- < \frac{2}{3} < \lambda_2^-$, $\lambda_1^+ < \lambda_2^+ < \frac{2}{3}$, with the help of Lemma A.0.1 in Appendix, we have the following estimates:

$$\begin{aligned} &\int_0^t \int_{-\infty}^{0} \mathbb{G}^-(\bar{x}-y+\frac{2}{3}(t-\tau),t-\tau)S_1(y,\tau)dyd\tau \\ &= \ O(1)\epsilon\mathcal{A}_1^{1,1}(\bar{x},t) + O(1)\epsilon e^{-(|\bar{x}|+t)/C}, \quad \bar{x}<0, \end{aligned}$$

$$\begin{aligned}
&\int_0^t \int_0^\infty \mathbb{G}^+(\bar{x} - y + \frac{2}{3}(t-\tau), t-\tau) S_1(y,\tau) dy d\tau \\
= \ & O(1)\epsilon e^{-(|\bar{x}|+t)/C}, \quad \bar{x} > 0.
\end{aligned}$$

Combining these two results, we get the estimate for the problem (4.35):

$$\|v(\bar{x},t)\| = O(1)\epsilon \frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(1+t)}}}{\sqrt{1+t}} + O(1)\epsilon \mathcal{A}_1^{1,1}(\bar{x},t) + O(1)\epsilon e^{-(|\bar{x}|+t)/C}.$$

The truncation error $z(\bar{x},t)$ generated by (4.35) is given by

$$\begin{cases} \partial_t z + (V - bI)\partial_{\bar{x}} z = L_{\varphi(\bar{x})} z + S(\bar{x},t), \\ z(\bar{x},0) = 0, \end{cases} \tag{4.38}$$

with $S(\bar{x},t) = (L_{\varphi(\bar{x})} - L_{H(\bar{x})})v(\bar{x},t)$ purely microscopic. In the solution $z(\bar{x},t)$, there are slowing decaying waves propagating in different directions. These multi-direction propagations prevent one from obtaining any exponentially sharp structure directly. However, the source term is purely microscopic, one can gain exponential sharp estimates in the compressive component around the shock front for large time.

4.3.3 Pointwise estimate of the approximate truncation error problem

The approximate problem for (4.38) is given as follows:

$$\begin{cases} \partial_t r + (V - bI)\partial_{\bar{x}} r = L_{\varphi_L(\bar{x})} r + S(x,t), \\ r(\bar{x},0) = z(\bar{x},0). \end{cases} \tag{4.39}$$

Following the framework, we split the whole space domain into three parts. The solvability of problem in the left and right far fields gives incoming-outgoing maps:

$$\begin{cases} \mathbb{L}_s[r_0](L+,s) & = & \mathbf{M}_2(\mathbb{L}_s[r_+](L+,s)) + I_1(L+,s) \\ & = & \frac{1}{2+60s}\mathbb{L}_s[r_+](L+,s) + I_1(L+,s), \\ \mathbb{L}_s[r_-](L+,s) & = & \mathbf{M}_3(\mathbb{L}_s[r_+](L+,s)) + I_2(L+,s) \\ & = & -\frac{1+6s}{5+150s}\mathbb{L}_s[r_+](L+,s) + I_2(L+,s), \\ \mathbb{L}_s[r_+](-L-,s) & = & \mathbf{M}_1(\mathbb{L}_s[r_0](-L-,s), \mathbb{L}_s[r_-](-L-,s) \\ & & +I_3(-L-,s) \\ & \equiv & -\frac{p_2(\mu_1,s)}{p_3(\mu_1,s)}\mathbb{L}_s[r_0](-L-,s) \\ & & -\frac{p_1(\mu_1,s)}{p_3(\mu_1,s)}\mathbb{L}_s[r_-](-L-,s) + I_3(-L-,s). \end{cases} \tag{4.40}$$

Apply the Laplace transform to the equation of r with respect to time variable t, multiply it with $G(\bar{x},s)$, integrate in the domain $(-L,L)$ by parts, with the help of C^1 continuity of r, we get

$$\begin{aligned} & G(L,s)(V-bI)\mathbb{L}_s[r](L,s) - G(-L,s)(V-bI)\mathbb{L}_s[r](-L,s) \\ = & \int_{-L}^{L} G(\bar{x},s)\mathbb{L}_s[S](\bar{x},s)d\bar{x}. \end{aligned} \tag{4.41}$$

Combine (4.40) and (4.41) together, one could have:

$$\begin{cases} \begin{pmatrix} A(s) & (1-b)\frac{p_2(\mu_1,s)}{p_3(\mu_1,s)} & (1-b)\frac{p_1(\mu_1,s)}{p_3(\mu_1,s)} \\ B(s) & b & 0 \\ C(s) & 0 & 1+b \end{pmatrix} \begin{pmatrix} \mathbb{L}_s[r_+](L+,s) \\ \mathbb{L}_s[r_0](-L-,s) \\ \mathbb{L}_s[r_-](-L-,s) \end{pmatrix} \\ \equiv K(s) \begin{pmatrix} \mathbb{L}_s[r_+](L+,s) \\ \mathbb{L}_s[r_0](-L-,s) \\ \mathbb{L}_s[r_-](-L-,s) \end{pmatrix} = \int_{-L}^{L} G(\bar{x},s)\mathbb{L}_s[S](\bar{x},s)d\bar{x}, \end{cases}$$

where

$$\begin{cases} A(s) = (1-b)G_{11}(L,s) - b\frac{1}{2+60s}G_{12}(L,s) + (1+b)\frac{1+6s}{5+150s}G_{13}(L,s), \\ B(s) = (1-b)G_{21}(L,s) - b\frac{1}{2+60s}G_{22}(L,s) + (1+b)\frac{1+6s}{5+150s}G_{23}(L,s), \\ C(s) = (1-b)G_{31}(L,s) - b\frac{1}{2+60s}G_{32}(L,s) + (1+b)\frac{1+6s}{5+150s}G_{33}(L,s). \end{cases}$$

This gives that

$$\left\|\begin{pmatrix}\mathbb{L}_s[r_+](L+,s)\\ \mathbb{L}_s[r_0](-L-,s)\\ \mathbb{L}_s[r_-](-L-,s)\end{pmatrix}\right\| = O(1)\epsilon\frac{adjK(s)}{detK(s)}\int_{-L}^{L} G(\bar{x},s)\frac{e^{-|\bar{x}|}}{\sqrt{s+C}}d\bar{x}. \quad (4.42)$$

When s is large enough, $\frac{adjK(s)}{detK(s)}$ is analytic and bounded. One could also generate a program in the Mathematica 8.0, that when $Re(s) > -C$ and finite, $\frac{adjK(s)}{detK(s)}$ is analytic and has no pole. Moreover, $detK(s) > 0$ when $Re(s) > -C$. Note that the right hand side of (4.42) decays to zero when $s \to \pm i\infty$. Therefore, by the complex analysis we have

$$\begin{aligned}&\left\|\begin{pmatrix}r_+(L+,t)\\ r_0(-L-,t)\\ r_-(-L-,t)\end{pmatrix}\right\|\\ = \;& O(1)\epsilon\frac{e^{-Ct}}{\sqrt{t}} * \int_R e^{i\eta t}\frac{adjK(i\eta - C)}{detK(i\eta - C)}\int_{-L}^{L} G(\bar{x}, i\eta - C)e^{-|\bar{x}|}d\bar{x}d\eta\\ = \;& O(1)\epsilon\frac{e^{-C_1 t}}{\sqrt{t}}.\end{aligned}$$

Once all the boundary data are clear, with the existence of the spectrum gap, one could have the following estimate in finite domain:

$$\sup_{x\in[-L,L]} |r(\bar{x},t)| = O(1)\varepsilon\frac{e^{-C_1 t}}{\sqrt{t}}. \quad (4.43)$$

Problem restricted to the left or right far field is just a constant coefficient problem in half space. Using Green's function for the linearized Broadwell model, one could obtain the estimate easily.

To summarize, we have:

$$\|r(\bar{x},t)\| = O(1)\epsilon\begin{cases}\frac{e^{-\frac{(\bar{x}+(2/3-\lambda_1^-)t)^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} < -L,\\ \frac{e^{-Ct}}{\sqrt{t}}, & \text{for } -L \le \bar{x} \le L,\\ e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} > L.\end{cases} \quad (4.44)$$

4.3.4 Iterated scheme

The error function $e(\bar{x},t) \equiv z(\bar{x},t) - r(\bar{x},t)$ of approximate problem (4.39) to (4.38) satisfies the initial value problem:

$$\begin{cases} \partial_t e + (V - bI)\partial_{\bar{x}} e = L_{\varphi(\bar{x})} e + S^0(\bar{x},t), \\ e(\bar{x},0) = 0, \end{cases} \tag{4.45}$$

where $S^0(\bar{x},t) = (L_{\varphi(\bar{x})} - L_{\varphi_L(\bar{x})}) r(\bar{x},t)$. From (4.44) and the choice of $L = O(1)|\ln\varepsilon|$, we have

$$\|S^0(\bar{x},t)\| = O(1)\varepsilon^2 \frac{e^{-C(\bar{x}+t)}}{\sqrt{t}}.$$

Using the same method of solving (4.39), we consider the following approximate problem:

$$\begin{cases} \partial_t r^1 + (V - bI)\partial_{\bar{x}} r^1 = L_{\varphi_L(\bar{x})} r^1 + S^0(\bar{x},t), \\ r^(\bar{x},0) = 0. \end{cases} \tag{4.46}$$

Similarly, one could obtain the following estimates:

$$\|r^1(\bar{x},t)\| = O(1)\epsilon^2 \begin{cases} \frac{e^{-\frac{(\bar{x}+(2/3-\lambda_1^-)t)^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} < -L, \\ \frac{e^{-Ct}}{\sqrt{t}}, & \text{for } -L \le \bar{x} \le L, \\ e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} > L. \end{cases}$$

Therefore, we introduce an iterated scheme to construct the solution of (4.38)

$$\begin{cases} r^0 = r, \\ e^0 = z, \\ e^1 \equiv e^0 - r^0, \end{cases} \tag{4.47}$$

for $k \ge 1$,

$$\begin{cases} \partial_t e^k + (V - bI)\partial_{\bar{x}} e^k = L_{\varphi(\bar{x})} e^k + S^{k-1}(\bar{x},t), \\ e^k(\bar{x},0) = 0, S^{k-1}(\bar{x},t) = (L_{\varphi(\bar{x})} - L_{\varphi_L(\bar{x})}) r^{k-1}(\bar{x},t). \end{cases} \tag{4.48}$$

$$\begin{cases} \partial_t r^k + (V - bI)\partial_{\bar{x}} r^k = L_{\varphi_L(\bar{x})} r^k + S^{k-1}(\bar{x}, t), \\ r^k(\bar{x}, 0) = 0, \end{cases} \tag{4.49}$$

$$e^{k+1} = e^k - r^k.$$

For each r^k, we have the estimates:

$$\|r^k(\bar{x}, t)\| = O(1)\epsilon^{k+1} \begin{cases} \frac{e^{-\frac{(\bar{x}+(2/3-\lambda_1^-)t)^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} < -L, \\ \frac{e^{-Ct}}{\sqrt{t}}, & \text{for } -L \le \bar{x} \le L, \\ e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} > L. \end{cases} \tag{4.50}$$

The solution of (4.38) can be written formally in terms of the iterated scheme:

$$z = \sum_{k=0}^{\infty} r^k.$$

From (4.50), the series is convergent, and

$$\|z(\bar{x}, t)\| = O(1)\epsilon \begin{cases} \frac{e^{-\frac{(\bar{x}+(2/3-\lambda_1^-)t)^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} < -L, \\ \frac{e^{-Ct}}{\sqrt{t}}, & \text{for } -L \le \bar{x} \le L, \\ e^{-C|\bar{x}|-Ct}, & \text{for } \bar{x} > L. \end{cases}$$

4.3.5 Summary on estimates of the linearized equation around shock layer

In this subsection, we summarize the linear estimates for the linearized equation:

$$\begin{cases} \partial_t h + (V - bI)\partial_{\bar{x}} h = L_{\varphi(\bar{x})} h, \\ h(\bar{x}, 0; y_0) = h_0(\bar{x}; y_0), \end{cases} \tag{4.51}$$

with respect to various type of confined initial data:

$$\begin{cases} \|h_0(\bar{x}; y_0)\| \le O(\epsilon), \\ h_0(\bar{x}; y_0) \equiv 0 \ \text{ for} |\bar{x} - y_0| \ge 1. \end{cases}$$

Theorem 4.3.5. *The solution of initial value problem* (4.51) *with the confined initial data satisfies the following cases.*

Case 1*: Purely Macroscopic Transverse initial Data around Shock, i.e., $y_0 = 0$, $h_0(\bar{x}; y_0)^c = 0$, $h_0(\bar{x}; y_0)^m = 0$ for all $\bar{x}$.*

$$\|h(\bar{x}, t; y_0)\| \leq O(1)\epsilon \frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(1+t)}}}{\sqrt{1+t}} + O(1)\epsilon \mathcal{A}_1^{1,1}(\bar{x}, t) + O(1)\epsilon e^{-(|\bar{x}|+t)/C}.$$

Case 2*: Purely Compressive initial Data around Shock, i.e., $y_0 = 0$, $h_0(\bar{x}; y_0)^t = 0$, $h_0(\bar{x}; y_0)^m = 0$ for all $\bar{x}$.*

$$\begin{aligned} &\|h(\bar{x}, t; y_0) - O(1)\epsilon \int_R (l_2(M^-), h(\bar{x}, 0)) d\bar{x} \psi(\bar{x})\| \\ \leq\ & O(1)\epsilon \mathcal{A}_1^{1,1}(\bar{x}, t) + O(1)\epsilon e^{-(|\bar{x}|+t)/C}. \end{aligned}$$

Case 3*: Purely Microscopic initial Data around Shock, i.e., $y_0 = 0$, $h_0(\bar{x}; y_0)^t = 0$, $h_0(\bar{x}; y_0)^c = 0$ for all $\bar{x}$.*

$$\|h(\bar{x}, t; y_0)\| \leq O(1)\epsilon \mathcal{A}_1^{1,1}(\bar{x}, t) + O(1)\epsilon e^{-(|\bar{x}|+t)/C}.$$

Case 4*: Purely Microscopic initial Data outside Shock, i.e., $y_0 < 0$, $h_0(\bar{x}; y_0)^t = 0$, $h_0(\bar{x}; y_0)^c = 0$ for all $\bar{x}$.*

$$\begin{aligned} \|h(\bar{x}, t; y_0)\| \ \leq\ & O(1)\epsilon \sum_{i=1}^{2} \frac{e^{-\frac{|\bar{x}-y_0+(\frac{2}{3}-\lambda_i^-)t|^2}{C(1+t)}}}{1+t} + O(1)\epsilon e^{-(|\bar{x}-y_0|+t)/C} \\ & + O(1)\epsilon \int_0^t \int_{-\infty}^0 \Big(\sum_{i=1}^{2} \frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_i^-)(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} + e^{-(|\bar{x}-y|+t-\tau)/C}\Big) \\ & \frac{e^{-\frac{|y_0+(\frac{2}{3}-\lambda_2^-)\tau|^2}{C(1+\tau)}}}{1+\tau} e^{-|y|/C} dy d\tau. \end{aligned}$$

Remark 3. *Only the second case containing a non-time decaying estimate around the shock layers. In the previous subsections, we have proved that for the general initial data, we have a non-decaying term around the shock layers: $diag(l(t))\psi(x)$, with the regularity estimate on the function $l(t)$:*

$$\|l(t)'\| \leq O(1) \frac{e^{-Ct}}{\sqrt{t}}.$$

The last case is used for the nonlinear coupling outside the shock layer region. Since the nonlinear collision operator is purely microscopic, we only considers the purely microscopic component for the nonlinear coupling. In the last case, we only consider the case of $y_0 < 0$ because in the right half region, which is the supersonic region, all wave around the shock near shock layer will dissipate exponentially fast in time. One can ignore the nonlinear coupling in this region.

4.4 Nonlinear stability of the shock profile

In this section, we prove the main Theorem 4.1.1 of this chapter. The perturbation $h(\bar{x},t) \equiv \tilde{F}(\bar{x},t) - \varphi(\bar{x}-x_0)$, of the Broadwell shock profile $\varphi(\bar{x})$ satisfies:

$$\begin{cases} \partial_t h + (V - bI)\partial_{\bar{x}} h = L_\varphi h + Q(h), \\ h(\bar{x},0) \le O(\epsilon) e^{-|\bar{x}|}, \\ \int_R h^c(\bar{x},0) d\bar{x} = 0. \end{cases} \tag{4.52}$$

The third condition in (4.52) is natural by choosing a suitable shock front $\varphi(\bar{x}-x_0)$ to monitor the decay of the perturbations. Thus, the total mass of the perturbation does not contain any compressive component. For simplicity, one may assume that $x_0 = 0$. The shock wave is stationary forward and the initial data of the perturbation are exponentially decaying in $\bar{x}$ so that one can expect the solution remains exponentially decaying in $\bar{x} > 0$.

The solution can be represented by the Green's function $\mathbb{G}(\bar{x}, y, t)$:

$$\begin{aligned} & h(\bar{x},t) \\ = & \int_{-\infty}^{\infty} \mathbb{G}(\bar{x} - y + \frac{2}{3}t, t) h(y,0) dy \\ & + \int_0^t \int_{-\infty}^{\infty} \mathbb{G}(\bar{x} - y + \frac{2}{3}(t-s), t-s)(\chi_- + \chi_-) Q(h)(y,s) dy ds \end{aligned}$$

$$
\begin{aligned}
&\equiv \mathbb{G}_{\varphi}^{t}[h(\cdot,0)]+\int_0^t \mathbb{G}_{\varphi}^{t-s}[\chi_+(\cdot)Q(h)(\cdot,s)]ds(\bar{x})\\
&\quad+\sum_{j=-\infty}^{\infty}\int_0^t \mathbb{G}_{\varphi}^{t-s}[\chi_{[2j-1,2j+1)}(\cdot)\chi_-(\cdot)Q(h)(\cdot,s)]ds
\end{aligned}
$$

where $\chi_{[2j-1,2j+1)}(\cdot)$ is the characteristic of the interval $[2j-1,2j+1)$, $\mathbb{G}_{\varphi}^{t}$ is the operator defined as $\mathbb{G}_{\varphi}^{t}[h(\cdot,0;y_0)] = h(\bar{x},t;y_0)$, see Theorem 4.3.5. $\chi_{\pm}$ are defined as

$$
\begin{aligned}
&\chi_-(\bar{x})=\begin{cases} 1, & \text{for } \ \bar{x}<-L,\\ 0, & \text{for } \ \bar{x}>L,\end{cases}\\
&\chi_+=1-\chi_-,\\
&0<\chi_-'(\bar{x})\le O(1)e^{-C|\bar{x}|}.
\end{aligned}
$$

Denote

$$
\begin{cases}
J_1=||\int_0^t \mathbb{G}_{\varphi}^{t-s}[\chi_+(\cdot)Q(h)(\cdot,s)]ds(\bar{x})||,\\
J_2=||\sum_{j=-\infty}^{\infty}\int_0^t \mathbb{G}_{\varphi}^{t-s}[\chi_{[2j-1,2j+1)}(\cdot)\chi_-(\cdot)Q(h)(\cdot,s)]ds||.
\end{cases}
$$

Since Q is purely microscopic, we first have the following estimate:

$$
||h(\bar{x},t)||=O(1)\epsilon\frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(1+t)}}}{\sqrt{1+t}}+O(1)\epsilon\mathcal{A}_1^{1,1}(\bar{x},t)+O(1)\epsilon e^{-(|\bar{x}|+t)/C}+J_1+J_2.
$$

Ansatz Assumption

$$
\begin{aligned}
\|h(\bar{x},t)\| &= O(1)\epsilon\frac{\chi_{[(\lambda_1^--\frac{2}{3})t+\sqrt{t},0]}(\bar{x})}{\sqrt{(|\bar{x}-(\lambda_1^--\frac{2}{3})t|+1)(|\bar{x}-(\lambda_2^--\frac{2}{3})t|+1)}}\\
&\quad+O(1)\epsilon\frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(1+t)}}}{\sqrt{1+t}}+O(1)\epsilon e^{-(|\bar{x}|+t)/C}. \qquad (4.53)
\end{aligned}
$$

We estimate J_1 and J_2 under the ansatz assumptions. Under this assumption, one has

$$
||\chi_+(\cdot)Q(h)(\bar{x},t)||\le O(1)\epsilon^2(\frac{e^{-ct}}{1+t}+\frac{1}{t^2})e^{-\bar{x}/C}. \qquad (4.54)
$$

With this, one can apply Theorem 4.3.5 to yield that for $\bar{x} < 0$,

$$
\begin{aligned}
& J_1(\bar{x}, t) \\
\leq \;& O(1)\epsilon^2 \int_0^t \left(\frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)(t-s)|^2}{C(1+t-s)}}}{1+t-s} + \mathcal{A}_1^{1,1}(\bar{x}, t-s) \right) \cdot \left(\frac{e^{-cs}}{1+s} + \frac{1}{s^2} \right) ds \\
\leq \;& O(1)\epsilon^2 (|\log(1+t)| \frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)(t)|^2}{2C(1+t)}}}{1+t} + \mathcal{A}_1^{1,1}(\bar{x}, t)).
\end{aligned}
$$

To estimate J_2, from Theorem 4.3.5 case 4, one has:

$$
\begin{aligned}
& J_2(\bar{x}, t) \\
\leq \;& O(1) \sum_{j=-\infty}^{\infty} \int_0^t \sum_{i=1}^{2} \int_{j-1}^{j} \left(\frac{e^{-\frac{|\bar{x}-z+(\frac{2}{3}-\lambda_i^-)(t-s)|^2}{C(1+t-s)}}}{1+t-s} + e^{-(|\bar{x}-z|+t-s)/C} \right) \\
& \qquad ||h(z,s)||^2 dz ds \\
& + O(1) \int_{j=-\infty}^{\infty} \int_0^t \sum_{i=1}^{2} \int_{j-1}^{j} \int_s^t \int_R \left(\frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_i^-)(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} + e^{-(|\bar{x}-y|+t-\tau)/C} \right) \\
& ||h(z,s)||^2 \cdot \frac{e^{-\frac{|z+(\frac{2}{3}-\lambda_2^-)(\tau-s)|^2}{C(1+\tau-s)}}}{1+\tau-s} e^{-|y|/C} dy d\tau dz ds \\
\leq \;& O(1) \sum_{i=1}^{2} \int_0^t \int_R \left(\frac{e^{-\frac{|\bar{x}-z+(\frac{2}{3}-\lambda_i^-)(t-s)|^2}{C(1+t-s)}}}{1+t-s} + e^{-(|\bar{x}-z|+t-s)/C} \right) ||h(z,s)||^2 dz ds \\
& + O(1) \sum_{i=1}^{2} \int_0^t \int_R \left(\left(\frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_i^-)(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} + e^{-(|\bar{x}-y|+t-\tau)/C} \right) e^{-|y|/C} \cdot \right. \\
& \qquad \left. \int_0^\tau \int_{-\infty}^{\infty} ||h(z,s)||^2 \frac{e^{-\frac{|z+(\frac{2}{3}-\lambda_2^-)(\tau-s)|^2}{C(1+\tau-s)}}}{1+\tau-s} dz ds \right) dy d\tau \equiv J_{2,1} + J_{2,2}.
\end{aligned}
$$

Using Lemma A.0.2, Lemma A.0.3 and Lemma A.0.4, we have

$$
J_{2,1} \leq O(1)\epsilon^2 \left(\frac{\chi_{[(\lambda_1^- - \frac{2}{3})t + \sqrt{t}, 0]}(\bar{x})}{\sqrt{(|\bar{x} - (\lambda_1^- - \frac{2}{3})t| + 1)(|\bar{x} - (\lambda_2^- - \frac{2}{3})t| + 1)}} + \frac{e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(1+t)}}}{\sqrt{1+t}} \right),
$$

and

$$
e^{-|y|/C} \int_0^\tau \int_{-\infty}^{\infty} ||h(z,s)||^2 \frac{e^{-\frac{|z+(\frac{2}{3}-\lambda_2^-)(\tau-s)|^2}{C(1+\tau-s)}}}{1+\tau-s} dz ds \leq O(1)\epsilon^2 \frac{e^{-|y|/C}}{\sqrt{\tau}}. \quad (4.55)
$$

From (4.55) and the definition of $J_{2,2}$, similar to the proof of Lemma A.0.1 or just use the conclusion for the propagation of damping waves in [20], one can yield that for $\bar{x} < 0$,

$$\begin{aligned} J_{2,2} &\leq O(1)\epsilon^2 \sum_{i=1}^{2} \int_0^t \int_R \left(\frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_i^-)(t-\tau)|^2}{C(1+t-\tau)}}}{1+t-\tau} + e^{-(|\bar{x}-y|+t-\tau)/C} \right) \frac{e^{-C|y|}}{\sqrt{\tau}} dy d\tau \\ &\leq O(1)\epsilon^2 \sum_{i=1}^{2} \frac{1}{(\sqrt{(\bar{x}-(\lambda_i^- - \frac{2}{3})t)^2+t+1}}. \end{aligned} \tag{4.56}$$

Thus, combing the estimate (4.55) and (4.56) together, the ansatz assumption (4.53) is true, and Theorem 4.1.1 is proved.

Appendix **A**

Wave interactions

In this appendix, we give the pointwise wave interaction in both space and in time. This kind of estimate is very important in our analysis, and originally started in [20].

Set

$$
\begin{aligned}
\Gamma^{\beta}(t) &\equiv \int_0^t (s+1)^{-\beta/2} ds \\
&= O(1)\begin{cases} 1, & \text{for } \beta > 2, \\ \log(t+1), & \text{for } \beta = 2, \\ (t+1)^{(2-\beta)/2}, & \text{for } \beta < 2. \end{cases}
\end{aligned}
$$

The first lemma is the result of the propagation of damping wave. With a small mirror modification in the proofs in [20], [34] , we have the following Lemma:

Lemma A.0.1. *Suppose that $\alpha, \beta \geq 0$, then there exists a positive constant C such that for all $|\frac{2}{3} - \lambda_i^{\pm}| = O(1)$, $i = 1, 2$,*

$$
\begin{aligned}
&\int_0^t \int_R \frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_1^-)(t-\tau)|^2}{C(1+t-\tau)}}}{(1+t-\tau)^{\frac{\beta}{2}}} \frac{e^{-C|y|-c\tau}}{(1+\tau)^{\alpha/2}} dy d\tau \\
&= O(1)\mathcal{A}_1^{\alpha,\beta}(\bar{x}, t)
\end{aligned}
$$

$$
\equiv \; O(1)\begin{cases}
\Big(\frac{1}{(|\bar{x}|+1)^{\frac{\beta-2}{2}}\sqrt{|\bar{x}|+1}|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^{\alpha/2}} + \frac{e^{-C|\bar{x}|}\Gamma^{\beta-1}(t+1)}{(t+1)^{\alpha/2}} \\
+\frac{e^{-C|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|}\Gamma^{\alpha}(t+1)}{(t+1)^{\frac{\beta-1}{2}}}\Big)e^{-C|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|} \\
\textit{for} \quad -(\frac{2}{3}-\lambda_1^-)(t+1)+\sqrt{t}<\bar{x}<0, \\
\frac{e^{-Ct}\Gamma^{\alpha}(t+1)+\Gamma^{\alpha}(\sqrt{t+1})(\sqrt{t+1})^{-1}}{(t+1)^{\frac{\beta-1}{2}}} + \frac{\Gamma^{\beta-1}(t+1)e^{-Ct}}{(t+1)^{\alpha/2}} \\
\textit{for} \;\; |\bar{x}+(\frac{2}{3}-\lambda_1^-)t|<\sqrt{t+1}, \\
(t+1)^{-\frac{\beta-1}{2}}\big[e^{-\frac{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^2}{C(t+1)}} + \frac{(t+1)^{1/2}e^{-C|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|}}{|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|^{\alpha/2}} \\
+\Gamma^{\alpha}(t+1)e^{-C|\bar{x}+(\frac{2}{3}-\lambda_1^-)t|}e^{-Ct}\big] \\
+(\bar{x}+(\frac{2}{3}-\lambda_1^-)t)^{(-2\beta+3)/2}e^{-C|\bar{x}-(\frac{2}{3}-\lambda_1^-)t|} \\
+\frac{\Gamma^{\beta-1}(t+1)e^{-C|\bar{x}-(\frac{2}{3}-\lambda_1^-)t|}e^{-Ct}}{(t+1)^{\alpha/2}} \\
\textit{for} \;\; |\bar{x}+(\frac{2}{3}-\lambda_1^-)t|>\sqrt{t+1}.
\end{cases}
$$

$$
\begin{aligned}
&\int_0^t\int_R \frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_2^-)(t-\tau)|^2}{C(1+t-\tau)}}}{(1+t-\tau)^{\frac{\beta}{2}}}\frac{e^{-C|y|-c\tau}}{(1+\tau)^{\alpha/2}}dyd\tau \\
= \;& O(1)(\Gamma^{\alpha}(t+1)(t+1)^{\frac{1-\beta}{2}}e^{-Ct}+\Gamma^{\beta-1}(t+1)(t+1)^{-\alpha/2})e^{-C|\bar{x}|-Ct} \\
& \textit{for} \;\; \bar{x}<0.
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t\int_R \frac{e^{-\frac{|\bar{x}-y+(\frac{2}{3}-\lambda_i^+)(t-\tau)|^2}{C(1+t-\tau)}}}{(1+t-\tau)^{\frac{\beta}{2}}}\frac{e^{-C|y|-c\tau}}{(1+\tau)^{\alpha/2}}dyd\tau \\
= \;& O(1)((\Gamma^{\alpha}(t+1)(t+1)^{\frac{1-\beta}{2}}e^{-Ct}+\Gamma^{\beta-1}(t+1)(t+1)^{-\alpha/2})e^{-C|\bar{x}|-Ct} \\
& \textit{for} \;\; \bar{x}>0.
\end{aligned}
$$

The following two lemmas are from [25], showing the interactions of the waves.

Define

$$
\begin{aligned}
&I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D) \\
\equiv &\int_0^t\int_{-\infty}^{\infty}(t-\tau)^{-(\beta+\gamma)/2}(t-\tau+1)^{-\gamma/2}e^{-\frac{[x-y-\lambda(t-\tau)]^2}{D(t-\tau)}}\theta^{\alpha}(y,\tau;\mu,D)dyd\tau,
\end{aligned}
$$

where

$$
\theta^{\alpha}(y,\tau;\mu,D)=(\tau+1)^{-\alpha/2}e^{-\frac{(y-\mu(\tau+1))^2}{D(\tau+1)}}.
$$

Lemma A.0.2. *Suppose that $\alpha > 0$, $\beta \geq \gamma \geq 0$, and $\beta - \gamma < 3$. Then*

$$\begin{aligned} I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\lambda,D) &= O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1) \\ &\quad (t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(t+1)]\theta(x,t;\lambda,D) \end{aligned}$$

In particular,

$$\begin{aligned} &I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\lambda,D) \\ &= \begin{cases} O(1)\theta(x,t;\lambda), & \text{for } \alpha \geq 3, \quad \beta = 1, \\ O(1)\theta^{3/2}(x,t;\lambda), & \text{for } \alpha \geq 2.5, \quad \beta = 2. \end{cases} \end{aligned}$$

Lemma A.0.3. *Suppose that $\alpha \geq 1$, $\beta \geq \gamma \geq 0$, and $\beta - \gamma < 3$ and that $\lambda < \mu$. Then, for any given constant $E > D$,*

$$\begin{aligned} &I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D) \\ = \ &O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\theta(x,t;\lambda,D) \\ &+O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\theta(x,t;\mu,D) \\ &\begin{cases} 0, \quad \text{for } x < \lambda(t+1) + \sqrt{t+1} \quad \text{or } x > \mu(t+1) - \sqrt{t+1}, \\ O(1)\left[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\lambda t)\theta(x,t;\lambda,E) + (\mu t - x)^{(-\beta+1)/2}\right. \\ \left. \cdot (x-\lambda t)^{(-\alpha+1)/2} + (t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\mu t - x)\theta(x,t;\mu,E)\right] \\ \text{for } \lambda(t+1) + \sqrt{t+1} < x < \mu(t+1) - \sqrt{t+1}. \end{cases} \end{aligned}$$

In particular,

$$\begin{aligned} &I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D) \\ &= \begin{cases} O(1)\psi^{1/2}(x,t;\lambda), & \text{for } \alpha = 2, \quad \beta = 1, \\ O(1)[\psi^{3/2}(x,t;\lambda) + \bar{\psi}^{3/2}(x,t;\mu)], & \text{for } \alpha = 3, \quad \beta = 2, \end{cases} \end{aligned}$$

here

$$\psi^{\alpha}(x,t;\lambda) = [(x-\lambda(t+1))^2 + t + 1]^{-\alpha/2},$$

$$\bar{\psi}^{\alpha}(x,t;\lambda) = [(x-\lambda(t+1))^3 + (t+1)^2]^{-\alpha/3}.$$

Consider the following wave interaction:

$$H(x,t;\mu_1,\mu_2,\mu_3;C)$$
$$\equiv \int_0^t \int_{\mu_1(\tau+1)+\sqrt{\tau+1}}^{\mu_2(\tau+1)-\sqrt{\tau+1}} \frac{e^{-\frac{(x-y-\mu_3(t-\tau+1))^2}{C(t-\tau+1)}}}{t-\tau+1} \frac{1}{(y-\mu_1\tau)(\mu_2\tau-y)} dy d\tau,$$

where $\mu_1 < \mu_2$. We quote Lemma 8.3 in [25].

Lemma A.0.4. *For* $|x-\mu_i t| = O(1)\sqrt{1+t}$,

$$H(x,t;\mu_1,\mu_2,\mu_3;C) \leq O(1)\frac{(\log(2+t))^2}{1+t}.$$

For $|x-\mu_i t| \geq O(1)\sqrt{1+t}$, *one has the following: If* $\mu_3 = \mu_2$, *then*

$$H(x,t;\mu_1,\mu_2,\mu_3;C)$$
$$= O(1)\begin{cases} \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_1 t)^2}{C(t+1)}}, & \text{for } x < \mu_1 t - \sqrt{t+1}, \\ \frac{\log(2+t)t^{-1/4}}{\sqrt{(x-\mu_1 t)(\mu_2 t - x)}}, & \text{for } \mu_1 t + \sqrt{t+1} < x < \mu_2 t - \sqrt{t+1}, \\ \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_2 t)^2}{C(t+1)}}, & \text{for } \mu_2 t + \sqrt{t+1} < x. \end{cases}$$

If $\mu_3 < \mu_1 < \mu_2$, *then*

$$H(x,t;\mu_1,\mu_2,\mu_3;C)$$
$$= O(1)\begin{cases} \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_3 t)^2}{C(t+1)}}, & \text{for } x < \mu_3 t - \sqrt{t+1}, \\ \frac{\log(2+t)t^{-1/4}}{\sqrt{(x-\mu_3 t)(\mu_1 t - x)}}, & \text{for } \mu_3 t + \sqrt{t+1} x < \mu_1 t - \sqrt{t+1}, \\ \frac{(\log(2+t)}{\sqrt{(x-\mu_2 t)t}}, & \text{for } \mu_1 t + \sqrt{t+1} < x < \mu_2 t + \sqrt{t+1}, \\ \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_2 t)^2}{C(t+1)}}, & \text{for } \mu_2 t + \sqrt{t+1} < x. \end{cases}$$

If $\mu_1 < \mu_1 < \mu_3$, *then*

$$H(x,t;\mu_1,\mu_2,\mu_3;C)$$
$$= O(1)\begin{cases} \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_1 t)^2}{C(t+1)}}, & \text{for } x < \mu_1 t - \sqrt{t+1}, \\ \frac{\log(2+t)t^{-1/4}}{\sqrt{(x-\mu_1 t)(\mu_3 t - x)}}, & \text{for } \mu_1 t + \sqrt{t+1} x < \mu_3 t - \sqrt{t+1}, \\ \frac{\log(2+t)t^{-1/4}}{\sqrt{(x-\mu_3 t)(\mu_2 t - x)}}, & \text{for } \mu_3 t + \sqrt{t+1} < x < \mu_2 t + \sqrt{t+1}, \\ \frac{(\log(2+t))^2}{1+t} e^{-\frac{(x-\mu_2 t)^2}{C(t+1)}}, & \text{for } \mu_2 t + \sqrt{t+1} < x. \end{cases}$$

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